Space Bounds for Reliable Storage: Fundamental Limits of Coding

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Abstract

We study the inherent space requirements of reliable storage algorithms in asynchronous distributed systems. A number of recent works have used codes in order to achieve a better storage cost than the well-known replication approach. However, a closer look reveals that they incur extra costs in certain scenarios. Specifically, if multiple clients access the storage concurrently, then existing asynchronous code-based algorithms may store a number of copies of the data that grows linearly with the number of concurrent clients. We prove here that this is inherent. Given three parameters, (1) the data size – D bits, (2) the concurrency level – c, and (3) the number of storage node failures that need to be tolerated – f, we show a lower bound of $\Omega(min(f, c) \cdot D)$ bits on the space complexity of asynchronous distributed storage algorithms. Intuitively, this implies that the asymptotic storage cost is either as high as with replication, namely $O(fD)$, or as high under concurrency as with the aforementioned code-based algorithms, i.e., $O(cD)$.

We further present a technique for combining erasure codes with replication so as to obtain the best of both. We present an adaptive $f - tolerant$ storage algorithm whose storage cost is $O(min(f, c) \cdot D)$. Together, our results show that the space complexity of providing reliable storage in asynchronous distributed systems is $\Theta(min(f, c) \cdot D)$.

1 Introduction

In recent years we see an exponential increase in storage capacity demands, creating a need for big data storage solutions. In this era, distributed storage plays a key role. Data is typically stored on a collection of nodes accessed asynchronously by clients over a network. By storing redundant information, data remains available following failures. The most common approach to achieve this is via replication [4]; in asynchronous settings, $2f + 1$ replicas are needed in order to tolerate f failures [4]. Given the immense size of data, the storage cost of replication is significant. Previous works have attempted to mitigate this cost via the use of erasure codes [3, 5, 9, 6, 15, 8].

Indeed, codes can reduce the storage cost as long as data is not accessed concurrently by multiple clients. For example, if the data size is D bits and a single failure needs to be tolerated, erasure-coded storage ideally requires $(k+2)D/k$ bits for some parameter $k > 1$ instead of the $3D$ bits needed for replication. But as concurrency grows, the cost of erasure-coded storage grows with it: when c clients access the storage concurrently, existing asynchronous code-based algorithms [5, 9, 6, 8] store $O(cD)$ bits in storage nodes or communication channels. Intuitively, this occurs because coded data cannot be reconstructed from a single storage node. Therefore, writing coded data requires coordination – old values cannot be deleted before ensuring that sufficiently many blocks of the new value are in place. This is in contrast to replication, where written values can always be read coherently from a single copy, and so old values may be safely overwritten without coordination.

In this work we prove that this extra cost is inherent: Given three problem parameters: f, c, and D, where f is the number of storage node failures tolerated (client failures are unrestricted), c is the concurrency allowed by the algorithm, and D is the data size, we prove that the storage complexity is $\Theta(min(f, c) \cdot D)$. Asymptotically, this means either a storage cost as high as that of replication, or as high as keeping as many versions of the data as the concurrency level.

Lower bound Our results are proven for emulations of a lock-free multi-reader multi-writer regular register [12, 14]; see Section 2 for definitions. (Interestingly, the lower bound does not hold for the weaker safe register semantics; see Appendix E). We consider algorithms that use (arbitrary) *black-box* encoding schemes, i.e., produce and manipulate code blocks of a given value independently of other values and meta-data; as formalized in Section 3. The storage consists of such code blocks, in addition to possibly unbounded data-independent meta-data, (e.g., timestamps), which we do not count as part of the storage cost. Our black-box assumption excludes storage-reduction techniques like de-duplication, which do require data-dependent meta-data. This assumption holds for many popular storage algorithms [3, 5, 9, 6, 8, 10]. Yet, the question whether there is a more storage-efficient algorithm that circumvents our result by taking stored values into consideration remains open; see further discussion in Sections 3 and 6.

We prove the bound in Section 4: we first use a fundamental pigeonhole argument to show that as long as no ongoing write operation contributes code blocks consisting of D or more bits to the storage, no write operation can complete. We then define a parameter $0 < \ell \leq D$. For a given ℓ, we devise a particular adversary behavior, which we prove drives the storage to a state where either (1) $f + 1$ storage nodes hold at least ℓ bits each, or (2) the storage holds more than $D - \ell + 1$ bits in distinct code blocks for each of c different operations. Now, picking $\ell = D/2$ implies our lower bound.

Algorithm To prove our bound tight, we present in Section 5 a reliable storage algorithm whose storage cost is $O(min(f, c) \cdot D)$. We achieve this by combining the advantages of replication and erasure coding. Our algorithm does not assume any a priori bound on concurrency; rather, it uses erasure codes when concurrency is low and adaptively switches to replication when it is high.

2 Model

We consider an asynchronous fault-prone shared memory system [2, 1, 11] consisting of set $B = \{bo_i, \ldots, bo_n\}$ of n base objects (typically residing at distinct storage nodes) supporting arbitrary atomic *read-modify-write* (RMW) access by clients from some infinite set Π (see Figure 1a). Any f out of n base objects and any number of clients may fail by crashing, for some predefined $f < n/2$.

We study algorithms that emulate a shared *register* [12], which stores a value v from some domain $\mathbb{V}$, where $D = \log_2 |\mathbb{V}|$. Initially, the register holds some initial value $v_0 \in \mathbb{V}$. Clients interact with the emulated register via high-level *read* and *write operations*. A client that performs a write operation is called a *writer*, and a client performing a read is a *reader*.

To distinguish the high-level emulated operations from low-level base object access, we refer to the latter as *RMWs*. We say that RMWs are *triggered* and *respond*, whereas operations are *invoked* and *return*. A (high-level) operation is emulated via a series of trigger and respond *actions* on base objects, starting with the operation's invocation and ending with its return. In the course of an operation, a client triggers RMWs separately on each $bo_i \in B$. The state of each $bo_i \in B$ changes atomically, according to the RMW triggered on it, at some point after the time when the RMW is triggered but no later than the time when the matching response occurs. To distinguish incomplete invocations to the emulated register from incomplete RMWs triggered on base objects, we refer to the former as *outstanding* operations and to the latter as *pending* RMWs.

A parameter c defines the write concurrency level, that is, at most c write operations are outstanding at a given time. We assume that $c < |\mathbb{V}|/2 = 2^{D-1}$. We use standard definitions of algorithms, runs, etc, which, due to space limitations, are deferred to Appendix A. The emulated register must satisfy the following two properties:

Lock-freedom If at some point in a fair run there is an outstanding operation of a correct client, then *some* operation eventually returns.

Regularity Our safety requirement is regularity, which is weaker than atomicity. There are a number of ways to extend Lamport's notion of regularity [12] to multi-writer registers [14]; we use the weakest one for our lower bound and the strongest for our algorithm, (called MWRegWeak and MWRegWO in [14], resp.), as defined in Appendix A. Intuitively, regularity means that a read r returns a value written by either (1) the last write w that completes before r is invoked, or (2) some write that is concurrent to r or to w, or (3) v_0 if no value is written before r.

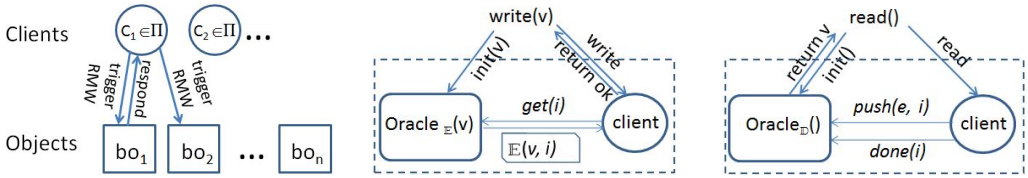
3 Coded Storage Algorithms

In Section 3.1 we present the model and assumptions under which our bound holds, and in Section 3.2 we discuss their relation to existing models and algorithms.

3.1 Model and assumptions

We first give a formal model for coded storage algorithms, then define the notion of storage cost in this model, and finally state our assumptions that the encoding is symmetric and algorithms use it as a black-box.

We consider algorithms that use (arbitrary) encoding schemes, which produce code blocks in some domain $\mathcal{E}$, so that each value is coded independently of other values. The coding scheme is based on two functions: The encoding function $\mathbb{E} : \mathbb{V} \times \mathbb{N} \rightarrow \mathcal{E}$ maps value/natural number pairs to code blocks. We denote the number of bits in block $e \in \mathcal{E}$ as $|e|$. The decoding function $\mathbb{D} : 2^{\mathcal{E}} \rightarrow \mathbb{V} \cup \{\perp\}$ takes as a parameter a set of code blocks and returns a value in $\mathbb{V}$, or $\perp$ in case no value can be decoded. For example, in a replication approach, each block e can be a full value v, so $\mathbb{D}(\{e\})$ simply returns v. Another example is *k-of-n* erasure codes, where for any value v and any subset S of size k of the set $\{e_i \mid e_i = E(v, i),\ 1 \leq i \leq n\}$, $\mathbb{D}(S) = v$. We capture



(a) Clients and base objects. (b) A writer and its oracle. (c) A reader and its oracle.

