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CHARACTERISTIC CLASSES AND OBSTRUCTION THEORY FOR INFINITE LOOP SPACES

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The classical extension problem is to determine whether or not a given map $g: A \to Y$, defined on a given subspace A of a space X, has an extension $X \to Y$. The present paper examines this question in the special case where the k-invariants of Y are cohomology classes of finite order (for instance if Y is an infinite loop space).

Introduction

Let (X, A) be a relative CW-complex and Y an $(m-1)$-connected simple CW-complex ($m \geq 1$). The classical obstruction theory describes the primary obstruction $\gamma^{m+1}(g) \in H^{m+1}(X, A; \pi_m Y)$ to extend a map $g: A \to Y$ to a map $X \to Y$, in term of the characteristic class $i^m(Y) \in H^m(Y; \pi_m Y)$ as follows : $\gamma^{m+1}(g) = (-1)^m \delta g^*(i^m(Y))$, where g^* is the homomorphism induced by g in cohomology and δ the coboundary operator of the cohomology sequence of the pair (X, A). If $\gamma^{m+1}(g) = 0$, then there is an extension of g to the $(m+1)$-dimensional skeleton of (X, A); but the vanishing of this primary obstruction is in general not sufficient in order to determine whether or not it is possible to extend the map g to X, and one must consider higher obstructions, which have a more difficult description.

The purpose of this paper is to provide such a description for the case where Y is a space with Postnikov k-invariants of finite order (for example an infinite loop space). For these spaces we define in Section 1 n-dimensional *characteristic classes* $j^n(Y) \in H^n(Y; \pi_n Y)$ for all positive integers n. Section 2 gives some examples of these characteristic classes, in connection with the cohomology of certain classical groups. Finally, Section 3 is devoted to the extension problem : for all positive integers n we define *obstruction classes* $\zeta^{n+1}(g) \in H^{n+1}(X, A; \pi_n Y)$ which are related to the characteristic classes of Y by the (similar) formula $\zeta^{n+1}(g) = (-1)^n \delta g^*(j^n(Y))$, and we show that, under suitable conditions (for instance after localization of the target space Y), the extensibility of the map g is equivalent to the vanishing of these classes.

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1. Characteristic classes

If Y is an $(m-1)$-connected space (with $\pi_1 Y$ abelian if $m = 1$), its characteristic class

$$i^m(Y) \in H^m(Y; \pi_m Y)$$

is classically defined to be the element of $H^m(Y; \pi_m Y)$ corresponding to the inverse of the Hurewicz isomorphism $\pi_m Y \xrightarrow{\cong} H_m(Y; \mathbb{Z})$ under the isomorphism $H^m(Y; \pi_m Y) \cong Hom(H_m(Y; \mathbb{Z}), \pi_m Y)$ given by the universal coefficient theorem (cf. [8, p.236]). The class i^m is natural in the following sense : if $h: Y \to Y'$ is a map between two $(m-1)$-connected spaces, then $h_*(i^m(Y)) = h^*(i^m(Y'))$, where the homomorphisms $h_*: H^m(Y; \pi_m Y) \to H^m(Y; \pi_m Y')$ and $h^*: H^m(Y'; \pi_m Y') \to H^m(Y; \pi_m Y')$ are induced by h. Our objective is to define, for certain spaces, characteristic classes in all dimensions.

Let us start by explaining our notation. All spaces we consider in this section are connected simple CW-complexes. For such a space Y and for any positive integer n, let $\alpha_n: Y \to Y[n]$ denote the n-th Postnikov section of Y (i.e., $Y[n]$ is a CW-complex obtained form Y by adjoining cells of dimension $\geq n+2$, with $\pi_i Y[n] = 0$ for $i > n$ and $(\alpha_n)_*: \pi_i Y \xrightarrow{\cong} \pi_i Y[n]$ for $i \leq n$), and $k^{n+1}(Y)$ the Postnikov k-invariant in $H^{n+1}(Y[n-1]; \pi_n Y)$: $k^{n+1}(Y)$ is a homotopy class of maps $Y[n-1] \to K(\pi_n Y, n+1)$ such that, if $K(\pi_n Y, n) \to PK(\pi_n Y, n+1) \xrightarrow{p} K(\pi_n Y, n+1)$ is the path-fibration over $K(\pi_n Y, n+1)$ and if W_n is the pull-back of $(k^{n+1}(Y), p)$, there exists a (non-unique) homotopy equivalence $\theta: Y[n] \xrightarrow{\simeq} W_n$.

