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Emergent Spirograph-like Patterns from Artificial Swarming

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Abstract

Computer simulations of a nonlinear planar system of first-order ODEs which we developed from a simple assumption that biological swarming is an outcome of aggregative behavior of the individuals in the swarm showed a surprising and novel outcome; the emergence of uniquely structured patterns that are intriguingly intricate, exquisite, symmetrical and regular. Some patterns look like flowers; others look like Spirograph curves and Guilloché patterns but with more intricate variations. Unlike Spirograph curves though, the swarm-induced patterns cannot be reproduced by any closed-form formula or by another pattern subjected to some resizing, translation, rotation and/or reflection.

Introduction

A way to model biological swarming is to assume that swarming is an interplay between long-range attraction and short-range repulsion between the individuals in the swarm [1]. It is a Lagrangian approach. In this paper, we develop a Lagrangian model via the Direct Method of Lyapunov, and show the surprising outcomes in the form of intriguing visual patterns.

A Two-Dimensional Swarm Model

Consider a swarm of $n \in \mathbb{N}$ individuals, taken as point masses. At time $t \geq 0$, let $(x_i(t), y_i(t))$, $i = 1, 2, \ldots, n$, be the planar position of the ith individual, which we define as a point mass residing in a disk of radius $r > 0$, i.e., $A_i := \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x_i)^2 + (z_2 - y_i)^2 \leq r^2\}$. We call the disk a *bin*, with its *bin size* being the radius r of the disk. Let us define the *centroid of the swarm* as $(x_C, y_C) := \left(\frac{1}{n}\sum_{k=1}^{n} x_k, \frac{1}{n}\sum_{k=1}^{n} y_k\right)$. At time $t \geq 0$, let $(v_i(t), w_i(t)) := (x_i'(t), y_i'(t))$ be the instantaneous velocity of the ith point mass. We have thus a system of first-order ODEs for the ith individual, assuming the initial condition at $t = t_0$:

$$x_i'(t) = v_i(t), \quad y_i'(t) = w_i(t), \quad x_{i0} := x_i(t_0), \quad y_{i0} := y_i(t_0), \quad t_0 \geq 0. \tag{1}$$

Suppressing t, we let $\mathbf{x}_i := (x_i, y_i) \in \mathbb{R}^2$ and $\mathbf{x} := (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^{2n}$ be our state vectors. Also, let $\mathbf{x}_0 := \mathbf{x}(t_0) = (x_{10}, y_{10}, \ldots, x_{n0}, y_{n0}) \in \mathbb{R}^{2n}$. If $\mathbf{g}_i(\mathbf{x}) := (v_i, w_i) \in \mathbb{R}^2$ and $\mathbf{G}(\mathbf{x}) := (\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_n(\mathbf{x})) \in \mathbb{R}^{2n}$, then our swarm of n individuals is governed by

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}), \quad \mathbf{x}_0 = \mathbf{x}(t_0), \quad t_0 \geq 0. \tag{2}$$

Our objective is to construct the instantaneous velocity $(v_i(t), w_i(t))$, $t \geq 0$, for every individual $i \in \mathbb{N}$ via a Lyapunov-like function that should also establish the boundedness and hence the cohesiveness of system (2). For long-range attraction between individuals, we will use $R_i := \left[(x_i - x_C)^2 + (y_i - x_C)^2\right]/2$, $i \in \mathbb{N}$.

For short-range repulsion, we will use $Q_{ij} := \left[(x_i - x_j)^2 + (y_i - y_j)^2 - (2r_V)^2\right]/2$. Let there be scalars $\gamma_i > 0$ and $\beta_{ij} > 0$, and consider as a tentative Lyapunov-like function for system (2),

$$L(\mathbf{x}) := \sum_{i=1}^{n} \left[\gamma_i R_i(\mathbf{x}) + \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{Q_{ij}(\mathbf{x})} \right]. \tag{3}$$

It is positive over the domain $D(L) := \left\{\mathbf{x} \in \mathbb{R}^{2n} : Q_{ij}(\mathbf{x}) > 0 \;\forall\, i, j \in \mathbb{N}, i \neq j\right\}$. The time-derivative of L along every solution of system (2) is, on suppressing $\mathbf{x}$,

$$\begin{aligned} \dot{L} \;&=\; \sum_{i=1}^{n} \left\{ \left(\gamma_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij}}{Q_{ij}} \right) \left(x_i - \frac{1}{n}\sum_{k=1}^{n} x_k \right) - 2 \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i}{Q_{ij}^2}(x_i - x_j) \right\} \dot{x}_i \\ &+\; \sum_{i=1}^{n} \left\{ \left(\gamma_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij}}{Q_{ij}} \right) \left(y_i - \frac{1}{n}\sum_{k=1}^{n} y_k \right) - 2 \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i}{Q_{ij}^2}(y_i - y_j) \right\} \dot{y}_i \,. \end{aligned}$$

