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Universal bounds for the exponent of stable homotopy groups

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Abstract

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Let X be an m-connected CW-complex and n an integer satisfying $2 \leq n \leq 2m$. We prove that if the nth integral homology group of X is of finite exponent, then the nth homotopy group of X has also finite exponent, and we give a universal bound for this exponent. This provides for instance universal bounds for the exponent of the stable homotopy groups of Moore spaces.

Keywords: Exponent of stable homotopy groups, Postnikov k-invariants, Moore spaces.

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Introduction

Let X be an m-connected CW-complex ($m \geq 1$) and $X[n]$ its nth Postnikov section; we proved in [2] that the k-invariants of X, $k^{n+1}(X) \in H^{n+1}(X[n-1]; \pi_n X)$, are cohomology classes of finite order for $2 \leq n \leq 2m$ and we gave upper bounds for their order. More precisely, there exist positive integers R_j ($j \in \mathbb{Z}$) such that $R_{n-m}k^{n+1}(X) = 0$ for any m-connected CW-complex X, if $2 \leq n \leq 2m$.

In this paper we exploit this fact in showing the following:

Theorem. *Let X be an m-connected CW-complex and n an integer satisfying $2 \leq n \leq 2m$. If the nth integral homology group of X is of finite exponent h_n, then the nth homotopy group of X has also finite exponent*: $R_{n-m}h_n\pi_n X = 0$.

If $H_n(X; \mathbb{Z}) = 0$, it turns out that $R_{n-m}\pi_n X = 0$; this is for instance the case if X is a k-dimensional m-connected CW-complex and $k+1 \leq n \leq 2m$. Observe that the upper bound R_{n-m} for the exponent of $h_n\pi_n X$ is universal, i.e., does not depend

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on X (only on m) and that we do not need any finiteness condition on X. Let us mention that our argument succeeds only in stable dimensions. However, if X is an r-fold loop space, it is possible to extend our results to dimensions $n \leqslant 2m+r$, even if $m=0$ (in particular to all values of n if X is an infinite loop space). Using the same method we are also able to estimate the exponent of the homotopy groups of finite H-spaces for all even dimensions in the stable range.

In the second part of the paper we consider the example of Moore spaces, i.e., simply connected spaces with only one nonvanishing integral homology group G: we obtain universal bounds (independent of G) for the exponent of the stable homotopy groups of Moore spaces.

1. k-invariants and exponents for homotopy groups

The purpose of the beginning of this section is to recall the main result of [2] on the order of certain k-invariants. We start by introducing some notation.

Definition 1.1. For $j \geqslant 1$, let L_j denote the product of all prime numbers p such that there exists a sequence of nonnegative integers $(a_1, a_2, a_3, \ldots)$ with

(a) $a_1 \equiv 0 \bmod 2p-2$, $a_i \equiv 1$ or $0 \bmod 2p-2$ for $i \geqslant 2$,

(b) $a_i \geqslant pa_{i+1}$ for $i \geqslant 1$,

(c) $\sum_{i=1}^{\infty} a_i = j$.

For example $L_1=1$, $L_2=2$, $L_3=2$, $L_4=6$, $L_5=6$, $L_6=2, \ldots$. Notice that L_j divides the product M_j of all primes $p \leqslant j/2+1$. These integers L_j occur in the study of the exponent of the stable homology groups of Eilenberg-MacLane spaces (cf. [6, exposé 11, Théorème 2]): for any Abelian group G and for each integer $s \geqslant 2$ one has $L_{i-s}H_i(K(G,s); \mathbb{Z})=0$ for $s<i<2s$.

Definition 1.2. $R_j := \prod_{k=2}^{j} L_k$ for $j \geqslant 2$ and $R_j := 1$ for $j \leqslant 1$.

When $j \geqslant 2$, remark that a prime number p divides R_j if and only if $p \leqslant j/2+1$. More precisely, the p-primary part $(R_j)_p$ of the integer R_j has the following properties:

(a) for $p=2$, $(R_j)_2 = 2^{j-1}$;

(b) for an odd prime p, $(R_j)_p = 1$ if $j<2p-2$ and $(R_j)_p \leqslant p^{j-2p+3}$ if $j \geqslant 2p-2$; for very large values of j (i.e., for $j \geqslant 2(p^{2p-3}+p^{2p-4}+\cdots+p^3+p^2-p+3))$ one has actually $(R_j)_p = p^{j-c}$, where c is a constant depending on p. But on the other hand, for any positive integer t, $(R_j)_p \leqslant p^{j/t}$ if j is sufficiently small; for instance, we will use later the following inequality for $p \geqslant 5$: $(R_j)_p < p^{[(j-1)/2]}$ at least if $3 \leqslant j \leqslant 2(p^{p-1}+p^{p-2}+\cdots+p^2+p)$ ([ ] denotes the integral part).

