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ANALYTIC FUNCTIONS WHICH OPERATE ON HOMOGENEOUS ALGEBRAS

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Abstract

It is well known that a complex-valued function ϕ, analytic on some open set Ω, extends to any commutative Banach algebra B so that the action of ϕ on B commutes with the action of the Gelfand transformation. In this paper, it is shown that if B is a homogeneous convolution Banach algebra over any compact group and if $0 \in \Omega$ is a fixed point of ϕ, then a similar result holds, with the Gelfand transformation replaced by the Fourier-Stieltjes transformation. Care is required, in that discussion of this relation usually requires simultaneous consideration of the extension of ϕ to B and to certain operator algebras.

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1. Introduction

Homogeneous convolution Banach algebras are notable for their similarity to closed translation-invariant subspaces of group algebras. Reiter presented a systematic account in [8] of the properties of Segal algebras (which share properties of the corresponding group algebra) over locally compact abelian or compact groups. The more general homogeneous Banach algebras have been discussed by Wang in [11] for locally compact abelian groups, and by the author in [13] and [14] for compact groups.

This paper continues the earlier study by the author.

It is well known that if ϕ denotes a complex-valued function which is analytic in some neighbourhood Ω in the complex plane then ϕ can be extended to any commutative Banach algebra B, with identity e, in such a way that for every $\lambda \in \Omega$

$$\phi(\lambda e) = \phi(\lambda)e. \tag{1}$$

Further, if $\phi(b)$ exists in B then its Gelfand transform $\phi(b)^\frown$ is related to the transform $\hat{b}$ of b by

$$\phi(b)^\frown = \phi \circ \hat{b}. \tag{2}$$

For a discussion of this theory see, for example, Larsen [6] or Rickart [9].

However, homogeneous convolution Banach algebras over compact groups need not be commutative (unless the group is commutative) and do not possess an identity (unless the group is finite). Nevertheless, we show that it is possible to produce an extension of ϕ. It is not usually possible to obtain an analogue of equation (1). However, we do show that in a sense ϕ does commute with the Fourier transformation to give an analogue of equation (2). Segal obtained this result for some functions in the group algebra in [10].

Conditions under which a similar result may be obtained for the nonhomogeneous measure algebras are discussed in [3] by Fountain, Ramsay and Williamson. The problem is by no means solved in this case.

Throughout this paper we use the notation of [4], [13] and [14]. Any unexplained notation may be found in these sources.

To begin with, let B denote any Banach algebra. As B need not contain an identity element, we are unable to determine inverse elements. Instead we use the notion of adverses as defined in Loomis [7]. If $bc - b - c = 0$ in B, then b is called a left adverse of c and c is called a right adverse of b. If b has both left and right adverses, then they must be equal and unique. This element is called the adverse of b.

A similar definition is to be found in Hille and Phillips [5], where an adverse is called a reverse or quasi-inverse.

The spectrum $\alpha(b)$ of an element b is the set of all complex numbers λ for which $\lambda^{-1}b$ does not have an adverse, together with $\lambda = 0$. In [5], 0 is sometimes excluded from $\alpha(b)$, but this does not cause us any confusion. For $\lambda \notin \alpha(b)$, we denote the adverse of $\lambda^{-1}b$ by $D(\lambda, b)$.

The basic properties of adverses are established in [5] and [7]. In particular, we see that if B does contain an identity element e, then the usual spectrum $\mathrm{sp}(b)$ of b contains the same non-zero numbers as $\alpha(b)$ (and so $\alpha(b)$ must be compact). Further, for $\lambda \notin \alpha(b)$,

$$(\lambda e - b)^{-1} = \frac{1}{\lambda}e - \frac{1}{\lambda}D(\lambda, b). \tag{3}$$

We will find particularly helpful the following result which gives a characterisation of those elements of B which possess left (right) adverses, and so, of those which possess adverses.