It is clear that the terms in the curly brackets are $\partial L/\partial x_i$ and $\partial L/\partial y_i$, respectively. Let there be scalars $\mu_i > 0$ and $\varphi_i > 0$ such that $\dot{x}_i = v_i := -\mu_i \dfrac{\partial L}{\partial x_i}$ and $\dot{y}_i = w_i := -\varphi_i \dfrac{\partial L}{\partial y_i}$. Then $\dot{L} \;\leq\; 0$ for all $\mathbf{x} \in D(L)$. Thus, for the ith individual, system (1) becomes

$$x_i'(t) = -\mu_i \frac{\partial L}{\partial x_i}, \;\; y_i'(t) = -\varphi_i \frac{\partial L}{\partial y_i}, \;\; x_{i0} = x_i(t_0), y_{i0} = y_i(t_0), \;\; t_0 \geq 0. \tag{4}$$

Define a $2n \times 2n$ diagonal matrix $H := \text{diag}(\mu_1, \varphi_1, \ldots, \mu_n, \varphi_n)$. Then our system (2) becomes the gradient system

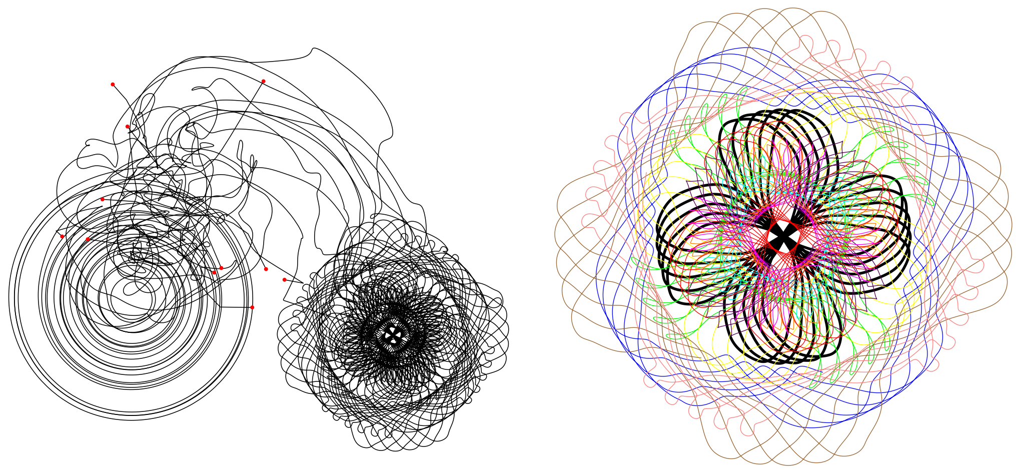
$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}) = -H\,(\nabla L(\mathbf{x})), \quad \mathbf{x}_0 := \mathbf{x}(t_0), \;\; t_0 \geq 0, \tag{5}$$

the ith terms of which are given by (4). From the Lyapunov-like function, we can conclude the boundedness of solutions of (5) about the centroid and, hence, the existence of equilibrium points or limit cycles in $D(L)$ via the Poincaré-Bendixson Theorem. This means that the trajectory of each individual in the swarm will either converge to a unique equilibrium point or to a unique limit cycle. Further, we can show that the parameters of the system determine whether we have equilibrium points or limit cycles. The unique patterns are the limit cycles emerging from such artificial swarming.

Computer Simulations

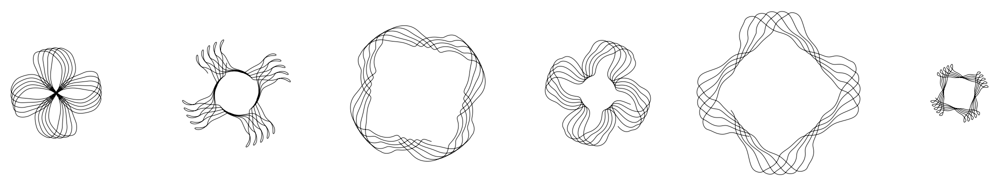
Our simulations involve $n = 9$, $n = 10$ or $n = 11$ individuals, shown as red disks, each of radius $1/2$ and randomly positioned initially within a square of side 60. The parameters γ_i, μ_i and φ_i are all fixed at 1, whereas β_{ij} are randomized between 10 and 30, leading to the variation in pattern. A simulation is performed over the time interval $[0, T]$, where T is indicated in each example. The entire trajectories are first shown in black. The emergent pattern is then isolated and also shown in color, with each trajectory assigned a different color. An attempt is made to name the portrait by referring to a part of the portrait that is familiar or can be named.

Example 1. Our first example shows an intriguing pattern by time $T = 1000$ (Fig. 1). The pattern is isolated by showing the trajectories over the time interval $[400, 1000]$. Each individual settles into its own unique pattern as shown in Fig. 2.

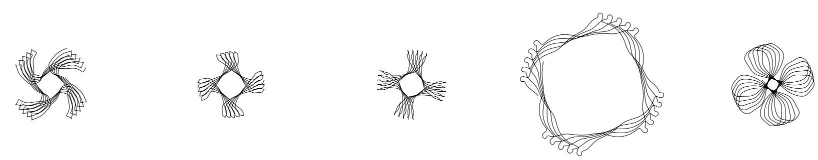


(a) Formation of a pattern by time $T = 1000$. (b) Color portrait.

