Salem numbers of trace -2, and a conjecture of Estes and Guralnick

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Abstract

In 1993 Estes and Guralnick conjectured that any totally real separable monic polynomial with rational integer coefficients will occur as the minimal polynomial of some symmetric matrix with rational integer entries. They proved this to be true for all such polynomials that have degree at most 4.

In this paper, we show that for *every* $d \geq 6$ there is a polynomial of degree d that is a counterexample to this conjecture. The only case still in doubt is degree 5.

One of the ingredients in the proof is to show that there are Salem numbers of degree $2d$ and trace -2 for every $d \geq 12$.

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1. Introduction

1.1. Salem numbers of trace -2

A *Salem number* is a real algebraic integer $\tau > 1$, conjugate to its reciprocal $1/\tau$, of degree at least 4, and with all conjugates other than τ and $1/\tau$ lying on the unit circle in the complex plane. See [20] for a recent survey. Smyth [19] considered the problem of finding Salem numbers of negative trace, and found examples that had trace -1 of every (even) degree greater than or equal to 8. He asked how small the trace could be. McMullen [18] raised the question of whether or not there are *any* Salem numbers of trace less than -1, still none being known at that time. The first examples having trace -2 were found by McKee and Smyth [12], and indeed they showed that there are Salem numbers of every trace [13].

In this paper we shall show that there are Salem numbers of trace -2 for every (even) degree greater than or equal to 24: Proposition 1 below. The key new idea is to use an interlacing construction from [12] to produce a finite number of infinite families of Salem numbers that between them cover all sufficiently large degrees.

1.2. A conjecture of Estes and Guralnick

Let A be an integer symmetric matrix, and let $m_A(x)$ be its minimal polynomial. Then certainly $m_A(x)$ is a monic integer polynomial with all roots real. Moreover it is separable, since A is diagonalisable over $\mathbb{Q}$. In [7, page 84] Estes and Guralnick make the conjecture 'that any totally real separable monic integral polynomial can occur as the minimal polynomial of a symmetric integral matrix'. In support of this conjecture, they prove it to be true if the polynomial in question has degree at most 4.

Dobrowolski [4] showed that there are infinitely many counterexamples to the conjecture, by obtaining a lower bound on the discriminant of any polynomial that appears as the minimal polynomial of an integer symmetric matrix and noting that infinitely many totally real separable monic integral polynomials have a discriminant that is lower than his bound. The smallest known degree for any of his counterexamples is 2880.

McKee [10] found counterexamples that had much lower degrees, including three of degree 6. This was based on a classification of all integer symmetric matrices such that the difference between the largest and smallest eigenvalues is less than 4. Recently [17] we found a sharp lower bound for the trace of the minimal polynomial of an integer symmetric matrix, and used this to provide some further counterexamples to the Estes-Guralnick conjecture.

The current paper finds counterexamples for every degree greater than or equal to 6. All sufficiently large degrees are covered by minimal polynomials of numbers of the form $\tau + 1/\tau + 2$, where τ is a Salem number of trace -2. Smaller degrees are dealt with by *ad hoc* arguments. It is still not known whether the conjecture is true or false for degree-5 polynomials.

1.3. Statement of results

We now list the main results of the paper, and deduce some immediate corollaries, leaving the proofs of the main results until later. We start with an existence theorem for Salem numbers of trace -2 for all large enough degrees.

Proposition 1. *For all $d \geq 12$ there is a Salem number of degree $2d$ and trace -2.*

The proof will be in Section 2. There are also three Salem numbers of degree $2d$ and trace -2 for $d = 10$ [12]. It is known that there are none for $d < 10$ [19], and none for $d = 11$ [8] .

If τ is a Salem number of degree $2d$ and trace t, then $\tau + 1/\tau + 2$ is a totally positive algebraic integer of degree d and trace $2d+t$. As an immediate corollary to Proposition 1 we have:

Corollary 2. *For all $d \geq 12$ there is a totally positive algebraic integer of degree d and trace $2d - 2$.*

In [17] we showed that the minimal polynomial of any totally positive algebraic integer of degree d and trace $< 2d - 1$ cannot be the minimal polynomial of integer symmetric matrix. Hence we have:

Corollary 3. *There are counterexamples to the conjecture of Estes and Guralnick for all degrees $d \geq 12$.*

Using small-span arguments, we find counterexamples for all degrees between 6 and 11 inclusive (Section 3), establishing our main result:

Theorem 4. *For all $d \geq 6$, there exists a totally real separable monic integer polynomial of degree d that is not the minimal polynomial of any integer symmetric matrix.*

Thus the conjecture of Estes and Guralnick has counterexamples for all degrees greater than or equal to 6. There remains the question of whether or not there exists a totally real separable monic integer polynomial of degree 5 that is not the minimal polynomial of any integer symmetric matrix.

