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Distance Methods

The Use of Distance Measurements in the Estimation of Seedling Density and Open Space Frequency

Avståndsmetoder

Användning av avståndsmätning för uppskattning av planttäthet och luckfrekvens

by

OLLE PERSSON

SKOGSHÖGSKOLAN

STOCKHOLM

CONTENTS

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Olle Persson

Chapter 1. Introduction

1.1. Previous work

Owing to increasing costs of planting in forestry, the control of planting success has become very important. Such control is generally carried out by means of sample surveys, which consist of inventories of the number of seedlings in sample plots of a certain size. The object of the present investigation is to compare this control method with that of using distance measurements for the estimation of seedling number and open space frequency.

The principle of estimating the number of individuals in a plant population by distance measurements is simple. If the population is dense, the distances between neighbouring individuals must, on an average, be short and conversely long if the number of individuals is small. The same conditions apply to the distance from a sample point to the nearest plant. These questions were discussed by BAUERSACHS (1942). In this paper BAUERSACHS deals with the distances that are often called "plant-to-plant distances". Other work on these problems has been done by foresters. Thus STOFFELS (1955) treated both the case of BAUERSACHS and the case of distances between sample points and plants. In the English literature the latter distances are generally called "point-to-plant distances". STOFFELS discussed comprehensivly the computation of standard errors of the estimates made in his work, but he studied only briefly the matter of bias. The importance of the population structure for obtaining unbiased estimates has, however, been discussed by ESSED (1957). In the work by BAUERSACHS empirical adjustments were made by comparing the estimates with known values. These adjustments indicate that the populations studied had a structure slightly more uniform than random. The word random is here used for a special random population and will be explained below. ESSED (1954) also used an empirical adjustment of the BAUERSACHS estimate. This adjustment seemed to indicate a slightly more even structure than the random one in the populations he used as a basis for his adjustment. MATÉRN (1959) computed the exact value of the adjustment to the BAUERSACHS estimate on the assumption of randomly distributed plants.

Parallel to the work of the foresters, ecologic literature in this field has developed. The problems studied by ecologists have often been slightly different from those of foresters. The central question has been whether a plant population can be considered grouped or random. To make clear what is meant by a random distribution, the author would like to quote CLARK & EVANS (1954): "In a random distribution of a set of points on a given area it is assumed that any point has had the same chance of occurring on any sub-area as any other point, that any sub-area of specified size has had the same chance of receiving a point as any other sub-area of that size, and that the placement of each point has not been influenced by that of any other point". This description agrees with the mathematical-statistical definition of a two-dimensional Poisson process; cf. FELLER (1957). Assuming that the population is generated by such a Poisson process of given density (=given mean number of individuals per unit area), we can easily compute the distributions and characteristics of the distances for both the "point-to-plant" and the "plant-to-plant" case, if the sample points are located uniformly. SKELLAM (1952) made this computation for the nearest plant in the "point-to-plant" case. He indicated that the hypothesis of the population being random can be tested by this distance distribution. THOMPSON (1956) treated the same problem more generally by considering plants of arbitrary order from the sample point. The order of a plant is the position it occupies when all plants are arranged according to their distances from the sample point. The same distributions were also computed by MORISITA (1954). Means and standard deviations are found in the work by THOMPSON. The distributions have been tabulated by ESSED (1957). He also computed the medians. The mean distances to the 1st, 2nd, 3rd, and 4th plant in the "plant-to-plant" case, still under the assumption of a Poisson process, have been computed by MATÉRN (1959). He also mentioned another type of estimate that is based on the harmonic mean of the squares of the distances. Such an expression has also been studied by MORISITA (1957). ESSED (1957) treated efficiency, taking into account the work required. Several of the results found in the mathematical works have previously been observed by sampling experimentally produced realisations of Poisson processes. Thus, COTTAM, CURTIS & HALE (1953) found the mean distance to the nearest plant in the "point-to-plant" case. A complete description of these investigations are found in COTTAM & CURTIS (1956). In addition to the works now mentioned concerning the Poisson case, still another may be mentioned, viz. CLARK & EVANS (1955), where the proportion of pairs of plants that are mutually closest neighbours is studied. Under the assumption of a Poisson process this proportion is independent of the density. It can therefore be used as a basis for testing the hypothesis that the plants are randomly distributed.

Another way of using distance measurements from sample points to plants has been suggested by STRAND (1954). He pointed out that the distribution of the distance from sample point to the nearest plant gives information on the proportion of circles of various sizes where plants are lacking. It is of economic importance to detect how the area is distributed according to open spaces of different sizes by seedling inventory in forestry. Inventories based on sample plots of fixed size and shape give information on open spaces of one type only. Investigations of the latter kind have been made for forest management purposes by ENEROTH (1945) and TIRÉN (1950). BRAATHE (1953) studied the importance of open spaces for the stand development. For further works on statistical methods in plant ecology, see GOODALL (1962).

1.2. Symbols and concepts

Some terms

random variable:	real valued function defined on a set of outcomes of a random experiment
random seedling distribution:	seedlings are distributed according to a Poisson process, cf. 1.1
randomly located sample point:	the sample point is located uniformly
seedling no. i:	the seedling standing in order i from the sample point
confidence interval:	an interval with end points which are functions of the sample values only and with a predetermined probability fall on either side of a parameter under estimate.

Some symbols

$F(x)$:	the distribution function of a random variable. If the random variable is denoted X, $F(x)$ is the probability that X takes on a value smaller than or equal to x, or using P to denote probability: $F(x) = P(X \leq x)$
$f(x)$:	the frequency function of X; $f(x) =$ the derivative of $F(x)$ if existing, else the steps of $F(x)$
$E(X)$:	the mean value of the random variable X; also denoted by m
$E(X\|H)$:	the mean value of X on the condition that H has occurred
$D(X)$:	standard deviation of X; also denoted by σ; the variance of $X = \sigma^2$
z:	the median of X, obtained from the equation $F(z) = \frac{1}{2}$
$\bar{x}$:	sample mean; $\bar{x} = \sum_{i=1}^{n} x_i/n$, where $x_1, \ldots, x_n$ are observed values of the random variable X

s: sample standard deviation; $s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1)$

z^*: sample median.

1.3. Survey of contents

The distributions of the distances between the sample point and the seedling no. 1, no. 2, and no. 3 in a population where the individuals are distributed in a square lattice, respectively randomly, are derived in chapter 2. In this chapter values of means, standard deviations and medians of the distributions are also given. These values are partly quoted from the literature. Tables showing the distributions are also given.

Chapter 3 treats of the distribution of the distance between seedling no. 1 and its nearest neighbour under the assumption that the seedlings are randomly distributed. The distribution has been tabulated for various numbers of seedlings per hectare. From the distribution the truncated distribution is computed for this distance. In the tables the point of truncation has been chosen as 0.8 metres. The means of both the original distribution and the truncated one have been tabulated for various numbers of seedlings. In this table the means of the distributions treated in chapter 2 have also been computed for the corresponding number of seedlings.

Chapter 4 is concerned with the standard deviation of the number of seedlings in a circular sample plot that has been randomly located both on a square lattice, and on a random population. The standard deviation has been tabulated for various numbers of seedlings per hectare and for the two sample plot sizes 5 sq.m. and 10 sq.m., respectively.

In chapter 5 the results arrived at in chapters 2, 3, and 4 are applied to data obtained by sampling 30 plantations in the South of Sweden. The data are based on observations of the number of seedlings in sample plots and of distances to seedlings nos. 1—3, and the distance from seedling no. 1 to its nearest neighbour. In addition to the estimates of the number of seedlings per hectare that can be obtained by means of the characteristics described in the chapters 2 and 3, some further estimates based on the distances to seedlings and mentioned in literature have been studied. The sample plot data are utilized for a comparison between various methods of estimating the number of seedlings. To show the occurrence and extent of open spaces, some of the fields have been treated as one group and a histogram has been drawn for the distances to seedling no. 1. Theoretical frequency functions have been fitted to the histogram.

Chapter 6 treats of an application of a computer programme for constructing and sampling models of uneven plantations designed for the Swedish computer FACIT EBB. Ten plantations of this kind have been constructed

Some of the estimates investigated in chapter 5 have been tested by sampling these models in the computer. The results largely agree with those obtained in chapter 5. In addition to the number of seedlings, the distributions of the distances to seedling no. 1 have also been estimated. These distributions show the occurrence of open spaces and they have been presented in figures for purposes of comparison with those of plantations with no seedlings lacking.

Chapter 2. Distributions of distances to seedlings from a randomly located point

2.1. Introduction

This chapter deals with distributions of distances from sample points to seedlings. Both square lattice and random seedling distributions are studied. Tables have been computed for the distribution functions and their characteristics. These tables are valid for a seedling number of 2,500 per hectare. If tables are wanted for other numbers of seedlings, such tables are easily obtained by a simple transformation of those given. An example of the procedure is given in connection with table 1. The distribution functions are to be found in fig. 3. Distribution functions for the random seedling distribution are also tabulated by ESSED (1957).

Means, medians and standard deviations of the distance distributions of both population structures are also presented by ESSED (1957). For the random distribution the characteristics are presented by THOMPSON (1956). Estimates of the number of seedlings per hectare can be constructed from these characteristics. Some additional proposals to such estimates are made under the assumption of random distribution by MATÉRN (1959). The properties of the estimates which are also discussed by ESSED (1957) and MATÉRN (1959) will be treated in chapter 5, where the results of a practical application of the theories are given.

2.2. Square lattice, frequency functions

Let P be a randomly located sample point in a plane and suppose that the seedlings are distributed there in a square lattice with meshes of size $2a \times 2a$. A_1, A_2, and A_3, respectively, are the seedlings nearest, second nearest and third nearest to the sample point. The distances $\overline{PA_1}$, $\overline{PA_2}$, and $\overline{PA_3}$ are random variables and are denoted by R_1, R_2, and R_3. A special outcome with the values r_1, r_2, and r_3 is shown in fig. 1. Fig. 2a—c is the basis of a computation of the frequency functions, denoted by $f_1(x)$, $f_2(x)$, and $f_3(x)$, respectively. Owing to reasons of symmetry it is sufficient to study only one of the meshes of the lattice. The area allowed for P, such that A_1 is the nearest seedling, is the square A_1BCD in fig. 2a. For P in connection with A_2 the allowed area is

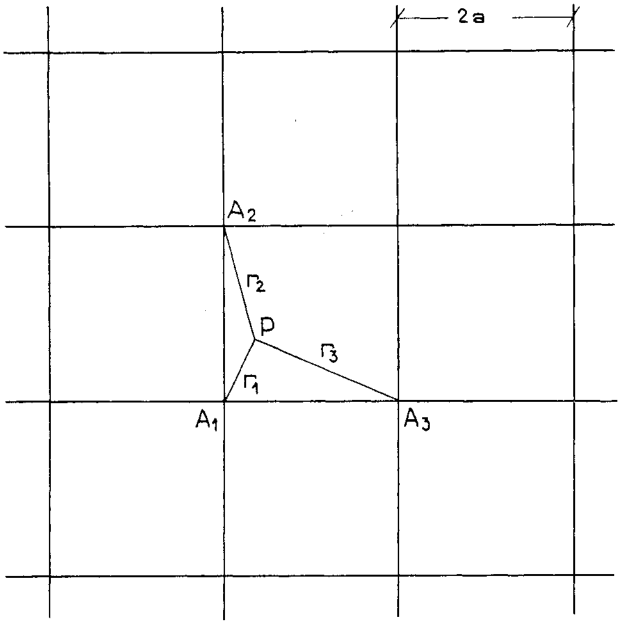

Fig. 1. Square lattice. P is a sample point and A_1, A_2, and A_3 are the seedlings no. 1, no. 2, and no. 3, respectively.

the triangle A_1BC in fig. 2b and in connection with A_3 the triangle A_1BC in fig. 2c. On account of the uniform distribution of P, the probability of P being located in a narrow strip of the width dx around the respective circle arcs EF and DE in the figures will be the ratio between the area of the strip and the area allowed. Such a probability is the probability element $f_i(x)dx$. The frequency functions are thus obtained directly from these surface ratios. As shown in the figures, the frequency functions in all three cases will have two different branches. Simple calculations lead to

$$2.2.1 \quad f_1(x) = \begin{cases} \dfrac{2x}{a^2} \cdot \dfrac{\pi}{4} & 0 < x \leq a \\ \dfrac{2x}{a^2}\left(\dfrac{\pi}{4} - \arccos\dfrac{a}{x}\right) & a < x \leq a\sqrt{2} \end{cases}$$

$$2.2.2 \quad f_2(x) = \begin{cases} \dfrac{2x}{a^2} \arccos\dfrac{a}{x} & a < x \leq a\sqrt{2} \\ \dfrac{2x}{a^2}\left(\arcsin\dfrac{a\sqrt{2}}{x} - \dfrac{\pi}{4}\right) & a\sqrt{2} < x \leq 2a \end{cases}$$

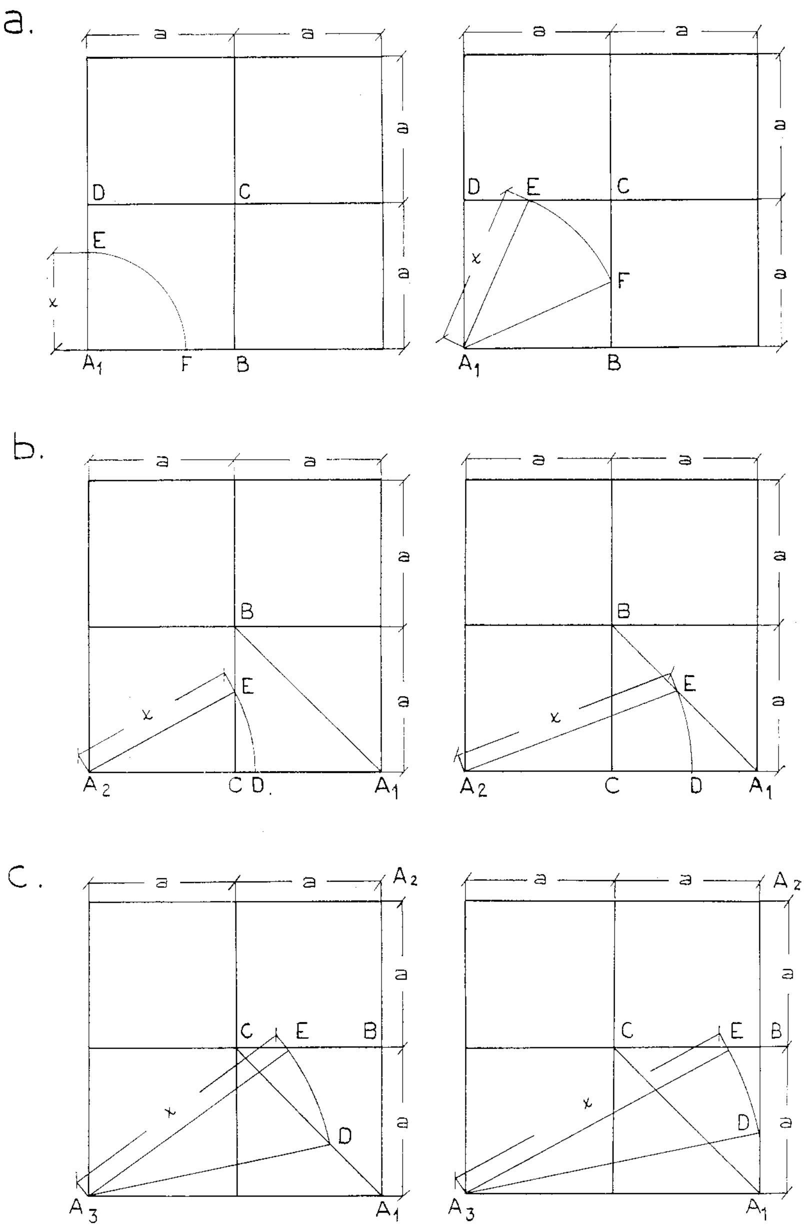

Fig. 2. Basic figures for computation of the distributions of the distances between sample point and seedlings; square lattice.

