PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 1, 1975

SETS OF p-SPECTRAL SYNTHESIS

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Let G be a Hausdorff locally compact Abelian group, Γ its character group. Certain closed subsets of Γ are introduced, these being closely related to sets of spectral synthesis for $L^1(G)^\wedge$. Some properties and examples of these sets are discussed, and then a Malliavin-type result is obtained.

In general we follow the notation used in [1]. We shall let λ, θ denote Haar measures on G, Γ respectively, chosen so that Plancherel's theorem holds.

1. The definition and some properties of S_p- and C_p-sets.

DEFINITION 1.1. Let Ξ be a closed subset of Γ. We shall call Ξ an S_p-set ($p \in [1,\infty)$) if, given $\epsilon > 0$ and $f \in L^1 \cap L^p(G)$ such that $\hat{f}$ vanishes on Ξ, there exists $g \in L^1 \cap L^p(G)$ such that $\hat{g}$ vanishes on a neighbourhood of Ξ and $\|f-g\|_p < \epsilon$. If such a g can be found of the form $h * f$, where $h \in L^1(G)$ and $\hat{h}$ vanishes on a neighbourhood of Ξ, then Ξ will be called a C_p-set. We also define S_∞- and C_∞-sets as above, with f, g in $L^1 \cap C_0(G)$ (rather than $L^1 \cap L^\infty(G)$).

Since, by [1], (33.12), $L^1(G)$ admits a bounded positive approximate identity $\{u_i\}_{i\in I}$ such that for each $i \in I$, $u_i \in L^1 \cap C_0(G)$ and $\text{supp}(\hat{u}_i)$ is compact, it follows (see [1], (32.33) (b) and (32.48) (a)) that we can (and shall) assume in Definition 1.1 that $f, g, h \in L^1 \cap C_0(G)$, where $\text{supp}(\hat{f})$ is compact and both $\text{supp}(\hat{g})$ and $\text{supp}(\hat{h})$ are compact and disjoint from Ξ ($p \in [1,\infty]$).

Clearly every C_p-set is an S_p-set. For the case $p = 1$ we just have the familiar S-set and C-set; see [3], 7.2.5 (a) and 7.5.1 respectively.

For $f \in L^\infty(G)$ the spectrum (written $\Sigma(f)$) will be defined as in [1], (40.21). For $f \in L^p(G)$ ($p \in [1,\infty)$), we define its spectrum by

$$\Sigma(f) = \cup\{\Sigma(\phi * f) \colon \phi \in C_{00}(G)\}$$

It is easily proved that for $f \in L^1(G)$, $\Sigma(f) = \text{supp}(\hat{f})$.

Given $\Xi \subset \Gamma$, we write

$$L^p_\Xi(G) = \{f \in L^p(G) \colon \Sigma(f) \subset \Xi\}.$$

We now have the following characterisation of S_p- and C_p-sets:

THEOREM 1.2. *Let $p \in [1,\infty)$ and suppose Ξ is a closed subset of Γ. Then*

(a) *Ξ is an S_p-set if and only if for all $l \in L^{p'}_{\Xi}(G)$ and for all $f \in L^1 \cap C_0(G)$ such that* $\mathrm{supp}(\hat{f})$ *is compact and $\hat{f}$ vanishes on Ξ, we have $l * f = 0$;*

(b) *Ξ is a C_p-set if and only if for all $f \in L^1 \cap C_0(G)$ such that* $\mathrm{supp}(\hat{f})$ *is compact and $\hat{f}$ vanishes on Ξ, and for all $l \in L^{p'}(G)$ such that $l * f \in L^{p'}_{\Xi}(G)$, we have $l * f = 0$.*

This result is known for the case $p = 1$ (see **[2]**, Chapter 7, 1.2 and 4.9). The proof is standard, and we shall not include it.

It is easy to adapt the proof of **[3]**, Theorem 7.5.2 to give:

THEOREM 1.3. *Let $p \in [1,\infty]$. Then*

(a) *every one-point subset of Γ is a C_p-set in Γ;*

(b) *finite unions of C_p-sets in Γ are C_p-sets in Γ;*

(c) *if the boundary of a closed set Ξ is a C_p-set, so is Ξ;*

(d) *if Ξ is a closed subset of a closed subgroup Λ of Γ, if $\partial_\Lambda(\Xi)$ is the boundary of Ξ relative to Λ, and if $\partial_\Lambda(\Xi)$ is a C_p-set in Γ then Ξ is also a C_p-set in Γ;*

(e) *each closed subgroup of Γ is a C_p-set in Γ.*

For $p \in [1, 2)$ it is not known whether the notions of C_p-set and S_p-set are identical (it appears in Theorem 2.1 that every closed set is a C_p-set for $p \geqq 2$). Furthermore we cannot say whether the union of two S_p-sets is itself an S_p-set. We can however obtain two partial results in this direction. Both these results (Theorem 1.4 (a), (b)) are known for the case $p = 1$ (see **[2]**, Chapter 2, 7.5).

