Dividends and the Time of Ruin under Barrier Strategies with a Capital-Exchange Agreement

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Abstract. We consider a capital-exchange agreement, where two insurers recapitalize each other in certain situations with funds they would otherwise use for dividend payments. We derive equations characterizing the expected time of ruin and the expected value of the respective discounted dividends until ruin, if dividends are paid according to a barrier strategy. In a Monte Carlo simulation study we illustrate the potential advantages of this type of collaboration.

1 Introduction

The identification of dividend payout strategies that balance safety and profitability is a classical topic of insurance risk theory. Whereas ruin theory focuses on the safety aspects (see e.g. Asmussen and Albrecher [2] for a survey), the de Finetti problem of maximizing expected discounted dividends over the lifetime of an insurance portfolio concentrates exclusively on the profitability aspect (see e.g. Azcue and Muler [4] for a recent overview of control problems arising from that). For control problems that address a balancing of the time of ruin against early dividend pay-outs, see e.g. [12, 13] and, in the form of a constraint on the ruin time, [11]. At the same time, there have recently been some research efforts to address the analysis of several surplus processes simultaneously, see e.g Chan et al. [7], Cai and Li [6], Gong et al. [10] and Avram et al. [3] on ruin-related measures and Badescu et al. [5] for a capital allocation problem. It is a natural

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question in this context whether certain forms of collaboration between two different companies can lead to a better overall profit and safety compromise than what the two can optimally achieve stand-alone. Gerber and Shiu [9] discuss the effects of merging two portfolios on optimal dividends according to barrier strategies, and Albrecher, Azcue and Muler [1] recently identified the optimal dividend strategy when two companies pay each other's deficit as long as it can be afforded.

In this paper we look at a different type of collaboration between two companies: whenever a company is in a sufficiently comfortable position to pay out capital, it first helps the other company to reach a well-capitalized position before it starts to pay out dividends to shareholders. Such a collaboration strategy clearly has a smoothing effect on the survival of both companies, while dividends are still paid out if the overall situation is sufficiently favorable. Within the family of barrier dividend strategies we look at the effects of such a type of collaboration on the expected ruin time and the resulting expected discounted dividend payments.

Section 2 introduces the model assumptions in detail. In Sections 3 and 4, we derive equations which are satisfied by the insurer's expected time of ruin and the expectation of the discounted dividends, respectively, under the capital-exchange agreement. In Section 5, we provide an efficient Monte Carlo algorithm which we apply in a simulation study. Finally, we aim to illustrate some decision criteria for when to enter such a capital-exchange agreement.

2 The Model

Let I_1 and I_2 be two insurers, and initially consider the situation where their surplus processes $C_i(t)$, $i = 1, 2$, are independent and each surplus follows a Cramér-Lundberg process,

$$C_i(t) = x_i + c_i t - S_i(t), \tag{1}$$

where x_i is the insurer's initial surplus, c_i is the premium (income) rate, and $S_i(t)$ is a compound Poisson process, representing the aggregate claims of I_i up to time t, with rate λ_i and individual claim size distribution function $F_{Y_i}(y)$ (density function $f_{Y_i}(y)$, respectively).

We now adjust this framework as follows. The two insurers enter into a *capital exchange agreement*, and each insurer sets a respective barrier b_i. Like under a classical dividend barrier strategy, I_i fully pays out its income as long as its current surplus is at barrier b_i (i.e. the surplus process of I_i is reflected at b_i). The capital-exchange agreement now defines that these pay-outs go to the other insurer if the surplus of that one is below its barrier level, and otherwise to the own shareholders in the form of dividends. Note that such a capital-exchange agreement introduces dependence on the adjusted surplus processes of the two insurers.

Let $D_i(t)$ be the aggregate dividend payments at time t of insurer i, and $A_i(t)$ are the aggregate payments insurer I_i has paid to the partner company under the agreement by time t. The *adjusted surplus* $U_i(t)$ of insurer i is then given by

$$\begin{aligned} U_1(t) &= C_1(t) - (D_1(t) + A_1(t)) + A_2(t), \\ U_2(t) &= C_2(t) - (D_2(t) + A_2(t)) + A_1(t) \end{aligned}$$

and we can write the dynamics as

$$\begin{aligned} dU_1(t) &= c_1 dt - dS_1 - dD_1 - dA_1 + dA_2, \\ dU_2(t) &= c_2 dt - dS_2 - dD_2 - dA_2 + dA_1. \end{aligned}$$

The time of ruin of insurer i in this framework can then be defined as a function of the two initial surplus levels and the barrier heights, and we write

$$\tau_i(x_1, x_2, b_1, b_2) = \inf\{t | U_i(t) < 0;\ U_1(0) = x_1,\ U_2(0) = x_2,\ b_1,\ b_2\}. \tag{2}$$

We define that the capital-exchange agreement ceases to exist once one of the two insurers is ruined, and the surviving insurer keeps operating its dividend barrier strategy with barrier b_i on a stand-alone basis.

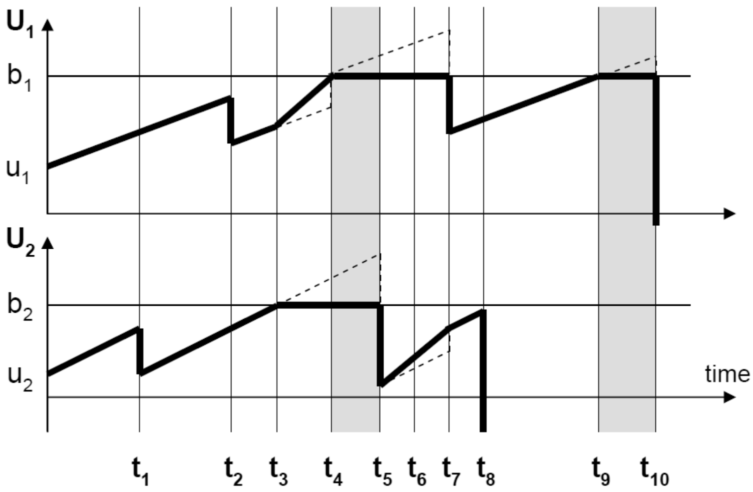

Figure 1: Sample path of (U_1,U_2).

Figure 1 depicts a sample path of U_1, U_2. Up to time t_3, both adjusted surplus processes run below their respective pay-out barriers b_1 and b_2, and the slope is c_1 and c_2, respectively. At time t_3, U_2 reaches its pay-out barrier b_2. Income at rate c_2 is from now on paid to the partner insurer (who now has an income rate of $c_1 + c_2$) up to time t_4 where also I_1 reaches its pay-out barrier b_1. Between times t_4 and t_5, any income is paid to the respective shareholders as dividends, because *both* adjusted surpluses now run at their pay-out barriers. At time t_5, a claim pulls the adjusted surplus of I_2 below b_2, so that it is now supported by I_1 up to time t_7 (during this period I_2 has an income rate of $c_1 + c_2$). Finally, at time t_8, I_2 suffers a large claim and is ruined. The capital-exchange agreement ceases to exist, and I_1 continues on its own. It now pays dividends whenever its adjusted surplus is at b_1 (between times t_9 and t_{10}) and is ruined once its adjusted surplus drops negative (at time t_{10}).

One observes that money is now generally kept longer in the system of the two insurers, as it is only released to shareholders once the adjusted surpluses of both insurers run at their respective barriers. Intuitively one expects this to have a positive impact on the lifetime, while the impact on expected dividends is not so obvious. In particular, a weak capital-exchange partner would most likely have a negative impact on dividend payments, while a strong partner might help to lift one's adjusted surplus faster so that dividend payments might (re-)start at an earlier time.
While the setup of the dependence structure is fairly straightforward to introduce, the implied mathematics are found to be challenging. Practically, the capital-exchange feature of the presented model can be extended to the case of n insurers, having in mind a holding company (HoldCo - for example, a financial investor) that owns a number of separate insurance undertakings, where well-performing entities support underperforming ones. Dividend payments at the HoldCo level are then only made if all entities are sufficiently well capitalised (as defined by their respective barriers b_i). The HoldCo could then assess such a capital exchange strategy by balancing the effects on the default risk and the dividend income, depending on the risk willingness of its shareholders. Note that within a classical insurance group, dividend clawback rules and financial assistance requirements might restrict the choice of implementable capital-exchange mechanisms as outlined here.

3 The Expected Time of Ruin

We observe that the surplus process U_i is bounded from above by the otherwise identical surplus process with income rate $c_1 + c_2$, adjusted by a dividend barrier strategy with barrier b_i, and conclude from the fact that Cramér-Lundberg-type surplus processes under a barrier strategy with finite barrier b have ruin probability one, that also $\mathbb{P}[\tau_i(x_1, x_2; b_1, b_2) < \infty] = 1$ for $i = 1, 2$. Hence, we turn to an alternative measure of risk. We assume the pay-out barriers b_1 and b_2 as fixed and define the *expected time of ruin* of insurer i as a function of the initial surplus levels

$$\gamma_i(x_1, x_2) = \mathbb{E}[\tau_i(x_1, x_2)]. \tag{3}$$

We restrict the support of $\gamma_i(x_1, x_2)$ to $0 \leq x_i \leq b_i$, $i = 1, 2$ (the situations $x_1 > b_1$ or $x_2 > b_2$ can be related to the considered case by defining how initial immediate lump sum payments are made to capital exchange partners and shareholders). Let us focus on the expected time of ruin for I_1 (by symmetry the situation of I_2 follows analogously).

