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On the non-torsion elements in the algebraic K-theory of rings of integers

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Introduction

The first goal of this paper is to investigate the problem of the image of non-torsion elements of $K_{2n-1}(O_F)$ in $K_{2n-1}(k_v)$ via the reduction map, where O_F is the ring of integers in a number field F, k_v a residue field, and n a positive integer. Our main result is:

Theorem 1. *For any number field F and for any integer* $n \geqq 1$, *the kernel of the reduction map*

$$K_{2n-1}(O_F) \to \prod_v K_{2n-1}(k_v)$$

is finite.

Sections 1 and 2 present a proof of this theorem. We then consider in Section 3 the behaviour of the non-torsion elements in the algebraic K-theory of number fields with respect to fields extensions. In particular, the following result is proved.

Theorem 2. *Let* E/F *be a finite Galois extension of number fields. Then for all integers* $n \geqq 2$, *the natural map*

$$K_{2n-1}(F)/\,\text{torsion} \to (K_{2n-1}(E)/\,\text{torsion})^{G(E/F)}$$

is injective and its cokernel is finite with the property that the order of its odd torsion subgroup is only divisible by prime numbers that divide the order of the torsion subgroup of $K_{2n-1}(E)$.

In Sections 4 and 5, we then observe that the argument of the proof of Theorem 1 provides a better understanding of the homotopy type of the space $BSL(O_F)^+$ in the case

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where O_F is the ring of integers $\mathbb{Z}$, because it enables us to get upper and lower bounds for the order of the Postnikov k-invariants of $BSL(\mathbb{Z})^+$ (see Propositions 5 and 6 and Corollaries 2 and 4); some of these results depend on the Quillen-Lichtenbaum conjecture for $\mathbb{Z}$. Furthermore, we study the image of the non-torsion elements of $K_i(\mathbb{Z})$ under the Hurewicz homomorphism

$$h_i: K_i(\mathbb{Z}) \to H_i(BSL(\mathbb{Z})^+; \mathbb{Z}) \cong H_i(SL(\mathbb{Z}); \mathbb{Z})$$

for all integers i for which $K_i(\mathbb{Z})$ is infinite (see Corollary 3). In particular, Section 6 gives a complete determination of the 5-dimensional Hurewicz homomorphism for the space $BSt(\mathbb{Z})^+$, where $St(\mathbb{Z})$ denotes the infinite Steinberg group over $\mathbb{Z}$:

Theorem 3. *There is a short exact sequence*

$$0 \to K_5(\mathbb{Z}) \xrightarrow{h_5} H_5(St(\mathbb{Z}); \mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

where h_5 is the Hurewicz homomorphism. Furthermore, the torsion subgroups of $K_5(\mathbb{Z})$ and of $H_5(St(\mathbb{Z}); \mathbb{Z})$ are isomorphic.

Throughout the paper, let us denote, for a prime number p, the p-primary part of a positive integer N by $|N|_p^{-1}$.

1. Basic K-theory and étale cohomology computations

Let F be a number field, $O_F = O$ its ring of integers and n an integer $\geqq 2$. Let $K_*(-)$ (respectively $K_*^{\text{ét}}(-)$) denote the Quillen (respectively étale) K-theory and $H_{\text{ét}}^*(-; \mathbb{Z}/M(n))$ (respectively $H_{\text{cont}}^*(-; \mathbb{Z}_p(n))$) denote the étale (respectively continuous, see [J]) cohomology. Choose an odd prime number p such that p does not divide the order of the torsion subgroup of $K_{2n-1}(O)$. The finite generation of $K_{2n-1}(O)$ and the choice of p give the free $\mathbb{Z}_p$-module

$$(K_{2n-1}(O)/\text{torsion}) \otimes \mathbb{Z}_p \cong K_{2n-1}(O) \otimes \mathbb{Z}_p.$$

Observe also that the natural map $K_{2n-1}(O)/\text{torsion} \to (K_{2n-1}(O)/\text{torsion}) \otimes \mathbb{Z}_p$ is an injection. Hence every non-torsion element in $K_{2n-1}(O)$ gives rise to a non-trivial element in $K_{2n-1}(O) \otimes \mathbb{Z}_p$.

Let x be an arbitrary non-torsion element in $K_{2n-1}(O)$ and denote in the same way its image in $K_{2n-1}(O) \otimes \mathbb{Z}_p$. Let p^s be the highest power of p such that there is an element y in $K_{2n-1}(O) \otimes \mathbb{Z}_p$ with $p^s y = x$. Choose an arbitrary (big in general) power of p, say $M = p^m$ such that $m > s$, and consider the quotient modulo M

$$K_{2n-1}(O)/M \cong (K_{2n-1}(O) \otimes \mathbb{Z}_p)/M$$

which is a free $\mathbb{Z}/M$-module. The image of x in this module (which will also be denoted by x) has order M/p^s.

It follows from Theorem 8.7 of [DF1] that the map $K_{2n-1}(O) \otimes \mathbb{Z}_p \to K_{2n-1}^{\text{ét}}(O[\frac{1}{p}])$ is surjective for any odd p. So, Theorem 1 of [S1] implies that the homomorphism

$$(K_{2n-1}(O) \otimes \mathbb{Z}_p)/\text{torsion} \to K_{2n-1}^{\text{ét}}(O[\tfrac{1}{p}])/\text{torsion}$$

is an isomorphism. Hence we have the following isomorphism (for our choice of p): $K_{2n-1}(O) \otimes \mathbb{Z}_p \xrightarrow{\cong} K^{\text{ét}}_{2n-1}(O[\frac{1}{p}])$. The localization exact sequence in algebraic K-theory (see Section 5 of [Q1] and Théorème 1 of [S2]) show that $K_{2n-1}(O) \cong K_{2n-1}(O_S)$ for any finite set of prime numbers S. We can also check (cf. [Ba], p. 290) that $K^{\text{ét}}_{2n-1}(O[\frac{1}{p}]) \cong K^{\text{ét}}_{2n-1}(O_S)$ for any finite set of prime numbers S containing p. Thus, the Dwyer-Friedlander spectral sequence (see [DF1], Remark 8.8) shows that $K^{\text{ét}}_{2n-1}(O_S) \cong H^1_{\text{cont}}(O_S; \mathbb{Z}_p(n))$ and this isomorphism is obviously functorial in S by morphisms of Dwyer-Friedlander spectral sequences.

Let E be the field $F(\mu_M)$, O_E its ring of integers, and let $G(E/F)$ denote the Galois group of E over F. Choose now a finite set of prime numbers S containing p such that $\operatorname{Pic}(O_{E,S}) = 0$ and consider the following commutative diagram:

Diagram 1

$$\begin{array}{ccc} H^1_{\text{cont}}(O_S; \mathbb{Z}_p(n))/M & \longrightarrow & H^1_{\text{ét}}(O_S; \mathbb{Z}/M(n)) \\ \downarrow & & \downarrow \\ H^1_{\text{cont}}(O_{E,S}; \mathbb{Z}_p(n))/M & \longrightarrow & H^1_{\text{ét}}(O_{E,S}; \mathbb{Z}/M(n)) \end{array} .$$

The horizontal arrows come from the Bockstein sequences in étale K-theory (later identified with continuous and étale cohomology respectively). They are obviously injective.