2. Proof of Proposition 1

Following [13], we say that a pair of relatively prime polynomials p and q satisfy the *circular interlacing condition* if they both have real coefficients and positive leading terms, and all their zeros interlace on the unit circle (progressing round the unit circle, zeros of p and q are encountered alternately). If m and n are coprime integers, then $q = (z^{m+n} - 1)/(z - 1)$ and $p = (z^m - 1)(z^n - 1)/(z - 1)$ satisfy the circular interlacing condition (see the first entry in [1, Table 8.3], or family 1 in [16, Table 1], or the interlacing quotient of $A_n(a, b)$ in [14, Table 7]). For $n \geq 1$, let $\Phi_n(z)$ be the minimal polynomial of the primitive nth root of unity $e^{2\pi i/n}$. It will prove convenient to use the terminology *cyclotomic polynomial* to mean any monic polynomial with integer coefficients that has all its roots on the unit circle. After Kronecker [9], a cyclotomic polynomial is a product of one or more of the Φ_n. A *Pisot number* is a real algebraic integer $\tau > 1$ such that all conjugates of τ (other than τ itself) have modulus strictly less than 1.

Applying Propositions 3.3 and 3.2(a) of [13] gives the following lemma.

Lemma 5. *Let p_1, p_2, p_3, p_4, p_5 and n be pairwise coprime integers, all at least 2. Put*

$$\frac{q(z)}{p(z)} = \frac{z^{p_1+p_2} - 1}{(z^{p_1} - 1)(z^{p_2} - 1)} + \frac{z^{p_3+p_4} - 1}{(z^{p_3} - 1)(z^{p_4} - 1)} + \frac{z^{p_5+n} - 1}{(z^{p_5} - 1)(z^n - 1)}, \quad (1)$$

where p and q are relatively prime. Then $(z^2 - 1)p(z) - zq(z) = f(z)g(z)$, where $f(z)$ is the minimal polynomial of a Salem number (or possibly a quadratic Pisot number), and $g(z)$ is either a cyclotomic polynomial or is equal to 1.

The distinction of the naming of n will become apparent in the next Lemma. We note that a difficulty with applying Lemma 5 to construct Salem numbers of specified trace is the possibility of cyclotomic factors appearing in $(z^2 - 1)p(z) - zq(z)$. We apply a method that was used in [12, Section 2.3] to find infinite families where the irreducibility of $(z^2 - 1)p(z) - zq(z)$ is guaranteed (in the notation of Lemma 5, $g(z) = 1$).

Lemma 6. *Let $(p_1, p_2, p_3, p_4, p_5)$ be one of the following* 5*-tuples:*

$$\begin{array}{ccc}
(2,3,5,7,11), & (2,3,5,7,13), & (2,3,5,11,13), \\
(2,3,5,11,19), & (2,3,5,13,17), & (2,3,5,13,19), \\
(2,3,5,17,19), & (2,3,7,11,13), & (2,3,7,11,17), \\
(2,3,7,11,19), & (2,3,7,13,17), & (2,3,7,13,19), \\
(2,3,11,13,19), & (2,3,11,17,19), & (2,3,13,17,19).
\end{array}$$

Then for every $n \geq 5$ such that $\gcd(n, p_1p_2p_3p_4p_5) = 1$*, and with $p(z)$ and $q(z)$ defined by Lemma 5, the polynomial $(z^2-1)p(z) - q(z)$ is the minimal polynomial of a Salem number of trace -2 and degree $n + p_1 + p_2 + p_3 + p_4 + p_5 - 3$.*

Before embarking on the proof, we make a few remarks.