All spaces we consider in this paper are connected simple CW-complexes. Let us call $X[n]$ the nth Postnikov section of a space X: $X[n]$ is a CW-complex obtained from X by adjoining cells of dimension $\geqslant n+2$, such that $\pi_j X[n]=0$ for $j>n$

and $\pi_j X[n] \cong \pi_j X$ for $j \leq n$. The Postnikov k-invariants $k^{n+1}(X)$ of X are maps $X[n-1] \to K(\pi_n X, n+1)$ and thus cohomology classes in $H^{n+1}(X[n-1]; \pi_n X)$, for $n \geq 2$. The following result provides universal bounds for the order of the first k-invariants of iterated loop spaces.

Theorem 1.3. *If X is an m-connected r-fold loop space $(m \geq 0,\ r \geq 0)$, then*

$$R_{n-m} k^{n+1}(X) = 0 \quad \text{for } 2 \leq n \leq 2m + r.$$

A proof of this assertion was given in [2, Theorem 1.6], but it used the integers M_j instead of L_j in the definition of the numbers R_j, because this approximation was sufficient for the purpose of [2]. Let us also recall the following:

Lemma 1.4 (cf. [1, Lemma 4]). *Let X be a connected simple CW-complex and assume that the k-invariant $k^{n+1}(X)$ is a cohomology class of finite order s in $H^{n+1}(X[n-1]; \pi_n X)$. Then there exists a map $f: X \to K(\pi_n X, n)$ such that the induced homomorphism $f_*: \pi_n X \to \pi_n X$ is multiplication by s.*

The existence of this map, together with the finiteness of the order of the k-invariants examined in Theorem 1.3, enables us to prove our main theorem which establishes a relationship between exponents for homology groups and exponents for homotopy groups.

Theorem 1.5. *Let X be an m-connected r-fold loop space $(m \geq 0,\ r \geq 0)$ and n an integer with $2 \leq n \leq 2m + r$. If there exists a positive integer h_n such that $h_n H_n(X; \mathbb{Z}) = 0$, then*

$$R_{n-m} h_n \pi_n X = 0.$$

Remark 1.6. The bound R_{n-m} does not depend on X; in particular, X is not necessarily a space of finite type.

Proof. According to Theorem 1.3, the order of $k^{n+1}(X)$ is finite and divides R_{n-m}. Then the map $f: X \to K(\pi_n X, n)$ given by Lemma 1.4 induces the following commutative diagram

$$\begin{array}{ccc} \pi_n X & \xrightarrow{f_*} & \pi_n K(\pi_n X, n) \\ \downarrow{\scriptstyle \text{Hu}} & & \cong \downarrow{\scriptstyle \text{Hu}} \\ H_n(X; \mathbb{Z}) & \xrightarrow{f_*} & H_n(K(\pi_n X, n); \mathbb{Z}), \end{array}$$

where Hu denotes the Hurewicz homomorphism: the composition $\text{Hu} \cdot f_*$ is multiplication by the order of $k^{n+1}(X)$. For any element α of $\pi_n X$, one has $\text{Hu}(h_n \alpha) = h_n \text{Hu}(\alpha) = 0$ by hypothesis. This implies $\text{Hu} \cdot f_*(h_n \alpha) = f_* \cdot \text{Hu}(h_n \alpha) = 0$ and consequently, $R_{n-m} h_n \alpha = 0$. □

Example 1.7. Let R be a ring with identity, $\mathrm{GL}(R)$ its infinite general linear group, $E(R)$ the subgroup of $\mathrm{GL}(R)$ generated by elementary matrices, and $BE(R)^+$ the space obtained by performing the plus construction on its classifying space; recall that $BE(R)^+$ is a simply connected infinite loop space and that $\pi_n BE(R)^+ = K_n R$ for $n \geq 2$. We deduce from Theorem 1.5 the following conclusion for each $n \geq 2$: if there exists an integer h_n such that $h_n H_n(E(R); \mathbb{Z}) = 0$, then $R_{n-1} h_n K_n R = 0$.