Figure 1: A model for code-based storage. Encoding and decoding are captured by oracles.

rateless codes [13], in which an encoder can generate a limit-less sequence of blocks, by using $\mathbb{N}$ as the domain for block numbers.

We encapsulate the encoder and decoder into two oracles, $oracle_{\mathbb{E}}$ and $oracle_{\mathbb{D}}$ as illustrated in Figure 1. The interaction with these oracles is as follows:

Definition 1 (Encoding/Decoding Oracles)**.** A *w=write(v)* (*read()*) invocation at a client c_i initializes an $oracle_{\mathbb{E}}(c_i, w)$ ($oracle_{\mathbb{D}}(c_i, w)$, resp.), which expires when w completes. $oracle_{\mathbb{E}}(c_i, w)$ exposes a $get(i)$ operation, which returns $\mathbb{E}(v, i)$ for $i \in \mathbb{N}$; and $oracle_{\mathbb{D}}(c_i, w)$ exposes two operations, $push(e, i)$ and $done(i)$, such that for all $i \in \mathbb{N}$, if c_i calls $done(i)$, then its read operation completes and returns $\mathbb{D}(\{e \mid push(e, i)$ previously occurred$\})$. We omit the parameters c_i, w when they are clear from the context.

Writers produce code blocks via $oracle_{\mathbb{E}}$ and store them in the storage, whereas readers try to obtain enough blocks to decode legal values via $oracle_{\mathbb{D}}$. In addition to code blocks, clients and base objects can store unbounded meta-data, e.g., program counters and timestamps. But to avoid trivializing the problem, the meta-data must be data-independent, as formally defined below.

Information is represented as list of code blocks and meta-data, $\langle e_1, e_2, \ldots, e_k; m\rangle$, where $\forall i$, $e_i \in \mathcal{E}$ and the meta-data m is from some arbitrary domain. The *state of a client* that has an outstanding operation consists of the information stored at the client as well the parameters of its pending RMWs that have not yet taken effect. The state of a client with no outstanding operation is empty. A *base object's state* consists of the information stored at the base object and all the responses of pending RMWs that took effect on it. For a base object bo_i (client c_i), we denote the list of code blocks in bo_i's (c_i's) state at time t in run r as $bo_i^r(t)$ (resp. $c_i^r(t)$).

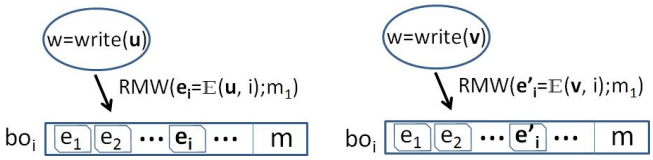
Let $\mathcal{S}$ be an ordered set including all base objects and clients, i.e., $B \cup \Pi$ ordered in some arbitrary way. For $S = \{bo_1, \ldots, bo_k, c_1, \ldots\} \subseteq \mathcal{S}$, $S^r(t)$ is the list of lists $bo_1^r(t), \ldots, bo_n^r(t), c_1^r(t), \ldots$ sorted according to their order in $\mathcal{S}$. A *block instance* $b \in S^r(t)$ is a triple $\langle i, j, e\rangle$ so that e is stored in the j^{th} position in the i^{th} list in S. We refer to the block contents as $b.e$.

Storage cost We count the number of bits stored in blocks in base objects as well as in clients, and neglect meta-data size. Note that oracle states are not counted as part of the storage cost, since we wish to measure the additional space required for making the data available for shared access, beyond its (trivial) existence at its sources and readers.

Definition 2 (Storage Cost)**.** The *storage cost at time t in a run r* is $\Sigma_{b \in \mathcal{S}^r(t)} |b.e|$. The *storage cost of an algorithm A* is the maximum storage cost at any point t in any run r of A.

Assumptions To make sure that the encoding does not leak information using block sizes, we assume *symmetry*, in the sense that output block sizes do not depend on input values. (Otherwise, we could for example, represent three values 0, 1, and 10 using a single coded block e_1 of size at most 1 bit by having $|e_1| = 0$ encode 10). Formally:

Definition 3 (Symmetric Encoding)**.** An encoding function $\mathbb{E}$ is *symmetric* if for every $v, v' \in \mathbb{V}$ and for all $i \in \mathbb{N}$, $|\mathbb{E}(v, i)| = |\mathbb{E}(v', i)|$. We denote $size(i) \triangleq |\mathbb{E}(v, i)|$.



(a) Run r (b) Run r_v

Figure 2: Black-box coding. Runs r and r_v have the same trace except that write w is invoked with u in r and with v in r_v; and each base object bo_i's state (blocks and meta-data) is identical at all times in both runs, except that blocks produced by w's oracle in r are replaced in r_v by the corresponding blocks of v.

Note that different block numbers (of all values) may have different sizes.

We next state our assumption that the storage treats the coding as a black-box. First, we define the notion of a source function, which we shall use to prohibit generation of code blocks by any source other than $oracle_{\mathbb{E}}$:

Definition 4 (Source Function)**.** A function is a *source function* for a run r if it maps every (b, t) s.t. $b \in \mathcal{S}^r(t)$ to a pair $\langle w, i\rangle$ s.t. $b.e$ was returned by $get(i)$ in $oracle_{\mathbb{E}}(w)$.

We use a source function to trace blocks in the storage to operations that produced them. To capture the restriction that the algorithm's decision what to store does not rely on block contents, we stipulate that we can replace the value written by a write operation w in a run r by an arbitrary value v, yielding the same sequence of states and actions, except that all stored block instances whose source is $\langle w, i\rangle$ are replaced with $\mathbb{E}(v, i)$. For clarity, we refer to the operation as w in both runs (see Figure 2).

Definition 5 (Black-Box Coding)**.** An algorithm A is *black-box coding* if for every run r there is a source function $source^r$ for r s.t. for every $w = write(v)$ operation in r, $\forall v \in \mathbb{V}$, there is run r_v satisfying the following:

1. r_v has the same sequences of invocations and returns as r except that w is replaced by $write(v)$ (possibly with no change) and return values of read operations may be different; and
2. client and base object states at every time t in r_v are the same as at time t in r except that the contents of every $b \in \mathcal{S}^r(t)$ s.t. $source^r(b, t) = \langle w, i\rangle$ for some i is replaced by $e' = \mathbb{E}(v, i)$.

In the following, we will only consider source functions satisfying Definition 5. In case multiple such source functions for r exist, we fix an arbitrary one and refer to it as $source^r$.

3.2 Relationship to existing models

Our model captures numerous existing distributed storage algorithms, including ones that use replication [4], and erasure codes [3, 5, 9, 6, 8, 10]. We note that some of them report a storage cost below $O(cD)$ [3, 5, 15, 8]. This is sometimes achieved by assuming periods of synchrony [3]. Other works shift the cost from storage nodes to the network and keep unbounded information in channels [8, 5]. However, since we define parameters and responses of pending RMWs to be part of clients' and base objects' states, information in channels is counted in our storage cost model and hence these algorithms are subject to our bound. The only non-black-box storage algorithm we are aware of is [15], where multiple values are encoded jointly, saving space, but also forfeiting

regular register semantics. It is as of now unclear whether lifting the black-box assumption suffices in order to circumvent our result.

The challenge of providing a lower bound on stored data when meta-data is potentially unbound was previously addressed in the context of byzantine storage [7]. That paper has shown that certain storage algorithms cannot be "amnesic", i.e., cannot "forget" values written to them. Like our black-box assumption, the notion of amnesia was defined in terms of runs. However, it did not yield explicit bound on storage cost.

4 Storage Lower Bound

We now show a lower bound of $O(min(f, c) \cdot D)$ bits on the storage cost of any lock-free algorithm that uses symmetric black-box coding to simulate a regular register:

Theorem 1. *Consider a lock-free algorithm A that uses symmetric black-box coding to simulate a regular register. The storage cost of A is $\Omega(min(f, c) \cdot D)$.*

For the sake of our proof, we quantify the blocks operation w by client c_i contributes to the storage, at base objects and clients other than c_i, and quantify the number of bits stored therein.

Definition 6. Let $S \subset \mathcal{S}$, and consider a time t and an operation w by client c_j in a run r. We define $S^r(t, w) \triangleq \{i \in \mathbb{N} \mid \exists b \in (S \setminus \{c_j\})^r(t)\text{: } source^r(b, t) = \langle w, i\rangle\}$, and $||S^r(t, w)|| \triangleq \Sigma_{i \in S^r(t,w)} size(i)$.

For $I \subseteq \mathbb{N}$, we say that two values $v' \neq v''$ in $\mathbb{V}$ are *I-colliding* if $\forall i \in I$, $\mathbb{E}(v', i) = \mathbb{E}(v'', i)$. We next use the pigeonhole argument and the assumption of symmetric black-box coding in order to show that write operations cannot return until some write stores enough bits in different blocks in every set of $n - f$ base objects.

