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The Wadge Hierarchy : Beyond Borel Sets

FOURNIER Kevin

FOURNIER Kevin, 2016, The Wadge Hierarchy : Beyond Borel Sets

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FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DE SYSTÈMES D'INFORMATION

The Wadge Hierarchy : Beyond Borel Sets

THÈSE DE DOCTORAT

présentée à la

Faculté des Hautes Etudes Commerciales
de l'Université de Lausanne
en cotutelle avec l'Université Paris Diderot

pour l'obtention du grade de
Docteur en Systèmes d'Information
et Docteur en Mathématiques

par

Kevin FOURNIER

Co-directeurs de thèse
Prof. Jacques DUPARC
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Jury

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La thèse est intitulée :

THE WADGE HIERARCHY : BEYOND BOREL SETS

Lausanne, le 14 janvier 2016

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Doctorat en Systèmes d'Information

Par la présente, je certifie avoir examiné la thèse de doctorat de

Kevin FOURNIER

Sa thèse remplit les exigences liées à un travail de doctorat.
Toutes les révisions que les membres du jury et le-la soussigné-e ont demandées durant le colloque de thèse ont été prises en considération et reçoivent ici mon approbation.

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Signature : W. Wadge Date : 26 JAN 2016

Prof. William W. WADGE
Membre externe du jury

The Wadge Hierarchy: Beyond Borel Sets

Kevin Fournier

February 2016

À la Lune et aux Amis.

"Mathematics is the only avant-garde remaining in the whole province of art. It's pure art, lad. Art and science. Art, science and language. Art as much as the art we once called art. It lost its wings after the Babylonians fizzled out. But emerged again with the Greeks. Went down in the Dark Ages. Moslems and Hindus kept it going. But now it's back, bright as ever."

Don DeLillo, *Ratner's Star*.

Acknowledgment

The romantic myth of the mathematician working alone, preferably by candlelight late at night before a duel, is still vivid nowadays. Maybe it is one of the numerous human flaws to seek tirelessly for baroque heroic figures – or simply a peculiarity carefully cultivated by the mathematic community itself. I have nonetheless to disappoint the reader right away: I try to keep my work in office hours, and my non-violent principles tend to keep me away from swords and pistols. Most importantly: this thesis is of course a personal and original work, but it should not overshadow the fact that I would not have been able to carry it out all by myself, alone. This thesis is the product of collaborations, interactions, and endless discussions. All the people I have met these last five years, mathematicians or not, have had an impact on my research, and if I take all the blames for any error, mistake or inaccuracy that could occur in this thesis, I should also share with them all the praises its positive outcomes could receive.

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Being a member of the board of the Swiss Graduate Society for Logic and Philosophy of Science has been a very rewarding adventure that I am very happy to have shared with Yann Pequignot, Marion Hämmerli and Dandolo Flumini. It was a lot of work, but it was worth it, and I am very grateful to Thomas Strahm and Thomas Studer for their support, and for believing in us youngsters. I wish to thank also other students, colleagues and senior researchers I met during these years, and who helped me and inspired me, in a way or another: Jérémie Cabessa, Fabrice Correia, Takako Nemoto, Daisuke Ikegami, Philipp Schlicht, Thilo Weinert, Zoltán Vidnyánszky, David Chodunský, Laura Fontanella, Sam van Gool, Kentaro Sato, Sylvain Cabanacq, Rafael Zamora, Jonas Rogger, Jan Walker, Alessandro Andretta, Riccardo Camerlo, Olivier Finkel, Matteo Viale and Silvia Steila.

They say teaching is the best way to learn. I do not know if it is a universal truth, but it worked for me and I would like to express my gratitude to all the students I had for their stimulating questions and our interesting discussions. More than the feeling of being helpful, I *learned*. I had also the chance to count on a solid team of teaching assistants: Chloé Morend, Jennifer Lopes, Nathalie Michel, Émilie Bruchez, Virginie Michaud and Aline de Montmollin. I thank them for their help, it was a great pleasure to work with them.

I thank my family for their constant love and care: they believed in me, and I owe them everything. I thank the fantastic people who were there to accompany me along this sometimes bumpy road: Fabien, Thomas, PP, Clément, Axel, Fabrizio, Alexandre, Mathias, Audrey, Barbara, Pierre, Alexis, Évelyne, Lutin, Grany, Annie, Camille, Charlotte, Clémentine, Aude, Hervé, Joy, Aline, Léa, Rebecca, Giuseppe and Marie – *pour tout et quoi qu'il advienne.* I am lucky I can call them friends.

Lausanne, June 2015
Kevin Fournier

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1 Introduction

This thesis is devoted to the study of non-Borel $\mathbf{\Delta}^1_2$ pointclasses of the Baire space, using reductions by continuous functions. This work is divided in three main parts. In the first one, we generalise results obtained by Duparc [26, 27] and Louveau [68] to provide a complete description of the Wadge hierarchy of the class $\mathrm{Diff}(\mathbf{\Pi}^1_1)$, the class of increasing differences of coanalytic sets, under some determinacy hypothesis. In a second part, we study some $\mathbf{\Delta}^1_2$ pointclasses above $\mathrm{Diff}(\mathbf{\Pi}^1_1)$, and give a fragment of the Wadge hierarchy for those classes. Finally, we apply our results and techniques to theoretical computer science and more precisely to the study of regular tree languages, that is sets of labeled binary trees that are recognized by tree automata.

1.1 The Borel sets and the emergence of descriptive set theory: A brief historical introduction

> "Descriptive set theory is the definability theory of the continuum, the study of the structural properties of definable sets of reals."
>
> Akihiro Kanamori, [51].

Over a century ago, the development of modern analysis by the french mathematicians Émile Borel, René Baire and Henri Lebesgue induced a fundamental interest in the study of *well-behaved* subsets of the real line. Topology, which developed about the same time, yielded the mathematical framework for such a study. *Borel sets* were introduced by Borel [15, pp. 46–47] in order to extend the notion of length of an interval to a measure on a wide class of subsets of the real line, and are defined by recursion as follows. Beginning with the intervals, we add at each stage sets whose complement was previously defined or which are the union of a countable family of previously defined set. The resulting family is stable under complementation and countable unions: it is the smallest σ-algebra containing the intervals. As soon as the Borel sets where introduced, they were set up in a natural hierarchy of height ω_1, the first uncountable ordinal. This hierarchy relies on counting the number of successive operations of countable unions and complementations that are necessary to produce a set, beginning with the class of all unions of open intervals ($\mathbf{\Sigma}^0_1$), and its dual the class of closed intervals ($\mathbf{\Pi}^0_1$). The class $\mathbf{\Sigma}^0_\alpha$, for α countable, is thus obtained by taking unions of a countable family of sets in $\mathbf{\Pi}^0_\beta$ with $\beta < \alpha$, and the class $\mathbf{\Pi}^0_\alpha$ is defined as its dual class, the class of sets whose complements are in $\mathbf{\Sigma}^0_\alpha$. Using a variant of Cantor's enumeration and diagonalization argument, Lebesgue [64] proved that this hierarchy is proper. Moreover, the Borel sets are well-behaved: they are measurable in the sense of Lebesgue [63], have the Baire property (BP) [11], which states that they each have a meager symmetric difference with some open set, and have the perfect set property [1], which states that they are all either countable or of the size of the continuum. This nice behaviour, along with their closure properties, made the Borel sets an appropriate domain for the mathematical practice and study, and they are now quite well-understood.

> "L'origine de tous les problèmes dont il va s'agir ici est une grossière erreur [...]. Fructueuse erreur, que je fus bien inspiré de la commettre!"
>
> Henri Lebesgue, preface to [73].

In a seminal article, Lebesgue [64] gave a demonstration of the fact that the projection of a Borel set of the plane is a Borel subset of the real line. A decade later however, it was discovered by Suslin [106] that the published proof was fallacious. The projection of a Borel set is not always Borel, so that the class of Borel sets is not stable under projection. This led to the definition of the class of *analytic* subsets ($\mathbf{\Sigma}^1_1$), which are the projections of Borel sets. Suslin proved that every Borel set is analytic, that there is an analytic set that is not Borel, and that a set of reals is Borel if and only if both it and its complement are analytic. Luzin and Sierpiński [74, 75] proved furthermore that the regular properties hold for the analytic sets, extending thus the natural domain of analysis. Luzin [72] and Sierpiński [102], building on the analytic sets, proposed another natural hierarchy above the Borel sets by alternating complementation with projection: the *projective* hierarchy. Its first level is this time formed by the class of analytic sets and the class of their complements, the coanalytic sets ($\mathbf{\Pi}^1_1$). On the second level, the class of projections of coanalytic sets is denoted by $\mathbf{\Sigma}^1_2$, and its dual class by $\mathbf{\Pi}^1_2$, and so on and so forth. This hierarchy is, as in the Borel case, proper. A natural question arose here: is it possible to extend the realm of analysis to these classes? Do they enjoy the same regularity properties as the Borel sets – or the analytic sets?

> "L'étude des ensembles analytiques a conduit naturellement à celle des ensembles projectifs, dont les propriétés extrêmement paradoxales nous obligent, à mon avis, à poser la question de la légitimité même de ces ensembles."
>
> Nikolai N. Luzin, [73].

These questions would need decades to be settled, and would only meet satisfactory answers in the second part of the twentieth century, thanks to modern set theory, the development of powerful metamathematical methods by Gödel and the invention of forcing by Cohen. Contrarily to the Borel case, the questions concerning the regularity properties of the projective classes are indeed independent of ZFC[1]. The frontier for the domain of analysis seems thus to be definitely, in ZFC, the class of analytic sets.

In this thesis, we take a look beyond this frontier, for our main interest is to study and describe non-Borel sets that are both in the classes $\boldsymbol{\Pi}^1_2$ and $\boldsymbol{\Sigma}^1_2$.

1.2 The Wadge hierarchy

One of the main concerns of descriptive set theory is the study of the complexity of subsets of the Baire space ω^ω, the "logician's reals", a space homeomorphic to the irrationals. A natural measure of the relative complexity of subsets of the Baire space is given by the *reducibility by continuous functions*. Given two subsets A and B of the Baire space, A is said to be reducible to B, and we write $A \leq_W B$, if and only if A is the preimage of B for some continuous function f from the Baire space to itself. If we understand the complexity of A to mean the difficulty of determining membership in A, we observe that if A is reducible to B then A is, in a certain sense, no more complicated than B: if we wonder whether x belongs to A, then we just have to compute $f(x)$ and see if it is in B or not. Given that computing the value of a continuous function is *topologically simple*, the second question is not more complicated than the first one, and thus the membership problem for A is not more complicated than the membership problem for B.

If A is reducible to its complement, we say that A is *self-dual*. The relation $\leq_W$ is merely by definition a preorder, and its initial segments are exactly the pointclasses of the Baire space, that is the classes of sets closed under continuous preimages. It thus refines all the well known hierarchies of pointclasses such as the Borel hierarchy, or the projective hierarchy. When restricted to a class with suitable determinacy properties, the partial order induced

[1]See for example Kanamori [51] for an exposition of these results.

by $\leq_W$ on its equivalence classes, known as the Wadge degrees, is in fact a well-quasi-ordering: the Wadge hierarchy. This follows from two important results: Wadge's Lemma [115] and the Martin-Monk Theorem [112]. Both are proved using a very powerful correspondence between the reducibility by continuous function and a certain infinite two players game called the *Wadge Game*. This is a two-player game with perfect information. In the Wadge game $W(A, B)$, with $A, B \subseteq \omega^\omega$, the players I and II take turn in choosing integers. The second player may skip, while the first one cannot, in such a way that after infinitely many moves, the first player has produced an infinite sequence of integers x, and the second one has played an infinite sequence y. The second player wins the game if and only if $(x \in A \leftrightarrow y \in B)$. As it turns out, the rules of the game were designed by Wadge so that a winning strategy for II immediately yields a continuous function that witnesses the reduction between A and B, and any continuous function f satisfying $(x \in A \leftrightarrow f(x) \in B)$ for all $x \in \omega^\omega$ can be turned into some winning strategy for II in this game.

The Wadge hierarchy of the Borel subsets of the Baire space has been thoroughly studied by Louveau [68] and Duparc [26, 27], in two different manners that were both initiated by Wadge in his PhD thesis [115]. The former relies on a Theorem proved by Wadge stating that all the non-self-dual Borel pointclasses can be obtained by ω-ary Borel boolean operations on open sets. Louveau's work provides a description of all the Borel pointclasses, and thus of the whole Wadge hierarchy on the Borel sets, by means of boolean operations. The use and study of boolean operations in this context led to prominent results concerning the consistency strength of the Wadge determinacy and structural properties of the Borel pointclasses by Louveau and Saint-Raymond [69, 70, 71].

The latter approach, followed by Duparc, aims to define and make use of operations on sets, such as the sum and the countable multiplication, in order to give, for each non-self-dual Wadge class of Borel subsets, a canonical complete set. It relies heavily on the peculiar characterization of the continuous reducibility relation made available through the Wadge game. In an effort to extend this approach, Duparc introduced the so-called *conciliatory sets*, namely subsets of $\omega^{\leq\omega}$, as an ansatz. The shift from infinite sequences to both finite and infinite sequences, and the definition of a preorder $\leq_c$ on the subset of $\omega^{\leq\omega}$ is motivated by natural game theoretic considerations, and in particular by the will to symmetrize the Wadge game. For two subsets A, B of $\omega^{\leq\omega}$, we indeed define $A \leq_c B$ to hold if and only if player II has a winning strategy in the variant of the Wadge game where both players can skip, and even stop playing after a finite number of moves. This ansatz places reliance on the definition of a mapping $C \longmapsto C^b$ from conciliatory sets to subsets of the Baire space together with the preorder $\leq_c$ on the subsets of $\omega^{\leq\omega}$ that does not arise from a reduction relation, but rather from a game. For each non-self-dual pointclass Γ of the Baire space, a class Γ_c of subsets of $\omega^{\leq\omega}$ is also defined. Duparc [26, 27] studies the conciliatory hierarchy, the hierarchy induced by $\leq_c$, when restricted to the class corresponding to the Borel pointclass, gives its complete description, and finally proves that the Wadge hierarchy restricted to the non-self-dual Borel degrees and the fragment of the conciliatory hierarchies coincide, via the function $C \longmapsto C^b$. Since there is a straightforward and uniform procedure to derive the structure of self-dual sets from the non-self-dual ones[2], one can study indistinctively the Wadge hierarchy or the conciliatory one. We give in Chapter 3, a topological interpretation of the conciliatory ansatz and a direct proof of the correspondence between the conciliatory and the Wadge hierarchy.

1.3 Wadge hierarchy and $\mathbf{\Delta}^1_2$ sets

1.3.1 The state of the art

The concept defined by Wadge has given rise to a flourishing area of research in descriptive set theory, with interesting applications to set theory and theoretical computer science. Aside from the authors and articles already cited, we survey here some of the most important pieces of work related to the Wadge hierarchy.

Until the end of the 80's, most of the research on continuous reducibility was concerned with the general theory of pointclasses of the Baire space under (AD), as illustrated by the works of Becker, Jackson, Kechris, Martin, Moschovakis and Steel [13, 49, 54, 81, 105]. In these papers, questions about closure and structural properties of arbitrary pointclasses of the Baire space are addressed, under the full axiom of determinacy (AD). Relationship between determinacy and the structure of the Wadge preorder have been also investigated by Harrington [42], Hjorth [45], and Andretta [2, 3]. Interest in generalizations of continuous reducibility on zero-dimensional Polish spaces has rapidly grown last decades, as both more general reduction notions and topological spaces outside the zero-dimensional Polish world were considered. Lecomte [65] studied for example the descriptive complexity of subsets of products of Polish spaces, whereas Andretta and Martin [5] defined and ana-

[2]Note that the converse is not true in general, so that the *essence* of the Wadge hierarchy appears to be completely captured by the study of the non-self-dual degrees.

lyzed the Borel-Wadge preorder on the Baire space, the preorder induced by Borel functions instead of continuous functions. Other generalizations of the Wadge hierarchy were recently explored by Ikegami, Motto Ros, Pequignot, Schlicht, Selivanov and Tanaka [47, 48, 83, 84, 93, 98, 100].

Regarding the Wadge hierarchy of the $\boldsymbol{\Delta}^1_2$ sets of the Baire space, or equivalently the pointclasses included in the $\boldsymbol{\Delta}^1_2$ class, not much is known. Under (AD), Martin and Steel [112] proved that the order type of the Wadge hierarchy restricted to the $\boldsymbol{\Delta}^1_2$ sets is the projective ordinal $\boldsymbol{\delta}^1_3$, which turns out to be $\aleph_{\omega+1}$ under this determinacy hypothesis by a result of Martin [79]. Some hierarchies of $\boldsymbol{\Delta}^1_2$ sets have been considered, such as the hierarchy of differences of coanalytic sets, the hierarchy of $\boldsymbol{C}$-sets of Selivanovski [101], and the hierarchy of $\mathcal{R}$-sets of Kolmogorov [57, 58], but none of them exhausts the $\boldsymbol{\Delta}^1_2$ class. Moreover, the only piece of information on their Wadge rank is given by a result from Kechris and Martin mentioned by Steel [105], which states that, under (AD), the order type of the Wadge hierarchy of the ω decreasing differences of co-analytic sets is $\aleph_2$. Regarding the height of the Wadge hierarchy of the Borel sets, Wadge [115] proved it to be $V^{\omega_1}(2)$, the second value of the ω_1-th Veblen function of basis ω_1. The *Veblen hierarchy of basis* ω_1 consists of functions $(V^\xi)_{\xi<\omega_2}$ from $\omega_2 \setminus \{0\}$ to ω_2 that are defined as follows:

(i) V^0 is almost the exponentiation of base ω_1:
   - $V^0(1) = 1$;
   - $V^0(\alpha+1) = V^0(\alpha) \cdot \omega_1$ for all $0 < \alpha < \omega_2$;
   - $V^0(\alpha) = \omega_1^\alpha$ for all $\alpha < \omega_2$ limit of cofinality ω_1;
   - $V^0(\alpha) = \omega_1^{\alpha+1}$ for all $\alpha < \omega_2$ limit of cofinality ω.

(ii) For $\lambda > 0$, V^λ is the function that enumerates the fixpoints of cofinality ω_1 of the Veblen functions of lesser degrees:
   - $V^\lambda(1) = 1$;
   - $V^\lambda(1+\alpha)$ is the α^{th} fixpoint of cofinality ω_1 of all V^ξ with $\xi < \lambda$.

So far, the general situation may thus be roughly depicted by Fig. 1.1. Note that the inclusions are all provably strict in ZF + DC.

1.3.2 Our contribution

Just like in the Borel case with the Borel hierarchy or the Hausdorff-Kuratowski hierarchies, the three classical hierarchies mentioned on the $\boldsymbol{\Delta}^1_2$ class are very coarse. If they provide benchmarks for the study of the Wadge hierarchy of the $\boldsymbol{\Delta}^1_2$ sets, they nonetheless leave tremendous gaps to explore. This explo-

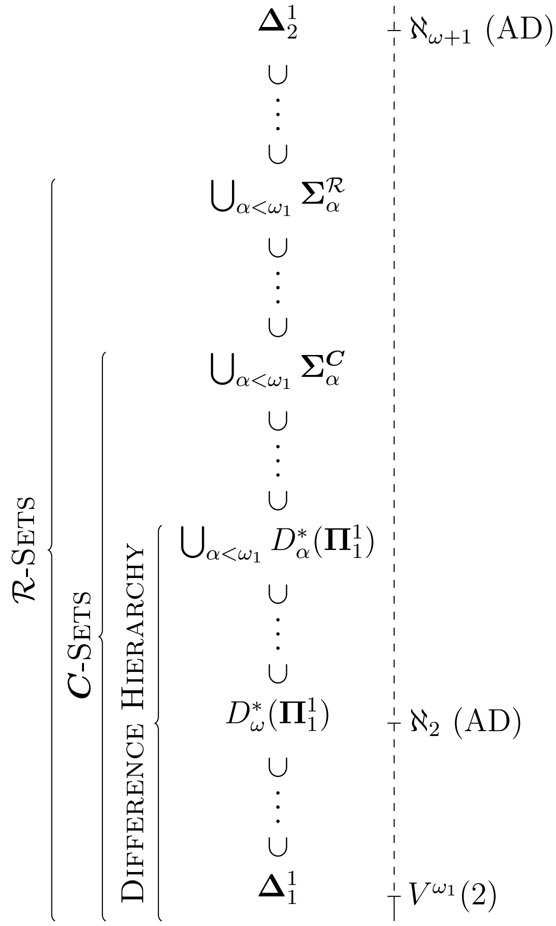


Figure 1.1: Wadge hierarchy of $\mathbf{\Delta}^1_2$ sets: the state of the art.

ration, or at least the beginning of this exploration, is the main subject of this thesis.

The difference hierarchies

It is well-known that *increasing* and *decreasing* differences do not coincide in general. For example, all countable increasing differences of open sets are included in the class of ω decreasing differences of open sets, which coincide with the $\mathbf{\Pi}^0_2$ class. This discrepancy relies on the fact that the open sets have the generalized reduction property, a structural property not shared by the closed sets. The situation is the same for the $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$ classes: the coanalytic sets, unlike the analytic sets, have the generalized reduction property, so that the increasing difference hierarchy of coanalytic sets is much finer than the decreasing difference hierarchy of coanalytic sets. For α countable, let $D_\alpha(\mathbf{\Pi}^1_1)$ and $D^*_\alpha(\mathbf{\Pi}^1_1)$ denote respectively the class of increasing and the class of decreasing α differences of coanalytic sets.

In Chapter 4, we study the increasing differences of coanalytic sets. Assuming coanalytic determinacy, our work provides the full description of the Wadge hierarchy of $\bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Pi}^1_1)$ sets, both in terms of pointclasses (*à la Louveau*) and complete sets (*à la Duparc*). Surprisingly enough, the set of operations and methods used in the Borel case is sufficient for this task, we so to speak only add the possibility for them to act on coanalytic sets. We compute the height of the Wadge hierarchy of $\bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Pi}^1_1)$, and we give another proof of a result due to Andretta and Martin [5] which states that the non-self-dual pointclasses closed under preimages by Borel functions included in $\bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Pi}^1_1)$ are exactly the classes $D_\alpha(\mathbf{\Pi}^1_1)$ and their duals. Zooming in the general picture, our contribution is depicted in Fig. 1.2.

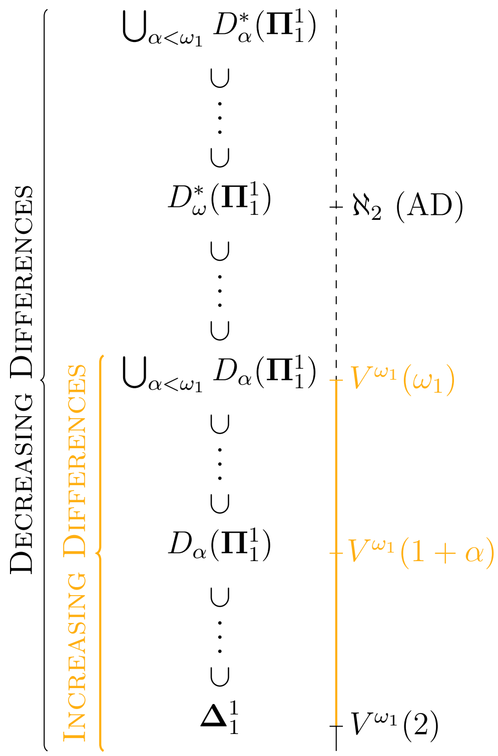


Figure 1.2: Our contribution to the description of the Wadge hierarchy of differences of coanalytic sets.

The $\boldsymbol{C}$-sets and the $\mathcal{R}$-sets

The $\boldsymbol{C}$-sets of Selivanovski constitute the smallest σ-algebra of subsets of the Baire space containing the open sets and closed under Suslin's operation $\boldsymbol{\mathcal{A}}$. They are set up in a hierarchy as follows:

- $\boldsymbol{\Sigma}_1^{\boldsymbol{C}} = \boldsymbol{\Sigma}_1^1$;
- for $0 < \alpha < \omega_1$, $\boldsymbol{\Pi}_\alpha^{\boldsymbol{C}} = \check{\boldsymbol{\Sigma}}_\alpha^{\boldsymbol{C}}$;
- for $1 < \alpha < \omega_1$, $\boldsymbol{\Sigma}_\alpha^{\boldsymbol{C}} = \boldsymbol{\mathcal{A}}\left(\bigcup_{\xi<\alpha} \boldsymbol{\Pi}_\xi^{\boldsymbol{C}}\right)$.

The class $\boldsymbol{\Sigma}_2^{\boldsymbol{C}}$ contains all the decreasing differences of coanalytic sets, and the class $\sigma D_2(\boldsymbol{\Pi}_1^1)$ of countable unions of differences of two coanalytic sets. From a complete coanalytic set, we define the operation $(D_2(\boldsymbol{\Pi}_1^{\boldsymbol{C}}), \cdot)$ which transforms complete open sets to $\sigma D_2(\boldsymbol{\Pi}_1^1)$-complete sets, preserves the Wadge ordering and is compatible with the operations already defined by Wadge and Duparc for the study of the Borel sets. This allows us to unravel an incomplete fragment of the Wadge hierarchy of the $\boldsymbol{\Sigma}_2^{\boldsymbol{C}}$ class. To climb further, we generalize this operation, beginning with a $\boldsymbol{\Pi}_\alpha^{\boldsymbol{C}}$-complete sets instead of a coanalytic set. This allows us to describe a cofinal but incomplete fragment of the Wadge hierarchy of Selivanovski's $\boldsymbol{C}$-sets.

The $\mathcal{R}$-sets of Kolmogorov are generated from the open sets by the operations of countable union and intersection, and closed under the transformation $\mathcal{R}$. They can be spread into a hierarchy of length ω_1 as follows

- $\boldsymbol{\Sigma}_1^{\mathcal{R}} = \boldsymbol{\Sigma}_1^1$;
- for $0 < \alpha < \omega_1$, $\boldsymbol{\Pi}_\alpha^{\mathcal{R}} = \check{\boldsymbol{\Sigma}}_\alpha^{\mathcal{R}}$;
- for $1 < \alpha < \omega_1$, $\boldsymbol{\Sigma}_\alpha^{\mathcal{R}} = R_\alpha(\boldsymbol{\Sigma}_1^0)$.

Where R_α denotes the α-th superposition of the $\mathcal{R}$-transform. The class $\boldsymbol{\Sigma}_2^{\mathcal{R}}$ contains all $\boldsymbol{C}$-sets and we can, as before, generalize the operation $(D_2(\boldsymbol{\Pi}_1^{\boldsymbol{C}}), \cdot)$ to unravel a cofinal but incomplete fragment of the Wadge hierarchy of Kolmogorov's $\mathcal{R}$-sets. Our general contribution to the study of the Wadge hierarchy of $\boldsymbol{\Delta}_2^1$ sets is depicted in Fig. 1.3.

1.4 From descriptive set theory to automata theory

> "Since the discovery of irrational numbers, the issue of impossibility has been one of the driving forces in mathematics. Computer science brings forward a related problem, that of difficulty. The mathematical expression of difficulty is complexity, the concept

which affects virtually all subjects in computing science, taking on various contents in various contexts."

André Arnold, Jacques Duparc, Filip Murlak, and Damian Niwiński, [8].

Theoretical computer science is the study, from a mathematical point of view, of *models of computation*. As such, its developments since the 1930s with the works of Church, Gödel, Klenne, Post and Turing, have always been strongly connected to mathematics and logic, with deep foundational questioning and motivations. In particular, the concept of *complexity* has become prominent, and among complexity measures topological complexity has grown to be more and more popular last decades [8, 41, 66, 95, 110]. One can indeed study and compare the expressive power, and thus in some sense the complexity of different models of computation by looking at the topological complexity of the tasks they can perform. Here we study tree automata, that is finite devices whose inputs are infinite labeled binary trees over a finite alphabet. For an automaton $\mathcal{A}$ and a tree t, $\mathcal{A}$ either accepts or rejects t, and the set of trees accepted by $\mathcal{A}$ is called the *language* of $\mathcal{A}$. A set of trees is called *regular* if it is the language of a certain automaton. If tree automata are finite devices that can appear to be quite rudimentary at first sight, their expressive power is nonetheless surprisingly and interestingly vast. Identifying the space of infinite trees with the Cantor space, Rabin [96] proved that all regular languages are in the $\mathbf{\Delta}^1_2$ class. The works of Gogacz, Michalewski, Mio and Skrzypczak [40], and Finkel, Lecomte, and Simonnet [36, 103] provide the exact bound for the complexity of regular languages: the class of $\mathcal{R}$-sets of finite ranks. We can thus use our knowledge developed on the $\mathcal{R}$-sets of the Baire space to study the class of all regular languages. Following and extending the works of Duparc, Facchini and Murlak [8, 32, 34, 85], we study the Wadge hierarchy of regular tree languages. To do so, we adapt the operations used in the descriptive set theory framework to construct a very long sequence (L_α) of strictly more and more complex regular languages. The length of this sequence is the ordinal $\varphi_\omega(0)$, where (φ_α) are the Veblen functions of basis ω. This sequence is moreover cofinal: for every regular language L, there exists an ordinal $\alpha < \varphi_\omega(0)$ such that $L \leq_W L_\alpha$.

We also investigate the difference between two subclasses of tree automata: the deterministic and the unambiguous automata. While the former class is now quite well understood, thanks in particular to the work of Murlak [8, 85], we do not know much about the expressive power of the latter. By definition, all deterministic automata are unambiguous, and it was shown by

Niwiński and Walukiewicz [91] that unambiguous automata do not recognize all regular languages. Building on an example due to Hummel [46] of an unambiguous automaton recognizing a $\mathbf{\Sigma}^1_1$-complete language, and adapting the operations used before, we prove that the height of the Wadge hierarchy restricted to unambiguously recognizable tree languages is at least $\varphi_2(0)$. Since Murlak [85] proved that the height of the Wadge hierarchy restricted to deterministically recognizable tree languages is $(\omega^\omega)^3+3$, our work illustrates the discrepancy between these two classes of tree automata and proves that unambiguous automata are much more complex than deterministic automata.

1.5 Organization of the thesis

Chapter 2: Preliminaries

This chapter is devoted to a quick recapitulation of descriptive set theory notions, and to a rapid presentation of the Wadge theory. Basic knowledge of set theory and topology is assumed, and can be found in books by Jech [50], Kunen [60], and Kuratowski [61]. Classical references for descriptive set theory are Moschovakis [80], Kechris [55], and Louveau [67]. Regarding the Wadge theory, in addition to Wadge's thesis [115] and Van Wesep [112] seminal paper on the subject, we refer the reader to the survey wrote by Andretta and Louveau [4] as an introduction to the third part of the Cabal seminar anthology [56].

Chapter 3: The Baire space and reductions by relatively continuous relations

In this chapter we prove that the conciliatory preorder is in fact induced by reductions by relatively continuous relations, as defined by Pequignot [93], when the set $\omega^{\leq\omega}$ is endowed with the prefix topology, and we show that under (AD) the conciliatory hierarchy and the Wadge hierarchy restricted to non-self-dual classes coincide via the mapping $C \mapsto C^b$. All the proofs in this chapter can be relativized to a pointclass with appropriate closure and determinacy properties, so that e.g. in ZF + DC the conciliatory hierarchy and the Wadge hierarchy restricted to non-self-dual Borel classes coincide, which gives a direct proof to a results of Duparc [26, Theorem 3]. Results in this chapter are part of a joint work with Jacques Duparc [29].

Chapter 4: Differences of coanalytic sets

This chapter is devoted to the extension of results obtained by Wadge [115], Louveau [68] and Duparc [26, 27] for the Borel sets to a wider pointclass: $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$, the class of increasing differences of coanalytic sets. We prove that, assuming $\mathrm{DET}(\boldsymbol{\Pi}^1_1)$ and adding the analytic class – or an analytic complete set, the operations and methods used in the Borel case give rise exactly to the Wage hierarchy of $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$. We give a full description of the Wadge hierarchy restricted to the class $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$, including its height. Results in this chapter concerning the *à la* Louveau approach will appear in an article by the author [38].

Chapter 5: A first glimpse above $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$

In this chapter, we offer a glimpse into $\boldsymbol{\Delta}^1_2$ pointclasses that lie above $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$. First we consider decreasing differences of coanalytic sets that coincide with the increasing differences only at the finite levels, but then become far more complex. In particular, one can prove that the class of ω decreasing differences of coanalytic sets contains $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$, and that under (AD) its Wadge rank is ω_2. Climbing further up, we consider the class of Selivanovski's $\boldsymbol{C}$-sets and the class of Kolmogorov's $\mathcal{R}$-sets. To unravel a fragment of their Wadge hierarchy, we define for each non-self-dual pointclass a new operation on sets denoted by $(D_2(\Gamma), \cdot)$. These new operations are designed to transform an open set into a set that is a countable union of $D_2(\Gamma)$ sets. For Γ with suitable closure properties, this operation preserves the Wadge ordering and behaves well with respect to the other operations used in the study of the Wadge hierarchy of Borel sets by Duparc. Using well chosen pointclasses Γ, we unravel a fragment of the Wadge hierarchy of $\mathcal{R}$-sets. More details and references on $\boldsymbol{C}$-sets can be found in Selivanovski [101], Burgess [19] and Louveau [67]; concerning $\mathcal{R}$-sets, we refer the reader to Kolmogorov [57, 58], Burgess [18, 20] and Kanovei [52].

Chapter 6: Application to Automata Theory

We transport some of the techniques we developed in the descriptive set theory framework to theoretical computer science and, more precisely, to *automata theory*. From definable subsets of the Baire space, we thus shift our attention to sets of full binary trees that are recognizable by automata. In this context, the use of topological tools has proved useful for the study of relative complexity and characterization of regular languages. After an introduction to this new framework and the formulation of relevant definitions and notations as well as classical results, we use operations on languages – inspired by the operations used in the Baire space case, to construct a cofinal sequence of strictly more and more complex regular tree languages. This fragment of the Wadge hierarchy of regular tree languages has length $\varphi_\omega(0)$, where (φ_α) are the Veblen functions of basis ω, which provides a lower bound for the height of this hierarchy. In the second part of this chapter, we study the discrepancy between deterministically and unambiguously recognizable languages by proving that the height of the Wadge hierarchy restricted to unambiguously recognizable tree languages is at least $\varphi_2(0)$, an ordinal tremendously larger than the height of the Wadge hierarchy restricted to deterministically recognizable languages which is $(\omega^\omega)^3+3$, as unraveled by Murlak [85]. Most of the results in this chapter are part of joint works with Duparc [29], Duparc and Hummel [31], and Duparc, Facchini and Michalewski [28].

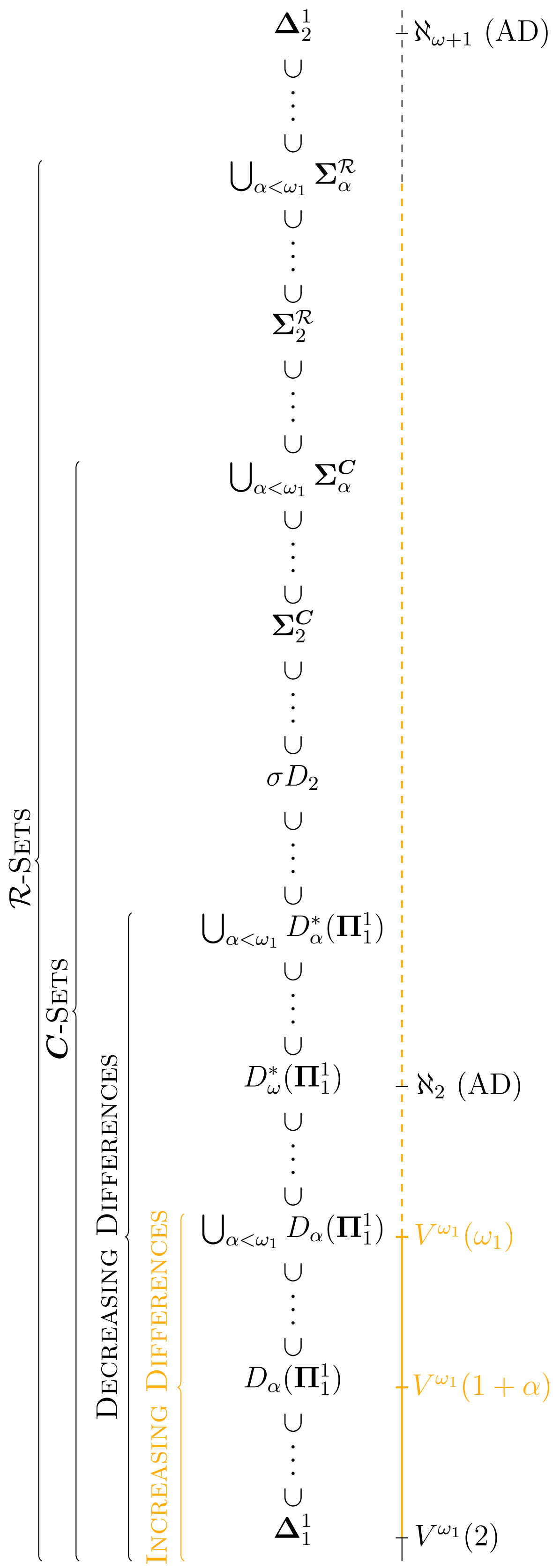


Figure 1.3: Our contribution to the study of the Wadge hierarchy of $\mathbf{\Delta}^1_2$ sets.

2 Preliminaries

2.1 The Borel and the Projective hierarchies

One of the main purposes of classical descriptive set theory is to describe and classify definable subsets of Polish spaces, i.e. second countable and completely metrizable topological spaces, by means of hierarchies, reducibilities and set-theoretic operations. Given a Polish space X, the σ-algebra of Borel sets $\mathbf{B}(X)$ is obtained from the open sets of X by the set-theoretical operations of complementation and countable unions. This class can then naturally be spread into a hierarchy of length ω_1, called the *Borel hierarchy*. More precisely, for every Polish space X and every countable ordinal $0 < \xi$, we define the classes $\mathbf{\Sigma}^0_\xi(X)$, $\mathbf{\Pi}^0_\xi(X)$ and $\mathbf{\Delta}^0_\xi(X)$ as follows:

- $\mathbf{\Sigma}^0_1(X)$ is the class of all the open subsets of X,
- $A \in \mathbf{\Pi}^0_\xi(X)$ if and only if $A^\complement \in \mathbf{\Sigma}^0_\xi(X)$,
- for $2 \leq \xi$, $A \in \mathbf{\Sigma}^0_\xi(X)$ if and only if there is a sequence $(A_n)_{n<\omega}$ of elements of $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(X)$ such that $A = \bigcup_{n<\omega} A_n$,
- $A \in \mathbf{\Delta}^0_\xi(X)$ if and only if $A \in \mathbf{\Sigma}^0_\xi(X)$ and $A \in \mathbf{\Pi}^0_\xi(X)$.