The Hochschild-Serre spectral sequence

$$E_2^{r,s} \cong H^r(G(E/F); H^s_{\text{ét}}(O_{E,S}; \mathbb{Z}/M(n))) \Rightarrow H^{r+s}_{\text{ét}}(O_S; \mathbb{Z}/M(n))$$

gives the exact sequence

$$0 \to H^1(G(E/F); \mathbb{Z}/M(n)) \to H^1_{\text{ét}}(O_S; \mathbb{Z}/M(n)) \to H^1_{\text{ét}}(O_{E,S}; \mathbb{Z}/M(n))^{G(E/F)} .$$

On the other hand, by Herbrand quotient we have

$$\# H^1(G(E/F); \mathbb{Z}/M(n)) = \# H^0_{\text{Tate}}(G(E/F); \mathbb{Z}/M(n)) .$$

In addition:

$$\begin{aligned} \# H^0_{\text{Tate}}(G(E/F); \mathbb{Z}/M(n)) &\leqq \# H^0(G(E/F); \mathbb{Z}/M(n)) = \# H^0(G(F(\mu_{p^\infty})/F); \mathbb{Z}/M(n)) \\ &\leqq \# H^0(G(F(\mu_{p^\infty})/F); \mathbb{Q}_p/\mathbb{Z}_p(n)) = |w_n(F)|_p^{-1} . \end{aligned}$$

However, the surjectivity of the map

$$K_{2n-1}(O) \otimes \mathbb{Z}_p \to H^1_{\text{cont}}(O[\tfrac{1}{p}]; \mathbb{Z}_p(n))$$

and some additional étale cohomology computations (see [S1], p. 376) show that we have a surjective map:

$$K_{2n-1}(O)_p \to H^0_{\text{ét}}(O[\tfrac{1}{p}]; \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong H^0(G(F(\mu_{p^\infty})/F); \mathbb{Q}_p/\mathbb{Z}_p(n)) .$$

So, by our choice of p, $K_{2n-1}(O)_p$ is trivial and consequently, $|w_n(F)|_p^{-1} = 1$. Hence $H^1(G(E/F); \mathbb{Z}/M(n)) = 0$ and the right vertical arrow in Diagram 1 is injective.

Let v be a place of F corresponding to a prime ideal not over any prime in S. Let w be a place of E over v. We have the following commutative diagram, where π_v and π_w are the reduction mod v and mod w respectively:

Diagram 2

$$\begin{array}{ccc}
K_{2n-1}(O_S)/M & \xrightarrow{\pi_v} & K_{2n-1}(k_v)/M \\
\downarrow\cong & & \downarrow \\
(K_{2n-1}(O_S)\otimes\mathbb{Z}_p)/M & \xrightarrow{\pi_v} & (K_{2n-1}(k_v)\otimes\mathbb{Z}_p)/M \\
\downarrow\cong & & \downarrow \\
H^1_{\mathrm{cont}}(O_S;\mathbb{Z}_p(n))/M & \xrightarrow{\pi_v} & H^1_{\mathrm{cont}}(k_v;\mathbb{Z}_p(n))/M \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(O_S;\mathbb{Z}/M(n)) & \xrightarrow{\pi_v} & H^1_{\text{ét}}(k_v;\mathbb{Z}/M(n)) \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(O_{E,S};\mathbb{Z}/M(n)) & \xrightarrow{\pi_w} & H^1_{\text{ét}}(k_w;\mathbb{Z}/M(n))\,.
\end{array}$$

As explained above, the third and fourth (from top) left vertical arrows are injections. Thus, we have proved the following

Lemma 1. *The composition of the four left vertical arrows in Diagram 2 is injective.*

Using now the exact sequence $0 \to \mathbb{Z}/M(1) \to G_m \to G_m \to 0$ of sheaves for étale topology on $\operatorname{spec} O_{E,S}$, we get an isomorphism $O_{E,S}^*/O_{E,S}^{*M} \xrightarrow{\cong} H^1_{\text{ét}}(O_{E,S}; \mathbb{Z}/M(1))$ because of the condition $\operatorname{Pic}(O_{E,S}) = 0$. Tensoring it with $\mathbb{Z}/M(n-1)$, we get the isomorphism

$$O_{E,S}^*/O_{E,S}^{*M} \otimes \mathbb{Z}/M(n-1) \xrightarrow{\cong} H^1_{\text{ét}}(O_{E,S}; \mathbb{Z}/M(n))\,,$$

and similarly

$$k_w^*/k_w^{*M} \otimes \mathbb{Z}/M(n-1) \xrightarrow{\cong} H^1_{\text{ét}}(k_w; \mathbb{Z}/M(n))\,.$$

Observe that the bottom horizontal arrow in Diagram 2 is easily identified with the map

$$O_{E,S}^*/O_{E,S}^{*M} \otimes \mathbb{Z}/M(n-1) \xrightarrow{\pi_w} k_w^*/k_w^{*M} \otimes \mathbb{Z}/M(n-1)\,.$$

2. An application of Tchebotarev's density theorem

Let ξ be a generator of $\mathbb{Z}/M(1)$. Then $\xi^{\otimes(n-1)}$ is a generator of $\mathbb{Z}/M(n-1)$. Let $\beta \otimes \xi^{\otimes(n-1)}$ be the image in $O_{E,S}^*/O_{E,S}^{*M} \otimes \mathbb{Z}/M(n-1)$ of the element x (introduced in Section 1) of $K_{2n-1}(O)/M$ via the left vertical arrows in Diagram 2. The element β above is considered to be as usual the image of an element (also denoted by) β from $O_{E,S}^*$. Lemma 1 shows that the order of $\beta \otimes \xi^{\otimes(n-1)}$ is still M/p^s. We want to prove that the image of $\beta \otimes \xi^{\otimes(n-1)}$ via the bottom horizontal arrow in Diagram 2 has also order M/p^s for some choices of w. To prove it, we use an argument of Rubin ([Ru], p. 319) in application of Tchebotarev's density theorem.

Observe that for our purpose it is enough to consider the map

$$\pi_w : O^*_{E,S} / O^{*M}_{E,S} \to k^*_w / k^{*M}_w .$$

The element β in $O^*_{E,S} / O^{*M}_{E,S}$ has order M/p^s. So it is enough to prove that the image of β via the above map π_w has order M/p^s. Let t be the highest power of p such that

$$\beta \in O^{*t}_{E,S} / O^{*M}_{E,S} .$$

Note that the map $O^*_{E,S}/O^{*M}_{E,S} \to E^*/E^{*M}$ induced by the inclusion $O_E \hookrightarrow E$ is an imbedding.

Lemma 2. *The integer t is also the highest power of p such that $\beta \in E^{*t}/E^{*M}$. In addition, β has order M/t in $O^*_{E,S}/O^{*M}_{E,S}$ and in E^*/E^{*M}. Hence $t = p^s$.*

Proof. Indeed, if t_1 is the maximum power of p such that $\beta \in E^{*t_1}/E^{*M}$ then $\beta = y^{t_1} z^M$ for some y and z in E^*. Hence $\beta = (yz^{M/t_1})^{t_1}$ and obviously $yz^{M/t_1} \in O^*_{E,S}$. So $t_1 | t$ by definition of t. On the other hand, $t | t_1$ by definition of t_1. Let some power of p (call it t_1 again) be the order of β in $O^*_{E,S}/O^{*M}_{E,S}$. Then $\beta^{t_1} = z^M$ for $z \in O^*_{E,S}$. So $\beta = z^{M/t_1}\eta$ where $\eta \in E^*$ and $\eta^{t_1} = 1$. But all roots of unity of order M are in E, so there is a root of unity $\xi \in E^*$ such that $\eta = \xi^{M/t_1}$. Hence $\beta = (z\xi)^{M/t_1}$, where obviously $z\xi \in O^*_{E,S}$. By definition of t we have $(M/t_1)|t$. Hence $(M/t)|t_1$. But M/t annihilates β hence $t_1|(M/t)$ by definition of t_1.

