Graphics calculators and the school mathematics curriculum: Perspectives and issues from three countries

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Abstract
Over the past two decades, graphics calculators have been prominent in many discussions of technology in mathematics education. This paper describes how they have become part of teaching, learning and assessment in school mathematics in each of three different countries: Australia, Singapore and the United States of America, as well as directions for future use. Critical issues associated with effective implementation of graphics calculators into the school mathematics curriculum are highlighted, including the nature of school mathematics, examination practices, Computer Algebra Systems, the support of teachers and students, curriculum change and development, the focus on learning, dealing with inherent limitations of graphics calculators, school and university differences, future technologies.

Introduction

While there are many aspects of technology that have been discussed over the first twenty years of Asian Technology Conferences in Mathematics (ATCM), few of them have been as prominent in typical secondary school classrooms as those concerned with calculators, and in particular graphics calculators. Many of the workshops offered to teachers attending the conferences have involved aspects of educational use of graphics calculators, and many papers have been published in successive conference proceedings regarding aspects of this technology, along with developments of it. Indeed, the inaugural conference in Singapore in 1995 included an introductory paper by the first author [7] on the significance of the technology and a workshop for local teachers regarding calculator use. A summary of educational arguments for a range of calculators is provided by the first author at the 12th ATCM. [8] In addition, calculator manufacturers have been prominent at successive conferences, offering information and workshop opportunities to attendees, and promoted the sharing of expertise from many international settings.

Graphics calculators have not been confined to the ATCM; their effects have been studied in many countries. A good summary of much empirical research into the effects of calculator use of various kinds [16] described overall positive results and a suggestion that attention be focused on how to best use the technologies rather than the questions of whether their use was valuable, which was no longer a matter of contention.

In this paper, we review the ways in which graphics calculators have been introduced and used in three different countries, to highlight some key issues and to provide advice to those in other countries in which the use of technology has not yet become prominent in standard classrooms. The three countries chosen have differing administrative structures for education and differing circumstances for teaching, but there are nonetheless many similarities which we expect will be helpful to colleagues in other countries.

Developments in three countries

The ways in which and the extent to which technology is adopted into mathematics education depends on the society and the culture concerned. In this section of the paper, we give a brief account of the history, major steps and issues associated with the use of graphics calculators in three different countries.

The Australian experience

Although its population is only around 23 million people, Australia consists of a federation of six states and two territories, each of which is constitutionally responsible for its own education system. In recent years, all states and territories have adopted an Australian Curriculum in mathematics for the first ten years of school, and a measure of agreement has been reached on suitable courses for the final two years of school. However, responsibility for the curriculum and its assessment in the final two years of school rests with individual states and territories and the national courses are not yet implemented. For these reasons, generalizations about the role of technology in the senior secondary years need to be interpreted cautiously. While there are differences between states, in general the mathematics curriculum in each state is developed through a collaborative process involving senior government officials, experienced mathematics teachers in schools and mathematicians in universities, and other interested parties, with no single group having ultimate authority.

Early Calculator Use

The use of technology by students in secondary school mathematics has been common in Australia since the late 1970s, when scientific calculators were approved for use in high stakes examinations at the end of secondary school, typically used by universities to select students into competitive courses. Accordingly, Australian school mathematics curricula have generally accepted a role for technology since that time, especially for computational purposes. Thus, since the 1980's and in the existing national curriculum [2], students are expected to use digital technologies, including calculators, as well as other computational methods throughout schooling up to Year 10. Similarly, each of the four senior secondary mathematics subjects officially endorsed nationally as an agreed and common base for mathematics subjects in Year 11 and 12 in the states includes an explicit aim for students to "choose and use technology appropriately" [2], although decisions about the details of the technology involved remain the responsibility of individual states.

Graphics calculators were first evident in Australian schools in the early 1990s, only a few years after their invention, but were not in widespread use until decisions were made to allow their use in the important high stakes examinations at the end of school. Decisions of that kind served three important purposes: ensuring a measure of coherence between the experiences of learning mathematics and the examination conditions; providing a solid minimum basis for adjusting school curricula to the promises and possibilities of technology; and supporting a concern that all students have equitable access to the same conditions for learning and doing mathematics. Although a major argument for the use of graphics calculators is to support teaching and learning of mathematics, their association with end of school examinations has inevitably resulted in some students and teachers regarding them principally as devices to support their responses to examination questions.

Equity Issues

Australian schools include both government schools and a significant proportion of private schools. All children are provided with a public education, although some parents are willing and able to pay significant fees for their children to attend private schools. While private schools also receive some government funding, and students attending government schools are still obliged to pay some fees, generally speaking private schools are regarded as better resourced than government schools and students at these schools are more likely to have economic advantages. Some private schools are notionally affiliated with particular religions, although the mathematics curriculum at schools is the same for all schools. At both government and private schools, parents of students are generally expected to pay for necessary learning materials, such as books and calculators, so that a technology that is affordable to most families is seen as preferable. (In some schools, however, family resources are inadequate, and other arrangements are made, such as the occasional use of a class set of calculators or schemes for lending students calculators.)