THEOREM 1.4. (a) *Suppose $\Xi = \Xi_1 \cup \Xi_2$, where Ξ_1 and Ξ_2 are disjoint closed subsets of Γ. Then, for $p \in [1,\infty)$, Ξ is an S_p-set if and only if both Ξ_1 and Ξ_2 are S_p-sets.*

(b) *Let $p \in [1,\infty)$ and suppose Ξ_1 is an S_p-set and Ξ_2 is a C_p-set. Then $\Xi = \Xi_1 \cup \Xi_2$ is an S_p-set.*

The final result of this section gives us an inclusion result between the set of C_p-sets (respectively S_p-sets) and the set of C_q-sets (respectively S_q-sets) for $1 \leqq p < q \leqq \infty$.

THEOREM 1.5. *Let $1 \leqq p < q \leqq \infty$. Then every C_p-set* (respectively S_p-set) *is a C_q-set* (respectively S_q-set).

Proof. Assume Ξ is a C_p-set. Suppose we are given $\epsilon > 0$ and $f \in L^1 \cap C_0(G)$ with $\mathrm{supp}(\hat{f})$ compact and $\hat{f}$ vanishing on Ξ. We can find $h \in L^1 \cap C_0(G)$ such that $\|f - h * f\|_q < \epsilon/2$. Since Ξ is a C_p-set there exists $g \in L^1(G)$ such that $\hat{g}$ has compact support disjoint from Ξ and $\|h\|_r \|f - g * f\|_p < \epsilon/2$, where $p^{-1} + r^{-1} - q^{-1} = 1$ (with the usual convention for the cases $p = 1$ and $q = \infty$). Now (see [1], (20.18))

$$\begin{aligned}\|f - h * g * f\|_q &\leqq \|f - h * f\|_q + \|h\|_r \|f - g * f\|_p \\ &< \epsilon.\end{aligned}$$

It remains only to note that $h * g \in L^1 \cap C_0(G)$ and $(h * g)^\wedge$ has compact support disjoint from Ξ.

The proof that every S_p-set is an S_q-set is similar.

2. Examples of S_p- and C_p-sets.

THEOREM 2.1. *For $p \in [2, \infty]$ every closed subset of Γ is a C_p-set.*

Proof. In view of Theorem 1.5 we need only prove the theorem for $p = 2$.

Let Ξ be a closed subset of Γ and suppose we are given $\epsilon > 0$ and $f \in L^1 \cap C_0(G)$ with $\mathrm{supp}(\hat{f})$ compact, $\hat{f}$ vanishing on Ξ and $\|f\|_1 \leqq 1$. Now $\Omega = \{\gamma \in \Gamma: \hat{f}(\gamma) \neq 0\}$ is a relatively compact open set, and hence there exists a compact set $\Upsilon \subset \Omega$ such that $\theta(\Omega \backslash \Upsilon) < \epsilon^2$. Choose an open set ∇ such that $\Upsilon \subset \nabla \subset \nabla^- \subset \Omega$, and (see [3], 2.6.1) $k \in L^1 \cap C_0(G)$ such that $\xi_\Upsilon \leqq \hat{k} \leqq \xi_\nabla$. Then, using Plancherel's theorem,

$$\begin{aligned}\|f - k * f\|_2 &= \left(\int_{\Omega\backslash\Upsilon} |1 - \hat{k}(\gamma)|^2 |\hat{f}(\gamma)|^2 \, d\theta(\gamma)\right)^{\frac{1}{2}} \\ &< \theta(\Omega\backslash\Upsilon)^{\frac{1}{2}} \\ &< \epsilon;\end{aligned}$$

and clearly, $\hat{k}$ has compact support disjoint from Ξ.

DEFINITION 2.2. Let Ω be a relatively compact open subset of Γ. We shall call Ω a β-symmetry set ($\beta > 0$) if there exist nets $\{\Upsilon_i\}_{i \in I}$ and $\{\nabla_i\}_{i \in I}$ such that each Υ_i is compact, $\{\nabla_i\}_{i \in I}$ is a base of symmetric open neighbourhoods of zero in Γ, partially ordered by

$$\nabla_i < \nabla_j \text{ if and only if } \nabla_i \supset \nabla_j,$$

$(\Upsilon_i + 2\nabla_i)^- \subset \Omega$ for each $i \in I$, and

$$\lim_{i \in I} \frac{\theta(\Omega \backslash \Upsilon_i)^\beta}{\theta(\nabla_i)} = 0.$$