Conditioning on the first arrival of a claim from either S_1 or S_2 within h time units (for h sufficiently small) and exploiting the Markov property of the bivariate process (U_1, U_2) gives the following equations for $x_i, b_i \geq 0$. In particular, for the interior points, $x_1 < b_1,\ x_2 < b_2$:

$$\begin{aligned}
\gamma_1(x_1, x_2) = {} & e^{-(\lambda_1+\lambda_2)h}(h + \gamma_1(x_1 + c_1 h, x_2 + c_2 h)) \\
& + \int_0^h e^{-\lambda_2 t} \lambda_1 e^{-\lambda_1 t} \left(\int_0^{x_1 + c_1 t} (t + \gamma_1(x_1 + c_1 t - z, x_2 + c_2 t)) f_{Y_1}(z) dz \right. \\
& \left. + t \cdot (1 - F_{Y_1}(x_1 + c_1 t)) \right) dt \\
& + \int_0^h e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 t} \left(\int_0^{x_2 + c_2 t} (t + \gamma_1(x_1 + c_1 t, x_2 + c_2 t - z)) f_{Y_2}(z) dz \right. \\
& \left. + (t + \gamma_1^{(0)}(x_1 + c_1 t)) \cdot (1 - F_{Y_2}(x_2 + c_2 t)) \right) dt,
\end{aligned} \tag{4}$$

where $\gamma_1^{(0)}(x)$ is the expected time of ruin of I_1 in the (classical) stand-alone case with initial surplus x.

Remark 1. In the case of exponential jump sizes with mean $\mathbb{E}[Y_1] = 1/\nu_1$, the expected time of ruin $\gamma_1^{(0)}(x)$ in the classical case is known explicitly (cf. Gerber (1979), p. 150) as

$$\left(c_1 - \frac{\lambda_1}{\nu_1} \right) \cdot \gamma_1^{(0)}(x) = \frac{e^{Rb_1}}{R} \left(\frac{\nu_1}{\nu_1 - R} - e^{-Rx} \right) - \frac{1}{\nu_1} - x, \tag{5}$$

where $R = \nu_1 - \frac{\lambda_1}{c_1}$ is the *adjustment coefficient*. ■

Furthermore, we find for the boundary $x_1 = b_1,\ x_2 < b_2$:

$$
\begin{aligned}
\gamma_1(b_1,x_2) &= e^{-(\lambda_1+\lambda_2)h}(h+\gamma_1(b_1,x_2+(c_1+c_2)h)) \\
&\quad + \int_0^h e^{-\lambda_2 t}\lambda_1 e^{-\lambda_1 t}\left(\int_0^{b_1}(t+\gamma_1(b_1-z,x_2+(c_1+c_2)t))f_{Y_1}(z)dz\right. \\
&\quad \left. +t\cdot(1-F_{Y_1}(b_1))\right) dt \\
&\quad + \int_0^h e^{-\lambda_1 t}\lambda_2 e^{-\lambda_2 t}\left(\int_0^{x_2+(c_1+c_2)t}(t+\gamma_1(b_1,x_2+(c_1+c_2)t-z))f_{Y_2}(z)dz\right. \\
&\quad \left. +(t+\gamma_1^{(0)}(b_1))\cdot(1-F_{Y_2}(x_2+(c_1+c_2)t))\right) dt, \qquad (6)
\end{aligned}
$$

for the boundary $x_1 < b_1,\ x_2 = b_2$:

$$
\begin{aligned}
\gamma_1(x_1,b_2) &= e^{-(\lambda_1+\lambda_2)h}(h+\gamma_1(x_1+(c_1+c_2)h,b_2)) \\
&\quad + \int_0^h e^{-\lambda_2 t}\lambda_1 e^{-\lambda_1 t}\left(\int_0^{b_1+(c_1+c_2)t}(t+\gamma_1(x_1+(c_1+c_2)t-z,b_2))f_{Y_1}(z)dz\right. \\
&\quad \left. +\ t\cdot(1-F_{Y_1}(x_1+(c_1+c_2)t))\right) dt \\
&\quad + \int_0^h e^{-\lambda_1 t}\lambda_2 e^{-\lambda_2 t}\left(\int_0^{b_2}(t+\gamma_1(x_1+(c_1+c_2)t,b_2-z))f_{Y_2}(z)dz\right. \\
&\quad \left. +(t+\gamma_1^{(0)}(x_1+(c_1+c_2)t))\cdot(1-F_{Y_2}(b_2))\right) dt, \qquad (7)
\end{aligned}
$$

and in the corner point $x_1 = b_1$, $x_2 = b_2$:

$$
\begin{aligned}
\gamma_1(b_1,b_2) &= e^{-(\lambda_1+\lambda_2)h}(h+\gamma_1(b_1,b_2)) \\
&\quad + \int_0^h e^{-\lambda_2 t}\lambda_1 e^{-\lambda_1 t}\left(\int_0^{b_1}(t+\gamma_1(b_1-z,b_2))f_{Y_1}(z)dz+t\cdot(1-F_{Y_1}(b_1))\right) dt \\
&\quad + \int_0^h e^{-\lambda_1 t}\lambda_2 e^{-\lambda_2 t}\left(\int_0^{b_2}(t+\gamma_1(b_1,b_2-z))f_{Y_2}(z)dz\right. \\
&\quad \left. +(t+\gamma_1^{(0)}(b_1))\cdot(1-F_{Y_2}(b_2))\right) dt.
\end{aligned}
$$

The function γ_1 is continuous in the interior of $[0,b_1]\times[0,b_2]$, which can be seen by approaching x_1,x_2 from arbitrary directions by taking $x_1=x_1+j\cdot h$ and $x_2=x_2+k\cdot h$, with $j,k\in\mathbb{R}$, and letting $h\to 0$ in (4) in each case. Comparison of (6) and (7) with (4) furthermore shows continuity at the boundaries $x_1=b_1$ and $x_2=b_2$.

Differentiating (4) w.r.t. h, we observe by symmetry that γ_1 is also differentiable w.r.t. x_1, x_2 in the interior. Applying the operator $\frac{d}{dh}$ to all of the above conditions and taking the limit $h\to 0$, we obtain a system of integro-differential equations,

$$\begin{aligned}
x_1<b_1,x_2<b_2 \quad &: \quad 0=-(\lambda_1+\lambda_2)\gamma_1(x_1,x_2)+1+c_1\frac{\partial\gamma_1}{\partial x_1}(x_1,x_2)+c_2\frac{\partial\gamma_1}{\partial x_2}(x_1,x_2)\\
&\quad +\lambda_1\int_0^{x_1}\gamma_1(x_1-z,x_2)f_{Y_1}(z)dz\\
&\quad +\lambda_2\left(\int_0^{x_2}\gamma_1(x_1,x_2-z)f_Y(z)dz+\gamma_1^{(0)}(x_1)\cdot(1-F_{Y_2}(x_2))\right), \qquad (8)\\
x_1=b_1,x_2<b_2 \quad &: \quad 0=-(\lambda_1+\lambda_2)\gamma_1(b_1,x_2)+1+(c_1+c_2)\cdot\frac{\partial\gamma_1}{\partial x_2}(b_1,x_2)\\
&\quad +\lambda_1\cdot\int_0^{b_1}\gamma_1(b_1-z,x_2)f_{Y_1}(z)dz\\
&\quad +\lambda_2\left(\int_0^{x_2}\gamma_1(b_1,x_2-z)f_{Y_2}(z)dz+\gamma_1^{(0)}(b_1)\cdot(1-F_{Y_2}(x_2))\right), \qquad (9)\\
x_1<b_1,x_2=b_2 \quad &: \quad 0=-(\lambda_1+\lambda_2)\gamma_1(x_1,b_2)+1+(c_1+c_2)\cdot\frac{\partial\gamma_1}{\partial x_1}(x_1,b_2)\\
&\quad +\lambda_1\int_0^{x_1}\gamma_1(x_1-z,b_2)f_{Y_1}(z)dz\\
&\quad +\lambda_2\left(\int_0^{b_2}\gamma_1(x_1,b_2-z)f_{Y_2}(z)dz+\gamma_1^{(0)}(x_1)\cdot(1-F_{Y_2}(b_2))\right), \qquad (10)
\end{aligned}$$

and again an equation in the corner point (b_1,b_2),

$$\begin{aligned}
0&=-(\lambda_1+\lambda_2)\gamma_1(b_1,b_2)+1\\
&+\lambda_1\cdot\int_0^{b_1}\gamma_1(b_1-z,b_2)f_{Y_1}(z)dz\\
&+\lambda_2\cdot\left(\int_0^{b_2}\gamma_1(b_1,b_2-z)f_{Y_2}(z)dz+\gamma_1^{(0)}(b_1)\cdot(1-F_{Y_2}(b_2))\right). \qquad (11)
\end{aligned}$$

Continuity of $\gamma_1(x_1,x_2)$ on the boundaries $x_1=b_1$ or $x_2=b_2$ and comparing (8) to (9) and (10), gives the boundary conditions

$$\frac{\partial\gamma_1}{\partial x_1}(b_1,x_2)=\frac{\partial\gamma_1}{\partial x_2}(b_1,x_2)\ \ \forall 0\le x_2<b_2, \tag{12}$$
$$\frac{\partial\gamma_1}{\partial x_1}(x_1,b_2)=\frac{\partial\gamma_1}{\partial x_2}(x_1,b_2)\ \ \forall 0\le x_1<b_1. \tag{13}$$

Similarly, approaching (b_1,b_2) from interior points gives
$c_1\frac{\partial\gamma_1}{\partial x_1}(b_1,b_2)+c_2\frac{\partial\gamma_1}{\partial x_2}(b_1,b_2)=0$, so that we have $\frac{\partial\gamma_1}{\partial x_1}(b_1,b_2)=\frac{\partial\gamma_1}{\partial x_2}(b_1,b_2)=0$.