Teachers and Curriculum Change

A key event in the implementation of graphics calculators into Australian curricula was a national conference convened by the professional body, the Australian Association of Mathematics Teachers. [11] The conference recognized the particular contribution of graphics calculators to secondary school mathematics teaching, learning and assessment and involved invited participants with expertise, experience and a diverse set of perspectives to identify the key issues and to share experiences with this technology. A key feature of the conference was to respect and to recognize professional mathematics teachers in schools, in addition to academics and researchers, to learn from the wisdom of good practice. The conference resulted in the publication of a communiqué [1], highlighting professional experience and opinion on the role of the calculators for a wider audience. The conference recognized that graphics calculators had been catalysts for pedagogical change in many cases and that the best classrooms reflected a strong and well-balanced curriculum with identifiable improvements to teaching and learning processes. A number of issues that needed attention were identified during the conference, including the importance of providing adequate support for teachers, for curriculum development, to enhancing teaching and learning with technology and for careful design of assessment. Improving equity of access to a sound curriculum for all students was also a key point, and it was recognized that, while the provision of graphics calculators to all students might be seen as a necessary step, it was not sufficient to ensure equity.

By 2000, most Australian states had included the use of graphics calculators as an integral component of their stronger senior secondary mathematics courses, including their use in examinations, and made some revisions to official courses to accommodate and support the sound use of calculators (and other technologies). Typical senior secondary courses in mathematics in Australia include a mixture of aspects of mathematics, including introductory calculus, trigonometry, algebra and probability and statistics. Graphics calculators have supported the study and practice of mathematics in many of these areas, in particular related to representing and using functions, the solution of equations, elementary differential and integral calculus, operations with complex numbers, statistical analysis and aspects of geometry, in addition to meeting students' computational needs. In recent years, curricula at all levels have increased a focus on the uses of mathematics in other fields and in everyday affairs, so that some emphasis on elementary mathematical modeling and applications of mathematics are also common, much more so than in previous generations. Graphics calculators were regarded by many teachers and students as helpful tools for solving practical problems as well as offering opportunities for learning mathematics in new ways. While course revisions to recognize the roles of graphics calculators in most cases were not extensive, typically there was less emphasis placed on students developing expertise with routine computation by hand of the kinds that a graphics calculator can undertake, and more emphasis on applications of mathematics that were supported by the technology.

Supporting Teachers

Support for teachers to use graphics calculators has been provided in various ways. In most cases, suggestions have been made in teacher guides to the curriculum regarding appropriate teaching and learning activities. In addition, advice has been generated and refined to help students make appropriate use of graphics calculators in formal assessment, encouraging students to be discriminating in their selection of methods of addressing problems (including the use or otherwise of graphics calculators). Textbooks published by commercial publishers to support courses have generally included advice on calculator use as well as activities that exploit their potential. Calculator manufacturers, or their representatives, have often provided professional development workshops for teachers to help them to understand the appropriate use of graphics calculators, both at conferences for teachers and at separate events arranged for the purpose of supporting the use of a particular company's products, as well as online materials of various kinds. In recent years, since most secondary school classrooms include a capacity for teachers to project computer screens for class viewing, teachers have made good use of software emulators of graphics calculators for teaching purposes. In most cases, teachers and students have found it convenient for all students in a class to use the same model of graphics calculator. In recent years, the use of graphics calculators (as well as other forms of technology) has been addressed in teacher education programs, so that newly graduating teachers are generally familiar with their operation and educational use.

A significant exception to these observations about the role of graphics calculators is the most populous state (New South Wales), in which graphics calculators were permitted for examination use in only a relatively less sophisticated mathematics subject, typically studied by students without strong interests in mathematical or scientific studies beyond secondary school. Although the more sophisticated curricula do not prevent students and teachers from using graphics calculators in classroom teaching, the examination restriction on the use of only a scientific calculator with limited capabilities seems to have resulted in few students with stronger interests in mathematics encountering graphics calculators in their studies. It is perhaps relevant to note also that senior school mathematics courses in New South Wales have changed relatively little in recent decades, compared with those of other states, so that they include significant work in formal geometry and relatively little attention to probability and statistics, in contrast to courses in other states.

Recent Developments

In the years since 2000, Australian schools and society have continued to give a prominent place to various technologies, especially computers, tablets and the Internet. While many teachers in all states have included a diverse range of technologies into their mathematics curriculum, graphics calculators have continued to be used and have become the major platform for many students to access technology in learning and doing mathematics. An important reason for this is probably related to the continuing significance of formal examinations at the end of secondary school; while experience has developed with the use of graphics calculators in mathematics examinations, examination authorities continue to be reluctant to permit more powerful technologies and technologies capable of accessing the Internet into examinations, because of continuing concerns with examination security and with equity of access by all students. By their nature and by design, graphics calculators have provided students with a strong set of minimal tools for learning and doing the mathematics typically encountered in senior secondary school.

In the last decade, some (but not all) Australian state education systems have adopted the use of graphics calculators with computer algebra systems (CAS) for senior secondary school. This development began in one state (Victoria), where mathematics education researchers studied the role and use of CAS in school mathematics in pilot programs; subsequently, the use of CAS has been approved on a wider scale, for other courses and in other states. One state that uses graphics calculators has consciously chosen to not encourage the use of CAS, however, and continues to not permit its use in formal assessment. The significant difference between CAS and other graphics calculators is the availability of symbolic algebra and calculus operations, in addition to typical graphics calculator functionalities, so that students can also use their calculators to support explorations involving symbolic manipulation and are no longer restricted to providing numerical answers to mathematical questions. To accommodate unease about possible declines in traditional mathematics skills, examinations for students using CAS calculators have generally included two separate components, one of which does not permit any use of technology at all, which reflects an intention in all curricula that students do not become excessively reliant on their graphics calculator for mathematical tasks that are better handled mentally or with traditional mathematical techniques by hand.