THEOREM 2.3. *Suppose we are given $\beta > 0$ and a closed subset Ξ of Γ with the property that for any relatively compact set $\Upsilon \subset \Xi^c$ there exists a β-symmetry set Ω such that $\Upsilon \subset \Omega \subset \Xi^c$. Then Ξ is a C_p-set for all $p \geqq (2+\beta)^{-1}(2+2\beta)$.*

Proof. Let $p = (2+\beta)^{-1}(2+2\beta)$. Suppose we are given $\epsilon > 0$ and $f \in L^1 \cap C_0(G)$, where $\operatorname{supp}(\hat{f})$ is compact, $\hat{f}$ vanishes on Ξ and $\|f\|_1 \leqq 1$. Now $\Upsilon = \{\gamma \in \Gamma: \hat{f}(\gamma) \neq 0\}$ is a relatively compact open subset of Ξ^c and hence, by assumption, there exists a relatively compact open set Ω such that $\Upsilon \subset \Omega \subset \Xi^c$, and nets $\{\Upsilon_i\}_{i\in I}$ and $\{\nabla_i\}_{i \in I}$ satisfying the conditions of Definition 2.2. Choose $i \in I$ such that Υ_i is nonvoid and

$$\left[\frac{\theta(\Omega\backslash\Upsilon_i)^{\beta}}{\theta(\nabla_i)}\right]^{\alpha/2} < 2^{-\alpha}\theta(\Omega)^{-\alpha/2}\epsilon,$$

where $\alpha = (1+\beta)^{-1}$. Define $k_i = \theta(\nabla_i)^{-1}g_i h_i$, where g_i, h_i in $L^2(G)$ are such that $\hat{g}_i = \xi_{\nabla_i}$ (cf. **[3]**, 2.6.1) $k_i \in L^1 \cap C_0(G)$, $\xi_{\Upsilon_i} \leqq \hat{k}_i \leqq \xi_{\Upsilon_i + 2\nabla_i}$ and

$$\|k_i\|_1 \leqq \left[\frac{\theta(\Upsilon_i + \nabla_i)}{\theta(\nabla_i)}\right]^{\frac{1}{2}} .$$

It follows from Hölder's inequality that

$$\begin{aligned}\|f - k_i * f\|_p &\leqq \|f - k_i * f\|_1^{\alpha}\, \|f - k_i * f\|_2^{1-\alpha} \\ &\leqq \|f\|_1^{\alpha}\left[1 + \left[\frac{\theta(\Upsilon_i + \nabla_i)}{\theta(\nabla_i)}\right]^{\frac{1}{2}}\right]^{\alpha} \theta(\Omega\backslash\Upsilon_i)^{(1-\alpha)/2} \\ &\leqq 2^{\alpha}\theta(\Upsilon_i + \nabla_i)^{\alpha/2}\,\frac{\theta(\Omega\backslash\Upsilon_i)^{(1-\alpha)/2}}{\theta(\nabla_i)^{\alpha/2}} \\ &< \epsilon\end{aligned}$$

(recall that $\alpha = (1+\beta)^{-1}$ and $p = (2+\beta)^{-1}(2+2\beta) = 2(1+\alpha^{-1})^{-1}$). Noting that $\hat{k}_i$ has compact support disjoint from Ξ we see that Ξ is a C_p-set, and the conclusion follows from Theorem 1.5.

We have two corollaries when G is a Euclidean space.

COROLLARY 2.4. *Let $m \geqq 1$ and suppose $\Xi \subset R^m$ is an open set with the property that for any relatively compact set $\Upsilon \subset R^m$ there exists a number κ_m $(= \kappa_m(\Upsilon))$ such that*

$$\theta((\partial(\Xi) \cap \Upsilon) + \nabla_n) \leqq \kappa_m n^{-1}$$

for all $n \in \{1, 2, \cdots\}$, where $\partial(\Xi)$ denotes the boundary of Ξ and

$$\nabla_n = \{x \in R^m : \|x\| < n^{-1}\}.$$

Then Ξ, Ξ^c and $\partial(\Xi)$ are C_p-sets for all $p > (2+m)^{-1}(2+2m)$.

Proof. By Theorem 1.3 (c) we need consider only $\partial(\Xi)$.

