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BOUNDARY NON-CROSSINGS OF ADDITIVE WIENER FIELDS [1]

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Abstract. Let $\{W_i(t), t \in \mathbb{R}_+\}, i = 1,2$ be two Wiener processes and let $W_3 = \{W_3(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^2\}$ be a two-parameter Brownian sheet, all three processes being mutually independent. We derive upper and lower bounds for the boundary non-crossing probability

$$P_f = P\{W_1(t_1) + W_2(t_2) + W_3(\mathbf{t}) + f(\mathbf{t}) \le u(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^2\},$$

where $f, u : \mathbb{R}_+^2 \to \mathbb{R}$ are two general measurable functions. We further show that, for large trend functions $\gamma f > 0$, asymptotically, as $\gamma \to \infty$, $P_{\gamma f}$ is equivalent to $P_{\gamma \underline{f}}$, where $\underline{f}$ is the projection of f on some closed convex set of the reproducing kernel Hilbert Space of the field $W(\mathbf{t}) = W_1(t_1) + W_2(t_2) + W_3(\mathbf{t})$. It turns out that our approach is applicable also for the additive Brownian pillow.

Keywords: ...Boundary non-crossing probability; reproducing kernel Hilbert space; additive Wiener field; polar cones; logarithmic asymptotics; Brownian sheet, Brownian pillow.
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1 INTRODUCTION

Calculation of boundary non-crossing probabilities of Gaussian processes is a key topic of interest for both theoretical and applied probability, see, e.g., [11, 22, 17, 20, 18, 8, 3, 5, 4, 6, 7, 14] and the references therein. Numerous applications concerned with the evaluation of boundary non-crossing probabilities relate to mathematical finance, risk theory, queueing theory, statistics, physics among many other fields. Also calculation of boundary non-crossing probabilities of random fields is of interest in various contexts, see e.g., [19, 10, 12, 21].
In this paper we are concerned with the investigation of boundary non-crossing probabilities of an additive Wiener field which is defined as the sum of a standard Brownian sheet and two independent Wiener processes. The choice of the model is quite natural since both the Wiener process and the Brownian sheet appear naturally as limiting processes when we consider the schemes in the domain of attraction of the Central Limit Theorem. One one hand,

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these processes have continuous trajectories and independent increments, which makes our model very tractable and flexible. On the other hand, arbitrary functions defined on the positive quadrant, can be decomposed uniquely into three components, two of them representing its behavior on the axes and the third component being zero on the axes. Hence any trend function that we can consider here is suitable for our model.

Definition 1.1 *Brownian sheet* $\widetilde{W} = \{\widetilde{W}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^2\}$ *is a Gaussian random field with zero mean and covariance function*

$$\mathbb{E}\left\{\widetilde{W}(\mathbf{t})\widetilde{W}(\mathbf{s}\right\} = (s_1 \wedge t_1)(s_2 \wedge t_2).$$

By the definition, the Brownian sheet is zero on the axes and in what follows we shall consider its continuous modification.

Let $W_i = \{W_i(t), t \in \mathbb{R}_+\}, i = 1, 2$ be two Wiener processes and let $W_3 = \{W_3(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^2\}$ be a Brownian sheet. For two measurable functions $f, u : \mathbb{R}_+^2 \to \mathbb{R}$ we shall investigate the boundary non-crossing probability

$$P_f = \mathbb{P}\left\{f(\mathbf{t}) + W(\mathbf{t}) \le u(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}_+^2\right\},$$

with W an additive Wiener field defined by

$$W(\mathbf{t}) = W_1(t_1) + W_2(t_2) + W_3(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}_+^2, \tag{1.1}$$

where we assume that W_1, W_2, W_3 are mutually independent. Clearly, the additive Wiener field W is a centered Gaussian random field with covariance function

$$\mathbb{E}\{W(\mathbf{s})W(\mathbf{t})\} = s_1 \wedge t_1 + s_2 \wedge t_2 + (s_1 \wedge t_1)(s_2 \wedge t_2), \quad \mathbf{s} = (s_1, s_2), \mathbf{t} = (t_1, t_2). \tag{1.2}$$

For our study we shall modify some techniques applied for Brownian pillow. To be more precise, we can not apply the methods proposed for Brownian pillow from [2, 3, 12] since they are based on the fact that it vanishes on some rectangle. Therefore, we modify essentially the methods to meet the properties of our model, and in that context some additional conditions are introduced in our main result.

As it is commonly the case for random fields, also for the additive Wiener field explicit calculations of boundary non-crossing probabilities are not available even for the case that both f, u are constants, see e.g., [10]. Therefore in our analysis we shall derive upper and lower bounds considering general measurable functions u and function f from the reproducing kernel Hilbert space (RKHS) of W denoted by $\mathcal{H}_{2,+}$. We shall consider some general measurable functions u and trend functions f from the RKHS of W denoted by $\mathcal{H}_{2,+}$.

In order to determine $\mathcal{H}_{2,+}$ we need to recall first the corresponding RKHS's of W_1, W_2 and W_3. It is well-known (see e.g., [1]) that the RKHS of the Wiener process W_1, denoted by $\mathcal{H}_1$, is characterized as follows

$$\mathcal{H}_1 = \Big\{h : \mathbb{R}_+ \to \mathbb{R} \big| h(t) = \int_{[0,t]} h'(s) ds, \quad h' \in L_2(\mathbb{R}_+, \lambda_1)\Big\},$$

with the inner product $\langle h, g\rangle = \int_{\mathbb{R}_+} h'(s)g'(s)ds$ and the corresponding norm $\|h\|^2 = \langle h, h\rangle$. It is also well-known that the RKHS of the Brownian sheet W_3, denoted by $\mathcal{H}_2$, is characterized as follows

$$\mathcal{H}_2 = \Big\{h : \mathbb{R}_+^2 \to \mathbb{R} \big| h(\mathbf{t}) = \int_{[0,\mathbf{t}]} h''(\mathbf{s}) d\mathbf{s}, \quad h'' \in L_2(\mathbb{R}_+^2, \lambda_2)\Big\},$$

with the inner product $\langle h, g\rangle = \int_{\mathbb{R}_+^2} h''(\mathbf{s})g''(\mathbf{s})d\mathbf{s}$ and the corresponding norm $\|h\|^2 = \langle h, h\rangle$. Here the symbols λ_1 and λ_2 stand for the Lebesgue measures in the $\mathbb{R}_+^1$ and in $\mathbb{R}_+^2$, respectively. As shown in Lemma 4 in Appendix the