Lemma 3. *There are infinitely many prime ideals w in O_E such that $\pi_w(\beta)$ has order M/t in k^*_w/k^{*M}_w.*

Proof. Let W denote the subgroup of $O^*_{E,S}/O^{*M}_{E,S}$ (or E^*/E^{*M}) generated by β. By Kummer pairing we have the following isomorphism:

$$G(E(\beta^{1/M})/E) \cong \mathrm{Hom}(W; \mathbb{Z}/M(1)) .$$

It shows that $G(E(\beta^{1/M})/E)$ is cyclic of order M/t by Lemma 2. Let L denote $E(\beta^{1/M})$, O_L its ring of integers and δ a generator of $G(E(\beta^{1/M})/E)$. By Tchebotarev's density theorem, there are infinitely many prime ideals w of E such that $Fr_w = \delta$. Let l be the prime number in $\mathbb{Z}$ below w (but consider only these prime numbers l not in S) and let $\tilde{w}$ be a prime ideal in L over w. Observe that Fr_w has order M/t and generates the Galois group $G(k_{\tilde{w}}/k_w)$ also of order M/t. For an element α of $O_{E,S}$ (respectively $O_{L,S}$) let us write $\bar{\alpha}$ for its image in k_w (respectively $k_{\tilde{w}}$). Let t_1 denote the order of $\bar{\beta}$ in k^*_w/k^{*M}_w. Hence $\bar{\beta}^{t_1} = \bar{z}^M$ for some $\bar{z} \in k^*_w$. Therefore, the element $(\beta^{1/M})^{t_1}$ of $O^*_{L,S}$ maps to an element of $k^*_{\tilde{w}}$ coming from k^*_w because the roots of unity of order M are in k^*_w. We have the following equalities in $k_{\tilde{w}}$:

$$\overline{(\beta^{1/M})^{t_1}} = Fr_w\big(\overline{(\beta^{1/M})^{t_1}}\big) = \overline{Fr_w(\beta^{1/M})^{t_1}} = \overline{\delta(\beta^{1/M})^{t_1}} = \overline{\xi^{t_1}(\beta^{1/M})^{t_1}} ,$$

where ξ is a primitive root of unity of order M/t in $O^*_{E,S}$ by definition of δ. These equalities show that $\bar{\xi}^{t_1} = 1$. But as is well known, $\bar{\xi}$ has also order M/t. Hence (M/t) divides t_1. In addition, the order of $\bar{\beta}$ in k^*_w/k^{*M}_w divides M/t. Consequently, $M/t = t_1$.

Diagram 2 and Lemmas 1 and 3 then imply:

Proposition 1. *Let O_F be the ring of integers in a number field F and n an integer $\geqq 2$. Then, for any non-torsion element x of $K_{2n-1}(O_F)$ and any odd prime number p such that $K_{2n-1}(O_F)$ contains no p-torsion, there are infinitely many prime ideals v of O_F such that the order of the cyclic group $K_{2n-1}(k_v)_p$ is arbitrarily large and such that x maps to a non-trivial element of order $(\# K_{2n-1}(k_v)_p)/p^s$ via the reduction map*

$$K_{2n-1}(O_F) \to K_{2n-1}(k_v)_p\,,$$

where p^s is described at the beginning of Section 1. *In particular, if x maps to a generator of an infinite cyclic direct summand of $K_{2n-1}(O_F)/$*torsion, *then $p^s = 1$: hence there are infinitely many prime ideals v such that x maps to a generator of $K_{2n-1}(k_v)_p$.*

Theorem 1. *Let O_F be the ring of integers in a number field F and n an integer $\geqq 1$. Then, the kernel of the reduction map*

$$K_{2n-1}(O_F) \to \prod_v K_{2n-1}(k_v)$$

is finite.

Proof. If $n \geqq 2$, this is an immediate consequence of the previous proposition. If $n = 1$, let $x \in K_1(O_F)$ be an element of the kernel of the reduction map; then $x \equiv 1 \bmod v$ for every prime ideal v, hence $x - 1$ is divisible by all v and therefore $x = 1$. (Unfortunately, we cannot obtain an assertion analogous to Proposition 1 for $n = 1$.)

Remark 1. In [A2], the first author considered the images of the non-torsion elements of $K_5(\mathbb{Z})$ in $K_5(\mathbb{F}_v)$ and proved that they may be non-trivial in general.

Remark 2. For an odd prime p which is not relatively prime to the order of the torsion subgroup of $K_{2n-1}(O_F)$, the reduction map $K_{2n-1}(O_F) \to \prod_v K_{2n-1}(k_v)_p$ has still finite kernel, though we are not able to prove in this way that we can map a non-torsion element to a generator of $K_{2n-1}(k_v)_p$ for this p as stated in Proposition 1.

3. Some remarks about non-torsion elements in odd K-groups of number fields

The main objective of this section is to prove the following result.

Theorem 2. *Let E/F be a finite Galois extension of number fields. Then for all integers $n \geqq 2$, the natural map*

$$K_{2n-1}(F)/\text{torsion} \to (K_{2n-1}(E)/\text{torsion})^{G(E/F)}$$

is injective and its cokernel is finite with the property that the order of its odd torsion subgroup is only divisible by prime numbers that divide the order of the torsion subgroup of $K_{2n-1}(E)$.

Proof. Let p be an odd prime. The extension E/F gives a map $i : \operatorname{spec} E \to \operatorname{spec} F$ which induces homomorphisms $i^* : K_m(F) \to K_m(E)$ and $i_* : K_m(E) \to K_m(F)$ for all integers $m \geqq 0$ such that

$$i_* i^* = [E:F] \cdot \mathrm{id} \quad \text{and} \quad i^* i_* = \sum_{\sigma \in G(E/F)} \sigma =: \mathrm{tr}_G .$$

Hence we see that

$$K_m(F) \otimes \mathbb{Q} \xrightarrow[\cong]{i^*} \mathrm{tr}_G(K_m(E) \otimes \mathbb{Q}) = (K_m(E) \otimes \mathbb{Q})^{G(E/F)} .$$

In addition,

$$(K_m(E) \otimes \mathbb{Q})^{G(E/F)} \cong K_m(E)^{G(E/F)} \otimes \mathbb{Q} ,$$

because it is actually true for any finitely generated free $\mathbb{Z}$-module instead of $\mathbb{Q}$ (with trivial $G(E/F)$-action) and consequently also for any abelian torsion-free group since such a group is a direct limit of its finitely generated subgroups (note that $G(E/F)$ is finite). So we obtain $\mathrm{rank}_{\mathbb{Z}}(K_m(F)/\mathrm{torsion}) = \mathrm{rank}_{\mathbb{Z}}(K_m(E)/\mathrm{torsion})^{G(E/F)}$ and deduce that the map

$$K_m(F)/\mathrm{torsion} \to (K_m(E)/\mathrm{torsion})^{G(E/F)}$$

is injective and has finite cokernel since it is an isomorphism after tensoring with $\mathbb{Q}$.

Now, take $m = 2n-1$ for $n \geqq 2$ and consider the commutative diagram

Diagram 3

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
A & \longrightarrow & B^{G(E/F)} \\
\downarrow & & \downarrow \\
K_{2n-1}(F) \otimes \mathbb{Z}_p & \longrightarrow & (K_{2n-1}(E) \otimes \mathbb{Z}_p)^{G(E/F)} \\
\downarrow & & \downarrow \\
H^1_{\mathrm{cont}}(F; \mathbb{Z}_p(n)) & \longrightarrow & H^1_{\mathrm{cont}}(E; \mathbb{Z}_p(n))^{G(E/F)} , \\
\downarrow & & \\
0 & &
\end{array}$$

where A and B are the kernels of the corresponding obvious maps. The spectral sequence (see [J])

$$E_2^{r,s} \cong H^r(G(E/F); (H^s_{\text{ét}}(E; \mathbb{Z}/p_k(n)))_k) \Rightarrow H^{r+s}_{\mathrm{cont}}(F; \mathbb{Z}_p(n))$$

gives the following exact sequence

$$0 \to H^1(G(E/F); 0) \to H^1_{\mathrm{cont}}(F; \mathbb{Z}_p(n)) \to H^1_{\mathrm{cont}}(E; \mathbb{Z}_p(n))^{G(E/F)} \to H^2(G(E/F); 0)$$

which shows that the bottom horizontal arrow in this diagram is an isomorphism and that the bottom right vertical map is surjective. By Theorem 1 of [S1] and the Dwyer-Friedlander spectral sequence, we know that $A \subset K_{2n-1}(F)_p$ and $B \subset K_{2n-1}(E)_p$ (see Section 1). Now, let us assume that p does not divide the order of the torsion subgroup of $K_{2n-1}(E)$. Then, $B = 0$ and we deduce from the snake lemma that the middle horizontal arrow in the above diagram is surjective. In the same way as before we see that

$$(K_{2n-1}(E) \otimes \mathbb{Z}_p)^{G(E/F)} \cong K_{2n-1}(E)^{G(E/F)} \otimes \mathbb{Z}_p .$$

Therefore, the map

$$(K_{2n-1}(F)/\text{torsion}) \otimes \mathbb{Z}_p \to (K_{2n-1}(E)/\text{torsion})^{G(E/F)} \otimes \mathbb{Z}_p$$

is surjective. This implies that the cokernel of the map

$$K_{2n-1}(F)/\text{torsion} \to (K_{2n-1}(E)/\text{torsion})^{G(E/F)}$$

has order (up to 2-torsion) divisible only by primes dividing the order of the torsion subgroup of $K_{2n-1}(E)$.