Let Υ be any relatively compact open subset of $\partial(\Xi)^c$. We shall show that for any $\epsilon > 0$ there exists an $(m+\epsilon)$-symmetry set Ω such that $\Upsilon \subset \Omega \subset \partial(\Xi)^c$. Since Υ is relatively compact in R^m there exists an integer $n_0 > 0$ such that

$$\Upsilon \subset \Delta_{n_0} = \{x \in R^m : \|x\| < n_0\}.$$

For each $n \in \{1, 2, \cdots\}$ define

$$\Upsilon_n = (\partial(\Xi) + \nabla_n)^c \cap (\Delta_{n_0} \backslash \Delta^-_{n_0 - n^{-1}})^c \cap \Delta_{n_0}.$$

Clearly Υ_n is compact and

$$(\Upsilon_n + 2\nabla_{3n})^- \subset \Delta_{n_0} \cap \partial(\Xi)^c.$$

Putting $\Omega = \Delta_{n_0} \cap \partial(\Xi)^c$ we have

$$\begin{aligned} \Omega \backslash \Upsilon_n &= (\Omega \cap (\partial(\Xi) + \nabla_n)) \cup (\Omega \cap (\Delta_{n_0} \backslash \Delta^-_{n_0 - n^{-1}})) \\ &= (\Delta_{n_0} \cap \partial(\Xi)^c \cap (\partial(\Xi) + \nabla_n)) \cup (\Delta_{n_0} \cap \partial(\Xi)^c \cap (\Delta_{n_0} \backslash \Delta^-_{n_0 - n^{-1}})) \\ &\subset (\Delta_{n_0} \cap (\partial(\Xi) + \nabla_n)) \cup (\Delta_{n_0} \backslash \Delta^-_{n_0 - n^{-1}}) \\ &\subset (((\Delta^{\cdot}_{n_0} + \nabla_1) \cap \partial(\Xi)) + \nabla_n) \cup (\Delta_{n_0} \backslash \Delta^-_{n_0 - n^{-1}}). \end{aligned}$$

Hence, since $\Delta_{n_0} + \nabla_1$ is relatively compact,

$$\theta(\Omega \setminus \Upsilon_n) \leqq \kappa_m (\Delta_{n_0} + \nabla_1) n^{-1} + O(n^{-1}).$$

Using the fact that

$$\theta(\nabla_{3n}) = \kappa'_m 3^{-m} n^{-m}$$

for some constant κ'_m, we have

$$\lim_{n \to \infty} \frac{\theta(\Omega \backslash \Upsilon_n)^{m+\epsilon}}{\theta(\nabla_{3n})} = 0,$$

and so Ω is an $(m+\epsilon)$-symmetry set for all $\epsilon > 0$.

Thus $\partial(\Xi)$ satisfies the conditions of Theorem 2.3 with $\beta = m + \epsilon$, and hence is a C_p-set for all $p > (2+m)^{-1}(2+2m)$.

COROLLARY 2.5. *Let $m \geqq 1$ and put*

$$\Xi = \{x \in R^m : \|x\| = 1\}.$$

Then Ξ is a C_p-set for all $p > (2+m)^{-1}(2+2m)$.

Proof. Let ∇ be any relatively compact set in R^m. Then

$$\begin{aligned} \theta((\Xi \cap \nabla) + \nabla_n) &\leqq \theta(\Xi + \nabla_n) \\ &= \kappa'_m((1+n^{-1})^m - (1-n^{-1})^m) \\ &= O(n^{-1}), \end{aligned}$$

where κ'_m is a constant. Now apply Corollary 2.4.

REMARK 2.6. For $m \geqq 3$, Corollary 2.5 gives an example of a C_p-set $((2+m)^{-1}(2+2m) < p < 2)$ which is not an S-set; cf. **[3]**, 7.3.2.

3. The failure of certain closed sets to be S_p-sets. In this section we use a proof along the lines of that of Malliavin's theorem (**[3]**, 7.6.1) to show that every nondiscrete Γ contains a closed set which is not an S_p-set for any $p \in [1, 2)$. As in the proof of **[3]**, Theorem 7.6.1, we first consider the cases:

(a) Γ is an infinite compact group;

(b) $\Gamma = R$.

THEOREM 3.1. *Let G be an infinite discrete group. Then there exists a closed set $\Xi \subset \Gamma$ which is not an S_p-set for any $p \in [1, 2)$.*

Proof. Using the notation of **[3]**, Theorem 7.8.6 we consider the function ϕ_1 on G defined by

$$\phi_1 : x \to (D^1 m_x)(\zeta).$$

It is easily proved from **[3]**, 7.6.4 and Theorem 7.8.6 that $f_0 \in L^1(G)$ and ϕ_1 (as above) can be chosen so that f_0 and ζ satisfy the hypotheses of **[3]**, 7.6.3 (Theorem) (with $f = f_0$ and $\xi = \zeta$) and $\phi_1 \in L^q(G)$ for all $q > 2$. Having thus chosen f_0 and ϕ_1 we shall prove that the closed set $\Xi = \{\gamma \in \Gamma : \hat{f}_0(\gamma) = \zeta\}$ is not an S_p-set for any $p \in [1, 2)$.