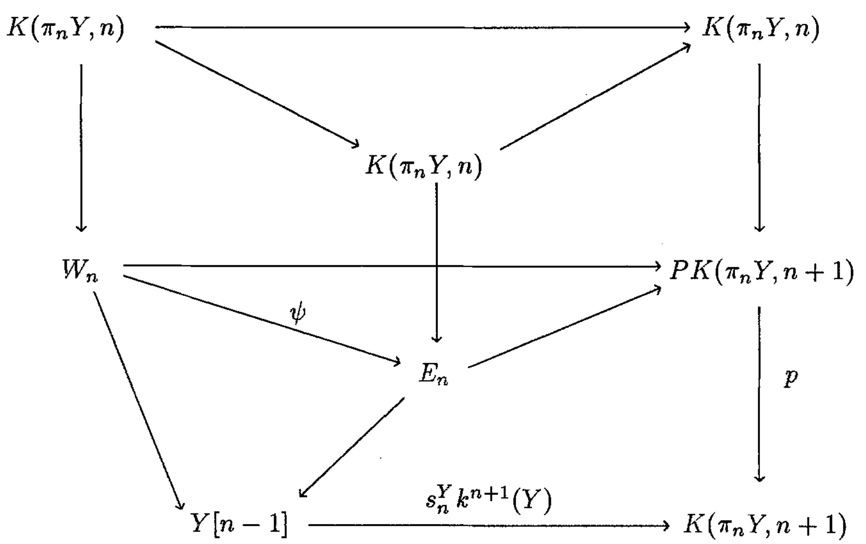
In order to introduce the notion of n-dimensional characteristic classes for a space Y, we must assume that n is a positive integer such that the k-invariant $k^{n+1}(Y)$ is an element of finite order, say s_n^Y, in the group $H^{n+1}(Y[n-1]; \pi_n Y)$

(notice that $k^{n+1}(Y)$ is trivial for all $n \leq m$ if Y is $(m-1)$-connected). For instance, this condition is satisfied for all $n \geq 1$ if Y is an H-space of finite type (cf. [1, Proposition 4.1]). Other examples are given in [3] where the following theorem is proved : there exist positive integers S_t ($t \in \mathbb{Z}$) such that $S_{n-m+1}k^{n+1}(Y) = 0$ for $n \leq r + 2m - 2$ if Y is an $(m-1)$-connected r-fold loop space ($m \geq 1, r \geq 0$); in particular, all k-invariants of an infinite loop space have finite order.

Under this hypothesis, it is possible to construct a map

$$f_n^Y : Y \to K(\pi_n Y, n)$$

which induces multiplication by s_n^Y on $\pi_n Y$ (cf. [2, Lemma 4]). Look at the commutative diagram



where each column is a fibration and E_n the pull-back of $(s_n^Y k^{n+1}(Y), p)$; this implies the existence of the map ψ, and the fact that $s_n^Y k^{n+1}(Y)$ is homotopic to the constant map produces a homotopy equivalence $E_n \simeq Y[n-1] \times K(\pi_n Y, n)$.

Observe that ψ induces multiplication by s_n^Y on $\pi_n Y$ since $s_n^Y k^{n+1}(Y)$ is actually the composition of $k^{n+1}(Y)$ with $s_n^Y \cdot$ identity: $K(\pi_n Y, n+1) \to K(\pi_n Y, n+1)$. We write f_n^Y for the composition

$$Y \xrightarrow{\alpha_n} Y[n] \underset{\simeq}{\xrightarrow{\theta}} W_n \xrightarrow{\psi} E_n \simeq Y[n-1] \times K(\pi_n Y, n) \xrightarrow{\pi} K(\pi_n Y, n) \underset{\simeq}{\xrightarrow{\eta}} K(\pi_n Y, n)$$

where π denotes the projection onto the second factor and η a map inducing an isomorphism on $\pi_n Y$ such that the induced homomorphism $(f_n^Y)_*: \pi_n Y \to \pi_n Y$ is *exactly* multiplication by s_n^Y. This map f_n^Y is *not unique.*

Definition 1.1. If Y is a connected simple CW-complex and n a positive integer such that $k^{n+1}(Y)$ is a cohomology class of finite order s_n^Y, an *n-dimensional characteristic map* for Y is a map

$$f_n^Y : Y \longrightarrow K(\pi_n Y, n)$$

which induces mutliplication by s_n^Y on $\pi_n Y$. The *n-dimensional characteristic class* of Y associated with f_n^Y is

$$j^n(Y) := (f_n^Y)^*(i^n(K(\pi_n Y, n))) \in H^n(Y; \pi_n Y) ,$$

where $(f_n^Y)^*$ is the homomorphism induced by f_n^Y in cohomology (in other words, $j^n(Y)$ is the cohomology class corresponding to the homotopy class of f_n^Y).