Remark 3. The assertion of Theorem 2 holds also if we replace the number fields by their rings of integers since $K_{2n-1}(O_F) \cong K_{2n-1}(F)$ for all $n \geqq 2$ according to the localization exact sequence in algebraic K-theory (see Section 5 of [Q1]).

Corollary 1. *Let E_i/F be a direct system of finite extensions of a number field F. Let $\tilde{E} = \varinjlim E_i$. Then the natural map*

$$K_{2n-1}(F)/\text{torsion} \to K_{2n-1}(\tilde{E})/\text{torsion}$$

is an injection for all integers $n \geqq 2$. (In particular if E_i goes through all finite extensions of F, then $\tilde{E} = \bar{F}$.)

We would like to finish this section with some results about the non-divisibility property for non-torsion elements in odd K-groups of number fields.

Proposition 2. *Let F be a number field, p an odd prime number and n an integer $\geqq 2$. Then, no non-torsion element of $K_{2n-1}(F(\mu_{p^\infty}))$ is divisible in this group.*

Proof. Let $M = p^m$ and $E = F(\mu_{p^k})$ for $k \geqq m$. Consider the following commutative diagram:

Diagram 4

$$\begin{array}{ccc}
K_{2n-1}(F)/M & \longrightarrow & K_{2n-1}(E)/M \\
\downarrow & & \downarrow \\
(K_{2n-1}(F) \otimes \mathbb{Z}_p)/M & \longrightarrow & (K_{2n-1}(E) \otimes \mathbb{Z}_p)/M \\
\downarrow & & \downarrow \\
H^1_{\text{cont}}(F; \mathbb{Z}_p(n))/M & \longrightarrow & H^1_{\text{cont}}(E; \mathbb{Z}_p(n))/M \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(F; \mathbb{Z}/M(n)) & \longrightarrow & H^1_{\text{ét}}(E; \mathbb{Z}/M(n)) .
\end{array}$$

The top left vertical arrow is an isomorphism. Observe that the middle left vertical arrow has finite kernel of order $\leqq \# K_{2n-1}(F)_p$. The bottom left vertical arrow is injective. In the same way as in Section 1, we may conclude that $\# H^1(G(E/F); \mathbb{Z}/M(n)) \leqq |w_n(F)|_p^{-1}$ and consequently that the bottom horizontal arrow has kernel of order $\leqq |w_n(F)|_p^{-1}$. Let x be a non-torsion element in $K_{2n-1}(F)$ and choose $M = p^m$ such that the image of x in $K_{2n-1}(F)/M$ has order bigger than the product $\# K_{2n-1}(F)_p |w_n(F)|_p^{-1}$. Then the image of x (through Diagram 4) in $H^1_{\text{ét}}(E; \mathbb{Z}/M(n))$ is non-trivial. Therefore, the image of x in $K_{2n-1}(E)/M$ is non-trivial for each $k \geqq m$.

4. The Postnikov invariants and the Hurewicz homomorphism for $BSL(\mathbb{Z})^+$

Let $BGL(\mathbb{Z})^+$, respectively $BSL(\mathbb{Z})^+$, be the space obtained by performing the plus construction on the classifying space of the infinite general, respectively special, linear group over the integers $\mathbb{Z}$. In order to investigate the homotopical properties of these spaces, it is sufficient to consider $BSL(\mathbb{Z})^+$ because of the homotopy equivalence

$$BGL(\mathbb{Z})^+ \simeq BSL(\mathbb{Z})^+ \times B\mathbb{Z}/2 .$$

In Section 2 of [A3], we showed that the Postnikov k-invariants of $BE(R)^+$ are torsion classes for any ring R and we provided universal upper bounds for their order (here $E(R)$ is the subgroup of $GL(R)$ generated by elementary matrices). The purpose of this section is to deduce from Proposition 1 a better description of the order of these k-invariants in the case when $R = \mathbb{Z}$ and to explain the consequences on the Hurewicz homomorphism between the algebraic K-theory of $\mathbb{Z}$ and the integral homology of the group $SL(\mathbb{Z})$.

If X is a simple m-connected CW-complex and i an integer $\geqq m+1$, let us denote by $X \to X[i]$ its i-th Postnikov section: $X[i]$ is a CW-complex obtained from X by adjoining cells of dimensions $\geqq i+2$ such that $\pi_k X[i] = 0$ for $k > i$ and $\pi_k X \xrightarrow{\cong} \pi_k X[i]$ for $k \leqq i$. For $i \geqq m+2$, the k-invariant $k^{i+1}(X)$ of X is a cohomology class in

$$H^{i+1}(X[i-1]; \pi_i X)$$

such that $X[i]$ is homotopic to the fibre of the map $X[i-1] \to K(\pi_i X, i+1)$ corresponding to $k^{i+1}(X)$. If $k^{i+1}(X)$ is a torsion element, the Hurewicz homomorphism $h_i : \pi_i X \to H_i(X; \mathbb{Z})$ has the following nice property (see Section 2 of [A3]).

Proposition 3. *If the k-invariant $k^{i+1}(X)$ is a cohomology class of finite order $\varrho_i(X)$ in $H^{i+1}(X[i-1]; \pi_i X)$, then there is a homomorphism $\theta_i : H_i(X; \mathbb{Z}) \to \pi_i X$ such that the composition $\pi_i X \xrightarrow{h_i} H_i(X; \mathbb{Z}) \xrightarrow{\theta_i} \pi_i X$ is multiplication by $\varrho_i(X)$.*

We shall need the following results on the order of the k-invariants of certain spaces.

Lemma 4. *Let v be a prime power. Then for any integer $n \geqq 2$, the k-invariant*

$$k^{2n}(BGL(\mathbb{F}_v)^+) \quad of \quad BGL(\mathbb{F}_v)^+ ,$$

where $\mathbb{F}_v$ *is the field with* v *elements, is a cohomology class of order* $\gcd((n-1)!, v^n - 1)$ *in* $H^{2n}(BGL(\mathbb{F}_v)^+[2n-2]; K_{2n-1}(\mathbb{F}_v))$.

Proof. See Theorem C of [H].

Lemma 5. *For any integer* $n \geqq 2$, *the k-invariant* $k^{2n}(U)$ *of the unitary group* U *is a cohomology class of order* $(n-1)!$ *in* $H^{2n}(U[2n-2]; \mathbb{Z})$.

Proof. Since $U \simeq \Omega BU$, the cohomology suspension

$$\sigma^* : H^{2n+1}(BU[2n-1]; \pi_{2n} BU) \to H^{2n}(U[2n-2]; \pi_{2n-1} U)$$

has the property that $\sigma^*(k^{2n+1}(BU)) = k^{2n}(U)$ (see [W], p. 438, Example 3). By Lemma 4.4 of [P], the order of $k^{2n+1}(BU)$ is $(n-1)!$: therefore, the order of $k^{2n}(U)$ divides $(n-1)!$. On the other hand, it is known that the image of a generator of $\pi_{2n-1} U \cong \mathbb{Z}$ under the Hurewicz homomorphism is $(n-1)!\, x_{n-1}$, where x_{n-1} is a generator of degree $2n-1$ of the exterior algebra $H_*(U; \mathbb{Z})$ (see for instance [D], Section III). Consequently, Proposition 3 implies that the order of $k^{2n}(U)$ is a positive multiple of $(n-1)!$.