RKHS corresponding to the covariance function of the additive Wiener field W given in (1.2) is

$$\mathcal{H}_{2,+} = \Big\{ h : \mathbb{R}_+^2 \to \mathbb{R} \Big| h(\mathbf{t}) = h_1(t_1) + h_2(t_2) + h_3(\mathbf{t}), \text{ where } h_i \in \mathcal{H}_1, i = 1,2 \text{ and } h_3 \in \mathcal{H}_2 \Big\} \tag{1.3}$$

equipped with the inner product

$$\langle h, g \rangle = \int_{\mathbb{R}_+} h_1'(s) g_1'(s) ds + \int_{\mathbb{R}_+} h_2'(s) g_2'(s) ds + \int_{\mathbb{R}_+^2} h''(\mathbf{s}) g''(\mathbf{s}) d\mathbf{s} \tag{1.4}$$

and the corresponding norm $\|h\|^2 = \langle h, h \rangle$. For simplicity we used the same notation for the norm and the inner product of $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_{2,+}$. Note that in the case when $h \in \mathcal{H}_2 \cap C^2(\mathbb{R}^2)$ we have that $h''(u,s) = \frac{\partial^2 h(u,s)}{\partial u \partial s}$, and it is the motivation for the notation h''.
As in [13], a direct application of Theorem 1' in [15] shows that for any $f \in \mathcal{H}_{2,+}$ we have

$$\Big| P_f - P_0 \Big| \quad \le \tfrac{1}{\sqrt{2\pi}} \|f\|. \tag{1.5}$$

Clearly, the above inequality provides a good bound for the approximation rate of P_f by P_0 when $\|f\|$ is small. Recall that P_0 cannot be calculated explicitly, however it can be determined with a given accuracy by using simulations. More generally, if we want to compare P_f and P_g for $g \in \mathcal{H}_{2,+}$ and $g \ge f$, we obtain further (by Theorem 1' in [15]) that

$$\Phi(\alpha - \|g\|) \le P_g \le P_f \le \Phi(\alpha + \|f\|), \tag{1.6}$$

where Φ is the distribution of an $N(0,1)$ random variable and $\alpha = \Phi^{-1}(P_0)$ is a finite constant. When $f \le 0$, then we can take always $g = 0$ above. If $f(\mathbf{t}_0) > 0$ for some $\mathbf{t}_0$ with non-negative components, then the last inequalities are useful when $\|f\|$ is large. Indeed, for any $g \ge f, g \in \mathcal{H}_{2,+}$ using (1.6) we obtain as $\gamma \to \infty$

$$\ln P_{\gamma f} \ge \ln \Phi(\alpha - \rangle \gamma g) \ge -(1 + o(1)) \frac{\gamma^2}{2} \|g\|^2,$$

hence

$$\ln P_{\gamma f} \ge -(1 + o(1)) \frac{\gamma^2}{2} \|\underline{f}\|^2, \quad \gamma \to \infty, \tag{1.7}$$

where $\underline{f}$ (which is unique and exists) satisfies

$$\min_{g, f \in \mathcal{H}_{2,+}, g \ge f} \|g\| = \|\underline{f}\| > 0. \tag{1.8}$$

In Section 2 we identify $\underline{f}$ with the projection of f on a closed convex set of $\mathcal{H}_{2,+}$, and moreover we show that

$$\ln P_{\gamma f} \sim \ln P_{\gamma \underline{f}} \sim -\frac{\gamma^2}{2} \|\underline{f}\|^2, \quad \gamma \to \infty. \tag{1.9}$$

Our results in this paper are of both theoretical and practical interest. Furthermore, our approach can be applied when dealing instead of the additive Wiener sheet W with the linear combinations of W_1, W_2, W_3. Additionally, the techniques developed in this contribution are applicable also for the evaluations of boundary non-crossing probabilities of the additive Brownian pillow, i.e., when W_1, W_2 are independent Brownian bridges and W_3 is a Brownian pillow. For the later case our results are more general than those in [12].
Organization of the paper is as follows: We continue below with preliminaries followed then by a section containing the main result. In Appendix we present three technical lemmas. Lemma 3 contains Itô's formula for the

product of two fields in the plane, one of them being the Brownian sheet and the another one having bounded variation. Lemma 4 states that the RKHS of W is determined uniquely, whereas Lemma 5 describes the asymptotic behavior of h'' for h from the closed convex subset of $\mathcal{H}_{2,+}$ that is used for projection.

2 Preliminaries

Recall that in this paper bold letters are reserved for vectors, so we shall write for instance $\mathbf{t} = (t_1, t_2) \in \mathbb{R}_+^2$ and λ_1 and λ_2 denote the Lebesgue measures on $\mathbb{R}_+$ and $\mathbb{R}_+^2$, respectively whereas ds and $d\mathbf{s}$ mean integration with respect to these measures.

2.1 Expansion of one-parameter functions

The results of this subsection were formulated in a different form in e.g., [2, 14, 12]. However we shall introduce some modifications (re-writing for instance V_1 below) which are important for the two-parameter case. From the derivations below it will become clear how to obtain expansion of multiparameter functions of two components, one of which is the "analog of the smallest concave majorant" and the other one is a negative function. Specifically, when studying the boundary crossing probabilities of the Wiener process with a deterministic trend $h \in \mathcal{H}_1$, then it has been shown (see [4]), that the smallest concave majorant of h solves (1.8) and determines the large deviation asymptotics of this probability. Moreover, as shown in [14] the smallest concave majorant of h, which we denote by $\underline{h}$, can be written analytically as the unique projection of h on the closed convex set

$$V_1 = \{h \in \mathcal{H}_1 \big| \, h'(s) \text{ is a non-increasing function}\}$$

i.e., $\underline{h} = Pr_{V_1} h$. Here we write $Pr_A h$ for the projection of h on some closed set A also for other Hilbert spaces considered below. In the following for a given real-valued function φ we denote its one-parameter increment $\Delta_s^1 \varphi(t) = \varphi(t) - \varphi(s)$, $0 \le s \le t < \infty$. With this notation we can re-write V_1 as

$$V_1 = \{h \in \mathcal{H}_1 \big| \, \Delta_s^1 h'(t) \le 0 \, , 0 \le s \le t < \infty\}.$$

Lemma 1. *Let* $\widetilde{V}_1 = \{h \in \mathcal{H}_1 \big| \, \langle h, f \rangle \le 0$ *for any* $f \in V_1\}$ *be the polar cone of* V_1 *and let* $h \in \mathcal{H}_1$.