The characteristic class $j^n(Y)$ is *not uniquely defined* since it depends on the map f_n^Y. The fibre of the Postnikov section $Y[n| \to Y[n-1]$ is $K(\pi_n Y, n)$ and we call ρ the inclusion map $K(\pi_n Y, n) \hookrightarrow Y[n]$; because of the isomorphism $(\alpha_n)^*: H^n(Y[n]; \pi_n Y) \xrightarrow{\cong} H^n(Y; \pi_n Y)$, we may consider the induced homomorphism $\rho^*: H^n(Y; \pi_n Y) \to H^n(K(\pi_n Y, n); \pi_n Y)$. It is then obvious that all n-dimensional characteristic classes $j^n(Y)$ of Y have the same image under ρ. In fact, we write ρ for the composition of this map with a self-equivalence of $K(\pi_n Y, n)$, such that

$$\rho^*(j^n(Y)) = s_n^Y i^n K(\pi_n Y, n)) .$$

Definition 1.2. $J^n(Y)$ is the image of any n-dimensional characteristic class $j^n(Y)$ of Y under the homomorphism $H^n(Y;\pi_nY) \to H^n(Y;\pi_nY)/\mathrm{Ker}\rho^*$. $J^n(Y)$ is *uniquely determined.*

The remainder of this section establishes some elementary properties of these classes.

Proposition 1.3. *If Y is an $(m-1)$-connected simple CW-complex, then $j^m(Y)$ is uniquely determined and $i^m(Y) = j^m(Y) = J^m(Y)$.*

Proof. If Y is $(m-1)$-connected, $k^{m+1}(Y)$ is trivial ($s_m^Y = 1$) and $Y[m] = K(\pi_mY, m)$; thus, ρ^* is an isomorphism, $j^m(Y)$ is unique and $j^m(Y) = J^m(Y)$. Since any m-dimensional characteristic map $f_m^Y: Y \to K(\pi_mY, m)$ induces identity on π_mY, the naturality of i^m implies :

$$j^m(Y) = (f_m^Y)^*(i^m(K(\pi_mY, m))) = i^m(Y).$$

Let us discuss the naturality of J^n.

Proposition 1.4. *Let Y and Y' be connected simple CW-complexes, n a positive integer such that $k^{n+1}(Y)$ and $k^{n+1}(Y')$ have finite order s_n^Y and $s_n^{Y'}$ respectively, $h: Y \to Y'$ a map, and*

$$h_*: H^n(Y;\pi_nY)/Ker\rho^* \to H^n(Y;\pi_nY')/Ker\rho^* \text{ and}$$
$$h^*: H^n(Y';\pi_nY')/Ker\rho^* \to H^n(Y;\pi_nY')/Ker\rho^*$$

the homomorphisms induced by h. Then

$$s_n^{Y'} h_*(J^n(Y)) = s_n^Y h^*(J^n(Y')) .$$

Proof. Look at the commutative diagram

$$\begin{array}{ccccc}
H^n(Y;\pi_nY) & \xrightarrow{h_*} & H^n(Y;\pi_nY') & \xleftarrow{h^*} & H^n(Y';\pi_nY') \\
\downarrow{\scriptstyle\rho^*} & & \downarrow{\scriptstyle\rho^*} & & \downarrow{\scriptstyle\rho^*} \\
H^n(K(\pi_nY,n);\pi_nY) & \xrightarrow{h_\#} & H^n(K(\pi_nY,n);\pi_nY') & \xleftarrow{h^\#} & H^n(K(\pi_nY',n);\pi_nY')
\end{array}$$

where the homomorphisms $h_*, h^*, h_\#, h^\#$ are induced by h. It follows from $h_\#(i^n(K(\pi_nY,n))) = h^\#(i^n(K(\pi_nY',n)))$ that $\rho^*(s_n^{Y'}h_*(j^n(Y))) = s_n^{Y'}h_\#\rho^*(j^n(Y)) = s_n^{Y'}h_\#(s_n^Y(i^n(K(\pi_nY,n)))) = s_n^Y h^\#(s_n^{Y'}(i^n(K(\pi_nY',n)))) = s_n^Y h^\#\rho^*(j^n(Y')) = \rho^*(s_n^Y h^*(j^n(Y')))$ for all $j^n(Y)$ and $j^n(Y')$. This becomes $s_n^{Y'}h_*(J^n(Y)) = s_n^Y h^*(J^n(Y'))$ in the quotient $H^n(Y;\pi_nY')/\mathrm{Ker}\,\rho^*$.