Now, we concentrate our attention to the order of the k-invariants $k^{2n}(BSL(\mathbb{Z})^+)$ for $n \geqq 2$. For the remainder of the paper, let us denote by ϱ_{2n-1} the order of

$$k^{2n}(BSL(\mathbb{Z})^+) \quad \text{in} \quad H^{2n}(BSL(\mathbb{Z})^+[2n-2]; K_{2n-1}(\mathbb{Z}));$$

observe that ϱ_{2n-1} is also the order of $k^{2n}(BGL(\mathbb{Z})^+)$ because of the homotopy equivalence $BGL(\mathbb{Z})^+ \simeq BSL(\mathbb{Z})^+ \times B\mathbb{Z}/2$.

Remark 4. Since $K_{2n-1}(\mathbb{Z}) \cong K_{2n-1}(\mathbb{Q})$ by the localization exact sequence in algebraic K-theory (see Section 5 of [Q1]), it is easy to check that the order of $k^{2n}(BSL(\mathbb{Q})^+)$ is a positive multiple of ϱ_{2n-1} for any integer $n \geqq 2$.

If O is the ring of integers in a number field, Dwyer and Friedlander defined for any prime p a space $\tilde{K}^{\text{ét}}(O[\frac{1}{p}])$ and a map $\phi : BGL(O[\frac{1}{p}])^+ \to \tilde{K}^{\text{ét}}(O[\frac{1}{p}])$, and reformulated the Quillen-Lichtenbaum conjecture as follows: the p-adic Quillen-Lichtenbaum conjecture for O is true if and only if ϕ is a p-adic homotopy equivalence (see [DF1], Remark 8.8 and [DF2], Section 3). The study of this space for $O = \mathbb{Z}[\zeta_p]$, where ζ_p is a primitive p-th root of unity, gives the next information on the order of the k-invariants of $BSL(\mathbb{Z})^+$.

Proposition 4. *If* p *is an odd regular prime and if the* p*-adic Quillen-Lichtenbaum conjecture for* $\mathbb{Z}[\zeta_p]$ *holds, then* $|\varrho_{2n-1}|_p^{-1}$ *divides* $|(n-1)!|_p^{-1}$ *for any integer* $n \geqq 2$.

Proof. According to Corollary 4.6 of [DF2], there are p-adic homotopy equivalences

$$BGL(\mathbb{Z}[\zeta_p, \tfrac{1}{p}])^+ \overset{p}{\simeq} \tilde{K}^{\text{ét}}(\mathbb{Z}[\zeta_p, \tfrac{1}{p}]) \overset{p}{\simeq} BGL(\mathbb{F}_v)^+ \times \prod_{(v-1)/2} U,$$

for a suitably chosen prime v, if one assumes the p-adic Quillen-Lichtenbaum conjecture for $\mathbb{Z}[\zeta_p]$. Then, it follows clearly from Lemmas 4 and 5 that the p-primary part of the order of $k^{2n}(BGL(\mathbb{Z}[\zeta_p, \frac{1}{p}])^+)$ is equal to $|(n-1)!|_p^{-1}$ and one can deduce from the locali-

1. Some known results about bundles on $\mathbb{P}^5$

Let G be a connected, simply connected, semisimple complex Lie group and let ϕ be the set of the roots of G. Let $\Delta = \{\alpha_1, \ldots, \alpha_k\}$ be a fundamental system of roots. We have the Cartan decomposition

$$\operatorname{Lie} G = \mathscr{G}_0 \oplus \sum_{\alpha \in \phi^-} \mathscr{G}_\alpha \oplus \sum_{\alpha \in \phi^+} \mathscr{G}_\alpha .$$

Let $\phi^+(i) = \{\alpha \in \phi^+ \mid \alpha = \sum n_j \alpha_j \text{ with } n_i = 0\}$ and let $P(\alpha_i) \subset G$ be the parabolic subgroup such that $\operatorname{Lie} P(\alpha_i) = \mathscr{G}_0 \oplus \sum_{\alpha \in \phi^-} \mathscr{G}_\alpha \oplus \sum_{\alpha \in \phi + (i)} \mathscr{G}_a$. Then $G/P(\alpha_i)$ is a rational homogeneous manifold with $\operatorname{Pic} = \mathbb{Z}$.

Let $\{v_1, \ldots, v_k\}$ be the fundamental weights with respect to Δ.

We will apply this construction to the cases

(i) $G = SL(6)$, $\Delta = \{\beta_1, \ldots, \beta_5\}$, $SL(6)/P(\beta_1) \simeq \mathbb{P}^5$; the reductive factor in the Levi decomposition of $P(\beta_1)$ is isomorphic to $SL(5) \cdot \mathbb{C}^*$. We denote in this case by $\{\mu_1, \ldots, \mu_5\}$ the fundamental weights.

(ii) $G = Sp(6)$, $\Delta = \{\sigma_1, \sigma_2, \sigma_3\}$, $Sp(6)/P(\sigma_1) \simeq \mathbb{P}^5$; the reductive factor in the Levi decomposition of $P(\sigma_1)$ is isomorphic to $Sp(4) \cdot \mathbb{C}^*$. We denote in this case by $\{\lambda_1, \lambda_2, \lambda_3\}$ the fundamental weights.

Let $\varrho(v)$ be the irreducible representation of $P(\alpha_i)$ whose restriction to the reductive factor has highest weight $v = \sum n_j v_j$ with $n_j \geqq 0$ for $j \neq i$. Let E^v be the homogeneous vector bundle over $G/P(\alpha_i)$ associated to $\varrho(v)$.

The *quotient bundle* Q on $\mathbb{P}^5 = \mathbb{P}(H)$, is defined by the Euler sequence

$$0 \to \mathcal{O}(-1) \to H \otimes \mathcal{O} \to Q \to 0 .$$

The bundle Q, as well as Q^*, is stable and $SL(6)$-invariant, precisely

$$Q \simeq E^{\mu_5}, \quad Q^* \simeq E^{\mu_2 - \mu_1} .$$

Remember also $\mathcal{O}(t) \simeq E^{t\mu_1} \forall t \in \mathbb{Z}$.

We list now some cohomological lemmas that are applications of Bott theorem [Bo] which will be used in the rest of the paper. For more details see [AO].

Lemma 1.1. $H^1(\operatorname{End} \wedge^2 Q(t)) = H^1(\operatorname{End} Q(t)) = \begin{cases} 0 & \text{for } t \neq -1, \\ H & \text{for } t = -1 . \end{cases}$

Lemma 1.2. $H^1(\wedge^2 Q \otimes \wedge^4 Q^*(t)) = 0 \ \forall t \in \mathbb{Z}$.