That is, the element b has a left (right) adverse in B if and only if given any maximal regular left (right) ideal M of B, there exists an element $c \in B$ such that $cb - b - c$ $(bc - b - c)$ is in M.

Throughout this paper, ϕ will represent a complex-valued function which is analytic in some zero-neighbourhood Ω and which has 0 as a fixed point. It is shown in [5] that for any $b \in B$ with $\alpha(b) \subseteq \Omega$, the Cauchy-type integral

$$-\frac{1}{2\pi i}\int_\Gamma \phi(\lambda)\frac{D(\lambda, b)}{\lambda}\, d\lambda, \tag{4}$$

where Γ is a suitable contour in Ω enclosing $\alpha(b)$, defines an element $\phi(b)$ in B. Further, the integral is independent of the choice of Γ. If B has an identity e, then, in view of equation (3), the integral reduces to

$$\frac{1}{2\pi i}\int_\Gamma \phi(\lambda)(\lambda e - b)^{-1}\, d\lambda, \tag{5}$$

which is a familiar integral in the theory of commutative Banach algebras.

At this point we assume that B is a homogeneous Banach convolution algebra over a compact group G. We let Σ denote the dual object of G. For each equivalence class $\sigma \in \Sigma$, U_σ denotes a continuous, irreducible unitary representation which acts on a (finite) d_σ-dimensional Hilbert space $\mathcal{H}_\sigma$. The Banach algebra of bounded linear operators acting on $\mathcal{H}_\sigma$ is denoted by $\mathcal{B}(\mathcal{H}_\sigma)$. The Fourier-Stieltjes transform $\hat{\mu}$ of a measure μ is defined by

$$\hat{\mu}(\sigma) = \int_G U_\sigma(x)^*\, d\mu(x),$$

for each $\sigma \in \Sigma$. Then $\hat{\mu}(\sigma) \in \mathcal{B}(\mathcal{H}_\sigma)$.

The arguments which follow are, in fact, valid when B is assumed to be a linear subspace of the space of pseudomeasures defined on G. However, we assume that all elements of B are measures. Then

DEFINITION 1. The convolution Banach algebra B is homogeneous if

(i) it is left translation invariant,

(ii) each left translation operator is continuous on B,

(iii) each left shift operator is continuous from G to B, and

(iv) the embedding J of B into the pseudomeasures is continous; that is, there exists a positive constant K such that

$$\sup_{\sigma\in\Sigma} \|\hat{\mu}(\sigma)\|_{\mathcal{B}(\mathcal{H}_\sigma)} \le K\|\mu\|_B$$

for all $b \in B$.

Examples include the usual function spaces $A(G), C(G), U^p(G)$ and $L^p(G)$ for $1 \leq p < \infty$, and any of their closed left-translation invariant subspaces. On the other hand, $L^\infty(G)$ and $M(G)$ are not homogeneous. Many more examples are given in [13]. We proved in [13] that if B is homogeneous, then for each $\sigma \in \Sigma$, there exists a subspace $\mathcal{E}_\sigma$ of $\mathcal{H}_\sigma$ satisfying

$$\{\hat{b}(\sigma) \colon b \in B\} = \{T \in \mathcal{B}(\mathcal{H}_\sigma) \colon E_\sigma \subseteq \ker T\}.$$

Denote the latter set by $\mathcal{J}(\mathcal{E}_\sigma)$. Further, the set of all trigonometric polynomials p, with $\hat{p}(\sigma) \in \mathcal{J}(\mathcal{E}_\sigma)$ for each $\sigma \in \Sigma$, forms a dense subset of B.

We use these properties of homogeneous convolution Banach algebras to obtain our results.

We will extend the function ϕ to the algebra B in two steps. Suppose that b is an element for which $\alpha(b) \subseteq \Omega$. We know that for each $\sigma \in \Sigma$, $\hat{b}(\sigma)$ is in $\mathcal{J}(\mathcal{E}_\sigma)$. Therefore, we first show that we can define $\phi(\hat{b}(\sigma))$ in $J(E_\sigma)$ for each σ, and then prove that the existence of each of the operators $\phi(\hat{b}(\sigma))$ ensures the existence of $\phi(b)$ in B.