Remark 1.5. Let Y be a connected simple CW-complex, n a positive integer with $k^{n+1}(Y)$ of finite order s_n^Y, R a subring of the field of rationals $\mathbb{Q}$ such that s_n^Y is invertible in R, $\ell\colon Y \to Y_R$ the localization map, and $\ell_*\colon H^n(Y;\pi_nY) \to H^n(Y;\pi_nY\otimes R)$ and $\ell^*\colon H^n(Y_R;\pi_nY\otimes R) \to H^n(Y;\pi_nY\otimes R)$ the homomorphisms induced by ℓ. The behaviour of J^n under the localization map is described by the previous proposition : since ℓ localizes the k-invariants (cf. [7, Theorem 2.3]), $k^{n+1}(Y_R)$ is trivial and $\ell_*(J^n(Y)) = s_n^Y\ell^*(J^n(Y_R))$.

The same is actually true for j^n : if $j^n(Y) \in H^n(Y;\pi_nY)$ is an n-dimensional characteristic class of Y, there exists an n-dimensional characteristic class $j^n(Y_R) \in H^n(Y_R;\pi_nY\otimes R)$ satisfying the relation

$$\ell_*(j^n(Y)) = s_n^Y\ell^*(j^n(Y_R)).$$

In order to prove this, consider an n-dimensional characteristic map $f_n^Y\colon Y \to K(\pi_nY,n)$ corresponding to $j^n(Y)$ and call $f_n^Y\otimes R\colon Y_R \to K(\pi_nY\otimes R,n)$ its localization. The composition of $f_n^Y\otimes R$ with a map $K(\pi_nY\otimes R,n) \to K(\pi_nY\otimes R,n)$ inducing multiplication by $1/s_n^Y$ on $\pi_nY\otimes R$ is an n-dimensional characteristic map for Y_R. Therefore, the induced homomorphism $(f_n^Y\otimes R)^*\colon H^n(K(\pi_nY\otimes R,n);\pi_nY\otimes R) \to H^n(Y_R;\pi_nY\otimes R)$ maps $i^n(K(\pi_nY\otimes R,n))$ onto $s_n^Y j^n(Y_R)$ for some n-dimensional characteristic class $j^n(Y_R)$ of Y_R. The commutative diagram

$$\begin{array}{ccccc}
H^n(K(\pi_nY,n);\pi_nY) & \xrightarrow{\ell_\#} & H^n(K(\pi_nY,n);\pi_nY\otimes R) & \xleftarrow{\ell^\#} & H^n(K(\pi_nY\otimes R,n);\pi_nY\otimes R) \\
\big\downarrow (f_n^Y)^* & & \big\downarrow (f_n^Y)^* & & \big\downarrow (f_n^Y\otimes R)^* \\
H^n(Y;\pi_nY) & \xrightarrow{\ell_*} & H^n(Y;\pi_nY\otimes R) & \xleftarrow{\ell^*} & H^n(Y_R;\pi_nY\otimes R)
\end{array}$$

completes the argument : $\ell_*(j^n(Y)) = (f_n^Y)^*\ell_\#(i^n(K(\pi_nY,n))) =$

$(f_n^Y)^* \ell^\#(i^n(K(\pi_n Y \otimes R, n))) = \ell^*(s_n^Y j^n(Y_R))$.

We determine finally the relationship between the characteristic classes of a space Y and those of its loop space ΩY. The cohomology suspension σ^* induces the commutative diagram

$$\begin{array}{ccc} H^n(Y;\pi_n Y) & \xrightarrow{\sigma^*} & H^{n-1}(\Omega Y;\pi_{n-1}\Omega Y) \\ \downarrow{\rho^*} & & \downarrow{\rho^*} \\ H^n(K(\pi_n Y,n);\pi_n Y) & \xrightarrow{\sigma^*} & H^{n-1}(K(\pi_{n-1}\Omega Y, n-1);\pi_{n-1}\Omega Y) \end{array}$$

and a homomorphism

$$\sigma^*: H^n(Y;\pi_n Y)/\mathrm{Ker}\rho^* \to H^{n-1}(\Omega Y;\pi_{n-1}\Omega Y)/\mathrm{Ker}\rho^* .$$

Proposition 1.6. *Let Y be a connected simple CW-complex and n a positive integer such that $k^{n+1}(Y)$ has finite order s_n^Y. Then*

$$\sigma^*(J^n(Y)) = (s_n^Y/s_{n-1}^{\Omega Y})J^{n-1}(\Omega Y) ,$$

where $s_{n-1}^{\Omega Y}$ is the order of $k^n(\Omega Y)$.