Altogether $\gamma_1(x_1,x_2)$ is characterised as the solution to the equation system (8) with boundary conditions (12) and (13).

Exponential claims. Assume that the claim sizes of I_1 are i.i.d. $\mathrm{Exp}(\nu_1)$ distributed, and the claim sizes of I_2 are $\mathrm{Exp}(\nu_2)$ distributed. Applying the operator $\left(\frac{d}{dx_1}+\nu_1\right)$ followed by the operator $\left(\frac{d}{dx_2}+\nu_2\right)$ to (8) yields a third-order PDE with constant coefficients,

$$\begin{aligned}
0 &= \nu_1\nu_2+\nu_2c_1\nu_1\left(1-\frac{\lambda_1}{c_1\nu_1}\right)\frac{\partial\gamma_1}{\partial x_1}(x_1,x_2)+\nu_1c_2\nu_2\left(1-\frac{\lambda_2}{c_2\nu_2}\right)\frac{\partial\gamma_1}{\partial x_2}(x_1,x_2)\\
&+ \left[c_1\nu_1\left(1-\frac{\lambda_1}{c_1\nu_1}\right)+c_2\nu_2\left(1-\frac{\lambda_2}{c_2\nu_2}\right)\right]\frac{\partial^2\gamma_1}{\partial x_1\partial x_2}(x_1,x_2)\\
&+ c_1\nu_2\frac{\partial^2\gamma_1}{\partial x_1^2}(x_1,x_2)+c_2\nu_1\frac{\partial^2\gamma_1}{\partial x_2^2}(x_1,x_2)\\
&+ c_1\frac{\partial^3\gamma_1}{\partial x_1^2\partial x_2}(x_1,x_2)+c_2\frac{\partial^3\gamma_1}{\partial x_1\partial x_2^2}(x_1,x_2),
\end{aligned} \tag{14}$$

so that the dynamics in the interior are now described locally. One observes that $\gamma_p(x_1,x_2)=A_1x_1+A_2x_2$ with the condition $1=\left(\frac{\lambda_1}{\nu_1}-c_1\right)A_1+\left(\frac{\lambda_2}{\nu_2}-c_2\right)A_2$ is a particular solution to this inhomogeneous PDE. Terms of the form $e^{-rx_1}e^{-sx_2}$ can appear in the solution to the homogeneous problem if they fulfil the characteristic equation

$$\begin{aligned} 0 \;=\; & -\nu_2 c_1 \nu_1 \left(1 - \frac{\lambda_1}{c_1\nu_1}\right) r - \nu_1 c_2 \nu_2 \left(1 - \frac{\lambda_2}{c_2\nu_2}\right) s \\ & + \left[c_1\nu_1 \left(1 - \frac{\lambda_1}{c_1\nu_1}\right) + c_2\nu_2 \left(1 - \frac{\lambda_2}{c_2\nu_2}\right)\right] rs \\ & + c_1\nu_2 r^2 + c_2\nu_1 s^2 - c_1 r^2 s - c_2 r s^2. \end{aligned} \tag{15}$$

A plot of (15) for a particular choice of parameters is shown in Figure 2.

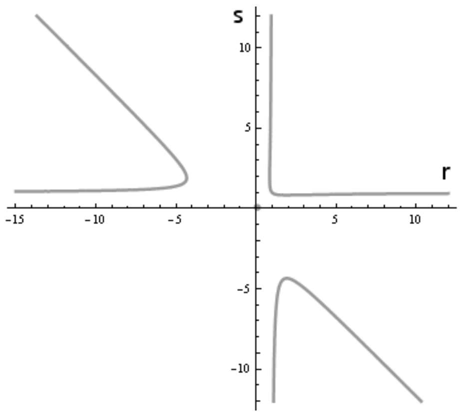

Figure 2: Plot of the implicit equation (15), with $c_1 = c_2 = 6$, $\lambda_1 = \lambda_2 = 5$ and $\nu_1 = \nu_2 = 1$.

While it turns out to be mathematically intricate to obtain an explicit solution for $\gamma_1(x_1, x_2)$ (which must also match the original IDE (8) - note that some terms cancelled when applying the differential operator, which now need to be recalibrated by a suitable combination of homogeneous solutions - and the boundary conditions), the above characterisation may be useful for setting up a numerical solution procedure or also a hybrid numerical procedure, where finite-difference methods are applied after simulating the boundaries, with the aim of achieving an improvement in run time over crude Monte Carlo simulation.

4 The Expected Sum of Discounted Dividends

Apart from the prolonging effect of the capital-exchange agreement on the expected time until ruin, we are interested in how the expected sum of discounted dividends until ruin is affected. We hence define

$$V_i(x_1, x_2; b_1, b_2) \tag{16}$$

as the expectation of the sum of discounted dividends until ruin paid to the shareholders of I_i, where we use a constant force of interest $\delta > 0$ for discounting. In the following we will assume that the pay-out barriers have been set and, thus, use the shortened notation $V_1(x_1, x_2) = V_1(x_1, x_2; b_1, b_2)$.

Proceeding as in Section 3, one can again condition on the occurrence of a jump event within h time units, h sufficiently small, to find that for the interior points $x_1 < b_1,\ x_2 < b_2$:

$$\begin{aligned}
V_1(x_1, x_2) = {} & e^{-(\lambda_1+\lambda_2)h} e^{-\delta h} V_1(x_1 + c_1 h, x_2 + c_2 h) \\
& + \int_0^h e^{-\lambda_2 t} \lambda_1 e^{-\lambda_1 t} \left(\int_0^{x_1+c_1 t} e^{-\delta t} V_1(x_1 + c_1 t - z, x_2 + c_2 t) f_{Y_1}(z) dz \right) dt \\
& + \int_0^h e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 t} \left(\int_0^{x_2+c_2 t} e^{-\delta t} V_1(x_1 + c_1 t, x_2 + c_2 t - z) f_{Y_2}(z) dz \right. \\
& \left. + e^{-\delta t} V_1^{(0)}(x_1 + c_1 t)) \cdot (1 - F_{Y_2}(x_2 + c_2 t) \right) dt,
\end{aligned} \tag{17}$$

where $V_1^{(0)}(x_1)$ is the stand-alone expectation of the sum of discounted dividends.

Remark 2. In the exponential claim size case with $\mathbb{E}[Y_1] = 1/\nu_1$, the expectation of the sum of discounted dividends in the classical case has an explicit form (cf. [8], p. 183),

$$V_1^{(0)}(x_1, b_1) = \frac{(\nu_1 + r_1)e^{r_1 x_1} - (\nu_1 + r_2)e^{r_2 x_1}}{r_1(\nu_1 + r_1)e^{r_1 b} - r_2(\nu_1 + r_2)e^{r_2 b}}, \tag{18}$$

where r_1 and r_2 are the solutions to the equation

$$\xi^2 + \left(\nu_1 - \frac{\lambda_1 + \delta}{c_1}\right)\xi - \frac{\nu_1 \delta}{c_1} = 0. \tag{19}$$

■

Similarly it follows for the boundary $x = b_1,\ x_2 < b_2$:

$$\begin{aligned}
V_1(b_1, x_2) = {} & e^{-(\lambda_1+\lambda_2)h} e^{-\delta h} V_1(b_1, x_2 + (c_1 + c_2)h) \\
& + \int_0^h e^{-\lambda_2 t} \lambda_1 e^{-\lambda_1 t} \left(\int_0^{b_1} e^{-\delta t} V_1(b_1 - z, x_2 + (c_1 + c_2)t) f_{Y_1}(z) dz \right) dt \\
& + \int_0^h e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 t} \left(\int_0^{x_2 + (c_1+c_2)t} e^{-\delta t} V_1(b_1, x_2 + (c_1 + c_2)t - z) f_{Y_2}(z) dz \right. \\
& \left. + e^{-\delta t} V_1^{(0)}(b_1) \cdot (1 - F_{Y_2}(x_2 + (c_1 + c_2)t)) \right) dt,
\end{aligned} \tag{20}$$

for the boundary $x_1 < b_1,\ x_2 = b_2$:

$$\begin{aligned}
V_1(x_1, b_2) = {} & e^{-(\lambda_1+\lambda_2)h} e^{-\delta h} V_1(x_1 + (c_1 + c_2)h, b_2) \\
& + \int_0^h e^{-\lambda_2 t} \lambda_1 e^{-\lambda_1 t} \left(\int_0^{b_1 + (c_1+c_2)t} e^{-\delta t} V_1(x_1 + (c_1 + c_2)t - z, b_2) f_{Y_1}(z) dz \right) dt \\
& + \int_0^h e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 t} \left(\int_0^{b_2} e^{-\delta t} V_1^{(0)}(x_1 + (c_1 + c_2)t, b_2 - z) f_{Y_2}(z) dz \right. \\
& \left. + e^{-\delta t} V_1^{(0)}(x_1 + (c_1 + c_2)t)) \cdot (1 - F_{Y_2}(b_2)) \right) dt
\end{aligned} \tag{21}$$

and in the corner point we obtain

$$
\begin{aligned}
&V_1(b_1,b_2) = e^{-(\lambda_1+\lambda_2)h}\left(v_1(h)+e^{-\delta h}V_1(b_1,b_2)\right)\\
&+\int_0^h e^{-\lambda_2 t}\lambda_1 e^{-\lambda_1 t}\left(\int_0^{b_1}\left(v_1(t)+e^{-\delta t}V_1(b_1-z,b_2)\right)f_{Y_1}(z)dz+v_1(t)\cdot(1-F_{Y_1}(b_1))\right)dt\\
&+\int_0^h e^{-\lambda_1 t}\lambda_2 e^{-\lambda_2 t}\left(\int_0^{b_2}\left(v_1(t)+e^{-\delta t}V_1(b_1-z,b_2)\right)f_{Y_2}(z)dz\right.\\
&\left.+\left(v_1(t)+e^{-\delta t}V_1^{(0)}(b_1)\right)\cdot(1-F_{Y_2}(b_2))\right)dt, \qquad (22)
\end{aligned}
$$

where $v_1(t) = \int_0^t c_1 e^{-\delta s}ds = \frac{c_1}{\delta}(1-e^{-\delta t})$ is the present value of the discounted dividends paid at rate c_1 over $[0,t)$.