By convention, we set $\mathbf{\Pi}^0_0(X) = \{X\}$ and $\mathbf{\Sigma}^0_0(X) = \{\emptyset\}$. Note that, for every $\xi < \omega_1$, the following holds:

$$\mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1} \qquad \text{and} \qquad \mathbf{\Pi}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}.$$

This hierarchy provides a bottom-up description of the Borel sets since for any Polish space

$$\mathbf{B}(X) = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi(X) = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi(X).$$

For X uncountable, this hierarchy is strict, i.e. for every countable ordinal $0 < \xi$, $\mathbf{\Sigma}^0_\xi \backslash \mathbf{\Pi}^0_\xi \neq \emptyset$. Above the Borel class lie the *projective sets*, which are obtained from the Borel sets by taking projection[1] and complementation. If $B \subseteq X \times Y$, we denote by $\pi''(B) = \{x \in X : \exists y(x, y) \in B\}$ the projection

[1] Or equivalently direct image by continuous functions.

of B. Analogously to the case of Borel sets, the class $\mathbf{P}(X)$ of all projective sets of a Polish space X ramifies in a hierarchy of length ω, starting with the *analytic* sets. A subset of a Polish space X is *analytic* if it is the projection of a Borel subset of $X \times \omega^\omega$, where ω^ω denotes the Baire space. For every Polish space X and every positive integer n, we define the classes $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$ and $\mathbf{\Delta}^1_n(X)$ as follows:

- $A \in \mathbf{\Sigma}^1_1(X)$ if and only if A is an analytic subset of X,
- $B \in \mathbf{\Pi}^1_n(X)$ if and only if $B^\complement \in \mathbf{\Sigma}^1_n(X)$,
- $B \in \mathbf{\Sigma}^1_{n+1}(X)$ if and only if there is $C \in \mathbf{\Pi}^1_n(X \times \omega^\omega)$ such that $B = \pi''(C)$,
- $B \in \mathbf{\Delta}^1_n(X)$ if and only if $B \in \mathbf{\Sigma}^1_n(X)$ and $B \in \mathbf{\Pi}^1_n(X)$.

Note that, by Suslin's theorem, $\mathbf{B}(X) = \mathbf{\Delta}^1_1(X)$, and that for every positive integer n, the following holds:

$$\mathbf{\Sigma}^1_n \subseteq \mathbf{\Delta}^1_{n+1} \qquad \text{and} \qquad \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}.$$

This hierarchy provides a bottom-up description of the projective sets since for any Polish space

$$\mathbf{P}(X) = \bigcup_{n<\omega} \mathbf{\Sigma}^1_\xi(X).$$

2.2 Games and Determinacy

First considered by Russian and Polish mathematicians in the period between the two world wars, infinite games have played a prominent role in the development of modern descriptive set theory. We refer the interested reader to Larson [62], Mycielski [88], and Telgársky [109] for a thorough historical account on the interplay between descriptive set theory and infinite games.

Definition 2.1 (Gale-Stewart [39])**.** Let A be a subset of the Baire space. The *Gale-Stewart game* $G(A)$ is the following two-player infinite game:

$$\begin{array}{llllllllll} \mathrm{I}: & a_0 & & a_2 & & \cdots & a_{2n} & & a_{2n+2} & \cdots \\ \mathrm{II}: & & a_1 & & a_3 & & & a_{2n-1} & & \end{array}$$

Player I plays $a_0 \in \omega$, II then plays $a_1 \in \omega$, etc. I wins if and only if $a = (a_0, a_1, \ldots) \in X$.

A strategy for player I is a map $\sigma : \omega^{<\omega} \longrightarrow \omega$. The beginning of a play where player I follows the strategy σ is

$$\begin{array}{lcccccccc} \mathrm{I}: & \sigma(\emptyset) & & \sigma(a_1) & & & \sigma(a_1, a_3) & & \\ & & & & & \dots & & & \dots \\ \mathrm{II}: & & a_1 & & a_3 & & & a_5 & \end{array}$$

We define mutatis mutandis a strategy for player II. A strategy for a player is *winning* if the player wins every time he follows it, whatever his opponent plays. We say that the game $G(A)$, or just the set A, is *determined*, if one of the two players has a winning strategy in this game. A pointclass is determined if and only if all its elements are determined.

Clearly, using the axiom of Dependent Choices (DC), two players cannot have both a winning strategy for the same game. But, contrarily to the case of finite games[2], Gale and Stewart [39] proved in their seminal article on the subject that assuming the axiom of choice (AC), there exists a subset of the Baire space which is not determined. Thus arose the questions of establishing which subsets and which pointclasses of the Baire space are determined. Whilst the first question might never meet a satisfying answer – other than a slight refinement of the Lapalissade *the class of determined subsets of the Baire space is the class of all subsets of the Baire space that are determined*, the second one has been nicely settled by a combination of works due to Martin and Harrington. First, Martin [78] proved that in ZFC all the Borel subsets of the Baire space are determined, concluding twenty years of cumulative work initiated by Gale and Stewart, and later pursued by Wolfe [117], Davis [24], and Paris [92]. Then, Harrington [42] showed that the determinacy of all analytic sets implied the existence of sharps, a large cardinal hypothesis independent from ZFC. The largest provably determined pointclass in ZFC is therefore the Borel sets.

From another perspective, assuming the determinacy of a pointclass with appropriate closure properties is sufficient to prove its regularity. Proof of these consequences of determinacy, such as measurability, the perfect set property and the Baire property, were first given in a sequence of papers by Mycielski [86, 87] and Mycielski and Świerczkowski [90]. The connection between determinacy and regularity properties led to the introduction of (AD), the Axiom of (full) Determinacy, by Mycielski and Steinhaus [89], which asserts that every subset of the Baire space is determined. This axiom

[2]See Morgenstern and Von Neumann [114].

contradicts (AC), but is consistent with ZF + DC and has the goodness of excluding annoying counterexamples, so that it has been argued to be a natural alternative framework for descriptive set theory.

Most of the results in this thesis depend on determinacy hypotheses, and these assumptions will be made clear every time they are needed. In general, our ambient theory is ZFC, to which we add the hypothesis that a certain pointclass Γ is determined, in symbol DET(Γ). Sometimes, the full axiom of determinacy (AD) is needed: in this case it is understood that we work in ZF + DC + AD. Although the tension between choice and determinacy will not be discussed here, it is worth noticing their subtle interaction all along this work.

2.3 The Wadge hierarchy, pointclasses and boolean operations

> "The Wadge Hierarchy is the ultimate analysis of $\mathcal{P}(\omega^\omega)$ in terms of topological complexity [...]"
>
> Alessandro Andretta, Alain Louveau, [4].

The Wadge theory is in essence the theory of *pointclasses*. Let X be a topological space. A pointclass is a collection of subsets of X that is closed under continuous preimages. For Γ a pointclass, we denote by $\check{\Gamma}$ its *dual* class containing all the subsets of X whose complements are in Γ, and by $\Delta(\Gamma)$ the ambiguous class $\Gamma \cap \check{\Gamma}$. If $\Gamma = \check{\Gamma}$, we say that Γ is self-dual.

We only consider the *Baire space* in this thesis, with the usual topology. The *Wadge preorder* $\leq_W$ on $\mathcal{P}(\omega^\omega)$ is defined as follows: for $A, B \subseteq \omega^\omega$, $A \leq_W B$ if and only if there exists $f : \omega^\omega \longrightarrow \omega^\omega$ continuous such that $f^{-1}(B) = A$. For $A, B \subseteq \omega^\omega$, we write $A <_W B$ if and only if $A \leq_W B$ but $B \nleq_W A$. The Wadge preorder induces an equivalence relation $\equiv_W$ whose equivalence classes are called the *Wadge degrees*, and denoted by $[A]_W$. We say that the set $A \subseteq \omega^\omega$ is *self-dual* if it is Wadge equivalent to its complement, that is if $A \equiv_W A^\complement$, and *non-self-dual* if it is not. We use the same terminology for the Wadge degrees.

A useful game characterization is provided by the *Wadge game*, a two players infinite game. Let $A, B \subseteq \omega^\omega$, in the Wadge game $W(A, B)$ player I plays first an integer x_0, II answers with an integer y_0, and so on and so forth. Player II has the possibility to skip, even ω times, provided she also plays infinitely often. At the end of the game, each player has constructed an infinite sequence, x for I and y for II. Player II wins the game if and only if $(x \in A \leftrightarrow y \in B)$. Noticing that strategies for II can be viewed as continuous functions, we have:

$$\text{II has a winning strategy in } W(A,B) \quad \longleftrightarrow \quad A \leq_W B.$$

Given a pointclass Γ with suitable closure properties, the assumption of the determinacy of Γ is sufficient to prove that Γ is semi-linearly ordered by $\leq_W$, denoted SLO(Γ), i.e. that for all $A, B \in \Gamma$,

$$A \leq_W B \qquad \text{or} \qquad B \leq_W A^{\complement}.$$

and that $\leq_W$ is well founded when restricted to sets in Γ.[3] Under these conditions, the Wadge degrees of sets in Γ with the induced order is thus a hierarchy called the *Wadge hierarchy*. There exists a unique ordinal, called the height of the Γ-Wadge hierarchy, and a mapping d_w^Γ from the Γ-Wadge hierarchy onto its height, called the *Wadge rank*, such that, for every A, B non-self-dual in Γ, $d_w^\Gamma(A) < d_w^\Gamma(B)$ if and only if $A <_W B$ and $d_w^\Gamma(A) = d_w^\Gamma(B)$ if and only if $A \equiv_W B$ or $A \equiv_W B^{\complement}$. The wellfoundedness of the Γ-Wadge hierarchy ensures that the Wadge rank can be defined by induction as follows:

- $d_w^\Gamma(\emptyset) = d_w^\Gamma(\emptyset^{\complement}) = 1$
- $d_w^\Gamma(B) = \sup\left\{d_w^\Gamma(A) + 1 : A \text{ is non-self-dual}, A <_W B\right\}$ for $A >_W \emptyset$.

Note that given two pointclasses Γ and Γ', for every $A \in \Gamma \cap \Gamma'$,

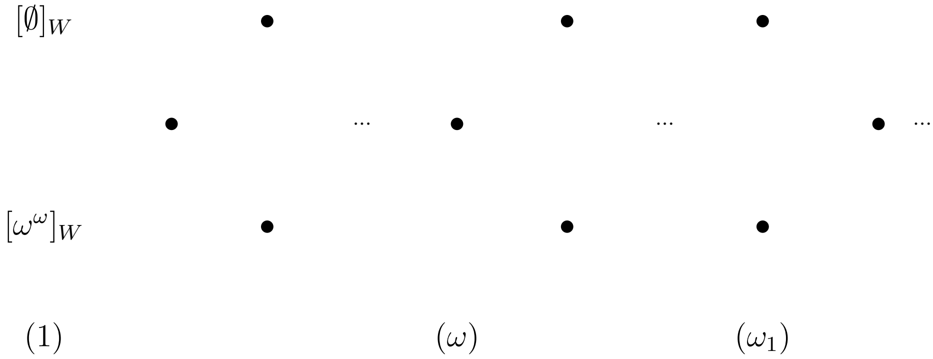
$$d_w^\Gamma(A) = d_w^{\Gamma'}(A).$$

Under sufficient determinacy assumptions, we can therefore safely speak of *the* Wadge rank of a subset of the Baire space, denoted by d_w, as its Wadge rank with respect to any topological class with suitable closure and determinacy properties including it.

The general diamond-like shape of the Wadge hierarchy is depicted below. At the bottom of the hierarchy lie the empty set and the whole Baire space, dual and mutually incomparable. Then self-dual and non-self-dual degrees alternate, with self-dual degrees at limit levels of cofinality ω, and non-self-

[3] For more details about the relation between determinacy and the Wadge preorder, see Andretta [2, 3].

dual degrees at limit levels of cofinality strictly greater.



There is a strong connection between pointclasses included in Γ and Wadge degrees of sets in Γ since all non-self-dual pointclasses are of the form

$$\{B \subseteq \omega^\omega : B \leq_W A\}$$

for some non-self-dual set A, while self-dual pointclasses are all of the form

$$\{B \subseteq \omega^\omega : B <_W A\},$$

for some $A \subseteq \omega^\omega$. We say that a pointclass is a *Wadge class* if it is of the form $\{B \subseteq \omega^\omega : B \leq_W A\}$ for some $A \subseteq \omega^\omega$. Observe that the non-self-dual Wadge classes are exactly the non-self-dual pointclasses. Moreover, notice that all self-dual pointclasses that are not Wadge classes are of the form $\Gamma' \cup \check{\Gamma}'$ for some non-self-dual pointclass Γ'.

We have a direct correspondence between $(\mathcal{P}(\omega^\omega), \leq_W)$ restricted to Γ and the pointclasses included in Γ with the inclusion: the pointclasses are exactly the initial segments of the Wadge hierarchy. The semi-linear ordering property becomes then: for any pointclasses Γ' and Γ'' included in Γ,

$$\Gamma' \subseteq \Gamma'' \text{ or } \check{\Gamma}'' \subseteq \Gamma'.$$

We can define the Wadge rank counterpart for pointclasses:

$$|\Gamma'|_w = \sup \{d_w(A) + 1 : A \text{ is non-self-dual and in } \Gamma'\}.$$

In his thesis, Wadge begins the analysis of the hierarchy when restricted to the Borel subsets of the Baire space and initiates two approaches for its study, based on the correspondence between $(\mathcal{P}(\omega^\omega), \leq_W)$ and the pointclasses with the inclusion. The first one, later completed by Louveau [68], relies on boolean operations.

An ω-ary operation $\mathcal{O}$ is a function:

$$\mathcal{O} : \mathcal{P}(\omega^\omega)^\omega \longrightarrow \mathcal{P}(\omega^\omega).$$

which assigns a set to a countable sequence of sets. The *truth table* $T_\mathcal{O}$ for an operation $\mathcal{O}$ is a subset of $\mathcal{P}(\omega)$ such that for any sequence of subsets of the Baire space $(A_n)_{n\in\omega}$, and for all $x \in \omega^\omega$,

$$x \in \mathcal{O}\left((A_n)_{n\in\omega}\right) \leftrightarrow \{n \in \omega : x \in A_n\} \in T_\mathcal{O}.$$

Not all operations admit a truth table, but each truth table completely determines an operation. Operations that admit a truth table are said to be (ω-ary) *boolean operations*, and were first defined and studied by Kantorovich and Livenson [53][4], and Lyapunov [76][5]. We say that a boolean operation is of a certain complexity (Borel, $\boldsymbol{\Pi}^1_1$, etc.) if its truth table is of this complexity as a subset of the Cantor space 2^ω.

Wadge proved that all the non-self-dual Borel pointclasses can be obtained by ω-ary Borel boolean operations on open sets – a result later generalized to all non-self-dual pointclasses of the Baire space by Van Wesep [111] under (AD), using of course arbitrary ω-ary boolean operations. Louveau's work provides a description of all the Borel Wadge classes, and thus of the whole Wadge hierarchy on the Borel sets, by means of boolean operations.

The second approach to the study of the Wadge hierarchy aims to define and make use of operations on sets, such as the sum and the countable multiplication, in order to give, for each non-self-dual Wadge class of Borel subsets, a canonical complete set. It relies heavily on the peculiar characterization of the continuous reducibility relation made available through the Wadge game. In an effort to extend this approach, Duparc [26, 27] introduces the so-called *conciliatory sets*, namely subsets of $\omega^{\leq\omega}$, as an ansatz. We give more details about this approach in the next section.

2.4 The conciliatory sets

Conciliatory sets are sets of finite or infinite sequences of integers, that is sets included in $\omega^{\leq\omega}$. From a conciliatory set $B \subseteq \omega^{\leq\omega}$, one defines the set $B^b \subseteq (\omega \cup \{b\})^\omega$ of infinite sequences by

$$B^b = \left\{\alpha \in (\omega \cup \{b\})^\omega : \alpha_{[\ /b]} \in B\right\},$$

[4] who call them *analytic operations*.
[5] who calls them *set-theoretical operations*.

where b is an extra symbol that stands for "blank", and $\alpha_{[\ /b]}$ is the sequence obtained from α once all occurrences of b have been removed. To lighten the notations, we write ω_b to denote the set $\omega \cup \{b\}$. Since $(\omega_b)^\omega$ endowed with the product of the discrete topology is homeomorphic to the Baire space, we identify these two spaces via the following isometry

$$\begin{aligned} h : (\omega_b)^\omega &\longrightarrow \omega^\omega \\ (x_0, x_1, \ldots) &\longmapsto (x'_0, x'_1, \ldots) \end{aligned}$$

where

$$x'_i = \begin{cases} 0 & \text{if } x_i = b; \\ x_i + 1 & \text{else.} \end{cases}$$

The use of h is always implicit and, depending on the context, we consider B^b to be a subset of $(\omega_b)^\omega$ or of ω^ω via this homeomorphism. We define no topology at all on conciliatory sets, but for any pointclass Γ we allow ourselves to say that $B \subseteq \omega^{\leq\omega}$ is in Γ_c if and only if $B^b \in \Gamma$.

Example 2.2. The conciliatory set $\{\langle 0 \rangle\} \in (\mathbf{\Sigma}^0_2)_c$, for

$$\{\langle 0 \rangle\}^b = \bigcup_{n \in \omega} b^n{}^\frown 0^\frown b^\omega$$

is $\mathbf{\Sigma}^0_2$.

The shift from infinite sequences to both finite and infinite sequences, and the definition of a preorder $\leq_c$ on the subsets of $\omega^{\leq\omega}$ is motivated by natural game theoretic considerations, and in particular by the will to symmetrise the Wadge game. Let A and B be two conciliatory sets. The conciliatory game $C(A, B)$ is the following: both players play integers, I begins, II answers, and so on and so forth. The winning conditions for II are: if the sequence of I is in A, then she has to produce a sequence in B, and if it is not in A, she has to produce a sequence not in B. But in the conciliatory game, I can also skip, and both players do not have to produce an infinite sequence, so that at the end of the game, they might even have played only finitely many integers. From this game we define the *conciliatory preorder*: for any conciliatory sets A and B, we say that $A \leq_c B$ if and only if II has a winning strategy in $C(A, B)$.

Lemma 2.3. *Let A and B be two conciliatory sets. The games $C(A, B)$ and $W(A^b, B^b)$ are equivalent, i.e. player* I *(respectively* II*) has a winning strategy in the game $C(A, B)$ if and only if player* I *(respectively* II*) has a winning strategy in the game $W(A^b, B^b)$.*

Proof. By replacing "skips" in the conciliatory game by "blanks" in the Wadge game, a winning strategy for a player in one game gives rise to a winning strategy for the same player in the other game. Notice that a winning strategy for II in $C(A, B)$ provides a winning strategy for II in $W(A^b, B^b)$ which never requires her to skip. $\square$

Hence the determinacy of the conciliatory game for a certain class Γ_c is equivalent to the determinacy of the Wadge game for the corresponding pointclass, and the map $C \longmapsto C^b$ is an embedding from $(\mathcal{P}(\omega^{\leq\omega}), \leq_c)$ to $(\mathcal{P}(\omega^\omega), \leq_W)$. Observe moreover that there is no self-dual set with respect to the conciliatory preorder, since player I always has the following strategy in the game $C(A, A^\complement)$: at first he skips, and then simply copies II's moves, so that the range of $C \longmapsto C^b$ is included in the non-self-dual degrees of the Wadge hierarchy. A main purpose of Chapter 3 is to prove Theorem 3.10 stating that, assuming (AD), all non-self-dual degrees are reached by $C \longmapsto C^b$. More precisely, we will show that $A \subseteq \omega^\omega$ is non-self-dual if and only if there exists some conciliatory set $B \subseteq \omega^{\leq\omega}$ such that $A \equiv_W B^b$. Therefore, thanks to the following fact, one can indistinctively study either the Wadge hierarchy or the conciliatory hierarchy.

Fact 2.4. *Let $A \subseteq \omega^\omega$ be such that $A \equiv_W A^\complement$. Then there exists a family $(A_n)_{n\in\omega}$ of non-self-dual subsets of the Baire space such that*

$$A \equiv_W \bigcup_{n\in\omega} n^\frown A_n = \bigcup_{n\in\omega} \left\{ n^\frown x \in \omega^\omega : x \in A_n \right\}.$$

The conciliatory counterpart of the Wadge rank is naturally defined as follows.

Definition 2.5. For any conciliatory set C, we define:

$$d_c(C) = \begin{cases} 1 & \text{iff } C \text{ or } C^\complement \text{ is the empty set;} \\ \sup\{d_c(C') + 1 : C' <_c C\} & \text{else.} \end{cases}$$

It is a consequence of Theorem 3.10 that the Wadge and the conciliatory ranks are compatible, i.e. $d_c(C) = d_w(C^b)$ for all conciliatory set C.

The aim of the approach initiated by Duparc is to give a complete set for each conciliatory degree of the considered pointclass. In order to do so, one can define set theoretical counterparts to the following ordinal operations on conciliatory sets: the sum, the multiplication by a countable ordinal and the countable supremum. First, denote by shift the map from $(\omega_b)^{\leq\omega}$ to

$(\omega_b \smallsetminus \{0\})^{\leq\omega}$ that shifts each entry of the sequence by one but keeps the blanks, i.e.

$$\begin{aligned}\text{shift} : (\omega_b)^{\leq\omega} &\longrightarrow (\omega_b \smallsetminus \{0\})^{\leq\omega}\\ x &\longmapsto \text{shift}(x)\end{aligned}$$

where shift(x) and x have the same length, and are such that for every n strictly smaller than the length of x,

$$\text{shift}(x)(n) = \begin{cases} b & \text{if } x_n = b;\\ x_n + 1 & \text{else.}\end{cases}$$

For all $C \subseteq \omega^{\leq\omega}$, let shift$(C)$ denote the set of all shifted sequences of C:

$$\text{shift}(C) = \{\text{shift}(x) : x \in C\}.$$

Note that for every $A, B \subseteq \omega^\omega$, $A \leq_W B$ if and only if shift$(A) \leq_W$ shift(B). Moreover, for any conciliatory set D,

$$\text{shift}(D^b) \equiv_W \text{shift}(D)^b.$$

We can now define the counterparts to the ordinal sum and countable supremum. Let $(A_i)_{i\in\omega}$ be a family of conciliatory sets.

(a) $A_0 + A_1 = \text{shift}(A_1) \cup \left\{u^\frown\langle 0\rangle^\frown x : u \in (\omega \smallsetminus \{0\})^{<\omega}, x \in A_0\right\}$;
(b) $\mathbf{sup}_{i\in\omega}\{A_i\} = \left\{\langle i\rangle^\frown x : x \in A_i\right\}$.

From these operations, we can define by induction the counterpart to the ordinal multiplication by a countable ordinal. Let A be a conciliatory set.

- $A \cdot 1 = A$;
- $A \cdot (\nu + 1) = (A \cdot \nu) + A$, for ν countable;
- $A \cdot \gamma = \mathbf{sup}_{i\in\omega} A \cdot \gamma_i$, for $\gamma = \sup_{i\in\omega} \gamma_i$ countable and limit.

These operations are defined to behave well with respect to the conciliatory rank.

Proposition 2.6 ([26, Theorem 4])**.** *Let $(A_i)_{i\in\omega}$ be any family of conciliatory sets, and $\nu < \omega_1$.*

(a) $d_c(A_0 + A_1) = d_c(A_0) + d_c(A_1)$;
(b) $d_c(A_0 \cdot \nu) = d_c(A_0) \cdot \nu$;
(c) $d_c(\mathbf{sup}_{i\in\omega}\{A_i\}) = \sup\{d_c(A_i) : i \in \omega\}$.

These operations allow us tu construct from the empty set the ω_1 first degrees of the conciliatory hierarchy. To go further, we need a counterpart to the exponentiation of basis ω_1 and beyond that, to the *Veblen functions*.

Definition 2.7. The Veblen hierarchy of base ω_1 consists of functions $(V^\xi)_{\xi<\omega_2}$ from $\omega_2 \smallsetminus \{0\}$ to ω_2 which are defined as follows:

(i) V^0 is almost the exponentiation of base ω_1:
  - $V^0(1) = 1$;
  - $V^0(\alpha+1) = V^0(\alpha)\cdot\omega_1$ for all $0 < \alpha < \omega_2$;
  - $V^0(\alpha) = \omega_1^\alpha$ for all $\alpha < \omega_2$ limit of cofinality ω_1;
  - $V^0(\alpha) = \omega_1^{\alpha+1}$ for all $\alpha < \omega_2$ limit of cofinality ω.

(ii) For $\lambda > 0$, V^λ is the function that enumerates the fixpoints of cofinality ω_1 of the Veblen functions of lesser degrees:
  - $V^\lambda(1) = 1$;
  - $V^\lambda(1+\alpha)$ is the α^{th} fixpoint of cofinality ω_1 of all V^ξ with $\xi < \lambda$.

The set theoretical counterparts to this ordinal hierarchy come from a generalization of the *eraser game* [26]. We denote by $\omega_{\twoheadleftarrow}$ the set $\omega \cup \{\twoheadleftarrow\}$, and from a set $A \subseteq \omega^{\leq\omega}$, we define the set $A^{\approx} \subseteq (\omega_{\twoheadleftarrow})^{\leq\omega}$ by

$$A^{\approx} = \left\{\alpha \in (\omega_{\twoheadleftarrow})^{\leq\omega} : \alpha^{\looparrowleft} \in A\right\}$$

where ${}^{\looparrowleft}$ is the operation that realizes "$\twoheadleftarrow$" into an eraser. It is inductively defined in the following way:

- $\varepsilon^{\looparrowleft} = \varepsilon$, where ε stands for the empty sequence;
- for α finite with $|\alpha^{\looparrowleft}| = k$:
  (i) $(\alpha^\frown i)^{\looparrowleft} = \alpha^{\looparrowleft\frown} i$, if $i \in \omega$;
  (ii) $(\alpha^\frown \twoheadleftarrow)^{\looparrowleft} = \alpha^{\looparrowleft} \restriction (k-1)$, if $k > 0$;
  (iii) $(\alpha^\frown \twoheadleftarrow)^{\looparrowleft} = \varepsilon$, if $k = 0$.
- for α infinite, $\alpha^{\looparrowleft} = \lim_{n\in\omega}(\alpha \restriction n)^{\looparrowleft}$.

We once again make a slight abuse of notation, identify $(\omega_{\twoheadleftarrow})^{\leq\omega}$ and $\omega^{\leq\omega}$, and write also $A^{\approx}$ for the corresponding conciliatory set. One can prove that for any conciliatory set O in $(\mathbf{\Sigma}^0_1)_c \smallsetminus (\mathbf{\Pi}^0_1)_c$,

$$O^{\approx} \in (\mathbf{\Sigma}^0_2)_c \smallsetminus (\mathbf{\Pi}^0_2)_c.$$

Setting ${}^{\approx}$ for ${}^{\approx_1}$, the iteration of this idea provides us with a family of operations ${}^{\approx_\xi}$, for $\xi < \omega_1$ which enjoys the following property.

Proposition 2.8 ([27, Proposition 34])**.** *Let O be a conciliatory set in $(\mathbf{\Sigma}^0_1)_c \smallsetminus (\mathbf{\Pi}^0_1)_c$. Then for all $0 < \xi < \omega_1$:*

$$O^{\approx_\xi} \in (\mathbf{\Sigma}^0_{1+\xi})_c \smallsetminus (\mathbf{\Pi}^0_{1+\xi})_c.$$

These operations preserve the conciliatory ordering: for any conciliatory sets A and B and any $0 < \xi < \omega_1$:

$$A \leq_c B \quad \Longrightarrow \quad A^{\approx_\xi} \leq_c B^{\approx_\xi};$$

and their compositions behave also well, since for every conciliatory A and $0 < \xi, \nu < \omega_1$:

$$(A^{\approx_\xi})^{\approx_\nu} \leq_c A^{\approx_{\xi+\nu}}.$$

Now setting

$$V_\nu(A) = A^{\approx_{\omega^\nu}}$$

for $A \subseteq \omega^{\leq\omega}$ and $\nu < \omega_1$, the following correspondence between the operations ${}^{\approx_\xi}$ and the ranks can be proved.

Proposition 2.9 ([27, Proposition 41])**.** *For all conciliatory set A and ordinal $\nu < \omega_1$:*

$$d_c\big(V_\nu(A)\big) = V^\nu\big(d_c(A)\big).$$

It is one of the main results of [27] that $(\oplus, (\odot\nu)_{\nu<\omega_1}, \mathbf{sup}, (V_\xi)_{\xi<\omega_1})$ generates, up to complement, the whole conciliatory hierarchy of Borel sets from the empty set.

For all $\xi < \omega_1$, there exists a kind of inverse for the ${}^{\approx_\xi}$ operation, denoted by ${}^{\sim_\xi}$, which is defined on the subsets of the Baire space. Recall that for any sequence of Borel subsets $B = (B_n)_{n\in\omega}$ of the Baire space, there exists a Polish zero-dimensional topology $\mathcal{T}'$ on ω^ω, finer than the original one but with the same Borel sets, such that each B_n is open in $(\omega^\omega, \mathcal{T}')$. Let φ_B be the continuous function from $(\omega^\omega, \mathcal{T}')$ to the Baire space given by the identity on the underlying set. For any $A \subseteq \omega^\omega$, we can observe the effect on A of the change of topology by looking at $\varphi_B^{-1}(A)$. Duparc [26, 27] defines *question trees* to encode the change of topology in a particular way that fits the game point of view, by means of auxiliary questions. To a ξ-question tree T is associated a family of Borel subsets $(T_n)_{n\in\omega} \subseteq \mathbf{\Sigma}^0_{1+\xi}$, and each sequence of $\mathbf{\Sigma}^0_{1+\xi}$ subsets of the Baire space is coded by a ξ-question tree. For T a ξ-question tree and $A \subseteq \omega^\omega$, we denote by A^T the set $\varphi_T^{-1}(A)$ obtained from A after the modification of the topology induced by T. We only give the formal definition of the 1-question trees here, the general case can be found in [27, Definition 21].

Definition 2.10 ([26, Definition 25])**.** Given $A \subseteq \omega^\omega$ and $(F_u)_{u\in(\omega^{<\omega}\smallsetminus\emptyset)}$ a family of closed subsets of the Baire space abbreviated by (F_u), we define $A^{(F_u)} \subseteq [T_{A^{(F_u)}}]$, where $T_{A^{(F_u)}}$ is a non-empty pruned tree on the alphabet

$$\Lambda = \omega \cup \left\{\langle 1\rangle^\frown v : v \in \omega^{<\omega}\right\}.$$

A sequence $x \in (\Lambda)^\omega$ is in $[T_{A^{(F_u)}}]$ if and only if, for all integer n,
(a) $x(2n) \in \omega$;
(b) $x(2n+1) = 0$ or $x(2n+1) = \langle 1 \rangle^\frown v$ with $v \in \omega^{<\omega}$.

Moreover, setting $u = \langle x(2i) : i \leq n \rangle$ and $x' = \langle x(2i) : i \in \omega \rangle$, the following conditions must hold:

– if $x(2n+1) = 0$, then $x' \in F_u$;
– if $x(2n+1) = \langle 1 \rangle^\frown v$, then $u^\frown v$ must be an initial segment of x' which verifies $u^\frown v^\frown y \notin F_u$ for any $y \in \omega^\omega$.

Now $A^{(F_u)} \subseteq [T_{A^{(F_u)}}]$ is defined by

$$x \in A^{(F_u)} \Leftrightarrow x' \in A.$$

For all $\xi < \omega_1$, we set:

$$A^{\not\approx_\xi} = \text{ a } <_W \text{-minimal element of } \{A^T : T \text{ is a } \xi\text{-question tree}\}.$$

It is well-defined and satisfies the following properties.

Proposition 2.11. *Let $\xi < \omega_1$ and $(B_n)_{n\in\omega}$ be a sequence of $\mathbf{\Sigma}^0_{1+\xi}$ subsets of the Baire space. Then for all $i \in \omega$,*

$$B_i^{\not\approx_\xi} \in \mathbf{\Sigma}^0_1,$$

and there exists a $\xi+1$-question tree T such that for all $i \in \omega$:

$$B_i^T \in \mathbf{\Sigma}^0_1.$$

These operations preserve the Wadge ordering. Indeed for every sets A, B and any ordinal $0 < \xi < \omega_1$:

$$A \leq_W B \quad \Longrightarrow \quad A^{\not\approx_\xi} \leq_W B^{\not\approx_\xi}.$$

The operation ${}^{\not\approx_\xi}$ is not exactly the inverse of ${}^{\approx_\xi}$, but the following holds.

Proposition 2.12 ([27, Proposition 28 and Proposition 31])**.** *Let $A \subseteq \omega^\omega$, $B \subseteq \omega^{\leq\omega}$ and $0 < \xi < \omega_1$. Then*

$$A^{\not\approx_\xi} \leq_W B^b \Longrightarrow A \leq_W \left(B^{\approx_\xi}\right)^b;$$

and

$$B^b \equiv_W \left((B^{\approx_\xi})^b\right)^{\not\approx_\xi}.$$

3 The Baire space and reductions by relatively continuous relations

In an attempt to give a meaning to the conciliatory approach, we use the concept of *admissible representation*, which is the starting point of the development of computable analysis from the Type-2 theory of effectivity point of view (see Weihrauch [116]). This simple yet fundamental idea arises from the wish to code points of a topological space by elements of the Baire space. In other words: to *represent* a topological space via the Baire space. In this chapter we prove that the conciliatory preorder is in fact induced by reductions by relatively continuous relations when the set $\omega^{\leq\omega}$ is endowed with the prefix topology, and we show that under (AD) the conciliatory hierarchy and the Wadge hierarchy restricted to non-self-dual classes coincide via the mapping $C \mapsto C^b$. Note that all the proofs in this chapter can be relativized to a pointclass with appropriate closure and determinacy properties, so that e.g. in ZF + DC the conciliatory hierarchy and the Wadge hierarchy restricted to non-self-dual Borel classes coincide. Results in this chapter are part of a joint work with Jacques Duparc [29].

3.1 Representations and reduction by continuous relations

Recall that a topological space X is *second countable* if it admits a countable basis of open sets, and that it is T_0 if for any two points x, y of X, there exists an open set that contains one of these points but not the other. Let X be a second countable T_0 space, and f, g two partial functions from the Baire space to X. We say that $f \preceq g$ if and only if there exists a continuous function $h : \mathrm{dom}(f) \to \mathrm{dom}(g)$ such that for all $x \in \mathrm{dom}(f)$, $g \circ h(x) = f(x)$. Notice that the set of partial continuous functions from ω^ω to X is downward closed with respect to $\preceq$.

Definition 3.1. Let X be a second countable T_0 space. A partial continuous function ρ from ω^ω to X is called an *admissible representation* of X if for every partial continuous function f from ω^ω to X, $f \preceq \rho$ holds.

Notice that any admissible representation of X must be onto, for it must be above all constant functions.

Fact 3.2. *Every second countable T_0 space admits an admissible representation.*

Admissible representations can be used to define the *reductions by relatively continuous relations*, a notion introduced by Pequignot [93] to generalize the Wadge hierarchy to second countable T_0 spaces.

Let X, Y be two second countable T_0 spaces, we say that $R \subseteq X \times Y$ is a *total relation* from X to Y if for all $x \in X$ there exists $y \in Y$ such that $(x, y) \in R$.

Definition 3.3. Let X, Y be two second countable T_0 spaces, and a total relation R from X to Y. We say that R is *relatively continuous* if for some admissible representations ρ_X of X and ρ_Y of Y, there exists a continuous *realizer* $f : \mathrm{dom}(\rho_X) \to \mathrm{dom}(\rho_Y)$ such that for every $z \in \mathrm{dom}(\rho_X)$ we have

$$\big(\rho_X(z), \rho_Y \circ f(z)\big) \in R.$$

Relatively continuous relations were first studied by Brattka and Hertling [17]. Observe that if R and S are total relations from X, to Y such that R is relatively continuous and $R \subseteq S$, then S is also relatively continuous.

Definition 3.4 ([93])**.** Let X, Y be two second countable T_0 spaces, $A \subseteq X$ and $B \subseteq Y$. We say that A is reducible to B, and write $A \preccurlyeq_W B$, if there exists a total relatively continuous relation R from X to Y which is a reduction of A to B, i.e.:

$$\forall x \in X \ \forall y \in Y \ \big[R(x, y) \to (x \in A \leftrightarrow y \in B)\big].$$

The relation $\preccurlyeq_W$ is merely by definition a quasi-order, and is strongly connected to $\leq_W$.

Fact 3.5. *Let X, Y be two second countable T_0 spaces with fixed admissible representations ρ_X and ρ_Y. For every $A \subseteq X$ and $B \subseteq Y$, the following are equivalent:*

(1) $A \preccurlyeq_W B$;

(2) there exists a continuous function $f : \mathrm{dom}(\rho_X) \to \mathrm{dom}(\rho_Y)$ *such that*

$$f^{-1}(\rho_Y^{-1}(B)) = \rho_X^{-1}(A).$$

The definition of $\preccurlyeq_W$ does not depend on the choice of the admissible representation. It is intrinsic to the topology, and provides us with a generalization of the Wadge hierarchy for the second countable T_0 spaces, as argued by Pequignot [93].

3.2 The conciliatory space

It cannot be found in the work of Duparc, but the set $\omega^{\leq\omega}$ can be endowed with a very natural topology – the so-called *prefix topology*. The resulting topological space, the *conciliatory space Conc*, appears in domain theory as a classical example of an ω-algebraic domain[1] and of a reflective complete countably based f_0-space[2].

Definition 3.6. The conciliatory space *Conc* is the topological space $(\omega^{\leq\omega}, \tau_{\mathcal{C}})$ whose topology $\tau_{\mathcal{C}}$ is induced by the basis $\{O_s : s \in \omega^{<\omega}\}$, where

$$O_s = \left\{t \in \omega^{\leq\omega} : s \subseteq t\right\}.$$

The conciliatory space is a second countable T_0 space, but it is not Hausdorff since every open set containing a finite sequence contains also all the sequences extending it. One can wonder if the conciliatory relation $\leq_c$ is induced by the continuous reductions for the prefix topology. By the work of Selivanov [100], it is not the case. Even though they are identical if we restrict ourselves to $\mathbf{\Delta}^0_2$ sets, and share the property that all degrees are non-self-dual, the Wadge hierarchy of *Conc* does not satisfies the Wadge duality principle SLO.

Fact 3.7 (Folklore)**.** *Let* $A \subseteq$ *Conc be the set of all sequences of length* 1*, and* $B \subseteq$ *Conc be the set of all infinite sequences. The sets* A*,* B *and their complements are pairwise Wadge incomparable.*

Proof. We only prove one of the twelve cases here, the others are similar. Suppose that there exists a reduction from B to A, i.e. a continuous function f : *Conc* $\to$ *Conc* such that $x \in B$ if and only if $f(x) \in A$. Let n and m be

[1] See e.g. Becher and Grigorieff [12].
[2] See e.g. Selivanov [99].

two integers and notice that the set $f^{-1}(\langle n,m\rangle^\frown\omega^{\leq\omega})$ has to be both open and contained in $\omega^{<\omega}$. Since the empty set is the only open set contained in $\omega^{<\omega}$, the image of f is a subset of $\omega^{\leq 1}$, the set of sequences of length at most 1. Now let $\langle n\rangle$ be in the image of f. Then $f^{-1}(\langle n\rangle^\frown\omega^{\leq\omega})$ has to be both open and contained in ω^ω, the set of infinite sequences. But there is no such open subset of the conciliatory space, which prevents any reduction of B to A from happening. □

Hence the conciliatory hierarchy does not coincide with the Wadge hierarchy of the conciliatory space.

Question. *Is there a topology on the set $\omega^{\leq\omega}$ such that the conciliatory relation coincides with the reduction by continuous functions?*

The conciliatory space allows us nonetheless to interpret in a satisfactory way the approach used by Duparc [26, 27].