Diagram 6

$$\begin{array}{ccc} k^{2n}(BSL(\mathbb{Z})^+) & \in & H^{2n}(BSL(\mathbb{Z})^+[2n-2]; K_{2n-1}(\mathbb{Z})) \\ & & \downarrow (\pi_v)_* \\ & & H^{2n}(BSL(\mathbb{Z})^+[2n-2]; K_{2n-1}(\mathbb{F}_v)) \\ & & \uparrow (\pi_v)^* \\ k^{2n}(BSL(\mathbb{F}_v)^+) & \in & H^{2n}(BSL(\mathbb{F}_v)^+[2n-2]; K_{2n-1}(\mathbb{F}_v)), \end{array}$$

where $(\pi_v)_*$ and $(\pi_v)^*$ are induced by the reduction mod v. The restriction of the homomorphism $(\pi_v)_*$ to p-torsion is just reduction mod p^m, and consequently injective (because $m \geqq \bar{m}$). Since $k^{2n}(BSL(\mathbb{Z})^+)$ is a torsion element, the p-primary part of the order of $(\pi_v)_*(k^{2n}(BSL(\mathbb{Z})^+)) = (\pi_v)^*(k^{2n}(BSL(\mathbb{F}_v)^+))$ (see [W], p. 424) is then still $|\varrho_{2n-1}|_p^{-1}$: this shows that $|\varrho_{2n-1}|_p^{-1}$ divides the p-primary part of the order of $k^{2n}(BSL(\mathbb{F}_v)^+)$ which is equal to $|(n-1)!|_p^{-1}$ by Lemma 4, according to our assumption $p^m \geqq |(n-1)!|_p^{-1}$.

If p is an odd prime and n an odd integer, then the p-adic Quillen-Lichtenbaum conjecture for $\mathbb{Z}$ implies that $|\# T|_p^{-1} = |w_n(\mathbb{Q})|_p^{-1} = 1$ and we get:

Corollary 2. *For any odd prime number p, if the p-adic Quillen-Lichtenbaum conjecture for $\mathbb{Z}$ holds, then $|\varrho_{2n-1}|_p^{-1}$ divides $|(n-1)!|_p^{-1}$ for any odd integer $n \geqq 3$.*

Remark 5. Notice that $(n-1)!$ is a better upper bound for the order of

$$k^{2n}(BSL(\mathbb{Z})^+)$$

than the integer R_{2n-2} introduced in [A3].

Now, consider the Hurewicz homomorphism

$$h_{2n-1}: K_{2n-1}(\mathbb{Z}) \to H_{2n-1}(BSL(\mathbb{Z})^+; \mathbb{Z}) \cong H_{2n-1}(SL(\mathbb{Z}); \mathbb{Z})$$

for odd integers $n \geqq 3$. We are actually interested in the homomorphism

$$K_{2n-1}(\mathbb{Z})/\text{torsion} \to H_{2n-1}(SL(\mathbb{Z}); \mathbb{Z})/\text{torsion}$$

induced by h_{2n-1}. Recall that it has been understood for the case $n = 3$ in Theorem 1.5 of [A1]: *The homomorphism $K_5(\mathbb{Z})/\text{torsion} \to H_5(SL(\mathbb{Z}); \mathbb{Z})/\text{torsion}$ induced by h_5 is multiplication by* 2. Furthermore, some general information on the Hurewicz homomorphism is given in [A3]. By [Bo], $H_*(SL(\mathbb{Z}); \mathbb{Q}) \cong \Lambda(u_5, u_9, \ldots, u_{4k+1}, \ldots)$ and $K_{2n-1}(\mathbb{Z}; \mathbb{Q}) \cong \mathbb{Q}$ for n odd $\geqq 3$, and the rational Hurewicz homomorphism maps $K_{2n-1}(\mathbb{Z}; \mathbb{Q})$ into $\Lambda(u_{2n-1})$. Thus, if ε_{2n-1} is a generator of the infinite cyclic direct summand of

$$K_{2n-1}(\mathbb{Z}) \cong \mathbb{Z} \oplus T$$

(T finite), we may conclude that $h_{2n-1}(\varepsilon_{2n-1}) = \eta_{2n-1} x_{2n-1} + t_{2n-1}$, where η_{2n-1} is a positive integer, x_{2n-1} a generator of an infinite cyclic direct summand of $H_{2n-1}(SL(\mathbb{Z}); \mathbb{Z})$ whose rationalization is u_{2n-1}, and t_{2n-1} a torsion element in $H_{2n-1}(SL(\mathbb{Z}); \mathbb{Z})$. In order to generalize the result on h_5, we deduce the next assertion from Propositions 3 and 5 and Corollary 2.

Corollary 3. *With the notation introduced above, the Hurewicz homomorphism*

$$h_{2n-1}: K_{2n-1}(\mathbb{Z}) \to H_{2n-1}(SL(\mathbb{Z}); \mathbb{Z})$$

has the following property for all odd integers $n \geqq 3$: if p is an odd prime number such that $K_{2n-1}(\mathbb{Z})$ contains no p-torsion (in particular, if the p-adic Quillen-Lichtenbaum conjecture for $\mathbb{Z}$ is true), then $|\eta_{2n-1}|_p^{-1}$ divides $|(n-1)!|_p^{-1}$.

5. The Postnikov invariants of the space $JK(\mathbb{Z})$

In this section, let us consider the prime 2. Bökstedt introduced in [Bök] the space $JK(\mathbb{Z})$ defined by the homotopy fibre square

Diagram 7

$$\begin{tikzcd} JK(\mathbb{Z}) \arrow[r] \arrow[d] & BO \arrow[d, "c"] \\ BGL(\mathbb{F}_3)^+ \arrow[r, "b"] & BU, \end{tikzcd}$$

where c is the complexification and b the Brauer lifting (see also [M], Section 4 or [DF 2], Section 4). The fibre of both horizontal maps is U and we shall denote by g the inclusion $U \hookrightarrow JK(\mathbb{Z})$ and by g' the inclusion $U \hookrightarrow BGL(\mathbb{F}_3)^+$. We obtain the following commutative diagram for any positive integer n:

Diagram 8

$$\begin{tikzcd} \pi_{2n}BO \arrow[r, "\partial"] \arrow[d, "c_*"] & \pi_{2n-1}U \arrow[r, "g_*"] \arrow[d, "\cong"] & \pi_{2n-1}JK(\mathbb{Z}) \arrow[r] \arrow[d] & \pi_{2n-1}BO \arrow[r] \arrow[d] & 0 \\ \pi_{2n}BU \arrow[r, "\cdot(3^n-1)"] & \pi_{2n-1}U \arrow[r, "g'_*"] & K_{2n-1}(\mathbb{F}_3) \arrow[r] & 0. & \end{tikzcd}$$

If n is odd, $\pi_{2n-1}BO$ and $\pi_{2n}BO$ are isomorphic to $\mathbb{Z}/2$ or 0 and ∂ is then trivial since $\pi_{2n-1}U \cong \mathbb{Z}$. Thus, we get the short exact sequence

$$0 \longrightarrow \underbrace{\pi_{2n-1}U}_{\cong \mathbb{Z}} \xrightarrow{g_*} \pi_{2n-1}JK(\mathbb{Z}) \longrightarrow \pi_{2n-1}BO \longrightarrow 0.$$

Since g'_* is surjective, it turns out that g_* is a split injection (in particular,

$$\pi_{2n-1}JK(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

if $n \equiv 1 \bmod 4$ and $\pi_{2n-1}JK(\mathbb{Z}) \cong \mathbb{Z}$ if $n \equiv 3 \bmod 4$). This provides the next statement.

Proposition 6. *For any odd integer $n \geqq 3$, the order of the k-invariant $k^{2n}(JK(\mathbb{Z}))$ in $H^{2n}(JK(\mathbb{Z})[2n-2]; \pi_{2n-1}JK(\mathbb{Z}))$ is a positive multiple of $(n-1)!$.*

Proof. Look again at the diagram explaining the naturality of the k-invariants:

Diagram 9

$$\begin{array}{ccc} k^{2n}(U) & \in & H^{2n}(U[2n-2]; \pi_{2n-1}U) \\ & & \downarrow g_* \\ & & H^{2n}(U[2n-2]; \pi_{2n-1}JK(\mathbb{Z})) \\ & & \uparrow g^* \\ k^{2n}(JK(\mathbb{Z})) & \in & H^{2n}(JK(\mathbb{Z})[2n-2]; \pi_{2n-1}JK(\mathbb{Z})), \end{array}$$

where $g_\#$ and g^* are induced by the map $g: U \to JK(\mathbb{Z})$. Because $g_\#$ is actually a split injection, the order of $g_\#(k^{2n}(U)) = g^*(k^{2n}(JK(\mathbb{Z})))$ is $(n-1)!$ by Lemma 5. Therefore, $(n-1)!$ divides the order of $k^{2n}(JK(\mathbb{Z}))$.