Figure 1 : *Cross-in-the-Lotus*



(a) $i = 1$ (b) $i = 2$ (c) $i = 3$ (d) $i = 4$ (e) $i = 5$ (f) $i = 6$



(g) $i = 7$ (h) $i = 8$ (i) $i = 9$ (j) $i = 10$ (k) $i = 11$

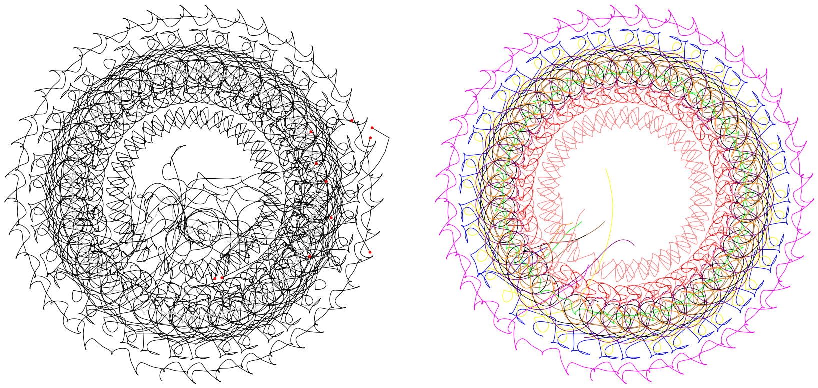
Figure 2 : *The trajectory of each individual taken over the time interval* $[400, 1000]$.

Example 2. Our second example shows a pattern that forms in the vicinity of the initial positions. The pattern eerily reminds one of a crown-of-thorns (Fig. 3). The black-and-white portrait is taken over $T \in [0, 1000]$ and color over $[100, 1000]$.

Example 3. From a crown-of-thorns to a more pleasant Mediterranean olive wreath – as it seems to appear in this third example (Fig. 4) – or a beautiful South Pacific lei or some exotic Asian chrysanthemum (Fig. 5).

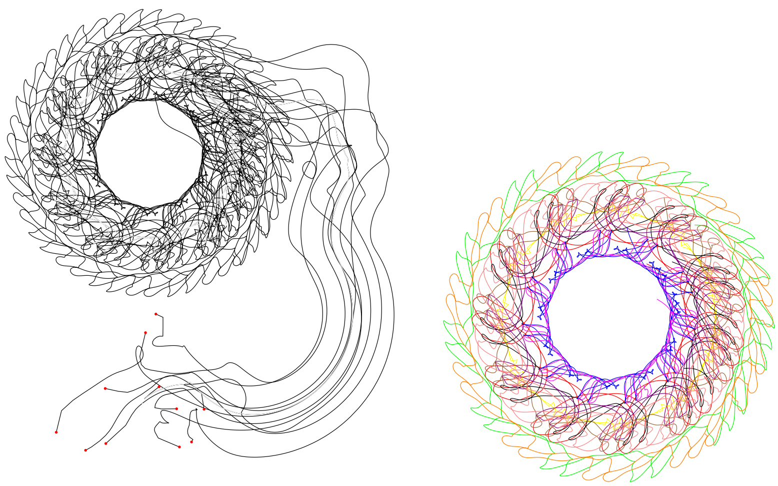
References

[1] V. Gazi and K.M. Passino. Stability analysis of social foraging swarms. *IEEE Transactions on Systems, Man and Cybernetics – Part B*, 34(1):539–557, 2004.



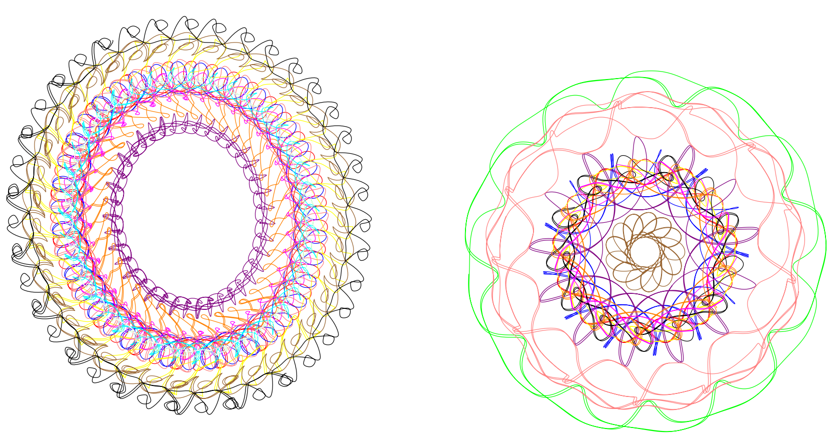
(a) Black-and-white portrait. (b) Color portrait.

Figure 3 : *Crown-of-Thorns*



(a) Black-and-white portrait. (b) Color portrait.

Figure 4 : *Olive Wreath*



(a) Color portrait. (b) Color portrait.

Figure 5 : *Lei and Chrysanthemum*