Lemma 3.8. *An admissible representation of the conciliatory space is given by the following application:*

$$\begin{aligned}\rho : (\omega_b)^\omega &\longrightarrow Conc\\ x &\longmapsto x_{[\ /b]}\end{aligned}$$

Proof. To show that it is admissible, we let $f:\omega^\omega\to Conc$ be a partial continuous function, and we define the reduction $g:\omega^\omega\to(\omega_b)^\omega$ via the map on the finite sequences g' it arises from. First set

$$\mathrm{dom}_{<\omega}(f)=\{s\in\omega^{<\omega}: s^\frown\omega^\omega\cap\mathrm{dom}(f)\neq\emptyset\},$$

and observe that, since f is continuous, for every finite sequence $s\in\mathrm{dom}_{<\omega}(f)$ there exists a finite sequence t such that $f(s^\frown\omega^\omega)\subseteq t^\frown\omega^{\leq\omega}$. Two cases arise: either $s^\frown\omega^\omega\cap\mathrm{dom}(f)$ is a singleton, and we denote by t_s its image by f, or it contains at least two points, and we set t_s to be the maximal finite sequence such that $f(s^\frown\omega^\omega)\subseteq t_s{}^\frown\omega^{\leq\omega}$. Now we define $g'(s)$ by induction on the length of s in $\mathrm{dom}_{<\omega}(f)$. Set $g'(\varepsilon)=\varepsilon$, and suppose s is of length $n+1$. Then

$$g'(s)=\begin{cases} g'(s\restriction_n)^\frown b & \text{if } t_s=g'(s\restriction_n)_{[\ /b]};\\ g'(s\restriction_n)^\frown t_s\left(\left|g'(s\restriction_n)_{[\ /b]}\right|+1\right) & \text{else .}\end{cases}$$

The map g is partial, continuous and realizes the reduction since for all $x\in\mathrm{dom}(f)$, $f(x)=g(x)_{[\ /b]}=\rho\circ g(x)$. The map ρ is thus an admissible representation of *Conc.* □

Notice that for $A \subseteq Conc$, $A^b = \rho^{-1}(A)$. The mapping $C \mapsto C^b$ that links the conciliatory sets to subsets of the Baire space can thus be seen as the inverse of an admissible representation for the space *Conc*. By reformulating Lemma 2.3, the conciliatory hierarchy is nothing but the generalized Wadge hierarchy of the conciliatory space.

Lemma 3.9. *Let $A, B \subseteq Conc$, then the following are equivalent:*

(1) $\rho^{-1}(A) \leq_W \rho^{-1}(B)$;

(2) II has a winning strategy in $C(A, B)$.

From Fact 3.5 we obtain that for all $A, B \subseteq Conc$, II has a winning strategy in the game $C(A, B)$ if and only if $A \preccurlyeq_W B$, so that $\leq_c$ and $\preccurlyeq_W$ coincide. Hence, the study of the conciliatory hierarchy is not only a technical formulation for the study of the Wadge hierarchy of the Baire space, but it also provides a description of the generalized Wadge hierarchy defined by Pequignot [93] for the conciliatory space.

3.3 The correspondence between the conciliatory and the Wadge hierarchies

Using Lemma 3.9, the inverse map of the representation gives us an embedding from $(\mathcal{P}(\omega^{\leq\omega}), \preccurlyeq_W)$ to $(\mathcal{P}(\omega^\omega), \leq_W)$. Since no set is self-dual with respect to the conciliatory preorder, its range is included in the non-self-dual degrees. We prove that under (AD) it is actually onto these degrees - modulo Wadge equivalence.

Theorem 3.10. *Let $A \subseteq \omega^\omega$ be non-self-dual. Then there exists $C \subseteq \omega^{\leq\omega}$ such that:*

$$A \equiv_W \rho^{-1}(C).$$

The proof is by induction on the Wadge rank of A. As we have

$$\emptyset \equiv_W \rho^{-1}(\emptyset) \qquad \text{and} \qquad \omega^\omega \equiv_W \rho^{-1}\left(\omega^{\leq\omega}\right),$$

the Theorem holds for the first Wadge degrees.

Let $A \subseteq \omega^\omega$ such that for every $A' \subseteq \omega^\omega$ non-self-dual with $A' <_W A$, there exists a conciliatory set $B' \subseteq \omega^{\leq\omega}$ such that $A' \equiv_W \rho^{-1}(B')$. Then we have two cases depending on wether A is initializable or not.

Definition 3.11. Let A and B be subsets of the Baire space. We define:

$$A \to B = \operatorname{shift}(A) \cup \left\{ u^\frown \langle 0 \rangle^\frown x : u \in (\omega \setminus \{0\})^{<\omega}, x \in B \right\}.$$

If $A \equiv_W A \to A$, we say that A is *initializable*. It is a reinforcement of the notion of non-self-dualness.

3.3.1 The uninitializable case.

We recall the following result concerning uninitializable subsets of the Baire space.

Fact 3.12 ([26, Proposition 14]). *Let $A \subseteq \omega^\omega$ be non-self-dual and uninitializable. Then there exists $B \subseteq \omega^\omega$ initializable, $I \subseteq \omega^{<\omega}$ a maximal antichain, and $(C_i)_{i \in I}$, each $C_i \subseteq \omega^\omega$ non self dual such that, for all $i \in I$, $C_i <_W A$, $B <_W A$ and*

$$A \equiv_W B \to \sum_{i \in I} C_i,$$

where:

$$B \to \sum_{i \in I} C_i = \operatorname{shift}(B) \cup \left\{ u^\frown \langle 0 \rangle^\frown i^\frown \alpha : u \in (\omega \setminus \{0\})^{<\omega}, i \in I, \alpha \in C_i \right\}.$$

Let $A \subseteq \omega^\omega$ be non-self-dual and uninitializable. By Fact 3.12 and our induction hypothesis, there exist conciliatory sets B' and C'_i for each $i \in I$ such that

$$B \equiv_W \rho^{-1}(B') \qquad \text{and} \qquad C_i \equiv_W \rho^{-1}(C'_i).$$

Now consider the conciliatory set $\tilde{A}$ defined as follows:

$$x \in \tilde{A} \leftrightarrow \begin{cases} x \in \operatorname{shift}(B') & \text{or} \\ x = u^\frown \langle 0 \rangle^\frown i^\frown x', \text{ where } u \in (\omega \setminus \{0\})^{<\omega}, i \in I, \text{ and } x' \in C'_i, & \text{or} \\ x \in (\operatorname{shift}(B') \cap \omega^{<\omega})^\frown \langle 0 \rangle, & \text{or} \\ x \in (\operatorname{shift}(B') \cap \omega^{<\omega})^\frown \langle 0 \rangle^\frown \{ i' : i' \in I' \}. \end{cases}$$

where $t \in I' \subseteq \omega^{<\omega}$ if and only if there exists $s \in I$ such that $t \subset s$, and t is not the empty sequence.

Claim.

$$B \to \sum_{i \in I} C_i \equiv_W \rho^{-1}\left(\tilde{A}\right).$$

Proof. We prove first that $B \to \sum_{i \in I} C_i \leq_W \rho^{-1}\left(\tilde{A}\right)$. We denote by σ_B the winning strategy for II in the game $W\big(\text{shift}(B), \rho^{-1}(\text{shift}(B'))\big)$, and for each $i \in I$, σ_i the winning strategy for II in the game $W(C_i, \rho^{-1}(C_i'))$. A winning strategy for II in the game $W\left(B \to \sum_{i \in I} C_i, \rho^{-1}\left(\tilde{A}\right)\right)$ is thus the following. As long as player I does not play "0", II follows σ_B. If I plays $\langle 0 \rangle ^\frown i$ with $i \in I$, II plays $\langle 0 \rangle ^\frown i$ and follows ultimately σ_i. Since σ_B and σ_i are winning, the strategy described is winning for II.

We prove now that $\rho^{-1}\left(\tilde{A}\right) \leq_W B \to \sum_{i \in I} C_i$. We denote by σ_B' the winning strategy for II in the game $W\big(\rho^{-1}(\text{shift}(B')), \text{shift}(B)\big)$, and for each $i \in I$, σ_i' the winning strategy for II in the game $W(\rho^{-1}(C_i'), C_i)$. A winning strategy for II in the game $W\left(\rho^{-1}\left(\tilde{A}\right), B \to \sum_{i \in I} C_i\right)$ is thus the following. As long as player I does not play "0", II follows σ_B'. If I plays "0", II continues to follow σ_B', interpreting "0" as a "b". As long as I does not play a sequence $i \in I$, II continues to follow σ_B', interpreting each move of her opponent as a "b". When I has constructed a sequence $i \in I$ (if he does), II plays $\langle 0 \rangle ^\frown i$ and then follows σ_i. □

Since $A \equiv_W B \to \sum_{i \in I} C_i$ holds, we have:

$$A \equiv_W \rho^{-1}\left(\tilde{A}\right),$$

which completes the proof of the uninitializable case.

3.3.2 The initializable case

If A is initializable, we distinguish two cases depending on whether A is strongly non-self-dual or not. The set A is strongly non-self-dual, if and only if player II has a winning strategy $\sigma_{\twoheadleftarrow}$ in the game $W\left(A^{\approx}, A\right)$. Because $\sigma_{\twoheadleftarrow}$ is winning, we have that, for any $x, x' \in (\omega_{\twoheadleftarrow})^{\omega}$:

$$x^{\hookleftarrow} \in A \iff \sigma_{\twoheadleftarrow}(x) \in A.$$

Take $a \in \omega$ and define the continuous mapping

$$\tau : (\omega_b)^{\omega} \longrightarrow (\omega_{\twoheadleftarrow})^{\omega}$$

induced by the following strategy. As long as player I does not play "b", player II copies his moves. If I plays "b", II answers by playing "a" repeatedly until player I stops to play "b". If it ever happens, and whatever I plays next, II plays enough "$\twoheadleftarrow$" to erase all the "a" she has just played, and then copies I moves until he plays a "b", and so on. It defines a continuous map from $(\omega_b)^\omega$ to $(\omega_\twoheadleftarrow)^\omega$ such that for all $x \in \omega^\omega$:

$$\tau(x)^{\hookleftarrow} = \begin{cases} \rho(x)^\frown a^\omega & \text{if } \rho(x) \text{ is finite;} \\ \rho(x) & \text{otherwise.} \end{cases}$$

Moreover, this application behaves well with regards to the interpretations of "b" and "$\twoheadleftarrow$", since we have that for all $x, x' \in \omega^\omega$:

$$\rho(x) = \rho(x') \Longrightarrow \tau(x)^{\hookleftarrow} = \tau(x')^{\hookleftarrow}.$$

In fact, we even have the equivalence if we restrict ourselves to sequences x such that $\rho(x)$ is infinite. Set $\tilde{A} = \rho \circ (\sigma_\twoheadleftarrow \circ \tau)^{-1}(A)$.

Claim.

$$A \equiv_W \rho^{-1}(\tilde{A}).$$

Proof. We have directly that $\rho^{-1}(\tilde{A}) \leq_W A$ holds since $\rho^{-1}(\tilde{A}) = (\sigma_\twoheadleftarrow \circ \tau)^{-1}(A)$, and $\sigma_\twoheadleftarrow \circ \tau$ is continuous. In order to prove that A reduces to $\rho^{-1}(\tilde{A})$, let $g : \omega^\omega \to (\omega_b)^\omega$ be the inclusion, and observe that for all $x \in \omega^\omega$:

$$\begin{aligned} g(x) \in \rho^{-1}(\tilde{A}) &\leftrightarrow \rho \circ g(x) \in \tilde{A} \\ &\leftrightarrow g(x) \in (\sigma_\twoheadleftarrow \circ \tau)^{-1}(A) \\ &\leftrightarrow \sigma_\twoheadleftarrow \circ \tau \circ g(x) \in A \\ &\leftrightarrow \sigma_\twoheadleftarrow(x) \in A \\ &\leftrightarrow x \in A. \end{aligned}$$

Hence $A \leq_W \rho^{-1}(\tilde{A})$ also holds, and the claim is proved. □

Suppose now that A is initializable but not strongly non-self-dual. Since A is initializable, $A^{\approx_1}$ is non-self-dual [26, Proposition 28].

Claim.

$$A^{\approx_1} <_W A$$

Proof. Since A is not strongly non-self-dual, player I has a winning strategy in the game $W(A^{\approx}, A)$. Therefore, player II has a winning strategy $\sigma_\twoheadleftarrow$ in the game $W\left(A^\complement, A^{\approx}\right)$. For any finite sequence $u \in (\omega_\twoheadleftarrow)^{<\omega}$, define the closed set $F_u = \sigma_\twoheadleftarrow^{-1}\left(\sigma_\twoheadleftarrow(u)^\frown \omega^\omega\right)$. We construct a winning strategy for II in $W\left((A^\complement)^{(F_u)}, A\right)$. In this game, each odd move of player I is devoted to answer a question. Suppose he has already played a main run $u \in \omega^{<\omega}$. If he answers 0, he guarantees that his main run will be in $\sigma_{\twoheadleftarrow}^{-1}\left(\sigma_{\twoheadleftarrow}(u)^\frown\omega^\omega\right)$, it means that in the game $W\left(A^\complement, A^\approx\right)$, if II follows $\sigma_{\twoheadleftarrow}$ and I has already played u, II won't need to use her eraser afterwards to win. Thus, supposing II has always skipped from the beginning, she has the following winning strategy: she plays $(\sigma_{\twoheadleftarrow}(u))^{\looparrowleft}$, and then follows $\sigma_{\twoheadleftarrow}$. If I does not answer 0, since $\sigma_{\twoheadleftarrow}$ is winning, we know that his main run will nevertheless reach a finite sequence $v_0 \in \omega^{<\omega}$ such that $(\sigma_{\twoheadleftarrow}(v_0))^{\looparrowleft}$ is not the empty sequence, and in the game $W\left(A^\complement, A^\approx\right)$, where player I has already played v_0 and II follows $\sigma_{\twoheadleftarrow}$, II will not erase her first move $(\sigma_{\twoheadleftarrow}(v_0))^{\looparrowleft}(0)$. Supposing II has always skipped from the beginning, she then has the following winning strategy: she plays $(\sigma_{\twoheadleftarrow}(v_0))^{\looparrowleft}(0)$, and then waits for I to play in his main run a finite sequence v_1 extending v_0 such that $(\sigma_{\twoheadleftarrow}(v_1))^{\looparrowleft}$ has length at least two, and such that II will not erase her second move $(\sigma_{\twoheadleftarrow}(v_1))^{\looparrowleft}(1)$ when she follows $\sigma_{\twoheadleftarrow}$ in the game $W\left(A^\complement, A^\approx\right)$, where player I has already played v_1, and so on, and so forth. Since $\sigma_{\twoheadleftarrow}$ is winning in $W\left(A^\complement, A^\approx\right)$, this strategy will provide as the game goes along an infinite sequence for II, which will be in A if and only if the main run of player I is in $A^\complement$, thus being a winning strategy for II.

By minimality of $(A^\complement)^{\nsim_1}$, we have $(A^\complement)^{\nsim_1} \leq_W A$. Since $(A^\complement)^{\nsim_1} \equiv_W (A^{\nsim_1})^\complement$,

$$(A^{\nsim_1})^\complement \leq_W A \qquad \text{and} \qquad A^{\nsim_1} \leq_W A,$$

so that $A^{\nsim_1} <_W A$. □

By induction hypothesis, there exists then a conciliatory set $B \subseteq \omega^{\leq\omega}$ such that

$$A^{\nsim_1} \equiv_W \rho^{-1}(B).$$

But this last statement is equivalent to

$$\rho^{-1}\left(B^\approx\right) \equiv_W A.$$

This completes the proof of Theorem 3.10.

Notice that Theorem 3.10 can be relativized to any pointclass with appropriate determinacy and closure properties. In particular, it gives a direct proof in ZF + DC of the equivalence between the conciliatory hierarchy and the Wadge hierarchy when restricted to non-self-dual Borel sets, as stated by Duparc [26, 27].

Corollary 3.13. *Let $A \subseteq \omega^\omega$ be non-self-dual. Then there exists $F \subseteq \omega^{<\omega}$ such that:*

$$A \equiv_W \rho^{-1}(A \cup F).$$

Proof. By Theorem 3.10, there exists a conciliatory set B such that $A \equiv_W \rho^{-1}(B)$. Let σ be a winning strategy for player II in $W(A, \rho^{-1}(B))$, we define F as follows:

- the empty sequence ε is in F if and only if $\varepsilon \in B$;
- a finite non-empty sequence u is in F if and only if $\sigma(u) \in B$.

The fact that $B \leq_c (A \cup F)$ is direct, and in fact does not depend on F. For the converse, a strategy for II in $C(A \cup F, B)$ is the following: II skips whenever I skips, and otherwise applies σ, replacing "blanks" by "skips". Thus, $B \equiv_c (A \cup F)$, and $A \equiv_W \rho^{-1}(A \cup F)$. □

One can also prove the following characterization of the Γ_c classes *inside* the conciliatory space.

Corollary 3.14. *A pointclass* Γ *is non-self-dual if and only if there exists a subset* $A \subseteq Conc$ *such that*

$$\Gamma_c = \{C \subseteq Conc : C \preccurlyeq_W A\}\,.$$

Proof. The first implication is essentially given by Theorem 3.10. Suppose Γ is a non-self-dual pointclass, then there exists $A \subseteq \omega^\omega$ non-self-dual such that

$$\Gamma = \{B \subseteq \omega^\omega : B \leq_W A\}\,.$$

By Theorem 3.10, there exists $A' \subseteq Conc$ such that $A \equiv_W \rho^{-1}(A')$. By Lemma 3.9 we then conclude that $\Gamma_c = \{C \subseteq Conc : C \preccurlyeq_W A'\}$.

Conversely, let $A \subseteq Conc$ and consider the pointclass

$$\Gamma = \left\{B \subseteq \omega^\omega : B \leq_W \rho^{-1}(A)\right\}.$$

Since $\rho^{-1}(A)$ is non-self-dual, it is a non-self-dual pointclass. We then use Lemma 3.9 to conclude that

$$\Gamma_c = \{C \subseteq Conc : C \preccurlyeq_W A\}\,.$$

□

This point of view allows us to define the Γ_c classes directly in the conciliatory space, i.e. without references to the Baire space, for the Borel and Hausdorff-Kuratowski classes. Here we use the following generalized definitions of the Borel classes for non-metrizable spaces, as defined by Tang [107, 108], Selivanov [100], and de Brecht [25].

Definition 3.15. Let X be a topological space. For each positive ordinal $\alpha < \omega_1$ we define by induction

$$\begin{aligned}
&\mathbf{\Sigma}^0_1(X) = \{O \subseteq X \mid \ O \text{ is open}\},\\
&\mathbf{\Sigma}^0_\alpha(X) = \left\{ \bigcup_{i\in\omega} B_i \cap C_i^{\complement} \;\middle|\; B_i, C_i \in \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta(X) \text{ for each } i \in \omega \right\},\\
&\mathbf{\Pi}^0_\alpha(X) = \left\{ A^{\complement} \mid A \in \mathbf{\Sigma}^0_\alpha(X) \right\},\\
&\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X).
\end{aligned}$$

Notice that if $\alpha > 2$, then

$$\mathbf{\Sigma}^0_\alpha(X) = \left\{ \bigcup_{i\in\omega} B_i \;\middle|\; B_i \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X) \text{ for each } i \in \omega \right\}.$$

And if X is metrizable the previous statement holds also for $\alpha = 2$, i.e.

$$\mathbf{\Sigma}^0_2(X) = \left\{ \bigcup_{i\in\omega} B_i \;\middle|\; B_i \in \mathbf{\Pi}^0_1(X) \text{ for each } i \in \omega \right\}.$$

Hence this definition of the Borel classes coincides with the classical one when we restrict ourselves to metrizable spaces.

Notice that every ordinal θ can be written as $\theta = \lambda + n$, where λ is limit and $n < \omega$. We call θ *even* if n is even, and *odd* if n is odd.

Definition 3.16. Let $(A_\eta)_{\eta<\theta}$ be an *increasing* sequence of subsets of the Baire space, with $\theta < \omega_1$. Define the set $D_\theta((A_\eta)_{\eta<\theta})$ by

$$D_\theta((A_\eta)_{\eta<\theta}) = \left\{ x \in \bigcup_{\eta<\theta} A_\eta \quad : \quad \text{the least } \eta < \theta \text{ with } x \in A_\eta \text{ has parity opposite to that of } \theta \right\}.$$

For $\theta < \omega_1$, and Γ a pointclass, let

$$D_\theta(\Gamma) = \{ D_\theta((A_\eta)_{\eta<\theta}) | A_\eta \in \Gamma, \eta < \theta \}.$$

It is also a pointclass.

Theorem 3.17 ([25, Theorem 78])**.** *For all non-zero countable ordinals η and ξ,*

$$\left(D_\eta(\mathbf{\Sigma}^0_\xi) \right)_c = D_\eta(\mathbf{\Sigma}^0_\xi)(Conc).$$

Note that the proof of Theorem 3.17 relies on a lemma proved by Saint-Raymond [97, Lemma 17], and does not require the Axiom of Determinacy.

4 Differences of coanalytic sets

In this chapter, we extend the results obtained by Wadge, Louveau and Duparc for the Borel sets to a wider pointclass: $\mathrm{Diff}(\mathbf{\Pi}^1_1)$, the class of increasing differences of coanalytic sets. From the works of Martin [77] and Harrington [42], we know that the class $D^*_{\omega^2}(\mathbf{\Pi}^1_1)$ of all decreasing ω^2 differences of coanalytic sets is determined under $\mathrm{DET}(\mathbf{\Pi}^1_1)$. Since $\mathrm{Diff}(\mathbf{\Pi}^1_1) \subseteq D^*_\omega(\mathbf{\Pi}^1_1)$ – see Proposition 5.4, this determinacy hypothesis is sufficient for our work and is assumed all along this chapter. Results in Section 4.1 will appear in an article by the author [38].

4.1 Pointclasses and boolean operations

4.1.1 General Observations

We begin with some general observations on the difference hierarchy of coanalytic sets. We denote the class of all countable differences of coanalytic sets by

$$\mathrm{Diff}(\mathbf{\Pi}^1_1) = \bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Pi}^1_1).$$

Merely by definition, we have $D_\alpha(\mathbf{\Pi}^1_1) \subseteq D_\beta(\mathbf{\Pi}^1_1)$ and $D_\alpha(\mathbf{\Pi}^1_1) \subseteq \check{D}_\beta(\mathbf{\Pi}^1_1)$ for all $\alpha < \beta$. Moreover, since there exists a ω^ω-universal set for $\mathbf{\Pi}^1_1$, the hierarchy does not collapse, i.e. for all $\alpha < \omega_1$, $D_\alpha(\mathbf{\Pi}^1_1) \smallsetminus \check{D}_\alpha(\mathbf{\Pi}^1_1) \neq \emptyset$. We have thus the following classical diamond-shape diagram:

$$\begin{array}{ccccccc}
\mathbf{\Pi}^1_1 & & D_2(\mathbf{\Pi}^1_1) & & D_3(\mathbf{\Pi}^1_1) & & \\
& \Delta(D_2(\mathbf{\Pi}^1_1)) & & \Delta(D_3(\mathbf{\Pi}^1_1)) & & \cdots & \\
\mathbf{\Sigma}^1_1 & & \check{D}_2(\mathbf{\Pi}^1_1) & & \check{D}_3(\mathbf{\Pi}^1_1) & &
\end{array}$$

where the pointclasses are strictly included in each other from the left to the right.

4.1.2 The ambiguous classes

To describe the pointclasses included in $\Delta(D_\alpha(\boldsymbol{\Pi}^1_1))$, we need a characterization of the $D_\alpha(\boldsymbol{\Pi}^1_1)$ classes.

The successor case

Proposition 4.1. *For every countable ordinal α, we have:*
(a) $D_{\alpha+1}(\boldsymbol{\Pi}^1_1) = \check{D}_\alpha(\boldsymbol{\Pi}^1_1) \cap \boldsymbol{\Pi}^1_1 = \left\{D \cap C \mid D \in \check{D}_\alpha(\boldsymbol{\Pi}^1_1) \textit{ and } C \in \boldsymbol{\Pi}^1_1\right\}$*;*
(b) $\check{D}_{\alpha+1}(\boldsymbol{\Pi}^1_1) = D_\alpha(\boldsymbol{\Pi}^1_1) \cup \boldsymbol{\Sigma}^1_1 = \{D \cup C \mid D \in D_\alpha(\boldsymbol{\Pi}^1_1) \textit{ and } C \in \boldsymbol{\Sigma}^1_1\}$*.*

Proof. We only prove the first assertion for the finite differences, the other follows by considering the complements, and the generalization to the transfinite is straightforward. Let $n = 2k$ for $k \geq 1$. Observe that for any increasing family $(A_i)_{i<n}$ of coanalytic subsets of the Baire space, we have:

$$D_n((A_i)_{i<n}) = A_{n-1} \smallsetminus D_{n-1}((A_i)_{i<n-1});$$

and therefore $D_n(\boldsymbol{\Pi}^1_1) \subseteq \check{D}_{n-1}(\boldsymbol{\Pi}^1_1) \cap \boldsymbol{\Pi}^1_1$.

For the other inclusion, let $D \in \check{D}_{n-1}(\boldsymbol{\Pi}^1_1)$ and $B \in \boldsymbol{\Pi}^1_1$. Then there exists an increasing family of coanalytic sets $(A_i)_{i<n-1}$ such that:

$$D = D_{n-1}((A_i)_{i<n-1})^{\complement}.$$

We obtain:

$$\begin{aligned} D \cap B &= \bigcap_{i=1}^{k-1}(A_{2i}^{\complement} \cup A_{2i-1}) \cap A_0^{\complement} \cap B \\ &= (B \cap A_{2k-2}^{\complement}) \cup \bigcup_{i=0}^{k-2}(B \cap A_{2i}^{\complement} \cap A_{2i+1}) \\ &= D_n((A_0 \cap B, A_1 \cap B, \ldots, A_{n-2} \cap B, B)), \end{aligned}$$

where the second equality relies on the fact that the family $(A_i)_{i<n-1}$ is increasing. Thus $D \cap B \in D_n(\boldsymbol{\Pi}^1_1)$ and $D_n(\boldsymbol{\Pi}^1_1) = \check{D}_{n-1}(\boldsymbol{\Pi}^1_1) \cap \boldsymbol{\Pi}^1_1$. The odd case is similar. $\square$

This result can be illustrated by the following diagram.

$$\begin{tikzcd}
\boldsymbol{\Sigma}^1_1 \arrow[drr, dashed, "\cap\boldsymbol{\Pi}^1_1"] & \subseteq & \check{D}_2(\boldsymbol{\Pi}^1_1) & \subseteq & \check{D}_3(\boldsymbol{\Pi}^1_1) & \subseteq & \cdots \\
\boldsymbol{\Pi}^1_1 & \subseteq & D_2(\boldsymbol{\Pi}^1_1) \arrow[urr, dashed, "\cup\boldsymbol{\Sigma}^1_1"] & \subseteq & D_3(\boldsymbol{\Pi}^1_1) & \subseteq & \cdots
\end{tikzcd}$$

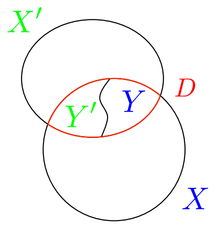
This inductive definition for the successor classes $D_\alpha(\mathbf{\Pi}^1_1)$ and $\check{D}_\alpha(\mathbf{\Pi}^1_1)$ allows us to adapt a result from Louveau [68].

Proposition 4.2 (Louveau's trick I)**.**
Let $\alpha < \omega_1$, and $D \in \Delta(D_{\alpha+1}(\mathbf{\Pi}^1_1))$. Then there exists $B \in \mathbf{\Delta}^1_1$, $X \in \check{D}_\alpha(\mathbf{\Pi}^1_1)$ and $Y \in D_\alpha(\mathbf{\Pi}^1_1)$ such that

$$D = (X \cap B) \cup (Y \smallsetminus B).$$

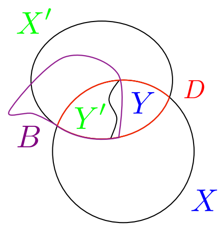
Proof. The set D is both in $D_{\alpha+1}(\mathbf{\Pi}^1_1)$ and in $\check{D}_{\alpha+1}(\mathbf{\Pi}^1_1)$. Proposition 4.1 gives $X' \in \mathbf{\Pi}^1_1$, $X \in \check{D}_\alpha(\mathbf{\Pi}^1_1)$, $Y' \in \mathbf{\Sigma}^1_1$ and $Y \in D_\alpha(\mathbf{\Pi}^1_1)$ such that

$$D = X' \cap X \qquad \text{and} \qquad D = Y' \cup Y.$$



In particular, we have that $Y' \cap X'^{\complement} = \emptyset$. By the separation property for the analytic sets, there exists a Borel subset B such that

$$Y' \subseteq B \qquad \text{and} \qquad B \cap X'^{\complement} = \emptyset.$$



Hence,

$$D = (X \cap B) \cup (Y \smallsetminus B).$$

$\square$

The limit case

A similar description of the ambiguous classes can be provided for the limit case, using a countable Borel partition instead of just one Borel set and its complement.

Proposition 4.3 (Louveau's trick II).
Let $D \subseteq \omega^\omega$ be in the $\Delta(D_\delta(\mathbf{\Pi}^1_1))$ class with $\delta < \omega_1$ limit. Then there exists a countable Borel partition $(C_i)_{i\in\omega}$ of the Baire space such that, for all $j < \omega$,

$$D \cap C_j \in D_{\alpha_j}(\mathbf{\Pi}^1_1),$$

with $\alpha_j < \delta$.

Proof. We only prove it for $\delta = \omega$. Let $D \subseteq \omega^\omega$ be in the $\Delta(D_\delta(\mathbf{\Pi}^1_1))$ class. By definition there exists two increasing families $(B_i)_{i\in\omega}$ and $(B'_i)_{i\in\omega}$ of coanalytic subsets of the Baire space such that

$$D = \bigcup_{i\in\omega}(B_{2i+1} \smallsetminus B_{2i}) \qquad \text{and} \qquad D^{\complement} = \bigcup_{i\in\omega}(B'_{2i+1} \smallsetminus B'_{2i}).$$

By the generalized reduction property of the class of coanalytic sets, there exists a disjoint coanalytic family $(C_i)_{i\in\omega}$ such that

- for all $i < \omega$, $C_{2i} \subseteq B_i$ and $C_{2i+1} \subseteq B'_i$, and
- $\bigcup_{i\in\omega} C_i = \bigcup_{i\in\omega} B_i \cup \bigcup_{i\in\omega} B'_i$.

Since $\bigcup_{i\in\omega} B_i \cup \bigcup_{i\in\omega} B'_i = D \cup D^{\complement} = \omega^\omega$, the family $(C_i)_{i\in\omega}$ is in fact an analytic, thus Borel, partition of the Baire space. In addition, the fact that $C_{2i} \subseteq B_i$ and $C_{2i+1} \subseteq B'_i$ hold for all $i \in \omega$ implies that $D \cap C_{2i}$ and $D^{\complement} \cap C_{2i+1}$ are in the class $D_{i+1}(\mathbf{\Pi}^1_1)$. To prove that our partition is indeed as required, it only remains to show that for all $i \in \omega$, $D \cap C_{2i+1}$ is a finite differences of coanalytic sets. Fix $i \in \omega$, we have

$$D \cap C_{2i+1} = C_{2i+1} \cap \left(D^{\complement} \cap C_{2i+1}\right)^{\complement}.$$

But $D^{\complement} \cap C_{2i+1}$ is a finite difference of analytic sets, so that $D \cap C_{2i+1}$ is also a finite difference of coanalytic sets. □

Louveau's tricks I and II provide a bottom up description of the ambiguous classes, and from them we can now derive the complete description *à la* Louveau of the Wadge hierarchy of the class $\mathrm{Diff}(\mathbf{\Pi}^1_1)$.

4.1.3 Boolean operations and descriptions

We recall the definitions of the operations used by Louveau in [68]. Besides the differences that we have already introduced, four more operations are needed.

(a) *Separated Unions*. Let Γ and Γ' be two pointclasses. The set A is in $\mathrm{SU}(\Gamma, \Gamma')$ if and only if there exists a disjoint family $(C_n)_{n\in\omega}$ of sets in Γ, and a family $(A_n)_{n\in\omega}$ of sets in Γ' such that

$$A = \mathrm{SU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}) = \bigcup_{n\in\omega} A_n \cap C_n.$$

(b) *One-sided Separated Unions*. Let Γ and Γ' be two pointclasses. The set A is in $\mathrm{Sep}(\Gamma, \Gamma')$ if there exists $C \in \Gamma$, $B_1 \in \check{\Gamma}'$ and $B_2 \in \Gamma'$ such that

$$A = \mathrm{Sep}(C, B_1, B_2) = (C \cap B_1) \cup (B_2 \smallsetminus C).$$

(c) *Two-sided Separated Unions*. Let Γ, Γ' and Γ'' be three pointclasses. The set A is in $\mathrm{Bisep}(\Gamma, \Gamma', \Gamma'')$ if there exists C_1, C_2 in Γ disjoint, $A_1 \in \check{\Gamma}'$, $A_2 \in \Gamma'$, and $B \in \Gamma''$ such that

$$A = \mathrm{Bisep}(C_1, C_2, A_1, A_2, B) = (C_1 \cap A_1) \cup (C_2 \cap A_2) \cup (B \smallsetminus (C_1 \cup C_2)).$$

If $\Gamma'' = \{\emptyset\}$, we just write $\mathrm{Bisep}(\Gamma, \Gamma')$.

(d) *Separated Differences*. Let Γ, Γ' and Γ'' be three pointclasses, and $\xi \geq 2$ be countable. The set A is in $\mathrm{SD}_\xi((\Gamma, \Gamma'), \Gamma'')$ if there is an increasing family $(C_\eta)_{\eta<\xi}$ in Γ, an increasing family $(A_\eta)_{\eta<\xi}$ in Γ' and $B \in \Gamma''$ such that, for all $\eta < \xi$, $A_\eta \subseteq C_\eta \subseteq A_{\eta+1}$ and

$$A = \mathrm{SD}_\xi((C_\eta)_{\eta<\xi}, (A_\eta)_{\eta<\xi}, B) = \bigcup_{\eta<\xi}(A_\eta \smallsetminus \bigcup_{\eta'<\eta} C_{\eta'}) \cup (B \smallsetminus \bigcup_{\eta<\xi} C_\eta).$$

These operations, combined and applied in certain ways to certain classes give us all the non-self-dual pointclasses included in $\mathrm{Diff}(\mathbf{\Pi}^1_1)$. But first we need to introduce some notation. Let $u_0, u_1 \in (\omega_1+1)^\omega$, we denote by $\langle u_0, u_1\rangle$ the sequence $u \in (\omega_1 + 1)^\omega$ such that, for all $n \in \omega$, $u(2n) = u_0(n)$, and $u(2n+1) = u_1(n)$. Similarly, if $(u_i)_{i\in\omega} \subseteq \big((\omega_1+1)^\omega\big)^\omega$, we denote by $\langle (u_i)_{i\in\omega}\rangle$ the sequence $u \in (\omega_1 + 1)^\omega$ such that for all $n, m \in \omega$, $u(\langle n, m\rangle) = u_n(m)$, where $(n, m) \mapsto \langle n, m\rangle$ is a bijection between $\omega \times \omega$ and ω. We now define inductively the set of *descriptions* $D \subseteq (\omega_1 + 1)^\omega$, and for each $u \in D$, the class Γ_u it describes.

Definition 4.4. The set of *descriptions* $D \subseteq (\omega_1+1)^\omega$ is the least satisfying the following conditions:

(a) If $u(0)=0$, then $u \in D$ and $\Gamma_u = \{\emptyset\}$.

(b) If $u(0)=\xi<\omega_1$ with $\xi \neq 0$, $u(1)=1$ and $u(2)=\eta<\omega_1$, then $u \in D$ and $\Gamma_u = D_\eta(\mathbf{\Sigma}^0_\xi)$.

(c) If $u(0)=\omega_1$, $u(1)=1$ and $u(2)=\eta<\omega_1$, then $u \in D$ and $\Gamma_u = D_\eta(\mathbf{\Pi}^1_1)$.

(d) If $u = \xi^\frown 2^\frown \eta^\frown u^*$, where $1 \leq \xi < \omega_1$, $1 \leq \eta \leq \omega_1$, $u^* \in D$ and $u^*(0) > \xi$, then $u \in D$ and $\Gamma_u = \mathrm{Sep}(D_\eta(\mathbf{\Sigma}^0_\xi), \Gamma_{u^*})$.

(e) If $u = \xi^\frown 3^\frown \eta^\frown \langle u_0, u_1\rangle$, where $1 \leq \xi < \omega_1$, $1 \leq \eta \leq \omega_1$, $u_0, u_1 \in D$, $u_0(0) > \xi$, $u_1(0) \geq \xi$ or $u_1(0)=0$, and $\Gamma_{u_1} \subset \Gamma_{u_0}$, then $u \in D$ and $\Gamma_u = \mathrm{Bisep}(D_\eta(\mathbf{\Sigma}^0_\xi), \Gamma_{u_0}, \Gamma_{u_1})$.

(f) If $u = \xi^\frown 4^\frown \langle (u_n)_{n\in\omega}\rangle$, where $1 \leq \xi < \omega_1$, each $u_n \in D$, and either $u_n(0) = \xi_1 > \xi$ for all $n \in \omega$, and the Γ_{u_n} are strictly increasing, or $u_n(0) = \xi_n$ and the ξ_n are strictly increasing with $\xi < \sup_{n\in\omega} \xi_n$, then $u \in D$ and $\Gamma_u = \mathrm{SU}(\mathbf{\Sigma}^0_\xi, \bigcup_{n\in\omega} \Gamma_{u_n})$.

(g) If $u = \xi^\frown 5^\frown \eta^\frown \langle u_0, u_1\rangle$, where $1 \leq \xi < \omega_1$, $2 \leq \eta \leq \omega_1$, $u_0, u_1 \in D$, $u_0(0) = \xi$, $u_0(1) = 4$, $u_1(0) \geq \xi$ or $u_1(0) = 0$, and $\Gamma_{u_1} \subset \Gamma_{u_0}$, then $u \in D$ and $\Gamma_u = \mathrm{SD}_\eta((\mathbf{\Sigma}^0_\xi, \Gamma_{u_0}), \Gamma_{u_1})$.

For a description $u \in D$, we call the first element $u(0)$ of u the *level* of the class Γ_u. Notice that compared to the Borel case, we only add the classes $D_\eta(\mathbf{\Pi}^1_1)$, which are given the level ω_1. Observe also that if the level seems to depend on the description of the class rather than on the class itself, it is not the case – see Louveau and Saint-Raymond [70].

Proposition 4.5. *Let $u \in D$ with $u(0) = \xi \neq 0$.*

(a) *If $\xi < \omega_1$, then*
  - *Γ_u is closed under union with a $\mathbf{\Delta}^0_\xi$ set.*
  - *$\mathrm{SU}(\mathbf{\Sigma}^0_\xi, \Gamma_u) = \Gamma_u$, and we say that Γ_u is closed under $\mathbf{\Sigma}^0_\xi - \mathrm{SU}$.*

(b) *If $\xi = \omega_1$, then*
  - *Γ_u is closed under union with a $\mathbf{\Delta}^1_1$ set.*
  - *$\mathrm{SU}(\mathbf{\Pi}^1_1, \Gamma_u) = \Gamma_u$, and we say that Γ_u is closed under $\mathbf{\Pi}^1_1 - \mathrm{SU}$.*

Proof. The only thing left to verify is the case where $\Gamma_u = D_\eta(\mathbf{\Pi}^1_1)$, the rest is by the same induction as in [68]. Let $0 < \eta < \omega_1$, we have to prove that the class $D_\eta(\mathbf{\Pi}^1_1)$ is closed under union with a Borel set, and under $\mathbf{\Pi}^1_1 - \mathrm{SU}$. The first comes from the fact that the class $\mathbf{\Pi}^1_1$ is closed under union with a Borel set. For the second, let $(C_n)_{n\in\omega}$ be a disjoint family of $\mathbf{\Pi}^1_1$ sets, and $(A_n)_{n\in\omega}$

a family of $D_\eta(\mathbf{\Pi}^1_1)$ sets. For all integer n, there exists a family $(A_n^\alpha)_{\alpha<\eta}$ such that $A_n = D_\eta((A_n^\alpha)_{\alpha<\eta})$. Thus:

$$\begin{aligned}
\mathrm{SU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}) &= \bigcup_{n\in\omega} A_n \cap C_n \\
&= \bigcup_{n\in\omega} D_\eta((A_n^\alpha)_{\alpha<\eta}) \cap C_n \\
&= \bigcup_{n\in\omega} D_\eta((A_n^\alpha \cap C_n)_{\alpha<\eta}) \\
&= D_\eta((\bigcup_{n\in\omega}(A_n^\alpha \cap C_n))_{\alpha<\eta}).
\end{aligned}$$

Notice that the last equality holds because for all $\alpha < \eta$ and all integers n and m, if $n \neq m$ then $(A_n^\alpha \cap C_n) \cap (A_m^\alpha \cap C_m) = \emptyset$. Hence $\mathrm{SU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}) \in D_\eta(\mathbf{\Pi}^1_1)$ and the class $D_\eta(\mathbf{\Pi}^1_1)$ is closed under $\mathbf{\Pi}^1_1 - \mathrm{SU}$. □

Proposition 4.6. *Let $u \in D$. Then Γ_u is a non-self-dual pointclass included in* $\mathrm{Diff}(\mathbf{\Pi}^1_1)$.