Claim 1. *Let w be a write operation invoked in a run r of A, and t be a point in r. Consider a set of values $U \subset \mathbb{V}$, $|U| < 2^{D-1}$, and a set of base objects $S \subset \mathcal{S}$. If $||S^r(t, w)|| < D$, then there are two $S^r(t, w)$-colliding values $u \neq u'$ in $\mathbb{V} \setminus U$.*

Proof. Since $|\mathbb{V} \setminus U| > 2^{D-1}$ and $||S^r(t, w)|| < D$, the claim follows from the pigeonhole argument. □

Lemma 1. *Consider a run r of algorithm A that begins with the invocation of c concurrent write operations. Let S be a set of at least $n - f$ base objects and assume that at every time t in r for every operation w in r, $||S^r(t, w)|| < D$. Then no write operation returns in r.*

Proof. Let $W_{ops} = \{w_1, \ldots, w_c\}$ be the set of c concurrent writes invoked in r. Assume by contradiction that there exists a complete write in W_{ops}. Let w be the first such write, and t be the time when it returns. Next we inductively build a sequence of sets of values $U_0, U_1, \ldots, U_c$, where $|U_i| = i$:

- $U_0 = \{\}$
- $\forall i \in \{0, \ldots, c-1\}$, we use U_i to build U_{i+1}. By the lemma premise, $||S^r(t, w_{i+1})|| < D$. Now since $|U_i| < c < 2^{D-1}$, by Claim 1, there are two $S^r(t, w_{i+1})$-*colliding* values $u_{w_{i+1}} \neq u'_{w_{i+1}}$ in $\mathbb{V} \setminus U_i$. We let $U_{i+1} = U_i \cup \{u_{w_{i+1}}\}$.

The set U_c contains exactly c (different) values s.t. for every operation $w_i \in W_{ops}$ there is a value $u_{w_i} \in U_c$ that has a $S^r(t, w_i)$-*colliding* value $u'_{w_i} \in \mathbb{V}$. By applying Definition 5 (c times), there is a run r' that begins with the invocation of c concurrent write operations, in which every operation $w_i \in W_{ops}$ writes u_{w_i} s.t. w returns at time t, and for every operation $w_i \in W_{ops}$,

$S^r(t, w_i) = S^{r'}(t, w_i)$. Next, let clients with outstanding operations and all base objects in $B \setminus S$ fail at time t in r' (note that by assumption $|S| \geq n - f$, so $|B \setminus S| \leq f$), and let some client c_j invoke a solo read operation at time $t + 1$. By lock-freedom, c_j's read operation completes, and by regularity, it returns a value $u \in U_c$ at some time $t' > t$.

Let w' be the operation that writes u in r'. Since u has a $S^r(t, w')$-*colliding* value u' and since $S^r(t, w') = S^{r'}(t, w')$, u and u' are $S^{r'}(t, w')$-*colliding*. By Definition 5, there is a run r'' with the same operations as in r' except that w' writes u' (instead of u) s.t. every client's and base object's state at time t in r' is identical to its state at time t in r'' (note that clients with outstanding operations and all base objects in $B \setminus S$ fail at time t) except that for every block instance $b \in \mathcal{S}^{r'}(t)$ s.t. $source^{r'}(b, t) = \langle w', i \rangle$, $b.e$ is replaced with a block $\mathbb{E}(u', i)$. In particular, states of base objects in S at time t are identical to their states at time t in r' except that for every block instance $b \in S^{r'}(t)$ s.t. $source^{r'}(b, t) = \langle w', i \rangle$, $b.e$ is replaced with a block $\mathbb{E}(u', i)$.

Now since u and u' are $S_{r'}(t, w')$-*colliding*, states of base objects in S at time t in r'' are identical to their states at time t in r'. In addition, since clients with outstanding operations and all base objects in $B \setminus S$ fail at time t, the solo reader c_j cannot distinguish between r' and r'', and thus, it pushes the same blocks to its oracle and calls *done* with the same number in r'' as in r', and therefore, its read operation returns u at time t'' in run r''. However, since the clients invoke write operations with different values in r', u is not written in r''. A contradiction to regularity.

□

Having shown a condition under which write operations cannot complete, we define an (unfair) adversary behavior that takes advantage of this in order to prevent progress. We introduce some notation, and then use it in order to define the adversary. We define a parameter $0 < \ell \leq D$, and for any time t in a run r of algorithm A we define the following sets, as illustrated in Figure 3 in Appendix B. For clarity, from now on we omit the superscript r.

- $C(t)$: the set of outstanding write operations at time t.
- $C^-_\ell(t) = \{w \in C(t) \mid ||\mathcal{S}(t, w)|| \leq D - \ell\}$: The set of write operations each of which has at most $D - \ell$ bits in blocks, produced by its oracle with different numbers, in the storage (excluding the client performing it) at time t.
- $C^+_\ell(t) = C(t) \setminus C^-_\ell(t)$.
- $F_\ell(t) = \{bo_i \in B \mid \Sigma_{b \in \{bo_i\}(t)} |b.e| \geq \ell\}$. Base objects that store blocks that consist (together) of more than ℓ bits at time t. These are base objects that we will "freeze" in our counter-example because they are already "full", i.e., consume enough space for our lower bound.

We fix the parameter ℓ throughout the proof and omit subscript ℓ from the notation. The next observation on storage cost immediately follows from the definitions.

Observation 1. *At any point t in every run r of A, the storage cost is at least $|C^+(t)|(D - \ell + 1)$.*

We next define a particular adversary behavior that schedules actions in a way that prevents progress. Note that the adversary controls the scheduling of client actions and RMW responses.

Definition 7. (*Ad*) At any time t, *Ad* schedules an action as follows:

1. If there is a pending RMW on a base object in $B \setminus F(t)$ by a client performing an operation in $C^-(t)$, then choose the longest pending of these RMWs, allow it to take effect on the corresponding base object, and schedule its response.

2. Else, choose in a fair order an operation by a client $c_i \in \Pi$ and schedule its action (trigger RMW, call its oracle, get response from its oracle, or return), without allowing it to affect the base object yet. By fair order we mean any order in which every client is chosen infinitely often (e.g., $c_1, c_1, c_2, c_1, c_2, c_3 \ldots$).

In other words, *Ad* delays RMWs triggered by operations in $C^+(t)$ (for which the storage already holds $D - \ell$ bits) as well as RMWs on "frozen" base objects in $F(t)$ (which store at least ℓ bits), and fairly schedules all other actions. We demonstrate *Ad*'s behavior in Figure 3 in Appendix B. Though this behavior may be unfair, in every infinite run of *Ad*, every correct client gets infinitely many opportunities to take steps. We use *Ad* to build an unfair run with no progress (no write returns), and then build an indistinguishable fair run to contradict lock-freedom. The following observation immediately follows from the adversary's freezing of base objects in F.

Observation 2. *Assume run r of algorithm A in which the environment behaves like Ad. For each base object bo, if $bo \in F(t)$ at some time t, then $bo \in F(t')$ for all $t' > t$ in r.*

Another consequence of *Ad*'s behavior is captured by the following:

Lemma 2. *Consider a run r of algorithm A. If the adversary behaves like Ad, then for every time t and for every write operation w in r, $||(\mathcal{S} \setminus F(t))(t, w)|| < D$.*

Proof. Assume by way of contradiction that there is time t and write operation w performed by client c_j s.t. $||(\mathcal{S} \setminus F(t))(t, w)|| \geq D$. The definition of $(\mathcal{S} \setminus F(t))(t, w)$ takes into account only blocks returned by w's oracle that are stored outside of $c_j(t)$. Thus, w triggered at least one RMW that has a matching response before time t' in r. Let $t' \leq t$ be the time when the last RMW triggered by w responded, and denote this RMW by rmw and the base object on which rmw was triggered by bo. By *Ad*, $w \in C^-(t' - 1)$, and therefore, by definition, $||(\mathcal{S} \setminus F(t' - 1))(t' - 1, w)|| \leq D - \ell$. Now consider two cases:

- First, rmw adds blocks (possibly overwriting other blocks) with less than ℓ bits to bo. In this case, since bo is the only storage component that changed at time t', $|(\mathcal{S} \setminus F(t'))(t', w)| < D$.
- Second, rmw adds blocks (possibly overwriting other blocks) with at least ℓ bits to bo. In this case, $bo \in F(t')$. Now since $||(\mathcal{S} \setminus F(t' - 1))(t' - 1, w)|| \leq D$, by Observation 2, $F(t' - 1) \subseteq F(t')$, and given $bo \in F(t')$ and it is the only storage component that changed at time t', we get $||(\mathcal{S} \setminus F(t'))(t', w)|| \leq ||(\mathcal{S} \setminus F(t' - 1))(t' - 1, w)|| < D$.