Corollary 1.8. *Let X be an m-connected r-fold loop space ($m \geq 0$, $r \geq 0$) and suppose that n is an integer satisfying $m+2 \leq n \leq 2m+r$ such that the CW-complex X has no n-dimensional cells. Then*

$$R_{n-m}\pi_n X = 0.$$

Proof. This follows directly from the previous theorem since $H_n(X; \mathbb{Z}) = 0$. □

Corollary 1.9. *Let X be an m-connected r-fold loop space ($m \geq 0$, $r \geq 0$) and assume that X is of finite dimension $k < 2m+r$. Then*

$$R_{n-m}\pi_n X = 0$$

for $k+1 \leq n \leq 2m+r$.

If X is a finite H-space we may extend this result to all even values of $n \leq 2m+r$ (cf. also [10] for other information on the homotopy groups of finite H-spaces).

Theorem 1.10. *Consider a finite m-connected H-space which is an r-fold loop space ($m \geq 0$, $r \geq 0$). Then*

$$R_{n-m}\pi_n X = 0$$

for all even dimensions n such that $2 \leq n \leq 2m+r$.

Proof. Let us look again at the diagram introduced in the proof of Theorem 1.5. According to [17, Corollary 2.2], the Hurewicz homomorphism $\mathrm{Hu}: \pi_n X \to H_n(X; \mathbb{Z})$ is zero if n is even, because X is a finite H-space; therefore the same is true for the composition $\mathrm{Hu} \cdot f_*$. The proof is then complete since $\mathrm{Hu} \cdot f_*: \pi_n X \to \pi_n X$ is multiplication by a divisor of R_{n-m} for $2 \leq n \leq 2m+r$. □

Remark 1.11. If $m = 0$ (and $r \geq 2$) in Theorems 1.3, 1.5, 1.10 or Corollaries 1.8, 1.9, it is possible (cf. [2, Theorem 2.4]) to replace the integer R_n by $\bar{R}_n$ which is defined as follows:

$$\bar{R}_n := \begin{cases} R_n/(n/2+1), & \text{if } n/2+1 \text{ is a prime,} \\ R_n, & \text{otherwise.} \end{cases}$$

For example, $\bar{R}_2 = 1$ and we get: if X is a connected double loop space such that either X has no 2-dimensional cells or X is a finite complex, then $\pi_2 X = 0$ (for the case of a finite H-space cf. also [5, Theorem 6.11] or [17, Corollary 2.2]).

Remark 1.12. The integers R_j (and $\bar{R}_j$) introduced in this section provide universal upper bounds for the exponent of stable homotopy groups, but we do not claim that these bounds are best possible; however, each prime p dividing R_j must divide the corresponding best possible universal bound (but may be the power of p dividing R_j is too big). In order to prove this assertion, we check, for any integer $j \geq 2$ and any prime p dividing R_j (i.e., $p \leq j/2+1$), the existence of integers n for which it is possible to construct an $(n-j)$-connected infinite loop space X satisfying $H_n(X;\mathbb{Z})=0$, $p\pi_n X=0$ but $\pi_n X \neq 0$. Indeed, let us consider any integer $n > \max(4p-6, j-1)$, define $s := n-2p+3$, and call X the fibre of the infinite loop map $K(\mathbb{Z}/p, s) \to K(\mathbb{Z}/p, n+1)$ corresponding to the Steenrod operation $P^1 : H^s(-;\mathbb{Z}/p) \to H^{n+1}(-;\mathbb{Z}/p)$; if $p=2$, we interpret P^1 as Sq^2. X is an $(s-1)$-connected infinite loop space (notice that $s-1 \geq n-j$) and the spectral sequence of this fibration produces the exact sequence

$$H_{n+1}(K(\mathbb{Z}/p, s);\mathbb{Z}) \to H_{n+1}(K(\mathbb{Z}/p, n+1);\mathbb{Z}) \to H_n(X;\mathbb{Z})$$
$$\to H_n(K(\mathbb{Z}/p, s);\mathbb{Z}) \to 0,$$

where the first homomorphism on the left is surjective since the cohomology class corresponding to P^1 is nontrivial ($s \geq 2$), and $H_n(K(\mathbb{Z}/p, s);\mathbb{Z})=0$ since $n<2s$, $n<s+2p-2$ (cf. [6, exposé 11]): consequently, $H_n(X;\mathbb{Z})=0$, but on the other hand $\pi_n X \cong \mathbb{Z}/p$.