By the same line of argument as in Section 3, one sees that $V_1(x_1,x_2)$ is continuous for all $(x_1,x_2)\in[0,b_1]\times[0,b_2]$. Given differentiability w.r.t. h, by symmetry we can establish differentiability of V_1 w.r.t x_1, x_2. Applying again the operator $\frac{d}{dh}$ to each equation and letting $h\to 0$ yields,

$$
\begin{aligned}
x_1<b_1,x_2<b_2 \quad : \quad & 0 = -(\lambda_1+\lambda_2+\delta)V_1(x_1,x_2)+c_1\frac{\partial V_1}{\partial x_1}(x_1,x_2)+c_2\frac{\partial V_1}{\partial x_2}(x_1,x_2)\\
&+\lambda_1\int_0^{x_1}V_1(x_1-z,x_2)f_{Y_1}(z)dz\\
&+\lambda_2\left(\int_0^{x_2}V_1(x_1,x_2-z)f_{Y_2}(z)dz+V(x_1)\cdot(1-F_{Y_2}(x_2))\right), \qquad (23)\\
x_1=b_1,x_2<b_2 \quad : \quad & 0 = -(\lambda_1+\lambda_2+\delta)V_1(b_1,x_2)+(c_1+c_2)\cdot\frac{\partial V_1}{\partial x_2}(b_1,x_2)\\
&+\lambda_1\cdot\int_0^{b_1}V_1(b_1-z,x_2)f_{Y_1}(z)dz\\
&+\lambda_2\left(\int_0^{x_2}V_1(b_1,x_2-z)f_{Y_2}(z)dz+V(b_1)\cdot(1-F_{Y_2}(x_2))\right), \qquad (24)\\
x_1<b_1,x_2=b_2 \quad : \quad & 0 = -(\lambda_1+\lambda_2+\delta)V_1(x_1,b_2)+(c_1+c_2)\cdot\frac{\partial V_1}{\partial x_1}(x_1,b_2)\\
&+\lambda_1\int_0^{x_1}V_1(x_1-z,b_2)f_{Y_1}(z)dz\\
&+\lambda_2\left(\int_0^{b_2}V_1(x_1,b_2-z)f_{Y_2}(z)dz+V(x_1)\cdot(1-F_{Y_2}(b_2))\right), \qquad (25)
\end{aligned}
$$

and we obtain an equation in the corner point $x_1 = b_1,\ x_2 = b_2$,

$$
\begin{aligned}
0 &= -(\lambda_1 + \lambda_2 + \delta)V_1(b_1, b_2) + c_1 \\
&+ \lambda_1 \cdot \int_0^{b_1} V_1(b_1 - z, b_2) f_{Y_1}(z) dz \\
&+ \lambda_2 \cdot \left(\int_0^{b_2} V_1(b_1, b_2 - z) f_{Y_2}(z) dz + V(b_1) \cdot (1 - F_{Y_2}(b_2)) \right). \qquad (26)
\end{aligned}
$$

As in Section 3, one can compare (23) to (24) and (25) to produce the following boundary conditions using the continuity of $V_1(x_1, x_2)$,

$$
\frac{\partial V_1}{\partial x_1}(b_1, x_2) = \frac{\partial V_1}{\partial x_2}(b_1, x_2) \ \ \forall\, 0 \le x_2 < b_2, \qquad (27)
$$

$$
\frac{\partial V_1}{\partial x_1}(x_1, b_2) = \frac{\partial V_1}{\partial x_2}(x_1, b_2) \ \ \forall\, 0 \le x_1 < b_1, \qquad (28)
$$

and comparing (23) to (26) finally yields in $(b_1, b_2):\ c_1 \frac{\partial V_1}{\partial x_1}(b_1, b_2) + c_2 \frac{\partial V_1}{\partial x_2}(b_1, b_2) = c_1$. The system of equations (23), (27) and (28) is solved by $V_1(x_1, x_2)$.

Exponential claims. Again we consider the exponential claim size case with $Y_1 \sim$Exp(ν_1) and $Y_2 \sim$Exp(ν_2), and applying $\left(\frac{d}{dx_1} + \nu_1\right)$, followed by the operator $\left(\frac{d}{dx_2} + \nu_2\right)$, transforms (23) into the PDE

$$
\begin{aligned}
0 \ &= \ -\delta \nu_1 \nu_2 V_1(x_1, x_2) + (c_1 \nu_1 \nu_2 - \nu_2(\delta + \lambda_1)) \frac{\partial V_1}{\partial x_1}(x_1, x_2) \\
&+ \ (c_2 \nu_1 \nu_2 - \nu_1(\delta + \lambda_2)) \frac{\partial V_1}{\partial x_2}(x_1, x_2) \\
&+ \ (c_1 \nu_1 + c_2 \nu_2 - \delta - \lambda_1 - \lambda_2) \frac{\partial^2 V_1}{\partial x_1 \partial x_2}(x_1, x_2) \\
&+ \ c_1 \nu_2 \frac{\partial^2 V_1}{\partial x_1^2}(x_1, x_2) + c_2 \nu_1 \frac{\partial^2 V_1}{\partial x_2^2}(x_1, x_2) \\
&+ \ c_1 \frac{\partial^3 V_1}{\partial x_1^2 \partial x_2}(x_1, x_2) + c_2 \frac{\partial^3 V_1}{\partial x_1 \partial x_2^2}(x_1, x_2). \qquad (29)
\end{aligned}
$$

In analogy to the expected time of ruin case, the dynamics of the function $V_1(x_1, x_2)$ are now defined locally in the interior of $[0, b_1] \times [0, b_2]$. The explicit solution of (29) together with the corresponding IDE and boundary conditions is of similar complexity as for γ_1.

5 A Simulation Study

In the following we suggest an efficient Monte Carlo algorithm to numerically compute $\gamma_1(x_1, x_2)$ and $V_1(x_1, x_2)$. We will then compare the results with the ones for the stand-alone case, for which explicit formulas are available for exponential claim sizes (cf. Remarks 1 and 2). The aim is to identify decision-theoretical aspects for the justification of a capital-exchange agreement, as compared to the performance in the stand-alone situation.

5.1 A Monte Carlo Algorithm

To set up an efficient algorithm for producing MC estimates of $\gamma_1(x_1, x_2)$ and $V_1(x_1, x_2)$, we observe the following:

- U_1 can only drop negative at a jump time of $S_1(t)$, hence we only have to check the surplus at jump times to see when to stop the process.
- In between any two claim arrivals (from either S_1 or S_2), the surplus processes grow at constant rates $\tilde{c}_1 \in \{0, c_1, c_1 + c_2\}$ and $\tilde{c}_2 \in \{0, c_2, c_1 + c_2\}$, respectively.

The Expected Time of Ruin. The two aggregate claim processes can be combined into one compound Poisson process with intensity $\lambda_1 + \lambda_2$ and a claim $\tilde{Y}_i$ comes with probability $\lambda_1/(\lambda_1 + \lambda_2)$ from I_1 with distribution function $F_{Y_1}(y)$ and with probability $\lambda_2/(\lambda_1 + \lambda_2)$ from I_2 with distribution function $F_{Y_2}(y)$. In the implementation, we hence generate jump times and jump sizes for this combined process. The triples $Z_j = (t_j, \eta_j, \tilde{Y}_j)$ reflect the claim arrival times t_j, a marker $\eta_j = 1$ if the claim is from I_1 and $\eta_j = 0$ otherwise, and the corresponding claim sizes $\tilde{Y}_j$.

Conditioning on $\{Z_j\}_{j\geq 1}$ we can write

$$\gamma_1(x_1, x_2) = \mathbb{E}\left[\inf\{t_j|\, U_1(t_j) < 0, \{Z_j\}_{j\geq 1}\}\right], \tag{30}$$

with $U_1(0) = x_1,\ U_2(0) = x_2$ and the recursions conditional on no prior ruin of the respective process can be written as

$$\begin{aligned} U_1(t_{j+1}) &= \min\Big[b_1, U_1(t_j) + c_1(t_{j+1} - t_j) \\ &\quad +1_{\{\tau_2 > t_j\}} \cdot c_2 \left((t_{j+1} - t_j) - \min\left(t_{j+1} - t_j, \frac{b_2 - U_2(t_j)}{c_2} \right)\right)\Big] - \eta_{j+1} \cdot \tilde{Y}_{j+1}, \end{aligned} \tag{31}$$

$$\begin{aligned} U_2(t_{j+1}) &= \min\Big[b_2, U_2(t_j) + c_2(t_{j+1} - t_j) \\ &\quad +1_{\{\tau_1 > t_j\}} \cdot c_1 \left((t_{j+1} - t_j) - \min\left(t_{j+1} - t_j, \frac{b_1 - U_1(t_j)}{c_1} \right)\right)\Big] - (1 - \eta_{j+1}) \cdot \tilde{Y}_{j+1}. \end{aligned} \tag{32}$$

For a set of samples $\{z_j^{(k)}\}_{j\geq 1}$, $1 \leq j \leq N$, of $\{Z_j\}_{j\geq 1}$, we then simply compute the MC estimate of $\gamma_1(x_1, x_2)$ as

$$\hat{\gamma}_1(x_1, x_2) = \frac{1}{N} \sum_{k=1}^{N} \left[\inf\{t_j|\, U_1(t_j) < 0, \{Z_j\}_{j\geq 1} = \{z_j^{(k)}\}_{j\geq 1}\}\right], \tag{33}$$

using the recursions (31) and (32).