So far we showed that $||(\mathcal{S} \setminus F(t'))(t', w)|| < D$. By Observation 2, and since no RMW by w takes effect after time t', $(\mathcal{S} \setminus F(t''))(t'', w) \subseteq (\mathcal{S} \setminus F(t'))(t', w)$, $\forall t'' \geq t'$. Therefore, we get $||(\mathcal{S} \setminus F(t))(t, w)|| < D$. A contradiction.

□

The next corollary uses Lemmas 1 and 2 in order to conclude that *Ad* can prevent progress of write operations.

Corollary 1. *Consider a run r of algorithm A that begins with the invocation of c concurrent write operations. If the adversary behaves like Ad and $|F(t)| \leq f$ for all t in r, then no write operation returns in r.*

Proof. By Lemma 2, for every time t for every write operation w in r, $||(\mathcal{S} \setminus F(t))(t, w)|| < D$. And since $B \subset \mathcal{S}$, for every time t for every write operation w in r, $||(B \setminus F(t))(t, w)|| < D$. Now since $|F(t)| \leq f$ for every time t in r, $|B \setminus F(t)| \geq n - f$. Therefore, by Lemma 1, no write operation returns in r.

□

We have shown that Ad can prevent completion of write operations in algorithms that store ℓ bits in less than $f+1$ base objects. However, this does not directly imply a storage bound, since Ad is nor fair. In the next lemma we use the fact that lock-freedom must be satisfied in fair runs, i.e., operations invoked by correct clients must eventually complete, in order to blow up the storage. We show that for every algorithm, we can build a run where at some point the algorithm either stores ℓ bits in each of $f+1$ base objects (namely, $\exists t : |F(t)| > f$), or there are c concurrent operations each of which adds at least $D - \ell + 1$ bits to the storage cost (i.e., $|C^+(t)| = c$).

Lemma 3. *There is a run r of A and a time t in r when $|C^+(t)| = c$ or $|F(t)| > f$.*

Proof. Assume by way of contradiction that there is no such run of algorithm A. We build a run r of A with c clients that concurrently write different values, in which the environment behaves like adversary Ad. By the contradiction assumption, $|C^+(t)| < c$ and $|F(t)| \leq f$ for all t in r. We start with the invocation of c concurrent write operations, and allow the run to proceed indefinitely according to Ad. We say that a client c, which performs write operation w, is *stuck* in r if there is a time t in r s.t. for all $t' \geq t$, $w \in C^+(t')$ (and so no RMWs triggered by c take effect after time t). By Observation 2 and the assumption that $|F(t)| \leq f$ for all t, there is a time t_1 in r s.t. for every time $t_2 \geq t_1$, $F(t_1) = F(t_2)$.

Now we build a run r' that is identical to r but every base object $bo \in F(t_1)$ fails at time t_1 ($|F(t_1)| \leq f$), and every stuck client fails after its last RMW takes effect. Since by Ad, RMWs do not take effect on base objects in $F(t_1)$ after time t_1, runs r and r' are indistinguishable to all correct clients and base objects. Now notice that by the adversary's behavior, each correct client in r' gets infinitely many opportunities to trigger RMWs. In addition, since (1) for every correct client c_i in r' there are infinitely many times t when $c_i \in C^-(t)$, (2) Ad picks responses from base objects not in $F(t)$ in the order they are triggered, and (3) there are no correct base objects in $F(t')$ for all $t' > t_1$, every RMW triggered by a correct client on a correct base object has a matching response in r'. Therefore, run r' is fair.

By the contradiction assumption $|C^+(t)| < c$ for all t in r. Therefore, there is at least one client that is not stuck in r, and thus, there is at least one client that is correct in r'. Hence, by lock-freedom, some client eventually completes its write operation in r'. Now since r and r' are indistinguishable to all clients that are correct in both, the same is true in r. However, by Corollary 1, no write operation completes in r. A contradiction.

□

So far we have shown that every algorithm has a run where at some point either ℓ bits are stored in $f+1$ base objects, or there are c concurrent operations each of which adds at least $D-\ell+1$ bits to the storage cost. We now combine this result with Observation 1 to conclude our lower bound:

Proof (Theorem 1). Let $\ell = D/2$. By Lemma 3, there is a run r of A and a time t in r when $|C^+(t)| = c$ or $|F(t)| > f$. If $|F(t)| > f$, then the storage cost at time t in r is $(f+1)\ell = (f+1)D/2 = \Omega(fD)$. Otherwise, $|C^+(t)| = c$, and so by Observation 1, the storage cost at time t in r is at least $c(D-\ell) = cD/2 = \Omega(cD)$. The theorem follows.

□

By picking $\ell = D$, we get a second conclusion from Lemma 3 and Observation 1.

Corollary 2. *The storage cost of any algorithm that uses a black-box coding scheme to simulate a regular lock-free register, and does not store D bits (enough to represent a full replica) in $f+1$ base objects, grows linearly with the concurrency.*

5 Regular Register Emulation

We present a storage algorithm that combines full replication with erasure coding in order to achieve the advantages of both. A k-of-n erasure code takes a value from $\mathbb{V}$ and produces a set S of n blocks from $\mathcal{E}$ s.t. the value can be restored from any subset of S that contains no less than k different blocks. We assume that the size of each block is D/k. $Oracle_{\mathbb{E}}$ and $Oracle_{\mathbb{D}}$ are encapsulated by two functions *encode* and *decode*, respectively: *encode* gets a value $v \in \mathbb{V}$ and returns a set of n ordered elements $W = \{\langle e_1, 1\rangle, \ldots, \langle e_n, n\rangle\}$, where $e_1, \ldots, e_n \in \mathcal{E}$, and *decode* gets a set $W' \subset \mathcal{E} \times \mathbb{N}$ and returns $v' \in \mathbb{V}$ s.t. if $|W'| \geq k$ and $W' \subseteq W$, then $v = v'$. We use $k = n - 2f$. Note that when $k = 1$, we get full replication.

The main idea behind our algorithm is to have base objects store blocks from at most k different *writes*, and then turn to store full replicas. Our algorithm satisfies strong regularity and FW-termination, which is a stronger liveness property than lock-freedom (see Appendix A). In Appendix D, we prove the following:

Theorem 2. *There is an FW-terminating algorithm that simulates a regular register, whose storage cost is* $min((c+1)(2f+k)D/k, (2f+k)2D)$ *bits. Moreover, in a run with a finite number of writes, if all the writers are correct, the storage is eventually reduced to* $(2f+k)D/k$ *bits.*

Notice that k is a parameter of the algorithm, and if we pick $k = f$, then asymptotically the storage cost of our algorithm is $O(min(cD, fD)) = O(min(c, f) \cdot D)$.

The algorithms pseudocode appears in Algorithms 1-3 (Appendix C). The algorithm uses a set of n shared base objects $bo_1, \ldots, bo_n$ each of which holds three fields V_p, V_f, and $storedTS$.
The V_p field holds a set of timestamped code blocks so that the i^{th} block of a value can be stored in the V_p field of object bo_i. The V_f field stores a timestamped replica of a *single* value, (represented as a set of k code blocks). And $storedTS$ holds a timestamp, as explained below.

Write operation and storage efficiency The write operation (lines 3–15) consists of 3 sequentially executed rounds: *read timestamp*, *update*, and *garbage collection*; and, the read consists of one or more sequentially executed *read* rounds. At each round, the client invokes RMWs on all base objects in parallel, and awaits responses from at least $n - f$ base objects. The read rounds of both write and read rely on the readValue routine (lines 23–31) to collect the contents of the V_p and V_f, fields from $n - f$ base objects, as well as to determine the highest $storedTS$ known to these objects. The implementations of the update and garbage collection rounds are given by the update (lines 32–39) and GC (lines 40–45) routines, respectively.

The write implementation starts by encoding v into k code blocks (line 4) and invoking the read round where the client uses the combined contents of the V_p, V_f and $storedTS$ fields returned by readValue to determine the timestamp ts to be stored alongside v's code blocks on the base object; ts is set to be higher than all returned timestamps thus ensuring that the order of the timestamps associated with the stored values is compatible with the order of their corresponding writes, (which is essential for regularity).

The client then proceeds to the update round where it attempts to store the i^{th} code block $\langle e, i\rangle$ of v in $bo_i.V_p$ if the size of $bo_i.V_p$ is less than k (lines 36), or its full replica in $bo_i.V_f$ if ts is higher than the timestamp associated with the value currently stored in $bo_i.V_f$ (line 38). Storing $\langle e, i\rangle$ in $bo_i.V_p$ coincides with an attempt to reduce its size by removing stale code blocks of values whose timestamps are smaller than $storedTS$ (line 36). This guarantees that the size of V_p never exceeds the number of concurrent writes, which is a key for achieving our adaptive storage bound. Lastly, the client updates $bo_i.storedTS$ so as its new value is at least as high as the one returned by the readValue routine. This allows the timestamp associated with the latest complete update to propagate to the base object being written, in order to prevent future writes of old blocks into this base object.