Let $p \in [1, 2)$ and put

$$I = \{f \in L^1(G) \colon \hat{f}(\Xi) = \{0\}\},$$

$$I_1 = \text{the closed ideal of } L^1(G) \text{ generated by } f_0 - \zeta\,\xi_{\{0\}},$$

$$I_2 = \text{the closed ideal of } L^1(G) \text{ generated by } (f_0 - \zeta\,\xi_{\{0\}})^{*2},$$

and $$J = \{f \in L^1(G) \colon \hat{f} \text{ vanishes on a neighbourhood of } \Xi\}^-.$$

Clearly

$$\Xi = Z(I) = Z(I_1) = Z(I_2) = Z(J)$$

(where $Z(I)$ denotes the zero set of the ideal I; see **[3]**, 7.1.3). Since I and J are respectively the largest and smallest closed ideals in $L^1(G)$ having Ξ as their zero set, we have that $J \subset I_2 \subset I_1 \subset I$.

As $\phi_1 \in L^{p'}(G)$ we can define a continuous linear functional T on $(L^1(G), \|\cdot\|_p)$ by

$$T(g) = \sum_{x \in G} g(-x)\phi_1(x)$$

(recall that G is discrete and hence $L^1(G) \subset L^p(G)$). By **[3]**, 7.6.3, T annihilates I_2 but not I_1.

Now suppose that Ξ is an S_p-set and let $h \in L^1 \cap C_0(G) = L^1(G)$ with $\hat{h}$ vanishing on Ξ. Then, given $\epsilon > 0$, there exists $h' \in J$ such that $\|h - h'\|_p < \epsilon$ and hence, since $T(h') = 0$, $|T(h)| = |T(h - h')| \leqq \epsilon\,\|\phi_1\|_{p'}$. As this holds for all $\epsilon > 0$ we must have that $T(h) = 0$; thus T annihilates I, a contradiction of the fact that T does not annihilate $I_1 \subset I$. It follows that Ξ is not an S_p-set for any $p \in [1, 2)$.

We shall now examine the case when Γ contains an infinite compact open subgroup. We require two lemmas for arbitrary Hausdorff locally compact Abelian groups.

LEMMA 3.2. *Let G be a Hausdorff locally compact Abelian group and suppose H is a closed subgroup of G. Then a continuous integrable function f on G is constant on cosets of H if and only if*

$$\operatorname{supp}(\hat{f}) \subset A(\Gamma, H)$$

(the annihilator of H in Γ).

Proof. The result follows readily from the property

$$({}_hf)^{\wedge}(\gamma) = \gamma(h)\hat{f}(\gamma)$$

for all $\gamma \in \Gamma$ (where ${}_hf \colon x \to f(x + h)$).

LEMMA 3.3. *Let G be a Hausdorff locally compact Abelian group and suppose Λ is an open subgroup of Γ. If Ξ is a closed subset of Λ which is not an S_p-set in Λ then Ξ is not an S_p-set in Γ.*

Proof. Put $H = A(G, \Lambda)$. By **[1]**, (23.24) (e), H is compact. Furthermore, in view of Theorem 2.1, we can assume that $p < \infty$.

Suppose, to the contrary, that Ξ is an S_p-set in Γ. Given $\epsilon > 0$ and $\dot{f} \in L^1 \cap C_0(G/H)$ such that $\text{supp}(\hat{\dot{f}})$ is compact and $\hat{\dot{f}}$ vanishes on Ξ, put $f = \dot{f} \circ \pi_H$, where π_H denotes the natural homomorphism of G onto G/H. Denoting the Haar measures on H, G/H by λ_H, $\lambda_{G/H}$ respectively (normalised as in **[2]**, Chapter 3, 3.3 (i) with $\lambda_H(H) = 1$) we have, by **[2]**, Chapter 3, 4.5,

$$\begin{aligned}\|f\|_p^p &= \int_{G/H} \left\{ \int_H |f(x+y)|^p \, d\lambda_H(y) \right\} d\lambda_{G/H}(\dot{x}) \\ &= \int_{G/H} \left\{ \int_H |\dot{f} \circ \pi_H(x+y)|^p \, d\lambda_H(y) \right\} d\lambda_{G/H}(\dot{x}) \\ &= \int_{G/H} |\dot{f}(\dot{x})|^p \, d\lambda_{G/H}(\dot{x}),\end{aligned}$$

that is,

$$\|f\|_p = \|\dot{f}\|_p. \tag{3.1}$$

It is easily seen that

$$\dot{f}(\dot{x}) = \int_H f(x+y) d\lambda_H(y)$$

and, by **[2]**, Chapter 4, 4.3 ((3.1) shows that $f \in L^1(G)$),

$$\hat{\dot{f}}(\gamma) = \hat{f}(\gamma) \tag{3.2}$$

for all $\gamma \in \Lambda$. Furthermore, since f is constant on cosets of H, Lemma 3.2 shows that $\text{supp}(\hat{f}) \subset A(\Gamma, H) = \Lambda$. As $\text{supp}(\hat{\dot{f}})$ is assumed to be compact it follows from (3.2) that $\text{supp}(\hat{f})$ is compact and hence (note that f is continuous) we see that $f \in C_0(G)$.