2. Maximal regular ideals

Crucial to the proof of our main theorem concerning the extension of ϕ is the knowledge of the maximal regular ideal structures, not only of the algebra B, but also of the operator algebras $\mathcal{J}(\mathcal{E}_\sigma)$. Fortunately the structures are closely related.

All the information that we require is included in earlier papers, [12] and [14], by the author. However, as Corollary 2.8 of [12] incorrectly describes the maximal regular left ideals of $\mathcal{J}(\mathcal{E}_\sigma)$, and so Proposition 4.7 of [14] incorrectly describes those of B, we will review their structures briefly here.

Given the algebra B, we write

$$\operatorname{sp}(B) = \{\sigma \in \Sigma \colon \mathcal{E}_\sigma \neq \mathcal{H}_\sigma\} \quad \text{and} \quad \overline{\operatorname{sp}(B)} = \{\sigma \in \Sigma \colon \mathcal{E}_\sigma = \{0\}\}.$$

If $\sigma \in \overline{\operatorname{sp}(B)}$ then $\mathcal{J}(\mathcal{E}_\sigma) = \mathcal{B}(\mathcal{H}_\sigma)$, while if $\sigma \notin \operatorname{sp}(B)$ then $\hat{b}(\sigma) = 0$ for every $b \in B$. Therefore, we need only consider $\sigma \in \operatorname{sp}(B)$. The maximal regular right ideals of $\mathcal{J}(\mathcal{E}_\sigma)$ are precisely the sets

$$\{T \in \mathcal{J}(\mathcal{E}_\sigma) \colon T(\mathcal{H}_\sigma) \subseteq Z_\sigma\}$$

for some $d_\sigma - 1$ dimensional subspace Z_σ of $\mathcal{H}_\sigma$. Further, the maximal regular right ideals of B are precisely the sets

$$\{b \in B \colon \hat{b}(\sigma) \in m\}$$

where m is a maximal regular right ideal of $\mathcal{J}(\mathcal{E}_\sigma)$ for some $\sigma \in \operatorname{sp}(B)$. (For proof of these facts refer to Propositions 2.9 and 4.8 of [12], and Theorem 4.1 of [14].)

The maximal regular left ideals of $\mathcal{J}(\mathcal{E}_\sigma)$ fall into one of two classes. If $\sigma \in \overline{\mathrm{sp}(B)}$ then they are precisely the sets

$$\{T \in \mathcal{J}(\mathcal{E}_\sigma) : \mathcal{Y}_\sigma \in \ker T\},$$

where $\mathcal{Y}_\sigma$ is a one-dimensional subspace of $\mathcal{H}_\sigma$. On the other hand, if $\sigma \in \mathrm{sp}(B)\backslash\overline{\mathrm{sp}(B)}$ then they are precisely the sets

$$\{T \in \mathcal{J}(\mathcal{E}_\sigma) : Th \in \mathcal{E}_\sigma\}$$

for some $h \in \mathcal{E}_\sigma^\perp$, the complement $\mathcal{E}_\sigma$ in $\mathcal{H}_\sigma$. It then follows from Theorem 4.1 of [14] that the maximal, regular left ideals of B are precisely the sets

$$\{b \in B : \hat{b}(\sigma) \in m\}$$

where m is a maximal, regular left ideal of $\mathcal{J}(\mathcal{E}_\sigma)$ for some $\sigma \in \mathrm{sp}(B)$. (The characterisation of the maximal regular left ideals of the algebras $\mathcal{J}(\mathcal{E}_\sigma)$ is contained in Theorem 2.7 of [12].)