In the write's garbage collection round, the client attempts to further reduce the storage usage by (1) removing all code blocks associated with timestamps lower than ts from both $bo_i.V_p$ and $bo_i.V_f$ (lines 41–42), and (2) replacing a full replica (if it exists) of its written value v in $bo_i.V_f$ with its i^{th} code block $\langle e, i \rangle$ (line 44). It is safe to remove the full replica and values with older timestamps at this point, since once the update round has completed, it is ensured that the written value or a newer written value is restoreable from any $n - f$ base objects. This mechanism ensures that all code blocks except the ones comprising the value written with the highest timestamp are eventually removed from all objects' V_p and V_f sets, which reduces the storage to a minimum in runs with finitely many writes, which all complete. The garbage collection round also updates the $bo_i.storedTS$ field to ensure its value is at least as high as ts.

Key Invariant and read operation The write implementation described above guarantees the following key invariant: at all times, a value written by either the latest complete write or a newer write is available from every set consisting of at least $n - f$ base objects (either in the form of k code blocks in the objects' V_p fields, or in full from one of their V_f fields). Therefore, a read will always be able to reconstruct the latest completely written or a newer value provided it can successfully retrieve k matching blocks of this value. However, a read round may sample different base objects at different times (that is, it does not necessarily obtain an atomic snapshot of the base objects), and the number of blocks stored in V_p is bounded. Thus, the read may be unable to see k matching blocks of any single new value, as long as new values continue to be written concurrently with the read.

Nevertheless, for FW-Termination, the reads are only required to return in runs where a finite number of writes are invoked. Our implementation of read (lines 16–22) proceeds by invoking consecutive rounds of RMWs on the base objects via the readValue routine. After each round, the reader examines the collection of returned values and timestamps to determine if any value has k code blocks and is also associated with a timestamp that is at least as high as $storedTS$ (line 18). If any such value is found, the one associated with the highest timestamp is returned (line 21). Otherwise, the reader proceeds to invoke another round of base object accesses. Note that returning values associated with older timestamps may violate regularity, since they may have been written earlier than the write with timestamp $storedTS$, which in turn may have completed before the read was invoked.

6 Discussion

We studied the inherent space requirements of reliable storage in asynchronous distributed settings. We proved an asymptotic bound of $\Omega(min(f, c) \cdot D)$ for any storage algorithm using a symmetric black-box coding scheme, which produces code blocks of values independently of other values. We then presented an algorithm that combines replication and erasure codes, whose storage cost is $O(min(f, c) \cdot D)$.

Our work leaves open questions for future work. First, it is unclear whether the same lower bound still applies when stored bits are allowed to depend on multiple concurrent write values. The main requirement for extending our proof to general coding is a model that correctly accounts for the information stored in the storage when the clients code jointly. Second, while asymptotically optimal, the constants in our bound are not tight, and it could be interesting to close this gap. Finally, we believe that our model and adversary definitions can yield additional lower bounds.

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A Formal definitions

An *algorithm* defines the behavior of clients as deterministic state machines, where state transitions are associated with actions such as RMW trigger/response. A *configuration* is a mapping to states from system components, i.e., clients and base objects. An *initial configuration* is one where all components are in their initial states.

A *run* of algorithm A is a (finite or infinite) alternating sequence of configurations and actions, beginning with some initial configuration, such that configuration transitions occur according to A. For a run r, *trace(r)* is the subsequence of r consisting of all the operation invocation and returns in r. We use the notion of time t during a run r to refer to the configuration reached after the t^{th} action in r. A *run fragment* is a contiguous subsequence of a run starting and ending with a configuration. We assume that runs are *well-formed*, in that each client's first action is an invocation, and a client has at most one outstanding operation at any time.

We say that a base object or client is *faulty* in a run r if it fails any time in r, and otherwise, it is *correct*. A run is *fair* if (1) for every RMW triggered by a correct client on a correct base object, there is eventually a matching response, (2) every correct client gets infinitely many opportunities to trigger RMWs.

Liveness There is a range of possible liveness conditions, which need to be satisfied in fair runs. A *wait-free* object is one that guarantees that every correct client's operation completes, regardless of the actions of other clients. A *lock-free* object guarantees progress: if at some point in a run there is an outstanding operation of a correct client, then *some* operation eventually completes. An *FW-terminating* [1] register is one that has wait-free *write* operations, and in addition, if there are finitely many *write* invocations in a run, then every *read* operation completes.

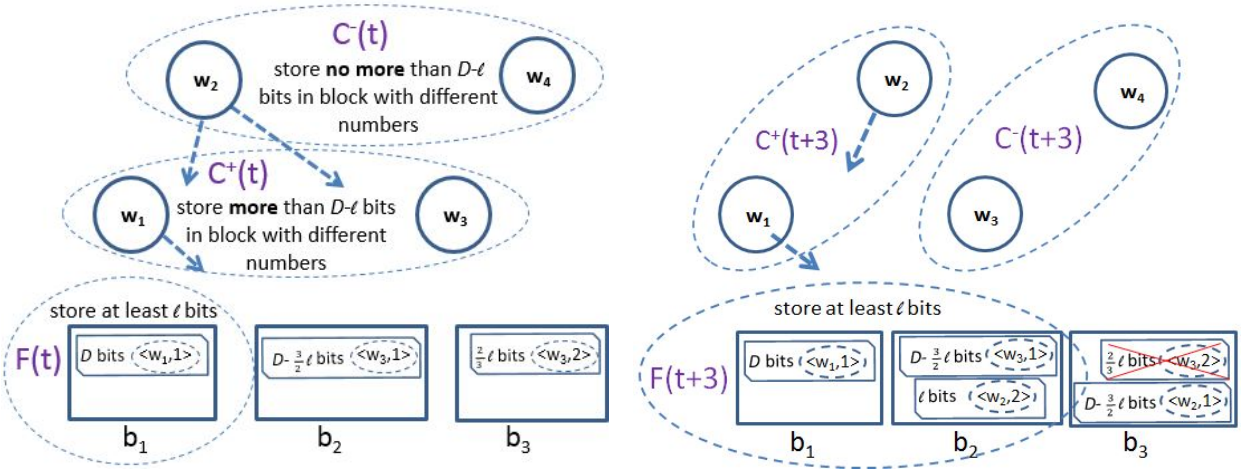
Safety In order to define regularity, we first introduce some terminology: Operation op_i *precedes* operation op_j in a run r, denoted $op_i \prec_r op_j$, if op_i's return occurs before op_j's invoke in r. Operations op_i and op_j are *concurrent* in a run r if neither one precedes the other. A run with no concurrent operations is *sequential*. Two runs are *equivalent* if every client performs the same sequence of operations in both, where operations that are outstanding in one can either be included in or excluded from the other. A *linearization* of a run r is an equivalent sequential run that preservers r's operation precedence relation and the object's sequential specification. The sequential specification for a register is as follows: A read returns the latest written value, or v_0 if none was written. A *write* w in a run r is *relevant* to a *read* rd in r [14] if $rd \not\prec_r w$; *rel-writes*(r, rd) is the set of all *writes* in r that are relevant to rd.

Following Lamport [12], we consider a hierarchy of safety notions. Lamport [12] defines *regular* and *safe* single-writer registers. Shao et al. [14] extend Lamport's notion of regularity to MWMR registers, and give four possible definitions. Here we use two of them. The first is the weakest definition, and we use it in our lower bound proof. The second, which we use for our algorithm, is the strongest definition that is satisfied by ABD [4] in case readers do not change the storage (no *write-back*):

A MWMR register is *weakly regular*, (called *MWRegWeak* in [14]), if for every run r and *read* rd that returns in r, there exists a linearization of the subsequence of r consisting of rd and the writes in r. A MWMR register is *strongly regular*, (called *MWRegWO* in [14]), if it satisfies weak regularity and the following condition: For all *reads* rd_1 and rd_2 that return in r, for all writes w_1 and w_2 in *rel-writes*$(r, rd_1) \cap$ *rel-writes*(r, rd_2), it holds that $w_1 \prec_{L_{rd_1}} w_2$ if and only if $w_1 \prec_{L_{rd_2}} w_2$.

We extend the safe register definition and say that a MWMR register is *strongly safe* if there exists a linearization σ_w of the subsequence of r consisting of the *write* operations in r, and for every *read* operation rd that has no concurrent *writes* in r, it is possible to add rd at some point in σ_w so as to obtain a linearization of the subsequence of r consisting of the write operations in r and rd.