(i) If $h \in \widetilde{V}_1$, *then* $h \le 0$.

(ii) We have $\langle Pr_{V_1} h, Pr_{\widetilde{V}_1} h \rangle = 0$ *and further*

$$h = Pr_{V_1} h + Pr_{\widetilde{V}_1} h. \tag{2.1}$$

(iii) If $h = h_1 + h_2$, $h_1 \in V_1$, $h_2 \in \widetilde{V}_1$ *and* $\langle h_1, h_2 \rangle = 0$, *then* $h_1 = Pr_{V_1} h$ *and* $h_2 = Pr_{\widetilde{V}_1} h$.

(iv) The unique solution of the minimization problem $\min_{g \ge h, g \in \mathcal{H}_1} \|g\|$ *is* $\underline{h} = Pr_{V_1} h$.

Proof Let $h \in \widetilde{V}_1$ and define $A = \{s \in \mathbb{R}_+ : h(s) > 0\}$. Fix $T > 0$ and consider the function v such that

$$v'(s) = \int_{[s,T]} h(u) 1_{u \in A} du 1_{s \le T}.$$

For any $0 \le s \le t < \infty$ we have $\Delta_s^1 v'(t) = -\int_{[s\wedge T, t\wedge T]} h(u)1_{u\in A}du \le 0$ and further

$$\begin{aligned}\int_{\mathbb{R}_+} |v'(s)^2| ds &= \int_{[0,T]} \Big(\int_{[s,T]} h(u)1_{u\in A}du\Big)^2 ds \\ &\le T^2 \int_{[0,T]} h^2(u)du \\ &= T^2 \int_{[0,T]} \Big(\int_{[0,u]} h'(s)ds\Big)^2 du \\ &\le T^4 \int_{\mathbb{R}_+} (h'(s))^2 ds \\ &< \infty.\end{aligned}$$

Consequently, $v' \in L_2(\mathbb{R}_+, \lambda_1), v(s) = \int_{[0,s]} v'(u)du \in \mathcal{H}_1$ and further $v \in V_1$. Therefore,

$$\begin{aligned}0 &\ge \langle h, v\rangle \\ &= \int_{\mathbb{R}_+} h'(s)v'(s)ds \qquad (2.2)\\ &= \int_{[0,T]} h'(s) \int_{[s,T]} h(u)1_{u\in A}duds \\ &= \int_{[0,T]} h(u)1_{u\in A} \int_{[0,u]} h'(s)dsdu \\ &= \int_{[0,T]} h^2(u)1_{u\in A}du \qquad (2.3)\end{aligned}$$

implying that $1_{u\in A} = 0$ a.e. λ_1, in other words, $h(u) \le 0$ a.e. λ_1. However, h is a continuous function and therefore $h(u) \le 0$ for any u.

Statements (ii) and (iii) follow immediately from [14] and are valid for any Hilbert space.

(iv) Write

$$f = h + \varphi = \underline{h} + \varphi + h - \underline{h} = \underline{h} + \varphi + Pr_{\widetilde{V}_1} h$$

and suppose that $f \in \mathcal{H}_1$ and $\varphi \ge 0$. Note that for any function $g \in V_1$ its derivative g' is non-increasing therefore g' is non-negative and $\lim_{t\to\infty} g'(t) = 0$. Since $\varphi \ge 0$, then for any sequence $t_n \to \infty$ we have

$$\lim_{n\to\infty} \varphi(t_n)\underline{h}'(t_n) \ge 0,$$

which implies

$$\begin{aligned}\langle \underline{h}, \varphi\rangle &= \int_{\mathbb{R}_+} \underline{h}'(u)\varphi'(u)du \\ &= \lim_{n\to\infty} \int_{[0,t_n]} \underline{h}'(u)\varphi'(u)du \\ &= \lim_{n\to\infty} \Big(\varphi(t_n)\underline{h}'(t_n) - \int_{[0,t_n]} \varphi(u)d(\underline{h}'(u))\Big) \\ &\ge \lim_{n\to\infty} \Big(-\int_{[0,t_n]} \varphi(u)d(\underline{h}'(u))\Big) \\ &\ge 0. \qquad (2.4)\end{aligned}$$

Consequently,

$$\begin{aligned}\|f\|^2 = \|h+\varphi\|^2 &= \|\underline{h}+\varphi+Pr_{\widetilde{V}_1}h\|^2\\ &= \|\underline{h}\|^2+2\langle\underline{h},\varphi\rangle+2\langle\underline{h},Pr_{\widetilde{V}_1}h\rangle+\|\varphi+Pr_{\widetilde{V}_1}h\|^2\\ &= \|\underline{h}\|^2+2\langle\underline{h},\varphi\rangle+\|\varphi+Pr_{\widetilde{V}_1}h\|^2\\ &\geq \|\underline{h}\|^2\end{aligned}$$

establishing the proof.