The Expected Discounted Dividends. We realise that no dividends are paid immediately after any claim arrival at t_j, since either the first or the second surplus process will drop below its pay-out barrier due to the claim. Hence, over the time interval $(t_j, t_{j+1}]$ in between two claim arrivals, either no dividends are paid or dividends are paid from a certain time t_j^{in} over the rest of that interval. Furthermore, we note that for $t_j^{\text{in}} < t_{j+1}$, one can write that

$$\int_{t_j^{\text{in}}}^{t_{j+1}} c_1 e^{-\delta s} ds = \frac{c_1}{\delta}\left(e^{-\delta t_j^{\text{in}}} - e^{-\delta t_{j+1}}\right). \tag{34}$$

Conditioning on the claim arrivals and sizes leads to

$$\begin{aligned} V_1(x_1, x_2) &= \mathbb{E}\left[\int_0^{\tau_1} c_1 \cdot 1_{\{A\}} \cdot e^{-\delta s} ds \middle| U_1(0) = x_1,\, U_2(0) = x_2\right] \\ &= \mathbb{E}\left[\sum_{j=1}^{n-1} \frac{c_1}{\delta}\left(e^{-\delta t_j^{\text{in}}} - e^{-\delta t_{j+1}}\right) \middle| t_0 = 0,\, t_n = \tau_1,\, \{Z_j\}_{j\leq 1}\right] \end{aligned} \tag{35}$$

with the event $A = \{[U_1(s) = b_1,\, U_2(s) = b_2,\, \tau_2 > s] \cup [U_1(s) = b_1, \tau_2 < s]\}$, the payment start times

$$t_j^{\text{in}} = \begin{cases} \min(t_j + \max((b_1 - U_1(t_j))/c_1, (b_2 - U_2(t_j))/c_2), t_{j+1}) & \text{if } \tau_2 > t_j \\ \min(t_j + (b_1 - U_1(t_j))/c_1, t_{j+1}) & \text{if } \tau_2 \leq t_j \end{cases},$$

and $U_1(t_j)$ and $U_2(t_j)$ are defined as in (31) and (32). For a set of samples $\{z_j^{(k)}\}_{k\geq 1}$ of $\{Z_j\}_{j\geq 1}$, $1 \leq j \leq N$, we compute the MC estimate of $V_1(x_1, x_2)$ as

$$\hat{V}_1(x_1, x_2) = \frac{1}{N}\sum_{k=1}^{N}\left[\sum_{j=1}^{n-1} \frac{c_1}{\delta}\left(e^{-\delta t_j^{\text{in}}} - e^{-\delta t_{j+1}}\right) \middle| \{Z_j\}_{j\geq 1} = \{z_j^{(k)}\}_{j\geq 1}\right]. \tag{36}$$

5.2 Specification and Results of the Simulation Study

We now consider the example where I_1 and I_2 have a similar insurance portfolio. We choose the income rate $c_1 = c_2 = 6$ and claims are produced according to $\lambda_1 = \lambda_2 = 5$ and $\nu_1 = \nu_2 = 1$. For I_1 we specify the barrier level $b_1 = 5$, while for I_2 we will test the behavior under a low, medium or high barrier, i.e. $b_2 = 1$, 5 or 20. Initially we consider the functions $\gamma_1(x_1, x_2)$ and $V_1(x_1, x_2)$, with $0 \leq x_1 \leq b_1$ and $0 \leq x_2 \leq b_2$. Due the symmetry reason, we only present the plots for I_1.

First, consider Figure 3 for the expected time of ruin of I_1. We observe that $\gamma_1(x_1, x_2)$ is monotonically increasing in both x_1 and x_2. This is as expected, since possible recapitalisation payments

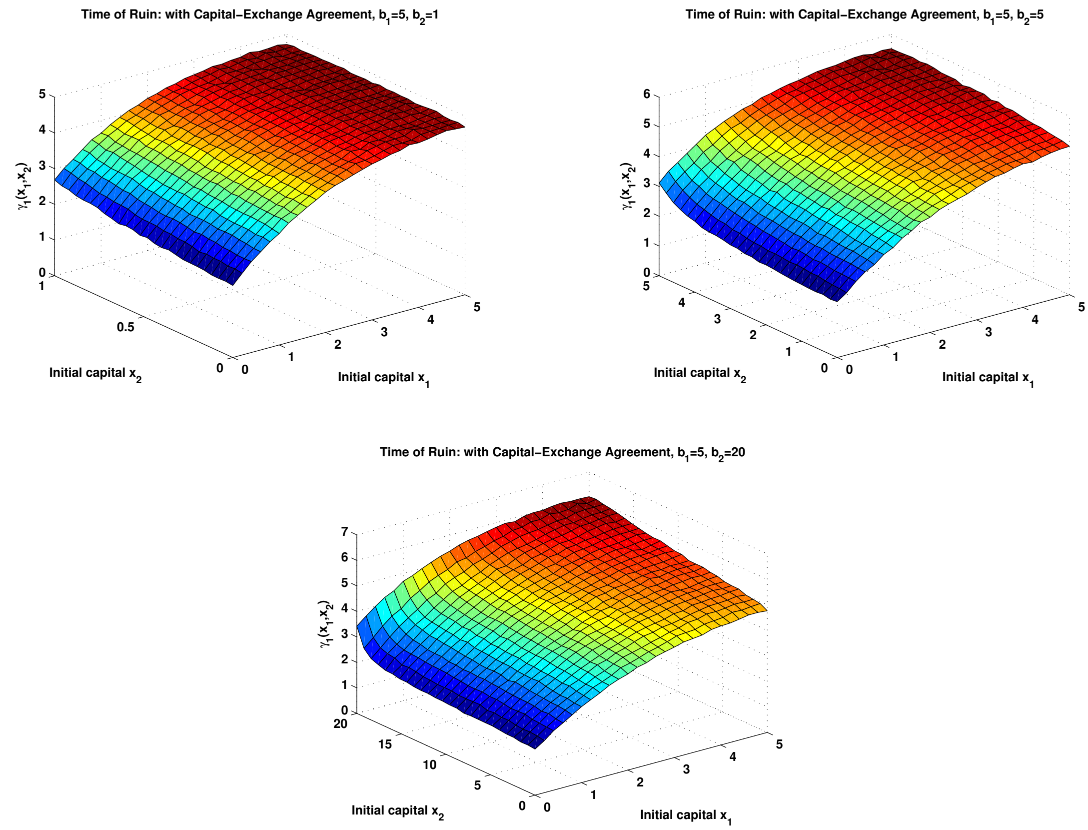

Figure 3: The $\gamma_1(x_1, x_2)$ surface for $b_1 = 5$, and $b_2 = 1, 5, 20$ (for $N = 50{,}000$ simulation runs).

from the partner increase the likelihood of survival, and a higher initial surplus of the partner leads to a higher probability of such payments being made in the future. In the case of the expectation of the sum of discounted dividends, as depicted in Figure 4, the plots are of a different shape. As b_2 increases, the expected discounted dividends become large as x_2 is either small or large. This reflects that dividend payments are blocked as long as U_2 moves within $[0, b_2)$. Early ruin of the partner brings a relative improvement as own profits lead to immediate dividend payments, and the other favorable situation is where I_2 has high surplus and reaches its barrier b_2 early. Otherwise we note that upon fixing x_2, V_1 is naturally an increasing function in x_1. Altogether, V_1 appears to generally decrease as b_2 grows.