B Illustration of Ad's behavior



(a) Time t (b) Time $t+3$

Figure 3: Example scenario in run of a storage algorithm with adversary Ad. Writers $c_1, \ldots, c_4$ perform $w_1, \ldots, w_4$; dashed lines represent triggered RMWs with no matching responses, and blocks are tagged (in dashed circles) by their sources. In this example, $2D/5 < \ell < D$. (a) Only w_2 and w_4 are in $C^-(t)$ at time t, where w_4 has no pending RMWs and w_2 has one triggered RMW on $b_1 \in F(t)$ and one triggered RMW on $b_3 \notin F(t)$. Therefore, by the first rule, Ad schedules the response on the RMW triggered by w_2 on b_3. In this example w_2 overwrites w_3's block in b_3, thus w_3 moves from C^+ to C^-. Then, at time $t+1$, no response can be scheduled by rule 1 (no operation in $C^-(t+1)$ has a pending RMW on a base object in $N \setminus F(t+1)$), so by rule 2, Ad chooses w_2 and lets it trigger an RMW on base object b_2. (b) Now since w_2 is the only operation that has a pending RMW on a base object not in $F(t+2)$, Ad schedules the response on the RMW triggered by w_2 on b_2 at time $t+2$. In this example w_2 adds a block with ℓ bits to b_2. Thus, c_2 is included in $C^+(t+3)$. In addition, b_2 stores more than ℓ bits at time $t+3$, so it belongs to $F(t+3)$.

C Definitions and Pseudocode

Here is the pseudocode of the algorithm in Section 5.

Algorithm 1 Definitions.

1: $TimeStamps = \mathbb{N} \times \Pi$, with selectors num and c, ordered lexicographically.
2: $Pieces = (\mathcal{E} \times \mathbb{N})$
3: $Chunks = Pieces \times TimeStamps$, with selectors val, ts
4: $encode : \mathbb{V} \to 2^{\mathcal{E} \times \{1,2,\ldots,n\}}$, $decode : 2^{\mathcal{E} \times \{1,2,\ldots,n\}} \to \mathbb{V}$
5: s.t. $\forall v \in \mathbb{V}$, $encode(v) = \{\langle *, 1\rangle, \ldots, \langle *, n\rangle\} \wedge$
6: $\forall W \in 2^{\mathcal{E} \times \mathbb{N}}$, if $W \subseteq encode(v) \wedge |W| \geq k$, then $decode(W) = v$

7: **base objects:**
8: $\forall i \in \{1, \ldots, n\}$, $bo_i = \langle storedTS, V_p, V_f\rangle$ s.t. $V_f, V_p \subset Chunks$, and $storedTS \in TimeStamps$,
9: initially $\langle\langle 0, 0\rangle, \{\langle\langle 0, 0\rangle, \langle v_{0_i}, i\rangle\rangle\}, \{\}\rangle$.

Algorithm 2 Strongly regular register emulation. Algorithm for client c_j.

1: **local variables:**
2: $storedTS, ts \in TimeStamp$, $WriteSet \in Pieces$
3: **operation** $Write(v)$
4: $WriteSet \leftarrow encode(v)$
5: $\langle storedTS, ReadSet\rangle \leftarrow readValue()$ ▷ **round 1:** read timestamps
6: $n \leftarrow max(storedTS.num,\ max\{n' \mid \langle\langle n', *\rangle, *\rangle \in ReadSet\})$
7: $ts \leftarrow \langle n+1, j\rangle$
8: || **for** i=1 to n ▷ **round 2:** update
9: $update(bo_i, WriteSet, ts, storedTS, i)$
10: **wait** for $n - f$ responses
11: || **for** i=1 to n ▷ **round 3:** garbage collect
12: $GC(bo_i, WriteSet, ts, i)$
13: **wait** for $n - f$ responses
14: **return** "ok"
15: **end**
16: **operation** $Read()$
17: $\langle storedTS, ReadSet\rangle \leftarrow readValue()$
18: **while** $\nexists ts \geq storedTS$ s.t. $|\{\langle ts, v\rangle \mid \langle ts, v\rangle \in ReadSet\}| \geq k$
19: $\langle storedTS, ReadSet\rangle \leftarrow readValue()$
20: $ts' \leftarrow \max_{ts \geq storedTS}(|\{\langle ts, v\rangle \mid \langle ts, v\rangle \in ReadSet\}| \geq k)$
21: **return** $decode(\{v \mid \langle ts', v\rangle \in ReadSet\})$
22: **end**

Algorithm 3 Functions used in strongly regular register emulation.

23: **procedure** $readValue()$
24: $ReadSet \leftarrow \{\}$, $T \leftarrow \{\}$
25: || **for** i=1 to n
26: $tmp \leftarrow read(bo_i)$
27: $ReadSet \leftarrow ReadSet \cup tmp.V_f \cup tmp.V_p$
28: $T \leftarrow T \cup \{tmp.storedTS\}$
29: **wait** for $n - f$ responses
30: **return** $\langle max(T), ReadSet\rangle$
31: **end procedure**
32: **update**$(bo, WriteSet, ts, storedTS, i) \triangleq$
33: **if** $ts \leq bo.storedTS$
34: **return**
35: **if** $|bo.V_p| < k$ ▷ write a piece and remove old pieces
36: $bo.V_p \leftarrow bo.V_p \setminus \{\langle ts', v\rangle \in bo.V_p \mid ts' < storedTS\} \cup \{\langle ts, \langle e, i\rangle\rangle \mid \langle e, i\rangle \in WriteSet\}$
37: **else if** $bo.V_f = \{\} \vee \exists ts' < ts : \langle ts', *\rangle \in bo.V_f$ ▷ write a full replica
38: $bo.V_f \leftarrow \{\langle ts, \langle e, j\rangle\rangle \mid \langle e, j\rangle \in WriteSet \wedge j \in \{1, \ldots, k\}\}$
39: $bo.storedTS \leftarrow max(bo.storedTS, storedTS)$
40: **GC**$(bo, WriteSet, ts, i) \triangleq$
41: $bo.V_p \leftarrow \{\langle ts', v\rangle \in bo.V_p | ts' \geq ts\}$ ▷ keep only new pieces
42: $bo.V_f \leftarrow \{\langle ts', v\rangle \in bo.V_f | ts' \geq ts\}$
43: **if** $\langle ts, *\rangle \in bo.V_f$ ▷ if V_f holds a full replica of my write
44: $bo.V_f \leftarrow \{\langle ts, \langle e, i\rangle\rangle \mid \langle e, i\rangle \in WriteSet\}$ ▷ keep only one piece of it
45: $bo.storedTS \leftarrow max(bo.storedTS, ts)$

D Algorithm's Correctness Proofs

Here we prove correctness of the algorithm in Section 5. Note that we prove here that the algorithm satisfies strong regularity and FW-termination, which are stronger safety and liveness properties than the one used in our lower bound (see Appendix A for details).

We start by proving the storage cost.

Observation 3. *For every run of the algorithm, for every base object bo_i, $bo_i.ts$ monotonically increasing.*

Lemma 4. *Consider a run r of the algorithm, and two* writes w_1, w_2*, where w_1 writes with timestamp ts_1. If $w_1 \prec_r w_2$, then w_2 sets its $\hat{ts}$, to a timestamp that is not smaller than ts_1.*

Proof. By Observation 3, for each base object bo, $bo.ts$ is monotonically increasing. Therefore, after w_1 finishes the garbage collection phase, there is a set S consisting of $n - f$ base objects s.t. for each $bo_i \in S$, $bo_i.ts \geq ts$. Recall that $n = 2f + k$, thus every two sets of $n - f$ base objects have at least one base object in common. Therefore, w_2 gets a response from at least one base object in S in its first phase, and thus sets $\hat{ts} = ts'$ s.t. $ts' \geq ts$.

□

Lemma 5. *For any run r of the algorithm, for any base object bo at any time t in r, $bo.V_p$ does not store more than one piece of the same write.*

Proof. The *writes* perform the second phase at most one time on each base object bo, and in each update they store at least one piece in $bo.V_p$. And since they does not store in $bo.V_p$ during the third phase, the lemma follows.

□

Lemma 6. *Consider a run r of the algorithm in which the maximum number of concurrent writes is $c < k - 1$. Then the storage at any time in r is not bigger than $(2f + k)(c + 1)D/k$ bits.*

Proof. Recall that we assume that $n = 2f + k$ and the size of each piece is D/k. Thus it suffices to show that there is no time t in r s.t. some base object stores more than $c + 1$ pieces at time t.

Assume by way of contradiction that the claim is false. Consider the time t when some $bo \in N$ stores $c + 2$ pieces for the first time. Notice that $|bo.V_p| \leq c + 1 < k$ till time t, and therefore, $bo.V_p$ does not contain more then one piece from the same write, and $bo.V_f = \perp$ till time t'. Now consider the write w that was invoked last among all the writes that store pieces in $bo.V_p$ at time t, denote its piece by p. Since bo stores $c + 2$ pieces at time t', by Lemma 6, there must be two writes w_1 and w_2 whose pieces p_1, p_2 are stored at time t in $bo.V_p$, and both returns before w is invoked. Denote their timestamps ts_1 and ts_2, and assume without loss of generality that $ts_1 > ts_2$. By Lemma 4, w sets its $\hat{ts}$ to ts' s.t. $ts' \geq ts_1 > ts_1$. Now consider two cases. First, if p was added before p_2, then $bo.ts > ts_2$ when p_2 was added. A contradiction. Otherwise, p was added after p_2. Thus, p_2 was deleted in line 36 of the update when p was added. A contradiction.