Proof. The classes Γ_u are pointclasses merely by definition, as results of Boolean operations on pointclasses. The fact that they are all in $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ is a consequence of the closure properties proved in Proposition 4.5. The existence of universals for the classes Γ_u provides the non-self-dualness. □

We now give to each description u a type. These types reveal information on the structural properties of the class described. For example the descriptions of type 1 share the property that the classes they describe can be written as $\mathrm{Bisep}(\mathbf{\Sigma}^0_\xi, \Gamma_{u'})$ or $\mathrm{Bisep}(\mathbf{\Pi}^1_1, \Gamma_{u'})$ for some ξ and some description u'; the descriptions of type 2 share the property that the classes they describe can be written as $\mathrm{SU}(\mathbf{\Sigma}^0_\xi, \bigcup_{n\in\omega}\Gamma_{u_n})$ or $\mathrm{SU}(\mathbf{\Pi}^1_1, \bigcup_{n\in\omega}\Gamma_{u_n})$ for some ξ and some family of descriptions $(u_n)_{n\in\omega}$, etc.

Definition 4.7. Let $u \in D$. The type $t(u)$ of u is 0 if $u(0) = 0$. If $u(0) \geq 1$ then the type $t(u)$ of u is

(a) 1 if:
- $u(1) = 1$ and $u(2)$ is successor;
- $u(1) = 3$, $t(u_1) = 0$ and $u(2)$ is successor;
- $u(1) = 3$, $t(u_1) = 1$ and $u_1(0) = u(0)$;
- $u(1) = 5$, $t(u_1) = 1$ and $u_1(0) = u(0)$.

(b) 2 if:

- $u(1) = 1$ and $u(2)$ is limit;
- $u(1) = 3$, $t(u_1) = 0$ and $u(2)$ is limit;
- $u(1) = 3$, $t(u_1) = 2$ and $u_1(0) = u(0)$;
- $u(1) = 4$;
- $u(1) = 5$ and $t(u_1) = 0$;
- $u(1) = 5$, $t(u_1) = 2$ and $u_1(0) = u(0)$.

(c) 3 if:
- $u(1) = 2$;
- $u(1) = 3$ and $u_1(0) > u(0)$;
- $u(1) = 3$, $t(u_1) = 3$ and $u_1(0) = u(0)$;
- $u(1) = 5$ and $u_1(0) > u(0)$;
- $u(1) = 5$, $t(u_1) = 3$ and $u_1(0) = u(0)$.

Thanks to these types, we can now sort the descriptions in four groups, depending on the position in which their associated class lies in the Wadge hierarchy. $D^0 = \{u \in D : t(u) = 0\}$ is the set of descriptions that code the class $\{\emptyset\}$, which is at the bottom of the hierarchy. $D^+ = \{u \in D : u(0) = 1 \text{ and } t(u) = 1\}$ is the set of descriptions that code classes which are at a successor position in the Wadge hierarchy. $D^\omega = \{u \in D : u(0) = 1 \text{ and } t(u) = 2\}$ is the set of descriptions that code classes which are at a limit of cofinality ω position in the Wadge hierarchy. $D^{\omega_1} = D \setminus (D^0 \cup D^+ \cup D^\omega) = \{u \in D : u(0) = 1 \text{ and } t(u) = 3\} \cup \{u \in D : u(0) > 1\}$ is the set of descriptions that code classes which are at a limit of cofinality ω_1 position in the Wadge hierarchy.

Theorem 4.8. *Let $\mathcal{W} = \{\Gamma_u : u \in D\} \cup \{\check{\Gamma}_u : u \in D\} \cup \{\Delta(\Gamma_u) : u \in D\}$. Then $\mathcal{W}$ is exactly the set of all Wadge classes included in* $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$.

The strategy for the proof is the same as the original one by Louveau [68], and relies on the determinacy of the class $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$. For each description u that is not in D^0, we find a code that describes the immediate predecessor of Γ_u if it is at a successor position, or a sequence of codes that describe a sequence of classes that is cofinal under Γ_u if it is at a limit position. Formally we have the following.

Lemma 4.9. *Let u be a description.*

(a) *If $u \in D^+$, there exists $\bar{u} \in D$ such that, for any Wadge class Γ, $\Gamma_{\bar{u}} \subset \Gamma \subset \Gamma_u$ implies that $\Gamma = \Delta(\Gamma_u)$.*

(b) *If $u \in D^\omega$, there exists a sequence of descriptions $(\bar{u}_n)_{n\in\omega}$ such that*

– *for all integer* n, $\Gamma_{\bar{u}_n} \subset \Gamma_u$; *and*
– *for any Wadge class* Γ, *if for all integer* n $\Gamma_{\bar{u}_n} \subset \Gamma \subset \Gamma_u$, *then* $\Gamma = \Delta(\Gamma_u)$.

(c) *If* $u \in D^{\omega_1}$, *there exists a set of descriptions* Q_u *of cardinality* ω_1 *such that* $\Delta(\Gamma_u) = \bigcup\{\Gamma_{\bar{u}} : \bar{u} \in Q_u\}$.

The proof of Theorem 4.8 now goes as follows. Suppose, towards a contradiction, that the collection $\tilde{\mathcal{W}}$ of Wadge classes included in $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$ that are not in $\mathcal{W}$ is not empty. Using our determinacy hypothesis, the SLO property holds for $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$, and the pointclasses included in $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$ are well-founded for the inclusion. By the definition of $\mathcal{W}$, there is thus either a self-dual Wadge class Γ that is the $\subset$-least class in $\tilde{\mathcal{W}}$, or a couple of non-self-dual classes Γ and $\check{\Gamma}$ such that for all $\Gamma' \in \tilde{\mathcal{W}}$, $\Gamma \subseteq \Gamma'$ or $\check{\Gamma} \subseteq \Gamma'$ holds. Both situations lead to the same argument: since the classes $D_\eta(\boldsymbol{\Pi}^1_1)$ are in $\mathcal{W}$ and cofinal in $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$, there exists a description u such that Γ_u is the least Wadge class described above Γ, and we have three cases:

– $u \in D^+$, and then $\Gamma = \Delta(\Gamma_u)$ or $\Gamma = \check{\Gamma}_{\bar{u}}$;
– $u \in D^\omega$, and then $\Gamma = \Delta(\Gamma_u)$;
– $u \in D^{\omega_1}$, and then $\Gamma = \Delta(\Gamma_u)$.

Thus $\Gamma \in \mathcal{W}$ in each case, and we reach a contradiction.

So the only thing left to prove here is Lemma 4.9. But most cases are already covered by the proofs in [68], or straightforward extension of those using Proposition 4.5. We do not go through them again here and only take care of the $D_\eta(\boldsymbol{\Pi}^1_1)$ classes.

4.1.4 The successor case

In this section we look at the classes $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$, with $\eta < \omega_1$. These classes are described by descriptions u such that $u(0) = \omega_1$, $u(1) = 1$ and $u(2) = \eta + 1$, and are of type 1.

Lemma 4.10. *Let* $\eta < \omega_1$ *and* u *be a description of the class* $D_{\eta+1}(\boldsymbol{\Pi}^1_1)$. *Then:*

(a) $\Gamma_u = \mathrm{Bisep}(\boldsymbol{\Pi}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$;
(b) $\Delta(\Gamma_u) = \mathrm{Bisep}(\boldsymbol{\Delta}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$.

Proof.

(a) By Proposition 4.1, we know that $D_{\eta+1}(\boldsymbol{\Pi}^1_1) = \check{D}_\eta(\boldsymbol{\Pi}^1_1) \cap \boldsymbol{\Pi}^1_1$ so that $D_{\eta+1}(\boldsymbol{\Pi}^1_1) \subseteq \mathrm{Bisep}(\boldsymbol{\Pi}^1_1, D_\eta(\boldsymbol{\Pi}^1_1))$. For the other inclusion, we use Proposition 4.5.

(b) By Proposition 4.2, we know that if $D \in \Delta(D_{\eta+1}(\mathbf{\Pi}^1_1))$, then there exists $B \in \mathbf{\Delta}^1_1$, $X \in \check{D}_\eta(\mathbf{\Pi}^1_1)$ and $Y \in D_\eta(\mathbf{\Pi}^1_1)$ such that

$$D = (X \cap B) \cup (Y \setminus B).$$

Thus $D = \text{Bisep}(B, B^\complement, X, Y, \emptyset)$, so that the inclusion from left to right holds. Since $D_{\eta+1}(\mathbf{\Pi}^1_1)$ is closed under $\mathbf{\Pi}^1_1 - \text{SU}$, the class$D_{\eta+1}(\mathbf{\Pi}^1_1)$ contains the class $\text{Bisep}(\mathbf{\Delta}^1_1, D_\eta(\mathbf{\Pi}^1_1))$. What remains to prove is that the dual class of $\text{Bisep}(\mathbf{\Delta}^1_1, D_\eta(\mathbf{\Pi}^1_1))$ is also included in $D_{\eta+1}(\mathbf{\Pi}^1_1)$. Let A be such that $A^\complement \in \text{Bisep}(\mathbf{\Delta}^1_1, D_\eta(\mathbf{\Pi}^1_1))$. There exists thus $A_1 \in \check{D}_\eta(\mathbf{\Pi}^1_1)$, $A_2 \in D_\eta(\mathbf{\Pi}^1_1)$, and B_1, B_2 two disjoint Borel sets such that:

$$A^\complement = (A_1 \cap B_1) \cup (A_2 \cap B_2).$$

Therefore $A = (B_1 \cap A_1^\complement) \cup (B_2 \cap A_2^\complement) \cup (B_1 \cup B_2)^\complement$ is in $D_{\eta+1}(\mathbf{\Pi}^1_1)$ since this class is closed under $\mathbf{\Pi}^1_1 - \text{SU}$, and the class $\Delta(D_{\eta+1}(\mathbf{\Pi}^1_1))$ contains the class $\text{Bisep}(\mathbf{\Delta}^1_1, D_\eta(\mathbf{\Pi}^1_1))$. □

This allows us to define the set Q_u for a description u of the class $D_{\eta+1}(\mathbf{\Pi}^1_1)$:

$$Q_u = \{\xi^\frown 3^\frown 1^\frown \langle \omega_1{}^\frown 1^\frown \eta^\frown 0^\omega, 0^\omega \rangle : \xi < \omega_1\}.$$

We prove now that the family of classes described by Q_u is cofinal below $\Delta(\Gamma_u)$.

Proposition 4.11. *Let $\eta < \omega_1$, and u be a description for the class $D_{\eta+1}(\mathbf{\Pi}^1_1)$. Then:*

$$\Delta(D_{\eta+1}(\mathbf{\Pi}^1_1)) = \bigcup_{u' \in Q_u} \Gamma_{u'}.$$

Proof. Using Lemma 4.10, we have to prove:

$$\text{Bisep}(\mathbf{\Delta}^1_1, D_\eta(\mathbf{\Pi}^1_1)) = \bigcup_{\xi < \omega_1} \text{Bisep}(\mathbf{\Sigma}^0_\xi, D_\eta(\mathbf{\Pi}^1_1)).$$

The inclusion from right to left is immediate since each $\text{Bisep}(\mathbf{\Sigma}^0_\xi, D_\eta(\mathbf{\Pi}^1_1))$ is included in $\text{Bisep}(\mathbf{\Delta}^1_1, D_\eta(\mathbf{\Pi}^1_1))$. For the other inclusion, we just have to come back to the definition of the operation Bisep. Let $A \in \text{Bisep}(\mathbf{\Delta}^1_1, D_\eta(\mathbf{\Pi}^1_1))$, then there exists C_1, C_2 two disjoint Borel sets such that:

$$A = (A \cap C_1) \cup (A \cap C_2),$$

with $A \cap C_1 \in \check{D}_\eta(\mathbf{\Pi}^1_1)$ and $A \cap C_2 \in D_\eta(\mathbf{\Pi}^1_1)$. But if C_1 and C_2 are Borel, there exists $\xi < \omega_1$ such that C_1 and C_2 are both in the class $\mathbf{\Sigma}^0_\xi$! Thus $A \in \text{Bisep}(\mathbf{\Sigma}^0_\xi, D_\eta(\mathbf{\Pi}^1_1))$, and the other inclusion holds. □

This finishes the successor case.

4.1.5 The limit case

In this section we look at the classes $D_\gamma(\mathbf{\Pi}^1_1)$, with $\gamma < \omega_1$ limit. These classes are described by descriptions u such that $u(0) = \omega_1$, $u(1) = 1$ and $u(2) = \gamma$, and are of type 2. First we define a new operation and give a reformulation of Louveau's Trick II.

Definition 4.12. Let Γ and Γ' be two pointclasses. The set A is in $\mathrm{PU}(\Gamma, \Gamma')$ if and only if there exists a partition $(C_n)_{n\in\omega}$ of sets in Γ, and a family $(A_n)_{n\in\omega}$ of sets in Γ' such that

$$A = \mathrm{PU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}) = \bigcup_{n\in\omega} A_n \cap C_n.$$

This operation is called the *Partitioned Union*. It is of course a special case of SU.

Lemma 4.13. *Let $\gamma < \omega_1$ be a limit ordinal, and u be a description of the class $D_\gamma(\mathbf{\Pi}^1_1)$. Then:*
(a) $\Gamma_u = \mathrm{SU}(\mathbf{\Pi}^1_1, \Gamma)$*;*
(b) $\Delta(\Gamma_u) = \mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma)$.
where $\Gamma = \bigcup_{\eta<\gamma} D_\eta(\mathbf{\Pi}^1_1)$.

Proof.
(a) Since $D_\gamma(\mathbf{\Pi}^1_1)$ is closed under $\mathbf{\Pi}^1_1 - \mathrm{SU}$, the inclusion from right to left is immediate. For the other one, let $(A_\alpha)_{\alpha<\gamma}$ be an increasing family of $\mathbf{\Pi}^1_1$ sets, and $D = D_\gamma((A_\alpha)_{\alpha<\gamma})$. By the generalized reduction property of the class of coanalytic sets, there exists a disjoint coanalytic family $(C_\alpha)_{\alpha<\gamma}$ such that
   - for all $\alpha < \gamma$, $C_\alpha \subseteq A_\alpha$;
   - $\bigcup_{\alpha<\gamma} C_\alpha = \bigcup_{\alpha<\gamma} A_\alpha$.

   Now we have $D \cap C_\alpha \subseteq A_\alpha$ for all $\alpha < \gamma$, and thus $D \cap C_\alpha \in \Gamma \cap \mathbf{\Pi}^1_1$. Since
$$D = \bigcup_{\alpha<\gamma} D \cap C_\alpha,$$
   $D \in \mathrm{SU}(\mathbf{\Pi}^1_1, \Gamma)$ and the second inclusion is proven.

(b) By Louveau's Trick II, we know that if $D \in \Delta(D_\gamma(\mathbf{\Pi}^1_1))$, there exists a countable Borel partition $(C_i)_{i\in\omega}$ of the Baire space such that, for all $j < \omega$,
$$D \cap C_j \in D_{\eta_j}(\mathbf{\Pi}^1_1),$$

with $\eta_j < \gamma$. Thus $D \in \mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma)$, so that $\Delta(D_\gamma(\boldsymbol{\Pi}^1_1)) \subseteq \mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma)$. Since $D_\gamma(\boldsymbol{\Pi}^1_1)$ is closed under $\boldsymbol{\Pi}^1_1 - \mathrm{SU}$, $\mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma) \subseteq D_\gamma(\boldsymbol{\Pi}^1_1)$. What remains to prove is that the dual class of $\mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma)$ is also included in $D_\gamma(\boldsymbol{\Pi}^1_1)$. Let A be such that $A^\complement \in \mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma)$. There exists a partition in coanalytic sets $(C_i)_{i\in\omega}$ such that

$$A^\complement = \bigcup_{i\in\omega} A^\complement \cap C_i,$$

with $A^\complement \cap C_i \in D_{\alpha_i}(\boldsymbol{\Pi}^1_1)$ and $\alpha_i < \gamma$. Notice that, for all integer i,

$$A \cap C_i = (A^\complement \cap C_i)^\complement \cap C_i.$$

By Proposition 4.1, $(A^\complement \cap C_i)^\complement \cap C_i \in D_{\alpha_i+1}(\boldsymbol{\Pi}^1_1)$ which is still included in $D_\gamma(\boldsymbol{\Pi}^1_1)$! The set A is therefore in $D_\gamma(\boldsymbol{\Pi}^1_1)$, and $\mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma) \subseteq \Delta(D_\gamma(\boldsymbol{\Pi}^1_1))$. $\square$

This allows us to define the set Q_u for a description u of the class $D_\gamma(\boldsymbol{\Pi}^1_1)$, it is the set of descriptions

$$\xi^\frown 4^\frown \langle (u'_n)_{n\in\omega} \rangle$$

for $\xi < \omega_1$, where $u'_n = \omega_1{}^\frown 1^\frown \gamma_n{}^\frown 0^\omega$, and $(\gamma_n)_{n\in\omega}$ is cofinal in γ. We prove now that the family of classes described by Q_u is cofinal below $\Delta(\Gamma_u)$.

Proposition 4.14. *Let $\gamma < \omega_1$ be limit, and u be a description for the class $D_\gamma(\boldsymbol{\Pi}^1_1)$. Then*

$$\Delta(D_\gamma(\boldsymbol{\Pi}^1_1)) = \bigcup_{u'\in Q_u} \Gamma_{u'}.$$

Proof. Using Lemma 4.13, we have to prove:

$$\mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma) = \bigcup_{\xi<\omega_1} \mathrm{SU}(\boldsymbol{\Sigma}^0_\xi, \Gamma'),$$

where $\Gamma = \bigcup_{\eta<\gamma} D_\eta(\boldsymbol{\Pi}^1_1)$ and $\Gamma' = \bigcup_{n\in\omega} D_{\gamma_n}(\boldsymbol{\Pi}^1_1)$. First notice that $\Gamma = \Gamma'$, so that we actually only have to prove:

$$\mathrm{PU}(\boldsymbol{\Pi}^1_1, \Gamma) = \bigcup_{\xi<\omega_1} \mathrm{SU}(\boldsymbol{\Sigma}^0_\xi, \Gamma).$$

For the first inclusion, from left to right, notice that any coanalytic countable partition is in fact a Borel and hence a $\boldsymbol{\Sigma}^0_\xi$ partition for a certain $\xi < \omega_1$. For the other inclusion, let $\xi < \omega_1$ and $D \in \mathrm{SU}(\mathbf{\Sigma}^0_\xi, \Gamma)$. By definition there exists a disjoint family $(C_n)_{n\in\omega}$ of $\mathbf{\Sigma}^0_\xi$ sets and a family $(A_n)_{n\in\omega}$ in Γ such that

$$D = \mathrm{SU}((C_n)_{n\in\omega}, (A_n)_{n\in\omega}).$$

But then we have

$$\begin{aligned} D &= \bigcup_{n\in\omega} C_n \cap A_n \\ &= \left((\bigcup_{n\in\omega} C_n)^{\complement} \cap \emptyset \right) \cup \bigcup_{n\in\omega} C_n \cap A_n \\ &= \mathrm{PU}((C'_n)_{n\in\omega}, (A'_n)_{n\in\omega}); \end{aligned}$$

where $C'_0 = (\bigcup_{n\in\omega} C_n)^{\complement}$, $C'_{n+1} = C_n$, $A'_0 = \emptyset$ and $A'_{n+1} = A_n$. Since $\emptyset \in \Gamma$ and $C'_0 \in \mathbf{\Pi}^0_\xi$, $D \in \mathrm{PU}(\mathbf{\Pi}^1_1, \Gamma)$ and the other inclusion follows. $\square$

This finishes the limit case, and the proof of Lemma 4.9.

4.2 Complete canonical sets

After the description à la Louveau of the pointclasses included in the class $\mathrm{Diff}(\mathbf{\Pi}^1_1)$, we turn our attention to complete elements of these classes. Following Duparc's approach and using the conciliatory sets, we define a complete set for each non-self-dual Wadge degree in the class $\mathrm{Diff}(\mathbf{\Pi}^1_1)$.

4.2.1 Canonical $D_\alpha(\mathbf{\Pi}^1_1)$-complete sets

In this section, we construct from a $\mathbf{\Pi}^1_1$-complete set A a family $(D^A_\xi)_{1<\xi<\omega_1}$ such that for any $1 < \alpha < \omega_1$, D^A_α is $D_\alpha(\mathbf{\Pi}^1_1)$-complete.

The finite levels

Definition 4.15. Let $m, n \in \omega$ be two integers such that $m < n$. Using the euclidian division, we define the m *modulo* n projection by

$$\begin{array}{rcl} \pi_{m(n)} : \omega^\omega & \longrightarrow & \omega^\omega \\ (u_j)_{j\in\omega} & \longmapsto & (u_{jn+m})_{i\in\omega}. \end{array}$$

Definition 4.16. Let $A \subseteq \omega^\omega$, and n be a positive integer. Define the sets $A_n^k \subseteq \omega^\omega$ for $k < n$ by

$$A_n^k = \bigcup_{i=0}^{k} \pi_{i(n)}^{-1}(A).$$

It forms an increasing family, and we will denote by D_n^A the set $D_n((A_n^k)_{0\leq k<n})$.

Notice that since for all $k < n-1$:

$$A_n^{k+1} \setminus A_n^k = \pi_{k+1(n)}^{-1}(A) \setminus A_n^k,$$

we have another expression for D_n^A:

$$D_n^A = \begin{cases} \bigcup_{2k+1<n} \left(\pi_{2k+1(n)}^{-1}(A) \setminus A_n^{2k}\right), & \text{if } n \text{ is even;} \\ \pi_{0(n)}^{-1}(A) \cup \bigcup_{0<2k<n} \left(\pi_{2k(n)}^{-1}(A) \setminus A_n^{2k-1}\right), & \text{if } n \text{ is odd.} \end{cases}$$

Fact 4.17. *Let $A \subseteq \omega^\omega$, and n be a positive integer. If A is $\mathbf{\Pi}_1^1$-complete, then for every $k < n$, A_n^k is $\mathbf{\Pi}_1^1$-complete.*

Proof. Fix $k < n$. It is clear that A_n^k is $\mathbf{\Pi}_1^1$. To prove that it is complete, we show that II has a winning strategy in the game $W(A, A_n^k)$. Suppose that I plays along a sequence $x = (x_0, x_1, \ldots)$, then II answers by the sequence

$$x' = (\overbrace{x_0, \ldots, x_0}^{k+1}, \underbrace{0, \ldots, 0}_{n-(k+1)}, \overbrace{x_1, \ldots, x_1}^{k+1}, \underbrace{0, \ldots, 0}_{n-(k+1)}, \ldots).$$

If $x \in A$, then $\pi_{i(n)}(x') \in A$ for all $i \leq k$, so that $x' \in A_n^k$. If $x \notin A$, then $\pi_{i(n)}(x') \notin A$ for all $i \leq k$, so that $x' \notin A_n^k$. This shows that the strategy is winning for player II, and completes the proof. □

Lemma 4.18. *Let n be a positive integer and $(f_i)_{i<n}$ be a family of n functions from the Baire space to itself. There exists a unique function $g : \omega^\omega \longrightarrow \omega^\omega$ such that for all $i < n$:*

$$f_i = \pi_{i(n)} \circ g.$$

We will denote it by $\mathrm{comb}((f_i)_{i<n})$*, and call it the combination of* $(f_i)_{i<n}$*.*

Proof. Define it by

$$\begin{aligned} \mathrm{comb}((f_i)_{i<n}) : \omega^\omega &\longrightarrow \omega^\omega \\ x &\longmapsto \bigcap_{i=0}^{n-1} \pi_{i(n)}^{-1}(f_i(x)), \end{aligned}$$

so that for all integer p, $x \in \omega^\omega$, and $j < n$

$$\mathrm{comb}((f_i)_{i<n})(x)(pn+j) = f_j(x)(p). \qquad \square$$

Lemma 4.19. *Let n be a positive integer and $(f_i)_{i<n}$ a family of n continuous functions from the Baire space to itself. Then $\mathrm{comb}((f_i)_{i<n})$ is also continuous.*

Proof. Let $s \in \omega^{<\omega}$ be a finite sequence, and $N_s = \{x \in \omega^\omega \mid s \subseteq x\}$ its associated basic clopen set. Then:

$$\mathrm{comb}((f_i)_{i<n})^{-1}(N_s) = \bigcup_{i<n} f_i^{-1}(N_{\pi_{i(n)}(s)}),$$

which is open since all the f_i are continuous. The function $\mathrm{comb}((f_i)_{i<n})$ is thus also continuous. $\square$

Proposition 4.20. *Let $A \subseteq \omega^\omega$ be $\mathbf{\Pi}^1_1$-complete, and n a positive integer. Then the set $D_n^A \subseteq \omega^\omega$ is $D_n(\mathbf{\Pi}^1_1)$-complete.*

Proof. We only do the proof for n even here, the odd case is similar. Consider $D = D_n((B_k)_{0\leq k<n}) \in D_n(\mathbf{\Pi}^1_1)$, with $(B_k)_{0\leq k<n}$ an increasing sequence of co-analytic subsets of the Baire space. We have, for all $0 \leq k < n$, a continuous function f_k from the Baire space to itself such that $B_k = f_k^{-1}(A)$. We show that the combination of the $(f_k)_{k<n}$ verifies:

$$D_n((B_k)_{0\leq k<n}) = \mathrm{comb}((f_k)_{k<n})^{-1}(D_n^A).$$

Let $x \in D_n((B_k)_{0\leq k<n})$. Thus $x \in B_{2k+1} \setminus B_{2k} = B_{2k+1} \setminus \bigcup_{j\leq 2k} B_j$ for some $2k+1 < n$. Then $\pi_{2k+1(n)} \circ \mathrm{comb}((f_i)_{i<n})(x) = f_{2k+1}(x) \in A$ and, for all $j \leq k$, $\pi_{j(n)} \circ \mathrm{comb}((f_i)_{i<n})(x) = f_j(x) \notin A$. Thus

$$\mathrm{comb}((f_i)_{i<n})(x) \in \pi^{-1}_{2k+1(n)}(A) \setminus \bigcup_{j\leq 2k} \pi^{-1}_{j(n)}(A) \subseteq D_n^A.$$

Now, let $x \notin D_n((B_k)_{0\leq k<n})$. There are three cases:

- $x \notin \bigcup_{k<n} B_k$,
- $x \in B_0$,
- $x \in B_{2k} \setminus B_{2k-1}$ for some $0 < 2k < n$.

If $x \notin \bigcup_{k<n} B_k$, then for all $k < n$, $\mathrm{comb}((f_i)_{i<n})(x) \notin \pi^{-1}_{k(n)}(A)$, and thus $\mathrm{comb}((f_i)_{i<n})(x) \notin D_n^A$. If $x \in B_0$, then $\mathrm{comb}((f_i)_{i<n})(x) \in \pi^{-1}_{0(n)}(A) = A_n^0$,

and thus $\operatorname{comb}((f_i)_{i<n})(x) \notin D_n^A$. If there exists $0 < 2k < n$ such that $x \in B_{2k} \smallsetminus B_{2k-1}$, then $x \in B_{2k} \smallsetminus \bigcup_{j\leq 2k-1} B_j$. Thus

$$\operatorname{comb}((f_i)_{i<n})(x) \in \pi_{2k(n)}^{-1}(A) \smallsetminus \bigcup_{j\leq 2k-1} \pi_{j(n)}^{-1}(A) = A_n^{2k} \smallsetminus A_n^{2k-1},$$

and $\operatorname{comb}((f_i)_{i<n})(x) \notin D_n^A$.

Hence $D_n((B_k)_{0\leq k<n}) = \operatorname{comb}((f_k)_{k<n})^{-1}(D_n^A)$, so that $D \leq_W D_n^A$, which proves that D_n^A is $D_n(\mathbf{\Pi}_1^1)$-complete. □

The transfinite levels

Set a bijection ϕ between ω and $\omega \times \omega$. It induces the following bijection φ:

$$\begin{aligned} \varphi : \omega^\omega &\longrightarrow \omega^{\omega\times\omega} \\ a &\longmapsto (a(\phi(i,j)))_{i,j\in\omega}. \end{aligned}$$

Let ξ be a countable ordinal, and ψ_ξ a bijection between ξ and ω. Then for any $\nu < \xi$ we have the following continuous function:

$$\begin{aligned} \varphi_\xi^\nu : \omega^\omega &\longrightarrow \omega^\omega \\ a &\longmapsto (\varphi(a)(\psi_\xi(\nu), j))_{j\in\omega}. \end{aligned}$$

This construction is a generalization of the projections defined before. They allow us indeed to partition an infinite sequence into ξ infinite sequences. Mutatis mutandis, we get the same results that in the finite case.

Definition 4.21. Let ξ be a countable ordinal, and A a subset of the Baire space. Define the sets $A_\xi^\nu \subseteq \omega^\omega$ for $\nu < \xi$ by

$$A_\xi^\nu = \bigcup_{\eta\leq\nu} (\varphi_\xi^\eta)^{-1}(A)$$

We will denote by D_ξ^A the set $D_\xi((A_\xi^\nu)_{\nu<\xi})$.

Lemma 4.22. *Let $\omega \leq \xi < \omega_1$ be a countable ordinal and $(f_\nu)_{\nu<\xi}$ be a family of functions from the Baire space to itself. There exists a unique function $g : \omega^\omega \to \omega^\omega$ such that for all $\nu < \xi$:*

$$f_\nu = \varphi_\xi^\nu \circ g.$$

We will denote it by $\operatorname{comb}((f_\xi)_{\nu<\xi})$, *and call it the combination of* $(f_\nu)_{\nu<\xi}$.

Proof. Define it by

$$\begin{array}{rcl} \mathrm{comb}((f_\nu)_{\nu<\xi}) : \omega^\omega & \longrightarrow & \omega^\omega \\ x & \longmapsto & \bigcap_{\nu<\xi} (\varphi_\xi^\nu)^{-1}(f_\nu(x)), \end{array}$$

so that

$$\varphi(\mathrm{comb}((f_\nu)_{\nu<\xi})(x))(\psi_\xi(\nu), j) = f_\nu(x)(j).$$

□

Likewise the finite case, one can show that if the functions of the family $(f_\nu)_{\nu<\xi}$ are all continuous, so is $\mathrm{comb}((f_\nu)_{\nu<\xi})$.

Proposition 4.23. *Let ξ be a countable ordinal, and A a $\mathbf{\Pi}_1^1$-complete subset of the Baire space. Then D_ξ^A is $D_\xi(\mathbf{\Pi}_1^1)$-complete.*

Proof. Consider $D = D_\xi((B_\nu)_{\nu<\xi})$, with $(B_\nu)_{\nu<\xi}$ an increasing family of co-analytic subsets of the Baire space. For each $\nu < \xi$, we have a continuous function f_ν from the Baire space to itself such that $B_\nu = f_\nu^{-1}(A)$. Observe that the combination of the $(f_\nu)_{\nu<\xi}$ verifies thus:

$$D_\xi((B_\nu)_{\nu<\xi}) = \mathrm{comb}((f_\nu)_{\nu<\xi})^{-1}(D_\xi^A).$$

Hence $D \leq_W D_\xi^A$, which proves that D_ξ^A is $D_\xi(\mathbf{\Pi}_1^1)$-complete. □

We can now sketch a first overview of the Wadge hierarchy above the Borel sets:

$$\begin{array}{ccccccc} [A]_W & & \left[D_2^A\right]_W & & \left[D_\alpha^A\right]_W & & \\ & \cdots & & \cdots & & & \cdots \\ [A^\complement]_W & & \left[(D_2^A)^\complement\right]_W & & \left[(D_\alpha^A)^\complement\right]_W & & \end{array}$$

where A is a $\mathbf{\Pi}_1^1$-complete set. What remains to describe is thus the $\Delta(D_\alpha(\mathbf{\Pi}_1^1))$ classes.

4.2.2 Cofinal sequences for the ambiguous classes

The description of the ambiguous classes provided by Louveau's Tricks, Propositions 4.2 and 4.3, will be our main tool to prove that the Veblen operations are sufficient to climb up the Wadge Hierarchy of the $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ sets. We begin with the successor case.

Let $\circleddash$ denote the empty set, and O the following $\mathbf{\Sigma}^0_1$-complete subset of the Baire space:

$$O = \{\alpha \in \omega^\omega \mid \exists n\ \alpha(n) = 0\}\,.$$

Proposition 4.24. *Let $A \subseteq \omega^\omega$ be a $\mathbf{\Pi}^1_1$-complete set, and $\theta < \omega_1$. Let $C \subseteq \omega^\omega$ be in the $\Delta(D_{\theta+1}(\mathbf{\Pi}^1_1))$ class. Then there exists $\xi < \omega_1$ such that*

$$C \leq_W ((D^A_\theta + \circleddash)^{\approx_{1+\xi}})^b.$$

Proof. By Louveau's Trick I, there exists a countable ordinal $\xi < \omega_1$ such that $C \in \mathrm{Sep}(\mathbf{\Sigma}^0_{1+\xi}, D_\theta(\mathbf{\Pi}^1_1))$. Thus, we have $X \in \check{D}_\theta(\mathbf{\Pi}^1_1))$, $Y \in D_\theta(\mathbf{\Pi}^1_1)$ and $B \in \mathbf{\Sigma}^0_{1+\xi}$ such that

$$C = (X \cap B) \cup (Y \smallsetminus B).$$

Notice that II has a winning strategy τ in the game $W(B^{\sim\xi}, O^b)$. II has also two winning strategies σ_1 and σ_2 respectively in the games $W(Y^T, D^A_\theta + \circleddash)$ and $W(X^T, D^A_\theta + \circleddash)$, where T is the ξ-tree such that $B^T = B^{\sim\xi}$. To prove that $C \leq_W ((D^A_\theta + \circleddash)^{\approx_{1+\xi}})^b$, we show that II has a winning strategy σ in the game $W(C^T, ((D^A_\theta + \circleddash)^{\approx})^b)$. Suppose I plays along a sequence of integers $x_0, x_1, \ldots$; II first follows σ_1 as long as $\tau(x_0, \ldots, x_k)$ is different of 0. If ever $\tau(x_0, \ldots, x_k) = 0$ for an integer k, then II erases everything she has played, and then follows σ_2. We show now that the strategy σ is winning. Suppose that $x \in C^T$, then we have two distinct cases: either $x \in B^T$, or it is not. If it is in B^T, then it is in C^T if and only if it is in X^T. But in this case, $(\sigma * x)^{\hookleftarrow} = \sigma_2 * x$, and since σ_2 is winning for II in the game $W(X^T, D^A_\theta + \circleddash)$, x is in C^T if and only if $(\sigma * x)^{\hookleftarrow}$ is in $D^A_\theta + \circleddash$. If x is not in B^T, then it is in C^T if and only if it is in Y^T. But in this case, $\sigma * x = \sigma_1 * x$, and since σ_1 is winning for II in the game $W(Y^T, D^A_\theta + \circleddash)$, x is in C^T if and only if $\sigma * x$ is in $D^A_\theta + \circleddash$. The strategy σ is thus winning in the game $W(C^T, ((D^A_\theta + \circleddash)^{\approx})^b)$.

$$C^T \leq_W ((D^A_\theta + \circleddash)^{\approx})^b,$$

which implies, by minimality:

$$C^{\sim\xi} \leq_W ((D^A_\theta + \circleddash)^{\approx})^b,$$

so that

$$C \leq_W ((D^A_\theta + \circleddash)^{\approx_{1+\xi}})^b.$$

□

Proposition 4.24 shows that the Veblen operations applied to $(D_\theta^A + \ominus)$ provide a complete backbone to the $\Delta(D_{\theta+1}(\mathbf{\Pi}_1^1))$ class. The same phenomenon occurs for the limit stages.

Proposition 4.25. *Let $D \subseteq \omega^\omega$ be in the $\Delta(D_\delta(\mathbf{\Pi}_1^1))$ class, with $\delta < \omega_1$ limit. Then there exists $\xi < \omega_1$ and a sequence $(\alpha_i)_{i\in\omega} \subset \delta$ such that*

$$D \leq_W \left(\left(\bigcup_{i\in\omega} i^\frown D_{\alpha_i}^A \right)^{\approx_\xi} \right)^b$$

where $A \subseteq \omega^\omega$ is $\mathbf{\Pi}_1^1$-complete.

Proof. By Louveau's Trick II, there exists a countable Borel partition $(C_i)_{i\in\omega}$ of the Baire space such that for all $j < \omega$:

$$D \cap C_j = D_j \in D_{\alpha_j}(\mathbf{\Pi}_1^1),$$

with $\alpha_j < \delta$. Since the partition is countable, there exists an ordinal $\xi < \omega_1$ such that $(C_i)_{i\in\omega} \subseteq \mathbf{\Sigma}_\xi^0$. Thus, there is a $\xi+1$-tree T such that for any $i \in \omega$:

$$C_i^T \leq_W O^b.$$

For all i, denote by σ_i the winning strategy for player II in the game $W(C_i^T, O^b)$, and by σ_{α_i} her winning strategy in the game $W(D_i^T, D_{\alpha_i}^A)$. To prove that $D \leq_W \left((\bigcup_{i\in\omega} i^\frown D_{\alpha_i}^A)^{\approx_\xi} \right)^b$, we prove first that II has a winning strategy τ in the game

$$W\left(D^T, \left(\bigcup_{i\in\omega} (i^\frown D_{\alpha_i}^A) \right)^b \right).$$

Suppose I plays along a sequence $(x_0, x_1, \ldots)$. As long as for all $i \in \omega$, $\sigma_i(x_0, x_1, \ldots, x_j)$ is not 0, II skips. If and when there is some $i \in \omega$ such that $\sigma_i(x_0, x_1, \ldots, x_j)$ is 0, player II plays along σ_{α_i}. Since all the σ_{α_i} are winning, τ is also winning. We have thus:

$$D^T \leq_W (\bigcup_{i\in\omega} i^\frown D_{\alpha_i}^A)^b,$$

which implies, by minimality:

$$D^{\sim\xi+1} \leq_W (\bigcup_{i\in\omega} i^\frown D_{\alpha_i}^A)^b,$$

so that:

$$D \leq_W \left(\left(\bigcup_{i \in \omega} i^\frown D^A_{\alpha_i} \right)^{\approx_{\xi+1}} \right)^b .$$

□

The situation is thus the best possible, since given $\ominus$ and the family $(D^A_\xi)_{\xi<\omega_1}$, the operations defined for the Borel sets by Duparc [26, 27] suffice at least to climb up the hierarchy on $\mathrm{Diff}(\mathbf{\Pi}^1_1)$. We will prove in the next section that it describes in fact exactly the whole hierarchy of the $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ sets.