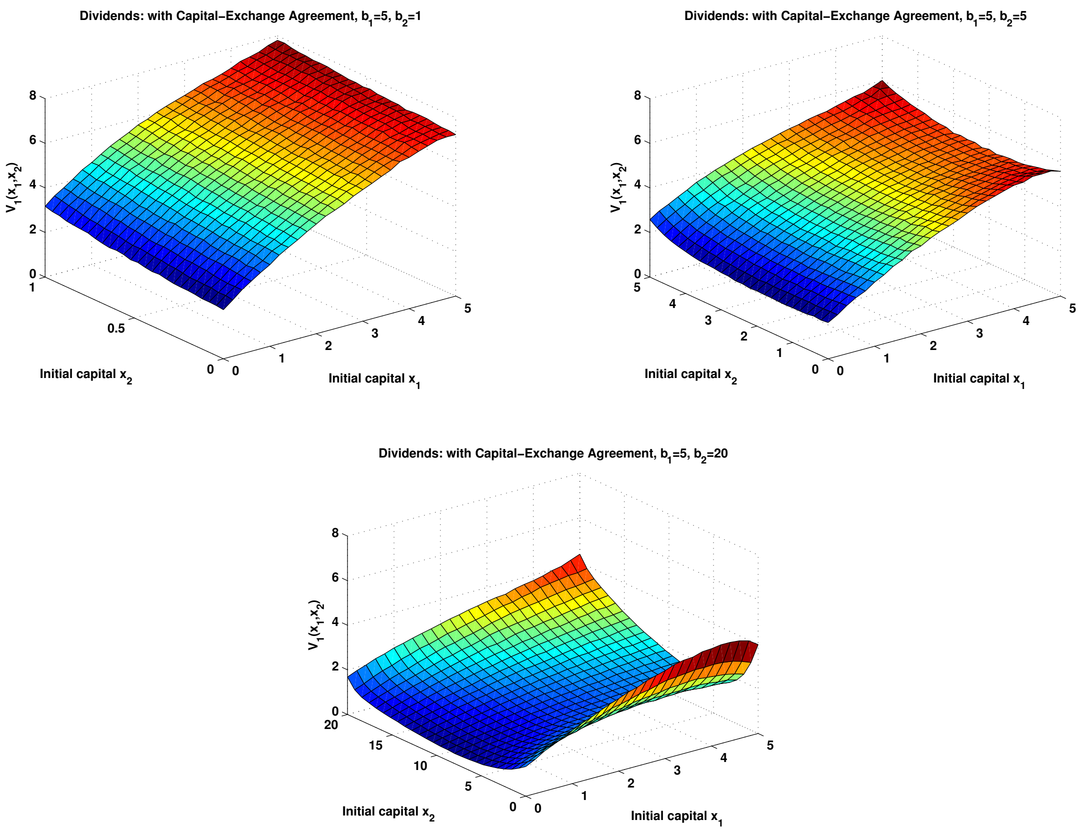

Figure 4: The $V_1(x_1, x_2)$ surface for $b_1 = 5$ and $b_2 = 1, 5, 20$, and $\delta = 0.1$ (for $N = 50{,}000$ simulation runs).

Comparison to the stand-alone case. We now compare the above results to the stand-alone case by considering the plots $\gamma_1(x_1, x_2) - \gamma_1^{(0)}(x_1)$ and $V_1(x_1, x_2) - V_1^{(0)}(x_1)$, in order to reason in what situations it would turn out profitable to enter into the capital-exchange agreement for the given barrier combinations.

Figure 5 confirms that an increase in the expected time of ruin is achieved across all (x_1, x_2) combinations. In the top left graph where b_2 is low, the effect is strongest when x_1 is small and x_2 is relatively high. However, note that $x_2 \leq 1$ is low due to $b_2 = 1$, so that the benefit from possible recapitalisation payments decreases as x_1 becomes larger relative to x_2. This features becomes

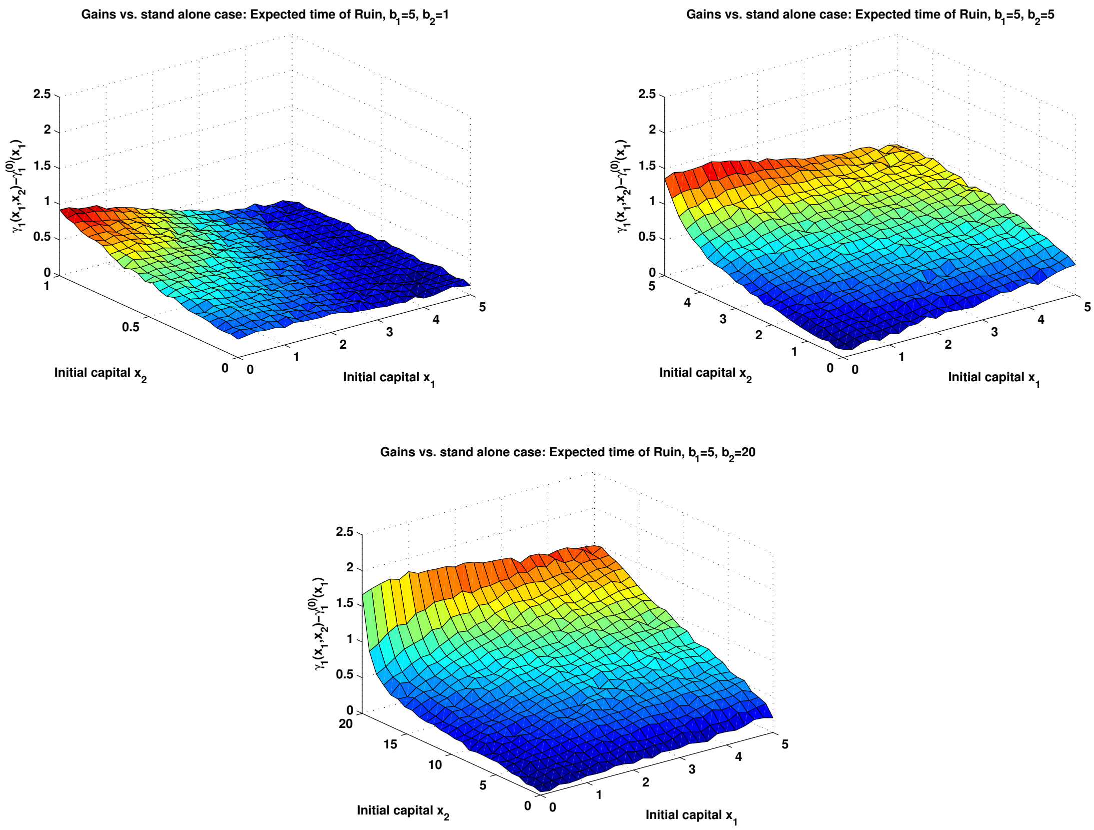

Figure 5: The $\gamma_1(x_1,x_2)-\gamma_1^{(0)}(x_1)$ surface for $b_1=5$, and $b_2=1,5,20$ (for $N=50{,}000$ simulation runs).

less and less pronounced as b_2 increases., and we observe in those cases that the gain is naturally largest when x_2 is high. Figure 6 then shows the change in the expected discounted dividends. In the case $b_2=1$, the gain is highest where x_1 is low and x_2 is close to b_2. This is justified, as I_2 is more likely to recapitalise I_1 so that it may reach its barrier faster. As x_1 approaches its own barrier b_1, the change in expected discounted dividends from the agreement drops negative, as now the risk of having to support I_2 instead paying early dividends becomes more pronounced. This notion of possibly having to support I_2 rather than paying dividends becomes so strong for $b_2=5$ and $b_2=20$, that the effect on the expected discounted dividends is negative for almost all cases, and large surplus levels x_1 produce the worst outcomes.

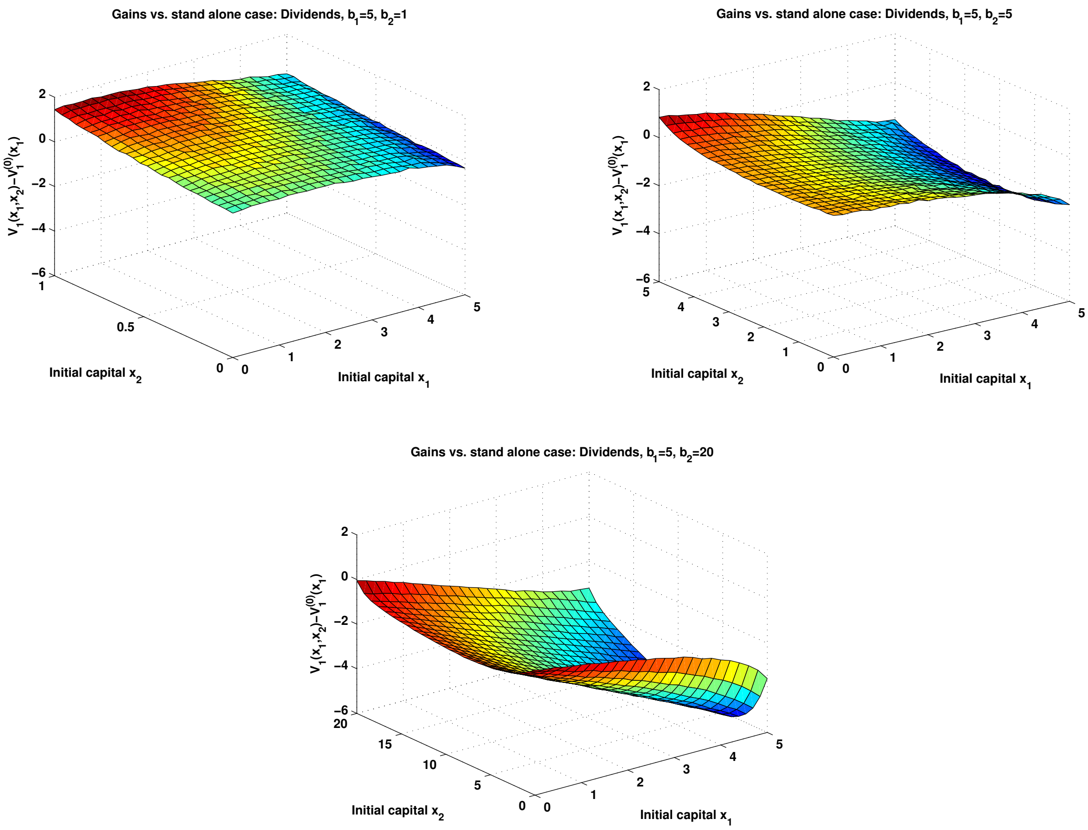

Figure 6: The $V_1(x_1, x_2) - V_1^{(0)}(x_1)$ surface for $b_1 = 5$ and $b_2 = 1, 5, 20$, and $\delta = 0.1$ (for $N = 50{,}000$ simulation runs).