□

Lemma 7. *The storage is never more than $(2f + k)2D$ bits at any time t in any run r of the algorithm.*

Proof. Each base object stores no more than $2k$ pieces at any time t in r. The lemma follows.

□

Lemma 8. *Consider a run r of the algorithm with finite number of writes, in which all writes correct. Then the storage is eventually reduced to $(2f + k)D/k$ bits.*

Proof. Consider a *write* w with the biggest timestamp ts in r. Since w is correct, and since *writes* are wait-free, w returns, and eventually performs *free* on every base object. Consider a base object bo s.t. w performs *free* on bo at time t. Notice that w deletes all pieces with smaller timestamps than ts and set $bo.ts = ts$ at time t. Now recall that bo ignore all updates with timestamp less than $bo.ts$, and therefore, bo store only w's piece at any time after time t. The lemma follows.

□

From Lemmas 6, 7, and 8 we get:

Corollary 3. *The storage of the algorithm is bounded by* $(2f+k)2D$ *bits, and in runs with at most* $c < k$ *concurrent writes the storage is bounded by* $(c+1)D/k$ *bits. Moreover, in a run with a finite number of writes, if all the writes are correct, the storage is eventually reduced to* $(2f+k)D/k$ *bits.*

We next prove the liveness property.

Lemma 9. *Consider a fair run r of the algorithm. Then every* write *w invoked by a correct client c_i eventually completes.*

Proof. Consider a correct client c_i. The *write* w is divided into three phase s.t. in each phase, c_i invokes operations on all the base objects, and waits for $n-f$ responses. The run r is fair, so every action invoked by c_i on a correct base object eventually returns, and no more than f base objects fail in r. Therefore, eventually c_i receives $n-f$ responses in each of the phases and returns.

□

Observation 4. *When a piece from $bo.V_p$ is deleted, $bo.ts$ is increased.*

Lemma 10. *If at time t, c_i completes the second phase of write with timestamp ts, then for every $t' > t$ for every $S \subseteq N$ s.t. $|S| \geq n-f$, exist write w with $ts' \geq ts$ s.t. at least k pieces of w are stored in S.*

Proof. Consider time t'. Let $\hat{ts}$ be the highest timestamp written by a write w that completed the second phase by time t. It is sufficient to show the lemma hold for $\hat{ts}$.

First note that $\forall bo$, $bo.ts \leq \hat{ts}$ before time t, because no write with a larger timestamp than $\hat{ts}$ started the third phase. This means that w's *update* left at lest one piece in which bo it occurred. Now consider a set S of $n-f$ base objects, and since $n = 2f+k$, w's *update* occurred in set S' that contains at least k base objects in S.

If w wrote to V_p, it was not overwritten by time t, because (1) no other write began *free* with timestamp bigger than $\hat{ts}$, and (2) since there is no base object bo s.t. $bo.ts \geq \hat{ts}$, no write delete w's piece in the second phase. Therefore if w wrote to V_p in all base objects in S', the lemma holds.

Otherwise, w wrote k pieces to V_f in base objects in some set $S'' \subseteq S'$. Consider two cases: First, there is base object $bo' \in S''$ s.t. some write overwritten w's pieces in $bo'.V_f$ before time t. Since there is no write with timestamp bigger than $\hat{ts}$ that started the third phase before time t, it is guarantee that k pieces with timestamp $ts' > \hat{ts}$ stored in $bo'.V_f$ at time t, and the lemma holds. Else, since w's pieces stored in $S' \setminus S''$ does not overwritten before time t, the lemma holds (no matter if w performed the third phase or not).

□

Invariant 1. *For any run r of the algorithm, for any time t in r, for any set S of $n-f$ base objects. Let $\hat{ts_s} = max\{bo.ts \mid bo \in S\}$. Then there is a timestamp $ts' \geq \hat{ts_s}$ s.t. there are at least k different pieces associated with ts' in S.*

Proof. We prove by induction. **Base:** the invariant holds at time 0. **Induction:** Assume that the induction holds before the t^{th} action is scheduled, we show that it holds also at time t. Assume that the t^{th} action is RMW on a base object bo, and consider any set S of $n-f$ base objects. If $bo \notin S$ then the invariant holds. Else, consider the two possible RMW actions:

- The t^{th} action is *update*. If no pieces are deleted, the invariant holds. If $bo.ts$ is increased, then consider the write with timestamp ts that is the the biggest timestamp among all writes that complete the second phase before time t. Notice that $bo.ts \leq ts$ at time t, and by Lemma 10, the invariant holds. The third option is that a piece p with timestamp $ts' > bo.ts$ of a *write* w is deleted and $bo.ts$ is not increased. Note that by Observation 4, such piece can be deleted only from $bo.V_f$, and since p is overwritten by k pieces with bigger timestamp, the invariant holds.
- The t^{th} action is *free*. If $bo.ts$ is not changes, then the invariant holds. Else, Consider the write with the biggest timestamp ts among all writes that complete the second phase before time t. Note that $bo.ts$ is set to a timestamp $ts' \leq ts$, so by Lemma 10, the invariant holds.

□

Lemma 11. *Consider a fair run r of the algorithm. If there is a finite number of write invocations in r, then every read operation rd invoked by a client c_i eventually returns.*

Proof. Assume by way of contradiction that rd does not return in r. By Lemma 9, the *writes* are wait-free, and since the number of *write* invocations in r is finite, there is a time t in r s.t. no *write* performs actions after time t. Therefore, any *read* that invokes $readValue()$ procedure after time t receives a set S of values that is stored in a set of $n-f$ base objects at time t. By invariant 1, there is a timestamp ts s.t. there is at least k different pieces in S associated with ts, and $ts > bo.ts$ for all $bo \in S$. Now since the every correct *read* rd invokes $readValue()$ infinitely many times in r, rd returns. A contradiction.

□

The next corollary follows from Lemmas 9, 11.

Corollary 4. *The algorithm satisfies the WF-termination property.*

We now prove that the algorithm satisfies strong regularity.

Definition 8. For every run r, σ_r is a sequential run s.t. the *writes* in r are ordered in σ_r by their timestamp, and every *read* in r that returns a value associate with timestamp ts, is ordered in σ_r immediately after the *write* that is associate with timestamp ts.

For simplicity we say the that v_0 was written by *write* w_0 that associated to timestamp 0 at time 0.

Lemma 12. *Consider a run r, and a read rd that returns a value v. Consider also the timestamp ts' that rd obtains in line 20 (Algorithm 2). Then v is the value written by a* write *associated with timestamp ts' or v_0 if $ts' = 0$.*

Proof. By the code, if $ts' = 0$, then rd returns v_0. Now notice that rd obtains at least k different pieces associated with timestamp ts', thus by decode definition, rd returns v.

□

Corollary 5. *For every run r, σ_r satisfies the sequential specification.*

Observation 5. *Consider a* write *w that obtains ts and $\hat{ts}$ in the first phase, then $ts > \hat{ts}$.*

Lemma 13. *For every run r, for every two writes w_1, w_2 with timestamp ts_1, ts_2. If w_2 was invoked after w_1 finished the second phase, then $ts_1 < ts_2$.*

Proof. First notice that for every base object *bo*, if a *write* w overwrites pieces of a *write* w' in bo, V_f, that w' timestamp is bigger than w''s. And by Observation 5, if w deletes w''s piece from $bo.V_p$, then it stores a piece with bigger timestamp than w''s timestamp. Therefore, the maximal timestamp in each base object is monotonically increasing. Now recall that in the second phase w_1 performed *update* on $n - f$ base object, and notice that after w_1 performs *update* on base object *bo* the maximal timestamp in *bo* is at lest as big as ts_1. Now since two sets of $n - f$ base object have at least one base object in common, w_2 picks $ts > ts_1$.

□

Lemma 14. *For every run r, for every two writes w_1, w_2 in r, if $w_1 \prec_r w_2$, then w_2 is not ordered before w_1 in σ_r.*

Proof. Follows immediately from Lemma 13.

□

Lemma 15. *For every run r, for every read rd and write w_1, if $rd \prec_r w_1$, then w_1 is not ordered before rd in σ_r.*

Proof. Assume that rd returns value that is associated with timestamp ts belonging to some *write* w, and w_1 is associated with timestamp ts_1. Since rd returns w's value, w begins the third phase before rd returns. And since w_1 was invoked after rd returns, w_1 was invoked after w's second phase. Therefore, by Lemma 13, $ts_1 > ts$, and thus w_1 is ordered after w in σ_r. Recall that by the construction of σ_r, rd is ordered immediately after w in σ_r, hence, rd is ordered before w_1 in σ_r.