Proof. Since $k^n(\Omega Y)$ is the image of $k^{n+1}(Y)$ under the cohomology suspension (cf. [8, p. 438]), it has finite order $s_{n-1}^{\Omega Y}$ dividing s_n^Y. Let $j^n(Y) \in H^n(Y;\pi_n Y)$ be an n-dimensional characteristic class of Y. It follows from $\rho^*(j^n(Y)) = s_n^Y i^n(K(\pi_n Y,n))$ that $\rho^*\sigma^*(j^n(Y)) = \sigma^*(s_n^Y i^n(K(\pi_n Y,n))) = s_n^Y i^{n-1}(K(\pi_{n-1}\Omega Y, n-1)) = (s_n^Y/s_{n-1}^{\Omega Y})\rho^*(j^{n-1}(\Omega Y))$ for any $(n-1)$-dimensional characteristic class of ΩY. Therefore we get : $\sigma^*(J^n(Y)) = (s_n^Y/s_{n-1}^{\Omega Y})J^{n-1}(\Omega Y)$.

2. Examples.

(a) Consider the infinite loop space BU. There exist characteristic classes $j^n(BU) \in H^n(BU;\pi_n BU)$ for all positive integers n, but if n is odd, the vanishing of $\pi_n BU$ implies clearly that $j^n(BU) = 0$; more interesting are the even dimensions since $\pi_{2t}BU \cong \mathbb{Z}$ for any $t \geq 1$. As usual let us call c_t the t-th universal Chern class in $H^{2t}(BU;\mathbb{Z})$.

Proposition 2.1. *Let t be a positive integer. If $j^{2t}(BU)$ is any $2t$-dimensional characteristic class of BU, then $j^{2t}(BU) = \pm c_t$ + decomposable elements.*

Proof. It is known that $k^{2t+1}(BU)$ has order $(t-1)$! (cf. [6, Lemma 4.4]). Thus, if $\rho^*: H^{2t}(BU;\mathbb{Z}) \to H^{2t}(K(\mathbb{Z},2t);\mathbb{Z})$ is the homomorphism defined in Section 1, $\rho^*(j^{2t}(BU)) = (t-1)!\ i^{2t}(K(\mathbb{Z},2t))$. On the other hand it is also proved in [6, Lemma 4.5] that $\rho^*(c_t) = \pm(t-1)!\ i^{2t}(K(\mathbb{Z},2t))$. Consequently, $j^{2t} \mp c_t$ belongs to the kernel of ρ^*, which is generated by products of the Chern classes $c_1, c_2, \ldots, c_{t-1}$.

(b) Let A be the field of rationals $\mathbb{Q}$ or the ring of integers $\mathbb{Z}$, $SL(A)$ its infinite special linear group, and $Y := BSL(A)^+$ the simply connected space obtained by performing the plus construction on the classifying space of $SL(A)$. It is known by Borel's computation [4] that the rational cohomology of Y is an exterior algebra generated by elements of degree $4t+1$, $t \geq 1$:

$$H^*(BSL(A)^+;\mathbb{Q}) = \Lambda(x_5, x_9, \ldots, x_{4t+1}, \ldots)\ .$$

Since Y is an infinite loop space, we may consider characteristic classes $j^n(Y) \in H^n(Y;\pi_n Y)$ for all $n \geq 1$. We want to show that the classes provide a description of the generators x_{4t+1}, $t \geq 1$. Consider the localization map $\ell: Y \to Y_{\mathbb{Q}}$ (i.e., the rational type of Y). According to [4], $\pi_n Y_{\mathbb{Q}} \cong \mathbb{Q}$ if $n \equiv 1 \bmod 4$, $n \geq 5$, and $\pi_n Y_{\mathbb{Q}} = 0$ otherwise. Therefore, the map ℓ induces the homomorphism $\ell_*: H^{4t+1}(Y;\pi_{4t+1}Y) \to H^{4t+1}(Y;\mathbb{Q})$ for $t \geq 1$.