4.2.3 A normal form

In the manner of Duparc [27], in order to complete the description of the Wadge Hierarchy of the $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ sets, we now define an application

$$\Omega : (V^{\omega_1}(\omega_1) \smallsetminus \{0\}) \longrightarrow \mathcal{P}(\omega^{\leq\omega})$$

which will verify that for all $0 < \alpha < V^{\omega_1}(\omega_1)$, $d_w(\Omega(\alpha)^b) = \alpha$, and that for all non-self-dual $B \in \mathrm{Diff}(\mathbf{\Pi}^1_1)$, there exists $\alpha < V^{\omega_1}(\omega_1)$ such that either $\Omega(\alpha)^b \equiv_W B$ or $\Omega(\alpha)^b \equiv_W B^\complement$. Before stating the definition of Ω, we need a technical result on the Veblen hierarchy.

Lemma 4.26. *Let $0 < \beta < V^{\omega_1}(\omega_1)$, and consider the set*

$$I_\beta = \left\{ \nu < \omega_1 \mid \exists \gamma < \omega_1^\beta \left(V^\nu(\gamma) = \omega_1^\beta \right) \right\}.$$

There exists $\alpha < \omega_1$ such that $\omega_1^\beta = V^{\omega_1}(1+\alpha)$ if and only if $\mathrm{cof}(\beta) \neq \omega$ and $I_\beta = \emptyset$.

Proof. Suppose first that $\omega_1^\beta = V^{\omega_1}(1+\alpha)$. By definition, $\mathrm{cof}(\omega_1^\beta) = \omega_1$, which implies that $\mathrm{cof}(\beta) \neq \omega$. Moreover, ω_1^β is also a fixpoint of all the Veblen functions V^ξ, with $\xi < \omega_1$. In other words, there exists no $\nu < \omega_1$ and no $\gamma < \omega_1^\beta$ such that $V^\nu(\gamma) = \omega_1^\beta$, proving that $I_\beta = \emptyset$.

Suppose now that $I_\beta = \emptyset$, and that $\mathrm{cof}(\beta) \neq \omega$. Then $\mathrm{cof}(\omega_1^\beta) = \omega_1$ and ω_1^β is a fixpoint of all the Veblen functions V^ξ, with $\xi < \omega_1$. Thus $\omega_1^\beta = V^{\omega_1}(1+\alpha)$, with $\alpha < \omega_1$ since $\beta < V^{\omega_1}(\omega_1)$. □

Definition 4.27. Let A be a $\mathbf{\Pi}^1_1$−complete subset of the Baire space,and $0 < \alpha < V^{\omega_1}(\omega_1)$. The ordinal α admits a unique Cantor Normal Form of base ω_1:

$$\alpha = \omega_1^{\alpha_n} \cdot \nu_n + \cdots + \omega_1^{\alpha_0} \cdot \nu_0,$$

with
$$V^{\omega_1}(\omega_1) > \alpha_n > \cdots > \alpha_0, \text{ and } 0 < \eta_i < \omega_1 \text{ for all } i \leq n.$$
Set:
$$\Omega(\alpha) = \Omega(\omega_1^{\alpha_n}) \cdot \eta_n + \cdots + \Omega(\omega_1^{\alpha_0}) \cdot \eta_0,$$
where $\Omega(\omega_1^\beta)$ is defined by

– If $\beta = 0$ then
$$\Omega(\omega_1^0) = \emptyset.$$

– If $\beta = \alpha + 1$ is successor, then $0 \in I_\beta$. Denote by γ_0 the ordinal such that $V^0(\gamma_0) = \omega_1^\beta$. Then set
$$\Omega(\omega_1^\beta) = V_0(\Omega(\gamma_0)).$$

– If β is limit of cofinality ω, there exists a sequence $(\beta_i)_{i\in\omega}$ of ordinals strictly less than β such that $\beta = \sup_{i\in\omega}\{\beta_i\}$. Then set:
$$\Omega(\omega_1^\beta) = \mathbf{sup}_{i\in\omega}\left\{\Omega(\omega_1^{\beta_i})\right\}.$$

– If β is limit of cofinality ω_1, then we have two cases: either I_β is empty, or not. If it is not empty, then as in the successor case, we set
$$\Omega(\omega_1^\beta) = V_{\nu_0}(\Omega(\gamma_0)),$$
where $\nu_0 < \omega_1$ and $\gamma_0 < \omega_1^\beta$ are the minimal ordinals such that $V^{\nu_0}(\gamma_0) = \omega_1^\beta$. If it is empty, then there exists $\xi < \omega_1$ such that $\omega_1^\beta = V^{\omega_1}(1+\xi)$ and we set
$$\Omega(\omega_1^\beta) = D_\xi^B,$$
where D_ξ^B is a conciliatory set such that $(D_\xi^B)^b \equiv_W D_\xi^A$.

Notice that the function Ω is the extension of the function defined by Duparc [27].

We now prove that the sets obtained by this function constitute a hierarchy that describes up to complement the whole Wadge hierarchy on Diff$(\mathbf{\Pi}_1^1)$. To do so, we prove that the family $(\Omega(\alpha))_{\alpha<V^{\omega_1}(\omega_1)}$ is well ordered by $<_c$, and that there is no gap in our construction.

Facts 4.28. *Let A be a $\mathbf{\Pi}_1^1-$complete subset of the Baire space, and, for all $\xi < \omega_1$, let D_ξ^B be a conciliatory set such that $(D_\xi^B)^b \equiv_W D_\xi^A$. Let also $(C_i)_{i\in\omega}$ be a family of conciliatory sets. The following hold:*

(a) *if $C_1, C_2 <_c D_\xi^B$, then $C_1 + C_2 <_c D_\xi^B$;*

(b) *if* $C_i <_c D^B_\xi$ *for all* $i \in \omega$, *then* $\mathbf{sup}_{i\in\omega} C_i <_c D^B_\xi$;
(c) *if* $C_1 <_c D^B_\xi$, *then* $V_\nu(C_1) <_c D^B_\xi$ *for all* $\nu < \omega_1$.

Proof. It is sufficient to notice that since each $D_\alpha(\mathbf{\Pi}^1_1)$ class is closed under preimages by Borel functions, the sets D^B_ξ are fixpoints for the Veblen operations, that is for all $0 < \xi, \nu < \omega_1$,

$$V_\nu(D^B_\xi) \equiv_c D^B_\xi. \qquad \square$$

Now we prove that the family $(\Omega(\alpha))_{\alpha < V^{\omega_1}(\omega_1)}$ is well ordered by $<_c$.

Lemma 4.29. *For* $0 < \beta < \alpha < V^{\omega_1}(\omega_1)$,

$$\Omega(\beta) \leq_c \Omega(\alpha) \qquad \text{and} \qquad \Omega(\beta)^\complement \leq_c \Omega(\alpha).$$

Proof. Since Ω is the extension of the function defined by Duparc [27], the only remaining case to check is if $\alpha = V^{\omega_1}(1+\xi)$, with $0 < \xi < \omega_1$, and if β is of the form $\omega_1^{\beta_0}$. By induction on ξ we have:

– If $\xi = 1$, then $\Omega(\alpha) = B$, and for all $\beta < \alpha$, $\Omega(\beta)$ and $\Omega(\beta)^\complement$ are Borel. Thus for all $\beta < \alpha$, $\Omega(\beta) <_c \Omega(\alpha)$.
– If $\xi = \mu + 1$, then $\Omega(\alpha) = D^B_{\mu+1}$. By the induction hypothesis and transitivity of $<_c$, we have that if $\beta \leq V^{\omega_1}(1+\mu)$, then $\Omega(\beta) <_c \Omega(\alpha)$. For $V^{\omega_1}(1+\mu) < \beta < V^{\omega_1}(1+\mu+1)$, we have two cases: either $\mathrm{cof}(\beta) = \omega$, or $I_{\beta_0} \neq \emptyset$. If $\mathrm{cof}(\beta) = \omega$, there exists a sequence ζ_i of ordinals such that $\sup_{i\in\omega} \zeta_i = \beta$, and then we use Facts 4.28 to conclude. If $I_{\beta_0} \neq \emptyset$, there exist $\nu < \omega_1$ and $\gamma < \beta$ such that $V^\nu(\gamma) = \beta$, and then we use Facts 4.28 to conclude.
– if $\xi = \gamma$ is limit, then $\Omega(\alpha) = D^B_\gamma$. By the induction hypothesis and transitivity of $<_c$, we have that if $\beta \leq V^{\omega_1}(1+\delta)$ with $\delta < \gamma$, then $\Omega(\beta) <_c \Omega(\alpha)$. If $V^{\omega_1}(1+\delta) < \beta < \Omega(\alpha)$ for all $\delta < \gamma$, then we are exactly in the same situation as before: either $\mathrm{cof}(\beta) = \omega$, or $I_{\beta_0} \neq \emptyset$, and we can conclude in the same way.

$\square$

We prove now that there is no gap in our construction.

Lemma 4.30. *Let* $C \subseteq \omega^\omega$ *be non-self-dual and* $0 < \alpha < V^{\omega_1}(\omega_1)$ *such that*

$$C \leq_W \Omega(\alpha)^b \qquad \text{and} \qquad C^\complement \leq_W \Omega(\alpha)^b.$$

Then there exists $\beta < \alpha$ *so that*

$$C \leq_W \Omega(\beta)^b \qquad \text{or} \qquad C^\complement \leq_W \Omega(\beta)^b.$$

Proof. Since Ω is the extension of the function defined by Duparc [27], the only remaining case to check is if $\alpha = V^{\omega_1}(1+\xi)$, with $0 < \xi < \omega_1$. By induction on ξ we have:

- If $\xi = 1$, then $\Omega(\alpha)^b \equiv_W A$ and if $C <_W \Omega(\alpha)^b$ then C is Borel and there exists $\beta < V^{\omega_1}(2)$ such that $C \equiv_W \Omega(\beta)^b$ or $C^{\complement} \equiv_W \Omega(\beta)^b$.[1]
- If $\xi = \mu + 1$, then $\Omega(\alpha)^b \equiv_W D^A_{\mu+1}$, and if $C <_W \Omega(\alpha)^b$ then $C \in \Delta(D_{\mu+1}(\mathbf{\Pi}^1_1))$ and by Proposition 4.24, there exists $\nu < \omega_1$ such that $C \leq_W V_\nu(D^A_\mu + \ominus)^b$. But $D^A_\mu \equiv_W \Omega(\omega_1^{\beta'})^b$, where $\omega_1^{\beta'} = V^{\omega_1}(1+\mu)$, so that $V_\nu(D^A_\mu + \ominus)^b = V_\nu(\Omega(\omega_1^{\beta'}) + \ominus)^b = \Omega(V^\nu(\omega_1^{\beta'} + 1))^b$ by definition of Ω. Notice that $V^\nu(\omega_1^{\beta'} + 1) < V^{\omega_1}(1+\xi)$, thus $C \leq_W \Omega(\beta)^b$ with $\beta < \alpha$.
- if $\xi = \gamma$ limit, then $\Omega(\alpha)^b \equiv_W D^A_\gamma$, and if $C <_W \Omega(\alpha)^b$ then $C \in \Delta(D_\gamma(\mathbf{\Sigma}^1_1))$, and by Proposition 4.25, there exists $\nu < \omega_1$ and a sequence $(\alpha_i)_{i\in\omega} \subset \delta$ such that $C \leq_W V_\nu\left(\bigcup_{i\in\omega} i^\frown D^A_{\alpha_i}\right)^b$. But for all $i \in \omega$ $D^A_{\alpha_i} \equiv_W \Omega(\omega_1^{\beta_i})^b$, where $\omega_1^{\beta_i} = V^{\omega_1}(1+\alpha_i)$, so that $V_\nu\left(\bigcup_{i\in\omega} i^\frown D^A_{\alpha_i}\right)^b = V_\nu\left(\mathbf{sup}_{i\in\omega}\left\{\Omega(\omega_1^{\beta_i})\right\}\right)^b = V_\nu\left(\Omega(\omega_1^{\sup_{i\in\omega}\{\beta_i\}})\right)^b = \Omega\left(V^\nu(\omega_1^{\sup_{i\in\omega}\{\beta_i\}})\right)^b$. Notice that $V^\nu(\omega_1^{\sup_{i\in\omega}\{\beta_i\}}) < V^{\omega_1}(\gamma)$, thus $C \leq_W \Omega(\beta)^b$ with $\beta < \alpha$.

□

The sets obtained by this function constitute a hierarchy that describes up to complement the whole Wadge hierarchy on $\mathrm{Diff}(\mathbf{\Pi}^1_1)$.

Theorem 4.31. *Let $C \in \mathrm{Diff}(\mathbf{\Pi}^1_1)$ be non-self dual. Then there exists a unique ordinal $0 < \alpha < V^{\omega_1}(\omega_1)$ such that:*

$$C \equiv_W \Omega(\alpha)^b \qquad \text{or} \qquad C^{\complement} \equiv_W \Omega(\alpha)^b.$$

Proof. Since C and $C^{\complement}$ are in $\mathrm{Diff}(\mathbf{\Pi}^1_1)$, there exists a countable ordinal δ such that both C and $C^{\complement}$ are in $D_\delta(\mathbf{\Pi}^1_1)$. Thus, $C \leq_W \Omega(V^{\omega_1}(1+\delta))$ and $C^{\complement} \leq_W \Omega(V^{\omega_1}(1+\delta))$. Using Lemma 4.30, take the least $\alpha < V^{\omega_1}(1+\delta)$ such that $C \leq_W \Omega(\alpha)^b$ or $C^{\complement} \leq_W \Omega(\alpha)^b$. By minimality, one has $C \not<_W \Omega(\alpha)^b$ or $C^{\complement} \not<_W \Omega(\alpha)^b$. Without loss of generality, assume that $C^{\complement} \not<_W \Omega(\alpha)^b$, then by determinacy we have $\Omega(\alpha)^b \leq_W C$. Therefore $C \equiv_W \Omega(\alpha)^b$. Uniqueness is an immediate consequence of the fact that $(\Omega(\alpha))_{\alpha<V^{\omega_1}(\omega_1)}$ is well ordered by $<_c$ as proved in Lemma 4.29. □

[1] See [27, Theorem 38].

Theorem 4.32. *Let $OT_{\mathrm{Diff}(\mathbf{\Pi}^1_1)}$ be the order type of the Wadge hierarchy on* $\mathrm{Diff}(\mathbf{\Pi}^1_1)$*. We have:*

$$OT_{\mathrm{Diff}(\mathbf{\Pi}^1_1)} = V^{\omega_1}(\omega_1).$$

This description of the Wadge hierarchy of $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ provides another proof of a result by Andretta and Martin [5] on Borel-Wadge degrees.

Corollary 4.33. *In the Baire space, the classes $D_\alpha(\mathbf{\Pi}^1_1)$ and $\check{D}_\alpha(\mathbf{\Pi}^1_1)$ are the only non-self-dual pointclasses below* $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ *that are closed under preimages by Borel functions.*

Proof. Suppose that Γ is a non-self-dual pointclass closed under preimages by Borel functions that is included in $\mathrm{Diff}(\mathbf{\Pi}^1_1)$, but none of the above. Then there exists $D \subseteq \omega^\omega$ non-self-dual of degree $d_w(D) = \delta$, such that

$$\Gamma = \{B \subseteq \omega^\omega \mid B \leq_W D\},$$

with $D <_W D^A_\alpha$ for some $\alpha < \omega_1$ and $D^A_\beta <_W D$ for all $\beta < \alpha$. Thus $V^{\omega_1}(1+\beta) < \delta < V^{\omega_1}(1+\alpha)$ for all $\beta < \alpha$. Since $V^{\omega_1}(1+\alpha)$ is limit, there exists an ordinal δ' such that:

$$D <_W (\Omega(\delta'))^b <_W D^A_\alpha$$

Suppose first that α is successor. Since

$$d_w((D^A_{\alpha-1} + \ominus)^b) = d_w(D^A_{\alpha-1}) + 1 \leq d_w(D),$$

$(D^A_{\alpha-1} + \ominus)^b$ is in Γ. By Proposition 4.24, there exists a $\xi < \omega_1$ such that $\Omega(\delta'))^b \leq_W ((D^A_\theta + \ominus)^{\approx_{1+\xi}})^b$. The set $((D^A_\theta + \ominus)^{\approx_{1+\xi}})^b$ is thus not in Γ, contradicting the fact that it is closed under preimages by Borel functions.

Suppose now that α is limit. By Proposition 4.25, there exists a $\xi < \omega_1$ and a sequence $(\alpha_i)_{i\in\omega} \subseteq \alpha$ such that

$$(\Omega(\delta'))^b \leq_W \left(\left(\bigcup_{i\in\omega} i^\frown D^A_{\alpha_i}\right)^{\approx_\xi}\right)^b.$$

The set $\left(\left(\bigcup_{i\in\omega} i^\frown D^A_{\alpha_i}\right)^{\approx_\xi}\right)^b$ is thus not in Γ. But since for all $\beta < \alpha$, $D^A_\beta <_W D$, we have:

$$\bigcup_{i\in\omega} i^\frown D^A_{\alpha_i} \leq_W D,$$

so that $\bigcup_{i\in\omega} i^\frown D^A_{\alpha_i}$ is in Γ, contradicting the fact that Γ is closed under preimages by Borel functions. $\square$

5 A first glimpse above $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$

The $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$ class is the limit we can reach by using the same operations and methods that were developed for the study of the Borel sets, and by just adding the coanalytic sets. In this chapter, we take a glimpse into $\boldsymbol{\Delta}^1_2$ pointclasses that lie above. First we consider decreasing differences of coanalytic sets that coincide with the increasing differences only at the finite levels, but then become far more complex. In particular, one can prove that the class of ω decreasing differences of coanalytic sets contains $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$, and that under (AD) its Wadge rank is ω_2. Climbing further up, we consider the class of Selivanovski's $\boldsymbol{C}$-sets and the class of Kolmogorov's $\mathcal{R}$-sets. To unravel a fragment of their Wadge hierarchy, we define for each non-self-dual pointclass a new operation on sets denoted by $(D_2(\Gamma), \cdot)$. These new operations are designed to transform an open set into a set that is a countable union of $D_2(\Gamma)$ sets. For Γ with suitable closure properties, this operation preserves the Wadge ordering and behaves well with respect to the Veblen operations used in the study of the Wadge hierarchy of Borel sets by Duparc. Using well chosen pointclasses Γ, we unravel a fragment of the Wadge hierarchy of $\mathcal{R}$-sets.

5.1 Decreasing differences of coanalytic sets

There is another standard way to introduce differences, namely by considering *decreasing* sequence of sets. If $(B_\eta)_{\eta<\theta}$ is a decreasing sequence of subsets of the Baire space, with $1 \leq \theta$, we define the set $D^*_\theta((B_\eta)_{\eta<\theta})$ by

$$D^*_\theta((B_\eta)_{\eta<\theta}) = \bigcup_{\substack{\eta<\theta \\ \eta \text{ even}}} (B_\eta \setminus B_{\eta+1})\,,$$

where if θ is odd, we let $B_\theta = \emptyset$ by convention.[1] These two definitions coincide up to a certain point.

[1] The notation $\theta - \boldsymbol{\Pi}^1_1$ can also be found in the literature for the class of θ decreasing differences of coanalytic sets.

Facts 5.1. *Let Γ be a pointclass.*

(a) *For every positive integer n, $D_n(\Gamma) = D_n^*(\Gamma)$.*
(b) *For every positive integer n, $D_{2n}(\Gamma) = D_{2n}(\check{\Gamma})$, and $D_{2n+1}(\Gamma) = \check{D}_{2n+1}(\check{\Gamma})$.*
(c) *For every ordinal $0 < \theta$,*

$$D_\theta^*(\check{\Gamma}) = \begin{cases} D_\theta(\Gamma), & \text{if } \theta \text{ is even;} \\ \check{D}_\theta(\Gamma), & \text{if } \theta \text{ is odd.} \end{cases}$$

In this section, we discuss the discrepancy between the pointclasses of differences using *increasing* sequences of coanalytic sets, and differences using *decreasing* sequences of coanalytic sets. We prove that the situation is the following:

$$\begin{array}{l} \bigcup_{n\in\omega} D_n(\boldsymbol{\Pi}_1^1) \\ \qquad\quad \| \qquad\qquad \subset\ D_\omega(\boldsymbol{\Pi}_1^1)\ \subset\ \ldots\ \subset\ \mathrm{Diff}(\boldsymbol{\Pi}_1^1)\ \subsetneq\ \Delta(D_\omega^*(\boldsymbol{\Pi}_1^1)) \\ \bigcup_{n\in\omega} D_n^*(\boldsymbol{\Pi}_1^1) \end{array}$$

This at first sight quite intriguing situation can be explained by a fundamental dissymmetry between the two classes of analytics and coanalytic sets. The latter enjoys indeed the generalized reduction property, whereas the former does not.

Lemma 5.2. *Let $(D_i)_{i\in\omega}$ be a family of subsets of the Baire space and $(\alpha_i)_{i\in\omega} \subseteq \omega_1$ such that:*

- *for all $i \in \omega$, $D_i = D_{\alpha_i}^*((A_\beta^i)_{\beta<\alpha_i}) \in D_{\alpha_i}^*(\boldsymbol{\Pi}_1^1)$;*
- *if $i \neq j$, then $A_0^i \cap A_0^j = \emptyset$.*

Then

$$\bigcup_{i\in\omega} D_i \in D_\alpha^*(\boldsymbol{\Pi}_1^1),$$

where $\alpha = \sup_{i\in\omega} \alpha_i$.

Proof. It is sufficient to notice that:

$$\bigcup_{i\in\omega} D_i = \bigcup_{\substack{\beta<\alpha \\ \beta \text{ odd}}} \left(\left(\bigcup_{i\in\omega} A_\beta^i \right) \setminus \left(\bigcup_{i\in\omega} A_{\beta+1}^i \right) \right).$$

□

In fact the classes $D_\alpha^*(\boldsymbol{\Pi}_1^1)$ are closed under $\boldsymbol{\Pi}_1^1 - \mathrm{SU}$.

Lemma 5.3. *For all* $0 < \alpha < \omega_1$, *the class* $D^*_\alpha(\mathbf{\Pi}^1_1)$ *is closed under* $\mathbf{\Pi}^1_1 - \mathrm{SU}$.

Proof. Let $(D_i)_{i\in\omega}$ be a family of $D^*_\alpha(\mathbf{\Pi}^1_1)$ sets. By definition, there exists for each integer i a decreasing family of $\mathbf{\Pi}^1_1$ sets $(A^i_\xi)_{\xi<\alpha}$ such that $D_i = D^*_\alpha((A^i_\xi)_{\xi<\alpha})$. Let now $(C_i)_{i\in\omega}$ be a disjoint family of $\mathbf{\Pi}^1_1$ sets.

$$\begin{aligned}\mathrm{SU}((C_i)_{i\in\omega},(D_i)_{i\in\omega}) &= \bigcup_{i\in\omega}(C_i \cap D_i)\\ &= \bigcup_{i\in\omega}(D^*_\alpha((C_i \cap A^i_\xi)_{\xi<\alpha}),\end{aligned}$$

And we conclude by Lemma 5.2. □

We now give the proof of the inclusion of the classes $D_\alpha(\mathbf{\Pi}^1_1)$ in the class $D^*_\omega(\mathbf{\Pi}^1_1)$.

Proposition 5.4. *For every* $\alpha < \omega_1$, $D_\alpha(\mathbf{\Pi}^1_1) \subseteq D^*_\omega(\mathbf{\Pi}^1_1)$.

Proof. We proceed by induction on $\alpha < \omega_1$. If α is finite, we conclude by Facts 5.1.

For ω, let $(A_i)_{i\in\omega}$ be an increasing sequence of coanalytic sets, and consider $D_\omega((A_i)_{i\in\omega})$. Using the generalized reduction property on the family $(A_{2i+1})_{i\in\omega}$, we get a new sequence of coanalytic sets $(B_i)_{i\in\omega}$ which is disjoint and such that:

- for all $i \in \omega$, $B_i \subseteq A_{2i+1}$;
- $\bigcup_{i\in\omega} B_i = \bigcup_{i\in\omega} A_{2i+1}$.

Thus, we have:

$$\begin{aligned}D_\omega((A_i)_{i\in\omega}) &= \bigcup_{i\in\omega}\left(B_i \cap \left(\bigcup_{j\in\omega} A_{2j+1} \setminus A_{2j}\right)\right)\\ &= \bigcup_{i\in\omega}(B_i \cap \underbrace{(\bigcup_{j\leq i} A_{2j+1} \setminus A_{2j})}_{\in D_{2i+1}\subseteq D^*_{2i+2}}).\end{aligned}$$

Since the coanalytic family $(B_i)_{i\in\omega}$ is disjoint, we conclude by Lemma 5.3. The general proof for $\gamma < \omega_1$ limit is mutatis mutandis the same.

Suppose now that there exists $\beta < \omega_1$ such that $D_\alpha(\mathbf{\Pi}^1_1) \subseteq D^*_\omega(\mathbf{\Pi}^1_1)$ for all $\alpha < \beta + 1$. Since the odd case is similar, we assume that $\beta + 1$ is even. Let $(A_\alpha)_{\alpha\in\beta+1}$ be an increasing sequence of coanalytic sets, and consider

$D_{\beta+1}((A_\alpha)_{\alpha<\beta+1})$. By our induction hypothesis, there exists a decreasing sequence of coanalytic sets $(B_i)_{i\in\omega}$ such that:

$$D_{\beta-1}((A_\alpha)_{\alpha<\beta-1}) = D^*_\omega((B_i)_{i\in\omega}).$$

In particular since the family $(A_\alpha)_{\alpha\in\beta+1}$ is increasing, $D_{\beta-1}((A_\alpha)_{\alpha<\beta-1})$, and thus $D^*_\omega((B_i)_{\in\omega})$ are included in $A_{\beta-1}$. Hence:

$$\begin{aligned} D_{\beta+1}((A_\alpha)_{\alpha<\beta+1}) &= D^*_\omega((B_i)_{i\in\omega}) \cup A_\beta \setminus A_{\beta-1} \\ &= D^*_\omega((B_i \cap A_{\beta-1})_{i\in\omega}) \cup A_\beta \setminus A_{\beta-1} \\ &= D^*_\omega((A_\beta, A_{\beta-1}, B_0 \cap A_{\beta-1}, B_1 \cap A_{\beta-1}, \ldots)), \end{aligned}$$

which completes the proof. □

Our determinacy hypothesis for Chapter 4 is therefore sufficient, since by the works of Martin [77] and Harrington [42] the co-analytic deteminacy implies the determinacy of the $D^*_\omega(\mathbf{\Pi}^1_1)$ class. Moreover, it follows from the composition of these works by Harrington and Martin that the determinacy of the Wadge games of coanalytic sets is equivalent to $\text{DET}(\mathbf{\Pi}^1_1)$, so that $\text{DET}(\mathbf{\Pi}^1_1)$ is in fact optimal for the work done on the class $\text{Diff}(\mathbf{\Pi}^1_1)$.

The gap between $\text{Diff}(\mathbf{\Pi}^1_1)$ and $\Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ has, to our knowledge, not been investigated yet. The only piece of information on that matter is given by a result from Kechris and Martin mentioned by Steel [105].

Theorem 5.5 (Kechris - Martin)**.**
Under (AD), the order type of the Wadge hierarchy on $\Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ *subsets of the Baire space is* ω_2.

Combined with our results, it appears thus that under (AD) the inclusion between $\text{Diff}(\mathbf{\Pi}^1_1)$ and $\Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ is *strict*.

Question. *Is the equality* $\text{Diff}(\mathbf{\Pi}^1_1) = \Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ *consistent under weaker determinacy hypotheses?*

5.2 The C-sets

5.2.1 Suslin's operation $\mathcal{A}$ and the hierarchy of C-sets

Let Γ be a non-self-dual pointclass. The class $\text{Bor}(\Gamma)$ is defined as the least σ-algebra containing all Γ sets, that is the least family containing the Γ sets and closed under countable unions and complements.

Definition 5.6. A *Suslin scheme* is a sequence $(A_s)_{s\in\omega^{<\omega}}$ of subsets of the Baire space, indexed by the set $\omega^{<\omega}$ of finite sequences of integers. The result of Suslin's operation $\boldsymbol{\mathcal{A}}$ on the Suslin scheme $(A_s)_{s\in\omega^{<\omega}}$, in notation $\boldsymbol{\mathcal{A}}((A_s))$, is defined by:

$$\boldsymbol{\mathcal{A}}((A_s)) = \bigcup_{\alpha\in\omega^\omega} \bigcap_n A_{\alpha\restriction_n}$$

i.e.

$$\boldsymbol{\mathcal{A}}((A_s)) = \left\{x \in X : \exists\alpha \in \omega^\omega \ \forall n \ x \in A_{\alpha\restriction_n}\right\}$$

For Γ a pointclass, we denote $\boldsymbol{\mathcal{A}}(\Gamma)$ the class of all $\boldsymbol{\mathcal{A}}((A_s))$, for $(A_s)_{s\in\omega^{<\omega}}$ a Suslin scheme with $A_s \in \Gamma$ for all $s \in \omega^{<\omega}$. Notice that $\boldsymbol{\mathcal{A}}$ is a ω-ary Boolean operation.

Facts 5.7 (Suslin [106]). *Let Γ be a pointclass. Then the following hold.*

(1) $\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}(\Gamma)) = \boldsymbol{\mathcal{A}}(\Gamma)$;

(2) $\boldsymbol{\mathcal{A}}(\Gamma)$ *is closed under countable unions and intersections;*

(3) $\boldsymbol{\mathcal{A}}(\boldsymbol{\Pi}^0_1) = \boldsymbol{\Sigma}^1_1$.

Building on this operation, we introduce a hierarchy of sets, called *Selinavovski's hierarchy of $\boldsymbol{C}$-sets*, by

- $\boldsymbol{\Sigma}^{\boldsymbol{C}}_1 = \boldsymbol{\Sigma}^1_1$;
- for $0 < \alpha < \omega_1$, $\boldsymbol{\Pi}^{\boldsymbol{C}}_\alpha = \check{\boldsymbol{\Sigma}}^{\boldsymbol{C}}_\alpha$;
- for $1 < \alpha < \omega_1$, $\boldsymbol{\Sigma}^{\boldsymbol{C}}_\alpha = \boldsymbol{\mathcal{A}}\left(\bigcup_{\xi<\alpha} \boldsymbol{\Pi}^{\boldsymbol{C}}_\xi\right)$;

and we let $\boldsymbol{C} = \bigcup_{\xi<\omega_1} \boldsymbol{\Sigma}^{\boldsymbol{C}}_\xi$. Let also $\boldsymbol{\Delta}^{\boldsymbol{C}}_\alpha$ denote the ambiguous class $\boldsymbol{\Pi}^{\boldsymbol{C}}_\alpha \cap \boldsymbol{\Sigma}^{\boldsymbol{C}}_\alpha$ for all $1 \le \alpha < \omega_1$.

Proposition 5.8.

(1) *The class $\boldsymbol{C}$ is the least σ-algebra containing the open sets and closed under operation $\boldsymbol{\mathcal{A}}$.*

(2) *For any $0 < \xi < \omega_1$, $\mathrm{Bor}(\boldsymbol{\Sigma}^{\boldsymbol{C}}_\xi) \subseteq \boldsymbol{\Delta}^{\boldsymbol{C}}_{\xi+1}$.*

(3) *The class $\boldsymbol{C}$ is closed under preimage by $\boldsymbol{C}$-measurable functions.*

(4) *For any $0 < \xi < \omega_1$, the class $\boldsymbol{\Sigma}^{\boldsymbol{C}}_\xi$ is closed under preimage by Borel functions.*

Proof.

(1) It is clear from the definition that any σ-algebra containing the open sets and closed under $\boldsymbol{\mathcal{A}}$ must contain all $\boldsymbol{C}$-sets. For the other direction, note that the family $(\boldsymbol{\Sigma}^{\boldsymbol{C}}_\xi)_{\xi<\omega_1}$ is increasing. Suppose now $(A_s)_{s\in\omega^{<\omega}}$ is a Suslin scheme of $\boldsymbol{C}$-sets. Then the sets A_s are in $\boldsymbol{\Sigma}^{\boldsymbol{C}}_\xi$

for some $0 < \xi < \omega_1$. Then $A_s^{\complement} \in \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi+1}$ and finally $\boldsymbol{\mathcal{A}}(A_s) \in \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi+2}$ is a $\boldsymbol{C}$-set. So $\boldsymbol{C}$ is closed under $\boldsymbol{\mathcal{A}}$ and complement. And as countable union is a particular case of the operation $\boldsymbol{\mathcal{A}}$, the class $\boldsymbol{C}$ is a σ-algebra.

(2) We already noticed that $\mathbf{\Sigma}^{\boldsymbol{C}}_{\xi} \subseteq \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi+1}$. Also clearly $\mathbf{\Pi}^{\boldsymbol{C}}_{\xi} \subseteq \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi+1}$. Now by Facts 5.7, the class $\mathbf{\Sigma}^{\boldsymbol{C}}_{\xi+1}$ is closed under operation $\boldsymbol{\mathcal{A}}$ and under countable unions and intersections. Observe that $\mathrm{Bor}(\mathbf{\Sigma}^{\boldsymbol{C}}_{\xi})$ is the least family containing $\mathbf{\Sigma}^{\boldsymbol{C}}_{\xi} \cup \mathbf{\Pi}^{\boldsymbol{C}}_{\xi}$ and closed under countable unions and intersections, hence it is included in $\mathbf{\Sigma}^{\boldsymbol{C}}_{\xi+1}$. Moreover $\mathrm{Bor}(\mathbf{\Sigma}^{\boldsymbol{C}}_{\xi})$ is self-dual, it is thus included in $\Delta(\mathbf{\Sigma}^{\boldsymbol{C}}_{\xi+1}) = \mathbf{\Delta}^{\boldsymbol{C}}_{\xi+1}$.

(3) Let $f : \omega^\omega \to \omega^\omega$ be $\boldsymbol{C}$-measurable, and $A \in \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi}$. We prove by induction on $0 < \xi < \omega_1$ that $f^{-1}(A) \in \boldsymbol{C}$. Suppose first $A \in \mathbf{\Sigma}^{\boldsymbol{C}}_{1}$, i.e. $A = \boldsymbol{\mathcal{A}}(A_s)$ with closed sets A_s. Then each $f^{-1}(A_s^{\complement})$ is a $\boldsymbol{C}$-set, hence for some $0 < \xi_1 < \omega_1$, $A_s^{\complement} \in \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi_1}$. But then $f^{-1}(A) = \boldsymbol{\mathcal{A}}(f^{-1}(A_s)) \in \boldsymbol{\mathcal{A}}(\mathbf{\Pi}^{\boldsymbol{C}}_{\xi_1}) = \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi+1}$. If now $A \in \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi}$, we get $A = \boldsymbol{\mathcal{A}}(A_s)$, with $A_s \in \mathbf{\Pi}^{\boldsymbol{C}}_{\eta(s)}$ for some $\eta(s) < \xi$. Then by the induction hypothesis $f^{-1}(A_s) \in \boldsymbol{C}$, and hence $f^{-1}(A) = \boldsymbol{\mathcal{A}}(f^{-1}(A_s))$ is in $\boldsymbol{C}$ too.

(4) Let $f : \omega^\omega \to \omega^\omega$ be a Borel function, and $A \in \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi}$. We prove by induction on $0 < \xi < \omega_1$ that $f^{-1}(A) \in \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi}$. Suppose first $A \in \mathbf{\Sigma}^{\boldsymbol{C}}_{1}$, i.e. $A = \boldsymbol{\mathcal{A}}(A_s)$ with closed sets A_s. Then each $f^{-1}(A_s^{\complement})$ is a Borel set, and

$$f^{-1}(A) = \boldsymbol{\mathcal{A}}(f^{-1}(A_s)) \in \boldsymbol{\mathcal{A}}(\mathbf{\Delta}^1_1) \subseteq \boldsymbol{\mathcal{A}}(\mathbf{\Sigma}^1_1) = \mathbf{\Sigma}^{\boldsymbol{C}}_1.$$

If now $A \in \mathbf{\Sigma}^{\boldsymbol{C}}_{\xi}$, we get $A = \boldsymbol{\mathcal{A}}(A_s)$, with $A_s \in \mathbf{\Pi}^{\boldsymbol{C}}_{\eta(s)}$ for some $\eta(s) < \xi$. Then by the induction hypothesis $f^{-1}(A_s) \in \mathbf{\Pi}^{\boldsymbol{C}}_{\eta(s)}$, and hence $f^{-1}(A) = \boldsymbol{\mathcal{A}}(f^{-1}(A_s))$ is in $\mathbf{\Sigma}^{\boldsymbol{C}}_{\xi}$ too. $\square$

5.2.2 Unions of $D_2(\mathbf{\Pi}^1_1)$ sets

As long as we deal with *finite* unions (or intersections) of $D_2(\mathbf{\Pi}^1_1)$ sets, we stay in the class Diff($\mathbf{\Pi}^1_1$).

Fact 5.9. *Let k be a non negative integer, and $(D_i)_{i<k}$ a family of $D_2(\mathbf{\Pi}^1_1)$ sets. Then there exists an integer n such that:*

$$\bigcup_{i<k} D_i \in D_n(\mathbf{\Pi}^1_1).$$

To be more precise:

$$D_{2k}(\mathbf{\Pi}^1_1) \subseteq \bigcup_{i<k} D_2(\mathbf{\Pi}^1_1) \subseteq D_{2(2^k-1)}(\mathbf{\Pi}^1_1);$$

where $\bigcup_{i<k} D_2(\mathbf{\Pi}^1_1) = \left\{\bigcup_{i<k} D_i \mid (D_i)_{i<k} \subset D_2(\mathbf{\Pi}^1_1)\right\}$.

It is not the same story when we go to countable operations. Let Γ be a non-self-dual pointclass, and let $\sigma D_2(\Gamma)$ the class of all countable reunions of $D_2(\Gamma)$ sets:

$$\sigma D_2(\Gamma) = \left\{\bigcup_{i\in\omega} D_i : D_i \in D_2(\Gamma)\right\}.$$

This is a pointclass which contains, merely by definition, all the classes $D_\alpha(\Gamma)$ and $D^*_\alpha(\Gamma)$ for $\alpha < \omega_1$, and which is closed under finite intersections and countable unions. Moreover, it is included in $\mathrm{Bor}(\Gamma)$. If we consider the case where Γ is $\mathbf{\Pi}^1_1$, one could wonder if it is the class just above $\bigcup_{\alpha<\omega_1} D^*_\alpha(\mathbf{\Pi}^1_1)$. It is a consequence of a theorem by Steel [105, Theorem 1.2] that under AD, this is not the case.