Overall, the use of graphics calculators in Australia has been most prominent in secondary schools. While this is no doubt partly because graphics calculators have been developed mostly to suit the mathematics of secondary school curricula, it seems also to reflect relatively little experience with this technology by professional mathematicians and statisticians and a focus on the use of technology as a professional tool, rather than as a learning tool. This difference is evident in the use of sophisticated mathematical and statistical software in undergraduate courses beyond the first year or so. Unlike schools, universities are educating professionals for workplaces in which technology is routinely used, so the different perspective on the place of graphics calculators is unsurprising.

The Singaporean experience

Unlike Australia and the USA, Singapore is a city-state, with a much smaller population of about five and a half million people. Accordingly, the mathematics curriculum in Singapore is developed and managed by a single government body, the Ministry of Education (MOE), and there are no regional differences of significance. National decisions regarding the content and nature of the school mathematics curriculum are made by the MOE, which is responsible for resourcing and managing the work of schools and teachers as well as ensuring suitable relationships are maintained between schools and post-school institutions. National examinations are very important in Singapore, being used for university admission decisions.

In Singapore, the use of graphics calculators in national examinations was first permitted in the years 2002 through 2006 when students who sat for the Further Mathematics papers in the advanced level (pre-university level, or Year 11 and 12) examinations were allowed to use graphics calculators without a built-in computer algebra system (CAS), sometimes referred to as non-CAS graphics calculators. However, in these examinations the questions were graphics calculator neutral in that they were constructed in such a way that students who did not use a graphics calculator during the examinations would not be disadvantaged. Further Mathematics was a subject taken by students who had a keen interest in, and an aptitude for, mathematics. The majority of pre-university students took the subject Mathematics but not Further Mathematics, and graphics calculators were not allowed in the examinations for Mathematics. All students who studied Further Mathematics also studied Mathematics as a subject.

Graphics Calculators-Integrated Curriculum

In 2006, a revised mathematics curriculum for pre-university level students was implemented. In the revised curriculum, which is still in use currently, students could take either Higher 1 (H1) Mathematics or Higher 2 (H2) Mathematics, where H2 Mathematics is a subject taken by the majority of pre-university students while H1 Mathematics is taken by students who are less mathematically inclined. In the revised curriculum, the use of graphics calculators is expected and assumed for both subjects in all assessments including national examinations. In other words, the examination papers are set with the assumption that candidates will have access to graphics calculators and are proficient in solving problems with the aid of graphics calculators under conditions of a timed examination.

In particular, as the understanding of concepts of functions is fundamental in the learning of mathematics, students are expected to use the graphics calculator to explore properties of graphs, determine the range of a function from its graph, and examine the conditions for the existence of inverse functions and composite functions. Apart from using the graphics calculator as a graphing tool to investigate behaviours of various types of functions (such as trigonometric, exponential and logarithmic functions) and the effects of transformations on these functions, students are also expected to know how to use a graphics calculator to solve problems on different topics in the syllabus. For example, students need to learn how to use a graphics calculator to solve inequalities by graphical methods, find a numerical solution of an equation, carry out addition, subtraction, multiplication and division of complex numbers, find the square roots, modulus and argument of a complex number, convert a complex number from Cartesian form to polar form and vice versa, evaluate values of the derivative of a given function at given points, locate maximum and minimum points of a function, investigate the relationship between the graph of a given function and that of its derivative, derive the Maclaurin series for a function and use the first few terms of the series to approximate the function, illustrate concepts of limits (particularly that of a Riemann sum), evaluate definite integrals, and sketch typical members of the family of solution curves of a differential equation. Students also need to learn to use the graphics calculator to compute probabilities pertaining to Binomial and Normal distributions.

Emphases of the Revised Curriculum

While the revised curriculum continued to emphasize the mastery of mathematical concepts and acquisition of mathematical skills, greater emphasis, as compared to the curriculum before 2006, was given to the development of students' abilities to conjecture, discover, reason and communicate mathematics with the aid of technological tools. To achieve this objective, teachers were required to make suitable adjustment to their classroom practices and pedagogical strategies. For instance, teachers were encouraged to plan lessons such that the graphics calculator would play a key role when students engage in stimulating discussions and activities in which they can explore possibilities and make connections. Indeed, the use of graphics calculators was intended to be integral to the learning of mathematics at the advanced level in the revised curriculum.

One other intent of the revised curriculum was that graphics calculators be used in ways which allow students to learn mathematics in practical and meaningful contexts, using analytic methods together with graphical and numerical techniques. The computational and graphing capabilities of graphics calculators should be used to enable students to engage in active learning through exploratory work and experiments. Students could work collaboratively with others, share ideas and discuss their findings. Graphics calculators should also be used to extend the range of problems accessible to students and enable them to execute routine procedures quickly and accurately, to make connections between algebraic and geometric concepts, and to switch swiftly between numerical, graphical and symbolic representations in mathematical explorations. In a nutshell, it was intended that the routine tasks be relegated to the graphics calculator, allowing students more time for thinking, reflection and discovery.