2.2.3 $$f_3(x) = \begin{cases} \dfrac{2x}{a^2}\left(\dfrac{\pi}{4} + \arcsin\dfrac{a}{x} - \arcsin\dfrac{a\sqrt{2}}{x}\right) & a\sqrt{2} < x \le 2a \\ \dfrac{2x}{a^2}\left(\arcsin\dfrac{a}{x} - \arccos\dfrac{2a}{x}\right) & 2a < x \le a\sqrt{5} \end{cases}$$

2.3. Square lattice, distribution functions

From the definition the distribution functions of R_1, R_2, and R_3, below denoted by $F_1(x)$, $F_2(x)$, and $F_3(x)$, are those proportions of the allowed areas which meet the respective conditions $R_1 \le x$, $R_2 \le x$, and $R_3 \le x$. They may be obtained either geometrically by computating the areas, or by integrating the frequency functions. For branch no. 1 of $F_1(x)$ is obtained e.g. directly from fig. 2a

$$F_1(x) = \frac{\pi}{4}\left(\frac{x}{a}\right)^2.$$

If instead the frequency function 2.2.1 is used for the computation of the same branch, we have

$$F(x) = \int_0^x f_1(u)\,du = \int_0^x \frac{2u}{a^2}\,\frac{\pi}{4}\,du = \frac{\pi}{4}\left(\frac{x}{a}\right)^2.$$

This latter type of computation is easier in more complicated cases. The integrals that occur are easily computed by means of methods found in textbooks of integral calculus or by special integral tables, e.g. Ryshik & Gradstein (1957). $F_2(x)$ and $F_3(x)$ are computed by means of calculations of this latter type. Since the calculations are rather spacious and uninteresting for the continued presentation, only the results are presented.

2.3.1 $$F_1(x) = \begin{cases} \dfrac{\pi}{4}\left(\dfrac{x}{a}\right)^2 & 0 < x \le a \\ \left(\dfrac{x}{a}\right)^2\left(\dfrac{\pi}{4} - \arccos\dfrac{a}{x}\right) + \sqrt{\left(\dfrac{x}{a}\right)^2 - 1} & a < x \le a\sqrt{2} \end{cases}$$

2.3.2 $$F_2(x) = \begin{cases} \left(\dfrac{x}{a}\right)^2 \arccos\dfrac{a}{x} - \sqrt{\left(\dfrac{x}{a}\right)^2 - 1} & a < x \le a\sqrt{2} \\ \left(\dfrac{x}{a}\right)^2\left(\arcsin\dfrac{a\sqrt{2}}{x} - \dfrac{\pi}{4}\right) - 1 + \sqrt{2\left(\dfrac{x}{a}\right)^2 - 4} & a\sqrt{2} < x \le 2a \end{cases}$$

$$2.3.3 \quad F_3(x) = \begin{cases} \left(\frac{x}{a}\right)^2 \left(\frac{\pi}{4} + \arcsin \frac{a}{x} - \arcsin \frac{a\sqrt{2}}{x}\right) - 1 + \\ \quad + \sqrt{\left(\frac{x}{a}\right)^2 - 1} - \sqrt{2\left(\frac{x}{a}\right)^2 - 4} & a\sqrt{2} < x \leq 2a \\ \left(\frac{x}{a}\right)^2 \left(\arcsin \frac{a}{x} - \arccos \frac{2a}{x}\right) - 3 + \\ \quad + \sqrt{\left(\frac{x}{a}\right)^2 - 1} + 2\sqrt{\left(\frac{x}{a}\right)^2 - 4} & 2a < x \leq a\sqrt{5} \end{cases}$$

Table 1 has been computed from these expressions.

2.4. Square lattice, means, medians, and standard deviations

The means of the distributions can be computed in different ways either by aid of the frequency functions in 2.2 or by the uniform distribution of P. The latter procedure has been applied by ESSED (1957). $E(R_1)$ is the mean distance from a randomly located point within a square with the side a to a given corner of the square. By integration over the square the following expression is obtained

$$\frac{1}{a^2} \int_0^a \int_0^a \sqrt{x^2 + y^2}\, dx\, dy$$

and after some calculations

$$E(R_1) = \frac{a}{3} \left[\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right].$$

The mean of R_2 can also be obtained by means of simple probability reasoning and by employing the expression for $E(R_1)$. Using the more complete designation $E(R_i|a)$ for the mean value of R_i, when the mesh side of the square lattice is $2a$, we obtain

$$E(R_1|a)\,\tfrac{1}{2} + E(R_2|a)\,\tfrac{1}{2} = E\left(R_1|a\sqrt{2}\right).$$

By introducing the values of $E(R_1|a)$ and $E\left(R_1|a\sqrt{2}\right)$

$$E(R_2) = \frac{a}{3}\left(\sqrt{8} - 1\right)\left[\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right].$$

$E(R_3)$ can be computed in a similar way if we use the mean distance to a given corner of a rectangle. For the purpose of further requirements we give both means and standard deviations which are easily obtained after computation of $E(R_1^2)$, $E(R_2^2)$, and $E(R_3^2)$; cf. further ESSED (1957). The values presented by ESSED correspond to $a = \frac{1}{2}$ with denotations used here.

2.4.1
$$\begin{cases} E(R_1) = \frac{a}{3}\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right] & = 0.7652\ a \\ E(R_1^2) = \frac{2}{3}\, a^2 & = 0.6667\ a^2 \\ D(R_1) & = 0.2849\ a \end{cases}$$

2.4.2
$$\begin{cases} E(R_2) = \frac{a}{3}\left(\sqrt{8}-1\right)\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right] & = 1.3991\ a \\ E(R_2^2) = 2\,a^2 & = 2.0000\ a^2 \\ D(R_2) & = 0.2062\ a \end{cases}$$

2.4.3
$$\begin{cases} E(R_3) = \frac{a}{3}\left\{\sqrt{80} + \frac{11}{3}\ln\left(\sqrt{5}+2\right) - \left(\sqrt{8}+1\right)\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right]\right\} & = 1.8164\ a \\ E(R_3^2) = \frac{10}{3}\, a^2 & = 3.3333\ a^2 \\ D(R_3) & = 0.1848\ a \end{cases}$$

The medians are obtained from the equations $F_1(z_1) = \frac{1}{2}$, $F_2(z_2) = \frac{1}{2}$, and $F_3(z_3) = \frac{1}{2}$. The expressions in 2.3 have been used for the numerical solution of the equations. The various characteristics are found in table 2; cf. further Essed (1957). For later use we also calculate $E(1/R_3^2)$. Using the same symbols as before, we obtain (cf. fig. 2c)

$$E\left(\frac{1}{R_3^2}\right) = \frac{2}{a^2}\int_{a}^{2a}\int_{2a-x}^{a}\frac{1}{x^2+y^2}\,dx\,dy.$$

After some calculations we have

$$E\left(\frac{1}{R_3^2}\right) = \frac{2}{a^2}\left[\left(\varphi_0 - \frac{\pi}{8}\right)\ln 2 + \int_{\varphi_0}^{\pi/4}\ln\cot\varphi\,d\varphi\right]$$

where $\varphi_0 = \operatorname{arctg}\frac{1}{2}$.

By expanding the integrand in series (see e.g. Dwight (1947)) we obtain

2.4.4
$$E\left(\frac{1}{R_3^2}\right) = \frac{0.3131}{a^2}$$

2.5. Randomly distributed seedlings, frequency functions

Assume that seedlings are distributed according to a Poisson process in the plane with a density of $(1/2a)^2$ seedlings per unit area; cf. chapter 1.1. We use the same symbols as above. The frequency function of the distance

R_i can then be obtained in the following way; cf. further KENDALL (1963). The probability that the distance to seedling no. i is located in the interval $(x, x+dx)$ is $f_i(x)dx$. This event occurs if $i-1$ seedlings are located within a circle with centre in P and radius x and if at least one seedling is located within the boundary ring of width dx. The number of seedlings within the circle has a Poisson distribution with the parameter $(1/2a)^2 \pi x^2$ and the number of seedlings occurring within the boundary ring has a Poisson distribution with the parameter $2(1/2a)^2 \pi x\,dx$. Neglecting terms of higher order than dx we find

$$f_i(x)\,dx = \frac{\left[\left(\frac{1}{2a}\right)^2 \pi x^2\right]^{i-1}}{(i-1)!} e^{-\left(\frac{1}{2a}\right)^2 \pi x^2} 2\left(\frac{1}{2a}\right)^2 \pi x\,dx$$

i.e. the frequency function becomes

2.5.1 $$f_i(x) = \frac{\left[\frac{\pi}{4}\left(\frac{x}{a}\right)^2\right]^{i-1}}{(i-1)!} e^{-\frac{\pi}{4}\left(\frac{x}{a}\right)^2} \frac{\pi}{2a^2} x.$$

2.6. Randomly distributed seedlings, distribution functions

These functions are obtained by integrating the frequency functions. A partial integration gives the following recursion formula

$$F_i(x) = F_{i-1}(x) - e^{-\frac{\pi}{4}\left(\frac{x}{a}\right)^2} \frac{\left[\frac{\pi}{4}\left(\frac{x}{a}\right)^2\right]^{i-1}}{(i-1)!}$$

with the initial function

$$F_1(x) = 1 - e^{-\frac{\pi}{4}\left(\frac{x}{a}\right)^2}.$$

From this is obtained

2.6.1 $$F_i(x) = \sum_{\nu=i}^{\infty} \frac{\left[\frac{\pi}{4}\left(\frac{x}{a}\right)^2\right]^{\nu}}{\nu!} e^{-\frac{\pi}{4}\left(\frac{x}{a}\right)^2}.$$

Table 3 has been computed from these expressions by means of MOLINA's (1947) table of the Poisson distribution. Alternatively, we might have interpolated in existing tables of the χ^2-distribution or the incomplete Γ-function, since $\pi R_i^2/2a^2$ is distributed as χ^2 with $2i$ degrees of freedom as is clear from the expression 2.5.1.

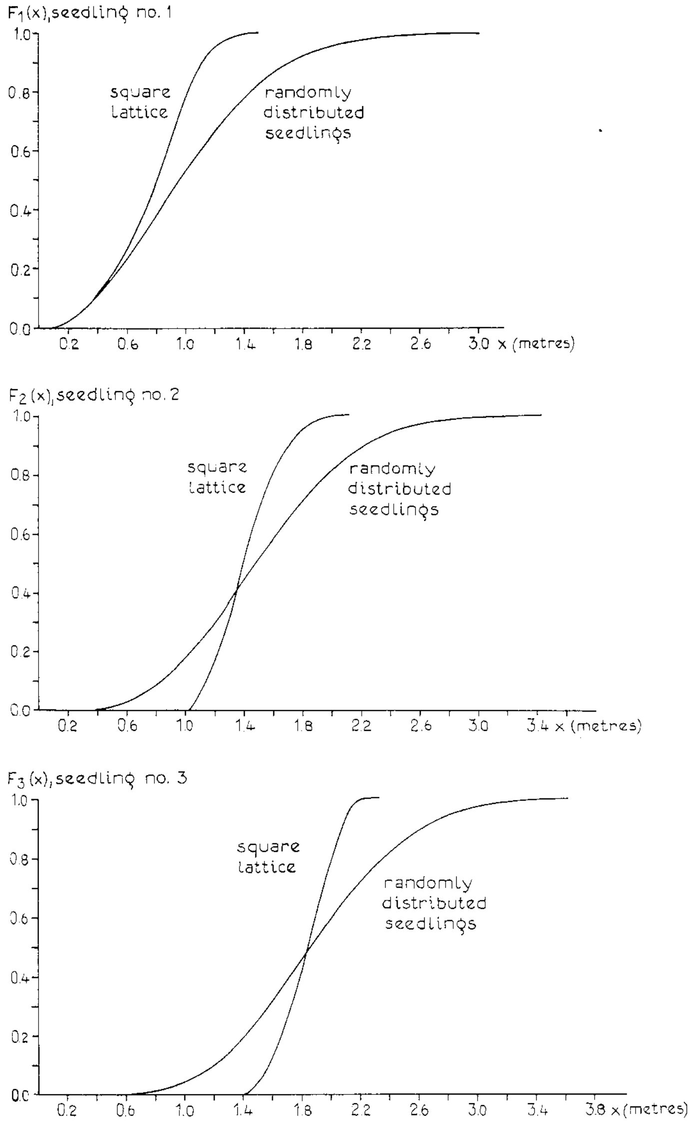

Fig. 3. Distribution functions of the distances between sample point and seedlings (2,500 seedlings/ha).