2.2 Expansion of two-parameter functions

For some given function $\varphi:\mathbb{R}_+^2\to\mathbb{R}$ we define

$$\Delta_{\mathbf{s}}\varphi(\mathbf{t})=\varphi(\mathbf{t})-\varphi(s_1,t_2)-\varphi(t_1,s_2)+\varphi(\mathbf{s}),$$

$$\Delta^1_{\mathbf{s}}\varphi(t_1,s_2)=\varphi(t_1,s_2)-\varphi(\mathbf{s}),\quad \Delta^2_{\mathbf{s}}\varphi(s_1,t_2)=\varphi(s_1,t_2)-\varphi(\mathbf{s}).$$

In our notation $\mathbf{s}=(s_1,s_2)\leq\mathbf{t}=(t_1,t_2)$ means that $s_1\leq t_1$ and $s_2\leq t_2$. Define the closed convex set

$$V_2=\{h\in\mathcal{H}_2\big|\Delta_{\mathbf{s}}h''(\mathbf{t})\geq 0,\ \Delta^1_{\mathbf{s}}h''(t_1,s_2)\leq 0,\ \Delta^2_{\mathbf{s}}h''(s_1,t_2)\leq 0 \text{ for any } \mathbf{s}\leq\mathbf{t} \text{ and } \mathbf{t}\in\mathbb{R}_+^2\} \tag{2.5}$$

and let $\widetilde{V}_2$ be the polar cone of V_2, namely

$$\widetilde{V}_2=\{h\in\mathcal{H}_2\big|\langle h,v\rangle\leq 0 \text{ for any } v\in V_2\}.$$

Below we derive the expansion for two-parameter functions. Since the results are very similar to the previous lemma, we shall prove only those statements that differ in details from Lemma 1.

Lemma 2. *(i) If $h\in\widetilde{V}_2$, then $h\leq 0$.*

(ii) For any $h\in\mathcal{H}_2$ we have $\langle Pr_{V_2}h,Pr_{\widetilde{V}_2}h\rangle=0$ and

$$h=Pr_{V_2}h+Pr_{\widetilde{V}_2}h.$$

(iii) If $h=h_1+h_2$, $h_1\in V_2$, $h_2\in\widetilde{V}_2$ and $\langle h_1,h_2\rangle=0$, then $h_1=Pr_{V_2}h$ and $h_2=Pr_{\widetilde{V}_2}h$.

(iv) For any $h\in\mathcal{H}_2$ the unique solution of the minimization problem $\min_{g\geq h,g\in\mathcal{H}_2}\|g\|$ is $\underline{h}=Pr_{V_2}h$.

Proof We prove only statement (i). Denote $\mathbf{T}=(T,T),T>0$ and consider the function v with

$$v''(\mathbf{s})=\int_{[\mathbf{s},\mathbf{T}]}h(\mathbf{u})1_{\mathbf{u}\in\mathbf{A}}d\mathbf{u}1_{\mathbf{s}\leq\mathbf{T}},$$

where $\mathbf{A}=\{\mathbf{s}\in\mathbb{R}_+^2\big|h(\mathbf{s})\geq 0\}$. Then for any $\mathbf{0}\leq\mathbf{s}\leq\mathbf{t}$

$$\begin{aligned}\Delta^1_{\mathbf{s}}v''(t_1,s_2) &= -\int_{[\mathbf{s}\wedge\mathbf{T},(t_1\wedge T,T)]}h(\mathbf{u})1_{\mathbf{u}\in\mathbf{A}}d\mathbf{u}\leq 0,\\ \Delta^1_{\mathbf{s}}v''(s_1,t_2) &= -\int_{[\mathbf{s}\wedge\mathbf{T},(T,t_2\wedge T)]}h(\mathbf{u})1_{\mathbf{u}\in\mathbf{A}}d\mathbf{u}\leq 0,\\ \Delta^2_{\mathbf{s}}v''(\mathbf{t}) &= \int_{[\mathbf{s}\wedge\mathbf{T},\mathbf{t}\wedge\mathbf{T}]}h(\mathbf{u})1_{\mathbf{u}\in\mathbf{A}}d\mathbf{u}\geq 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{R}_+^2} |v''(\mathbf{s})^2| d\mathbf{s} &= \int_{[\mathbf{0},\mathbf{T}]} \Big(\int_{[\mathbf{s},\mathbf{T}]} h(\mathbf{u}) 1_{\mathbf{u}\in\mathbf{A}} d\mathbf{u} \Big)^2 d\mathbf{s} \\ &\le T^4 \int_{[\mathbf{0},\mathbf{T}]} h^2(\mathbf{u}) d\mathbf{u} \\ &= T^4 \int_{[\mathbf{0},\mathbf{T}]} \Big(\int_{[\mathbf{0},\mathbf{u}]} h''(\mathbf{s}) d\mathbf{s} \Big)^2 d\mathbf{u} \\ &\le T^8 \int_{\mathbb{R}_+^2} (h''(\mathbf{s}))^2 d\mathbf{s} \\ &< \infty.\end{aligned}$$

Consequently,

$$v'' \in L_2(\mathbb{R}_+^2, \lambda_2), \quad v(\mathbf{s}) = \int_{[\mathbf{0},\mathbf{s}]} v''(\mathbf{u}) d\mathbf{u} \in \mathcal{H}_2$$

and further $v \in V_2$. Similarly to (2.2) we conclude that $1_{\mathbf{u}\in\mathbf{A}} = 0$ a.e. λ_2. Other details follow as in the proof of Lemma 1.

Since we are going to work with functions f in $\mathcal{H}_{2,+}$ we need to consider the projection of such f on a suitable closed convex set. In the following we shall write $f = f_1 + f_2 + f_3$ meaning that $f(\boldsymbol{t}) = f_1(t_1) + f_2(t_2) + f_3(\boldsymbol{t})$ where $f_1, f_2 \in \mathcal{H}_1$ and $f_3 \in \mathcal{H}_2$. Note in passing that this decomposition is unique for any $f \in \mathcal{H}_{2,+}$. Define the closed convex set

$$V_{2,+} = \{h = h_1 + h_2 + h_3 \in \mathcal{H}_{2,+} \big| h_1, h_2 \in V_1, h_3 \in V_2\}$$

and let $\widetilde{V_{2,+}}$ be the polar cone of $V_{2,+}$ given by

$$\widetilde{V_{2,+}} = \{h \in \mathcal{H}_{2,+} \big| \langle h, v\rangle \le 0 \text{ for any } v \in V_{2,+}\}$$

with inner product from (1.4). It follows that for any $h = h_1 + h_2 + h_3 \in \widetilde{V}_2$ we have $h_i \le 0, i = 1,2$ and $h_3 \le 0$. Furthermore, $\langle Pr_{V_{2,+}} h, Pr_{\widetilde{V_{2,+}}} h\rangle = 0$ and

$$h = Pr_{V_{2,+}} h + Pr_{\widetilde{V_{2,+}}} h. \tag{2.6}$$

Analogous to Lemma 2 we also have that for $h = f + g$, $f \in V_{2,+}$, $g \in \widetilde{V_{2,+}}$ such that $\langle f, g\rangle = 0$, then $f = Pr_{V_{2,+}} h$ and $g = Pr_{\widetilde{V_{2,+}}} h$. Moreover, the unique solution of (1.8) is

$$\underline{h} = Pr_{V_{2,+}} h = Pr_{V_1} h_1 + Pr_{V_1} h_2 + Pr_{V_2} h_3. \tag{2.7}$$