Use of Graphics Calculators in Examinations

With regard to the use of graphics calculator in examinations, as a general rule, unsupported answers obtained from graphics calculator are acceptable unless the question states otherwise. Where unsupported answers from graphics calculator are not allowed, candidates are required to present the mathematical steps using mathematical notations and not calculator commands. For questions in which graphs are used to find a solution, candidates should sketch these graphs as part of their answers. Incorrect answers without sufficient mathematical explanation will receive no marks. However, if there is written evidence of using graphics calculators correctly, marks may be awarded for the correct method used. [10] Candidates are expected to be aware that there are limitations inherent in graphics calculators. For example, answers obtained by tracing along a graph to find roots of an equation may not produce the required accuracy. To ensure that no candidate will have any unfair advantage over other students by pre-installing any additional applications, candidates are required to clear all memory (and not merely RAM) to erase all applications. Instructions are read out by invigilators before the start of each exam to ensure that graphics calculators are reset.

Training for Teachers

Given the wide range of skills pertaining to the use of graphics calculators which students were required to master, teachers were expected to provide sufficient guidance to help the students develop accurate conceptual understanding of the graphics calculator and its functionality. Indeed, teachers teaching mathematics at the pre-university level were required to be proficient in utilising the graphics calculator and adept at facilitating students' use of the graphics calculator to meet the new assessment requirements. To this end, it was imperative that training on the use of a graphics calculator, with an emphasis on solving typical examination problems, be provided to the teachers. Equipped with the basic skills in solving problems with the graphics calculator, the teachers could then be in a position to examine and adjust their classroom practices so as to optimise the educational benefits of integrating the use of a graphics calculator in the mathematics curriculum. To train the teachers in using the graphics calculators, the second author was requested by the MOE to conduct in-service courses to meet the requirement (see [13] for details of the courses). The second author also wrote two books to provide further support to the teachers. [14], [15].

Another area of concern at the initial stage of the implementation stems from the limitations of graphics calculators. For instance, the initial graph provided by a graphics calculator may not display the complete graph, and thus window settings may need to be adjusted. To address this concern, teachers were alerted to the inherent limitations of graphics calculators through sharing sessions and workshops.

Design of Test Items

To prepare the teachers in developing meaningful and appropriate test items, dialogue sessions with teachers on question design were conducted by the MOE. The general guiding principle for question design is that the use of graphics calculator should create more time and space for higher order thinking tasks or investigation during assessment, by freeing candidates from the drudgery of tedious computations, as well as provide candidates with opportunities to solve problems arising from a context or application or those which were previously inaccessible without the use of a graphics calculator. Neat solutions may not be available but the graphics calculator could provide the numerical answer required to take the question further or reach a meaningful conclusion. For instance, candidates can compute definite integrals of functions whose integrals are not in closed form and take the question further. Questions that allow candidates to carry out simple but otherwise tedious investigations or trials to form hypotheses could also be set.

More specifically, design of a question is determined by the assessment objective. If the objective is to assess whether students have understood a particular mathematical concept, then the question has to be designed in a way that the use of the graphics calculator will not eliminate the intent of the question. For instance, a question on calculus could assess either the techniques of integration or understanding of the concept of integration. For the former, the question could require the candidate to evaluate an integral and for the latter, the candidates could be asked to compute a definite integral to find the area under a graph. That is, since the numerical value of a definite integral can be obtained from a graphics calculator easily, if the assessment objective is to test a student's integration techniques, the question may ask the candidates to compute an indefinite integral instead of a definite integral. For instance, to assess skills in applying integration techniques, candidates may be asked to find $\int \frac{3x^2}{x-4}$ instead of $\int_{-2}^{5} \frac{3x^2}{x-4}$, a definite integral easily obtained from a graphics calculator.

The mark scheme should also be crafted accordingly to correspond to the objective of the question. For instance, in the previous example, if the aim is to assess student's understanding of the relationship between a definite integral and area under a graph, then the question may be designed to ask students to find the area under the graph by formulating the appropriate definite integral. In this case, the mark scheme need not award marks for the method of evaluating the integral.

Similarly, for an assessment objective to assess students' application of logarithmic properties, consider the question, "Given that $\ln k = 8500$, find the value of $\ln(ke^9)$". The use of a graphics calculator does not eliminate this intent since the value of k cannot be calculated with the graphics calculator as it is too large. Students will have to evaluate $\ln(ke^9)$ using logarithmic properties.

Looking Back

It has been almost a decade since the revised Singapore advanced level mathematics curriculum was implemented. We end this subsection with a reflection on the incorporation of the use of graphics calculators into this curriculum. First, based on anecdotal evidence, benefits brought about by the implementation of the graphics calculator-integrated curriculum include allowing teachers access to a wider range of problems (e.g. non-standard functions and matrices of higher orders); opportunities to engage students in active learning through exploratory work and experiments, sharing ideas, discussing findings, and working collaboratively; more time spent on discovery, thinking, reflection and making inferences; learning in meaningful contexts, and allowing students quick and accurate execution of routine procedures (e.g. calculations and graphics); connections between algebraic and geometric ideas; and ease of switching between different representations. Second, most teachers are now sufficiently skilled in using the graphics calculator and more teachers are using them in ways that would encourage students to explore mathematics concepts on their own. However, many teachers still use the graphics calculator mainly as a computational tool and a graphing tool rather than as a pedagogical tool. Third, it is observed that there is a shift in teachers' concerns from those which are assessment-related to those related to student misconceptions or learning difficulties as they realize that the use of graphics calculators may bring undesirable outcomes if students misuse the graphics calculator, and that conceptual understanding and awareness of the inherent limitations of the graphics calculator are required for the appropriate use of the graphics calculator.