Proposition 5.10. *Suppose AD and let Γ be a non-trivial pointclass closed under intersection with a $\mathbf{\Pi}^1_1$ and under union with a $\mathbf{\Sigma}^1_1$. Then the cofinality of the Wadge rank of Γ is at least ω_2.*

Proof. The proof we present here is due to Louveau. Given its closure properties, Γ is not self-dual, and its Wadge rank is not successor. Thus $\mathrm{cof}(|\Gamma|_W)$ is greater or equal to ω_1. Suppose it is ω_1, and let $\varphi : \omega_1 \to |\Gamma|_W$ be cofinal. Let $W \subseteq \omega^\omega \times \omega^\omega$ be a universal set for Γ, and consider the following Solovay game: player I plays $\alpha \in \omega^\omega$, II plays $(\beta, \gamma) \in \omega^\omega \times \omega^\omega$, and II wins if and only if

$$\alpha \in \mathrm{WO} \to \Big(W_\beta = W^\complement_\gamma \wedge \varphi(|\alpha|) \le d_w(W_\beta)\Big),$$

where WO is the set of codes of wellorderings, and $|\alpha|$ denote the ordinal coded by α.[2] Suppose first that I has a winning strategy σ. Then $\sigma(\omega^\omega)$ is a $\mathbf{\Sigma}^1_1$ in WO, and by the boundedness theorem, there exists a $\xi < \omega_1$ such that $\sup\{|x| : x \in \sigma(\omega^\omega)\} \le \xi$. But $\varphi(\xi) < |\Gamma|_W$! Player II can thus play (β, γ) such that $W_\beta = W^\complement_\gamma$ and $d_w(W_\beta) > \varphi(\xi)$, and win the game. Hence, I does not have a winning strategy in this game. Suppose now that II has a winning strategy τ. We define the set $R \subseteq \omega^\omega \times \omega^\omega$ such that:

$$\begin{aligned} R(\alpha, x) &\longleftrightarrow \alpha \in \mathrm{WO} \wedge x \in W_{(\tau*\alpha)_0} \\ &\longleftrightarrow \alpha \in \mathrm{WO} \wedge x \notin W_{(\tau*\alpha)_1}. \end{aligned}$$

By the closure properties of Γ, R is both in Γ and $\check{\Gamma}$, so that R is in Δ. But $d_w(R)$ is greater than or equal to $\sup\{d_w(R_\alpha) : \alpha \in \mathrm{WO}\}$, so that $d_w(R) \ge \sup\varphi$. Hence φ is not cofinal, a contradiction. $\square$

[2] See e.g. Moschovakis [80, Chapter 4].

The Wadge rank of $\sigma D_2(\mathbf{\Pi}^1_1)$ is thus, under AD, of cofinality at least ω_2, so that

$$\Delta(\sigma D_2(\mathbf{\Pi}^1_1)) \neq \bigcup_{\alpha<\omega_1} D^*_\alpha(\mathbf{\Pi}^1_1).$$

Observe that we can construct a $\sigma D_2(\mathbf{\Pi}^1_1)$-complete set fairly easily from a complete coanalytic set. Let $\phi : \omega \times \omega \to \omega$ be the Cantor bijection between $\omega \times \omega$ and ω defined by:

$$\phi(n,m) = \frac{(n+m)(n+m+1)}{2} + 1.$$

It induces a bijection:

$$\begin{aligned} \varphi : \omega^\omega &\longrightarrow \omega^{\omega\times\omega} \\ a &\longmapsto a(\phi(i,j))_{i,j\in\omega}. \end{aligned}$$

For all $i \in \omega$, we denote by φ_i the i-th projection of φ:

$$\begin{aligned} \varphi_i : \omega^\omega &\longrightarrow \omega^\omega \\ a &\longmapsto a(\phi(i,j))_{j\in\omega}. \end{aligned}$$

Define for all $A \subseteq \omega^\omega$:

$${}_\sigma D^A_2 = \left\{x \in \omega^\omega \mid \exists i \in \omega(\varphi_i(x) \in D^A_2)\right\}.$$

Fact 5.11. *Let Γ be a non-self-dual class and A be a subset of the Baire space complete for this class. Then the set ${}_\sigma D^A_2$ is $\sigma D_2(\Gamma)$ complete.*

5.2.3 The D_2 unfolded game

Inspired by the definition of the set ${}_\sigma D^A_2$ from A, we define a new operation on conciliatory sets. Let φ be a bijection between ω^ω and $\omega^{\omega\times\omega}$ as above, and denote by x^+ the final subsequence $(x(1), x(2), \ldots)$ of x in $\omega^{\leq\omega}$. We denote by x_- the set of all sequences such that $(x_-)^+ = x$. The $(D_2(\Gamma), \cdot)$ operation is the following.

Definition 5.12. Let Γ be a non-self-dual pointclass, $A \subseteq \omega^\omega$ be complete in Γ, and B a conciliatory set. Let $(D_2(\Gamma), B)$ denote the following conciliatory set:

$$(D_2(\Gamma), B) = \left\{x \in \omega^{\leq\omega} \mid (x^*)_{[\ /b]} \in B\right\}$$

where

$$x^*(i) = \begin{cases} \varphi_i(x)(0) & \text{, if } \varphi_i(x)^+ \in D^A_2; \\ b & \text{, else.} \end{cases}$$

If $\varphi_i(x)^+ \notin D^A_2$, we say that the i-th column is killed.

Observe that this operation is designed so that if O is a conciliatory set such that $O^b \in \boldsymbol{\Sigma}^0_1$, then $(D_2(\Gamma), O)^b$ is in the $\sigma D_2(\Gamma)$ class. Moreover, if Γ is closed under finite unions and intersections, and if its complete sets are initializable, then the operation $(D_2(\Gamma), \cdot)$ preserves the conciliatory preorder $\leq_c$.

Theorem 5.13. *Let Γ be a non-self-dual pointclass closed under finite unions and intersections, and such that its complete sets are initializable. Let B_0 and B_1 be conciliatory sets. The following hold:*
(i) $(D_2(\Gamma), B_0^{\complement}) \equiv_c (D_2(\Gamma), B_0)^{\complement}$.
(ii) *If $B_0 \leq_c B_1$, then $(D_2(\Gamma), B_0) \leq_c (D_2(\Gamma), B_1)$.*
(iii) *If $B_0 <_c B_1$, then $(D_2(\Gamma), B_0) <_c (D_2(\Gamma), B_1)$.*

Proof. The proof of the first statement is straightforward, while the demonstration of the second relies on a variation of the *remote control* technique first introduced by Duparc [26]. Let $(\beta_i)_{i\in\omega}$ be an enumeration of $4^{<\omega} \setminus \{\epsilon\}$ such that for any integers n and m, if $n \leq m$, then $|\beta_n| \leq |\beta_m|$. We call β_i the i-th *bet*. For any integer i, we also define the sets:

$$\beta_i^j = \{n \in \omega \mid \beta_i(n) = j\}$$

for $j = 0, \ldots, 3$. A bet encodes information on the auxiliary moves of player I: its length determines the number of columns it takes into account, and its n-th value, whether the $D_2(\boldsymbol{\Pi}^1_1)$ condition is true or not in the n-th column, and why. The value 0 means that the n-th column is not killed, i.e. $\varphi_n(x)^+ \in D_2^A$, where x is the sequence played by player I. The value 1 means that $\pi_0(\varphi_n(x)^+) \notin A$ and $\pi_1(\varphi_n(x)^+) \notin A$, the value 2 means that $\pi_0(\varphi_n(x)^+) \in A$ and $\pi_1(\varphi_n(x)^+) \in A$, and the value 3 means that $\pi_0(\varphi_n(x)^+) \notin A$ and $\pi_1(\varphi_n(x)^+) \in A$. We say that a bet β_j is fulfilled if for all $n < |\beta_j|$, $\beta_j(n)$ is true. Notice that it is, so to speak, a $D_2(\Gamma)$ condition; β_j is indeed fulfilled by the sequence x if and only if

$$x \in \bigcap_{l\in\beta_j^0\cup\beta_j^2} \varphi_l^{-1}(\pi_0^{-1}(A)_-) \cap \bigcap_{k\in\beta_j^2\cup\beta_j^3} \varphi_l^{-1}(\pi_1^{-1}(A)_-) = A_1^{\beta_j}$$

and

$$x \in \bigcap_{l\in\beta_j^3\cup\beta_j^1} \varphi_l^{-1}(\pi_0^{-1}(A^{\complement})_-) \cap \bigcap_{k\in\beta_j^0\cup\beta_j^1} \varphi_l^{-1}(\pi_1^{-1}(A^{\complement})_-) = A_0^{\beta_j}.$$

If we consider A, $A_0^{\beta_j}$ and $A_1^{\beta_j}$ in ω^ω, then $A_0^{\beta_j}$ is in $\check{\Gamma}$ and $A_1^{\beta_j}$ is in Γ. Thus

$$A_0^{\beta_j} \cap A_1^{\beta_j} \leq_W D_2^A,$$

and, since D_2^A is initializable, II has a winning strategy σ'_j that never requires her to skip in the Wadge game $C(A_0^{\beta_j} \cap A_1^{\beta_j}, D_2^A)$. For any finite sequence $u \in \omega^{<\omega}$ and any integer i such that $|u| \leq |\beta_i|$, we denote u^{β_i} the finite sequence $u \upharpoonright \beta_i^0$, that is u with all the killed values, according to the bet, deleted.

Suppose now that II has a winning strategy σ in $C(B_0, B_1)$, we describe a winning strategy σ' for II in $C((D_2(\Gamma), B_0), (D_2(\Gamma), B_1))$. Without loss of generality, we can consider that σ is a winning strategy in the conciliatory game where I can play finite sequences. Suppose I plays along a sequence $x \in \omega^{\leq\omega}$. On her main run, II follows σ, modulo the bets and the skips, i.e her $\varphi_i(0)$ move is $\sigma(u^{\beta_i})$ if it is not a skip, where u is the beginning of the main run already played by I. If $\sigma(u^{\beta_i})$ is a skip, then II plays 0 and kills the column. For the i-th column, if it is not killed to avoid a skip in the main run, σ' realizes the i-th bet by playing along σ_i. At the end of the game, II has played an infinite sequence. She has killed exactly all the columns which correspond to bets that are not fulfilled by I, so that on the main run remains $\sigma((x^*)_{[\ /b]})$. Since σ is winning, σ' is winning too.

The proof of the last statement relies on the fact that mutatis mutandis strategies for I can also be remote controlled. □

If moreover Γ contains Borel sets and is closed under preimage by Borel functions, $(D_2(\Gamma), \cdot)$ turns out to produce fixpoints of the Veblen operations.

Theorem 5.14. *Let* Γ *be a non-self-dual pointclass containing the Borel sets, closed under finite unions and intersections, closed under preimage by Borel functions, and such that its complete sets are initializable. Then for every conciliatory set* B *and every ordinal* $0 < \xi < \omega_1$*, the following holds:*

$$(V_\xi\left((D_2(\Gamma), B)\right))^b \leq_W (D_2(\Gamma), B)^b.$$

Proof. Let A be Γ-complete. Notice first that in the game

$$C((V_\xi\left((D_2(\Gamma), B)\right)), (D_2(\Gamma), B)),$$

to know whether the column i is killed or not for a certain integer i is a $D_2(\Gamma)$ condition: it is indeed equivalent to knowing whether the play x of I belongs to $\varphi_i^{-1}(V_\xi(D_2^A)_-)$ or not. Furthermore, recall that to know whether a given finite sequence u will be erased or not is a Borel condition. Now we describe a strategy σ for II in the game

$$C((V_\xi\left((D_2(\Gamma), B)\right)), (D_2(\Gamma), B)).$$

Suppose I plays along a sequence $x \in (\omega \cup \{\twoheadleftarrow_\iota \colon \iota < \xi\})^{\leq\omega}$. On her main run II plays along $x_{[\ /\{\twoheadleftarrow_\iota:\iota<\xi\}]}$, just removing the arrows from the play of her opponent. On her auxiliary moves, say on column i, she checks whether, after the application of the erasers, both:

– the corresponding move in her main run will appear or not in the main run of her opponent, and

– its column will be killed or not.

For example, assume the first move of I that is not an eraser is a_0. Then II plays a_0, and follows along the first column a winning strategy she has in the game

$$C\left(\varphi_0^{-1}\left(V_\xi(D_2^A)_-\right) \cap V_\xi[a_0], D_2^A\right),$$

where $V_\xi[a_0]$ is the set of sequences in which a_0 is not erased, so that her first column will be killed if a_0 is erased by I or if he kills his first column. Suppose now that the second move of I that is not an eraser is a_1. Then II plays a_1, and follows along the second column a winning strategy she has in the game

$$C\left(\left(\varphi_0^{-1}\left(V_\xi(D_2^A)_-\right) \cap V_\xi[a_1]\right) \cup \left(\varphi_1^{-1}\left(V_\xi(D_2^A)_-\right) \cap V_\xi[a_0,a_1]\right), D_2^A\right),$$

where $V_\xi[a_1]$ and $V_\xi[a_0,a_1]$ are respectively the set of sequences where a_1 and (a_0,a_1) are the first sequences not erased, so that her second column will be killed if a_1 is erased by I or if he kills the column where it appears. Continuing this way, we get a strategy σ such that for every $x \in (\omega \cup \{\twoheadleftarrow_\iota \colon \iota < \xi\})^{\leq\omega}$:

$$\sigma(x)^{*} = (x^{\hookleftarrow})^{*}.$$

Hence σ is winning. $\square$

Corollary 5.15. *Let Γ be a non-self-dual pointclass with the same properties as above. Let B, C be conciliatory sets, and $0 < \xi$ be a countable ordinal.*

(i) *If $C^b \leq_W (D_2(\Gamma), B)^b$, then $(V_\xi(C))^b \leq_W (D_2(\Gamma), B)^b$.*

(ii) *If $C^b <_W (D_2(\Gamma), B)^b$, then $(V_\xi(C))^b <_W (D_2(\Gamma), B)^b$.*

Observe that the classes $\boldsymbol{\Pi}_\alpha^{\boldsymbol{C}}$ contain the Borel sets, and are closed under countable unions and intersections. Moreover, as they are closed under preimage by Borel function, their complete sets are initializable. Hence the operations $(D_2(\boldsymbol{\Pi}_\alpha^{\boldsymbol{C}}), \cdot)$ preserve the conciliatory preorder and their images are fixpoints of the Veblen operations.

Proposition 5.16. *Let α and β be non-zero countable ordinals such that $\alpha < \beta$, and let B be a conciliatory sets. Then the following holds*

$$\left(D_2(\boldsymbol{\Pi}_\alpha^{\boldsymbol{C}}), (D_2(\boldsymbol{\Pi}_\beta^{\boldsymbol{C}}), B)\right) \leq_c (D_2(\boldsymbol{\Pi}_\beta^{\boldsymbol{C}}), B).$$

Proof. The argument is very similar to the one used in the proof of Theorem 5.14, and we omit it here. $\square$

Corollary 5.17. *Let α and β be non-zero countable ordinals such that $\alpha < \beta$, and let B_0 and B_1 be conciliatory sets. If $B_0 \leq_c (D_2(\mathbf{\Pi}^{\boldsymbol{C}}_\beta), B_1)$, then*

$$(D_2(\mathbf{\Pi}^{\boldsymbol{C}}_\alpha), B_0) \leq_c (D_2(\mathbf{\Pi}^{\boldsymbol{C}}_\beta), B_1).$$

These results allow us to combine the operations $(D_2(\mathbf{\Pi}^{\boldsymbol{C}}_\alpha), \cdot)$ with the ones we used to describe the Wadge hierarchy of the class Diff$(\mathbf{\Pi}^1_1)$, and to extend the function Ω to $\boldsymbol{C}$-sets. Notice that it does not provide us with the whole Wadge hierarchy this time, but only with a fragment of it – consider for example the potentially huge untouched interval between the classes Diff$(\mathbf{\Pi}^1_1)$ and $\sigma D_2(\mathbf{\Pi}^1_1)$.

Definition 5.18. Let A be a $\mathbf{\Pi}^1_1$−complete subset of the Baire space and $0 < \alpha < V^{\omega_1+\omega_1}(2)$. The ordinal α admits a unique Cantor Normal Form of base ω_1:

$$\alpha = \omega_1^{\alpha_n} \cdot \nu_n + \cdots + \omega_1^{\alpha_0} \cdot \nu_0,$$

with

$$V^{\omega_1+\omega_1}(2) > \alpha_n > \cdots > \alpha_0, \text{ and } 0 < \eta_i < \omega_1 \text{ for all } i \leq n.$$

Set:

$$\Omega(\alpha) = \Omega(\omega_1^{\alpha_n}) \bullet \eta_n + \cdots + \Omega(\omega_1^{\alpha_0}) \bullet \eta_0,$$

where $\Omega(\omega_1^\beta)$ is defined as follows.

- If $\beta = 0$ then
$$\Omega(\omega_1^0) = \emptyset.$$
- If $\beta = \alpha + 1$ is successor, then $0 \in I_\beta$. Denote by γ_0 the ordinal such that $V^0(\gamma_0) = \omega_1^\beta$. Then set
$$\Omega(\omega_1^\beta) = V_0(\Omega(\gamma_0)).$$
- If β is limit of cofinality ω, there exists a sequence $(\beta_i)_{i\in\omega}$ of ordinals strictly less than β such that $\beta = \sup_{i\in\omega}\{\beta_i\}$. Then set:
$$\Omega(\omega_1^\beta) = \mathbf{sup}_{i\in\omega}\left\{\Omega(\omega_1^{\beta_i})\right\}.$$
- If β is limit of cofinality ω_1, then we have two cases: either I_β is empty, or not. If it is not empty, then as in the successor case, we set
$$\Omega(\omega_1^\beta) = V_{\nu_0}(\Omega(\gamma_0)),$$

where $\nu_0 < \omega_1$ and $\gamma_0 < \omega_1^\beta$ are the minimal ordinals such that $V^{\nu_0}(\gamma_0) = \omega_1^\beta$. If it is empty, then there are again two cases: either there exists $\xi < \omega_1$ such that $\omega_1^\beta = V^{\omega_1}(1+\xi)$ and we set

$$\Omega(\omega_1^\beta) = D_\xi^B,$$

where D_ξ^B is a conciliatory set such that $(D_\xi^B)^b \equiv_W D_\xi^A$; or there exist some ordinals $0 < \beta' < \beta$ and $\zeta < \omega_1$ such that $\omega_1^\beta = V^{\omega_1+\zeta}(\beta')$ and we set

$$\Omega(\omega_1^\beta) = (D_2(\boldsymbol{\Pi}^{\boldsymbol{C}}_{1+\zeta}), \Omega(2+\beta')).$$

One can prove that the sets defined are well-ordered with respect to the conciliatory hierarchy.

Proposition 5.19. *Assume the determinacy of all $\boldsymbol{C}$-sets. For $0 < \beta < \alpha < V^{\omega_1+\omega_1}(2)$,*

$$\Omega(\beta) \leq_c \Omega(\alpha) \qquad \textit{and} \qquad \Omega(\beta)^\complement \leq_c \Omega(\alpha).$$

Proof. It is mutatis mutandis the same as the proof of Lemma 4.29. The remaining cases are treated using Theorems 5.13 and 5.14 and Proposition 5.16. □

Observe that it is a consequence of Proposition 5.16 that for all $0 < \alpha < \beta < \omega_1$ and any conciliatory set B such that $B^b \in \sigma D_2(\boldsymbol{\Pi}^{\boldsymbol{C}}_\beta)$, we have

$$(D_2(\boldsymbol{\Pi}^{\boldsymbol{C}}_\alpha), B)^b \in \sigma D_2(\boldsymbol{\Pi}^{\boldsymbol{C}}_\beta),$$

so that the operations $(D_2(\boldsymbol{\Pi}^{\boldsymbol{C}}_\alpha), \cdot)$ preserve the $\sigma D_2(\boldsymbol{\Pi}^{\boldsymbol{C}}_\beta)$ classes.

Question. *Let $0 < \beta < \omega_1$, and B a conciliatory set such that $B^b \in \boldsymbol{\Pi}^{\boldsymbol{C}}_\beta$. Do we have*

$$(D_2(\boldsymbol{\Pi}^{\boldsymbol{C}}_\alpha), B)^b \in \boldsymbol{\Pi}^{\boldsymbol{C}}_\beta$$

for every $\alpha < \beta$? In other words, do the operations $(D_2(\boldsymbol{\Pi}^{\boldsymbol{C}}_\alpha), \cdot)$ preserve the $\boldsymbol{\Pi}^{\boldsymbol{C}}_\beta$ classes, with $\alpha < \beta$?

5.3 The Kolmogorov hierarchy of $\mathcal{R}$-sets

The $\mathcal{R}$-sets were introduced by Kolmogorov [57, 58] as the family generated from the closed sets by the operations of countable union and intersection, and closed under the transformation $\mathcal{R}$. This class of sets is strictly included in $\boldsymbol{\Delta}^1_2$, and contains all the $\boldsymbol{C}$-sets.

5.3.1 The δs-operations and their transformations

Recall that an ω-ary operation $\mathcal{O}$ is a function:

$$\mathcal{O} : \mathcal{P}(\omega^\omega)^{\mathbb{A}_\mathcal{O}} \longrightarrow \mathcal{P}(\omega^\omega),$$

where $\mathbb{A}_\mathcal{O}$, the *arena* of $\mathcal{O}$, is a countable set of indices. The operation $\mathcal{O}$ assigns a set to a countable sequence of sets indexed by $\mathbb{A}_\mathcal{O}$. A *basis* for the operation $\mathcal{O}$ is a set $N_\mathcal{O} \subseteq \mathcal{P}(\mathbb{A}_\mathcal{O})$ such that

$$\mathcal{O}\left((A_s)_{s\in\mathbb{A}_\mathcal{O}}\right) = \bigcup_{S\in N_\mathcal{O}} \bigcap_{s\in S} A_s,$$

for every sequence $(A_s)_{s\in\mathbb{A}_\mathcal{O}}$ of subsets of the Baire space. Observe that not all operations have a basis, but that a basis completely determines an operation. The operations that admit a basis are called the δs-operations, and were introduced independently by Kolmogorov [57] and Hausdorff [44].

Example 5.20. Both the operation countable union $\bigcup$ and the operation countable intersection $\bigcap$ are δs-operations, with arena $\mathbb{A}_\bigcup = \mathbb{A}_\bigcap = \omega$ and basis

$$N_\bigcup = \{\{n\} : n \in \omega\} \qquad \text{and} \qquad N_\bigcap = \{\omega\}.$$

Notice that all δs-operations are boolean operations, but some boolean operations are not δs-operations: "taking the complement" for example is not a δs-operation. Kantorovich and Livenson [53] proved that the δs-operations are exactly the *positive* boolean operations, that is the boolean operations which preserve the inclusion.

There are three classical transformations of δs-operations: passage to the *complementary* (or *dual*) operation, *superposition*, and the $\mathcal{R}$-transform, all introduced by Kolmogorov [57, 58].

The complementary operation

Let $\mathcal{O}$ be a δs-operation with arena $\mathbb{A}_\mathcal{O}$ and basis $N_\mathcal{O}$. We define the dual, or the complementary, operation co-$\mathcal{O}$ with the same arena and with basis

$$N_{\text{co-}\mathcal{O}} = \{S \subseteq \mathbb{A}_\mathcal{O} : \text{ for all } T \in N_\mathcal{O}, T \cap S \neq \emptyset\}.$$

It satisfies the following property: for every family $(A_s)_{s\in\mathbb{A}_\mathcal{O}}$ of subsets of the Baire space

$$\text{co-}\mathcal{O}\left((A_s^\complement)_{s\in\mathbb{A}_\mathcal{O}}\right) = A^\complement \longleftrightarrow \mathcal{O}\left((A_s)_{s\in\mathbb{A}_\mathcal{O}}\right) = A.$$

Observe that the operations $\bigcup$ and $\bigcap$ are mutually complementary:

$$\text{co-}\bigcup = \bigcap \qquad \text{and} \qquad \text{co-}\bigcap = \bigcup.$$

The composition and the superposition

Given two operations $\mathcal{O}$ and $\mathcal{O}'$, their composition $\mathcal{O}' \circ \mathcal{O}$ is the operation with arena $\mathbb{A}_{\mathcal{O}} \times \mathbb{A}_{\mathcal{O}'}$ defined as:

$$\mathcal{O}' \circ \mathcal{O} \left((A_{(s,s')})_{(s,s') \in \mathbb{A}_{\mathcal{O}} \times \mathbb{A}_{\mathcal{O}'}} \right) = \mathcal{O}' \left((\mathcal{O}((A_{(s,s')})_{s \in \mathbb{A}_{\mathcal{O}}}))_{s' \in \mathbb{A}_{\mathcal{O}'}} \right)$$

Notice that the composition of two δs-operations is a δs-operation.

Example 5.21. The composition of $\bigcap$ and $\bigcup$, the operation $\bigcup \circ \bigcap$ with arena $\omega \times \omega$, is the following:

$$\bigcup \circ \bigcap \left((A_{i,j})_{(i,j) \in \omega \times \omega} \right) = \bigcup_{i \in \omega} \bigcap_{j \in \omega} A_{i,j}.$$

Its basis is given by

$$N_{\bigcup \circ \bigcap} = \{\{i\} \times \omega : i \in \omega\}.$$

A generalization of the composition transform is given by the superposition. Let $\mathcal{O}$ be a δs-operation, and let $(\mathcal{O}_s)_{s \in \mathbb{A}_{\mathcal{O}}}$ be a family of δs-operations. The superposition of $(\mathcal{O}_s)_{s \in \mathbb{A}_{\mathcal{O}}}$ with outer operation $\mathcal{O}$ is a δs-operation $(\mathcal{O} \mid (\mathcal{O}_s)_{s \in \mathbb{A}_{\mathcal{O}}})$, with arena

$$\mathbb{A}_{\left(\mathcal{O} \mid (\mathcal{O}_s)_{s \in \mathbb{A}_{\mathcal{O}}}\right)} = \left\{ (k,l) \in \left(\mathbb{A}_{\mathcal{O}} \times \bigcup_{s \in \mathbb{A}_{\mathcal{O}}} \mathbb{A}_{\mathcal{O}_s} \right) : l \in \mathbb{A}_{\mathcal{O}_k} \right\}$$

and defined as follows

$$(\mathcal{O} \mid (\mathcal{O}_s)_{s \in \mathbb{A}_{\mathcal{O}}}) \left((A_{(k,l)})_{(k,l) \in \mathbb{A}_{(\mathcal{O} \mid (\mathcal{O}_s)_{s \in \mathbb{A}_{\mathcal{O}}})}} \right) = \mathcal{O} \left(\left(\mathcal{O}_s \left((A_{(k,l)})_{l \in \mathbb{A}_{\mathcal{O}_s}} \right) \right)_{s \in \mathbb{A}_{\mathcal{O}}} \right).$$

Observe that if for all $s \in \mathbb{A}_{\mathcal{O}}$, the operation $\mathcal{O}_s$ is identically some fixed δs-operation $\mathcal{O}'$, the superposition of $(\mathcal{O}_s)_{s \in \mathbb{A}_{\mathcal{O}}}$ with outer operation $\mathcal{O}$ is exactly the composition of $\mathcal{O}'$ and $\mathcal{O}$:

$$(\mathcal{O} \mid (\mathcal{O}_s)_{s \in \mathbb{A}_{\mathcal{O}}}) = \mathcal{O} \circ \mathcal{O}'.$$

The $\mathcal{R}$-transform

The $\mathcal{R}$-transform is considered to be one of the most important contribution of Kolmogorov in the field of operations on set[3], and arose as a variant of the infinite iteration of the substitution. The $\mathcal{R}$-transform of a δs-operation $\mathcal{O}$ is the δs-operation $\mathcal{R}\mathcal{O}$ with arena $(\mathbb{A}_{\mathcal{O}})^{<\omega}$ and basis:

$$N_{\mathcal{R}\mathcal{O}} = \left\{ S \subseteq (\mathbb{A}_{\mathcal{O}})^{<\omega} : \exists T \subseteq S, \varepsilon \in T \wedge \forall t \in T \ \{v \in \mathbb{A}_{\mathcal{O}} : t^\frown v \in T\} \in N_{\mathcal{O}} \right\}.$$

[3] See for example Kanovei [52].

Example 5.22. Suslin's operation $\boldsymbol{\mathcal{A}}$ is the $\mathcal{R}$-transform of $\bigcup$:

$$\boldsymbol{\mathcal{A}} = \mathcal{R}\bigcup.$$

5.3.2 The Kolmogorov hierarchy of $\mathcal{R}$-sets

For any countable ordinal ξ, let $\bigcap_\xi$ denote the δs-operation which produces the intersection of a sequence of length ξ of sets. We define a sequence of δs-operations $(R_\alpha)_{\alpha<\omega_1}$ by induction.

- $R_0 = \bigcup$,
- $R_\alpha = \mathcal{R}\left(\bigcap_\alpha \mid (\text{co-}R_\gamma)_{\gamma<\alpha}\right)$, for $0 < \alpha$.

We denote by $\boldsymbol{\mathcal{R}}$ the class of $\mathcal{R}$-sets, which is the smallest class of subsets of the Baire space containing all open sets and closed under all the operations R_α and co-R_α. They can be naturally spread in a hierarchy, the *Kolmogorov hierarchy of $\mathcal{R}$-sets*. For every $\alpha < \omega_1$, we set

$$\boldsymbol{\Sigma}^{\mathcal{R}}_\alpha = R_\alpha(\boldsymbol{\Sigma}^0_1) \qquad \text{and} \qquad \boldsymbol{\Pi}^{\mathcal{R}}_\alpha = \text{co-}R_\alpha(\boldsymbol{\Pi}^0_1),$$

and denote by $\boldsymbol{\Delta}^{\mathcal{R}}_\alpha$ their ambiguous class:

$$\boldsymbol{\Delta}^{\mathcal{R}}_\alpha = \boldsymbol{\Sigma}^{\mathcal{R}}_\alpha \cap \boldsymbol{\Pi}^{\mathcal{R}}_\alpha.$$

Observe that $\boldsymbol{\Sigma}^{\mathcal{R}}_0$ and $\boldsymbol{\Pi}^{\mathcal{R}}_0$ coincide respectively with the classes of open and closed sets, and that $\boldsymbol{\Sigma}^{\mathcal{R}}_1 = \boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^{\mathcal{R}}_1 = \boldsymbol{\Pi}^1_1$.

Facts 5.23.

(1) *If $\gamma < \alpha < \omega_1$, then $\boldsymbol{\Sigma}^{\mathcal{R}}_\gamma \cup \boldsymbol{\Pi}^{\mathcal{R}}_\gamma \subseteq \boldsymbol{\Delta}^{\mathcal{R}}_\gamma$.*
(2) *Every $\mathcal{R}$-set belongs to one of the classes $\boldsymbol{\Sigma}^{\mathcal{R}}_\alpha$, $\boldsymbol{\Pi}^{\mathcal{R}}_\alpha$.*
(3) *All the $\boldsymbol{C}$-sets are included in the class $\boldsymbol{\Delta}^{\mathcal{R}}_2$*
(4) *For all $\alpha < \omega_1$, the classes $\boldsymbol{\Sigma}^{\mathcal{R}}_\alpha$ and $\boldsymbol{\Pi}^{\mathcal{R}}_\alpha$ are pointclasses closed under the operations R_α and $co - R_\alpha$. In particular, for $0 < \alpha$, the classes $\boldsymbol{\Sigma}^{\mathcal{R}}_\alpha$ and $\boldsymbol{\Pi}^{\mathcal{R}}_\alpha$ are closed under countable union and intersection.*
(5) *For all $\alpha < \omega_1$, the classes $\boldsymbol{\Sigma}^{\mathcal{R}}_\alpha$ and $\boldsymbol{\Pi}^{\mathcal{R}}_\alpha$ are closed under preimages by Borel functions.*

Proof. Folklore. See for example Kanovei [52]. □

It was proved by Lyapunov [76] that all the $\mathcal{R}$-sets are in the class $\boldsymbol{\Delta}^1_2$, and by Burgess [18, 19, 20, 21] that the class of all $\mathcal{R}$-sets and the $\boldsymbol{\Delta}^1_2$ class do not coincide.

5.3.3 A fragment of the Wadge hierarchy of $\mathcal{R}$-sets

The properties of the classes $\mathbf{\Pi}^{\mathcal{R}}_{\alpha}$ allow us, like in the $\boldsymbol{C}-$sets case, to define the operations $(D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), \cdot)$.

Proposition 5.24. *Let α and ξ be non-zero countable ordinals, and let B_0 and B_1 be conciliatory sets. The following hold:*

(i) $(D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0^{\complement}) \equiv_c (D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0)^{\complement}$.

(ii) *If $B_0 \leq_c B_1$, then $(D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0) \leq_c (D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_1)$.*

(iii) *If $B_0 <_c B_1$, then $(D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0) <_c (D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_1)$.*

(iv) $\left(V_{\xi}\left((D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0)\right)\right)^b \equiv_W (D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0)^b$.

(v) *For every $0 < \beta < \alpha$,*

$$\left(D_2(\mathbf{\Pi}^{\mathcal{R}}_{\beta}), \left((D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0)\right)\right)^b \equiv_W (D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0)^b.$$

(vi) *For every countable $0 < \beta$,*

$$\left(D_2(\mathbf{\Pi}^{\boldsymbol{C}}_{\beta}), \left((D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0)\right)\right)^b \equiv_W (D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), B_0)^b.$$

Hence the operations $(D_2(\mathbf{\Pi}^{\mathcal{R}}_{\alpha}), \cdot)$ behave well with respect to the Wadge preorder. Moreover, they do not overlap, neither with themselves nor with the Veblen operations, nor with the $(D_2(\mathbf{\Pi}^{\boldsymbol{C}}_{\alpha}), \cdot)$ operations. Now we can extend the fragment of the Wadge hierarchy defined in Section 5.2, and give a first description of the Wadge hierarchy of $\mathcal{R}$-sets.

Definition 5.25. Let $0 < \alpha < V^{\omega_1+\omega_1+\omega_1}(2)$. The ordinal α admits a unique Cantor Normal Form of base ω_1:

$$\alpha = \omega_1^{\alpha_n} \cdot \nu_n + \cdots + \omega_1^{\alpha_0} \cdot \nu_0,$$

with

$$V^{\omega_1+\omega_1+\omega_1}(2) > \alpha_n > \cdots > \alpha_0, \text{ and } 0 < \eta_i < \omega_1 \text{ for all } i \leq n.$$

Set:

$$\Omega(\alpha) = \Omega(\omega_1^{\alpha_n}) \bullet \eta_n + \cdots + \Omega(\omega_1^{\alpha_0}) \bullet \eta_0,$$

where $\Omega(\omega_1^{\beta})$ is defined as follows.

- If $\beta = 0$ then
$$\Omega(\omega_1^0) = \emptyset.$$
- If $\beta = \alpha + 1$ is successor, then $0 \in I_{\beta}$. Denote by γ_0 the ordinal such that $V^0(\gamma_0) = \omega_1^{\beta}$. Then set
$$\Omega(\omega_1^{\beta}) = V_0(\Omega(\gamma_0)).$$

– If β is limit of cofinality ω, there exists a sequence $(\beta_i)_{i\in\omega}$ of ordinals strictly less than β such that $\beta = \sup_{i\in\omega}\{\beta_i\}$. Then set:
$$\Omega(\omega_1^\beta) = \mathbf{sup}_{i\in\omega}\left\{\Omega(\omega_1^{\beta_i})\right\}.$$
– If β is limit of cofinality ω_1, then we have two cases: either I_β is empty, or not. If it is not empty, then as in the successor case, we set
$$\Omega(\omega_1^\beta) = V_{\nu_0}(\Omega(\gamma_0)),$$
where $\nu_0 < \omega_1$ and $\gamma_0 < \omega_1^\beta$ are the minimal ordinals such that $V^{\nu_0}(\gamma_0) = \omega_1^\beta$. If it is empty, then there are three cases: either there exists $\xi < \omega_1$ such that $\omega_1^\beta = V^{\omega_1}(1+\xi)$ and we set
$$\Omega(\omega_1^\beta) = D^B_\xi,$$
where D^B_ξ is a conciliatory set such that $(D^B_\xi)^b \equiv_W D^A_\xi$; or there exist some ordinals $0 < \beta' < \beta$ and $\zeta < \omega_1$ such that $\omega_1^\beta = V^{\omega_1+\zeta}(\beta')$ and we set
$$\Omega(\omega_1^\beta) = (D_2(\mathbf{\Pi}^{\boldsymbol{C}}_{1+\zeta}), \Omega(2+\beta'));$$
or there exist some ordinals $0 < \beta' < \beta$ and $\zeta < \omega_1$ such that $\omega_1^\beta = V^{\omega_1+\omega_1+\zeta}(\beta')$ and we set
$$\Omega(\omega_1^\beta) = (D_2(\mathbf{\Pi}^{\mathcal{R}}_{2+\zeta}), \Omega(2+\beta')).$$

One can prove that the sets defined are well-ordered with respect to the conciliatory hierarchy.

Proposition 5.26. *Assume the determinacy of all $\mathcal{R}$-sets. For $0 < \beta < \alpha < V^{\omega_1+\omega_1}(2)$,*
$$\Omega(\beta) \leq_c \Omega(\alpha) \qquad \textit{and} \qquad \Omega(\beta)^{\complement} \leq_c \Omega(\alpha).$$

Proof. It is mutatis mutandis the same as the proof of Proposition 5.19. The remaining cases are treated using Proposition 5.24. □

Like in the $\boldsymbol{C}$ sets case, notice that for all $0 < \alpha < \beta < \omega_1$ and all conciliatory set B such that $B^b \in \sigma D_2(\mathbf{\Pi}^{\mathcal{R}}_\beta)$, we have
$$(D_2(\mathbf{\Pi}^{\mathcal{R}}_\alpha), B)^b \in \sigma D_2(\mathbf{\Pi}^{\mathcal{R}}_\beta),$$
so that the operations $(D_2(\mathbf{\Pi}^{\mathcal{R}}_\alpha), \cdot)$ preserve the $\sigma D_2(\mathbf{\Pi}^{\mathcal{R}}_\beta)$ classes.

Question. *Let $0 < \beta < \omega_1$, and B a conciliatory set such that $B^b \in \mathbf{\Pi}^{\mathcal{R}}_\beta$. Do we have*
$$(D_2(\mathbf{\Pi}^{\mathcal{R}}_\alpha), B)^b \in \mathbf{\Pi}^{\mathcal{R}}_\beta$$
for every $\alpha < \beta$? In other words, do the operations $(D_2(\mathbf{\Pi}^{\mathcal{R}}_\alpha), \cdot)$ preserve the $\mathbf{\Pi}^{\mathcal{R}}_\beta$ classes, with $\alpha < \beta$?

6 Application to Automata Theory

> “ I was, oh, better than some. But no hope for true greatness. Mathematics is the wrong discipline for people doomed to non-greatness. However, that’s not why I switched. I didn’t switch to computers because I missed the world or because I was haunted by my own inadequacy per se. It was all too occult for me. I’m the type of person who’s willing to confront moderately awesome phenomena. Beyond that I lose my bearings. Chipping away at gigantic unproved postulates. Investigating the properties of common whole numbers and ending up in the wilds of analysis. Intoxicating theorems. Nagging little symmetries. The secrets hidden deep inside the great big primes. The way one formula or number or expression keeps turning up in the most unexpected places. The infinite. The infinitesimal. Glimpsing something, then losing it. The way it slides off the eyeball. The unfinished nature of the thing. ”
>
> Don DeLillo, *Ratner’s Star*.

In this chapter, we transport some of the techniques we developed in the descriptive set theory framework to theoretical computer science and, more precisely, to *automata theory*. From definable subsets of the Baire space, we thus shift our attention to sets of full binary trees that are recognizable by automata. In this context, the use of topology tools has proved useful for the study of relative complexity and characterization of regular languages.

After an introduction to this new framework and the formulation of relevant definitions and notations as well as classical results, we use operations on languages – inspired by the operations used in the *sequence* case, to construct a sequence of strictly more and more regular tree languages. This fragment of the Wadge hierarchy of regular tree languages has length $\varphi_\omega(0)$, where (φ_k) are the Veblen functions of basis ω, which provides a lower bound for the height of this hierarchy. In the second part of this chapter, we study the discrepancy between deterministically and unambiguously recognizable languages by proving that the height of the Wadge hierarchy restricted to unambiguously recognizable tree languages is at least $\varphi_2(0)$, an ordinal tremendously larger than the height of the Wadge hierarchy restricted to deterministically recognizable languages which is $(\omega^\omega)^3 + 3$, as unraveled by Murlak [85].

Results in Sections 6.2 and 6.3 are part of a joint work with Jacques Duparc [30]. Results in Section 6.4 are part of a joint work with Jacques Duparc and Szczepan Hummel [31].