Now $\hat{f}$ vanishes on $\Xi \cup \Lambda^c$ and, since by Theorem 1.4 (recall that Λ^c is open and closed) $\Xi \cup \Lambda^c$ is an S_p-set, there exists $g \in L^1 \cap C_0(G)$ such that $\hat{g}$ has compact support disjoint from $\Xi \cup \Lambda^c$ and $\|f - g\|_p < \epsilon$. By Lemma 3.2 again g is constant on cosets of H and we have the existence of $\dot{g} \in L^1 \cap C_0(G/H)$ such that $g = \dot{g} \circ \pi_H$ ($\dot{g} \in C_0(G/H)$ since, by **[2]**, Chapter 3, 1.8 (vii), $\dot{g}$ is continuous and by (3.2), $\hat{\dot{g}}$ has compact support). From (3.1) $\|\dot{f} - \dot{g}\|_p < \epsilon$, and (3.2) shows that $\hat{\dot{g}}$ vanishes on a neighbourhood of Ξ. Hence Ξ is shown to be an S_p-set in Λ, contrary to assumption.

COROLLARY 3.4. *Let G be a Hausdorff locally compact Abelian group, Γ its character group. If Γ contains an infinite compact open subgroup then there exists a closed subset of Γ which is not an S_p-set for any $p \in [1, 2)$.*

Proof. Combine Theorem 3.1 and Lemma 3.3.

Before considering the case $\Gamma = R$ we need to extend the result in **[3]**, Theorem 2.7.6.

THEOREM 3.5. *Suppose $f \in l^1(Z)$, $\delta \in (0, \pi)$ and $\hat{f}(\exp(ix)) = 0$ for $x \in [\pi - \delta, \pi + \delta]$. Let u be defined on R by*

$$u(x) = \begin{cases} \hat{f}(\exp(ix)) & (|x| \leqq \pi) \\ 0 & (|x| > \pi). \end{cases}$$

Then $u = \hat{g}$ for some $g \in L^1(R)$. Moreover, given $p \in [1, \infty]$, there exists a positive number $\kappa_p (= \kappa_p(\delta))$ such that

$$\|f\|_p \leqq \kappa_p \|g\|_p.$$

Proof. The first part of Theorem 3.5 is proved in **[3]**, 2.7.6.

Let $p \in [1, \infty]$. Consider the linear operator T from $L^1 \cap L^\infty(R)$ to $l^1(Z)$, defined by

$$(T(k))(n) = k * \hat{h}(n), \tag{3.3}$$

where $n \in Z$, and $h \in L^1(R)$ is defined as in **[3]**, 2.7.6. The argument at the end of the proof of **[3]**, 2.7.6 shows that there is a constant $\kappa_1 = \kappa_1(\delta)$ such that $\|T(k)\|_1 \leqq \kappa_1 \|k\|_1$. It is clear from (3.3) that $\|T(k)\|_\infty \leqq \kappa_2 \|k\|_\infty$, where $\kappa_2 = \|\hat{h}\|_1$. By the Riesz-Thorin convexity theorem T is continuous as

$$(L^1 \cap L^\infty(R), \|\cdot\|_{p_\alpha}) \xrightarrow{T} (l^1(Z), \|\cdot\|_{p_\alpha})$$

(recall that $l^1(Z) \subset l^\infty(Z)$), where $\alpha \in (0, 1)$, $p_\alpha = (1-\alpha)^{-1}$ and $\|T\|_{(\alpha)} \leqq \kappa_1^{1-\alpha} \kappa_2^{\alpha}$. In particular, choosing $\alpha \in [0, 1)$ such that $p_\alpha = p$ (and $\alpha = 1$ if $p = \infty$) and noting that $g \in L^1 \cap L^\infty(R)$ and (see **[3]**, 2.7.6, (5)) $f(n) = g * \hat{h}(n)$ for all $n \in Z$, we have

$$\|f\|_p \leqq \kappa_1^{1-\alpha} \kappa_2^{\alpha} \|g\|_p,$$

as required.