Students who lack conceptual understanding and the ability to analyze graphs are unable to recognize the limitations of the graphics calculator, thus resulting in poor performance in questions pertaining to graphing. Most of the learning difficulties faced or common errors made by students are due to their superficial level of mathematical content knowledge and the lack of an analytical habit of the mind as they blindly accept what they see on the graphics calculator screen. Teachers could do more to advise the students against relying too much on graphics calculator for performing simple mathematical tasks that could be done easily by hand, and promote in-depth thinking with the use of the graphics calculator.

In conclusion, to reap more educational benefits from the use of graphics calculator, teachers should use the graphics calculator as a pedagogical tool to develop conceptual understanding so that students use the graphics calculator as a learning tool, as opposed to using it merely as a computational tool or graphing tool. To help teachers develop new pedagogical skills, professional development is important and achievable through a systemic approach to planning pre-service training, in-service training and professional sharing.

The United States of America experience

The USA is the largest of the three countries being considered, with almost 320 million people and a complex structure of fifty states, over 16 000 school districts, and approximately 88 000 elementary and secondary schools. Control of education is mainly at the local school district level. Although there is no national mathematics curriculum, there are national teaching standards established by the independent teacher organization, the National Council of Teachers of Mathematics. [12] In addition, most of the states have adopted the Common Core State Standards for Mathematics. [4] Finally, the College Board Advanced Placement (AP) program for both Calculus and Statistics represent a set of national standards for the teaching of these courses in a secondary setting. A large number of students undertake AP examinations, which are recognized by many colleges and universities to provide incoming students with college credit if they are successful.

Graphics calculators first appeared in the USA in 1985, pioneered by CASIO and followed quickly by Sharp, Hewlett-Packard, and then Texas Instruments [17] and were used in a number of curriculum projects funded by the National Science Foundation from 1986. In the United States of America, this appearance coincided with a national call for the reform of the teaching of calculus, catalyzed by a 1986 conference at Tulane University, *Toward a Lean and Lively Calculus*, held in conjunction with the Joint Math Meetings of the Mathematical Association of America (MAA) and the American Mathematical Society (AMS) [5]. At the time, two different groups of mathematics educators were involved in the teaching of calculus: university professors and high school teachers involved in the Advanced Placement program. The latter quickly saw the graphics calculator as a game-changer, a tool that could help them reform their teaching in precisely the ways that the Tulane conference envisioned. AP Calculus teachers began adopting graphics calculators in large numbers starting in 1991 and by 1995 graphics calculators were required for the AP Calculus examinations. Similar developments happened in the case of statistics, for which a graphics calculator was quickly recognized as an important teaching and learning tool, facilitating both data analysis and elementary statistical inference, and the graphics calculator was also a required tool for students taking the AP Statistics examination from 1998.

Pedagogy and Research

The relatively quick adoption of graphics calculators by mathematics educators in the USA may come as a surprise in a field that is known for its resistance to change. But the pedagogical value of graphics calculators was clear to the leadership of this community of practice, and in less than a decade, graphics calculators became a staple of the mathematics classroom. As there is no national curriculum in the USA, the teaching community of practice has a strong voice in matters of pedagogy and curriculum.

The Power of Visualization

The power of visualization emerged as the single greatest strength of graphics calculators [6]. Able to plot graphs and tables of values of functions, as well as return the derivative of a function at a point and a numerical approximation to a definite integral, the graphics calculator became an essential tool for visualizing foundational concepts in AP Calculus.

In Figure 1, the graphics calculator easily gives the student a wealth of information to make sense of the notion of derivative as a slope function, comparing the slope value given with a visual estimate of the slope of the drawn tangent.