Proposition 2.2. *For $t \geq 1$, it is possible to choose*

$$x_{4t+1} = \ell_*(j^{4t+1}(BSL(A^+)))\ .$$

Proof. For any $t \geq 1$, let $j^{4t+1}(Y)$ be a $(4t+1)$-dimensional charactersitic class of Y and $j^{4t+1}(Y_{\mathbb{Q}})$ the corresponding characteristic class of $Y_{\mathbb{Q}}$ given by Remark 1.5. Since the k-invariants $k^{n+1}(Y)$ have finite order s_n^Y for all $n \geq 1$, $Y_{\mathbb{Q}}$ is a product of Eilenberg-MacLane spaces : $Y_{\mathbb{Q}} = \prod_{t=1}^{\infty} K(\mathbb{Q}, 4t+1)$. Its rational cohomology is then an exterior algebra generated by the classes $j^{4t+1}(Y_{\mathbb{Q}})$, $t \geq 1$. Using the isomorphism $\ell^*: H^*(Y_{\mathbb{Q}};\mathbb{Q}) \xrightarrow{\cong} H^*(Y;\mathbb{Q})$, we may choose $x_{4t+1} = s^Y_{4t+1}\ell^*(j^{4t+1}(Y_{\mathbb{Q}}))$ and deduce from Remark 1.5. that $x_{4t+1} = \ell_*(j^{4t+1}(Y))$ for $t \geq 1$.

Notice that $\ell_*(j^n(BSL(A)^+)) = 0$ if $n \not\equiv 1 \bmod 4$.

The same argument produces analogous assertions for the generators of the rational cohomology of $Sp(A)$ and $O(A)$ (cf.[4]) :

$$H^*(BSp(A)^+;\mathbb{Q}) = \mathbb{Q}[y_2, y_6, \ldots, y_{4t-2}, \ldots] ,$$
$$H^*(BO(A)^+;\mathbb{Q}) = \mathbb{Q}[z_4, z_8, \ldots, z_{4t}, \ldots] .$$

Proposition 2.3. *For $t \geq 1$, it is possible to choose*

$$y_{4t-2} = \ell_*(j^{4t-2}(BSp(A)^+)) \quad \textit{and} \quad z_{4t} = \ell_*(j^{4t}(BO(A)^+)) .$$

Remark finally that similar results are obtained when A is an imaginary quadratic number field or its ring of integers.

3. Obstruction theory.

The classical obstruction theory (cf. [5] or [8, §V. 5-6]) examines the following problem : let (X, A) be a relative CW-complex, Y a connected simple CW-complex and g a map $A \to Y$; the question is to determine whether or not g can be extended over X.

If Y is $(m-1)$-connected $(m \geq 1)$, it is possible to extend g over X_m, the m-dimensional skeleton of (X, A). If $\bar{g}: X_m \to Y$ is such an extension, one defines a cocycle $c^{m+1}(\bar{g}) \in H^{m+1}(X_{m+1}, X_m; \pi_m Y)$ whose vanishing corresponds to the extensibility of $\bar{g}$ over X_{m+1}, and one shows that if $\bar{g}$ and $\tilde{g}: X_m \to Y$ are extensions of g, then $c^{m+1}(\bar{g}) \sim c^{m+1}(\tilde{g})$: consequently, there is a uniquely defined element

$$\gamma^{m+1}(g) \in H^{m+1}(X, A; \pi_m Y)$$

which is the cohomology class of $c^{m+1}(\bar{g})$ for any extension $\bar{g}: X_m \to Y$ of g; $\gamma^{m+1}(g)$ is called the primary obstruction to extending g. It is related to the characteristic class $i^m(Y) \in H^m(Y; \pi_m Y)$ of the target space Y by the formula

$$\gamma^{m+1}(g) = (-1)^m \delta g^*(i^m(Y)) ,$$

where g^* denotes the homomorphism $H^*(Y;-) \to H^*(A;-)$ induced by g, and $\delta: H^*(A;-) \to H^{*+1}(X,A;-)$ the coboundary operator of the cohomology sequence of the pair (X,A). The primary obstruction $\gamma^{m+1}(g)$ gives a partial solution to the extension problem : $\gamma^{m+1}(g)$ is trivial if and only if g can be extended over X_{m+1}. But in general, there exist higher obstructions to extending g over X, and it is hard to describe them.