3 Main Result

Consider two measurable two-parameter functions $f, u : \mathbb{R}_+^2 \to \mathbb{R}$. Suppose that $f(\mathbf{0}) = 0$ and set

$$f_1(t_1) := f(t_1, 0),\ f_2(t_2) := f(0, t_2),\ f_3(\mathbf{t}) := f(\mathbf{t}) - f(t_1, 0) - f(0, t_2)),$$

hence we can write $f(\mathbf{t}) = f(t_1, 0) + f(0, t_2) + (f(\mathbf{t}) - f(t_1, 0) - f(0, t_2))$. Let $f_i \in \mathcal{H}_1,\ i = 1,2$ and $f_3 \in \mathcal{H}_2$. Recall their representations $f_i(t) = \int_{[0,t]} f_i'(s) ds, \quad f_i' \in L_2(\mathbb{R}_+, \lambda_1), i = 1,2,$ and $f_3(\mathbf{t}) = \int_{[0,\mathbf{t}]} f_3''(\mathbf{s}) d\mathbf{s}, \quad f_3'' \in L_2(\mathbb{R}_+^2, \lambda_2)$. We shall estimate the boundary non-crossing probability

$$P_f = \mathbb{P}\left\{f(\mathbf{t}) + W(\mathbf{t}) \le u(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}_+^2\right\}.$$

In the following we set $\underline{f_i} = Pr_{V_1} f_i, i = 1, 2$ and $\underline{f_3} = Pr_{V_2} f, \underline{f} = Pr_{V_{2,+}} f$ and define

$$\underline{f_{13}}(t) = \underline{f_1}'(t) - \underline{f_3}''(t, 0), \quad \underline{f_{23}}(t) = \underline{f_2}'(t) - \underline{f_3}''(0, t).$$

Note that due to the definition of the set V_2, see (2.5),

$$\Delta_{\mathbf{s}} \underline{f_3}''(\mathbf{t}) \geq 0,\ \Delta^1_{\mathbf{s}} \underline{f_3}''(t_1, s_2) \leq 0,\ \Delta^2_{\mathbf{s}} \underline{f_3}''(s_1, t_2) \leq 0 \text{ for any } \mathbf{s} \leq \mathbf{t} \text{ and } \mathbf{t} \in \mathbb{R}^2_+.$$

In general, the choice of the set V_2 is the key point of the whole work because we can easily integrate w.r.t. $\underline{f_3}''$ both in each one-parameter direction and in the plane in Riemann-Stieltjes sense. Indeed, $\underline{f_3}''$ is decreasing in each coordinate and is increasing in two-parameter sense. We state next our main result:

Theorem 1. *Let the following conditions hold:*

(i) both functions $\underline{f_{13}}(t)$ and $\underline{f_{23}}(t)$ are non-increasing in their arguments;
Riemann-Stieltjes integrals $\int_{[0,x]} u(x, t) d_t(\underline{f_3}''(x, t))$, $\int_{[0,x]} u(s, x) d_s(\underline{f_3}''(s, x))$, $\int_{\mathbb{R}_+} u(t, 0) d\underline{f_{13}}(t)$, $\int_{\mathbb{R}_+} u(0, t) d\underline{f_{23}}(t)$ and $\int_{\mathbb{R}^2_+} u(\mathbf{t}) d\underline{f_3}''(\mathbf{t})$ exist (as the integrals with respect to monotonic functions);

(ii)

$$\lim_{t\to\infty} u(t, 0)\underline{f_{13}}(t) = \lim_{t\to\infty} u(0, t)\underline{f_{23}}(t) = 0, \quad \lim_{t_1,t_2\to\infty} u(\mathbf{t})\underline{f_3}''(\mathbf{t}) = 0, \tag{3.1}$$

$$\lim_{x\to\infty} \int_{[0,x]} u(x, t) d_t(\underline{f_3}''(x, t)) = \lim_{x\to\infty} \int_{[0,x]} u(s, x) d_s(\underline{f_3}''(s, x)) = 0. \tag{3.2}$$

Then we have

$$P_f \leq P_{f-\underline{f}} \exp\Big(- \int_{\mathbb{R}_+} u(t, 0) d\underline{f_{13}}(t) - \int_{\mathbb{R}_+} u(0, t) d\underline{f_{23}}(t) + \int_{\mathbb{R}^2_+} u(\mathbf{t}) d\underline{f_3}''(\mathbf{t}) - \frac{1}{2}\|\underline{f}\|^2 \Big).$$

Remark 1. Any function $f \in \mathcal{H}_{2,+}$ starts from zero. Therefore f can not be a constant unless $f \equiv 0$ but this case is trivial.

Remark 2. Condition (ii) of the theorem means that asymptotically the shifts and their derivatives are negligible in comparison with function u. It is the generalization of the corresponding conditions for the Brownian bridge and Brownian pillow that are defined on a compact sets so that the corresponding condition holds automatically.