3. Extending analytic functions

The operator algebra $\mathcal{J}(\mathcal{E}_\sigma)$ does not contain the identity operator I_σ on $\mathcal{H}_\sigma$ unless $\sigma \in \overline{\mathrm{sp}(B)}$, in which case $\mathcal{J}(\mathcal{E}_\sigma)$ is equal to the whole of the algebra $\mathcal{B}(\mathcal{H}_\sigma)$. However we do sometimes find it useful, and possible, to work with $\mathcal{B}(\mathcal{H}_\sigma)$ instead of $\mathcal{J}(\mathcal{E}_\sigma)$. This follows from the following proposition.

PROPOSITION 1. *Let $T \in \mathcal{J}(\mathcal{E}_\sigma)$. Then T has an adverse in $\mathcal{J}(\mathcal{E}_\sigma)$ if and only if it has an adverse in $\mathcal{B}(\mathcal{H}_\sigma)$.*

Note that since adverses are unique when they exist, the adverse of T in $\mathcal{B}(\mathcal{H}_\sigma)$ must belong to $\mathcal{J}(\mathcal{E}_\sigma)$.

PROOF. As $\mathcal{J}(\mathcal{E}_\sigma)$ is a subspace of $\mathcal{B}(\mathcal{H}_\sigma)$ it is obvious that the adverse of T in $\mathcal{J}(\mathcal{E}_\sigma)$ must be its adverse in $\mathcal{B}(\mathcal{H}_\sigma)$.

On the other hand, suppose that T has an adverse in $\mathcal{B}(\mathcal{H}_\sigma)$, S say. Then $ST = S + T = TS$. It follows easily that S is in $\mathcal{J}(\mathcal{E}_\sigma)$ because for any $h \in \mathcal{E}_\sigma$, $Sh = STh - Th = 0$ since $\mathcal{E}_\sigma \subseteq \ker T$. Thus S is the adverse of T in $\mathcal{J}(\mathcal{E}_\sigma)$.

An operator T in $\mathcal{B}(\mathcal{H}_\sigma)$ has an adverse if and only if $I_\sigma - T$ is invertible. Consequently, the non-zero complex numbers in its spectrum $\alpha(T)$ in $\mathcal{J}(\mathcal{E}_\sigma)$ coincide with those in its spectrum $\mathrm{sp}(T)$ in $\mathcal{B}(\mathcal{H}_\sigma)$. The two spectra are equal if T is not invertible.

To begin our discussion we must be able to relate the spectra $\alpha(b)$ and $\mathrm{sp}(\hat{b}(\sigma))$, for $\sigma \in \mathrm{sp}(B)$; otherwise we have little chance of deciding when ϕ can be extended to the various algebras being considered.

THEOREM 1. *Let λ be a non-zero complex number and b an element of B. Then $\lambda \in \alpha(b)$ if and only if $\lambda \in \mathrm{sp}(\hat{b}(\sigma))$ for some $\sigma \in \mathrm{sp}(B)$.*

DISCUSSION. For any non-zero λ we have $D(\lambda, b) = D(1, \lambda^{-1}b)$. Therefore we need only consider Theorem 1 for the case $\lambda = 1$.

PROOF OF THEOREM 1. If $1 \notin \alpha(b)$ then b has an adverse in B, say c. Using the definition of an adverse, we can write

$$\hat{c}(\sigma)\hat{b}(\sigma) = \hat{c}(\sigma) + \hat{b}(\sigma) = \hat{b}(\sigma)\hat{c}(\sigma)$$

for each $\sigma \in \Sigma$, or equivalently

$$(I_\sigma - \hat{c}(\sigma))(I_\sigma - \hat{b}(\sigma)) = I_\sigma = (I_\sigma - \hat{b}(\sigma))(I_\sigma - \hat{c}(\sigma)).$$

Thus $1 \notin \mathrm{sp}(\hat{b}(\sigma))$ for each $\sigma \in \mathrm{sp}(B)$.