Note that as $b_2 \to \infty$, $V_1(x_1, x_2)$ will tend to zero as excess capital will only go to the partner. We conclude that for low own surplus levels and low barriers b_2 of the partner, the capital-exchange agreement can appear attractive in certain situations, while for larger own surplus levels, the possibility of having to recapitalise the partner clearly outweighs the effect from possible incoming support payments.

As the effects of the agreement on the expected time of ruin and the expectation of the discounted dividends are in opposite directions across most of the here considered cases, one will generally

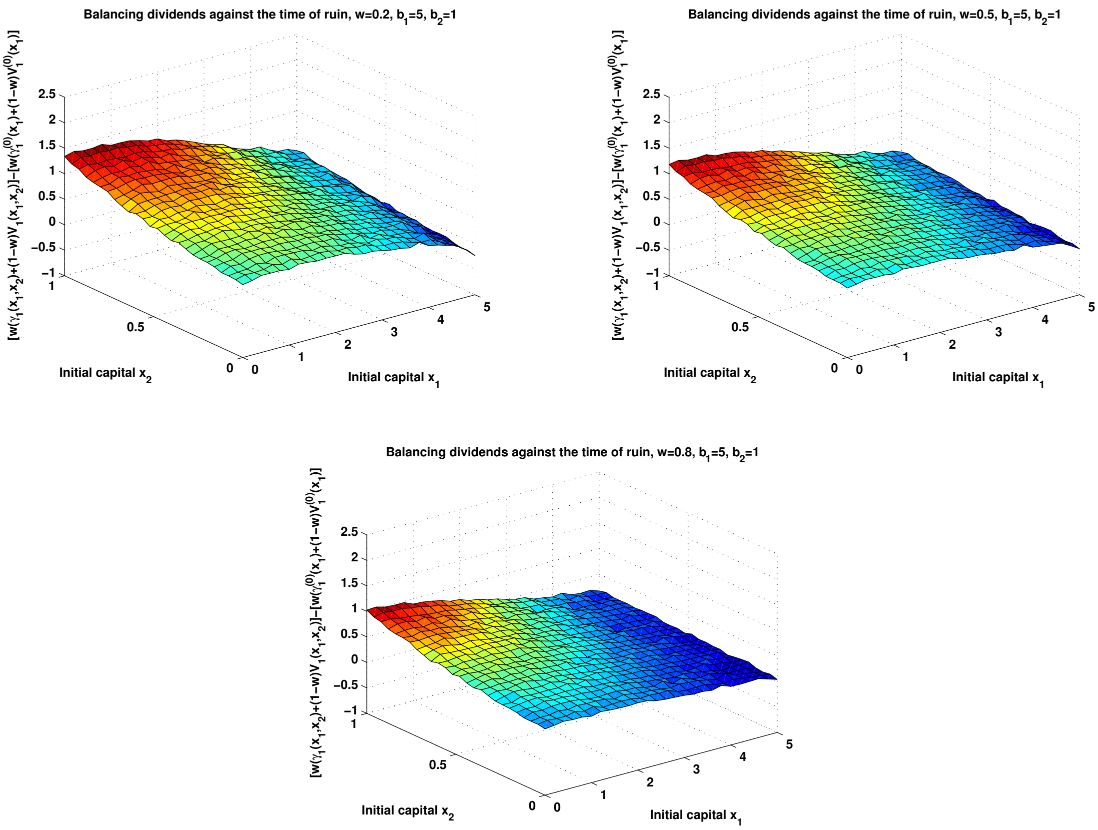

Figure 7: Balancing dividends and ruin time: $w = 0.2$, 0.5 and 0.8, with $b_1 = 5$ and $b_2 = 1$, and $\delta = 0.1$ (for $N = 50{,}000$ simulation runs).

have to balance the wish for a long lifetime against the desire to receive early dividends. This is illustrated in Figures 7 to 9, where low, medium or high weight is given to the expected time of ruin. It is especially for high barriers b_2 of the partner and high own surplus levels, that the preference for an expected increase in lifetime must be strong in order to justify entering the agreement, which becomes particularly clear from Figure 9.

Effect on the system of the two insurers. Finally, we investigate the effect of the capital-exchange agreement on the system of the two insurers against the stand-alone case. Hereby, we choose to compare the sums of the expected ruin times (one could also choose a different

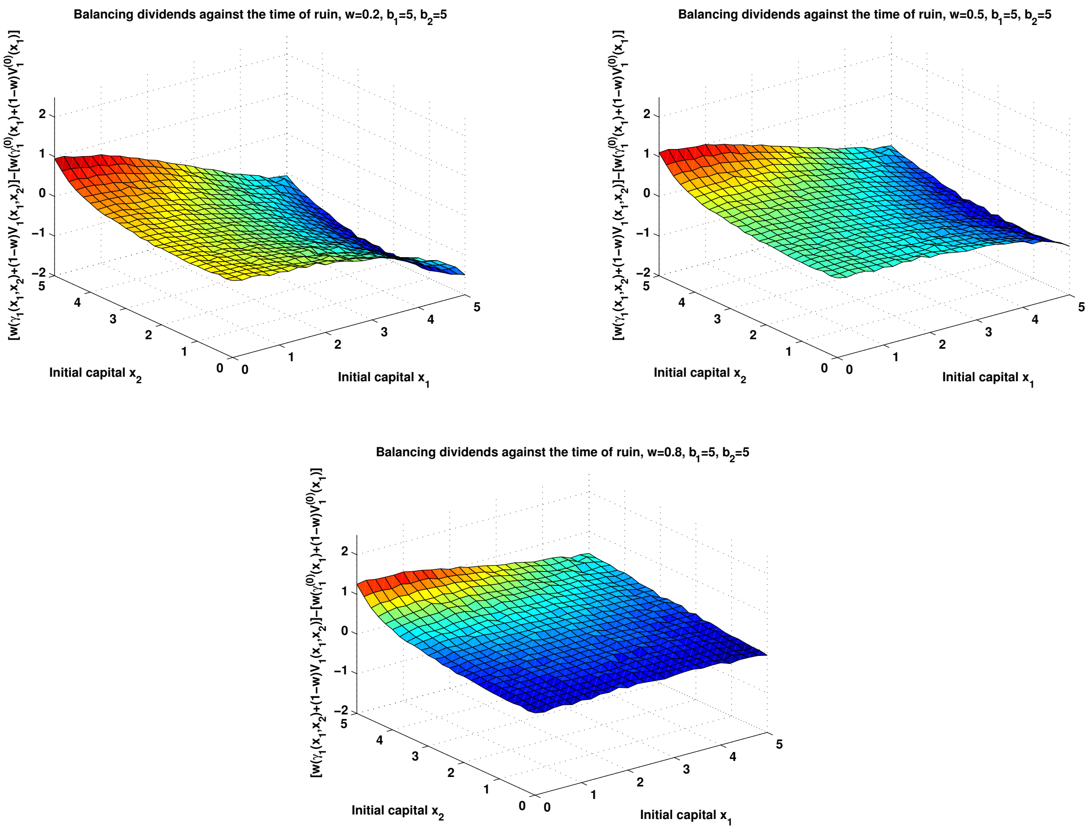

Figure 8: Balancing dividends and ruin time: $w = 0.2$, 0.5 and 0.8, with $b_1 = 5$ and $b_2 = 5$, and $\delta = 0.1$ (for $N = 50{,}000$ simulation runs).

criterion, such as the maximum time of ruin of the two) and the expected discounted dividends, respectively. In particular, we evaluate $\gamma_1(x_1, x_2) + \gamma_2(x_1, x_2) - \gamma_1^{(0)}(x_1) - \gamma_2^{(0)}(x_2)$ and $V_1(x_1, x_2) + V_2(x_1, x_2) - V_1^{(0)}(x_1) - V_2^{(0)}(x_2)$.

Regarding the system gains for the expected ruin times, as depicted in Figure 10, naturally all cases return positive results. In the case where one barrier is larger than the other (i.e. $b_1 = 5, b_2 = 1$ and $b_1 = 5, b_2 = 20$) the positive effect appears to be largest in those cases where the process with the lower barrier starts close to the barrier while the process with the larger barrier has little initial surplus. For the expectation of the sum of the discounted dividends, as shown in Figure