Proof Denote by $\widetilde{P}$ a probability measure that is defined via its Radon-Nikodym derivative

$$\frac{dP}{d\widetilde{P}} = \prod_{i=1,2} \exp\Big(- \frac{1}{2}\|f_i\|^2 + \int_{\mathbb{R}_+} f_i'(t) dW^0_i(t)\Big) \exp\Big(- \frac{1}{2}\|f_3\|^2 + \int_{\mathbb{R}^2_+} f_3''(\mathbf{t}) dW^0_3(\mathbf{t})\Big).$$

According to the Cameron-Martin-Girsanov theorem, $W^0_i(t) = W_i(t) + \int_{[0,t]} f_i'(s) ds,\ i = 1, 2$ are independent Wiener processes and $W^0_3(\mathbf{t}) = W_3(\mathbf{t}) + \int_{[\mathbf{0},\mathbf{t}]} f_3''(\mathbf{s}) d\mathbf{s}$ is a Brownian sheet w.r.t. the measure $\widetilde{P}$ being further independent of W^0_1, W^0_2. Denote $1_u\{X\} = 1\{X(\mathbf{t}) \leq u(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}^2_+\}$ and

$$W^0(\mathbf{t}) = W^0_1(t_1) + W^0_2(t_2) + W^0_3(\mathbf{t}).$$

Since $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2$, then using further (2.6) and (2.7) we obtain

$$
\begin{aligned}
&P_f \\
&= \mathbb{E}\left\{1_u\Big(\sum_{i=1,2}(W_i(t)+f_i(t)) + W_3(\mathbf{t}) + f_3(\mathbf{t})\Big)\right\} \\
&= \mathbb{E}_{\widetilde{P}}\left(\frac{dP}{d\widetilde{P}}1_u\Big(W^0(\mathbf{t})\Big)\right) \\
&= \exp\Big(-\frac{1}{2}\|f\|^2\Big)\mathbb{E}\left\{\exp\Big(\int_{\mathbb{R}_+} f_1'(t)dW_1^0(t) + \int_{\mathbb{R}_+} f_2'(t)dW_2^0(t) + \int_{\mathbb{R}_+^2} f_3''(\mathbf{t})dW_3^0(\mathbf{t})\Big)1_u\Big(W^0(\mathbf{t})\Big)\right\} \\
&= \exp\Big(-\frac{1}{2}\|\underline{f}\|^2\Big) \\
&\quad\times\mathbb{E}\left\{\prod_{i=1,2}\exp\Big(-\frac{1}{2}\|Pr_{\widetilde{V}_1}f_i\|^2 + \int_{\mathbb{R}_+} Pr_{\widetilde{V}_1}f_i'(t)dW_i^0(t)\Big)\exp\Big(-\frac{1}{2}\|Pr_{\widetilde{V_2}}f_3\|^2 + \int_{\mathbb{R}_+^2} Pr_{\widetilde{V_2}}f_3''(\mathbf{t})dW_2^0(\mathbf{t})\Big)\right. \\
&\quad\left.\times\exp\Big(\sum_{i=1,2}\int_{\mathbb{R}_+}\underline{f_i}'(t)dW_i^0(t) + \int_{\mathbb{R}_+^2}\underline{f_3}''(\mathbf{t})dW_2^0(\mathbf{t})\Big)1_u\Big(W^0(\mathbf{t})\Big)\right\}.
\end{aligned}
$$

Now we only need to re-write

$$
\sum_{i=1,2}\int_{\mathbb{R}_+}\underline{f_i}'(t)dW_i^0(t) + \int_{\mathbb{R}_+^2}\underline{f_3}''(t)dW_3^0(\mathbf{t}) = \sum_{i=1,2}\int_{\mathbb{R}_+}\underline{f_i}'(t)dW_i^0(t) + \int_{\mathbb{R}_+^2}\underline{f_3}''(t)dW^0(\mathbf{t}).
$$

In order to re-write $\int_{\mathbb{R}_+}\underline{f_1}'(t)dW_1^0(t)$, we mention that in this integral $dW_1^0(t) = d_1W_1^0(t) = d_1(W^0(t,0))$, therefore on the indicator $1_u\{\sum_{i=1,2}W_i^0(t) + W_3^0(\mathbf{t})\} = 1_u\{W^0(\mathbf{t})\}$ under conditions of the theorem we have the relations

$$
\begin{aligned}
\int_{\mathbb{R}_+}\underline{f_1}'(t)dW_1^0(t) &= \lim_{n\to\infty}\int_{[0,n]}\underline{f_1}'(t)dW_1^0(t) \\
&= \lim_{n\to\infty}\Big(\underline{f_1}'(n)W^0(n,0) + \int_{[0,n]}W^0(t,0)d(-\underline{f_1}')(t)\Big).
\end{aligned}
\tag{3.3}
$$

Similarly,

$$
\int_{\mathbb{R}_+}\underline{f_2}'(t)dW_2^0(t) = \lim_{n\to\infty}\Big(\underline{f_2}'(n)W^0(0,n) + \int_{[0,n]}W^0(0,t)d(-\underline{f_2}')(t)\Big).
\tag{3.4}
$$

Further, by Lemma 3, for $\mathbf{n} = (n_1, n_2)$

$$
\begin{aligned}
\int_{\mathbb{R}_+^2}\underline{f_3}''(\mathbf{t})dW^0(\mathbf{t}) = \lim_{n_1,n_2\to\infty}\Big(&\underline{f_3}''(\mathbf{n})W^0(\mathbf{n}) - \underline{f_3}''(n_1,0)W^0(n_1,0) - \underline{f_3}''(0,n_2)W^0(0,n_2) \\
&+\int_{[\mathbf{0},\mathbf{n}]}W^0(\mathbf{t})d\underline{f_3}''(\mathbf{t}) + \int_{[0,n_1]}W^0(s,n_2)d_s(-\underline{f_3}''(s,n_2)) + \int_{[0,n_2]}W^0(n_1,t)d_t(-\underline{f_3}''(n_1,t)) \\
&+\int_{[0,n_1]}W^0(s,0)d_s(\underline{f_3}''(s,0)) + \int_{[0,n_2]}W^0(0,t)d_t(\underline{f_3}''(0,t))\Big).
\end{aligned}
\tag{3.5}
$$