The purpose of this section is to consider the extension problem in the following special situation : we assume that $g: A \to Y$ has an extension $\bar{g}: X_n \to Y$ and that the k-invariant $k^{n+1}(Y)$ of Y has finite order s_n^Y (but we do not assume that Y is $(n-1)$-connected). The basic idea is to apply the classical theory to the composition of g (respectively $\bar{g}$) with any n-dimensional characteristic map $f_n^Y: Y \to K(\pi_n Y, n)$ introduced in Section 1.

Lemma 3.1. $c^{n+1}(f_n^Y \circ \bar{g}) = s_n^Y c^{n+1}(\bar{g}) \in H^{n+1}(X_{n+1}, X_n; \pi_n Y)$.

Proof. An elementary property of the cocycle c^{n+1} is that $c^{n+1}(f_n^Y \circ \bar{g}) = (f_n^Y)_*(c^{n+1}(\bar{g}))$, where $(f_n^Y)_*: H^{n+1}(X_{n+1}, X_n; \pi_n Y) \to H^{n+1}(X_{n+1}, X_n; \pi_n K(\pi_n Y, n))$ is induced by f_n^Y. But, by definition, $(f_n^Y)_*$ is multiplication by s_n^Y: $c^{n+1}(f_n^Y \circ \bar{g}) = s_n^Y c^{n+1}(\bar{g})$.

It follows from this lemma that $c^{n+1}(f_n^Y \circ \bar{g})$ does not depend on the choice of f_n^Y. If $\bar{g}$ and $\tilde{g}: X_n \to Y$ are extensions of g, then $c^{n+1}(f_n^Y \circ \bar{g}) \sim c^{n+1}(f_n^Y \circ \tilde{g})$ since $K(\pi_n Y, n)$ is $(n-1)$-connected. Thus, we may give the following

Definition 3.2. $\zeta^{n+1}(g) \in H^{n+1}(X, A; \pi_n Y)$ is the cohomology class of $s_n^Y c^{n+1}(\bar{g})$ for any extension $\bar{g}: X_n \to Y$ of $g: A \to Y$. It turns out that $\zeta^{n+1}(g) = \gamma^{n+1}(f_n^Y \circ g)$ for any n-dimensional characteristic map f_n^Y. Observe that this *obstruction class* $\zeta^{n+1}(g)$ is well defined, although f_n^Y (respectively $j^n(Y)$) is not uniquely determined.

Proposition 3.3. $\zeta^{n+1}(g) = (-1)^n \delta g^*(j^n(Y)) \in H^{n+1}(X, A; \pi_n Y)$ *for any n-dimensional characteristic class* $j^n(Y) \in H^n(Y; \pi_n Y)$.

Proof. If f_n^Y is an n-dimensional characteristic map corresponding to $j^n(Y)$, then $\zeta^{n+1}(g) = \gamma^{n+1}(f_n^Y \circ g) = (-1)^n \delta g^*(f_n^Y)^*(i^n(K(\pi_n Y, n))) = (-1)^n \delta g^*(j^n(Y))$.

Remark 3.4. The obstruction class ζ^{n+1} has the following properties.

a) If $g_0, g_1 \colon A \to Y$ are homotopic maps, extensible over X_n, then $\zeta^{n+1}(g_0) = \zeta^{n+1}(g_1)$.

b) Let Y' be another connected simple space with $k^{n+1}(Y')$ of finite order $s_n^{Y'}$, h a map $Y \to Y'$ and $h_* \colon H^{n+1}(X, A; \pi_n Y) \to H^{n+1}(X, A; \pi_n Y')$ the homomorphism induced by h. Then $s_n^Y \zeta^{n+1}(h \circ g) = s_n^{Y'} h_*(\zeta^{n+1}(g)) \in H^{n+1}(X, A; \pi_n Y')$ because both terms are equal to the cohomology class of $s_n^Y s_n^{Y'} c^{n+1}(h \circ \bar{g}) = s_n^Y s_n^{Y'} h_*(c^{n+1}(\bar{g}))$.

c) If h' is a cellular map $(X', A') \to (X, A)$, then $\zeta^{n+1}(g \circ h'|_{A'}) = (h')^*(\zeta^{n+1}(g)) \in H^{n+1}(X', A'; \pi_n Y)$, where $(h')^*$ is the homomorphism induced by h' in cohomology.

Our objective is now to exhibit the relationships between the obstruction classes $\zeta^{n+1}(g)$ and the solution of the extension problem.