6.1 Tree languages and tree automata

6.1.1 Full and conciliatory trees

A *conciliatory* binary tree over a finite set Σ is a partial function $t : \{0,1\}^{<\omega} \to \Sigma$ with a prefix-closed domain. Those trees can have both infinite and finite branches. A tree is called *full* if $\mathrm{dom}(t) = \{0,1\}^*$. Let $\mathcal{T}_\Sigma^{\leq\omega}$ and $\mathcal{T}_\Sigma$ denote, respectively, the set of all conciliatory binary trees and the set of full binary trees over Σ. Given $x \in \mathrm{dom}(t)$, we denote by t_x the subtree of t rooted at x. Let $\{0,1\}^n$ denote the set of words over $\{0,1\}$ of length n, and let t be a conciliatory tree over Σ. We denote by $t[n]$ the finite initial binary tree of height $n+1$ given by the restriction of t to $\bigcup_{0\leq i\leq n}\{0,1\}^i$. A subset of $\mathcal{T}_\Sigma$ is called a (full) *language* over Σ.

The space $\mathcal{T}_\Sigma$ equipped with the standard Cantor topology is a Polish space and is in fact homeomorphic to the Cantor space[1]. We can thus adapt the tools developed and the results obtained on the Wadge hierarchy for the Baire space to the space of full trees. As in the *sequences* case, one can define a very useful game characterization for the Wadge preorder. Let $L, M \subseteq \mathcal{T}_\Sigma$, the two-player infinite game $W(L, M)$ is defined as follows. Along the play, each player builds a tree, say t_{I} and t_{II}. At every round, player I plays first, and both players add a finite number of children to the terminal nodes of their tree. Player II is allowed to skip its turn, but has to produce a tree in $\mathcal{T}_\Sigma$ at the end of the game. Player II wins the game if and only if $t_I \in L \Leftrightarrow t_{II} \in M$. This game is designed in such a way that $L \leq_W M$ if and only if player II has a winning strategy in the game $W(L, M)$. In this chapter, we write $A <_W B$ when II has a winning strategy in $W(A, B)$ *and* I has a winning strategy in $W(B, A)$. This is in general stronger than the usual $A <_W B$ if and only if $A \leq_W B$ and $B \not\leq_W A$, but the two definitions coincide when the classes considered are determined.

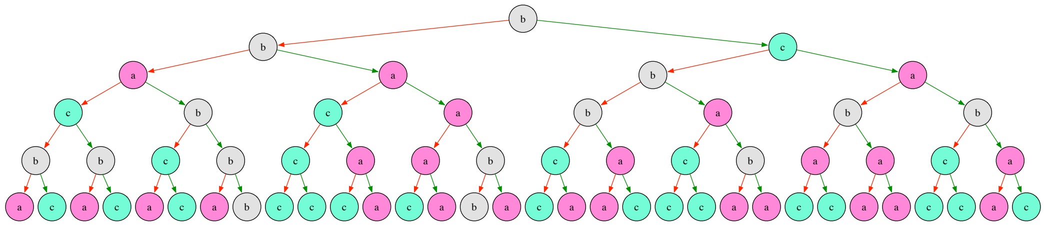
[1] See for example [8].

For conciliatory languages L, M we define the *conciliatory* version of the Wadge game: $C(L, M)$ [32]. The rules are similar, except for the fact that both players are now allowed to skip and to produce trees with finite branches – or even finite trees. For conciliatory languages L, M we use the notation $L \leq_c M$ if and only if II has a winning strategy in the game $C(L, M)$. In this chapter, we write $A <_c B$ when II has a winning strategy in $C(A, B)$ *and* I has a winning strategy in $C(B, A)$.

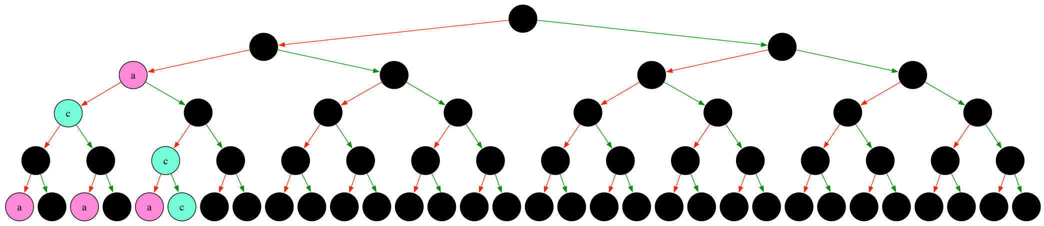
As in the sequences case, there is a strong connection between conciliatory and full languages. From a conciliatory language L over Σ, one defines the corresponding language L^b of full trees over $\Sigma \cup \{b\}$ by

$$L^b = \left\{ t \in T_{\Sigma \cup \{b\}} : t_{[\ /b]} \in L \right\},$$

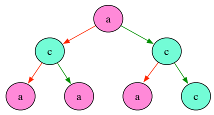
where b is an extra symbol that stands for "blank", and $t_{[\ /b]}$, the *undressing* of t, is informally the conciliatory tree over Σ obtained once all the occurrences of b have been removed in a top-down manner. More precisely, if there is a node v such that $t(v) = b$, we ignore this node and replace it with $v0$. If, for every integer n, $t(v0^n) = b$, then $v \notin \mathrm{dom}(t_{[\ /b]})$. This process is illustrated by Fig. 6.1.



(a) A tree t with blanks



(b) The blanks are deleted in a top-down manner.



(c) The resulting tree $t_{[\ /b]}$.

Figure 6.1: The undressing process.

Formally, for each $v \in \{0,1\}^*$ we consider two (possibly infinite) sequences (w_i) and (u_i) in $\{0,1\}^{<\omega}$:

- $w_0 = \varepsilon$, $u_0 = v$,
- for $0 \leq i$:
  - if $t(w_i) = b$, we set $u_{i+1} = u_i$ and $w_{i+1} = w_i0$;
  - if $t(w_i) \neq b$ and $u_i = au'_i$ for $a \in \{0,1\}$, we set $u_{i+1} = u'_i$ and $w_{i+1} = w_ia$;
  - if $t(w_i) \neq b$ and $u_i = \varepsilon$, we halt the construction at step i.

If the construction is halted at some step i, then $v \in \mathrm{dom}(t_{[\ /b]})$ and $t_{[\ /b]}(v) = t(w_i)$. Otherwise, $v \notin \mathrm{dom}(t_{[\ /b]})$. If Γ is a pointclass of full trees, we say that a conciliatory language L is in Γ if and only if L^b is in Γ.

Lemma 6.1. *Let L and M be conciliatory languages. Then*

$$L \leq_c M \text{ if and only if } L^b \leq_W M^b.$$

Proof. A strategy in one game can be translated directly into a strategy in the other game: arbitrary skipping in $C(L,M)$ gives the same power as the b labels in $W(L^b, M^b)$. In particular, in $W(L^b, M^b)$, II does not need to skip at all. □

The mapping $L \mapsto L^b$ gives thus a natural embedding of the preorder $\leq_c$ restricted to conciliatory sets in Γ into the Γ-Wadge hierarchy, exactly like in the Baire space case.

6.1.2 Automata on trees

In this chapter, we are not interested in *all* the subset of the space of full trees, not even in all the definable ones, but only in those that are called *regular*, i.e. those who can be defined via finite devices called *tree automata*.

Parity games

We choose to work with automata that have the parity condition as acceptance condition, and whose semantics can be defined in terms of *parity games*, a sort of game that has been introduced by Emerson and Jutla [33] and independently by Mostowski [82]. A parity game is a game between two players, $\exists$ and $\forall$, and is defined by a bipartite labeled graph $\mathcal{G} = (V, V_\exists, V_\forall, E, p_0, \mathrm{r})$. The sets of vertices $V_\exists$ and $V_\forall$ are disjoints sets of positions for $\exists$ and $\forall$ respectively, with $V = V_\exists \cup V_\forall$. The relation $E \subseteq V \times V$ is the set of edges, the relation of possible moves. The initial position in a play is p_0. The ranking function is $\mathrm{r} : V \to \omega$, and its range is finite. A vertex $v \in V$ is a successor of a vertex $v' \in V$ if $(v', v) \in E$. The game proceeds as follows: the player who owns the initial vertex chooses a successor of p_0, say p_1. Then the player who owns p_1 chooses one of its successors, and so on, and so forth. If they reach a position with no successor the game stops, otherwise they keep playing. A play is thus a sequence, possibly infinite, of positions $(p_0, p_1, \ldots)$. If the play is infinite, $\exists$ wins the game if and only if $\limsup_{n\to\infty} \mathrm{r}(p_n)$ is even. If the play is finite, $\exists$ wins the game if and only if the rank of the last position visited is even. It is well known that parity games are determined, and that a positional winning strategy can always be found for those games. For more details, see e.g. [41].

Parity tree automata

A *nondeterministic parity tree automaton* $\mathcal{A} = \langle \Sigma, Q, I, \delta, \mathrm{r} \rangle$ consists of a finite input alphabet Σ, a finite set Q of states, a set of initial states $I \subseteq Q$, a transition relation $\delta \subseteq Q \times \Sigma \times Q \times Q$ and a priority function $\mathrm{r} : Q \to \omega$. A run of the automaton $\mathcal{A}$ on a binary conciliatory input tree $t \in \mathcal{T}_\Sigma^{\leq\omega}$ is a conciliatory tree $\rho_t \in \mathcal{T}_Q^{\leq\omega}$ with $\mathrm{dom}(\rho_t) = \{\varepsilon\} \cup \{va : v \in \mathrm{dom}(t) \wedge a \in \{0, 1\}\}$ such that the root of this tree is labeled with a state $q \in I$, and for each $v \in \mathrm{dom}(t)$, transition $(\rho_t(v), t(v), \rho_t(v_1), \rho_t(v_1)) \in \delta$. The run ρ_t is *accepting* if parity condition is satisfied on each infinite branch of ρ_t, i.e. if the highest rank of a state occurring infinitely often on the branch is even, and if the rank of each leaf node in ρ_t is even. We say that a parity tree automaton $\mathcal{A}$ *accepts* a conciliatory tree t if it has an accepting run on t. The language *recognized* by $\mathcal{A}$, denoted $L(\mathcal{A})$ is the set of trees accepted by $\mathcal{A}$. We let $L^\omega(\mathcal{A})$ denote the set of full trees recognized by $\mathcal{A}$, i.e. $L^\omega(\mathcal{A}) = L(\mathcal{A}) \cap \mathcal{T}_\Sigma$. We use the following conventions in the diagrams. Nodes represent states of the automaton, and labels correspond to state ranks. A red edge shows the state that is assigned to the left successor node of a transition, and a green edge goes to the right successor node. In order to lighten the notation, transitions that are not depicted on a diagram lead to some all-accepting state. Given automata $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \leq_c \mathcal{B}$ for $L(\mathcal{A}) \leq_c L(\mathcal{B})$, and same with $<_c, \leq_W, <_W$. A parity tree automaton is called *deterministic* if the transition relation is the graph of a total function from $Q \times \Sigma$ to $Q \times Q$. Observe that given a deterministic parity tree automaton $\mathcal{A}$ and a tree t, there is a *unique* possible run of $\mathcal{A}$ on t.

Notice that as the set of states is finite, the priority function is bounded. Moreover, shifting all ranks by an even number does not change the language recognized by a parity tree automaton. It is thus sufficient to consider parity tree automata whose priorities are restricted to intervals $[\iota, \kappa]$, for $\iota \in \{0, 1\}$. We say that an automaton is of index $[\iota, \kappa]$ if its priorities are restricted to intervals $[\iota, \kappa]$. A language is of index $[\iota, \kappa]$ if there is an automaton of index $[\iota, \kappa]$ that recognizes it. For an index $[\iota, \kappa]$, we denote by $\overline{[\iota, \kappa]}$ the dual index, i.e. $\overline{[0, \kappa]} = [1, \kappa+1]$ and $\overline{[1, \kappa]} = [0, \kappa-1]$. Let us define the following partial order on indices:

$$[\iota, \kappa] \sqsubseteq [\iota', \kappa'] \text{ if and only if } \begin{cases} \{\iota, \dots, \kappa\} \subseteq \{\iota', \dots, \kappa'\} & \text{or;} \\ \{\iota + 2, \dots, \kappa + 2\} \subseteq \{\iota', \dots, \kappa'\}. \end{cases}$$

The above ordering induces a hierarchy, the *nondeterministic* Mostowski-Rabin index hierarchy. If a language L is recognized by a nondeterministic parity tree automaton of index $[\iota, \kappa]$ and $[\iota, \kappa] \sqsubseteq [\iota', \kappa']$ then L is also recognized by a nondeterministic parity tree automaton of index $[\iota', \kappa']$. Moreover, if a language is of index $[\iota, \kappa]$, then its complement is of index $\overline{[\iota, \kappa]}$.

Corollary 6.2. *The mapping $L \mapsto L^b$ embeds the conciliatory hierarchy for $\mathbf{\Delta}^1_2$-sets restricted to languages of index $[\iota, \kappa]$ into the $\mathbf{\Delta}^1_2$-Wadge hierarchy restricted to languages of index $[\iota, \kappa]$.*

Proof. By Lemma 6.1 it is enough to prove that each automaton A can be transformed into an automaton A' such that $L^\omega(A') = L(A)^b$. Given any automaton A, this is done by adding an all-accepting state $\top$ to the set of states Q_A, and the set $\{(q, b, q, \top) : q \in Q_A\}$ to the transition relation δ_A, as depicted in Fig. 6.2. The obtained automaton A' is such that $L^\omega(A') = L(A)^b$. □

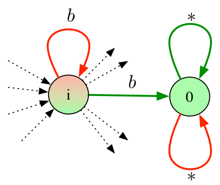


Figure 6.2: The added b transitions.

Alternating parity tree automata

The *alternating parity tree automata* are a generalization of the nondeterministic parity tree automata where the set of states Q is divided into two parts $Q = Q_\exists \cup Q_\forall$. The acceptance by such an automaton $\mathcal{A}$ of a conciliatory tree

t is defined via a parity game $\mathcal{P}(\mathcal{A}, t)$ between two players, $\exists$ and $\forall$. During a play, players construct a run of an automaton. If a given node is labelled by a state from $Q_\exists$, during this construction, then the next transition is chosen by $\exists$. Otherwise it is chosen by $\forall$. Player $\exists$ wins the game if in the constructed run on each infinite branch the parity condition holds, and if the rank of each leaf node is even. A tree t is accepted by the automaton $\mathcal{A}$ if and only if $\exists$ has a winning strategy in the game $\mathcal{P}(\mathcal{A}, t)$. As in the nondeterministic case, we say that an alternating parity tree automaton is of index $[\iota, \kappa]$ if its priorities are restricted to intervals $[\iota, \kappa]$, which gives rise to the *alternating* version of the Mostowski-Rabin index hierarchy. If a language L is recognized by an alternating parity tree automaton of index $[\iota, \kappa]$ and $[\iota, \kappa] \sqsubseteq [\iota', \kappa']$ then L is also recognized by an alternating parity tree automaton of index $[\iota', \kappa']$. Moreover, if a language is of alternating index $[\iota, \kappa]$, then its complement is of alternating index $\overline{[\iota, \kappa]}$. Note that nondeterministic automata are special cases of alternating parity tree automata, so that a language of index $[\iota, \kappa]$ is also of alternating index $[\iota, \kappa]$.

Game languages

Consider the alphabet $\Sigma_{[\iota,\kappa]} = \{\exists, \forall\} \times \{\iota, \ldots, \kappa\}$ with $\iota \in \{0, 1\}$ and $\iota \leq \kappa$. For each tree $t \in \mathcal{T}_{\Sigma_{[\iota,\kappa]}}$ we define the parity game $\mathcal{G}_t = (V, V_\exists, V_\forall, E, p_0, \mathrm{r})$ as follows:

- $V_\exists = \{v \in \{0,1\}^{<\omega} : t(v)_0 = \exists\}$;
- $V_\forall = \{v \in \{0,1\}^{<\omega} : t(v)_0 = \forall\}$;
- $E = \{(w, wi) : w \in \{0,1\}^{<\omega}$ and $i \in \{0, 1\}\}$;
- $p_0 = \varepsilon$;
- $\mathrm{r}(v) = t(v)_1$, for each $v \in \{0,1\}^{<\omega}$.

The *game language* of index $[\iota, \kappa]$, denoted by $W_{[\iota,\kappa]}$, corresponds to the class of full trees in $\mathcal{T}_{\Sigma_{[\iota,\kappa]}}$ for which player $\exists$ has a winning strategy in the corresponding parity game $\mathcal{G}_t$. Each language $W_{[\iota,\kappa]}$ is recognized by a nondeterministic parity tree automaton of index $[\iota, \kappa]$. It is worth noticing that the game languages witness the strictness of the index hierarchy of alternating tree automata [16], and that if $[\iota, \kappa] \sqsubseteq [\iota', \kappa']$, then $W_{[\iota,\kappa]} \leq_W W_{[\iota',\kappa']}$ [9]. Moreover, they are in some sense *complete* for their class of index, i.e. if $\mathcal{A}$ is an alternating parity tree automaton of index $[\iota, \kappa]$, then $L(\mathcal{A}) \leq_W W_{[\iota,\kappa]}$ [7, 9].

Topological complexity of regular languages

The topological approach to the complexity problem for regular languages has proved extremely fruitful. We recall some basic facts binding automata and classical topological hierarchies.

First, observe that nondeterministic parity tree automata recognize some sets that are neither analytic, nor coanalytic, but their expressive power is bounded by the second level of the projective hierarchy. Namely, by Rabin's complementation result [96], all nondeterministic languages are in the $\mathbf{\Delta}^1_2$ class.

Theorem 6.3. *Each language recognized by a parity tree automaton is in the class* $\mathbf{\Delta}^1_2$*.*

One can compute the exact topological complexity of the first non-trivial game languages.

Fact 6.4. *The languages* $W_{[0,1]}$ *and* $W_{[1,2]}$ *are respectively* $\mathbf{\Pi}^1_1$*-complete and* $\mathbf{\Sigma}^1_1$*-complete.*

It has been proved by Finkel and Simonnet [37] that every game language $W_{[\iota,\kappa]}$ with $[\iota,\kappa]$ strictly above $[0,1]$ is not in the class $D^*_\alpha(\mathbf{\Pi}^0_1)$ for all $\alpha < \omega^\omega$. A stronger and more precise result by Gogacz, Michalewski, Mio and Skrzypczak [40] relates game languages to the Kolmogorov hierarchy of $\mathcal{R}$-sets.

Theorem 6.5. *Let* $k > 0$*, the game language* $W_{[k-1,2k-1]}$ *is complete for the* k*-th level of the Kolmogorov hierarchy of* $\mathcal{R}$*-sets.*

Hence we can refine Theorem 6.3 to get a sharp complexity bound for regular languages.

Corollary 6.6. *Each language recognized by a parity tree automaton is in the class* $\bigcup_{n\in\omega}\mathbf{\Sigma}^{\mathcal{R}}_n$*.*

This last result was independently established by Simonnet [103] and discussed by Finkel, Lecomte, and Simonnet [36], but with other methods that did not provide the sharpness of the bound. Since the $\mathcal{R}$-sets are strictly included in the $\mathbf{\Delta}^1_2$ class, the bound given by Rabin's complementation result was not optimal.

6.2 Operations on languages and their automatic counterparts

In this section, we present classical operations defined by Duparc and Murlak [32] on conciliatory tree languages that allow us to construct more and more complicated languages. Without loss of generality, we may choose the alphabet $\Sigma = \{a, c\}$.

6.2.1 The sum

For $L, M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$, we define $L + M$ (the *sum* of L and M) as the language formed of all those trees $t \in \mathcal{T}_\Sigma^{\leq\omega}$ such that one of the following conditions holds:

- $t(10^n) = a$ for each integer n and $t_0 \in M$;
- the node 10^n is the first on the path $10^{<\omega}$ labeled with c and either $t(10^n 0) = a$ and $t_{10^n 00} \in L$, or $t(10^n 0) = c$ and $t_{10^n 00} \in L^\complement$.

This operation behaves well regarding the conciliatory hierarchy.

Facts 6.7 ([26, 32]). *Given L, M, and M' any conciliatory tree languages over Σ,*

(1) $(L + M)^\complement \equiv_c L + M^\complement$.

(2) *The operation $+$ preserves the conciliatory ordering: if $M' \leq_c M$, then*

$$L + M' \leq_c L + M.$$

(3) *Assuming enough determinacy:*

$$d_c(L + M) = d_c(L) + d_c(M).$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two automata that recognize, respectively, the conciliatory languages M and L. Then the automaton $\mathcal{B} + \mathcal{A}$ depicted in Fig. 6.3 recognizes the sum of L and M. In this picture, $\mathcal{C}$ is any automaton that recognizes a language equivalent to $L^\complement$, and the parity i and j are defined as follows:

- $i = 0$ if and only if the empty tree is accepted by $\mathcal{A}$;
- $j = 1$ if and only if $L(\mathcal{A})$ is equivalent to $L(\mathcal{A}) \to \circleddash$, where $\circleddash$ denotes any automaton that rejects all trees.[2]

[2]A player in charge of $L(\mathcal{A}) \to \circleddash$ in a conciliatory game is like a player in charge of $L(\mathcal{A})$, but with the extra possibility at any moment of the play to reach a definitively rejecting position.

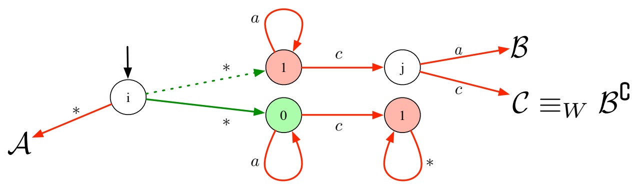


Figure 6.3: The automaton $\mathcal{B} + \mathcal{A}$ that recognizes $L(\mathcal{B}) + L(\mathcal{A})$. The values of i and j depend on properties of $\mathcal{A}$.

Notice that if $\mathcal{A}$ and $\mathcal{B}$ are parity tree automata of index $[0, 2]$ such that $L(\mathcal{B})^{\complement}$ can be recognized by an automaton of index $[0, 2]$, then $\mathcal{B} + \mathcal{A}$ is a parity tree automaton of index $[0, 2]$.

Lemma 6.8. *Let L, L', M and M' be conciliatory languages such that $L <_c L'$ and $M \leq_c M'$. Then the following hold.*

(1) $M + L <_c M' + L'$;
(2) $M <_c M + L$.

Proof.

(1) It is clear that $M + L \leq_c M' + L'$, what remains to prove is thus that I has a winning strategy in $C(M' + L', M + L)$. Let τ be the winning strategy for I in $C(L', L)$. Observe that, since $M \leq_c M'$, player I has a winning strategy τ' in $C(M', M^{\complement})$. A strategy σ for I in the game $C(M' + L', M + L)$ is the following. First I plays a on the node ε, and then, as long as player II does not play a c on the branch $10^{<\omega}$, I follows τ on the left subtree $0\{0, 1\}^{<\omega}$. If ever II plays a c on a node 10^n, then I copies II's moves for the branch 10^n0, and then follows τ' on the subtree $10^n0\{0, 1\}^{<\omega}$. Since τ and τ' are winning, σ is a winning strategy for I in $C(M' + L', M + L)$. Thus $M + L <_c M' + L'$.
(2) It is clear that $M \leq_c M + L$: a winning strategy for II in $C(M, M + L)$ is indeed to play a at ε, c at the node 1, a at the node 010, and then copy I's moves in the subtree $010\{0, 1\}^{<\omega}$. The winning strategy σ for I in the game $C(M + L, M)$ is similar. First, I plays a at ε, c at the node 1, c at the node 010, and then copy I's moves in the subtree $010\{0, 1\}^{<\omega}$.

□

6.2.2 Multiplication by a countable ordinal

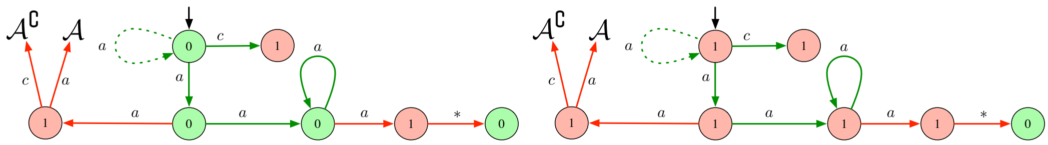
In order to define the multiplication of a language by a countable ordinal, we first introduce the operation $\sup_{n<\omega}$. Let $(L_n)_{n\in\omega} \subseteq \mathcal{T}_\Sigma^{\leq\omega}$ be a countable family of conciliatory languages. Define $\sup_{n<\omega} L_n$ as the conciliatory tree language containing all of those trees $t \in \mathcal{T}_\Sigma^{\leq\omega}$ such that one of the following conditions holds:

- $t(1^n) = a$ for all integer n;
- the node 1^n is the first on the path $1^{<\omega}$ labeled with c and $t_{1^n0} \in L_n$.

The multiplication by a countable ordinal is now defined as an iterated sum. For $L \subseteq \mathcal{T}_\Sigma^{\leq\omega}$, we define:

- $L \cdot 1 = L$;
- $L \cdot (\alpha + 1) = (L \cdot \alpha) + L$;
- $L \cdot \lambda = \sup_{\alpha<\lambda} L \cdot \alpha$, for λ limit.

Let $\mathcal{A}$ be an automaton that recognizes the conciliatory languages L. Then the automaton $\mathcal{A} \bullet \omega$ depicted in Fig. 6.4a recognizes a language equivalent to $L \cdot \omega$. In this picture, $\mathcal{C}$ is any automaton that recognizes a language equivalent to $L^{\complement}$. The automaton $\mathcal{A} \mathbin{\overline{\bullet}} \omega$ that recognizes the complement of $L(\mathcal{A} \bullet \omega)$, and thus a language equivalent to the complement of $L \cdot \omega$, is depicted in Fig. 6.4b. Notice that if $\mathcal{A}$ is of index $[0, 2]$, and if there exists an automaton that recognizes $L(\mathcal{A})^{\complement}$ of index $[0, 2]$, then both $\mathcal{A} \bullet \omega$ and $\mathcal{A} \mathbin{\overline{\bullet}} \omega$ are parity tree automata of index $[0, 2]$. Hence, for every ordinal $0 < \alpha < \omega^\omega$



(a) The automaton $\mathcal{A} \bullet \omega$.

(b) The automaton $\mathcal{A} \mathbin{\overline{\bullet}} \omega$.

Figure 6.4: Automata that recognize respectively a language equivalent to $L \cdot \omega$ and a language equivalent to its complement.

and for every automaton $\mathcal{A}$, there exists an automaton $\mathcal{A} \bullet \alpha$ that recognizes $L(\mathcal{A}) \cdot \alpha$. Moreover, if $\mathcal{A}$ is of index $[0, 2]$, and if there exists an automaton that recognizes $L(\mathcal{A})^{\complement}$ of index $[0, 2]$, then $\mathcal{A} \bullet \alpha$ is a parity tree automaton of index $[0, 2]$.

As a corollary of Lemma 6.8 and Facts 6.7, the multiplication by a countable ordinal behaves well regarding the conciliatory hierarchy.

Corollary 6.9. *Let L and M be conciliatory languages such that $L <_c M$. Then for every countable ordinals $0 < \alpha < \beta < \omega^\omega$:*

(1) $L \cdot \alpha <_c L \cdot \beta$;
(2) $L \cdot \alpha <_c M \cdot \alpha$.

6.2.3 The pseudo-exponentiation

Let $P \subseteq \mathcal{T}_\Sigma^{\leq\omega}$ be a conciliatory tree language. For $t \in \mathcal{T}_\Sigma^{\leq\omega}$, let:

$$i^P(t)(a_1, a_2, \ldots, a_n) = \begin{cases} t(a_1, 0, a_2, 0, \ldots, 0, a_n, 0), & \text{if } t_{a_1,0,a_2,0,\ldots,0,a_n,1} \in P; \\ b, & \text{otherwise.} \end{cases}$$

This process is illustrated in Fig. 6.5. The nodes in blue are called the *main run*. The blue arrows denote the dependency of a node of the main run on a subtree of auxiliary moves. If the auxiliary subtree of a main run node is not in P, then we say that the node is *killed*.

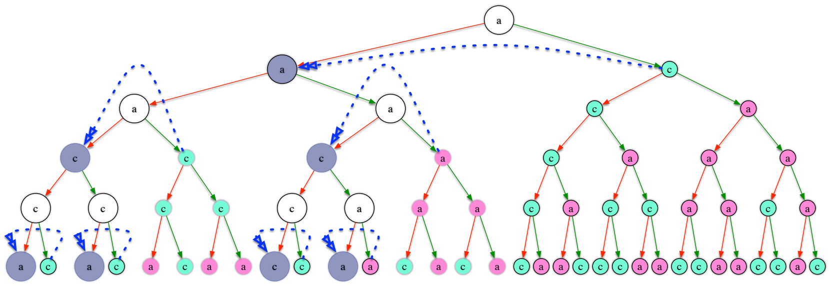


Figure 6.5: Main run and auxiliary moves.

Let $L \subseteq \mathcal{T}_\Sigma^{\leq\omega}$, we define the *action* of P on L, in symbols (P, L), by

$$\left\{t \in \mathcal{T}_\Sigma^{\leq\omega} : i^P(t)_{[\ /b]} \in L\right\}.$$

Let $P_{\mathbf{\Pi}_1^0}$ be the complete closed set of all full trees over Σ with all nodes on the leftmost branch $0^{<\omega}$ labelled by a. For $L \subseteq \mathcal{T}_\Sigma^{\leq\omega}$, we denote by $(\mathbf{\Pi}_1^0, L)$ the action of $P_{\mathbf{\Pi}_1^0}$ on L. This operation $(\mathbf{\Pi}_1^0, \cdot)$ behaves well regarding the conciliatory hierarchy.

Facts 6.10 ([26, 32]). *Let L and M be conciliatory tree languages over Σ. Then the following hold.*
(1) $(\mathbf{\Pi}_1^0, L)^\complement \equiv_c (\mathbf{\Pi}_1^0, L^\complement)$.
(2) *If* $L \leq_c M$, *then* $(\mathbf{\Pi}_1^0, L) \leq_c (\mathbf{\Pi}_1^0, M)$.
(3) *If* $L <_c M$, *then* $(\mathbf{\Pi}_1^0, L) <_c (\mathbf{\Pi}_1^0, M)$.

(4) *Assuming enough determinacy,* $d_c((\mathbf{\Pi}^0_1, L)) = \omega_1^{d_c(L)+\varepsilon}$*, where*

$$\varepsilon = \begin{cases} -1 & \text{if } d_c(L) < \omega; \\ 0 & \text{if } d_c(L) = \beta + n \text{ and } \operatorname{cof}(\beta) = \omega_1; \\ 1 & \text{if } d_c(L) = \beta + n \text{ and } \operatorname{cof}(\beta) = \omega. \end{cases}$$

Without assuming any determinacy hypothesis, we can nonetheless prove the following Proposition that links $(\mathbf{\Pi}^0_1, \cdot)$ to $+$.

Proposition 6.11. *Let L, L' and M be conciliatory languages such that $L <_c (\mathbf{\Pi}^0_1, M)$ and $L' <_c (\mathbf{\Pi}^0_1, M)$. Then*

(1) $L + L' <_c (\mathbf{\Pi}^0_1, M)$*;*

(2) $L \cdot \alpha <_c (\mathbf{\Pi}^0_1, M)$*, for any $\alpha < \omega^\omega$.*

Proof. We only prove the first part of the Proposition here, the other is mutatis mutandis the same. The fact that $L + L' \leq_c (\mathbf{\Pi}^0_1, M)$ is clear: if σ_0, σ_1 and σ' are winning strategies respectively in the games $C(L, (\mathbf{\Pi}^0_1, M))$, $C(L^\complement, (\mathbf{\Pi}^0_1, M))$ and $C(L', (\mathbf{\Pi}^0_1, M))$, a winning strategy for II in $C(L + L', (\mathbf{\Pi}^0_1, M))$ is the following. As long as player I does not play a c on the branch $10^{<\omega}$, II does not kill any nodes and follows σ' to what I plays in the subtree $0\{0,1\}^{<\omega}$ to get her main run. If ever II plays a c on a node 10^n, then II kills all the nodes of the main run she had already played (by playing c on the leftmost branches of appropriate auxiliary subtrees), and begins to play along a tree not in M in her main run, without killing any node. If I plays a on the node 10^n0, she kills every node in the main run she had already played, and then follows σ_0 on the subtree $10^n0\{0,1\}^{<\omega}$. If I plays c on the node 10^n0, she kills every node in the main run she had already played, and then she follows σ_1 on the subtree $10^n0\{0,1\}^{<\omega}$. The proof that I has a winning strategy τ in the game $C((\mathbf{\Pi}^0_1, M), L + L')$ is mutatis mutandis the same, given that I has a winning strategy for each of the games $C((\mathbf{\Pi}^0_1, M), L)$, $C((\mathbf{\Pi}^0_1, M), L^\complement)$ and $C((\mathbf{\Pi}^0_1, M), L')$. □

Given any automaton $\mathcal{A}$ recognizing $L \subseteq \mathcal{T}_\Sigma^{\leq\omega}$, the conciliatory language $(\mathbf{\Pi}^0_1, L)$ is recognized by the automaton $(\omega^\omega)^\mathcal{A}$ defined from $\mathcal{A}$ by replacing each state of $\mathcal{A}$ by a "gadget", as depicted in Fig. 6.6. By replacing a state by a gadget we mean that all transitions ending in this state should now end in the initial state of the gadget, and that all the transitions leaving this state should now leave from the final state of the gadget. This sort of gadget first appeared in [32]. Notice that if $L \subseteq \mathcal{T}_\Sigma^{\leq\omega}$ is of index $[0,2]$, then $(\mathbf{\Pi}^0_1, L)$ is also of index $[0,2]$. Observe also that for each positive integer n, the game languages $W_{[0,n]}$ is a fixed points for pseudo-exponentiation, i.e.

$$(\mathbf{\Pi}^0_1, W_{[0,n]})^b \equiv_W W_{[0,n]}.$$

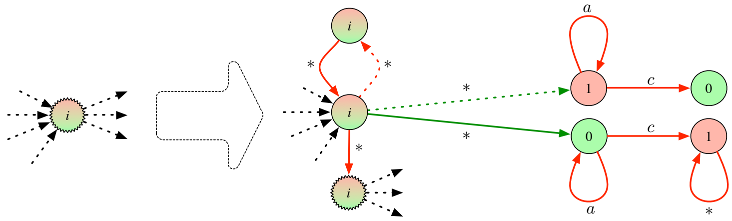
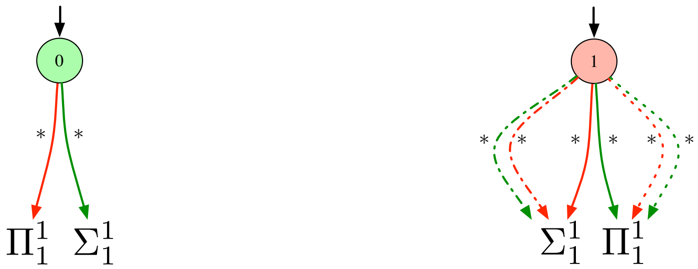


Figure 6.6: The gadget to replace a state in $\mathcal{A}$

6.2.4 The operation $(D_2(\boldsymbol{\Pi}_1^1), \cdot)$

The $D_2(\boldsymbol{\Pi}_1^1)$ class

We define a conciliatory language that is $D_2(\boldsymbol{\Pi}_1^1)$-complete and such that its complement and itself are both recognizable by parity tree automata whose priorities are restricted to $\{0, 1, 2\}$. Their definitions are given via the automata that recognize them. The abstract idea behind our construction is depicted by Fig. 6.7 which represents a general form of automata that would recognize languages that are $D_2(\boldsymbol{\Pi}_1^1)$-complete (Fig. 6.7a), and $\check{D}_2(\boldsymbol{\Pi}_1^1)$-complete (Fig. 6.7b).



(a) $\mathcal{A}_1$ (b) $\mathcal{A}_2$

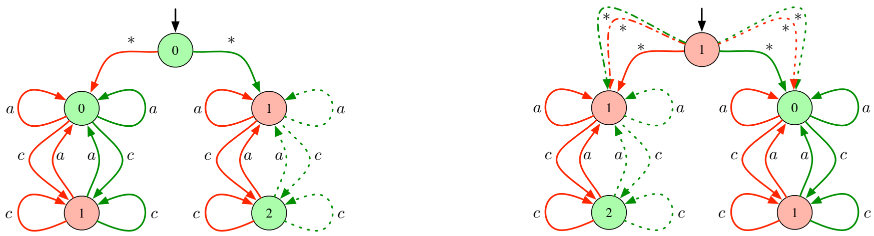
Figure 6.7: Sketch of automata, where $\boldsymbol{\Pi}_1^1$ and $\boldsymbol{\Sigma}_1^1$ denote automata that recognize respectively a $\boldsymbol{\Pi}_1^1$-complete language and the complement of this language.

The automaton $\mathcal{A}_1$, indeed, recognizes a tree $t \in \mathcal{T}_\Sigma^{\leq\omega}$ if and only if t_0 is in a given conciliatory $\boldsymbol{\Pi}_1^1$-complete language (say A) and t_1 is in its complement which is $\boldsymbol{\Sigma}_1^1$-complete. Since the maps $t \mapsto t_0$ and $t \mapsto t_1$ are continuous, the language recognized by the automaton is thus $D_2(\boldsymbol{\Pi}_1^1)$. Moreover, if $M \in \boldsymbol{\Pi}_1^1$ and $M' \in \boldsymbol{\Sigma}_1^1$, $M \cap M' \leq_c L(\mathcal{A}_1)$: a winning strategy for player II in the game $C(M \cap M', L(\mathcal{A}_1))$ is indeed to glue together her winning strategies in the games $C(M, A)$ and $C(M', A^{\complement})$. Hence, the language recognized by $\mathcal{A}_1$ is $D_2(\mathbf{\Pi}_1^1)$-complete. The reasoning for $\mathcal{A}_2$ is similar. We now define two automata: the first one recognizes a $\mathbf{\Sigma}_1^1$-complete language, and the other one recognizes the complement of the first one, i.e. a $\mathbf{\Pi}_1^1$-complete language. They are depicted in Fig. 6.8.

(a) $\mathcal{A}_{\mathbf{\Sigma}_1^1}$ (b) $\mathcal{A}_{\mathbf{\Pi}_1^1}$

Figure 6.8: Automata that recognize respectively a $\mathbf{\Sigma}_1^1$-complete and its complement

We denote by $A_{\mathbf{\Sigma}_1^1}$ and $A_{\mathbf{\Pi}_1^1}$ the conciliatory languages recognized respectively by $\mathcal{A}_{\mathbf{\Sigma}_1^1}$ and $\mathcal{A}_{\mathbf{\Pi}_1^1}$. Combining these constructions, we can now define an unambiguously recognizable conciliatory language that is $D_2(\mathbf{\Pi}_1^1)$-complete (Fig. 6.9a) and such that its complement (Fig. 6.9b) is also recognizable, via the automata that recognize each of them.



(a) $\mathcal{A}_{D_2(\mathbf{\Pi}_1^1)}$ (b) $\mathcal{A}_{\check{D}_2(\mathbf{\Pi}_1^1)}$

Figure 6.9: Automata that recognize respectively a $D_2(\mathbf{\Pi}_1^1)$-complete and its complement.

We denote by $A_{D_2(\mathbf{\Pi}_1^1)}$ and $A_{\check{D}_2(\mathbf{\Pi}_1^1)}$ the conciliatory languages recognized respectively by $\mathcal{A}_{D_2(\mathbf{\Pi}_1^1)}$ and $\mathcal{A}_{\check{D}_2(\mathbf{\Pi}_1^1)}$.

The operation $(D_2(\mathbf{\Pi}^1_1), \cdot)$

The operations defined in Section 6.2 are *Borel* in the sense that when we apply them to Borel languages, the resulting language is still Borel. In order to describe the most of the Wadge hierarchy of languages recognized by parity tree automata we need to climb higher.