THEOREM 3.6. *The real line R contains a closed set which is not an S_p-set for any $p \in [1, 2)$.*

Proof. It appears from Theorem 3.1 that there exists a closed set $\Xi_1 \subset T$ (the circle group) which is not an S_p-set for any $p \in [1, 2)$. By translation if necessary we can assume that $-1 \not\in \Xi_1$ and that Ξ_1 is disjoint from Ξ_2 for some closed arc $\Xi_2 \subset T$ containing -1. Put

$$\Upsilon_1 = \{x \in (-\pi, \pi): \exp(ix) \in \Xi_1\},$$

$$\Upsilon_2 = \{x \in (-\pi, \pi): \exp(ix) \in \Xi_2\} \cup [\pi, \infty) \cup (-\infty, -\pi],$$

$$\Xi = \Xi_1 \cup \Xi_2 \text{ and } \Upsilon = \Upsilon_1 \cup \Upsilon_2.$$

Let $p \in [1, 2)$ and suppose Υ_1 is an S_p-set. By Theorem 1.4, Υ is an S_p-set. Given $f \in l^1(Z)$ with $\hat{f}(\Xi) = \{0\}$ define $g \in L^1 \cap C_0(R)$ by

$$\hat{g}(x) = \begin{cases} \hat{f}(\exp(ix)) & (|x| \leqq \pi) \\ 0 & (|x| > \pi) \end{cases}$$

(see Theorem 3.5). Clearly $\hat{g}$ vanishes on Υ and hence, since Υ is an S_p-set, there exists a sequence $(g_n) \subset L^1 \cap C_0(R)$ such that each $\hat{g}_n$ vanishes on a neighbourhood of Υ and

$$\|g - g_n\|_p \to 0. \tag{3.4}$$

If, for each $x \in (-\pi, \pi]$, we define $f_n \in l^1(Z)$ by

$$\hat{f}_n(\exp(ix)) = \hat{g}_n(x)$$

(see **[3]**, Theorem 2.7.6) then Theorem 3.5 applied to (3.4) gives $\|f - f_n\|_p \to 0$ (note that each $\hat{f}_n$ vanishes on a neighbourhood of Ξ). Hence Ξ and consequently (see Theorem 1.4) Ξ_1 would be an S_p-set, contradicting our choice of Ξ_1. It follows that Υ_1 is not an S_p-set for any $p \in [1, 2)$.

We require two lemmas before proving the main result of this section.

LEMMA 3.7. *Let G, H be Hausdorff locally compact Abelian groups and suppose $k \in L^1 \cap C_0(G \times H)$ is such that $\Upsilon = \operatorname{supp}(\hat{k})$ is compact. Then the function $y \to k(x, y)\,(x \to k(x, y))$ is integrable over*

H for every $x \in G$ (over G for every $y \in H$). *Furthermore the functions*

$$\phi_1 \colon x \to \int_H k(x, y) d\lambda_H(y), \quad \phi_2 \colon y \to \int_G k(x, y) d\lambda_G(x)$$

are continuous.

Proof. Since k is continuous the function $y \to k(x, y)$ is continuous, and hence measurable, for every $x \in G$.

Choose $k_1(k_2)$ in $L^1 \cap C_0(G)(L^1 \cap C_0(H))$ such that $\hat{k}_1 = 1$ ($\hat{k}_2 = 1$) on a neighbourhood $\nabla_1(\nabla_2)$ of $\Upsilon_G(\Upsilon_H)$, where Υ_G, Υ_H are the projections of Υ onto G, H respectively. If we define h on $G \times H$ by $h[(x, y)] = k_1(x)k_2(y)$ then **[1]**, (31.7) (b) shows that $\hat{h} = 1$ on $\nabla_1 \times \nabla_2$, a neighbourhood of Υ. Thus $h * k = k$ 1.a.e. and, since $h * k$ and k are continuous,

$$h * k = k. \tag{3.5}$$

Now the map ν_x on $H \times G \times H$, defined by

$$\nu_x[(y, s, t)] = h(x - s, y - t)k(s, t),$$

is continuous for every $x \in G$. Applying **[1]**, (13.4) to $|\nu_x|$, considered as a function on $H \times (G \times H)$, it follows that ν_x is integrable and, using (3.5), that the function $y \to k(x, y)$ is integrable over H for every $x \in G$. Furthermore, since ν_x is integrable on $H \times (G \times H)$, we can use (3.5) and **[1]**, (13.8) to deduce that

$$\phi_1(x) = \int_H k_2(y) d\lambda_H(y) \int_{G \times H} k_1(x - s)k(s, t) d\lambda_G \times \lambda_H(s, t).$$

As $k \in L^1(G \times H)$, $k_2 \in L^1(H)$ and k_1 is uniformly continuous it follows that ϕ_1 is continuous.