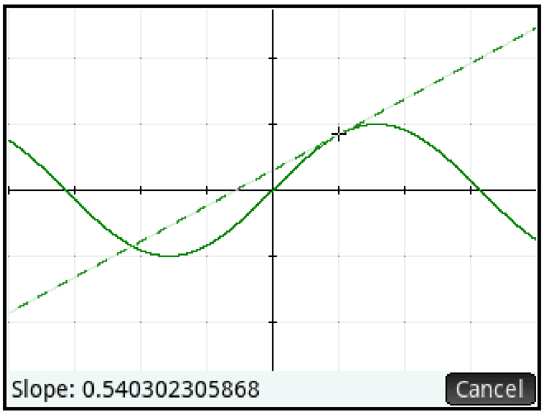

Figure 1: A function and its tangent

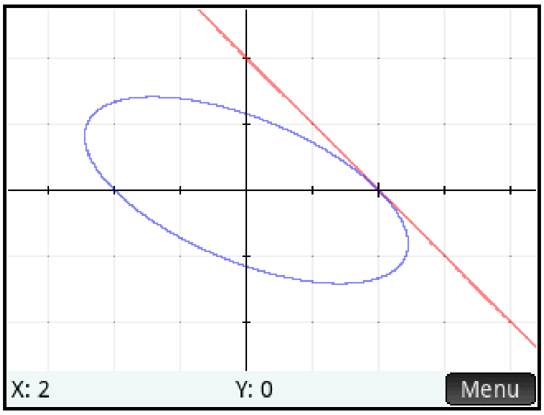

Figure 2: A tangent to a non-function

In addition, some rote problems in AP Calculus now became more interesting because the solutions could be tested graphically. For example, a common implicit differentiation problem is to find the equation of the line tangent to the curve $x^2 + 2xy + 3y^2 - 4 = 0$ through the point on the curve whose coordinates are (2,0). When the candidate solution can be verified by graphing, as in Figure 2, the problem becomes less of a rote process; the ability to visualize the solution gives such problems more meaning.

Multiple Representations and Representational Fluency

The availability of multiple representations of a mathematical object, as well as the ability to interact directly with one or more related representations (representational fluency), is another recognized benefit of graphics calculator use in mathematics education in the United States of America [18]. Here, representational fluency refers not only to the ease with which one moves from one representation to another, but also to the interactions one has with those representations, and the perceived effects that actions taken on one representation have on another related representation. In all of this, mathematical fidelity of the representations as they undergo changes and transformations is paramount to their use in mathematics education. Figure 3 and Figure 4 show an example of a transformation on a graphics calculator, in which a change to the symbolic representation results in the appropriate change to the graphical representation.

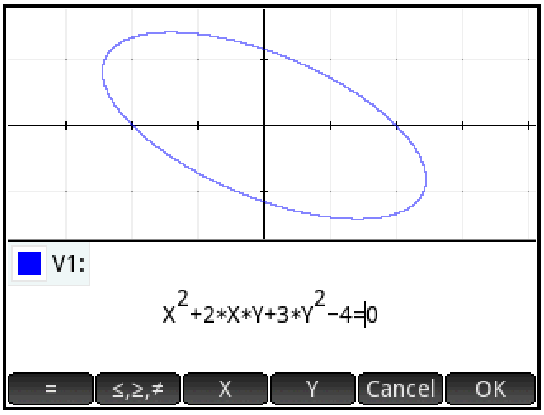

Figure 3: Change a symbolic object

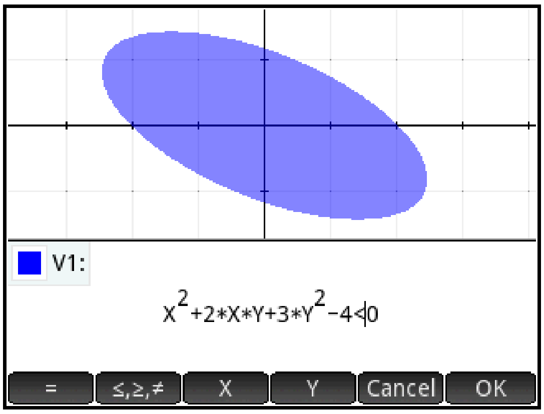

Figure 4: The change is reflected in the graph

With the advent of touch-screen technologies, the ways in which students can interact directly with mathematical representations have greatly increased. As these opportunities for representational fluency evolve, student conceptual understanding of mathematics will evolve as well. For example, as we move from mouse-centered interactions to direct finger-tip interactions, the notion of abduction via wandering dragging, *lieu muet* dragging, and the dragging test that are familiar features of dynamic geometry packages may become common features in other areas of the graphics calculator and hence, in the teaching of other topics in mathematics [3]. In Figure 5, the student drags Point B along the curve and observes the trail of Point D. The x-coordinate of point D is the same as the abscissa of Point B while its y-coordinate is the same as the slope of the tangent line through B. Thus, point D is a point on the slope function of the curve. The touch screen lets the student act directly on Point B rather than having the action mediated by a mouse.

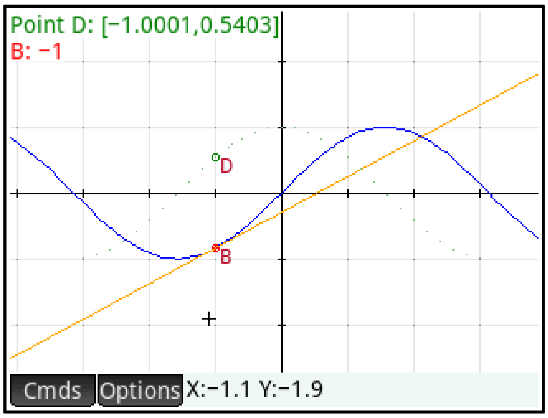

Figure 5: Exploring a slope by dragging

Teaching Standards and Large-Scale Assessments

Although it was the community of practice of AP Calculus teachers who began the adoption of graphics calculators and their integration into the curriculum, they were not alone in seeing the benefits of these tools and their positive affect on student achievement. As noted, the AP Statistics community also quickly adopted the technology and, by the year 2000, graphics calculators were embedded in the National Council of Teachers of Mathematics *Principles for School Mathematics* as essential tools for the teaching and learning of mathematics [12]. The focus on procedural fluency shifted to the realization that a conceptual understanding can lead to enhanced procedural fluency – and that technologies such as graphics calculators can support the development of those conceptual understandings. Thus graphics calculators are also referenced in the *Common Core State Standards for Mathematical Practice* [4], developed through a collaboration of US states and territories. The shift to using conceptual understanding to drive procedural fluency is documented in the recent NCTM publication on implementing the Common Core State Standards: *From Principles to Actions* [9].