□

Lemma 16. *For every run r, for every read rd and write w_1, if $w_1 \prec_r rd$, then rd is not ordered before w_1 in σ_r.*

Proof. Consider a *write* w_1 with timestamp ts_1 and a *read* rd s.t. $w_1 \prec_r rd$. Assume by way of contradiction that rd is ordered before w_1 in σ_r. Then rd returns a value with a timestamp ts that is associated with a *write* w that is ordered before w_1 in σ_r. By the construction of σ_r, $ts_1 > ts$. Now since w_1 completed the third phase before rd invoked, and since by Observation 3, for each *bo*, *bo.ts* is monotonically increasing, when rd invoked, for every set S of $n - f$ base objects, the maximal *bo.ts* of all $bo \in S$ is bigger than or equal to ts_1, and thus bigger than ts. Therefore rd set $\hat{ts}$, in the first phase, to timestamp bigger than ts, and thus does not return w's value. A contradiction.

□

The next corollary follows from Corollary 5, and Lemmas 14, 15, 16.

Corollary 6. *The algorithm simulates a strongly regular register.*

The following theorem stems from Corollaries 3, 4, and 6.

Theorem 2. *There is a FW-terminating algorithm that simulates a strongly regular register, which storage is bounded by $(2f+k)2D$ bits, and in runs with at most $c < k$ concurrent writes, the storage is bounded by $(c+1)D/k$ bits. Moreover, in a run with a finite number of writes, if all the writes are correct, the storage is eventually reduced to $(2f + k)D/k$ bits.*

E A (Simple) Safe and Wait-free Algorithm

We present here a simple storage-efficient algorithm that ensures *safe* semantics, but not *regularity*. Although this algorithm has no practical use, it shows that the impossibility result of Section 4 does not apply to a weaker safety property.

E.1 Algorithm

This algorithm simulates a wait-free and strongly safe MWMR register (see Appendix A) using erasure codes (see Section 5). It stores exactly n pieces of the data, one in each base object. The algorithm's definitions are presented in Algorithm 4, and the algorithm of client c_j can be found in Algorithm 5.

We define $Timestamps$ to be the set of timestamps $\langle num, c\rangle$, s.t. $num \in \mathbb{N}$ and $c \in \Pi$, ordered lexicographically. We define $Pieces$ to be the set of pairs consisting of an element from $\mathbb{E}$ (possible outputs of the *encode* function) and a number, and $Chunks = Pieces \times Timestamps$. Each base object bo_i stores exactly one value from $Chunks$, initially $\langle\langle v_{0_i}, i\rangle, \langle 0, 0\rangle\rangle$, where v_{0_i} is the i^{th} piece of v_0.

Since memory is fault-prone, actions are triggered in parallel on all base objects. This parallelism is denoted using ||**for** in the code. Operations then wait for $n - f$ base objects to respond. Recall that $n = 2f + k$, so every two sets of $n - f$ base objects have at least k pieces in common. Thus, if a write completes after storing pieces on $n - f$ base objects, a subsequent read accessing any $n - f$ base objects finds k pieces of the written value (as needed for restoring the value), provided that they are not over-written by later writes.

A *write*(v) operation (lines 1–9) first produces n pieces from v using *encode*, then reads from $n - f$ base objects to obtain a new timestamp, and finally, tries to store every piece together with the timestamp at a different base object. For every base object *bo*, c_j triggers the *update* RMW function, which overwrites *bo* only if c_j's timestamp is bigger than the timestamp stored in *bo*.

A *read* (lines 13–19) reads the values stored in $n - f$ base objects, and then tries to restore valid data as follows. If c_j reads at least k values with the same timestamp, it uses the *decode* function, and returns the restored value. Otherwise, it returns v_0. The latter occurs only if there are outstanding *writes*, that had updated fewer than $n - f$ base objects before the reader has accessed them. Therefore, these *writes* are concurrent with c_j's *read*, and by the safety property, any value can be returned in this case. The algorithm's correctness is formally proven in Appendix E.2.

Algorithm 4 Definitions.

1: $TimeStamps = \mathbb{N} \times \Pi$, with selectors num and c, ordered lexicographically.
2: $Pieces = (\mathbb{E} \times \mathbb{N})$
3: $Chunks = Pieces \times TimeStamps$, with selectors val, ts
4: $encode : \mathbb{V} \to 2^{\mathbb{E}\times\{1,2,\ldots,n\}}$, $decode : 2^{\mathbb{E}\times\{1,2,\ldots,n\}} \to \mathbb{V}$
5: s.t. $\forall v \in \mathbb{V}$, $encode(v) = \{\langle *, 1\rangle, \ldots, \langle *, n\rangle\} \wedge$
6: $\forall W \in 2^{\mathbb{E}\times\mathbb{N}}$, if $W \subseteq encode(v) \wedge |W| \geq k$, then $decode(W) = v$

Algorithm 5 Safe register emulation. Algorithm for client c_j.

```
1: operation write(v)
2:     W ← encode(v)
3:     R ← readValue()
4:     ts ← ⟨max({ts | ⟨ts, ∗⟩, ∗⟩ ∈ R}) + 1, j⟩
5:     || for all ⟨v, i⟩ ∈ W
6:       update(bo_i, ⟨v, i⟩, ts) ▷ trigger RMW on bo_i
7:     wait for n − f responses
8:     return "ok"
9: end
10: update(bo, w, ts) ≜
11:     if ts > bo.ts
12:       bo ← ⟨w, ts⟩
13: operation read()
14:     R ← readValue()
15:     if ∃ts s.t. |{v | ⟨ts, v⟩ ∈ R}| ≥ k
16:       ts′ ← ts s.t. |{v | ⟨ts, v⟩ ∈ R}| ≥ k
17:       return decode({v | ⟨ts′, v⟩ ∈ R})
18:     return v_0
19: end
20: procedure readValue()
21:     R ← {}
22:     || for i=1 to n
23:       R = R ∪ read(bo_i)
24:     wait until |R| ≥ n − f
25:     return R
26: end procedure
```

E.2 Correctness proof

Lemma 17. *The storage of the algorithm is nD/k.*

Proof. The size of each piece is D/k. We have n base objects, and each base object stores exactly one piece.

□

Lemma 18. *The algorithm is wait-free.*

Proof. There are no loops in the algorithm, and the only blocking instructions are the waits in lines 7 and 24. In both cases, clients wait for no more than $n - f$ responses, and since no more than f base objects can fail, clients eventually continue. Therefore, a client that gets the opportunity to perform infinitely many actions completes its operations.

□

We now prove that the algorithm satisfies strongly safety. We relay on the following single observation.

Observation 6. *The timestamps in the base objects are monotonically increasing.*

Definition 9. For every run r, we define the sequential run σ_{w_r} as follows: All the completed write operations in r are ordered in σ_{w_r} by their timestamp.

Lemma 19. *For every run r, the sequential run σ_{w_r} is a linearization of r.*

Proof. Since σ_{w_r} has no read operations, the sequential specification is preserved in σ_{w_r}. Thus, we left to show the real time order: For every two completed *writes* w_i, w_j in r, we need to show that if $w_i \prec_r w_j$, then $w_i \prec_{\sigma_r} w_j$.

Denote w_i's timestamp by ts. By Observation 6, at any point after w_i's return, at least $n - f$ base objects store timestamps bigger than or equal to ts. When w_j picks a timestamp, it chooses a timestamp bigger than those it reads from $n - f$ base objects. Since, $n > 2f$, w_j picks a timestamp bigger than ts, and therefore w_j is ordered after w_i in σ_{rd}.

□

Definition 10. For every run r, for every *read* rd that has no concurrent *write* operations in r, we define the sequential run $\sigma_{r_{rd}}$ by adding rd to σ_{w_r} after all the writes that precede it in r.

In order to show that the algorithm simulates a safe register, we proof in Lemmas 20 and 21 that the real time order and sequential specification respectively, are preserved in $\sigma_{r_{rd}}$.

Lemma 20. *For every run r, for every read rd that has no concurrent write operations in r, $\sigma_{r_{rd}}$ preserves r's operation precedence relation (real time order).*

Proof. By Lemma 19, the order between the writes in $\sigma_{r_{rd}}$ are preserved, and by construction of σ_{rd} the order between rd and write operations is also preserved.

□

Lemma 21. *Consider a run r and any read rd that has no concurrent writes in r. Then rd returns the value written by the write with the biggest timestamp that precedes rd in r, or v_0 if there is no such* write.

Proof. In case there is no *write* before rd in r, since there are also no writes concurrent with rd, rd reads pieces with timestamp $\langle 0, 0\rangle$ from all base objects, and thus, returns v_0. Otherwise, let w be the *write*(v) associated with the biggest timestamp ts among all the *writes* invoked before rd in r. Let t be the time when rd is invoked. Recall that rd has no concurrent *writes*, so all the writes invoked before time t complete before time t and store there pieces in $n - f$ base objects unless the base objects already hold a higher timestamp. By Observation 6 and the fact that w has the highest timestamp by time t, we get that at time t there are at least $n - f$ base objects that store a piece of v. Since $n = 2f + k$, every two sets of $n - f$ base objects have at least k base objects in common. Therefore, rd reads at least k pieces of v, and thus, restores and returns v.

□

Corollary 7. *There exists an algorithm that simulates a safe wait-free MWMR register with a worst-case storage cost of $nD/k = (2f/k + 1)D$.*