2.7. Randomly distributed seedlings, means, medians, and standard deviations

As in the case of a square lattice we list here the means, the medians and the standard deviations. These characteristics are easily obtained from the functions 2.5.1 and 2.6.1. Since the parameters are found in Thompson (1956) and ESSED (1957), we restrict ourselves to presenting the characteristics adapted to a number of seedlings of 2,500 per hectare, ($a=1$).

2.7.1
$$\begin{cases} E(R_1) = \dfrac{1}{2}\dfrac{1}{\sqrt{\left(\dfrac{1}{2a}\right)^2}} & = 1.0000\,a \\ E(R_1^2) = \dfrac{1}{\pi\left(\dfrac{1}{2a}\right)^2} & = 1.2732\,a^2 \\ D(R_1) & = 0.5227\,a \end{cases}$$

2.7.2
$$\begin{cases} E(R_2) = \dfrac{3}{4}\dfrac{1}{\sqrt{\left(\dfrac{1}{2a}\right)^2}} & = 1.5000\,a \\ E(R_2^2) = \dfrac{2}{\pi\left(\dfrac{1}{2a}\right)^2} & = 2.5465\,a^2 \\ D(R_2) & = 0.5445\,a \end{cases}$$

2.7.3
$$\begin{cases} E(R_3) = \dfrac{15}{16}\dfrac{1}{\sqrt{\left(\dfrac{1}{2a}\right)^2}} & = 1.8750\,a \\ E(R_3^2) = \dfrac{3}{\pi\left(\dfrac{1}{2a}\right)^2} & = 3.8197\,a^2 \\ D(R_3) & = 0.5514\,a \end{cases}$$

As above a must be put equal to $\frac{1}{2}$ to make the results comparable with those of ESSED (1957) and MATÉRN (1959).

The medians are computed as before by solving numerically the equations $F_i(z_i)=\frac{1}{2}$. They are given in table 4. We also report the means $E(1/R_3^2)$ and $E(1/R_3^4)$ for later use

2.7.4
$$E\left(\frac{1}{R_3^2}\right) = \frac{\pi}{8a^2}$$

2.7.5
$$E\left(\frac{1}{R_3^4}\right) = \frac{\pi^2}{32a^4}.$$

Chapter 3. Distribution of the distance between seedling no. 1 and its nearest neighbour

3.1. Introduction

In a square lattice, the distance between seedling no. 1 and its nearest neighbour is constantly equal to $2a$. For this reason the distribution of the distance between seedling no. 1 and its nearest neighbour is derived under the assumption of a Poisson process only; cf. further KENDALL (1963). Tables of the distributions corresponding to various assumptions on the numbers of seedlings per hectare have been prepared by means of the computer MERCURY at the Royal Institute of Technology. The results are given in table 5. The table has been checked by computing the means of the distributions by numerical integration. The values agree with those given by MATÉRN (1959). They are found in table 7, where the mean distances according to the distributions derived in chapter 2 have also been presented.

The seedlings recorded in the inventory of a regeneration stand are subject to certain conditions, one of which is that they must be located at a certain minimum distance from each other. For this reason the distance distribution corresponding to a random distribution of seedlings has been truncated. Distances $< k$ have been omitted. The distribution thus obtained does not correspond to the process of selecting those seedlings which were recorded. It will, however, be investigated for estimating the number of seedlings per hectare. In chapter 5 the estimates obtained by means of this truncated distribution are examined in greater detail. The point of truncation has been put equal to 0.8 metres for this purpose, a requirement set forthe inventories; cf. chapter 5. The truncated distance distribution has been computed numerically from table 5 and is found in table 6. The mean of the truncated distance, too, has been computed and is shown graphically in fig. 5 for the same numbers of seedlings as in the tables. Values for seedling numbers not tabulated have been taken from the figure. These mean distances, too, have been entered in table 7 and are printed in italics.

The distance between seedling no. 1 and its nearest neighbour also shows the evenness of the population. In a population with the individuals in clusters, these distances become shorter than those in a randomly distributed population of equal density. If the number of individuals is known, the empirical mean distance can be compared with the mean distance of a randomly distributed population of the same size. This has been proposed by CLARK & EVANS (1954) and it has also been studied by COOPER (1961) who found that young stands were more or less randomly distributed while the seedlings of young natural regeneration stands were grouped in clusters.

3.2. The original distribution

Assume that P is a randomly located point in the plane and that the seedlings are randomly distributed with a density of λ seedlings per unit area. According to 2.5.1 the frequency function of the distance from P to A_1 is

$$f_1(x) = e^{-\lambda\pi x^2}\, 2\lambda\pi x.$$

If we initially let A_1 be located in a circular ring with breadth dx and the fixed radius y, cf. fig. 4, then no other seedling can be located within the dashed region in the figure if the distance R_{11} to the nearest neighbour G_1 is to be greater than x (two cases). According to the assumption of a Poisson process the number of seedlings within the dashed region has a Poisson distribution with the parameter $\lambda Y(x, y)$ where $Y(x, y)$ is the area of the region. From the definition the distribution function $F_{11}(x|y)$ for the distance R_{11} conditioned by fixed y thus becomes

$$F_{11}(x|y) = 1 - e^{-\lambda Y(x, y)}.$$

After weighing with the distribution of y, the distribution function of R_{11} thus becomes

$$F_{11}(x) = \int_0^\infty [1 - e^{-\lambda Y(x, y)}]\, e^{-\lambda\pi y^2}\, 2\lambda\pi y\, dy.$$

When determining the area $Y(x, y)$ there are two cases, fig. 4a and b. If $2y \leq x$, we immediately obtain, cf. fig. 4a

$$Y(x, y) = \pi x^2 - \pi y^2 = \pi y^2 \left[\left(\frac{x}{y}\right)^2 - 1\right].$$

By elementary evaluations of the areas of triangles and sectors of circles for the case $2y \geq x$, cf. fig. 4b, we obtain

$$Y(x, y) = y^2 \left[\frac{\pi}{2}\left(\frac{x}{y}\right)^2 - 2\arcsin\frac{x}{2y} + \left(\frac{x}{y}\right)^2 \arcsin\frac{x}{2y} + \frac{x}{y}\sqrt{1 - \left(\frac{x}{2y}\right)^2}\,\right].$$

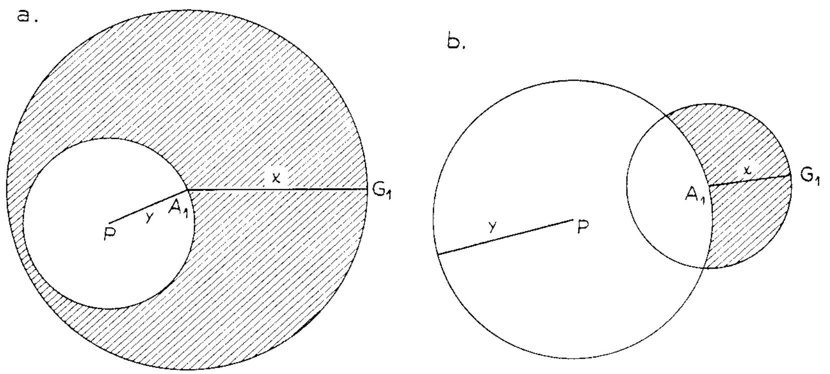

Fig. 4. Basic figures for computation of the distribution of the distance between seedling no. 1 and its nearest neighbour; randomly distributed seedlings.

Now, if $y\sqrt{\lambda}=v$ and $\frac{x}{2y}=u$ and we utilize both the expressions of $Y(x, y)$, the integral can be written

3.2.1 $$F_{11}(x)=1-\frac{\pi x^2\lambda}{4}e^{-\pi x^2\lambda}-\int_{\frac{x\sqrt{\lambda}}{2}}^{\infty}e^{-v^2\left[\pi+f\left(\frac{x\sqrt{\lambda}}{2v}\right)\right]}2\pi v\,dv$$

where $f(u)=2\pi u^2-2\arcsin u+4u^2\arcsin u+2u\sqrt{1-u^2}$.

The integral has been computed for $\lambda=1{,}500,\ 2{,}000,\ \ldots\ 5{,}000$ seedlings per hectare and is tabulated for $x=0,\ 0.1,\ \ldots\ 4.0$ metres. The results are to be found in table 5.

3.3. The truncated distribution

The distribution function is

3.3.1 $$F_{11}(x|R_{11}>k)=\frac{F_{11}(x)-F_{11}(k)}{1-F_{11}(k)}.$$

It is found in table 6 for various numbers of seedlings.

Finally, we show how the means have been computed. From the definition

$$E(R_{11}|R_{11}>k)=\int_k^{\infty}\frac{x f_{11}(x)\,dx}{1-F_{11}(k)}.$$

The integral $\int_{0.8}^{\infty}x\,f_{11}(x)\,dx$ can be approximated in the following way

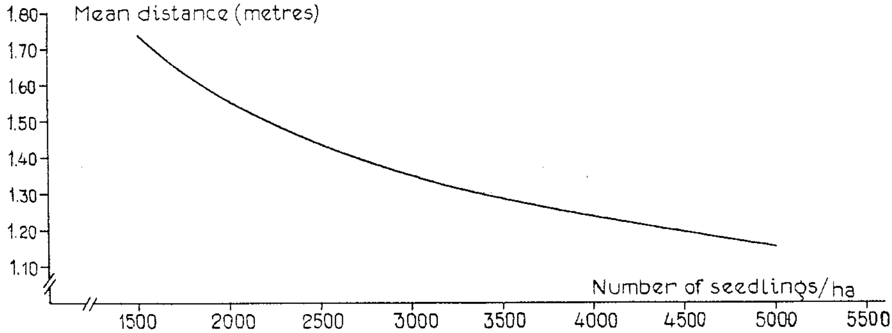

Fig. 5. Mean values of the truncated distance between seedling no. 1 and its nearest neighbour. Point of truncation $k = 0.8$ metres.

$$\int_{0.8}^{\infty} x f_{11}(x)\,dx \approx \sum_{i=0}^{n} (0.85 + 0.1\,i)\,[F_{11}(0.8 + 0.1\,(i+1)) - F_{11}(0.8 + 0.1\,i)].$$

The computations have been carried out according to this approximate expression by means of the differences in table 5. As mentioned in the introduction to the chapter, the values have been drawn and connected to a curve in fig. 5 from which the values in table 7 have been obtained.

Chapter 4. Characteristics of the number of seedlings in a randomly located circular plot

4.1. Introduction

In this chapter standard deviations of the number of seedlings in a randomly located circular sample plot will be treated. As in chapter 2, we treat the two cases square lattice and random distribution. It is a well-known fact that an unbiased estimate of the true mean is obtained by the sample plot mean if the plots are located randomly. For that reason the average number of seedlings per hectare of the sample plot will be used in chapter 5 for comparing the estimates based on distance measurements. For the square lattice there is an exact formula for the standard deviation of the number of seedlings of the circular sample plot; cf. KENDALL (1948). If the seedlings are randomly distributed, the number of seedlings in the sample plot will have a Poisson distribution. It is then possible to find the moments of every order.

Tables 8 and 9 show the standard deviations for plots of 5 and 10 sq.m. and various values of the true numbers of seedlings per hectare. The values are shown graphically in fig. 7.

4.2. Square lattice, standard deviations

As in the previous chapters, we assume that the lattice has square meshes of the size $2a \times 2a$, and that P is the centre of a randomly located circular sample plot with the radius r. The number of seedlings in the sample plot will depend on r and a. When a is larger than r, only 0 or 1 seedling occurs, and when a is small in comparison with r the number of seedlings may be large. If the distance a and the size of the plot are fixed, the areas within which P is to be located for different numbers of seedlings in the sample plot are also fixed. When the ratio between a and r varies, the appearance and size of these areas will vary. Fig. 6 shows the types of configurations that concern a circular sample plot of 10 sq. m. and numbers of seedlings up to 3,927 per hectare. For higher numbers of seedlings and this plot size the figures become more complicated. In the areas the number of seedlings in the sample plot has been entered. The standard deviations can be computed from the distributions of the number of seedlings which are obtained by

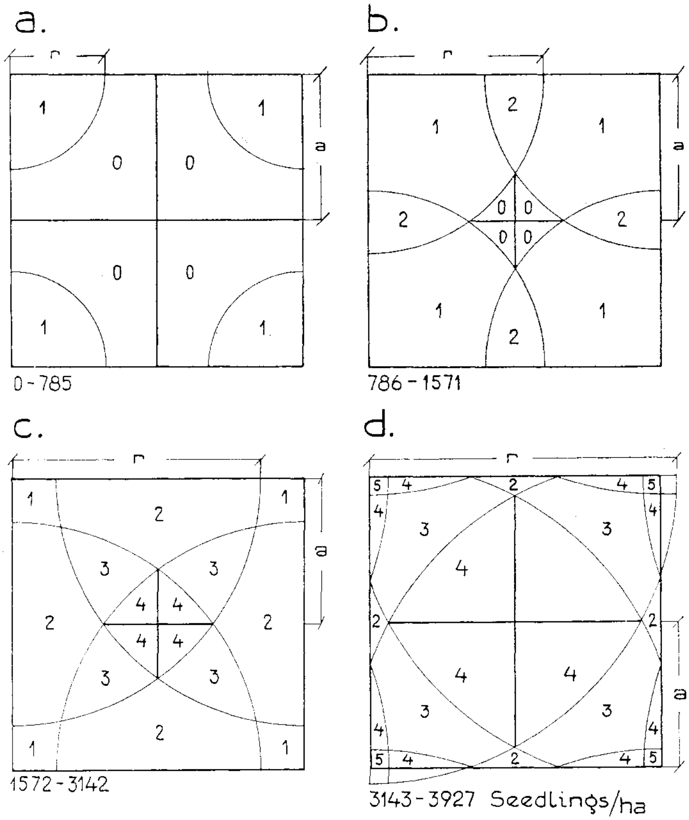

Fig. 6. Various types of configurations for the centre of a circular sample plot corresponding to different numbers of seedlings in the plot. In the respective areas the number of seedlings in the sample plot has been entered.

calculations of the areas. On account of the mechanism of location, it follows that the probabilities of respective numbers of seedlings occurring in the sample plot are determined by the ratio between the total area of figures allowed for the sample plot centre and the area of the lattice square. If the probability of i seedlings occurring in the sample plot is denoted by p_i and if $4Y_i$ is the area of the space allowed for the sample plot centre, we obtain for a 10 sq. m. sample plot from fig. 6a and b e.g.