The coprimeness condition ensures that in each of the three fractions on the right of (1), the only common factor in the numerator and denominator is $z-1$. Hence the degree of p (and q) is $n + \sum(p_i - 1) = n - 5 + \sum p_i$. The leading coefficient of q is 3, and $p(z) = z^d + 5z^{d-1} + \cdots$, where $d = n - 5 + \sum p_i$. Put $f(z) = (z^2-1)p(z) - zq(z)$. If $f(z)$ is irreducible, then any root of $f(z)$ has degree $n - 3 + \sum p_i$, and trace $-5 + 3 = -2$. After Lemma 5, we would have that $f(z)$ is the minimal polynomial of a Salem number. All that remains to be proved, therefore, is that $f(z)$ is irreducible for all the claimed values of n and $(p_1, \ldots, p_5)$.

The restriction on n implies that it is odd, and with $p_1 = 2$ and the other p_i odd, we see that the stated degree is even. The proof technique will not work (and the result is not always true) for all 5-tuples, which explains some of the gaps in the list of 15 given. The proof *does* work for the 5-tuples $(2, 3, 5, 7, 17)$ and $(2, 3, 7, 17, 19)$, but neither of these is needed for the sequel.

Proof. After our previous remarks, all that remains to be shown is that $f(z) = (z^2-1)p(z) - zq(z)$ is irreducible for all the advertised values of $(p_1, \ldots, p_5)$ and with $n \geq 5$ coprime to their product. After Lemma 5, we need merely exclude cyclotomic factors. To this end we use a trick from [2]: if ζ is a root of unity, then one of $-\zeta$, ζ^2, or $-\zeta^2$ is a Galois conjugate of ζ. Hence if ζ is a root of unity that is a zero of a certain integer polynomial, then at least one of $-\zeta$, ζ^2, or $-\zeta^2$ is a zero of the same polynomial.

Put

$$\frac{Q(y,z)}{P(y,z)} = \frac{z^{p_1+p_2}-1}{(z^{p_1}-1)(z^{p_2}-1)} + \frac{z^{p_3+p_4}-1}{(z^{p_3}-1)(z^{p_4}-1)} + \frac{yz^{p_5}-1}{(z^{p_5}-1)(y-1)}\,,$$

in lowest terms. Note that $\frac{Q(z^n,z)}{P(z^n,z)} = \frac{q(z)}{p(z)}$, but that the left hand side here will not be in lowest terms: both numerator and denominator will be divisible by $z-1$.

We look for cyclotomic points on the curve $C : (z^2-1)P(y,z) - zQ(y,z) = 0$ (that is, points where both y and z are roots of unity), and, if convenient, we use that $y = z^n$. In particular, if z is replaced by $-z$ (respectively z^2 or

$-z^2$), then y is replaced by $-y$ (respectively y^2 or $-y^2$), since n must be odd ($\gcd(n, p_1) = 1$). This implies that if (y, z) is a cyclotomic point on C with $y = z^n$, then one of $(-y, -z)$, (y^2, z^2), $(-y^2, -z^2)$ is a cyclotomic point on C. Hence (y, z) lies on both C and one of $C_1 : (z^2-1)P(-y, -z)+zQ(-y, -z) = 0$, $C_2 : (z^4 - 1)P(y^2, z^2) - z^2Q(y^2, z^2) = 0$, or $C_3 : (z^2 - 1)P(-y^2, -z^2) + z^2Q(-y^2, -z^2) = 0$.

For each of the fifteen 5-tuples $(p_1, p_2, p_3, p_4, p_5)$, and each of the three pairs (C, C_1), (C, C_2), (C, C_3) we eliminate y to get a single-variable polynomial in z which restricts z to a finite set. Similarly one can eliminate z to limit y to a finite set.

For example, with $(p_1, p_2, p_3, p_4, p_5) = (2, 3, 13, 17, 19)$ one finds immediately that there are no cyclotomic points on the intersection of C and C_1, or of C and C_3, as the single-variable polynomial in z obtained by eliminating y has no cyclotomic roots. But eliminating y between C and C_2 gives a polynomial with cyclotomic factors $\Phi_2(z)$, $\Phi_3(z)$, $\Phi_{13}(z)$, $\Phi_{17}(z)$ and $\Phi_{19}(z)$. These cyclotomic polynomials, however, are all factors of p but not q, so cannot divide $f(z)$. Thus there are no cyclotomic points on C that correspond to solutions to $(z^2 - 1)p(z) - zq(z) = 0$, for this choice of $(p_1, p_2, p_3, p_4, p_5)$.