Combining (3.3)–(3.5), using conditions (i)-(ii) and Lemma 5 we conclude that all values $\underline{f_3}''(\mathbf{n})$, $\underline{f_{13}}(n) = \underline{f_1}'(n) - \underline{f_3}''(n,0)$ and $\underline{f_{23}}(n) = \underline{f_2}'(n) - \underline{f_3}''(0,n)$ are non-negative, therefore we get that on the same indicator

$$
\begin{aligned}
&\sum_{i=1,2}\int_{\mathbb{R}_+}\underline{f_i}'(t)dW_i^0(t)+\int_{\mathbb{R}_+^2}\underline{f_3}''(t)dW^0(\mathbf{t})\le\lim_{n_1,n_2\to\infty}\Bigg(\underline{f_3}''(\mathbf{n})u(\mathbf{n})+\underline{f_{13}}(n_1)u(n_1,0)\\
&+\underline{f_{23}}(n_2)u(0,n_2)+\int_{[\mathbf{0},\mathbf{n}]}u(\mathbf{t})d\underline{f_3}''(\mathbf{t})+\int_{[0,n]}u(s,n)d_s(-\underline{f_3}''(s,n))+\int_{[0,n_2]}u(n_1,t)d_t(-\underline{f_3}''(n_1,t))\\
&+\int_{[0,n_1]}u(s,0)d_s(-\underline{f_{13}})(s)+\int_{[0,n_2]}u(0,t)d_t(-\underline{f_{23}})(t)\Bigg)\\
&\le\int_{\mathbb{R}_+^2}u(\mathbf{t})d\underline{f_3}''(\mathbf{t})+\int_{\mathbb{R}_+}u(s,0)d_s(-\underline{f_{13}})(s)+\int_{\mathbb{R}_+}u(0,t)d_t(-\underline{f_{23}})(t).
\end{aligned}
\tag{3.6}
$$

Further conclusions are similar to [2].

If u is bounded, then according to Lemma 5 condition (ii) above is satisfied. Hence, applied for $u(s,t)=u>0, s,t\geq 0$ combined with (1.7) the above theorem implies the following result.

Corollary 1. *If $f\in\mathcal{H}_{2,+}$ is such that $f(\mathbf{t}_0)>0$ for some $\mathbf{t}_0$ with non-negative components, then* (1.9) *holds, provided that condition (i) is valid.*

Remark 3. a) Our results can be generalized to higher dimensions. We only mention that in the case of n-parameter functions we have to define similarly all the differences $\Delta_{\mathbf{s}}^k f(\mathbf{t}),\ 1\le k\le n$ and the space

$$V_n=\{h\in\mathcal{H}_n^2\big|(-1)^k\Delta_{\mathbf{s}}^k h(\mathbf{t})\ge 0, \text{for any } \mathbf{s}\le\mathbf{t},\ 1\le k\le n\}.$$

b) The case of linear combinations of W_i's can be treated with some obvious modifications.

c) Consider the additive Brownian pillow

$$B(t_1,t_2)=B_1(t_1)+B_2(t_2)+B_3(t_1,t_2),\quad t_1,t_2\in[0,1],$$

which is constructed similarly to the additive Wiener field; here B_1,B_2 are two independent Brownian bridges and B_3 is a Brownian pillow being further independent of B_1,B_2. The RKHS's of B,B_1,B_3 are almost the same as W,W_1,W_3 with the only differences that the corresponding functions are defined on $[0,1]^2$ or $[0,1]$ and the functions are zero on the boundaries of these intervals. The closed convex spaces V_1,V_2 and V_3 are then defined similarly as in Section 2, and thus all the results above hold for the additive Brownian pillow by simply changing the conditions for f and u accordingly. Note that compared to [12] we do not need to put restrictions on $\underline{f}$. Thus the results obtained by our approach here are more general.

4 Appendix

Let $A\in\mathcal{H}_2$ be a two-parameter non-random function. If $A\in\widetilde{V_2}$, then A is non-increasing as function of any one-parameter variable and non-decreasing as a function of two variables. Then for the additive Wiener field $W=\{W(\mathbf{t})=W_1(t_1)+W_2(t_2)+W_3(\mathbf{t}),\mathbf{t}\in\mathbb{R}_+^2\}$ and for any $\mathbf{T}=(T,T)$ there exist two integrals of the first kind (according to the classification from the papers [9, 23] and [24]), $\int_{[\mathbf{0},\mathbf{T}]}A(\mathbf{u})dW(\mathbf{u})$ that is standard integral of non-random function with respect to a Gaussian process, or Itô integral, which are the same in this case because

$$\int_{[\mathbf{0},\mathbf{T}]}A(\mathbf{u})dW(\mathbf{u})=\int_{[\mathbf{0},\mathbf{T}]}A(\mathbf{u})dW_3(\mathbf{u}),$$

and $\int_{[\mathbf{0},\mathbf{T}]} W(\mathbf{u})dA(\mathbf{u})$ that is the Riemann-Stieltjes integral. We argue only for the existence of the integral $\int_{[\mathbf{0},\mathbf{T}]} A(\mathbf{u})dW(\mathbf{u})$ because the existence of the integral $\int_{[\mathbf{0},\mathbf{T}]} W(\mathbf{u})dA(\mathbf{u})$ is evident, due to the continuity of the trajectories of the Wiener field. Indeed, such function A attains its maximal value at $\mathbf{0}$. Therefore $\int_{[\mathbf{0},\mathbf{T}]} A^2(\mathbf{s})d\mathbf{s} \le A(\mathbf{0})T^2$ which implies that $\int_{[\mathbf{0},\mathbf{T}]} A(\mathbf{u})dW_3(\mathbf{u})$ is correctly defined as Itô integral. Moreover, denote the increments

$$\Delta^1_{ik,n}X = \Delta^1_{\left(\frac{T(i-1)}{n},\frac{T(k-1)}{n}\right)}X\Big(\frac{Ti}{n},\frac{T(k-1)}{n}\Big)$$

and

$$\Delta^2_{ik,n}X = \Delta^1_{\left(\frac{T(i-1)}{n},\frac{T(k-1)}{n}\right)}X\Big(\frac{T(i-1)}{n},\frac{Tk}{n}\Big),$$

where $X = A, W$. Then there exist two integrals of the second kind

$$\int_{[\mathbf{0},\mathbf{T}]} d_iA(\mathbf{u})d_jW(\mathbf{u}), \quad i = 1,2, j = 3-i$$

that are defined as the limits in probability of integral sums where for example,

$$\int_{[\mathbf{0},\mathbf{T}]} d_1A(\mathbf{u})d_2W(\mathbf{u}) = \lim_{n\to\infty}\sum_{1\le i,k\le n}\Delta^1_{ik,n}A\Delta^2_{ik,n}W.$$