4.2.1 $$\begin{cases} p_0 = \dfrac{Y_0}{a^2} = 1 - \dfrac{\pi}{4}\left(\dfrac{r}{a}\right)^2 \\ p_1 = \dfrac{Y_1}{a^2} = \dfrac{\pi}{4}\left(\dfrac{r}{a}\right)^2 \end{cases}$$ 0—785 seedlings per hectare

4.2.2 $$\begin{cases} p_0 = \dfrac{Y_0}{a^2} \\ p_1 = \dfrac{Y_1}{a^2} \\ p_2 = \dfrac{Y_2}{a^2} \end{cases}$$ 786—1,571 seedlings per hectare

By computation of p_i for various numbers of seedlings per hectare, the standard deviations have been obtained directly from the definition. The calculations become very tedious even for a figure of the type 6d. By means of quite different, mathematically more advanced methods, KENDALL (1948) computed the exact expression of the standard deviation sought; cf. also MATÉRN (1960) and KENDALL (1963). By means of the methods described by KENDALL (1948) the standard deviations computed according to the first of the procedures mentioned above have been checked. A standard deviation for 4,000 seedlings per hectare has also been computed with this method. For the purpose of completeness the expression is quoted from KENDALL (1948) as adapted to the symbols used in the present context.

4.2.3 $$\sigma^2 = \pi\left(\frac{r}{2a}\right)^2\left(1-\pi\left(\frac{r}{2a}\right)^2\right)+\sum_{t\leq\frac{r}{a}}\left(\frac{r}{2a}\right)^2(2\varphi-\sin 2\varphi)$$

where $\varphi = \arccos\left(\frac{at}{r}\right)$ and $t = 1,\ \sqrt{2},\ 2,\ \sqrt{5},\ \ldots$

The summation is carried out over those points of a lattice with meshes 1×1 which are situated in a circle with the radius r/a and centre in a point of the lattice. The centre point, is however not included in the total. For points of the lattice at a distance of 1 from the centre $t=1$ and for a distance of $\sqrt{2}$, $t=\sqrt{2}$ etc. The results are found in table 8.

4.3. Randomly distributed seedlings, standard deviations

From the assumptions stated in 2.5 it follows that the number of seedlings in the sample plot is distributed according to the Poisson law. If the area of the sample plot is A sq.m. and the density of seedlings is λ per sq.m., the mean value (= the parameter) of the Poisson distribution is λA, i.e. the mean number of seedlings per sq.m. of the sample plot is an unbiased estimate of λ. Since the variance of a Poisson distribution equals the mean, we thus obtain directly the standard deviations of the numbers of seedlings in plots 5 and 10 sq.m.

$\sigma = \sqrt{5\lambda}$ plots 5 sq.m.

$\sigma = \sqrt{10\lambda}$ » 10 » .

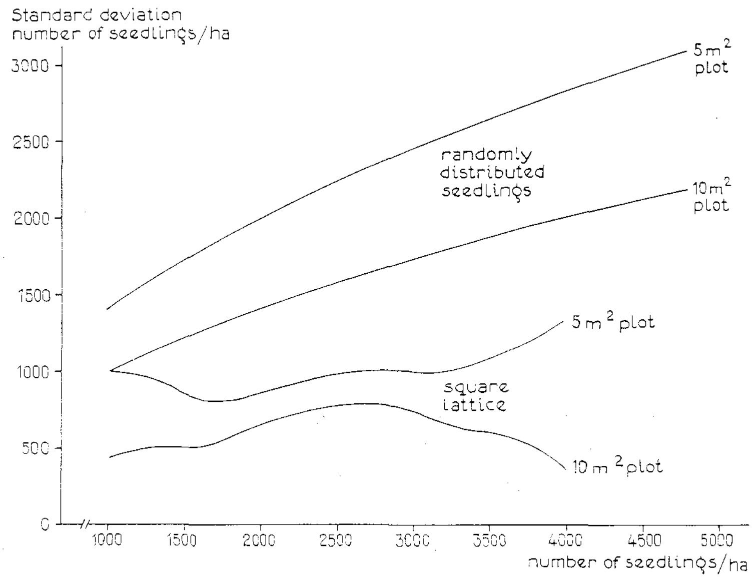

Fig. 7. Standard deviations of the number of seedlings in the circular sample plot.

These standard deviations have been tabulated for the same number of seedlings as in section 4.2 and they are shown in table 9. The standard deviations have been drawn in fig. 7.

Chapter 5. Estimation of the number of seedlings per hectare and a brief discussion of the occurrence of open spaces

5.1. Introduction

In this chapter it will first be shown how the observations of distances described previously can be used for estimating the number of seedlings per hectare. We shall also discuss the use of measurements of the distance to seedling no. 1 in order to illustrate the occurrence of open spaces. The data have been supplied by the Boxholm Company Ltd. and consist of results from ordinary sample plot inventories of 30 plantations in the South of Sweden. They have been supplemented with distance measurements from the sample plot centre to seedlings nos. 1—3, and from seedling no. 1 to its nearest neighbour. The seedlings recorded are so-called main seedlings and they meet the requirement that the distance between them should be $>$ 0.8 metres. This requirement has been established by the Company chief forester. Requirements of this type are connected with the problems of the optimum forest management and they should reasonably depend on local conditions. These problems will not be dealt with here. The circular plots are 10 sq.m. in size and have been distributed systematically. Their number has been chosen in such a way that, according to previous experiency the mean number of seedlings is estimated with a standard error of 10 per cent. As shown in column 6 of table 11, the standard error is considerably smaller in several cases in the present investigation. This is probably due to the fact that the sample plots are larger than those used in previous works and that the earlier empirical figures are partly obtained from natural regeneration stands; cf. further fig. 7 and the investigation in chapter 4. In addition to the estimates based on mean and median distances, some other estimates based on the squares of the distances have also been studied. These estimates are described by Matérn (1959). Essed (1957) studied the efficiency of estimates of the number of trees per hectare based on means and medians of the distances concerned. His investigation is concerned with old stands. In discussing the efficiency of various measurements he also considered the labour involved in determining which tree is no. 3, no. 4 etc., counted from the sample point.

According to ESSED the median distance from the sample point to tree no. 4 gives a practically acceptable estimate of the number of trees per hectare in the interval "systematic forests" to "random forests". We shall return to this matter in the following presentation. In the work by MATÉRN the Fisherian efficiency (cf. CRAMÉR (1945)), has been investigated also for the estimates based on the squares of the distances. It is there pointed out that an estimate should be "robust", i.e. it should have the property of giving reasonable results even for great deviations from the Poisson case. An expression based on the harmonic mean of the squares of the distances is surmised to possess this property. In 5.3 we shall discuss this expression further.

Another factor worthy of consideration is the technique of computation. It should be possible to calculate the estimate of the number of seedlings speedily, preferably in the field. For this reason two simple types of estimates utilizing the average distance to either seedling no. 1, no. 2, or no. 3 have been investigated. The two types are based on the assumptions that the seedlings are distributed at random and in square lattice, respectively. A discussion of the possibility of bias follows. The true population average is not known for any field. Therefore the estimate based on the average number of seedlings in the sample plots has been used as a substitute, with which the other estimates are compared. As mentioned in chapter 4, this average gives unbiased estimates if the sample plots are at random. The standard deviations of the sample plot means have been computed according to the formulas of simple random sampling. According to investigations by COCHRAN (cf. MATÉRN (1960) and COCHRAN (1963)) this procedure will probably overestimate the standard deviation of estimates from samples of the kind studied here. The risk that the method used for locating the sample plots will cause bias, is negligible since the net of sample plots is never entirely regular. If a distance of e.g. 20 metres between the plots is required and a boundary occurs after, say, 7 metres, the sample plot in the next line will have to be placed 13 metres from this boundary. If the population is periodic, which is the case if all the seedlings of a square lattice still exist, bias can occur if a completely regular net of sample plots happen to concur with the population; cf. COCHRAN (1963).

In addition to the works mentioned in chapter 1 concerning procedures for testing the hypothesis of random distribution of individuals, the following ones may be mentioned: PIELOU (1959) and GREIG-SMITH (1957). These works apply to ecological investigations, and are chiefly concerned with contagious distributions. Contagious distributions are those where the individuals of a population have a tendency to form clusters. Models of this kind have been used for descriptions of natural plant populations and animal populations. A comprehensive treatment of some contagious distributions is given by NEYMAN (1939), FELLER (1943), and WARREN (1962). In this presentation we shall not examine in detail the methods of testing hypotheses on the population structure but only compare the field data by means of the results arrived at in chapters 2—4. It will then appear that the structure of many of the plantations investigated at the time of estimation roughly equalled the random distribution; cf. the results of COOPER (1961) cited in the introduction to chapter 3. SKELLAM (1952) also gave some examples of changes of the same type in plant populations which were originally regular; cf. e.g. the discussion of Carex arenaria (L) in SKELLAM's work. See also TIRÉN (1950).

5.2. Some estimates for the number of seedlings per hectare

In 2.4.1 we found $E(R_1) = 0.7652a$. If the sample mean $\bar{r}_1$ is assumed to be sufficiently close to this value, we can use the ratio $\bar{r}_1/0.7652$ to estimate a. The total number of seedlings per hectare which is $2{,}500/a^2$ can then be estimated as

$$2{,}500\left(\frac{0.7652}{\bar{r}_1}\right)^2 = \frac{1{,}464}{\bar{r}_1^2}.$$

In a similar way estimates can be obtained from the other expressions of chapter 2. We now give a list of these estimates and also report the other ones mentioned in 5.1. These latter expressions are based on the means $\bar{w}_i$ of the squares of the observed distances to seedling no. i, and of the mean $\bar{h}_3$ of the reciprocals of the squares of the distances to seedling no. 3. The sample median of R_3 will in the list be denoted by z_3^*. The standard errors of the estimates are also given in the list. They are computed by methods described in 5.3 below.

Estimates of the number of seedlings per hectare based on the assumption of a square lattice of seedlings:

	Estimate based on		standard error:
5.2.1	$\bar{r}_1$ (metres):	$2{,}500\left(\frac{0.7652}{\bar{r}_1}\right)^2 = \frac{1{,}464}{\bar{r}_1^2}$;	$\frac{1{,}862}{a^2\sqrt{n}}$
5.2.2	$\bar{r}_2$:	$2{,}500\left(\frac{1.3991}{\bar{r}_2}\right)^2 = \frac{4{,}894}{\bar{r}_2^2}$;	$\frac{737}{a^2\sqrt{n}}$
5.2.3	$\bar{r}_3$:	$2{,}500\left(\frac{1.8164}{\bar{r}_3}\right)^2 = \frac{8{,}248}{\bar{r}_3^2}$;	$\frac{509}{a^2\sqrt{n}}$
5.2.4	z_3^* :	$2{,}500\left(\frac{1.8306}{z_3^*}\right)^2 = \frac{8{,}378}{(z_3^*)^2}$;	$\frac{999}{a^2\sqrt{n}}$

randomly distributed seedlings:

	Estimate based on		standard error:
5.2.5	$\bar{r}_1$ (metres):	$2{,}500\left(\frac{1.0000}{\bar{r}_1}\right)^2 = \frac{2{,}500}{\bar{r}_1^2}$;	$\frac{2{,}614}{a^2\sqrt{n}}$
5.2.6	$\bar{r}_2$:	$2{,}500\left(\frac{1.5000}{\bar{r}_2}\right)^2 = \frac{5{,}625}{\bar{r}_2^2}$;	$\frac{1{,}815}{a^2\sqrt{n}}$
5.2.7	$\bar{r}_3$:	$2{,}500\left(\frac{1.8750}{\bar{r}_3}\right)^2 = \frac{8{,}789}{\bar{r}_3^2}$;	$\frac{1{,}470}{a^2\sqrt{n}}$
5.2.8	z_3^* :	$2{,}500\left(\frac{1.8459}{z_3^*}\right)^2 = \frac{8{,}518}{(z_3^*)^2}$;	$\frac{1{,}899}{a^2\sqrt{n}}$
5.2.9	$\bar{w}_2$ (sq. m.):	$\frac{20{,}000}{\pi\bar{w}_2} = \frac{6{,}366}{\bar{w}_2}$;	$\frac{1{,}768}{a^2\sqrt{n}}$
5.2.10	$\bar{w}_3$:	$\frac{30{,}000}{\pi\bar{w}_3} = \frac{9{,}549}{\bar{w}_3}$;	$\frac{1{,}443}{a^2\sqrt{n}}$
5.2.11	$\bar{h}_3$:	$\frac{20{,}000}{\pi}\bar{h}_3 = 6{,}366\bar{h}_3$;	$\frac{2{,}500}{a^2\sqrt{n}}$.