On the other hand, suppose that $1 \notin \mathrm{sp}(\hat{b}(\sigma))$ for each $\sigma \in \mathrm{sp}(B)$. To show that $1 \notin \alpha(b)$ we need only find, for each maximal regular right (left) ideal M of B, an element c for which $b * c - b - c$ $(c * b - b - c)$ is in M.

We noted in Section 2 that there exists σ in $\mathrm{sp}(B)$ and a maximal regular right (left) ideal m of $\mathcal{J}(\mathcal{E}_\sigma)$ such that

$$M = \{b \in B \colon \hat{b}(\sigma) \in m\}.$$

Further, it follows from Proposition 1 tht $\hat{b}(\sigma)$ has an adverse in $\mathcal{J}(\mathcal{E}_\sigma)$. Consequently, there exists an operator S in $\mathcal{J}(\mathcal{E}_\sigma)$ such that $\hat{b}(\sigma)S - \hat{b}(\sigma) - S$ $(S\hat{b}(\sigma) - \hat{b}(\sigma) - S)$ is in m.

Define the trigonometric polynomial p by

$$p(x) = d_\sigma \mathrm{tr}(SU_\sigma(x)^*)$$

for each $x \in G$. Then it is easy to see that $p \in B$ (since $\hat{p}(\eta) \in \mathcal{J}(\mathcal{E}_\eta)$) and that p has the properties required of c.

COROLLARY 1. *$\phi(b)$ exists in B if and only if $\phi(\hat{b}(\sigma))$ exists in $\mathcal{B}(\mathcal{H}_\sigma)$ for each $\sigma \in \Sigma$.*

DISCUSSION. If $\sigma \notin \mathrm{sp}(B)$ then $\phi(\hat{b}(\sigma))$ is the zero operator in $\mathcal{B}(\mathcal{H}_\sigma)$.

PROOF OF COROLLARY 1. In view of Theorem 1, the existence of $\phi(b)$ and $\phi(\hat{b}(\sigma))$ depend only on the existence of a suitable contour Γ in Ω.

It is clear that if $\phi(b)$ exists in B then any contour Γ in Ω which encloses $\alpha(b)$ must enclose $\mathrm{sp}(\hat{b}(\sigma))$ and so can be used in the integral (5) to define $\phi(\hat{b}(\sigma))$ in $\mathcal{B}(\mathcal{H}_\sigma)$ for each $\sigma \in \Sigma$.

On the other hand if it is known that $\phi(\hat{b}(\sigma))$ exists then there is a contour, Γ_σ say, which encloses $\mathrm{sp}(\hat{b}(\sigma))$ and over which the integral (5) is evaluated. The Γ_σ need not be identical, and so it is not clear that any one of them will enclose $\alpha(b)$. However, $\alpha(b)$ is a compact subset of Ω and thus Ahlfors has shown in [1] that there does exist a suitable contour in Γ which encloses $\alpha(b)$. It can be used to define $\phi(b)$ by the integral (4). Since the integral (5) is independent of the choice of contour, Γ can also be used in place of Γ_σ to define $\phi(\hat{b}(\sigma))$.

COROLLARY 2. *$\phi(b)$ exists in B if and only if $\phi(\hat{b}(\sigma))$ exists in $\mathcal{J}(\mathcal{E}_\sigma)$ for each $\sigma \in \Sigma$.*

PROOF. Is an immediate consequence of Proposition 1, Corollary 1 and the equality of integrals (4) and (5) in $\mathcal{B}(\mathcal{H}_\sigma)$.

4. Transform equation

Suppose that $\phi(b)$ exists in B. What is its Fourier-Stieltjes transform? We show that for each σ, $\phi(b)^\frown(\sigma)$ is, in fact, the operator $\phi(\hat{b}(\sigma))$ in $\mathcal{J}(\mathcal{E}_\sigma)$.