Theorem 3.5. *Let (X, A) be a relative CW-complex, Y a connected simple CW-complex, g a map $A \to Y$, and n a positive integer such that the k-invariant $k^{n+1}(Y)$ has finite order s_n^Y.*

(a) If g can be extended over X_{n+1}, then $\zeta^{n+1}(g) = 0$.

(b) Assume that g has an extension $\bar{g} \colon X_n \to Y$, that multiplication by s_n^Y on $H^{n+1}(X, A; \pi_n Y)$ is injective, and that $\zeta^{n+1}(g) = 0$, then g can be extended over X_{n+1}.

Proof. Assertion (a) is obvious since the extensibility of g (and consequently of $f_n^Y \circ g$, for any n-dimensional characteristic map f_n^Y) over X_{n+1} implies the vanishing of $\gamma^{n+1}(f_n^Y \circ g) = \zeta^{n+1}(g)$. In order to prove (b), we deduce from the hypothesis and Definition 3.2 that $s_n^Y c^{n+1}(\bar{g}) \sim 0$, and therefore that $c^{n+1}(\bar{g}) \sim 0$. It is then a consequence of [5, Extension theorem I] that $\bar{g}|_{X_{n-1}}$ can be extended over X_{n+1}.

Corollary 3.6. *Let (X, A) be a d-dimensional relative CW-complex, Y an $(m-1)$-connected simple CW-complex $(m \geq 1)$, and g a map $A \to Y$. Assume for $m+1 \leq n \leq d-1$ that $k^{n+1}(Y)$ has finite order s_n^Y and that multiplication* *by s_n^Y on $H^{n+1}(X,A;\pi_n Y)$ is injective. Then g can be extended over X if and only if $\zeta^{n+1}(g)=0$ in $H^{n+1}(X,A;\pi_n Y)$ for $m \le n \le d-1$.*

Proof. Suppose that $\zeta^{n+1}(g)=0$ for $m \le n \le d-1$. Observe first that $\gamma^{m+1}(g)=(-1)^m\delta g^*(i^m(Y))=0$ by Propositions 1.3 and 3.3 : thus g may be extended over X_{m+1}. We then apply inductively (for $n=m+1,m+2,\dots,d-1$) assertion (b) of the previous theorem and obtain an extension of g over $X_d = X$. The converse is trivial.

We consider finally the extension problem in the case where Y is an $(m-1)$-connected infinite loop space : the obstruction classes $\zeta^{n+1}(g)$ may be defined because each k-invariant $k^{n+1}(Y)$ of the space Y is a cohomology class of finite order s_n^Y (note that any prime p dividing s_n^Y satisfies the inequality $p \le (n-m+3)/2$ according to [3, Corollary 1.9]). For a positive integer t, let us call M_t the product of all primes $p \le t/2+1$.

Corollary 3.7. *Let (X,A) be a d-dimensional relative CW-complex, Y an $(m-1)$-connected infinite loop space ($m \ge 1$), and g a map $A \to Y$. Let R denote the ring $\mathbb{Z}[1/M_{d-m}]$, ℓ the localization map $Y \to Y_R$, and $\ell_*: H^{n+1}(X,A;\pi_n Y) \to H^{n+1}(X,A;\pi_n Y\otimes R)$ the homomorphism induced by ℓ. Then the composition $\ell\circ g: A \to Y_R$ is extensible over X if and only if $\ell_*(\zeta^{n+1}(g))=0$ in $H^{n+1}(X,A;\pi_n Y\otimes R)$ for $m \le n \le d-1$. In particular, if $j^n(Y)$ is an n-dimensional characteristic class satisfying $\delta g^*(j^n(Y))=0$ in $H^{n+1}(X,A;\pi_n Y)$ for $m \le n \le d-1$, then $\ell\circ g$ has an extension $X \to Y_R$.*

Proof. Since the map ℓ localizes the k-invariants, one has $s_n^{Y_R}=1$ for $n \le d-1$. The previous corollary asserts that $\ell\circ g$ can be extended over X if and only if $\zeta^{n+1}(\ell\circ g)=0$ in $H^{n+1}(X,A;\pi_n Y\otimes R)$ for $m \le n \le d-1$; but $\zeta^{n+1}(\ell\circ g)$ vanishes if and only if $\ell_*(\zeta^{n+1}(g)) = s_n^Y\zeta^{n+1}(\ell\circ g)=0$. (cf. Remark 3.4 (b)). Finally, if $\delta g^*(j^n(Y))=0$, then $\ell_*(\zeta^{n+1}(g))=(-1)^n\ell_*(\delta g^*(j^n(Y)))=0$.

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