5.3. The properties of the estimates

Firstly it should be noted that the estimates 5.2.1—5.2.10 are all asymptotically unbiased. The meaning of this statement is that when the number of sample values n is large, the bias of an estimate is small. For proving this property we use the expression of a function $g(x)$ in a Taylor series

$$g(x) = g(m) + (x-m)g'(m) + \frac{(x-m)^2}{2}g''(m) + \ldots$$

If X is a positive random variable with the mean m and the standard deviation σ, we can apply this Taylorian formula and get

5.3.1 $$E(g(X)) \approx g(m) + \sigma^2\frac{g''(m)}{2}$$

5.3.2 $$D(g(X)) \approx \sigma|g'(m)|.$$

We now use the formula 5.3.1 on the estimate 5.2.1.

$$E\left(2{,}500\left(\frac{0.7652}{\bar{r}_1}\right)^2\right) = 2{,}500 \cdot (0.7652)^2 E\left(\frac{1}{\bar{r}_1^2}\right).$$

Here the function $g(x)=1/x^2$. By putting the values $E(\bar{r}_1)=0.7652a$ and $D(\bar{r}_1)=\frac{0.2849a}{\sqrt{n}}$, cf. table 2, in 5.3.1 we get

$$E\left(\frac{1}{\bar{r}_1^2}\right)\approx\left(\frac{1}{0.7652a}\right)^2+\left(\frac{0.2849a}{\sqrt{n}}\right)^2\cdot\frac{3}{(0.7652a)^4}=\left(\frac{1}{0.7652a}\right)^2+0\left(\frac{1}{n}\right).$$

The approximate mean of the estimate thus becomes

$$E\left(2{,}500\left(\frac{0.7652}{\bar{r}_1}\right)^2\right)\approx\frac{2{,}500}{a^2}.$$

In the same way we can show the asymptotical unbiasedness of all the other estimates. By evaluating the means of the expressions 5.2.4 and 5.2.8 we use the fact that the median z^* of a sample of n values from a population with a median z and the frequency function $f(x)$ has the asymptotical mean z and asymptotical standard deviation $1/(2f(z)\sqrt{n})$; cf. Cramér (1945). Also it should be noted that $E(\bar{w}_i)=E(R_i^2)$ and $D(\bar{w}_i)=D(R_i^2)/\sqrt{n}$. Further $R_i^2=(2a^2/\pi)y$, where y is χ^2-distributed with $2i$ degrees of freedom under the assumption of a Poisson process. According to these properties we thus have $E(\bar{w}_i)=(2a^2/\pi)2i$ and $D(\bar{w}_i)=4a^2\sqrt{i}/(\pi\sqrt{n})$. For computing the mean of 5.2.11 we can use the exact expression for $E(1/R_3^2)=\pi/8a^2$ (cf. 2.7.4), so this estimate is unbiased also for a finite sample. The idea of 5.2.11 is the following: If a population is composed of a number of regions, each of which is random but with different densities, this type of estimate is still unbiased.

Now we will give some examples of how the standard errors of the estimates have been computed. By applying the expression 5.3.2 on 5.2.1 it follows that

$$D\left(2{,}500\left(\frac{0{,}7652}{\bar{r}_1}\right)^2\right)\approx 2{,}500\cdot(0.7652)^2\cdot\frac{0.2849a}{\sqrt{n}}\cdot\frac{2}{(0.7652a)^3}=\frac{1{,}862}{a^2\sqrt{n}}.$$

The standard error of 5.2.11 is obtained exactly by means of 2.7.4 and 2.7.5. We get

$$D(\bar{h}_3)=D(1/R_3^2)/\sqrt{n}=\sqrt{[E(1/R_3^4)-(E(1/R_3^2))^2]/n}=\pi/(8a^2\sqrt{n})$$

and

$$D\left(\frac{20{,}000}{\pi}\,\bar{h}_3\right)=\frac{2{,}500}{a^2\sqrt{n}}.$$

It may be of some interest to study the bias which would be obtained according to 5.2.5 – 5.2.11 if the seedlings are distributed in a square lattice. By applying the formulas 5.3.1 and 2.4.1 – 2.4.3 we obtain, after neglecting the second term, the following

$$E\left(\frac{2{,}500}{\bar{r}_1^2}\right) \approx 2{,}500\left(\frac{1}{0.7652\,a}\right)^2 = \frac{4{,}270}{a^2}$$

$$E\left(\frac{5{,}625}{\bar{r}_2^2}\right) \approx 5{,}625\left(\frac{1}{1.3991\,a}\right)^2 = \frac{2{,}874}{a^2}$$

$$E\left(\frac{8{,}789}{\bar{r}_1^2}\right) \approx 8{,}789\left(\frac{1}{1.8164\,a}\right)^2 = \frac{2{,}664}{a^2}$$

$$E\left(\frac{8{,}518}{(z_3^*)^2}\right) \approx 8{,}518\left(\frac{1}{1.8306\,a}\right)^2 = \frac{2{,}542}{a^2}$$

$$E\left(\frac{6{,}366}{\bar{w}_2}\right) \approx 6{,}366 \cdot \frac{1}{2a^2} = \frac{3{,}183}{a^2}$$

$$E\left(\frac{9{,}549}{\bar{w}_3}\right) \approx 9{,}549 \cdot \frac{3}{10a^2} = \frac{2{,}865}{a^2}.$$

Using the result 2.4.4 we obtain the mean of 5.2.11

$$E\left(\frac{20{,}000}{\pi}\bar{h}_3\right) = \frac{20{,}000}{\pi} E\left(\frac{1}{R_3^2}\right) = \frac{1{,}993}{a^2}.$$

From these expressions it follows that we can expect large bias by some of the estimates if the assumption on population structure is wrong. As was noted before, the true population mean is $2{,}500/a^2$ in all cases.

For later use we now also calculate a confidence interval for the number of seedlings per hectare based on the sample median z_3^* and on the assumption of randomly distributed seedlings. It was mentioned above that z_3^* has the asymptotical mean z_3 and the asymptotical standard deviation $1/(2f_3(z)\sqrt{n})$. Moreover the distribution of z_3^* is asymptotically normal; cf. CRAMÉR (1945). By applying the frequency function of R_3 according to 2.5.1

$$f_3(x) = 2\left(\frac{1}{2a}\right)^2 \pi x \frac{\left[\left(\frac{1}{2a}\right)^2 \pi x^2\right]^2}{2} e^{-\left(\frac{1}{2a}\right)^2 \pi x^2}$$

and the median $z_3 = 1.8459\ a$ according to table 4, we get the standard deviation of z_3^*

$$D(z_3^*) = 0.7009a/\sqrt{n}$$

The asymptotical normality of z_3^* gives the following approximate relation

$$P\left(1{,}846\,a - 2\frac{0{,}7009\,a}{\sqrt{n}} < z_3^* < 1{,}846\,a + 2\frac{0.7009\,a}{\sqrt{n}}\right) \approx 0.95.$$

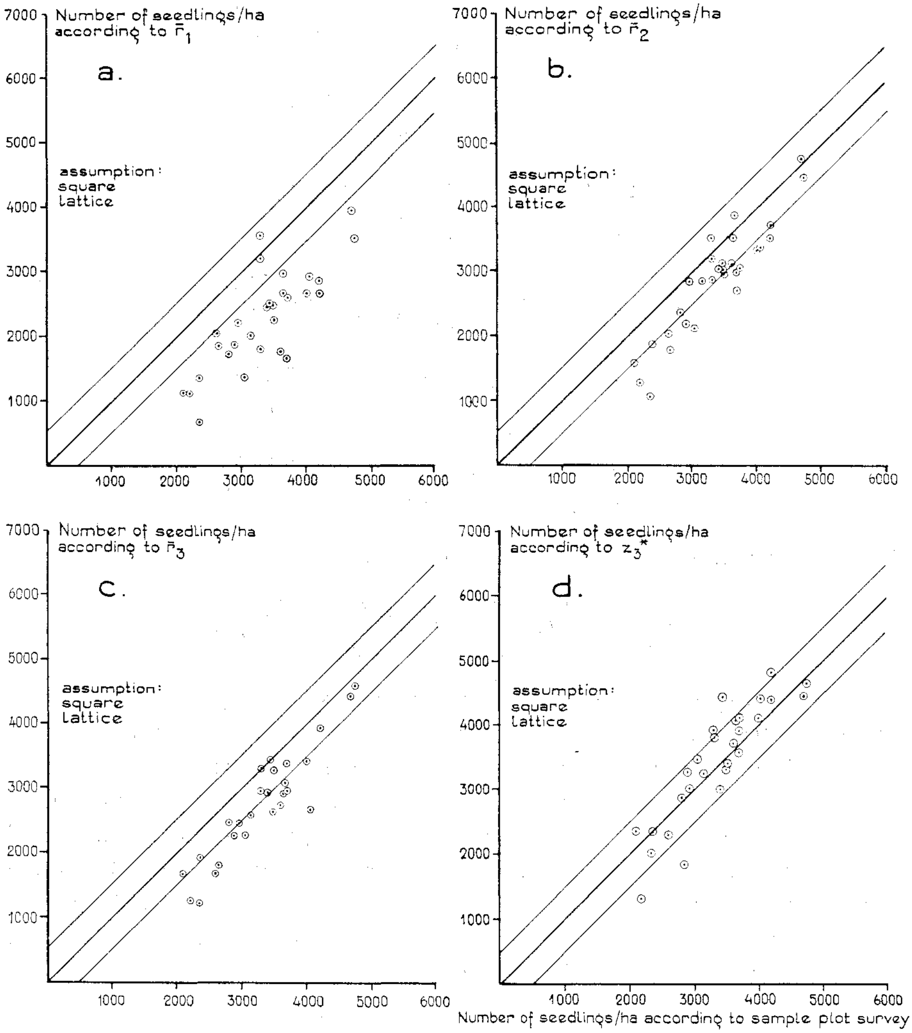

Fig. 8. Number of seedlings/ha according to different estimates; sample survey mad by the Boxholm Company Ltd.

After some transformations we have the approximate 95 % confidence interval for the number of seedlings per hectare

5.3.3 $$\left(\left(1.846-\frac{1.402}{\sqrt{n}}\right)\Big/ z_3^*\right)^2 2{,}500, \quad \left(\left(1.846+\frac{1.402}{\sqrt{n}}\right)\Big/ z_3^*\right)^2 2{,}500.$$

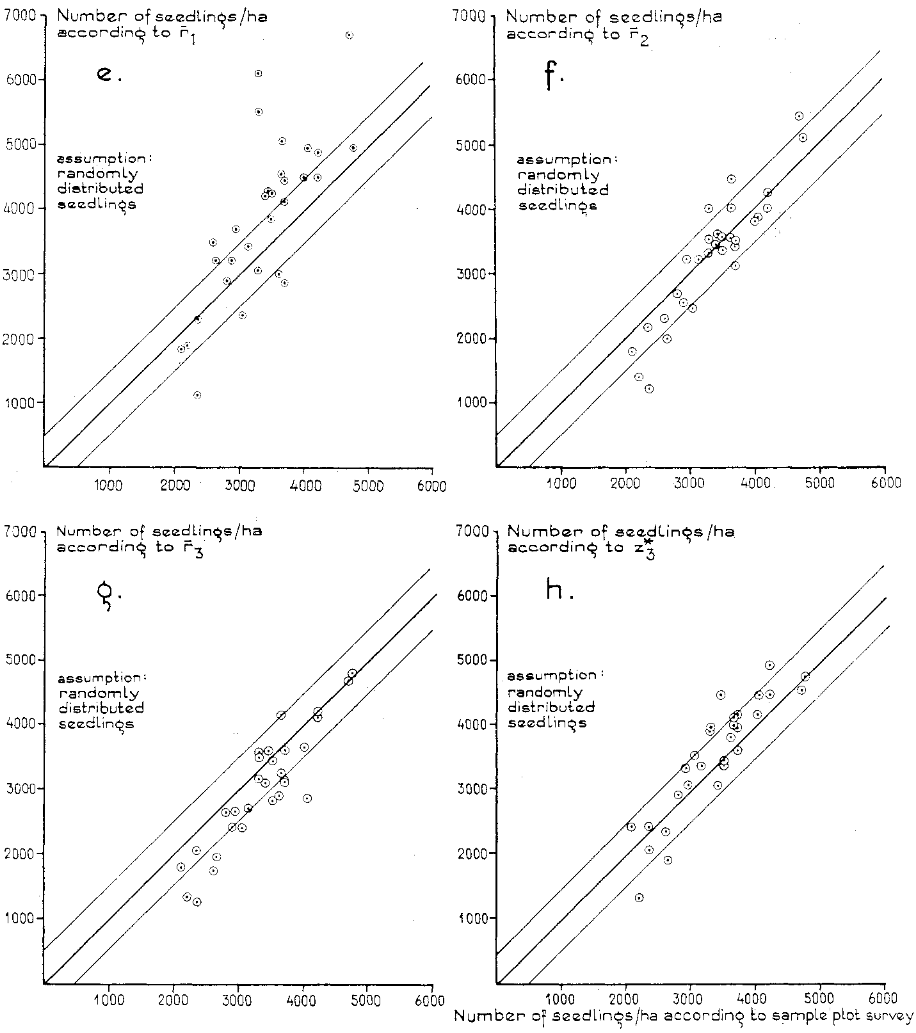

Fig. 8e—h.

5.4. Results of estimating the number of seedlings per hectare by using observed distances between sample points and seedlings

Now some results of the investigations of the data from the inventories described at the beginning of 5.1 will be given. In table 10 the mean distances to seedlings nos. 1—3 from the sample plot centre in each field are given. The table also gives the means $\overline{w}_i$ of the squares of the observed distances, and the median z_3^* of the distance from the sample plot centre to seedling no. 3. The median has been interpolated graphically in a diagram showing the cumu-

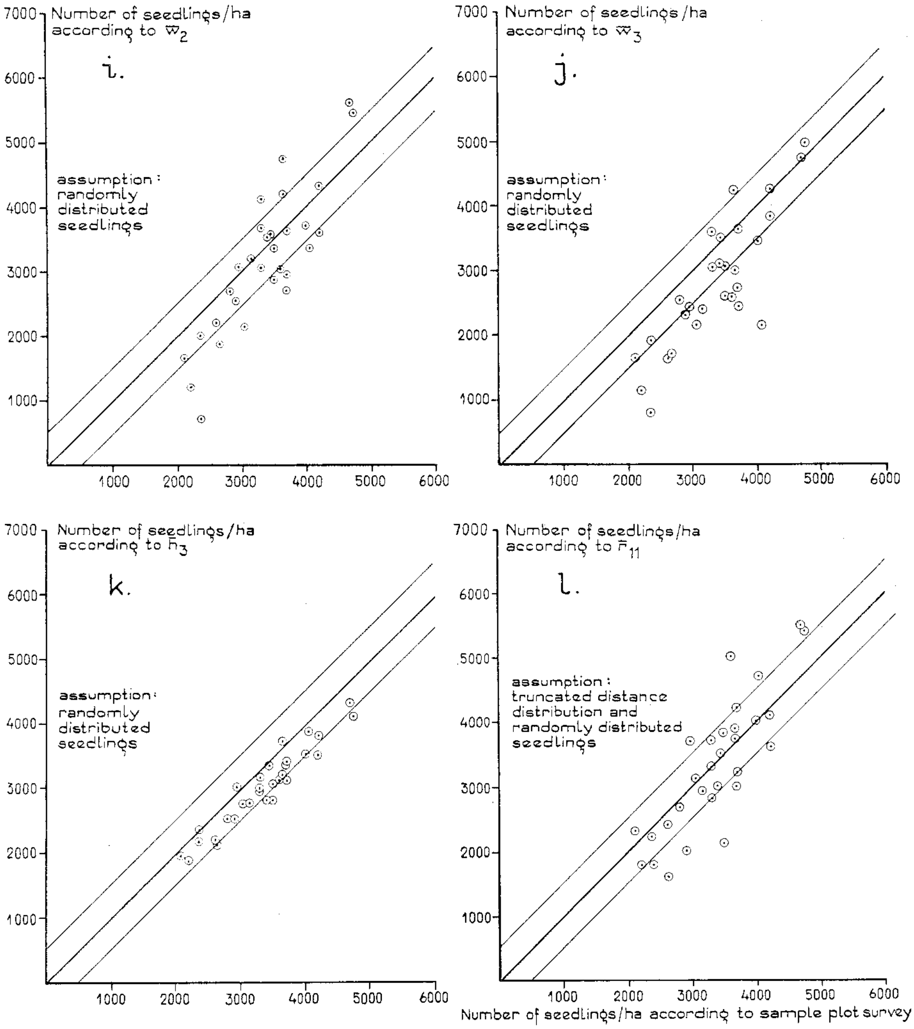

Fig. 8i—l.