First, however, observe that if equation (3) is applied to any operator T in $\mathcal{B}(\mathcal{H}_\sigma)$ then, for $\lambda \notin \alpha(T)$,

$$D(\lambda, T) = I_\sigma - \lambda(\lambda I_\sigma - T)^{-1}.$$

Take $\lambda \notin \alpha(b)$. Then, it follows from the definition of $D(\lambda, b)$, on taking Fourier-Stieltjes transforms, that for each σ,

$$\lambda^{-1}\hat{b}(\sigma)D(\lambda,b)^\frown(\sigma) = \lambda^{-1}\hat{b}(\sigma) + D(\lambda,b)^\frown(\sigma) = D(\lambda,b)^\frown(\sigma)\lambda^{-1}\hat{b}(\sigma).$$

But adverses are unique, when they exist, and so $D(\lambda,b)^\frown(\sigma) = D(\lambda,\hat{b}(\sigma))$. Therefore, for any σ,

$$D(\lambda,b)^\frown(\sigma) = I_\sigma - \lambda(\lambda I_\sigma - \hat{b}(\sigma))^{-1}.$$

We use this relation to determine the Fourier-Stieltjes transform of $\phi(b)$.

THEOREM 2. *For each $\sigma \in \Sigma$,*

$$\phi(b)^\frown(\sigma) = \phi(\hat{b}(\sigma)).$$

PROOF. Theorem 1 and its corollary ensure the existence of $\phi(\hat{b}(\sigma))$ in $\mathcal{J}(\mathcal{E}_\sigma)$ for each σ. A simple sequence of calculations yields the required result. For,

$$\begin{aligned}
\phi(\hat{b}(\sigma)) &= \frac{1}{2\pi i}\int_\Gamma \phi(\lambda)(\lambda I_\sigma - \hat{b}(\sigma))^{-1}\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \phi(\lambda\left[\frac{1}{\lambda}I_\sigma - \frac{1}{\lambda}D(\lambda,b)\hat{}(\sigma)\right]\,d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\lambda)}{\lambda}I_\sigma\,d\lambda - \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\lambda)}{\lambda}\left[\int_G D(\lambda,b)(x)U_\sigma(x)^*\,dx\right]\,d\lambda \\
&= \int_G\left[-\frac{1}{2\pi i}\int_\Gamma \frac{\phi(\lambda)}{\lambda}D(\lambda,b)\,d\lambda\right]U_\sigma(x)^*\,dx \\
&\qquad\qquad \left(\text{since } \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\lambda)}{\lambda}\,d\lambda = \phi(0) = 0\right) \\
&= \int_G \phi(b)U_\sigma(x)^*\,dx \\
&= \phi(b)\hat{}(\sigma).
\end{aligned}$$

COROLLARY 3. *Let $b \in B$ and $\lambda \notin \alpha(b)$. Then there exists $c \in B$ such that for each $\sigma \in \Sigma$*

$$\hat{b}(\sigma)[\lambda I_\sigma - \hat{b}(\sigma)]^{-1} = \hat{c}(\sigma) = [\lambda I_\sigma - \hat{b}(\sigma)]^{-1}\hat{b}(\sigma).$$

PROOF. Apply Theorem 2 to $\phi(\xi) = \xi/(\xi - \lambda)$.

This corollary generalizes, to the non-abelian compact case, a result first obtained for $L^1(\mathbf{R})$ by Wiener in [15], and extended to $L^p(\mathbf{T}), 1 \le p \le \infty$, and $C^k(\mathbf{T})$, $k \in \mathbf{N}$, by Edwards in [2].

5. A topological property of the domain of the extension

The extensions of ϕ, determined by the integrals (4) and (5), to B and to $\mathcal{B}(\mathcal{H}_\sigma)$ respectively, have as their domains the sets

$$D = \{b \in B\colon \alpha(b) \subseteq \Omega\} \quad\text{and}\quad \mathcal{D}_\sigma = \{T \in \mathcal{B}(\mathcal{H}_\sigma)\colon \mathrm{sp}(T) \subseteq \Omega\}.$$

Theorem 1 and its corollaries ensure that an element b of B is in D if and only if $\hat{b}(\sigma) \in \mathcal{D}_\sigma$ for each σ.