A more complicated example arises with $(p_1, p_2, p_3, p_4, p_5) = (2, 3, 11, 17, 19)$, and four other cases. Here eliminating y between C and C_1 gives a polynomial with cyclotomic factor $\Phi_{12}(z)$, which cannot be excluded as trivially as in the previous example. In this case we resort to eliminating z between C and C_1, and find that there is a cyclotomic factor $\Phi_4(y) = y^2 + 1$. Since $y = z^n$ and the only awkward cases are where z is a primitive 12-th root of unity, we must have n divisible by 3, contradicting coprimeness to $p_2 = 3$.

Similar calculations were performed for each of the advertised 5-tuples, and in all cases it was established that $(z^2-1)p(z)-zq(z)$ has no cyclotomic roots, for any value of n coprime to $p_1p_2p_3p_4p_5$. As remarked before the proof, there are 5-tuples, such as $(2, 3, 5, 7, 19)$, for which this process fails. For example, when $(p_1, \ldots, p_5) = (2, 3, 5, 7, 19)$ and $n = 43$, the polynomial $f(z)$ is not irreducible.

$\square$

Now as n varies over positive integers prime to $p_1, \ldots, p_5$, where $(p_1, \ldots, p_5)$ is one of the 5-tuples in the above Lemma, one finds Salem numbers of degree $n+p_1+p_2+p_3+p_4+p_5-3$ and trace -2. This gives infinitely many degrees for Salem numbers of trace -2, lying in residue classes that repeat modulo $p_1p_2p_3p_4p_5$, and in particular they repeat modulo $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times$

$17 \times 19 = 9699690$.

Lemma 7. *There are Salem numbers of trace -2 and degree $2d$ for all $d \geq 21$.*

Proof. Each of the fifteen infinite families of Salem numbers of trace -2 provided by the previous Lemma gives examples of degrees that lie in certain residue classes modulo 9699690, with degrees at least $p_1+p_2+p_3+p_4+p_5+2$ (we need $n \geq 5$ since 2 and 3 are among our p_i in all cases). A computation shows that all even residue classes modulo 9699690 are covered, and that all even degrees greater than or equal to 42 are covered. □

For example, suppose we wish to find a Salem number of degree 1000 and trace -2 (or indeed degree $1000 + 9699690t$ for any t). We cannot use $(2, 3, 5, 7, 11)$, as we would need $n = 1000 - 2 - 3 - 5 - 7 - 11 + 3 = 975$, which is not prime to all of 2, 3, 5, 7, 11. But we can use $(2, 3, 7, 11, 13)$, as then $n = 1000 - 2 - 3 - 7 - 11 - 13 + 3 = 967$ is prime to all of 2, 3, 7, 11, 13. We produce the polynomial $z^{1000} + 2z^{999} - 2z^{998} - 19z^{997} + \cdots + 2z + 1$.

The given fifteen infinite families form a minimal covering set in the sense that each of the families contributes to at least one residue class modulo 9699690 that is not covered by any of the others. No covering set exists using only the primes 2, 3, 5, 7, 11, 13, 17: we are forced to use primes up to 19.

To finish the proof of Proposition 1, we need to find Salem numbers of degree $2d$ and trace -2 for $12 \leq d \leq 20$. For $d = 19$ we can use the family corresponding to $(2, 3, 5, 7, 11)$, with $n = 13$. For $d \in \{13, 14, 16, 20\}$ we appeal to another interlacing argument.

Write

$$\frac{q(z)}{p(z)} = \frac{(z-1)(z^8 + z^7 - z^5 - z^4 - z^3 + z + 1)}{(z+1)(z^3-1)(z^5-1)} + \frac{(z^{p_1+p_2} - 1)}{(z^{p_1} - 1)(z^{p_2} - 1)}$$

where the fraction on the left is in lowest terms, and p_1, p_2 are distinct primes greater than 5. Then p and q satisfy the circular interlacing condition (see [15, §9.2]). If it is irreducible, the polynomial $(z^2 - 1)p(z) - zq(z)$ is the minimal polynomial of a Salem number of trace -2 and degree $p_1 + p_2 + 8$. Taking $(p_1, p_2) \in \{(7, 11), (7, 13), (11, 13), (13, 19)\}$ give us examples of Salem numbers of trace -2 and degrees 26, 28, 32, 40. (Sadly for $(7, 19)$ and $(11, 17)$ the polynomial $(z^2 - 1)p(z) - zq(z)$ is not irreducible.)