The relatively quick adoption of graphics calculators by the community of AP mathematics teachers could not have occurred without high-quality teacher professional development. CalcNet, TICAP, and other teacher professional development institutes were crucial in providing teachers opportunities to explore and learn the benefits of these tools.

Graphics calculators not only affected the ways in which mathematics was being taught; it also affected what mathematics was taught and what was no longer taught. As some topics faded into the background and others came to the forefront in graphics-calculator-enhanced classrooms, assessments had to change accordingly. For example, recent free-response sections of the AP Calculus exams bear little resemblance to their pre-graphics-calculator counterparts.

As mentioned earlier, graphics calculators were required on the AP Calculus exams starting in 1995, and on the AP Statistics exam from its inception in 1998. The Scholastic Aptitude Test (SAT), another College Board examination, recommends graphics calculators but does not require them. Other US college entrance examinations also allow, recommend, or require graphics calculators, though some may restrict the use of Computer Algebra Systems (CAS) or other specific features of various graphics calculators. Some of these exams have both calculator and non-calculator sections. Many state examinations in mathematics require a graphics calculator as well, but the details are too varied to report here. The stance adopted by the examination boards in general reflect the belief that these valuable tools would be less-used in the classroom if they were not permitted on the examinations.

Critical issues

When graphics calculators are included into the educational program, a number of issues need to be taken into account and suitable decisions made about them. In this section, we highlight some of the issues that have been relevant to the three countries described here, expecting that mathematics educators in other countries will find these important also, without necessarily reaching similar conclusions regarding their resolution.

The nature of school mathematics

School mathematics differs in detail from country to country, but it seems that the use of graphics calculators has been important for allowing students to address mathematical modelling questions and practical applications, without undue and premature focus on formal mathematical proof and argument. In each of the three countries described in this paper, there is an increasing emphasis on the use of mathematics to model everyday situations, and the graphics calculator has been a significant tool to facilitate this. In most countries, the place of probability and statistics is increasingly important and is now more likely to be regarded as part of the mathematics curriculum than previously, so that the capacity of graphics calculators to support work in data analysis and probability simulation has been important. This is evident in the AP Statistics courses in the USA, and also in Australian senior secondary courses, many of which include a significant statistics component.

Examinations

In all three countries, it is widely believed that, although schools and teachers might use a variety of technology in classrooms, they are most likely to focus on the technology that is sanctioned for use in official examinations. Experience has supported this contention in practice. Contrariwise, when technology is not permitted for examination use, it is rarely available on a wide scale to pupils, in both schools and the early undergraduate years, so that only a select few students with innovative teachers or access to resources are likely to enjoy the advantages of learning with personal technology. It is clear in all countries that there is a significant 'backwash' effect of examinations on beliefs and practices in classrooms, so that great care is needed to ensure that this effect is benign. It is widely held that the experience of the mathematics classroom and the experience of formal high stakes examinations should not be more different from each other than is manageable, and ideally should be as coherent as possible. The graphics calculator still represents the best compromise between the use of educational technology and the constraints of exams.

Computer Algebra Systems

Some graphics calculators include a Computer Algebra System (CAS), which allows exact arithmetic, symbolic algebraic manipulation, and other advanced operations. Although CAS has been available on hand-held calculators for more than twenty years now, opinion still seems to be divided amongst professionals and researchers in different countries on their appropriate use in school mathematics education. Although secondary mathematics teachers in the USA are almost unanimous in their support for graphics calculator use, the case for CAS is not as clear to all. That a CAS can be used to visualize arithmetic and symbolic processes in a manner similar to how the graphics calculator is used to visualize graphs and tables does not seem to resonate uniformly throughout the community of teaching practice. Similarly, CAS is permitted in some Australian states, and not in others. Advocates see merit in offering students an opportunity to explore powerful symbolic tools and to consider general, and not only particular, solutions to problems; they detect advantages for students to use a single integrated device that has diverse capabilities. Others are more cautious, and prefer a traditional focus on formal symbolic manipulation as a key to understanding mathematical argument, or are concerned with some of the difficulties of making best educational use of CAS capabilities. This question does not seem to be of concern in later undergraduate years, where the professional use of CAS (on computers, generally, rather than graphics calculators) is widespread and frequently reported at ATCM conferences.

Supporting teachers and students

In all three countries, it is widely recognized that it is critical to provide adequate support for school mathematics teachers and their students if graphics calculators are to be used well. Experience suggests that, without sound support, many teachers will struggle to make effective use of the technology with their students and may even unwittingly focus on the technology as an alternative to mathematical thinking, rather than a powerful supplement to it. Support can be provided in various forms, including teacher workshops, discussions with colleagues, publications for teachers, online advice from manufacturers and publishers or specific official curriculum guidance, among others. Students also need support, which of course firstly comes from their teachers, but which might also be provided through textbooks written with a view to sound educational use of graphics calculators as well as advice on appropriate and judicious use of the devices for formal examinations.