Our final result shows that a much stronger statement can be made about these domains.

Let C and $\mathcal{C}_\sigma$ denote the connected components containing the zero element of each of the domains respectively. Let $b \in B$.

THEOREM 3. *$b \in C$ if and and only if $\hat{b}(\sigma) \in \mathcal{C}_\sigma$ for each $\sigma \in \Sigma$.*

PROOF. Suppose that there exists $c \in C$ and $\sigma \in \Sigma$ for which $\hat{c}(\sigma) \notin \mathcal{C}_\sigma$. Then there exists open subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{D}_\sigma$, with $0 \in \mathcal{A}_1$ and $\hat{c}(\sigma) \in \mathcal{A}_2$.

It follows from (iv) of Definition 1 that the sets A_1 and A_2 given by

$$A_i = \{b \in B \colon \hat{b}(\sigma) \in \mathcal{A}_i\}, \qquad i = 1, 2,$$

are open in B. Further, since Ω is open, each of the sets

$$A_i' = \{b \in B \colon \alpha(b) \subseteq \Omega \text{ and } \hat{b}(\sigma) \in \mathcal{A}_i\}, \qquad i = 1, 2,$$

is also open in B. Clearly they must be disjoint, and $0 \in A_1'$ while $c \in A_2'$. But this is impossible because c was chosen from C, and so no such σ exists.

On the other hand, assume that $\hat{b}(\sigma) \in \mathcal{C}_\sigma$ for each $\sigma \in \Sigma$. Then Theorem 1 ensures that $\alpha(b) \subseteq \Omega$. If we note that C must be closed in D and recall that the set of trigonometric polynomials contained in B is dense in B, we see that it is sufficient to show that $b \in C$ when b is a trigonometric polynomial. This can be achieved by proving that b may be arcwise connected to 0 in C.

Now let γ_σ denote a continuous map from $[0, 1]$ to $\mathcal{C}_\sigma$ with

$$\gamma_\sigma(0) = 0 \quad \text{and} \quad \gamma_\sigma(1) = \hat{b}(\sigma).$$

This map must exist because $\mathcal{B}(\mathcal{H}_\sigma)$ is finite dimensional. Moreover, because b is a trigonometric polynomial, all but a finite number of the γ_σ may be assumed to be trivially 0.

Define the map $\gamma \colon [0, 1] \to B$ by

$$\gamma(t)(x) = \sum_{\sigma \in \Sigma} d_\sigma \operatorname{tr}(\gamma_\sigma(t) U_\sigma(x)^*)$$

for all $x \in G$ and $t \in [0, 1]$. Then for each t, $\gamma(t)$ is a trigometric polynomial. Moreover,

$$\gamma(t)\hat{}(\sigma) = \gamma_\sigma(t)$$

and so $\gamma(t)\hat{}(\sigma) \in \mathcal{J}(\mathcal{E}_\sigma)$. Thus $\gamma(t) \in B$. Clearly $\gamma(0) = 0$ and $\gamma(1) = b$. Observe that γ must be continuous, and so it remains only to show that $\alpha(\gamma(t)) \subseteq \Omega$ for each $t \in [0, 1]$. This fact follows easily from Theorem 1 because

$$\begin{aligned}\alpha(\gamma(t)) &= \{0\} \cup U\{\operatorname{sp}(\gamma(t)\hat{}(\sigma)) \colon \sigma \in \Sigma\} \\ &= \{0\} \cup U(\operatorname{sp}(\gamma_\sigma(t)) \colon \sigma \in \Sigma\} \\ &\subseteq \Omega.\end{aligned}$$

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