Lemma 3. *Let $A \in \widetilde{V}_2$ be a two-parameter non-random function and let $W = \{W(\mathbf{t}), \mathbf{t} \in \mathbb{R}^2_+\}$ be an additive Wiener field. Then for any $\mathbf{T} = (T,T)$ we have the following version of integration-by-parts formula:*

$$\begin{aligned}\int_{[\mathbf{0},\mathbf{T}]} A(\mathbf{s})dW(\mathbf{s}) \;&=\; A(\mathbf{T})W(\mathbf{T}) - A(T,0)W(T,0) - A(0,T)W(0,T) + \int_{[\mathbf{0},\mathbf{T}]} W(\mathbf{s})dA(\mathbf{s})\\ &+\int_{[0,T]} W(s,T)d_s(-A(s,T)) + \int_{[0,T]} W(T,t)d_t(-A(T,t))\\ &+\int_{[0,T]} W_1(s)d_s(A(s,0)) + \int_{[0,T]} W_2(t)d_s(A(0,t)).\end{aligned}$$

Proof The standard one-parameter Itô formula yields

$$\int_{[0,T]} A(s,T)d_sW(s,T) = A(\mathbf{T})W(\mathbf{T}) - A(0,T)W(0,T) - \int_{[0,T]} W(s,T)d_sA(s,T).$$

Using further the generalized two-parameter Itô formula (see e.g., [16]), we obtain

$$\int_{[0,T]} A(s,T)d_sW(s,T) = \int_{[0,T]} A(s,0)dW_1(s) + \int_{[\mathbf{0},\mathbf{T}]} A(\mathbf{s})dW(\mathbf{s}) + \int_{[\mathbf{0},\mathbf{T}]} d_1W(\mathbf{t})d_2A(\mathbf{t}),$$

and similarly

$$\int_{[0,T]} W(T,t)d_tA(T,t) = \int_{[0,T]} W(0,t)d_tA(0,t) + \int_{[\mathbf{0},\mathbf{T}]} W(\mathbf{s})dA(\mathbf{s}) + \int_{[\mathbf{0},\mathbf{T}]} d_1W(\mathbf{t})d_2A(\mathbf{t}).$$

From the last three equalities above we immediately get that

$$\begin{aligned}\int_{[\mathbf{0},\mathbf{T}]} A(\mathbf{s})dW(\mathbf{s}) &= \int_{[0,T]} A(s,T)d_sW(s,T) - \int_{[\mathbf{0},\mathbf{T}]} d_1W(\mathbf{t})d_2A(\mathbf{t}) - \int_{[0,T]} A(s,0)dW_1(s)\\ &= \int_{[0,T]} A(s,T)d_sW(s,T) - \int_{[0,T]} W(T,t)d_tA(T,t)\\ &+ \int_{[\mathbf{0},\mathbf{T}]} W(\mathbf{s})dA(\mathbf{s}) + \int_{[0,T]} W(0,t)d_tA(0,t) - \int_{[0,T]} A(s,0)dW_1(s)\\ &= A(\mathbf{T})W(\mathbf{T}) - A(T,0)W(T,0) - A(0,T)W(0,T) + \int_{[\mathbf{0},\mathbf{T}]} W(\mathbf{s})dA(\mathbf{s})\\ &\quad + \int_{[0,T]} W(s,T)d_s(-A(s,T)) + \int_{[0,T]} W(T,t)d_t(-A(T,t))\\ &\quad + \int_{[0,T]} W_1(s)d_s(A(s,0)) + \int_{[0,T]} W_2(t)d_s(A(0,t))\end{aligned}$$

establishing the proof.

Lemma 4. *The RKHS of the covariance function of the additive Wiener field W coincides with $\mathcal{H}_{2,+}$ given in* (1.3).

Proof If the function $h : \mathbb{R}_+^2 \to \mathbb{R}$ admits the representation

$$h(\mathbf{t}) = \sum_{i=1,2} h_i(t_i) + h_3(\mathbf{t}), \tag{4.1}$$

where $h_i \in \mathcal{H}_1, i = 1,2$ and $h_3 \in \mathcal{H}_2$, then the representation (4.1) is unique. This claim follows immediately if we put $t_i = 0, i = 1,2$. In view of (1.2) the claim follows by Theorem 5, p.24 in [1].
Consider the subspace $V_1 = \{h \in \mathcal{H}_1 \big| \Delta_s^1 h'(t) \le 0 \ , 0 \le s \le t < \infty\}$. Clearly, for any $h \in V_1$ we have that $h'(t) \downarrow 0$ as $t \to \infty$. Now we establish similar fact for the subspace

$$V_2 = \{h \in \mathcal{H}_2 \big| \Delta_{\mathbf{s}} h''(\mathbf{t}) \ge 0,\ \Delta_{\mathbf{s}}^1 h''(t_1, s_2) \le 0,\ \Delta_{\mathbf{s}}^2 h''(s_1, t_2) \le 0 \text{ for any } \mathbf{s} \le \mathbf{t} \text{ and } \mathbf{t} \in \mathbb{R}_+^2\}.$$

Lemma 5. *If $h \in V_2$ is such that $\int_{\mathbb{R}_+} (h''(s,0))^2 ds < \infty$ and $\int_{\mathbb{R}_+} (h''(0,t))^2 dt < \infty$, then $h''(s,t) \downarrow 0$ as $s \to \infty$ for any $t \in R_+$, $h''(s,t) \downarrow 0$ as $t \to \infty$ for any $s \in R_+$, and $h''(s,t) \downarrow 0$ as $s,t \to \infty$.*

Proof Note that it is sufficient to establish the first claim. Since $h \in V_2$, then $\int_{\mathbb{R}_+^2} (h''(s,t))^2 dsdt < \infty$ implying that $\int_{\mathbb{R}_+} (h''(s,t))^2 ds < \infty$ for a.e. t. Furthermore, $h''(s,t)$ is non-increasing in s therefore for such t we have $h''(s,t) \downarrow 0$ as $s \to \infty$ and it follows from the assumption that $h''(s,0) \downarrow 0$ as $s \to \infty$. Since it is non-increasing in t, we get such convergence for any t, hence the claim follows.
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