For $M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$, we denote by $(D_2(\mathbf{\Pi}^1_1), M)$ the action of $L(\mathcal{A}_{D_2(\mathbf{\Pi}^1_1)})$ on M. Observe that this operation is highly non-Borel, since if we apply it to a $\mathbf{\Sigma}^0_1$-complete conciliatory language, the resulting language will be complete for the pointclass of all the countable unions of $D_2(\mathbf{\Pi}^1_1)$ languages. The operation $(D_2(\mathbf{\Pi}^1_1), \cdot)$ behaves well with respect to $\leq_c$.

Theorem 6.12. *Let $M, M' \subseteq \mathcal{T}_\Sigma^{\leq\omega}$. If $M \leq_c M'$, then*

(1) $(D_2(\mathbf{\Pi}^1_1), M)^\complement \equiv_c (D_2(\mathbf{\Pi}^1_1), M^\complement)$;

(2) $(D_2(\mathbf{\Pi}^1_1), M) \leq_c (D_2(\mathbf{\Pi}^1_1), M')$.

Proof. The first point holds merely by definition of the operation $(D_2(\mathbf{\Pi}^1_1), \cdot)$. The proof of the second point relies on a variation of the *remote control* strategy and is the *tree* counterpart of the demonstration of Theorem 5.13. Let t be a finite binary tree over $\{0, 1, 2, 3\}$. We say that t is *coherent* if for every node $v \in \mathrm{dom}(t)$, $t(v) \in \{1, 2, 3\}$ implies that all the nodes in $v1\{0,1\}^{<\omega} \cap \mathrm{dom}(t)$ have the same label, $t(v)$. Let $(\beta_n)_{n\in\omega}$ be an enumeration of the set of coherent trees, such that if t_i is a subtree of t_j, then $i \leq j$. We call β_i the i-th *bet*. A bet encodes informations on the auxiliary moves of I in the game $C((D_2(\mathbf{\Pi}^1_1), M), (D_2(\mathbf{\Pi}^1_1), M'))$: its underlying binary tree determines the part of the main run taken into account, and the values at the nodes whether this node will be killed or not, and how. Suppose I plays a conciliatory tree t. For $v = v_0 \dots v_j \in \mathrm{dom}(\beta_i)$, $\beta_i(v) = 0$ means that the node $0v_00v_1 \dots 0v_j$ stays alive, i.e. that $t_{0v_00v_1\dots0v_j1} \in A_{D_2(\mathbf{\Pi}^1_1)}$. The value 1 means that the node $0v_00v_1 \dots 0v_j$ is killed because $t_{0v_00v_1\dots0v_j10}$ and $t_{0v_00v_1\dots0v_j11}$ belong to $A_{\mathbf{\Pi}^1_1}$, so that $t_{0v_00v_1\dots0v_j1} \in A_{\check{D}_2(\mathbf{\Pi}^1_1)}$. The value 2 means that the node $0v_00v_1 \dots 0v_j$ is killed because $t_{0v_00v_1\dots0v_j10} \in A_{\mathbf{\Sigma}^1_1}$ and $t_{0v_00v_1\dots0v_j11} \in A_{\mathbf{\Pi}^1_1}$, and the value 3 means that it is killed because both $t_{0v_00v_1\dots0v_j10}$ and $t_{0v_00v_1\dots0v_j11}$ belong to $A_{\mathbf{\Sigma}^1_1}$. We say that a bet β_i is fulfilled if at the end of the game, for all $v \in \mathrm{dom}(\beta_i)$, $\beta_i(v)$ is true with respect to the conciliatory tree played by I. Notice that it is a $D_2(\mathbf{\Pi}^1_1)$ condition (it is a finite intersection of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ sets), so that II can check if a bet is fulfilled or not with an auxiliary move.

Suppose now that II has a winning strategy σ in $C(M, M')$. We describe a winning strategy σ' for II in the game $C((D_2(\mathbf{\Pi}^1_1), M), (D_2(\mathbf{\Pi}^1_1), M'))$. Each level of II's main run correspond to a bet: suppose at some point I has constructed a finite tree t for his main run, and let β_i be a bet such that $\mathrm{dom}(t) = \mathrm{dom}(\beta_i)$. On the level i of her main run, II follows σ modulo β_i, in the sense that she plays along σ as if at all the levels $j < i$ of her main run such that β_j is not a subtree of β_i, the nodes were killed, and she checks with her auxiliary moves for the nodes of the main run at this level whether β_i is fulfilled or not, so that all the nodes of her main run at this level are killed if the bet is not fulfilled. At the end of the game, a unique sequence of bets forming a chain for the inclusion is fulfilled, which contains all information about the way player I used his auxiliary moves, and which nodes he killed. Hence,

$$i^{A_{D_2(\mathbf{\Pi}^1_1)}}(\sigma' * t)_{[\ /b]} = \sigma * i^{A_{D_2(\mathbf{\Pi}^1_1)}}(t)_{[\ /b]}.$$

What completes the proof. □

A winning strategy for I in in $C(M, M')$ can also be "remote controlled" to a winning strategy for I in $C((D_2(\mathbf{\Pi}^1_1), M), (D_2(\mathbf{\Pi}^1_1), M'))$, so that the following holds.

Corollary 6.13. *Let M and M' be conciliatory languages such that $M <_c M'$. Then*

$$(D_2(\mathbf{\Pi}^1_1), M) <_c (D_2(\mathbf{\Pi}^1_1), M')$$

The operation $(D_2(\mathbf{\Pi}^1_1), \cdot)$ is much stronger than $(\mathbf{\Pi}^0_1, \cdot)$, and is in some sense a fixed point of it.

Proposition 6.14. *Let $M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$. Then*

$$\big(\mathbf{\Pi}^0_1, (D_2(\mathbf{\Pi}^1_1), M)\big) \equiv_c (D_2(\mathbf{\Pi}^1_1), M).$$

Let $\mathcal{A}$ be an automaton that recognizes $M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$. Then the conciliatory tree language $(D_2(\mathbf{\Pi}^1_1), M)$ is recognized by the automaton $\varepsilon_{\mathcal{A}}$ defined from $\mathcal{A}$ by replacing each state of $\mathcal{A}$ by a "gadget", as depicted in Fig. 6.10. As in the pseudo-exponentiation case, by replacing a state by the gadget we mean that all transitions ending in this state should now end in the initial state of the gadget, and that all the transitions starting from this state should now start from the final state of the gadget. Notice that if $M \subseteq \mathcal{T}_\Sigma^{\leq\omega}$ is of index $[0,2]$, then $(D_2(\mathbf{\Pi}^1_1), M)$ is also of index $[0,2]$, and that $W_{[0,2]}$ is a fixed point of this operation. In particular the game language $W_{[0,2]}$ is above all the differences of coanalytic sets, which is a strengthening of a result obtained by Finkel and Simonnet [37].

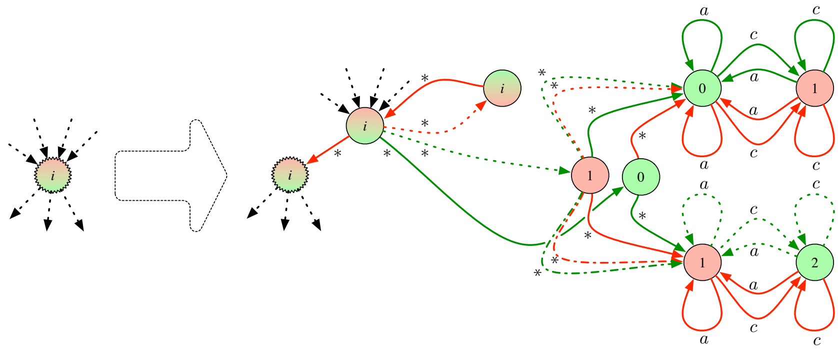


Figure 6.10: The gadget to replace a state in $\mathcal{A}$.

6.2.5 The operations $(D_2([W_{[0,n]}]), \cdot)$

Let L, M be two conciliatory languages over Σ, and let $L \Box M$ denote the *conjunctive product* of L and M, that is

$$L \Box M = \left\{ t \in \mathcal{T}_\Sigma^{\leq \omega} : t_0 \in L \text{ and } t_1 \in M \right\}.$$

Notice that the conjunctive product preserves the regularity and the index of languages, so that L and M are, respectively, of index $[\iota, \kappa]$ and $[\iota', \kappa']$, then $L \Box M$ is of index $[\min\{\iota, \iota'\}, \max\{\kappa, \kappa'\}]$.

Recall that the game languages $W_{[0,1]}$ and $W_{[1,2]}$ are, respectively, $\mathbf{\Pi}_1^1$-complete and $\mathbf{\Sigma}_1^1$-complete. The language $W_{[0,1]} \Box W_{[1,2]}$ is thus $D_2(\mathbf{\Pi}_1^1)$-complete. Hence the operation $(D_2(\mathbf{\Pi}_1^1), \cdot)$ is equivalent to the application of $W_{[0,1]} \Box W_{[1,2]}$. Building on this idea, we define for each positive integer n the operation $(D_2([W_{[0,n]}]), \cdot)$ as the application of $W_{[0,n]} \Box W_{[1,n+1]}$.

Lemma 6.15. *For every positive integer n, the following holds.*

$$W_{[0,n]} \Box W_{[0,n]} \equiv_W W_{[0,n]} \qquad \textit{and} \qquad W_{[1,n+1]} \Box W_{[1,n+1]} \equiv_W W_{[1,n+1]}$$

Proof. It is clear that $W_{[0,n]} \leq_W W_{[0,n]} \Box W_{[0,n]}$. For the converse, notice that the language $W_{[0,n]} \Box W_{[0,n]}$ is of index $[0, n]$, so that by *completeness* of $W_{[0,n]}$, we have also $W_{[0,n]} \Box W_{[0,n]} \leq_W W_{[0,n]}$. The proof for $W_{[1,n+1]}$ is similar. $\Box$

Thanks to this Lemma, we can generalize the results proved for the operation $(D_2(\mathbf{\Pi}_1^1), \cdot)$ by adapting the proof of Theorem 6.12.

Proposition 6.16. *Let $M, M' \subseteq \mathcal{T}_\Sigma^{\leq \omega}$, and n a positive integer. If $M <_c M'$, then*

(1) $(D_2(W_{[0,n]}), M)^{\complement} \equiv_c (D_2(W_{[0,n]}), M^{\complement})$;
(2) $(D_2(W_{[0,n]}), M) <_c (D_2(W_{[0,n]}), M')$;
(3) $\big(\mathbf{\Pi}^0_1, (D_2(W_{[0,n]}), M)\big) \equiv_c (D_2(W_{[0,n]}), M)$;
(4) $\big(D_2(W_{[0,m]}), (D_2(W_{[0,n]}), M)\big) \equiv_c (D_2(W_{[0,n]}), M)$, *for all* $0 < m < n$.

Let $\mathcal{A}$ be an automaton that recognizes $L \subseteq \mathcal{T}_\Sigma^{\leq\omega}$. Then the conciliatory tree language $(D_2([W_{[0,n]}]), L)$ is recognized by an automaton defined from $\mathcal{A}$ by replacing each state of $\mathcal{A}$ by a "gadget", as depicted in Fig. 6.11, where $\mathcal{W}_{[0,n]}$ and $\mathcal{W}_{[1,n+1]}$ are automata that recognize respectively the game languages $W_{[0,n]}$ and $W_{[1,n+1]}$. As in the pseudo-exponentiation case, by replacing a state by the gadget we mean that all transitions ending in this state should now end in the initial state of the gadget, and that all the transitions starting from this state should now start from the final state of the gadget. Observe

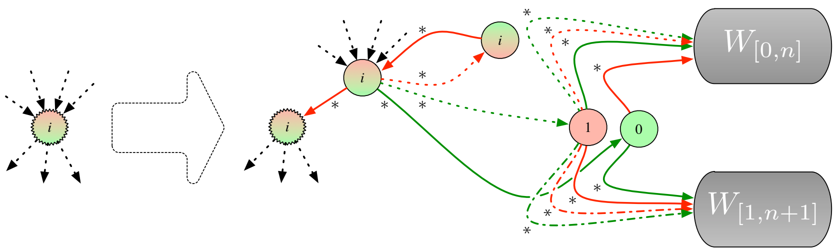


Figure 6.11: The gadget to replace a state in $\mathcal{A}$.

moreover that if L is of index $[\iota, \kappa]$ with $\iota \in \{0, 1\}$, then $(D_2([W_{[0,n]}]), L)$ is of index $[0, \max\{\kappa, n+1\}]$. In particular, languages of index $[\iota, \kappa]$ with $\iota \in \{0, 1\}$ are preserved by all the operations $(D_2([W_{[0,n]}]), L)$, with $n < \kappa$.

6.3 A fragment of the Wadge hierarchy of regular tree languages

Thanks to the operations defined above, we construct a sequence of strictly more and more complex regular languages. First, we recall the definition of the Veblen hierarchy of base ω, as introduced by Veblen [113].

Definition 6.17. The Veblen hierarchy of base ω consists of functions $(\varphi_\xi)_{\xi<\omega_1}$ from ω_1 to itself which are defined as follows:

(i) φ_0 is the exponentiation of base ω:
- $\varphi_0(0) = 1$;

– $\varphi_0(\alpha) = \omega^\alpha$ for all $0 < \alpha < \omega_1$.

(ii) For $\lambda > 0$, φ_λ is the function that enumerates the fixed points of the Veblen functions of lesser degrees. The ordinal $\varphi_\lambda(\alpha)$ is the $(1+\alpha)$-th fixed point of all φ_ξ with $\xi < \lambda$.

Every ordinal $\alpha > 0$ admits a unique Cantor normal form of base ω^ω which is an expression of the form

$$\alpha = (\omega^\omega)^{\alpha_k} \cdot \nu_k + \cdots + (\omega^\omega)^{\alpha_0} \cdot \nu_0,$$

where $k < \omega$, $0 < \nu_i < \omega^\omega$ for any $i \leq k$, and $\alpha_0 < \cdots < \alpha_k < \alpha$.

For every ordinal $0 < \alpha < \varphi_\omega(0)$, we inductively define a pair of languages $(L_\alpha, \overline{L}_\alpha)$ that are incomparable through the conciliatory ordering. If the Cantor normal form of α is $\alpha = (\omega^\omega)^{\alpha_k} \cdot \nu_k + \cdots + (\omega^\omega)^{\alpha_0} \cdot \nu_0$, we set

$$L_\alpha = L_{(\omega^\omega)^{\alpha_k}} \cdot \nu_k + \cdots + L_{(\omega^\omega)^{\alpha_0}} \cdot \nu_0,$$

and

$$\overline{L}_\alpha = L_{(\omega^\omega)^{\alpha_k}} \cdot \nu_k + \cdots + \overline{L}_{(\omega^\omega)^{\alpha_0}} \cdot \nu_0,$$

where $L_{(\omega^\omega)^{\alpha_i}}$ and $\overline{L}_{(\omega^\omega)^{\alpha_i}}$ are respectively

- the empty language $\ominus$ and the full language $\oplus$ if $\alpha_i = 0$;
- $(\mathbf{\Pi}^0_1, L_{\alpha_i})$ and $(\mathbf{\Pi}^0_1, \overline{L}_{\alpha_i})$ if $\alpha_i < (\omega^\omega)^{\alpha_i}$;
- $(D_2([W_{[0,k]}]), L_{2+\beta})$ and $(D_2([W_{[0,k]}]), \overline{L}_{2+\beta})$ if $\alpha_i = (\omega^\omega)^{\alpha_i}$ holds and $\alpha_i = \varphi_k(\beta)$ for some $0 < \beta < \alpha_i$;
- $W_{[0,k]}$ and $W_{[1,k+1]}$ if $\alpha_i = \varphi_k(0)$ for some positive integer k.

Lemma 6.18. *For $0 < \alpha < \beta < \varphi_\omega(0)$, we have*

(1) $L_\alpha \not\leq_c \overline{L}_\alpha$ *and* $\overline{L}_\alpha \not\leq_c L_\alpha$.

(2) $L_\alpha <_c L_\beta$; $\overline{L}_\alpha <_c L_\beta$; $L_\alpha <_c \overline{L}_\beta$ *and* $\overline{L}_\alpha <_c \overline{L}_\beta$.

Proof. The proof of the first part of the lemma, by induction on α, relies on the fact that the operations considered "commute" with taking the complement, see Facts 6.7 and 6.10, Proposition 6.16, and Corollary 6.9.

The proof of the second part of the lemma is also by induction on α and β, and relies on the fact that the operations preserve the relation $<_c$, see Lemma 6.8, Facts 6.10, and Corollary 6.13, and on the fact that they do not "overlap", see Propositions 6.11, 6.14 and 6.16. □

Applying the embedding $L \mapsto L^b$, we have thus generated a family $\left(L^b_\alpha\right)$ of regular languages that respects the strict Wadge ordering. Hence the main result follows.

Theorem 6.19. *There exists a family $\left(L^b_\alpha\right)_{\alpha<\varphi_\omega(0)}$ of regular tree languages such that $\alpha < \beta$ holds if and only if $L^b_\alpha <_W L^b_\beta$ holds as well.*

The height of the Wadge hierarchy restricted to regular languages is thus at least $\varphi_\omega(0)$.

6.4 Unambiguity vs. Determinism

An unambiguous automaton is a nondeterministic automaton that admits at most one accepting run on each input. By definition, the class of languages recognized by unambiguous automata includes the class of languages recognized by deterministic automata and is included in the class of languages recognized by nondeterministic automata. Depending on the context, some of these inclusions may be strict. For example, in the case of finite automata on finite words, none of these inclusions is strict, because every regular language is recognized by a deterministic finite automaton. The picture is still trivial for infinite words if we consider the parity condition, but becomes more interesting for Büchi automata. While not every regular language is recognized by deterministic Büchi automaton, it always can be recognized by an unambiguous automaton [6].

On the one hand, it is easy to observe that unambiguous automata are more expressive than the deterministic ones in this context: consider for example the language "exists exactly one branch with infinitely many labels a". On the other hand, it took a while to determine whether there are languages that are inherently ambiguous: it was shown by Niwiński and Walukiewicz [91] (later described in [22] and [23]) that unambiguous automata do not recognize all nondeterministic languages.

It is well-known that deterministic parity tree automata recognize only coanalytic sets. Since nondeterministic automata recognize some sets that are neither analytic, nor coanalytic, we use the Wadge hierarchy, the finest topological complexity measure, to address the question of the position of unambiguous languages in between deterministic and nondeterministic. By showing that some of the operations we defined, or slight modifications of them, preserve unambiguousness, we construct a sequence of unambiguous automata that recognize strictly more and more complex languages, and whose length is far beyond $(\omega^\omega)^3 + 3$, which is the height of the Wadge hierarchy of deterministic tree languages uncovered by Murlak [85]. Results in this section are a joint work with Jacques Duparc and Szczepan Hummel [31].

6.4.1 Operations and unambiguity

The sum and the pseudo-exponentiation

The operation sum is *in itself* unambiguous, so that if $\mathcal{A}$ and $\mathcal{B}$ are unambiguous, and if there exists an unambiguous $\mathcal{C}$ equivalent to the complement of $\mathcal{B}$, their sum $\mathcal{B} + \mathcal{A}$ is equivalent to an unambiguous language that we will denote by $\mathcal{B} +^u \mathcal{A}$. Moreover, if L and M are unambiguously recognizable conciliatory languages, and if the complement of M is equivalent to an unambiguously recognizable language $\check{M}$, the complement of $L + M$ is equivalent to $L + \check{M}$, which is unambiguously recognizable. We also define for every positive integer n:

$$\mathcal{B} \bullet^u n = \underbrace{\mathcal{B} +^u \cdots +^u \mathcal{B}}_{n \text{ times}}.$$

Notice that if $\mathcal{A}$ is unambiguous, then $(\omega^\omega)^{\mathcal{A}}$ is also unambiguous, so that the operation $(\mathbf{\Pi}^0_1, \cdot)$ preserves the unambiguity of tree languages.

The operation $(D^u_2(\mathbf{\Pi}^1_1), \cdot)$

Regarding the operation $(D_2(\mathbf{\Pi}^1_1), \cdot)$, we have to modify the conditions on the auxiliary moves to make it unambiguous preserving. To do so, we use the unambiguous automaton G defined by Hummel [46] and depicted in Fig. 6.12 that recognizes a $\mathbf{\Sigma}^1_1$-complete language. Notice that the complement of the language recognized by G is itself unambiguous.

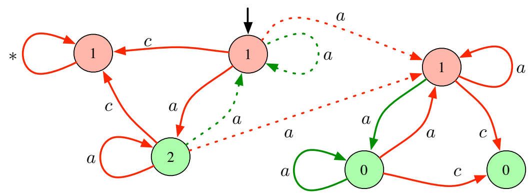


Figure 6.12: The automaton G.

From these automata we construct, in the same way that in Section 6.2.4, two unambiguous automata $\mathcal{A}^u_{D_2(\mathbf{\Pi}^1_1)}$ and $\mathcal{A}^u_{\check{D}_2(\mathbf{\Pi}^1_1)}$ that recognize respectively a $D_2(\mathbf{\Pi}^1_1)$-complete language and its complement. Now we define the operation $(D^u_2(\mathbf{\Pi}^1_1), \cdot)$ as the application of $L(\mathcal{A}^u_{D_2(\mathbf{\Pi}^1_1)})$. This operation is equivalent to $(D_2(\mathbf{\Pi}^1_1), \cdot)$, i.e. if L and M are two languages equivalent with respect to the conciliatory preorder, then $(D^u_2(\mathbf{\Pi}^1_1), L)$ and $(D_2(\mathbf{\Pi}^1_1), M)$ are also equivalent. Thus all the results proved in Section 6.2.4 hold for $(D^u_2(\mathbf{\Pi}^1_1), \cdot)$. If $\mathcal{A}$

is an unambiguous automaton that recognizes the language L, we denote by ${}^{u}\varepsilon_{\mathcal{A}}$ the unambiguous automaton that recognizes the language $(D_2^u(\mathbf{\Pi}_1^1), L)$ obtained from $\mathcal{A}$ by replacing each state of $\mathcal{A}$ by a "gadget", as depicted in Fig. 6.13.

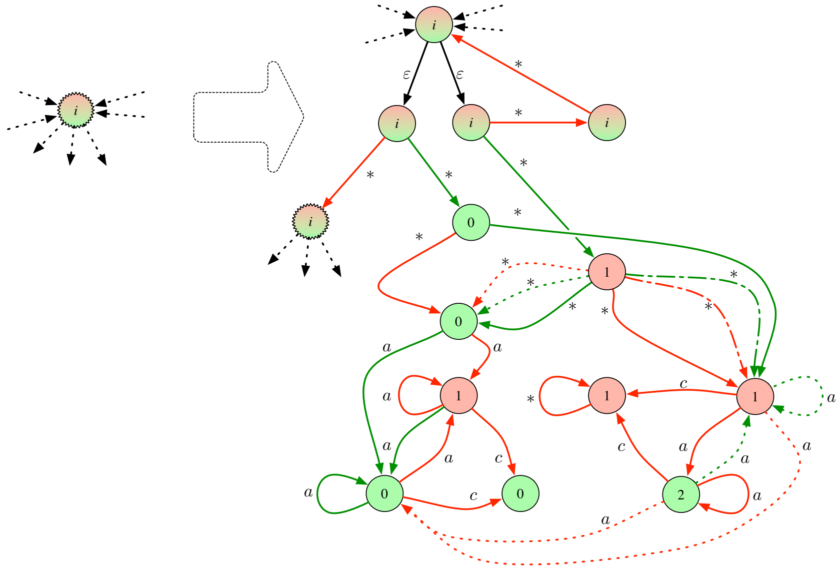


Figure 6.13: The gadget to replace a state in $\mathcal{A}$.

6.4.2 A fragment of the Wadge hierarchy of unambiguous tree languages

For every ordinal $0 < \alpha < \varphi_2(0)$, we inductively define a pair of unambiguous automata $(\mathcal{B}_\alpha, \bar{\mathcal{B}}_\alpha)$ whose languages are both non-self-dual and incomparable through the conciliatory ordering. If the CNF of α is

$$\alpha = \omega^{\alpha_k} \cdot n_k + \cdots + \omega^{\alpha_0} \cdot n_0,$$

we set

$$\mathcal{B}_\alpha = \mathcal{B}_{\omega^{\alpha_k}} \bullet^u n_k +^u \cdots +^u \mathcal{B}_{\omega^{\alpha_0}} \bullet^u n_0$$

and

$$\bar{\mathcal{B}}_\alpha = \mathcal{B}_{\omega^{\alpha_k}} \bullet^u n_k +^u \cdots +^u \bar{\mathcal{B}}_{\omega^{\alpha_0}} \bullet^u n_0,$$

where $\mathcal{B}_{\omega^{\alpha_i}}$ and $\bar{\mathcal{B}}_{\omega^{\alpha_i}}$ are respectively

- the unambiguous automaton $\ominus$ that recognizes $\ominus$, and the unambiguous automaton $\oplus$ that recognizes $\oplus$ if $\alpha_i = 0$;
- $(\omega^\omega)^{\mathcal{B}_{\alpha_i}}$ and $(\omega^\omega)^{\bar{\mathcal{B}}_{\alpha_i}}$ if $\alpha_i < \omega^{\alpha_i}$;
- ${}^u\varepsilon_{\mathcal{B}_{2+\beta}}$ and ${}^u\varepsilon_{\bar{\mathcal{B}}_{2+\beta}}$ if $\alpha_i = \omega^{\alpha_i}$ and $\alpha_i = \varepsilon_\beta$ for some $\beta < \alpha_i$.

Lemma 6.20. *Let* $0 < \alpha < \beta < \varphi_2(0)$*,*
(1) $\mathcal{B}_\alpha \not\leq_c \bar{\mathcal{B}}_\alpha$ *and* $\bar{\mathcal{B}}_\alpha \not\leq_c \mathcal{B}_\alpha$*.*
(2) $\mathcal{B}_\alpha <_c \mathcal{B}_\beta$ *;* $\bar{\mathcal{B}}_\alpha <_c \mathcal{B}_\beta$ *;* $\mathcal{B}_\alpha <_c \bar{\mathcal{B}}_\beta$ *and* $\bar{\mathcal{B}}_\alpha <_c \bar{\mathcal{B}}_\beta$*.*

Proof. It is essentially the same as the proof of Lemma 6.18. □

Applying the embedding $L \mapsto L^b$ which preserves trivially the unambiguity of languages, we have thus generated a family $\left(\mathcal{B}^b_\alpha\right)_{\alpha<\varphi_2(0)}$ of unambiguous automata that respects the strict Wadge ordering: $\alpha < \beta$ if and only if $\mathcal{B}^b_\alpha <_W \mathcal{B}^b_\beta$.

Theorem 6.21. *There exists a family* $\left(\mathcal{B}^b_\alpha\right)_{\alpha<\varphi_2(0)}$ *of unambiguous parity tree automata whose priorities are restricted to* $\{0,1,2\}$ *such that*
(1) *they recognize languages of full trees over the alphabet* $\{a,b,c\}$*;*
(2) $\alpha < \beta$ *holds if and only if* $\mathcal{B}^b_\alpha <_W \mathcal{B}^b_\beta$ *holds as well.*

Even though the exact Wadge rank of this family is unknown, this fragment of the $\mathbf{\Delta}^1_2$-Wadge hierarchy restricted to unambiguously recognizable languages climbs far above the $\mathbf{\Sigma}^1_1$ class. Moreover its length, $\varphi_2(0)$ is tremendously larger than $(\omega^\omega)^3 + 3$, which is the height of the Wadge hierarchy of deterministic tree languages uncovered by Murlak [85]. The gap between the respective topological complexity of the two considered classes of languages, measured by the difference between the height of their respective Wadge hierarchies, illustrates the discrepancy between these classes.

7 Conclusion

Throughout this thesis, our aim has been to extend the fine topological analysis of the Baire space beyond the Borel world. If a complete description of the whole Wadge hierarchy of the $\mathbf{\Delta}^1_2$ sets seems far out of reach for the time being, we have nonetheless been able to unravel the complete Wadge hierarchy of the class of increasing differences of coanalytic sets, and to give a fragment of the Wadge hierarchy of $\mathcal{R}$-sets. Moreover, we have provided an interpretation of the conciliatory ansatz used by Duparc, and applied the tools developed in the framework of descriptive set theory to automata theory. A lot of questions emerged from this work, as each answer leads us to new interrogations. A selection of those which seem to be the most immediate and promising is proposed below.

Reductions by relatively continuous relations

We proved in Chapter 3 that the conciliatory preorder is induced by reductions by relatively continuous relations, as defined by Pequignot [93], when the set $\omega^{\leq\omega}$ is endowed with the prefix topology, and that it is *not* induced by reductions by continuous functions.

Question 1. *Is there a* natural *topology on the set $\omega^{\leq\omega}$ such that the conciliatory relation would coincide with the reduction by continuous functions?*

The conciliatory hierarchy and the Wadge hierarchy restricted to non-self-dual classes coincide, so that one can also wonder if this is a characteristic property of *Conc*, or if it is shared by a wider class of topological spaces. We say that a topological space $\mathcal{X}$ has the property $(\mathcal{C})$ if for every non-self-dual Borel subset A of the Baire space, there exists a subset A' of $\mathcal{X}$ such that

$$A \preccurlyeq_W A' \qquad \text{and} \qquad A' \preccurlyeq_W A.$$

Question 2. *Does the class of all topological spaces that have the property $(\mathcal{C})$ coincide with a* natural *class of topological spaces?*

Pequignot conjectured in his PhD thesis [94, Problem 6] that all uncountable quasi-Polish spaces have the property $(\mathcal{C})$.

Another question arising from our analysis of *Conc* is whether we can generalize Theorem 3.17 to all the Borel Wadge classes. Does the result due to Wadge and which links non-self-dual pointclasses with boolean operations still hold if we move from pointclasses to initial segments for $\preccurlyeq_W$?

Question 3. *Let $\mathcal{X}$ be a second countable T_0 space. Can every non-self-dual initial segment for $\preccurlyeq_W$ be defined in terms of Boolean operations on open sets?*

The gap between $\mathrm{Diff}\,(\mathbf{\Pi}^1_1)$ and $D^*_\omega(\mathbf{\Pi}^1_1)$

Remember that from the works of Martin [77] and Harrington [42], we know that the class $D^*_{\omega^2}(\mathbf{\Pi}^1_1)$ is determined under $\mathrm{DET}(\mathbf{\Pi}^1_1)$. Hence the determinacy hypothesis $\mathrm{DET}(\mathbf{\Pi}^1_1)$ is sufficient to *fix* the structure of the Wadge hierarchy of the class $D^*_{\omega^2}(\mathbf{\Pi}^1_1)$. The fact that the natural extension of the methods used in the Borel case provides the full description of the Wadge hierarchy of the class $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ only might thus seem quite disappointing. It is nonetheless worth mentioning that the only result about the Wadge hierarchy above $\mathrm{Diff}(\mathbf{\Pi}^1_1)$, that is Theorem 5.5 due to Kechris and Martin and which states that the Wadge rank of the class $D^*_\omega(\mathbf{\Pi}^1_1)$ is $\aleph_2$, relies on (AD). It could thus very well be the case that even though the general structure of the Wadge hierarchy of the class $D^*_{\omega^2}(\mathbf{\Pi}^1_1)$ is fixed under $\mathrm{DET}(\mathbf{\Pi}^1_1)$, the precise content of the Wadge degrees and the ranks depend on the amount of determinacy and/or choice assumed. In particular, one can wonder whether a consequence of Theorem 5.5 regarding the gap between the classes $\mathrm{Diff}(\mathbf{\Pi}^1_1)$ and $D^*_\omega(\mathbf{\Pi}^1_1)$ still holds if we assume the axiom of choice and weak determinacy hypotheses.

Question 4. *Is the equality* $\mathrm{Diff}(\mathbf{\Pi}^1_1) = \Delta(D^*_\omega(\mathbf{\Pi}^1_1))$ *consistent under weaker determinacy hypothesis?*

The game quantifier

We recall the definition of the game quantifier $⅁$. Let $P \subseteq \omega^\omega \times \omega^\omega$, we put

$$⅁P = \{x \in \omega^\omega : \text{ player I has a winning strategy in the game } G(P_x)\}\,,$$

where $P_x = \{y \in \omega^\omega : (x, y) \in P\}$ is the projection of P along x. For Γ a pointclass we set

$$\Game\Gamma = \{\Game P : P \in \Gamma\}.$$

Observe that $\Game\Gamma$ is also a pointclass, and that $\Game$ preserves the inclusion of pointclasses. Moreover, if $\Gamma \subset \Gamma'$ are Borel non-self-dual pointclass, then $\Game\Gamma$ and $\Game\Gamma'$ are non-self-dual pointclasses, and $\Game\Gamma \subset \Game\Gamma'$ (see for example [36]).

This operation on sets, also called the game-theoretical projection, links together the class of Borel sets and the class of $\boldsymbol{\Delta}^1_2$ sets.

Theorem 7.1 (Burgess [18, 19, 20]).
- $\Game\boldsymbol{\Delta}^0_1 = \boldsymbol{\Delta}^1_1$;
- $\Game D_\eta(\boldsymbol{\Sigma}^0_1) = \boldsymbol{\Pi}^{\boldsymbol{C}}_\eta$, *for all* $0 < \eta < \omega_1$;
- $\Game\boldsymbol{\Delta}^0_2 = \boldsymbol{C}$;
- $\Game D_\eta(\boldsymbol{\Sigma}^0_2) = \boldsymbol{\Pi}^{\mathcal{R}}_{1+\eta}$, *for all* $0 < \eta < \omega_1$;
- $\Game\boldsymbol{\Delta}^0_3 = \boldsymbol{\mathcal{R}}$;
- $\Game\boldsymbol{\Delta}^1_1 \subset \boldsymbol{\Delta}^1_2$.

Note that the last inclusion is strict. This correspondence provides us with a new formulation of our work, but also opens other perspectives. First, we can wonder whether the fragment of the Wadge hierarchy of $\mathcal{R}$-sets we describe in Section 5.3 allows us to complete the correspondence for all the $\boldsymbol{\Delta}^0_3$ non-self-dual pointclasses.

Question 5. *Is there, for every $\boldsymbol{\Delta}^0_3$ non-self-dual pointclass Γ that contains the classes $\boldsymbol{\Sigma}^0_2$ and $\boldsymbol{\Pi}^0_2$, an ordinal $0 < \alpha < V^{\omega_1+\omega_1+\omega_1}(2)$ such that $\Omega(\alpha)^b$ or $(\Omega(\alpha)^b)^\complement$ is $\Game\Gamma$-complete?*

If the answer to this question is negative, we could add these "new" pointclasses to our fragment, and maybe apply our operations to them to generate a finer fragment of the Wadge hierarchy of $\mathcal{R}$-sets. Applying the game quantifier to the Borel pointclasses above $\boldsymbol{\Delta}^0_3$ would also provide benchmarks for the study of the Wadge hierarchy of the $\boldsymbol{\Delta}^1_2$ sets above the $\mathcal{R}$-sets.

Automata theory and regular languages

In Chapter 6, we have unraveled a fragment of the Wadge hierarchy restricted to regular languages. This fragment is incomplete: one can prove[1] indeed that the non-self-dual language D_2 defined by Arnold and Santocanale [10], and

[1] See Duparc, Facchini, Fournier and Michalewski [28].

which is strictly below $W_{[0,2]}$ but not in the class $Comp_1$, is not reached by our construction. One can nonetheless conjecture that our bound for the Wadge rank of the class of regular languages is sharp.

Question 6. *Is the Wadge rank of all the regular languages exactly $\varphi_\omega(0)$?*

A lot of questions remain open on the class of regular tree languages. For example, Skurczyński [104] proved that, for every positive integer n, there exist a $\mathbf{\Sigma}^0_n$-complete and a $\mathbf{\Pi}^0_n$-complete regular tree languages, but it is still an open question to know whether there exists some regular tree languages which are of infinite Borel rank. Concerning decidability, Bojańczyk and Place [14] proved that one can decide whether a language recognized by a tree automaton is a Boolean combination of open sets, a result extended to the $\mathbf{\Delta}^0_2$ class by Facchini and Michalewski [35], but the question remains open for larger pointclasses.

Regarding unambiguous languages, the lack of examples of higher complexity, and in particular of index greater than $[0,2]$, blocks our investigations. But if some unambiguous languages above our constructions were to be found, we could apply our methods and operations to derive a longer sequence of unambiguous regular languages. Our constructions nonetheless provide benchmarks for the study of unambiguous languages, and could lead to algorithmic results for this class. It might, for example, help determine whether the unambiguity of a given language is decidable. The result also could contribute to the resolution of the unambiguous index problem as it can help in characterizing unambiguous languages of index $[0,2]$.

Game languages of infinite rank

Finally let us consider the following variant of the parity game where the codomain of the ranking function r is ω_1, and its range may be infinite. If a play in this game is infinite, player $\exists$ wins the game if and only if the *lowest* rank occurring infinitely often is even. We call these games *generalized parity games*, and they allow us to construct game languages of infinite rank.

Consider the alphabet $\Sigma_{[\iota,\kappa]} = \{\exists,\forall\} \times \{\iota,\dots,\kappa\}$ with $\iota \in \{0,1\}$ and $\iota \leq \kappa < \omega_1$. For each tree $t \in \mathcal{T}_{\Sigma_{[\iota,\kappa]}}$ we define the generalized parity game $\mathcal{G}'_t = (V, V_\exists, V_\forall, E, p_0, \mathrm{r})$ as follows:

- $V_\exists = \{v \in \{0,1\}^* : t(v)_0 = \exists\}$;
- $V_\forall = \{v \in \{0,1\}^* : t(v)_0 = \forall\}$;
- $E = \{(w, wi) : w \in \{0,1\}^* \text{ and } i \in \{0,1\}\}$;

– $p_0 = \varepsilon$;
– $\mathrm{r}(v) = t(v)_1$, for each $v \in \{0,1\}^*$.

The generalized game language of index $[\iota,\kappa]$, denoted by $W'_{[\iota,\kappa]}$, corresponds to the class of full trees in $\mathcal{T}_{\Sigma_{[\iota,\kappa]}}$ for which player $\exists$ has a winning strategy in the corresponding generalized parity game $\mathcal{G}'_t$. Notice that for κ finite we have

$$W_{[\iota,\kappa]} = W'_{\overline{[\iota,\kappa]}},$$

and that for κ finite or infinite, if $[\iota,\kappa] \sqsubset [\iota',\kappa']$, then $W'_{[\iota,\kappa]} <_W W'_{[\iota',\kappa']}$.

Of course, the languages $W'_{[\iota,\kappa]}$ are not regular when κ is infinite. It would be nonetheless interesting to study their topological complexity. One could guess that they are all in the $\mathbf{\Delta}^1_2$ class, but we can hope for more and maybe to generalize the result of Gogacz, Michalewski, Mio and Skrzypczak [40].

Question 7. *Let $\kappa > 0$, is the game language $W'_{[0,\kappa]}$ complete for the κ-th level of the Kolmogorov hierarchy of $\mathcal{R}$-sets?*

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“Che io forse abbia amato tanto la sigaretta per poter riversare su di essa la colpa della mia incapacità? Chissà se cessando di fumare io sarei divenuto l’uomo ideale e forte che m’aspettavo? Forse fu tale dubbio che mi legò al mio vizio perché è un modo comodo di vivere quello di credersi grande di una grandezza latente.”

Italo Svevo, *La coscienza di Zeno.*

“What we’re really doing is imposing our own conceptual limitations on a subject that defies inclusion within the borders of our present knowledge. We’re talking around it. We’re making sounds to comfort ourselves. We’re trying to peel skin off a rock. But this [...] is simply what we do to keep from going mad.”

Don DeLillo, *Ratner’s Star.*