lated frequencies of 10 cm classes. The column number 2 from the far right of the table contains the mean $\bar{h}_3$ of the reciprocals of the squares of the distances from the sample plot centre to seedling no. 3. From these data estimates of the number of seedlings per hectare have been obtained according to the expressions in 5.2. The results of applying the various estimates to the present data are found in table 11. The columns 4, 5, and 6 contain the numbers of seedlings and their estimated standard errors according to the sample plot inventory. All the cases investigated are shown graphically in fig. 8 a—k,

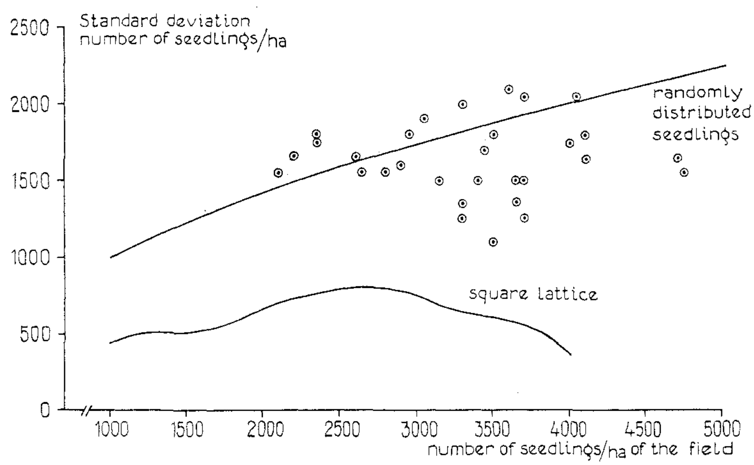

Fig. 9. Standard deviations of the number of seedlings of a 10 sq. m. circular sample plot; sample survey made by the Boxholm Company Ltd.

where the numbers of seedlings according to the sample plot survey have been used as abscissas. The outer lines in the figures demarcate a region obtained in the following way. About twice the average standard error of the sample plot mean is added to, respectively subtracted from the bisectris of the axes of the coordinate system. In fig. 9 the sample standard deviations of the numbers of seedlings of the individual sample plots have been presented together with the hypothetical ones according to the calculations carried out in chapter 4.

5.5. Discussion of the results of estimation

We proceed to a discussion of the results of the application of the various estimates; cf. table 11. Provided that the sample plot survey produced a correct result, the estimates based on the assumption of square lattice and on the mean distances have apparently underestimated the number of seedlings. The underestimation is greatest in the case with the nearest seedling.

The result obtained from an estimate based on the median of R_3 and under the assumption of square lattice seems to be almost unbiased. As shown by a comparison between the expressions 5.2.4 and 5.2.8, the difference between the estimated numbers of seedlings per hectare according to the assumptions of square lattice and random distribution is not great if the median is used; cf. also Essed (1957). Since the median can be easily estimated and since the estimated number of seedlings can then be rapidly obtained in the field, it is appropriate to study more closely the estimates based on the median. We shall therefore use the confidence interval 5.3.3 for the number

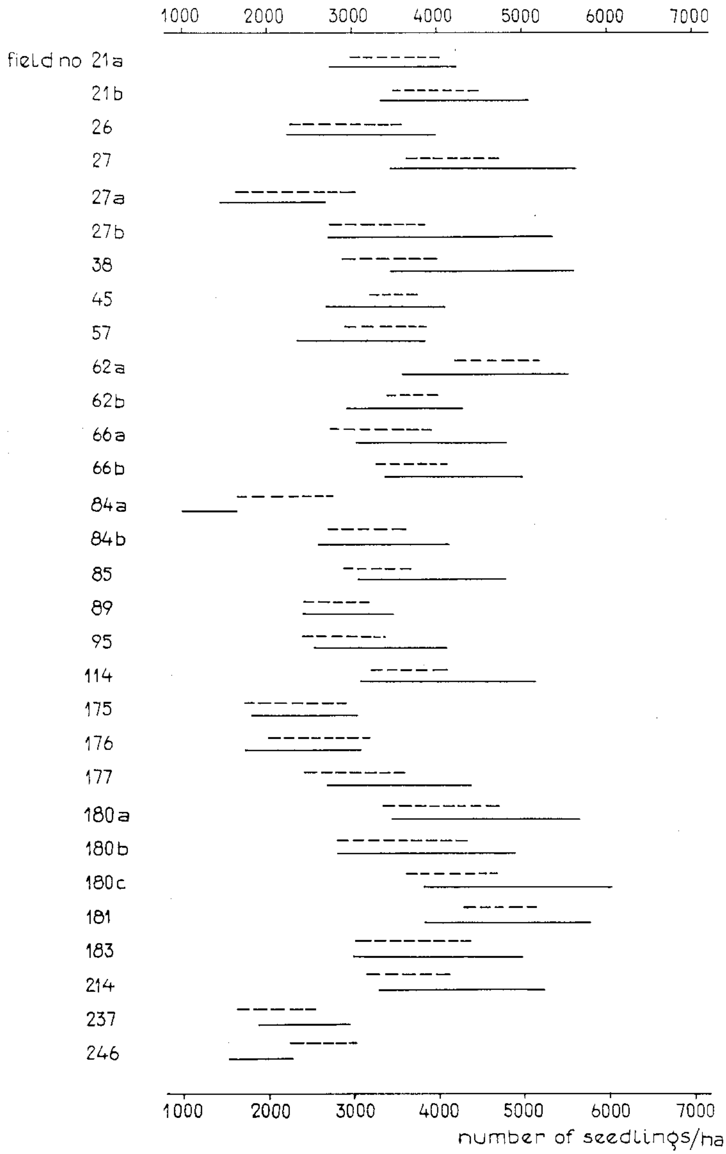

Fig. 10. Confidence intervals for the number of seedlings/ha; sample survey made by the Boxholm Company Ltd.
— — — — — according to sample plot survey.
———————— according to median distance between sample point and seedling no. 3 and the assumption of randomly distributed seedlings.

of seedlings per hectare. Table 12 shows such confidence intervals for the various fields and the corresponding ones according to the sample plot survey, $\bar{x} \pm 2\, s/\sqrt{n}$. They have also been drawn graphically in fig. 10 which shows that the ones based on the sample plots are about half as long as those based on the median.

Even in the cases with estimates based on the assumption of random distribution, the mean distance to the nearest seedling has given a rather erratic result although the bias is not as large as under the assumption of square lattice. In most cases the distances to seedlings nos. 2 – 3 give acceptable results. Deviations exceeding 1,000 seedlings relative to the sample plot survey do occur; cf. the standard errors in 5.2. The estimates 5.2.9 and 5.2.10 which are based on the maximum likelihood method (cf. CRAMÉR (1945)) have produced remarkably meagre results. A deviation from the assumption of a Poisson process results in over or underestimations. The estimate 5.2.11 has given the better result. As was mentioned at the beginning of the chapter, this estimate was surmised to be "robust". The investigation also corroborates this assumption; cf. fig. 8k. A weak tendency to underestimate the number of seedlings can however be discerned; cf. the bias of 5.2.11 computed in 5.3 under the assumption of a square lattice. Some of the estimates will be further discussed in chapter 6.

5.6. Results of estimating the number of seedlings per hectare by using the mean distance between seedling no. 1 and its nearest neighbour

Chapter 3 treated the truncated distribution of the distance between seedling no. 1 and its nearest neighbour. We shall now study the estimates of the number of seedlings according to the mean distance of this distribution. In some way we should utilize the knowledge of the seedlings not growing closer than 0.8 metres. Table 13 shows the estimated mean distances $\bar{r}_{11}$ of $E(R_{11})$ as well as the number of seedlings to be expected if the truncated distribution according to chapter 3 were to be applied. The numbers of seedlings have been taken from the curve in fig. 5. For purposes of comparison the results obtained in the sample plot survey have also been entered. Finally, the results are presented as above in fig. 8 l. According to this method the estimation of the number of seedlings is obviously uncertain.

5.7. Extent of open spaces

In chapter 1 it was mentioned that the distribution of the distance from sample point to seedling no. 1 describes the proportion of blank circles of various sizes. To get an idea of this in the material investigated, some of the fields inventoried (85, 114, 180 b, 183, and 214) have been treated as one

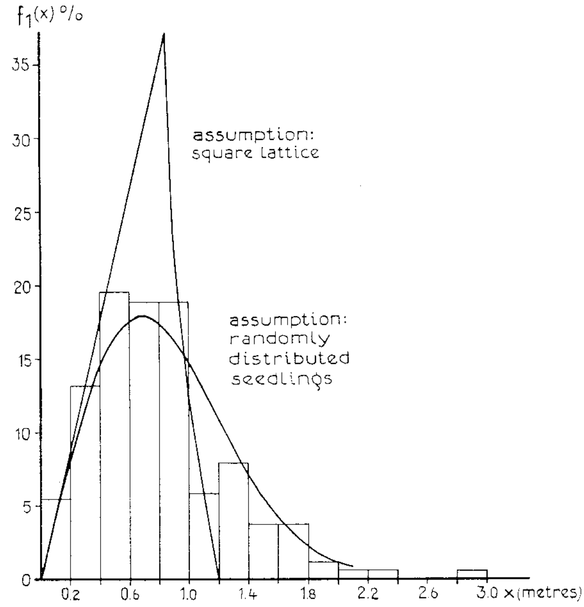

Fig. 11. Distribution of the distance between sample point and seedling no. 1 in a group of five fields; sample survey made by the Boxholm Company Ltd. The fitted frequency functions are theoretical ones according to the assumption of square lattice and of randomly distributed seedlings, respectively.

group. These fields are about equally large in area, ranging from 1.5 to 2.3 hectares and with numbers of seedlings estimated to range from 3,250 to 3,750 per hectare. By pooling them together a more accurate estimate of the distribution has been obtained. In table 14 the distances have been grouped into 20-cm classes and the relative frequencies have been computed. The theoretical frequencies according to the hypotheses of square lattice and random seedling distribution have also been presented for 3,500 seedlings per hectare.Histogram and theoretical distributions are to be found in fig. 11. As shown by both table and figure, the extent of open spaces better agrees with that found in a population of randomly distributed seedlings than in a square lattice.

Chapter 6. Some additional models of plantations

6.1. Introduction

In order to study how some of the estimates described in chapter 5 are affected by seedling mortality in a plantation with initially complete square lattice, a programme has been designed for the computer FACIT EDB. This programme vill now be briefly described before the results of the application are presented.

In the investigation by STRAND (1954) mentioned above p.8 maps have been used to exemplify tests of randomness. There is also the possibility of generating map figures corresponding to some model mechanism for the distribution of the seedlings and subsequently on these maps to apply manually the methods of sampling to be studied. This procedure will be very tedious. Such maps have been used by many of the ecologists; cf. e.g. COTTAM, CURTIS & HALE (1953), and PIELOU (1960).

Thanks to the modern electronic computers with large stores, entirely new possibilities are made available. In these stores the entire population can be placed, and the sampling is done very quickly. This technique was applied in a work by PALLEY & O'REGAN (1961) for comparison between point sampling and line sampling. Their investigation dealt with characteristics of growing timber. Various properties of the trees were stored in an IBM 701 computer where the computations were then carried out.

6.2. Model construction and programming

The approximate appearance of the seedling mortality has been studied on some maps of plantations that have been regularly observed by the Department of Regeneration at the College of Forestry. It has then been considered feasible to make possible both individual mortality in the lattice and a combination of individual and group mortality. Groups comprising three seedlings in a row or five seedlings crosswise have been chosen. The individual mortality has been obtained by uniform allotment one by one of the seedlings lost and the group mortality by a corresponding allotment of the clusters. The groups of three seedlings should cause more elongated open spaces than do the groups of five seedlings crosswise. The elongated open spaces correspond to situations where planting has failed for some reason. Rather more round open spaces, may however be thought of as representing damages caused by animals, diseases or other agents. In the combination of individual and grouped mortality one half of the seedlings lost have been removed one by one. The total seedling mortality percentage may be chosen arbitrarily. In the cases with group mortality, the individual seedlings have been allotted first, and then the groups. If a group happens to cover seedlings already dead, new groups have been allotted until the total mortality desired is obtained. Large calamity open spaces are of minor interest from the point of sampling methodology since they are detected without sampling, but they can be studied by a choice of high seedling mortality.

The programme has been constructed after entering the undisturbed plantation into a system of coordinates with the lattice mesh side as a unit. By a choice of this unit lattices of varying density up to 900 seedlings can be obtained corresponding to various numbers of seedlings per hectare. Each seedling in the lattice has its fixed place in the machine store. By letting e.g. a store cell be empty or contain a figure, a dead or alive seedling respectively may be indicated. By means of a standard programme for producing random numbers, the places of the dead seedlings have been determined. In the groups the choice concerned the centre seedling. A map of the model can be written by the typewriter of the computer. By producing a new set of random numbers, sampling points have then been located at a number of maximum 200. On the same model maximum 15 samples can be obtained. Circular plots of various sizes can be located around the sampling points, the largest one determining the area of locating the random numbers; cf. 6.3 (below). For these sample plots the machine then records the number of seedlings and computes their mean and variance. Furthermore the distances to seedling nos. 1—3 are computed. Also the mean of these distances as well as their variances are computed. The distances to seedling no. 1 are finally grouped into 10-cm classes and the observed distribution function is printed. To test the programme samples have been taken on an undisturbed lattice of 2×2 sq.m. by means of 25, 50, 100, and 200 points. The result shows good agreement with previous, theoretically computed values and is presented in table 15. Fig. 12 shows the observed distribution function as well as the theoretical one obtained from chapter 2.