The 2-adic Quillen-Lichtenbaum conjecture for $\mathbb{Z}$ implies a 2-adic homotopy equivalence $BGL(\mathbb{Z}[\frac{1}{2}])^+ \overset{2}{\simeq} JK(\mathbb{Z})$ (see Section 4 of [DF2] or the introduction of [M]) and it follows from the localization exact sequence in algebraic K-theory (see Section 5 of [Q1]) that $BGL(\mathbb{Z})^+ \overset{2}{\simeq} BGL(\mathbb{Z}[\frac{1}{2}])^+$. Consequently, we get the following result on the order ϱ_{2n-1} of $k^{2n}(BSL(\mathbb{Z})^+)$.

Corollary 4. *If the 2-adic Quillen-Lichtenbaum conjecture for $\mathbb{Z}$ is true, then $|\varrho_{2n-1}|_2^{-1}$ is a positive multiple of $|(n-1)!|_2^{-1}$ for any odd integer $n \geqq 3$.*

Corollaries 2, 3 and 4 enable us to formulate the following conjectures.

Conjecture 1. *For any odd integer $n \geqq 3$, the order of the k-invariant $k^{2n}(BSL(\mathbb{Z})^+)$ in $H^{2n}(BSL(\mathbb{Z})^+[2n-2]; K_{2n-1}(\mathbb{Z}))$ is $\varrho_{2n-1} = (n-1)!$.*

Conjecture 2. *For any odd integer $n \geqq 3$, the Hurewicz homomorphism*

$$h_{2n-1}: K_{2n-1}(\mathbb{Z}) \to H_{2n-1}(SL(\mathbb{Z}); \mathbb{Z})$$

has the property that $\eta_{2n-1} = (n-1)!$ (where η_{2n-1} is explained in the paragraph before Corollary 3).

6. The Hurewicz homomorphism $K_5(\mathbb{Z}) \to H_5(St(\mathbb{Z}); \mathbb{Z})$

Let us first determine exactly the order ϱ_i of the k-invariants $k^{i+1}(BSL(\mathbb{Z})^+)$ of $BSL(\mathbb{Z})^+$ in small dimensions.

Proposition 7. $\varrho_3 = 2$ *and* $\varrho_4 = 1$.

Proof. Since $BSL(\mathbb{Z})^+$ is simply connected, its first k-invariant is

$$k^4(BSL(\mathbb{Z})^+) \in H^4(K(K_2(\mathbb{Z}), 2); K_3(\mathbb{Z})) \cong H^4(K(\mathbb{Z}/2, 2); \mathbb{Z}/48) \cong \mathbb{Z}/4.$$

By Section 2 of [A3], we know that $2k^4(BSL(\mathbb{Z})^+)=0$. On the other hand, if $k^4(BSL(\mathbb{Z})^+)$ were trivial, then we would get

$$H_3(BSL(\mathbb{Z})^+;\mathbb{Z}) \cong H_3(K(\mathbb{Z}/2,2)\times K(\mathbb{Z}/48,3);\mathbb{Z}) \cong \mathbb{Z}/48:$$

but this is wrong since $H_3(BSL(\mathbb{Z})^+;\mathbb{Z}) \cong \mathbb{Z}/24$ (see the introduction of [A1]). Consequently, $\varrho_3=2$. The second k-invariant is $k^5(BSL(\mathbb{Z})^+) \in H^5(BSL(\mathbb{Z})^+[3];K_4(\mathbb{Z}))$. The vanishing of $K_4(\mathbb{Z})$ (see [Ro2]) implies that $k^5(BSL(\mathbb{Z})^+)=0$, in other words that $\varrho_4=1$.

For the next dimension, let us consider the space $BSt(\mathbb{Z})^+$ instead of $BSL(\mathbb{Z})^+$, where $St(\mathbb{Z})$ denotes the infinite Steinberg group over $\mathbb{Z}$; $BSt(\mathbb{Z})^+$ is actually the fibre of the second Postnikov section $BSL(\mathbb{Z})^+ \to K(K_2\mathbb{Z},2)$. Our last objective is to understand exactly the Hurewicz homomorphism

$$h_5: K_5(\mathbb{Z}) \to H_5(BSt(\mathbb{Z})^+;\mathbb{Z}) \cong H_5(St(\mathbb{Z});\mathbb{Z})\,.$$

Recall that $K_5(\mathbb{Z}) \cong \mathbb{Z}\oplus T$ and $H_5(St(\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z}\oplus T'$, where T and T' are finite abelian groups by [Bo] and that we proved the following assertion in [A1], Theorem 1.5.

Proposition 8. *The homomorphism $K_5(\mathbb{Z})/T \to H_5(St(\mathbb{Z});\mathbb{Z})/T'$ induced by h_5 is multiplication by* 2.

This enables us to investigate the 6-th dimensional k-invariant.

Proposition 9. *$k^6(BSt(\mathbb{Z})^+)$ is an element of order* 2 *in $H^6(BSt(\mathbb{Z})^+[4];K_5(\mathbb{Z}))$.*

Proof. Since $BSt(\mathbb{Z})^+$ is 2-connected and $\pi_i BSt(\mathbb{Z})^+ \cong K_i(\mathbb{Z})$ for $i \geqq 3$, it follows from the vanishing of $K_4(\mathbb{Z})$ (see [Ro2]) that $BSt(\mathbb{Z})^+[4] \simeq K(\mathbb{Z}/48,3)$ and $k^6(BSt(\mathbb{Z})^+)$ belongs to the group $H^6(K(\mathbb{Z}/48,3);K_5(\mathbb{Z}))$ which is, by the universal coefficient theorem, isomorphic to $\operatorname{Hom}(H_6(K(\mathbb{Z}/48,3);\mathbb{Z}),K_5(\mathbb{Z})) \oplus \operatorname{Ext}(H_5(K(\mathbb{Z}/48,3);\mathbb{Z}),K_5(\mathbb{Z}))$. Using [C], it is possible to calculate that

$$H_5(K(\mathbb{Z}/48,3);\mathbb{Z}) \cong \mathbb{Z}/2 \quad \text{and} \quad H_6(K(\mathbb{Z}/48,3);\mathbb{Z}) \cong \mathbb{Z}/2\,.$$

Therefore, $2k^6(BSt(\mathbb{Z})^+)=0$. Propositions 3 and 8 then conclude the proof.

The main result of this section is:

Theorem 3. (a) *The Hurewicz homomorphism h_5 is injective and there is a short exact sequence $0 \to K_5(\mathbb{Z}) \xrightarrow{h_5} H_5(St(\mathbb{Z});\mathbb{Z}) \to \mathbb{Z}/2 \to 0$.*

(b) *If T and T' denote the torsion subgroups of $K_5(\mathbb{Z})$ and $H_5(St(\mathbb{Z});\mathbb{Z})$ respectively, the restriction of $h_5: T \to T'$ is an isomorphism.*

Proof. Consider the fibration given by the definition of $k^6(BSt(\mathbb{Z})^+)$.

$$BSt(\mathbb{Z})^+[5] \longrightarrow BSt(\mathbb{Z})^+[4] \xrightarrow{k^6(BSt(\mathbb{Z})^+)} K(K_5(\mathbb{Z}),6)$$

and observe that $BSt(\mathbb{Z})^+[4] \simeq K(\mathbb{Z}/48, 3)$ and that

$$H_5(BSt(\mathbb{Z})^+[5]; \mathbb{Z}) \cong H_5(St(\mathbb{Z}); \mathbb{Z}).$$

The Hochschild-Serre spectral sequence for integral homology of this fibration provides an exact sequence:

$$\underbrace{H_6(K(\mathbb{Z}/48,3);\mathbb{Z})}_{\cong \mathbb{Z}/2} \xrightarrow{\psi} \underbrace{H_6(K(K_5(\mathbb{Z}),6);\mathbb{Z})}_{\cong K_5(\mathbb{Z})} \xrightarrow{h_5} H_5(St(\mathbb{Z});\mathbb{Z}) \longrightarrow \underbrace{H_5(K(\mathbb{Z}/48,3);\mathbb{Z})}_{\cong \mathbb{Z}/2} \to 0.$$