2. Exponents for the stable homotopy groups of Moore spaces

In this section we apply our results to Moore spaces, in particular to spheres, in order to deduce universal bounds for the exponent of their stable homotopy groups. Let $M(G,k)$ be a Moore space of type (G,k), i.e., a simply connected space satisfying $\tilde{H}_j(M(G,k);\mathbb{Z}) \cong G$ if $j=k$ and 0 if $j \neq k$; here G is any Abelian group and k an integer ≥ 3. The conclusion of Theorem 1.5 for the space $M(G,k)$ (with $m=k-1$ and $r=0$) is:

$$R_{n-k+1}\pi_n M(G,k)=0 \quad \text{if } k+1 \leq n \leq 2k-2.$$

It is possible to define stable homotopy groups for Moore spaces: $\pi_i^s(G) := \varinjlim_k \pi_{i+k} M(G,k) = \pi_{i+k} M(G,k)$ for $k \geq i+2$; we shall use the notation π_i^s for $\pi_i^s(\mathbb{Z}) = \varinjlim_k \pi_{i+k} S^k$. The next theorem is a direct consequence of the above assertion for $k=i+2$ and $n=2i+2$.

Theorem 2.1. *For any Abelian group G and for each integer $i \geq 1$ one has*:

$$R_{i+1}\pi_i^s(G)=0.$$

Example 2.2. Let us consider the space $S^k \cup_{p^t} e^{k+1} = M(\mathbb{Z}/p^t, k)$, where p is a prime number and t a positive integer; our argument implies:

$$(R_{n-k+1})_p \pi_n M(\mathbb{Z}/p^t, k)=0 \quad \text{for } k \geq 3,\ k+1 \leq n \leq 2k-2$$

and

$$(R_{i+1})_p \pi_i^s(\mathbb{Z}/p^t) = 0 \quad \text{for } i \geq 1.$$

The important point is that the integer $(R_{i+1})_p$ is independent of t. Several results have been obtained on this question (cf. [3, 7, 9, 14]): for $i \geq 1$ it is known that $p^t \pi_i^s(\mathbb{Z}/p^t) = 0$ if p is odd or $p = 2$ and $t \geq 2$, and $4\pi_i^s(\mathbb{Z}/2) = 0$. Thus, our result produces a better bound for the exponent of $\pi_i^s(\mathbb{Z}/p^t)$ if i is small in comparison to t.

Example 2.3. Finally, look at the k-dimensional sphere $S^k = M(\mathbb{Z}, k)$ and conclude that

$$R_{n-k+1}\pi_n S^k = 0 \quad \text{for } k \geq 3,\ k+1 \leq n \leq 2k-2$$

and

$$R_{i+1}\pi_i^s = 0 \quad \text{for } i \geq 1.$$

Since our bound is universal, we actually do not use here the fact that the space we are looking at is a sphere: therefore, we do not hope to get a good estimate for the exponent of stable homotopy groups of spheres (cf. also Remark 1.12), but we mention it as an example. However, let us compare this bound R_{i+1} with the information given in the literature (cf. [4, 7, 8, 11, 13, 15, 16]).

If $i < 2p-3$, we obtain the well-known fact (cf. [16, § IV, Proposition 11]) that $(\pi_i^s)_p$ is trivial, since $(R_{i+1})_p = 1$.

If $p = 2$, then $(R_{i+1})_2 = 2^i$ and our assertion becomes $2^i(\pi_i^s)_2 = 0$: this was proven in [11]; [15] produces a better bound and an even better estimate for the exponent of $(\pi_i^s)_2$ may be given by the following argument. Consider the homotopy commutative diagram

$$\begin{array}{ccc} \Omega_0^{2k+1} S^{2k+1} & \longrightarrow & \Omega^\infty \Sigma^\infty \mathbb{R}P^{2k} \\ \big\downarrow & & \big\downarrow {\scriptstyle \Omega^\infty\Sigma^\infty \text{ (inclusion)}} \\ \Omega_0^\infty S^\infty & \longrightarrow & \Omega^\infty \Sigma^\infty \mathbb{R}P^\infty, \end{array}$$

using a stable Hopf invariant, and the 2-local retraction of $\Omega^\infty \Sigma^\infty \mathbb{R}P^\infty$ to $\Omega_0^\infty S^\infty$ established in [12]: the image of $(\pi_{i+2k+1} S^{2k+1})_2$ in $(\pi_i^s)_2$ is then annihilated by the suspension order of the identity for $\Sigma^{\text{large}} \mathbb{R}P^{2k}$.

For odd primes p, it follows from [8] and [13] that $p^{[i/2]}(\pi_i^s)_p = 0$; according to Definition 1.2, notice that $p^{[i/2]} < (R_{i+1})_p$ for very large values of i, but on the other hand that $(R_{i+1})_p < p^{[i/2]}$ for i sufficiently small (at least for $1 < i < 2(p^{p-1} + p^{p-2} + \cdots + p)$, if $p \geq 5$): in this case, our result provides a better bound although we do not claim it is the best possible.

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