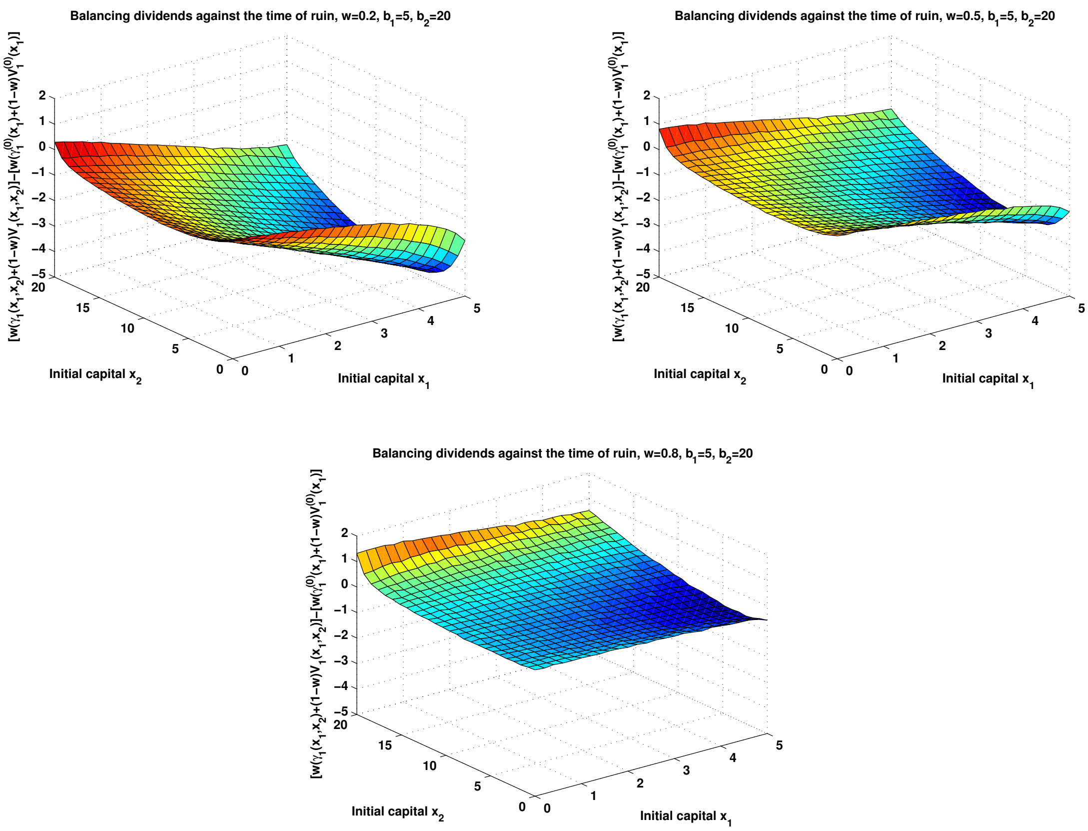

Figure 9: Balancing dividends and ruin time: $w = 0.2$, 0.5 and 0.8, with $b_1 = 5$ and $b_2 = 20$, and $\delta = 0.1$ (for $N = 50{,}000$ simulation runs).

11, a negative effect from the agreement is observed throughout, with the highest relative impact when both initial surplus levels are high. Again, it seems that in view of the whole system of the two insurers, putting a capital-exchange agreement in place must mostly be justified by a strong preference of extending the expected lifetimes of the insurers.

6 Concluding Remarks

In this paper we investigated the impact of a capital-exchange agreement on the expected time of ruin and the expected discounted dividends of insurers. Such an agreement could for example

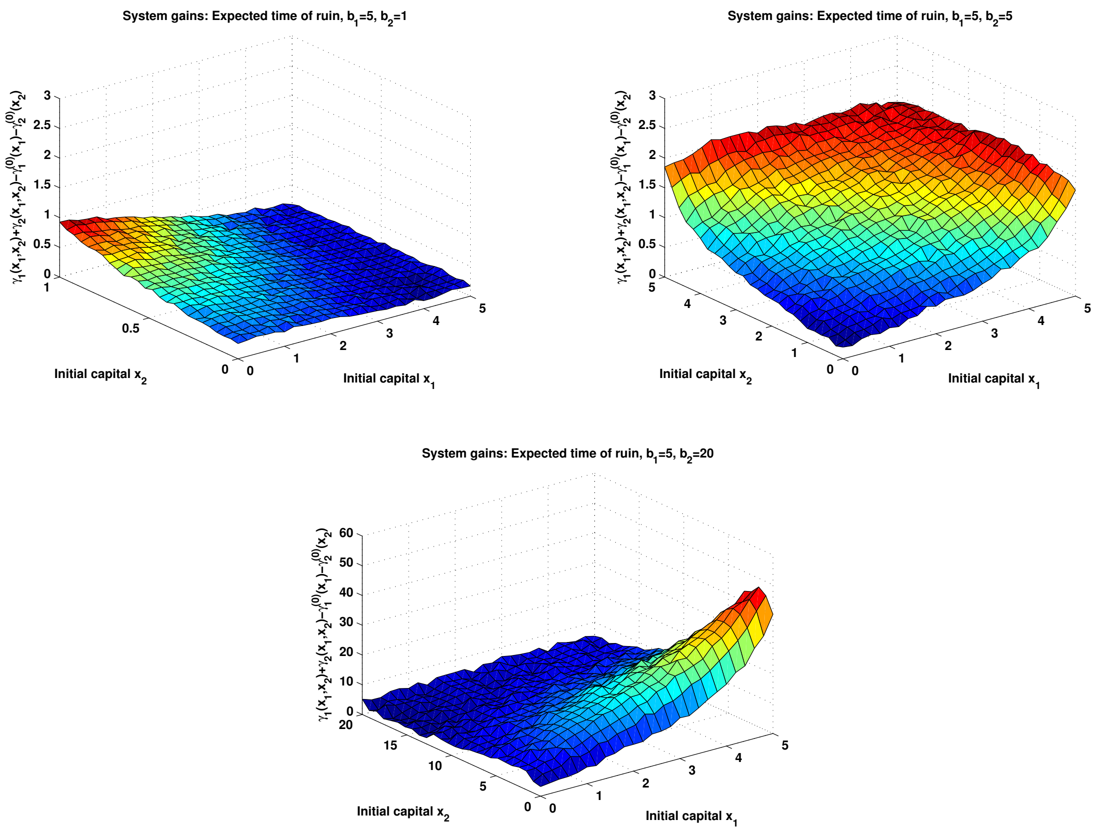

Figure 10: System gains in the expected ruin time for $b_1 = 5$, and $b_2 = 1, 5, 20$ (for $N = 50{,}000$ simulation runs).

exist among entities within an insurance group, as they recapitalise each other until the surplus processes of all subsidiaries run at some satisfactory levels; only then dividends are released to the shareholders. We have characterised the expected ruin time and the expected discounted dividends in this setup by deriving a set of equations in each case. The results of a Monte Carlo simulation study finally illustrated that the agreement naturally improves the expected time of ruin. This is the case from the viewpoint of each insurer that has assumed the agreement, and hence, also improves the expected ruin time across the system of the participating insurers. The effect on the dividends is found to be twofold. For low barrier levels of the partner, a positive effect is observed if one's own initial surplus is low. As either the partner's barrier increases, or the insurer own initial

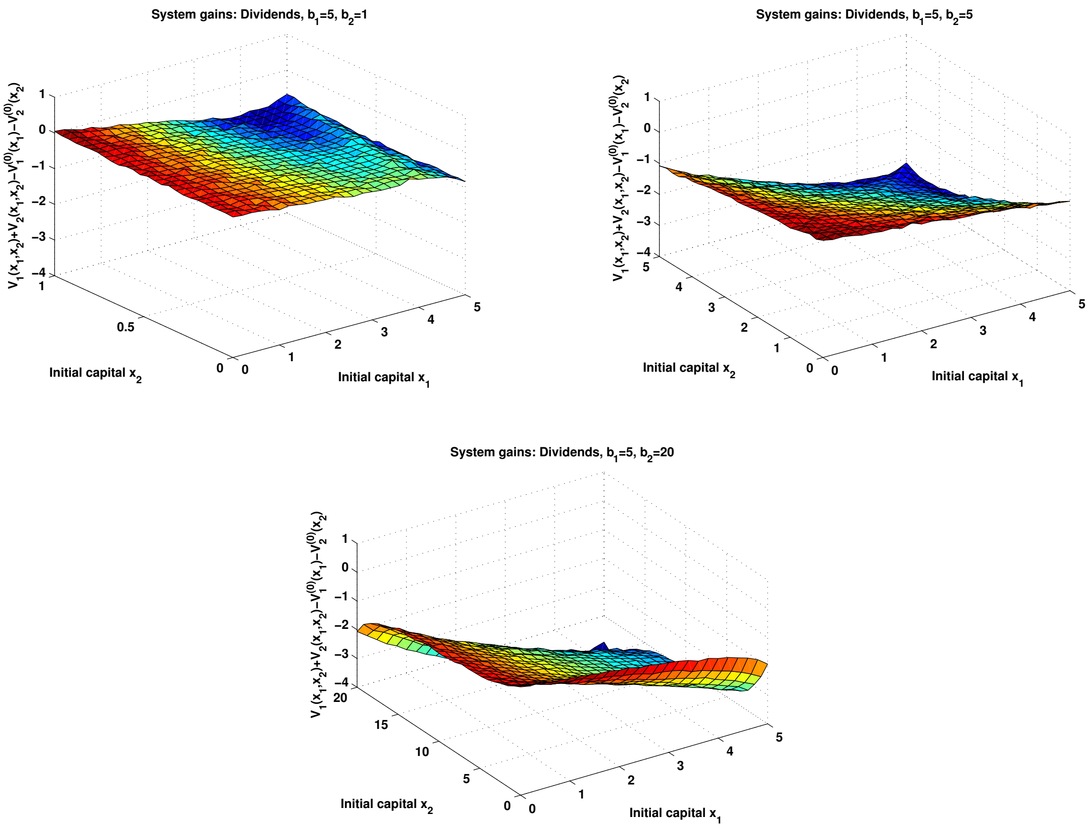

Figure 11: System gains in the expectation of total dividends for $b_1 = 5$ and $b_2 = 1, 5, 20$, and $\delta = 0.1$ (for $N = 50{,}000$ simulation runs).

surplus is close to its barrier level, the effect on the expected discounted dividends appears negative. Asymptotically for some $b_i \to \infty$, the expected discounted dividends of all participating companies tends to zero, as dividend payments are blocked due to their recapitalisation obligation for insurer i. We conclude that in many situations a strong preference for an increase in the expected lifetime is required to justify entering the capital-exchange agreement. This effect is observed for single insurers, as well as from a systemic point of view.

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