The smallest degree of a Salem number of trace -2 is degree 20 [12]: there are exactly three such Salem numbers. It is known that there are none of degree 22 [8]. In [11], a method was given for computing totally positive algebraic integers of small trace. In particular, some 209 examples of degree 12 and trace 22 were computed. Many of these correspond to Salem numbers of degree 24 and trace -2, for example the larger real root of the palindromic polynomial

$$\begin{gathered} z^{24}+2z^{23}-4z^{22}-28z^{21}-72z^{20}-116z^{19}-116z^{18}-27z^{17}+166z^{16} \\ +431z^{15}+701z^{14}+900z^{13}+973z^{12}+900z^{11}+701z^{10}+\cdots+1\,. \end{gathered}$$

In [5], 321 totally positive monic integer polynomials of degree 15 and trace 28 were found. Of these, 6 correspond to Salem numbers of trace -2 and degree 30.

To complete the proof of Propostion 1, we need to find examples for $d = 17$ and $d = 18$. Applying the technique of [11], we found examples for $d = 17$ and $d = 18$:

$$\begin{aligned} &z^{34}+2z^{33}+z^{32}-3z^{31}-8z^{30}-12z^{29}-14z^{28}-15z^{27}-15z^{26}-14z^{25} \\ &-13z^{24}-13z^{23}-13z^{22}-12z^{21}-9z^{20}-4z^{19}+z^{18}+3z^{17}+z^{16} \\ &-4z^{15}-9z^{14}+\cdots-3z^{3}+z^{2}+2z+1\,, \end{aligned}$$

and

$$\begin{aligned} &z^{36}+2z^{35}+z^{34}-3z^{33}-8z^{32}-12z^{31}-13z^{30}-11z^{29}-8z^{28}-7z^{27}-9z^{26} \\ &-13z^{25}-17z^{24}-20z^{23}-21z^{22}-19z^{21}-15z^{20}-12z^{19}-11z^{18}-12z^{17} \\ &-15z^{16}+\cdots-3z^{3}+z^{2}+2z+1\,. \end{aligned}$$

The methods of [5] and [6] would presumably also be effective in searching for such examples.

This completes the proof of Proposition 1.

Whenever there is a Salem number of trace -2 and degree $2d$, there is a totally positive algebraic integer of degree d and trace $2d-2$ (this is the case $t=-2$ of the remark before Corollary 2). By [17] these give counterexamples to the conjecture of Estes and Guralnick. We therefore have counterexamples for degree 10, and for all degrees ≥ 12. Moreover these examples can be explicitly constructed: given any particular degree there is a finite process to produce a counterexample. In the next section we fill in most of the gaps for other degrees: the only outstanding case being degree 5.

3. Smaller degrees

The *span* of a totally real algebraic integer is the difference between the largest and smallest conjugates. If a polynomial has all its roots real, then its *span* is defined to be the difference between the largest and smallest roots. Thus the span of a totally real algebraic integer equals the span of its minimal polynomial. The *span* of an integer symmetric matrix is defined to be the difference between the largest and smallest eigenvalues, i.e., the span of its characteristic polynomial. Any of these three things is said to have *small span* if its span is strictly less than 4. For small-span algebraic integers/polynomials/matrices one can assume that all conjugates/roots/eigenvalues lie in the interval $[-2, 2.5)$ (for algebraic integers θ, replace θ by $\pm\theta + c$ for some $c \in \mathbb{Z}$; for polynomials apply the transformation $x \mapsto \pm x + c$; for matrices apply the transformation $A \mapsto \pm A + cI$).

In [10], a complete description was given of all small-span integer symmetric matrices. As a consequence, two arguments were given [10, §5] for potentially determining if a small-span polynomial is the *minimal* polynomial of an integer symmetric matrix. If all the roots lie in the interval $[-2, 2]$, then a 'growing' procedure allows one to bound the size of the matrix, and hence bound the search: it was in this way that three degree-6 counterexamples to the Estes-Guralnick conjecture were found. A simpler argument applies if not all the roots are in the interval $[-2, 2]$ (but with all the roots in the interval $[-2, 2.5)$). Then one needs only to check matrices up to 12×12 (Theorem 3 of [10]). From the table at the end of §3 of [10], we see that there are counterexamples of degrees 7, 8, 9, 10, 11, 12, 13. These are already more than enough to complete the proof of Theorem 4, although we remark that Capparelli et al. [3] give small-span examples of degrees 14, 15 and 16 that do not correspond to Salem numbers but give further counterexamples to the Estes-Guralnick conjecture.

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