6.3. Border effect on the result of estimation

On account of the limited capacity of the machine only plantations with a maximum of 900 planted seedlings have been studied. If the lattice is dense, the planted area will be small in relation to the circular plots. A certain influence of the border zone can then be expected. One way of avoiding

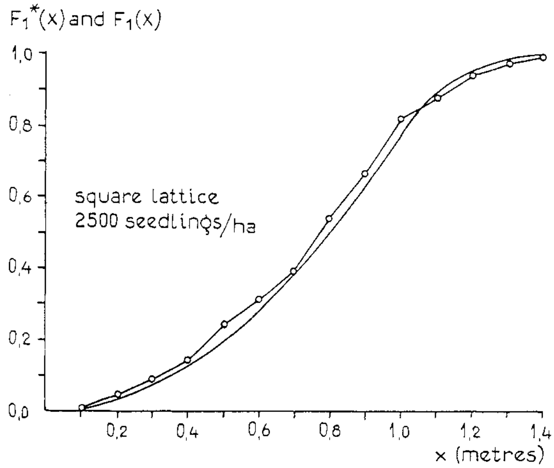

Fig. 12. Sample distribution function $F_1^*(x)$ for the distance between sample point and seedling no. 1 (the polygon line) and the corresponding theoretical one, $F_1(x)$, according to chapter 2 (the smooth curve). Result from testing the programme for artificial sampling in the computer FACIT EDB (2,500 seedlings/ha).

bias in the sample plot surveys on account of this influence of the border is to choose a large area around the area of investigation and allot the sample points uniformly over this larger area; cf. e.g. MASUYAMA & SENGUPTA (1955). To avoid a special investigation of the marginal zone plots, we have chosen to locate the sampling points randomly inside a zone with a breadth of the radius of the 10 sq.m. plot. Bias caused by this procedure of allotment in the cases investigated below has been found to underestimate the number of seedlings by less than 200 seedlings per hectare, probably considerably less.

No theoretical investigations have been made in this paper concerning the influence of the border zone on the mean distance to the seedlings from the sampling points. Bias (if any) in the results obtained in estimating the number of seedlings per hectare according to these distances may thus be influenced by the border zone.

6.4. Results of the model sampling

As already mentioned, ten different models have been constructed. All the populations consist of 900 seedlings in square lattice before mortality. The first four populations have meshes of the square lattice of 2×2 sq.m. and in the other six the seedlings grow in a lattice with meshes of 1.6734×1.6734 sq.m. In all the models about 30 per cent of the seedlings have been "killed", which results in about 1,750 seedlings per hectare in the first four models and about 2,500 seedlings per hectare in the other six. A summary of

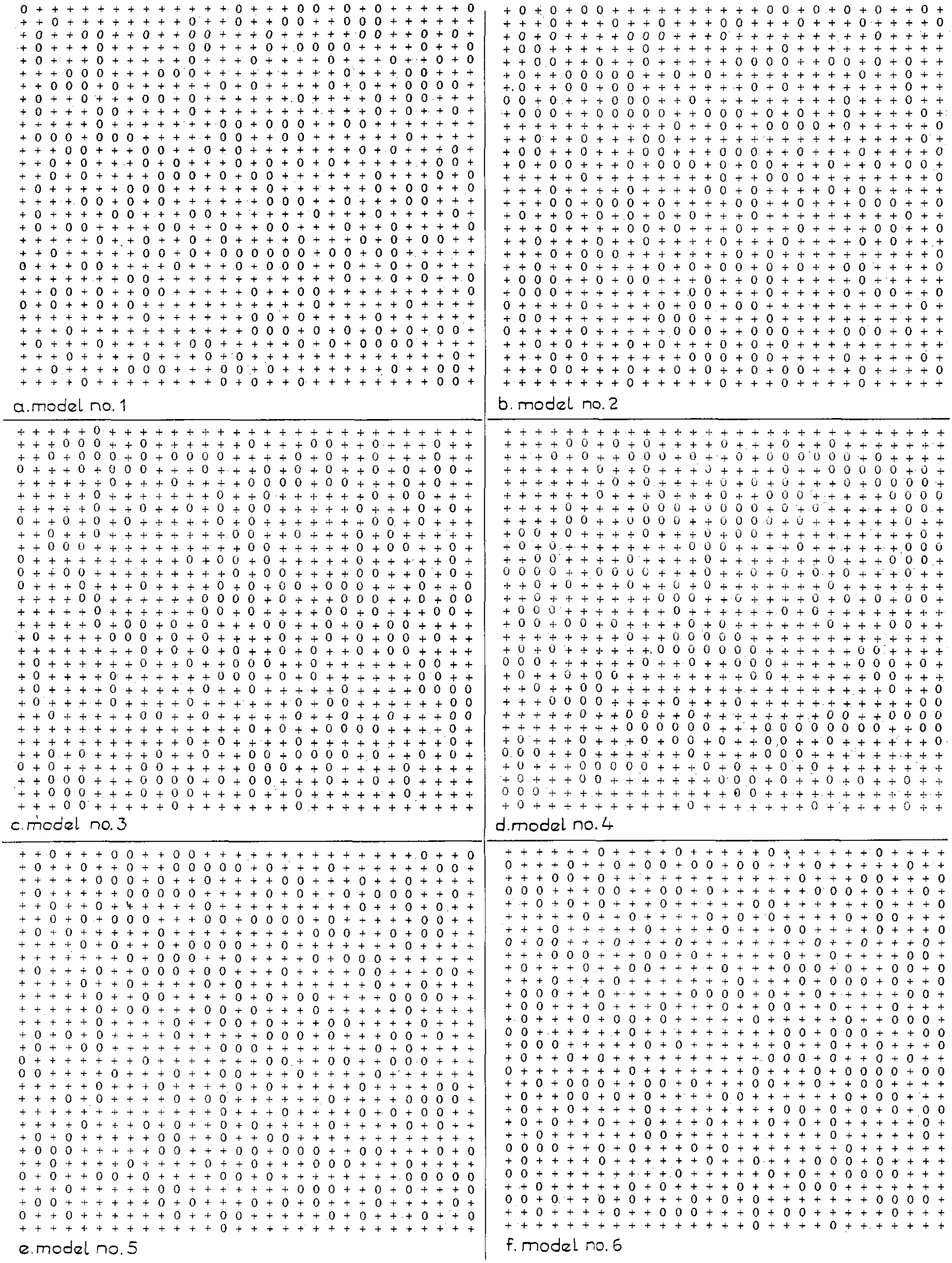

Fig. 13. Maps printed by the computer FACIT EDB (artificial sampling).

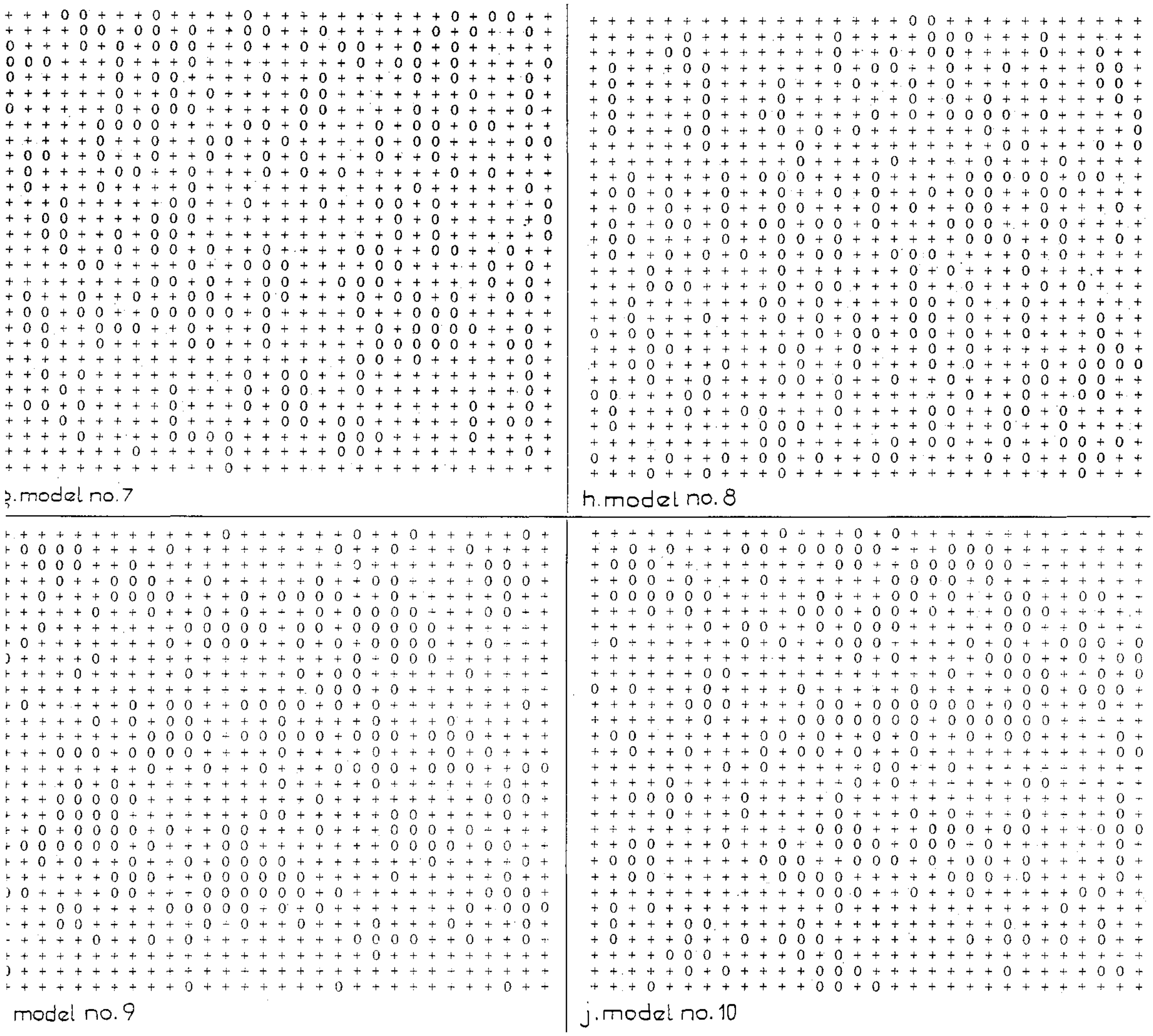

Fig. 13. g—j.

the models and a presentation of the seedling mortality are given in table 16. The maps printed by the machine are found in fig. 13 a—j (0 means dead seedling).

The models nos. 1, 3, 4, 5, 7, and 9 have been studied by 15 samples each comprising 25 randomly selected sampling points. The selection was made as described in section 6.3. The purpose is still to study the mean number of seedlings per hectare obtained according to a sample plot survey and according to the mean distances to the seedlings nos. 1—3 under the assumptions of square lattice and random distribution. In table 17 these assumptions are called 1 and 2, respectively. The models nos. 2, 6, 8, and 10 have been studied by 10 samples comprising 25 sampling points and 5 samples of 200 points. The large samples have been used in order to give a picture of how the distribution function for the distance to seedling no. 1 is changed by the open spaces in the plantation. This distribution function has also been estimated for the small samples. The estimated numbers of seedlings are found in table 17. Only group means and ranges of the sample means of the models have been presented. The distribution functions of the large samples are given in table 18 and they are drawn in fig. 14 a—d, as well as the corresponding theoretical ones for a square lattice of 2×2 sq.m. without open spaces. Only the means of the five samples are shown. A 95 per cent confidence region for the distribution function of the distance to seedling no. 1 for 200 sample points has also been drawn for model no. 2. This presentation has been made by demarcating twice the standard deviation

$$\sqrt{\frac{F_1^*(x)(1-F_1^*(x))}{200}}$$

both upward and downward from the mean curve observed. Table 19 shows observed values of percentages of open spaces. They are obtained from samples of 25 points.