In order to determine the image of ψ, let us consider the map $B\Sigma_\infty^+ \to BGL(\mathbb{Z})^+$ induced by the inclusion of the infinite symmetric group Σ_∞ into $GL(\mathbb{Z})$. We denote by $\widetilde{B\Sigma_\infty^+}$ the (2-connected) fibre of the second Postnikov section $B\Sigma_\infty^+ \to B\Sigma_\infty^+[2]$. The above map lifts to a map $f\colon \widetilde{B\Sigma_\infty^+} \to BSt(\mathbb{Z})^+$ and produces, because of the naturality of the k-invariants (see [W], p. 424), the following map of fibrations:

Diagram 10

$$\begin{array}{ccccc}
\widetilde{B\Sigma_\infty^+}[5] & \longrightarrow & \widetilde{B\Sigma_\infty^+}[4] & \xrightarrow{k^6(B\Sigma_\infty^+)} & K(\pi_5 B\Sigma_\infty^+, 6) \\
\downarrow & & \downarrow & & \downarrow \\
BSt(\mathbb{Z})^+[5] & \longrightarrow & BSt(\mathbb{Z})^+[4] & \xrightarrow{k^6(BSt(\mathbb{Z})^+)} & K(K_5(\mathbb{Z}), 6).
\end{array}$$

It is well known that for any positive i, $\pi_i \widetilde{B\Sigma_\infty^+} \cong \pi_i^s$, the i-th stable homotopy group of spheres. Since $\pi_3^s \cong \mathbb{Z}/24$, $\pi_4^s = 0$ and $\pi_5^s = 0$, we obtain the homotopy equivalence $\widetilde{B\Sigma_\infty^+}[4] \simeq K(\mathbb{Z}/24, 3)$ and f provides the commutative diagram

Diagram 11

$$\begin{array}{ccc}
\mathbb{Z}/2 \cong H_6(K(\mathbb{Z}/24,3);\mathbb{Z}) & \longrightarrow & \pi_5^s = 0 \\
\downarrow f_* & & \downarrow f_* \\
\mathbb{Z}/2 \cong H_6(K(\mathbb{Z}/48,3);\mathbb{Z}) & \xrightarrow{\psi} & K_5(\mathbb{Z}),
\end{array}$$

where the composition ψf_* is obviously trivial. Here f_* is induced by the homomorphism $f_*: \pi_3^s \to K_3(\mathbb{Z})$ which is injective according to [Q2]: thus, if α generates $\pi_3^s \cong \mathbb{Z}/24$, $f_*(\alpha) = 2a$ where a is a generator of $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$. However, the computation (following [C]) of $H_6(K(G,3);\mathbb{Z})$ for a finite cyclic group G tells us that $H_6(K(\mathbb{Z}/24,3);\mathbb{Z}) \cong \mathbb{Z}/2$ is generated by 12α and that $H_6(K(\mathbb{Z}/48,3);\mathbb{Z}) \cong \mathbb{Z}/2$ is generated by $24a$. Therefore, f_* is an isomorphism and ψ is trivial. Consequently, we get the short exact sequence

$$0 \longrightarrow K_5(\mathbb{Z}) \xrightarrow{h_5} H_5(St(\mathbb{Z});\mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and the assertion (b) follows from Proposition 8.

If the "connectivity conjecture" for $\mathbb{Z}$ is true, then T is a finite 2-group of order at most 8 (see [Ro1], Conjecture 1.2 and Theorem 1.3). Furthermore, if the 2-adic

Quillen-Lichtenbaum conjecture for $\mathbb{Z}$ holds, then the 2-torsion subgroup of T' is trivial (see [A3], Remark 1.11). Consequently, we can formulate the following

Conjecture 3. $K_5(\mathbb{Z}) \cong \mathbb{Z}$.

References

[A1] *D. Arlettaz*, On the algebraic K-theory of $\mathbb{Z}$, J. Pure Appl. Algebra **51** (1988), 53–64.

[A2] *D. Arlettaz*, On the homomorphism $K_5\mathbb{Z} \to K_5\mathbb{F}_p$, Comm. Algebra **16**(1) (1988), 171–184.

[A3] *D. Arlettaz*, The Hurewicz homomorphism in algebraic K-theory, J. Pure Appl. Algebra **71** (1991), 1–12.

[Ba] *G. Banaszak*, Generalization of the Moore exact sequence and the wild kernel for higher K-groups, Comp. Math. **86** (1993), 281–305.

[Bök] *M. Bökstedt*, The rational homotopy type of $\Omega Wh^{\mathrm{Diff}}(*)$, in: Algebraic Topology, Aarhus 1982, Lect. Notes Math. **1051**, Springer (1984), 25–37.

[Bo] *A. Borel*, Cohomologie réelle stable de groupes S-arithmétiques classiques, C. R. Acad. Sci. Paris (A) **274** (1972), 1700–1702.

[C] *H. Cartan*, Algèbres d'Eilenberg-MacLane et homotopie, Sém. H. Cartan Ecole Norm. Sup. (1954/1955), exposé 11.

[D] *A. Douady*, Périodicité du groupe unitaire, Sém. H. Cartan Ecole Norm. Sup. (1959/1960), exposé 11.

[DF1] *W. Dwyer* and *E. Friedlander*, Algebraic and étale K-theory, Trans. Amer. Math. Soc. **292** (1985), 247–280.

[DF2] *W. Dwyer* and *E. Friedlander*, Conjectural calculations of the general linear group homology, in: Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Contemp. Math. **55**, Part I (1986), 135–147.

[H] *J. Huebschmann*, The homotopy type of $F\Psi^q$. The complex and symplectic cases, in: Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Contemp. Math. **55**, Part II (1986), 487–518.

[J] *U. Jannsen*, Continuous étale cohomology, Math. Ann. **280** (1988), 207–245

[M] *S. Mitchell*, On the plus construction for $BGL(\mathbb{Z}[\frac{1}{2}])$ at the prime 2, Math. Z. **209** (1992), 205–222.

[P] *F.P. Peterson*, Some remarks on Chern classes, Ann. Math. **69** (1959), 414–420.

[Q1] *D. Quillen*, Higher Algebraic K-theory: I, in: Algebraic K-theory 1, Higher K-Theories, Lect. Notes Math. **341**, Springer (1973), 85–147.

[Q2] *D. Quillen*, Letter from Quillen to Milnor on $\mathrm{Im}(\pi_i O \to \pi_i^s \to K_i\mathbb{Z})$, in: Algebraic K-theory, Evanston 1976, Lect. Notes Math. **551**, Springer (1976), 182–188.

[Ro1] *J. Rognes*, Approximating $K_*(\mathbb{Z})$ through degree five, K-Theory **7** (1992), 175–200.

[Ro2] *J. Rognes*, $K_4(\mathbb{Z})$ is the trivial group, preprint.

[Ru] *K. Rubin*, Kolyvagin's systems of Gauss sums, in: Arithmetic Algebraic Geometry, ed. by G. van der Geer et. al., Progr. Math. **89**, Birkhäuser (1991), 309–324.

[S1] *C. Soulé*, On higher p-adic regulators, in: Algebraic K-theory, Evanston 1980, Lect. Notes Math. **854**, Springer (1981), 372–401.

[S2] *C. Soulé*, Groupes de Chow et K-théorie des variétés sur un corps fini, Math. Ann. **268** (1984), 317–345.

[W] *G.W. Whitehead*, Elements of Homotopy Theory, Grad. Texts Math. **61**, Springer (1978).

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