Curriculum change and development

The mechanisms for curricula to adapt to the availability of new technologies depends on the structure of the educational systems in each country. An appropriate balance is needed between changing too slowly, and thus losing educational opportunities, or changing too rapidly, and risking changes to which teachers and their communities cannot adjust. Ideally, curriculum development involves many interested parties, including at least professional mathematicians in universities, professional mathematics teachers in schools and official bodies such as Ministries or Departments of Education. The best decisions about suitable use of graphics calculators are made by people who are themselves well-versed in the use of graphics calculators with students, so that it may be necessary for those without such experience to learn from their colleagues, or better, to explore the use of the technology personally and in their classrooms before forming strong opinions about it. In each of the three countries, curricula have adapted to the graphics calculator by changing the emphasis for some mathematical topics, such as including a study of numerical solution to equations rather than only exact solutions, emphasizing data analysis more than mathematical statistics or including probability simulation in addition to formal probability to model events involving randomness. The best curriculum change occurs when teachers are adequately supported to understand the changes and helped to develop the necessary mathematical, calculator and pedagogical expertise, so these aspects need to be considered as part of a change process.

A focus on learning

In all three countries described in this paper, and in many others, the main arguments for the use of graphics calculators have been related to improving student learning; graphics calculators are essentially learning tools, not merely calculators with a focus on generating answers. Calculator development over the past thirty years has been concerned with improving the mathematics learning opportunities for students in typical school curricula. Accordingly, advice for teachers and students in various forms is best focused on how to use them for learning, not only how to use them as calculators, particularly in the context of examinations. A misunderstanding of this distinction accounts for some of the different approaches to the use of graphics calculators at the end of secondary school and the early undergraduate years in each of the three countries. Experience suggests that a focus on learning is easier to attain if teachers are given good tools to support the use of graphics calculators in pedagogically effective ways. This includes in part the use of classroom devices, such as emulators that can focus a class attention on the same screen, and in part on the availability of good resources for classrooms and for teachers that emphasize the role of the calculators for learning purposes.

Teaching and learning limitations

Graphics calculators have some inherent limitations, which need to be accommodated somehow by the teacher in the classroom. For example, calculator screens have a discrete collection of pixels and so are ultimately unable to represent a continuous function precisely. Similarly, irrational numbers cannot be represented faithfully on a device with a limited number of decimal places on the screen or in working memory. In each of these cases, powerful computers also have similar limitations, but the graphics calculator provides a powerful and suggestive approximation to mathematical concepts such as these. Students need to understand some of these sorts of limitations and to learn to be judicious and thoughtful users of technology; indeed, there are opportunities for students to learn about important mathematical ideas by exploring the limitations of a small hand-held digital device and thinking about what they see on the screen. Teachers also need to understand calculator limitations and help students appreciate the need for thoughtful use, rather than uncritical trust of their calculator. One way to ensure that calculator limitations are addressed is to make specific reference to them in official curriculum documents, recognizing a need for classroom attention.

School and university differences

Although the graphics calculator has become a common sight in mathematics courses (e.g., from the start of formal Algebra on in the USA, and during the final school years in Australia and Singapore) this is only true in the secondary program. Tertiary mathematics education has yet to fully embrace the graphics calculator as the technology of choice. A number of college and university professors either prefer applications that run on a computer or they eschew electronic technology altogether for the teaching of mathematics. The transition for students in each of these three countries from the secondary school classroom to the college classroom can be smooth or bumpy depending on the philosophy of their professors.

Future technologies

A rapidly increasing number of students are arriving at schools with laptops, tablets and smartphones. In addition, some schools are now providing either tablets or Chromebooks for their students. In either case, there are full-featured graphics calculator apps available for these devices. If and when these devices replace the traditional graphics calculator may be largely a function of examination board approvals for devices that often have cameras and connect to the internet. In any event, the user interface of graphics calculators in general will be positively affected in the future by a desire to benefit from the affordances of a high-quality touch screen display.

Conclusions

Despite substantial differences in the size, educational structure and cultures of Australia, Singapore and the United States of America, there are many similarities of their adoption of the graphics calculator as an important tool for students learning secondary school mathematics. Over the twenty years of the Asian Technology Conferences on Mathematics, graphics calculators have continued to feature prominently and continue to provide significant opportunities for curriculum enhancement and development for the region. It seems likely that countries considering graphics calculators for secondary school or early undergraduate use will find the experiences and perspectives of these three countries valuable for their own consideration.

While concerns about the use of graphics calculators as mere computational devices are persistent, a common feature of the successful implementation of this technology into secondary school mathematics in the three countries is the development by teachers of new approaches to learning mathematics. In each country, mathematics teachers have used the opportunities afforded by graphics calculators to help students to explore mathematics in more conceptually powerful ways than previously. Rather than undermining mathematical thinking, as some in each country originally feared, the graphics calculator has now been recognized as a device that offers new ways of developing and supporting mathematical thinking. While intrinsic components of the success of the graphics calculator have been their small size and the opportunities they provide for personal student interaction, many of the new ways of learning rely on the capability of the calculators to represent mathematical concepts in visual ways, others exploit the capability for data analysis, and still others make use of their computational power. In each country, expertise in the educational use of graphics calculators has been shared amongst communities of teachers and mathematics educators, while, beyond each country, productive partnerships between mathematics teachers and manufacturers have supported the continuing development of a technology that is designed to meet the unique needs of mathematics education.

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