The other part of the lemma is proved similarly.

LEMMA 3.8. *Suppose G, H are Hausdorff locally compact Abelian groups, with character groups Γ, Λ respectively. If $p \in [1, 2)$ and the closed set $\Xi' \subset \Gamma$ is not an S_p-set, then $\Xi = \Xi' \times \Lambda$ is not an S_p-set in $\Gamma \times \Lambda$.*

Proof. Suppose to the contrary that Ξ is an S_p-set in $\Gamma \times \Lambda$. Let $f \in L^1 \cap C_0(G)$ with $\text{supp}(\hat{f})$ compact and $\hat{f}$ vanishing on Ξ, and choose $g \in L^1 \cap C_0(H)$ such that $\text{supp}(\hat{g})$ is compact and $|g(y)| \geqq 1$ for all y in some neighbourhood V of zero in H. Define h on $G \times H$ by $h[(x, y)] = f(x)g(y)$. Then, by **[1]**, (31.7) (b), $\text{supp}(\hat{h})$ is compact and

$$\hat{h}([\gamma_1, \gamma_2]) = \hat{f}(\gamma_1)\hat{g}(\gamma_2) = 0$$

for all $[\gamma_1, \gamma_2] \in \Xi$.

Let $\epsilon > 0$ be given. Since Ξ is assumed to be an S_p-set we can find $k \in L^1 \cap C_0(G \times H)$ such that $\text{supp}(\hat{k})$ is compact and disjoint from Ξ, and

$$\|h - k\|_p < \epsilon \lambda_H(V)^{1/p}. \tag{3.6}$$

Thus, for all γ_1 in some neighbourhood ∇ of Ξ' and for all $\gamma_2 \in \Lambda$, we have (see **[1]**, (13.8))

$$\begin{aligned}\int_H \left\{ \int_G k(x, y)\bar{\gamma}_1(x) d\lambda_G(x) \right\} \bar{\gamma}_2(y) d\lambda_H(y) \\ = \int_{G \times H} k(x, y)([\gamma_1, \gamma_2])^-(x, y) d\lambda_G \times \lambda_H(x, y) \\ = 0.\end{aligned}$$

Since $\gamma_2 \in \Lambda$ was chosen arbitrarily

$$\int_G k(x, y)\bar{\gamma}_1(x) d\lambda_G(x) = 0 \qquad \lambda_H - \text{a.e.}.$$

Now

$$\psi: (x, y) \to k(x, y)\bar{\gamma}_1(x)$$

is continuous and integrable, and $\text{supp}(\hat{\psi})$ is compact. Hence, by Lemma 3.7, the function ϕ on H defined by

$$\phi(y) = \int_G \psi(x, y) d\lambda_G(x)$$

is continuous and so, for all $y \in H$ and $\gamma_1 \in \nabla$,

$$\int_G k(x, y)\bar{\gamma}_1(x) d\lambda_G(x) = 0. \tag{3.7}$$

Using (3.6) we see that

$$W = \left\{ y \in V: \int_G |h(x, y) - k(x, y)|^p d\lambda_G(x) < \epsilon^p \right\}$$

has the property that $\lambda_H(V\backslash W) < \lambda_H(V)$, that is, $\lambda_H(W) > 0$. Choose any $y_0 \in W$ (W is nonempty). Then

$$(3.8) \qquad \int_G |f(x) - g(y_0)^{-1}k(x, y_0)|^p \, d\lambda_G(x) < \epsilon^p \, |g(y_0)|^{-1} \leqq \epsilon^p$$

and so, defining $f_1 \in L^1 \cap C_0(G)$ by $f_1(x) = g(y_0)^{-1}k(x, y_0)$, (3.7) shows that $\hat{f}_1$ vanishes on ∇ and, from (3.8), $\|f - f_1\|_p < \epsilon$; thus we have a contradiction of the assumption that Ξ' is not an S_p-set.

THEOREM 3.9. *Let G be a Hausdorff noncompact locally compact Abelian group, Γ its character group. Then Γ contains a closed set which is not an S_p-set for any $p \in [1, 2)$.*

Proof. By **[1]**, (24.30), Γ is topologically isomorphic with $R^n \times \Gamma_0$, where Γ_0 is a Hausdorff locally compact Abelian group containing a compact open subgroup.

If $n \geqq 1$ then Theorem 3.6 and Lemma 3.8 combine to show that $R^n \times \Gamma_0$ contains a closed set which is not an S_p-set for any $p \in [1, 2)$.

If $n = 0$ then Γ contains a compact open subgroup (with is infinite since Γ is nondiscrete) and the result follows from Corollary 3.4.

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Received September 11, 1974.

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