6.5. Discussion of the results of estimation

We now proceed to a brief summary of the information given in tables 17 – 19. With respect to the number of seedlings, the sample plot surveys have produced nearly unbiased estimates. The dispersion of the means has been estimated by means of the range, a division into small samples being made. Such a calculation reveals that the standard deviation of the mean values varies from about 200 seedlings to about 400 seedlings per hectare. For sample plots of 10 sq.m. size, the standard deviation is more uniform—about 200 seedlings per hectare. There is no evidence that dispersion would be essentially greater in cases where half of the seedling mortality was in groups. The estimates based on the distance to the seedlings nos. 1—3, largely confirm the results of the investigations in chapter 5, underestimations by use of seedling no. 1 and the assumption of square lattice, overestimations under the assumption of random distribution. Seedling no. 3 has produced a slight bias and the standard deviations are remarkably small i.e. about 200 seedlings per hectare. As far as can be judged by these sampling experiments, the mean distance to seedling no. 3 and some of the assumptions

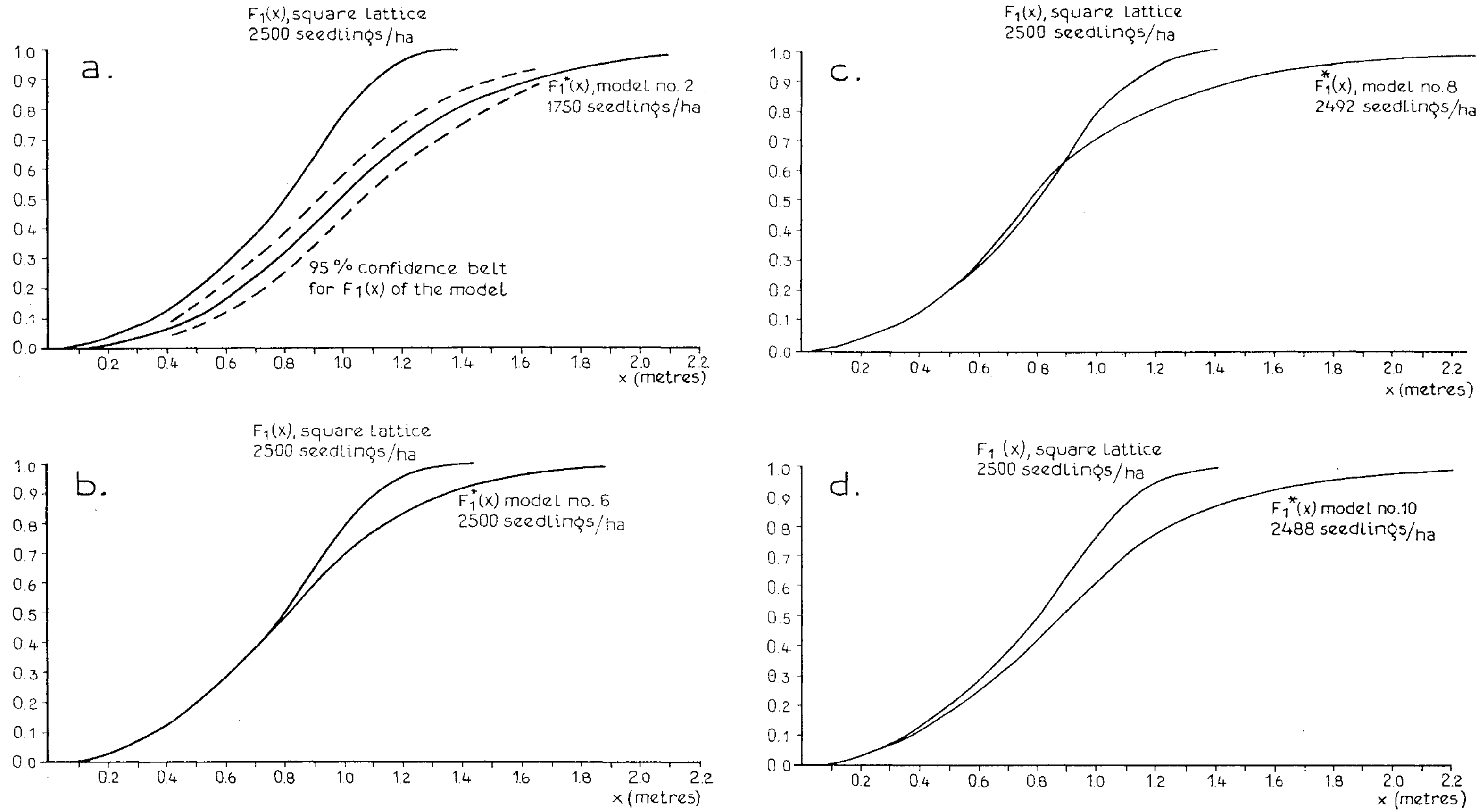

Fig. 14. Sample distribution functions $\overset{*}{F}_1(x)$ for the distance between sample point and seedling no. 1 and the theoretical distribution function $F_1(x)$ for this distance in original square lattice; results from artificial sampling in the computer FACIT EDB.

studied should be advantageously used as a basis for estimating the number of seedlings per hectare if the seedling mortality is not more uneven than that in the present models. The distribution functions largely show what was expected, viz. the distances increase when there are open spaces. The observed relative frequencies of open spaces in table 19 show that in order to obtain dependable estimates of the frequency of open spaces we are in need of large samples; cf. also fig. 14a.

Sammanfattning

Principen för uppskattning av individantalet i en växtpopulation genom avståndsmätning är enkel. Om populationen är individrik bör avstånden mellan närstående individer bli korta, åtminstone i genomsnitt, är individantalet litet bör de bli långa. Samma sak gäller avstånden från sampelpunkter till närstående plantor. För att undersöka möjligheterna att tillämpa dessa principer vid kontroll av planteringar har denna undersökning utförts.

I kapitel 1 redogöres för ett antal arbeten som behandlar frågor rörande avståndsmätning i växtpopulationer. I de skogligt orienterade arbetena har den centrala frågan varit att uppskatta antalet plantor eller träd per arealenhet. En annan grupp av arbeten, med ekologisk inriktning, har däremot behandlat frågor om hur plantorna står fördelade, gruppställdhet kontra s. k. slumpmässig plantfördelning.

Till grund för de studier som gjorts i detta arbete ligger två olika antagan den om plantornas fördelning i rummet. I det ena antages att plantorna står i ett kvadratgitter, den fördelning man eftersträvat vid planteringen, i det andra att de är fördelade enligt en s. k. Poissonprocess. Denna senare fördelning överensstämmer med den som brukar kallas slumpmässig. I bl. a. ekologiska arbeten finns beskrivet hur från början regelbundet fördelade växtpopulationer efter konkurrens och annan påverkan övergår i mer eller mindre slumpmässiga populationer. Det är bl. a. av detta skäl den slumpmässiga plantfördelningen studerats. Ett annat är att den leder till enkla räkningar.

I kapitel 2 behandlas fördelningarna för avstånden från en slumpmässigt vald sampelpunkt till närmaste, näst närmaste och tredje planta i ordning från sampelpunkten. Dessa plantor kallas i fortsättningen planta 1, 2 resp. 3. I kvadratgitterfallet har karakteristikor till dessa fördelningar beräknats av Essed (1957). Under antagandet om slumpmässig plantfördelning har både fördelningar och karakteristikor till de olika avstånden behandlats av bl. a. Skellam (1952), Morisita (1954) och Thompson (1956). De i kapitlet härledda fördelningarna och några av deras karakteristikor har tabellerats och uppritats för ett plantantal av 2 500/ha. Se tabell 1, 2, 3 och 4 samt figur 3.

Kapitel 3 behandlar fördelningen för avståndet mellan planta 1 och dess närmaste grannplanta under antagandet om slumpmässig plantfördelning. Enligt denna fördelning har tabell 5 uppgjorts med hjälp av datamaskin. Ur tabellen har den stympade fördelningen för detta avstånd beräknats och sammanställts i tabell 6. I tabell 7 har även medelavstånden för såväl den ursprungliga som stympade fördelningen sammanställts. Medelavstånd för den ursprungliga fördelningen har beräknats tidigare av MATÉRN (1959).

I kapitel 4 behandlas standardavvikelsen till plantantalet på en slumpmässigt utplacerad cirkulär provyta. Därvid utnyttjas bl. a. ett resultat av KENDALL (1948). Standardavvikelserna under de båda antagandena om populationsstrukturen och olika plantantal/ha har sammanställts i tabell 8 och 9 samt figur 7.

Kapitel 5 behandlar en tillämpning av resultaten från kapitel 2, 3 och 4 vid uppskattningen av plantantalet/ha i några planteringar. Materialet utgör resultaten från samplingundersökningar av 30 planteringar vid Boxholms bruk. Vid samplingen har planträkning på provytor utförts. Dessutom har avstånden mellan provytecentrum och planta 1, 2 resp. 3 samt mellan planta 1 och dess närmaste grannplanta mätts. Utöver de uppskattningar som kan erhållas genom utnyttjande av medelavstånd har ytterligare några på avståndens kvadrater grundade prövats. Dessa finns beskrivna i tidigare litteratur. Vid bearbetningen har provyteresultaten använts för jämförelsen av de olika uppskattningarna. I tabell 11 och figur 8 a – l är resultaten sammanställda.

Här skall nu några resultat av undersökningarna i kapitel 5 anges. På avstånden till planta 1 grundade uppskattningar är mycket osäkra. Ett felaktigt antagande om populationsstruktur kan ge en »bias» på över 40 %. Om däremot avstånden till planta 3 används, finns goda möjligheter till praktiskt taget medelvärdesriktiga uppskattningar av tätheten i populationer av det slag som studerats i detta arbete. Detta bekräftar tidigare undersökningsresultat av ESSED (1957). Speciellt lämpliga tycks uppskattningar grundade på medianavståndet till planta 3 vara. Dessa uppskattningar uppvisar emellertid en stor variation. Se tabell 12 och figur 10. En ökning av antalet sampelpunkter till det fyrdubbla av det vid användande av 10 m² provytor torde vara nödvändigt om samma noggrannhet som i provyteuppskattningarna skall erhållas. Uppskattningar grundade på medeltalet av avståndens inverterade kvadrater har gett mycket goda resultat. Denna typ av uppskattning har föreslagits av MORISITA (1957) och MATÉRN (1959). Till sist har uppskattningar grundade på den stympade fördelningen för avståndet mellan planta 1 och dess närmaste grannplanta gjorts. Dessa är, liksom de på avstånden mellan sampelpunkt och planta 1 grundade, osäkra. Som avslutning på kapitel 5 ingår ett studium av luckornas utbredning. Därvid har avståndet mellan sampelpunkten och planta 1 använts. Fem likartade planteringar har sammanslagits och histogram över avståndsfördelningen uppritats. Till detta har anpassats frekvensfunktioner grundade på antagandet om dels kvadratgitter, dels slumpmässig plantfördelning. Resultaten finns sammanställda i tabell 14 och figur 11. Som framgår överensstämmer den observerade fördelningen bäst med den som skulle väntas i en population med slumpmässig plantfördelning.

Kapitel 6 behandlar en tillämpning av ett program som uppgjorts för datamaskinen FACIT EDB. Med hjälp av detta kan modeller av luckiga planteringar konstrueras. Genom artificiell sampling på några sådana modeller i maskinen har några av de i kapitel 5 undersökta uppskattningsuttrycken prövats. Resultaten överensstämmer i stort sett med dem från kapitel 5. Se tabell 17. Utöver plantantalen har även fördelningarna för avstånden till planta 1 uppskattats genom frekvenskvoter. Se tabell 18 och figur 14. Osäkerheten i en sådan uppskattning har åskådliggjorts i figur 14 a. För säkra uppskattningar av luckornas utbredning krävs ett ansenligt antal sampelpunkter. Osäkerheten vid små stickprov framgår dessutom av tabell 19.

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TABLES

Tab. 1. Distribution functions of the distances between sample point and seedlings; square lattice (2,500 seedlings/ha).

Note: If the table is used for other numbers of seedlings/ha, the argument x must be changed. E.g. for 3,600 seedlings/ha $a = \frac{1}{2}\sqrt{1/0{,}36} = 0.8333$ and all the x-values must be multiplied by 0.8333.

x (metres)	$F_1(x)$	$F_2(x)$	$F_3(x)$
0.0	0.0000		
0.1	0.0079		
0.2	0.0314		
0.3	0.0707		
0.4	0.1257		
0.5	0.1963		
0.6	0.2827		
0.7	0.3848		
0.8	0.5027		
0.9	0.6362		
1.0	0.7854	0.0000	
1.1	0.8886	0.0617	
1.2	0.9509	0.1801	
1.3	0.9865	0.3408	
1.4	0.9998	0.5396	
1.5		0.7096	0.0503
1.6		0.8230	0.1544
1.7		0.9038	0.2884
1.8		0.9584	0.4467
1.9		0.9898	0.6266
2.0		1.0000	0.8264
2.1			0.9496
2.2			0.9967

Tab. 2. Characteristics of the distances between sample point and seedlings; square lattice.

	R_1	R_2	R_3
$E(R_i)$	0.7652 a	1.3991 a	1.8164 a
σi	0.2849 a	0.2062 a	0.1848 a
$z i$	0.7979 a	1.3815 a	1.8306 a

Tab. 3. Distribution functions of the distances between sample point and seedlings; randomly distributed seedlings.

x (metres)	$F_1(x)$	$F_2(x)$	$F_3(x)$
0.0	0.0000	0.0000	0.0000
0.1	0.0078	0.0000	0.0000
0.2	0.0309	0.0005	0.0000
0.3	0.0682	0.0024	0.0001
0.4	0.1181	0.0073	0.0003
0.5	0.1783	0.0169	0.0011
0.6	0.2463	0.0332	0.0031
0.7	0.3194	0.0575	0.0071
0.8	0.3951	0.0910	0.0146
0.9	0.4707	0.1340	0.0268
1.0	0.5441	0.1860	0.0453
1.1	0.6134	0.2460	0.0714
1.2	0.6773	0.3123	0.1059
1.3	0.7348	0.3828	0.1492
1.4	0.7855	0.4553	0.2011
1.5	0.8292	0.5273	0.2606
1.6	0.8661	0.5969	0.3262
1.7	0.8967	0.6621	0.3959
1.8	0.9215	0.7218	0.4676
1.9	0.9413	0.7749	0.5389
2.0	0.9568	0.8210	0.6078
2.1	0.9687	0.8602	0.6724
2.2	0.9777	0.8927	0.7313
2.3	0.9843	0.9191	0.7837
2.4	0.9892	0.9401	0.8291
2.5	0.9926	0.9564	0.8674
2.6	0.9951	0.9688	0.8991
2.7	0.9967	0.9781	0.9246
2.8	0.9979	0.9848	0.9447
2.9	0.9986	0.9897	0.9602
3.0	0.9991	0.9931	0.9719
3.1		0.9955	0.9805
3.2		0.9971	0.9867
3.3		0.9982	0.9911
3.4		0.9989	0.9942
3.5		0.9993	0.9962
3.6			0.9976
3.7			0.9985
3.8			0.9991

Tab. 4. Characteristics of the distances between sample point and seedlings; randomly distributed seedlings.

	R_1	R_2	R_3
$E(R_i)$	1.0000 a	1.5000 a	1.8750 a
σi	0.5227 a	0.5445 a	0.5514 a
zi	0.9394 a	1.4618 a	1.8459 a

Tab. 5. Distribution function of the distance between seedling no. 1 and its nearest neighbour; randomly distributed seedlings.

x (metres)	$F_{11}(x)$ at various numbers of seedlings/ha							
	1,500	2,000	2,500	3,000	3,500	4,000	4,500	5,000
0.1	0.0024	0.0032	0.0040	0.0049	0.0057	0.0065	0.0074	0.0082
0.2	0.0099	0.0132	0.0166	0.0200	0.0235	0.0269	0.0303	0.0338
0.3	0.0266	0.0303	0.0381	0.0459	0.0538	0.0616	0.0695	0.0774
0.4	0.0407	0.0547	0.0686	0.0827	0.0966	0.1106	0.1245	0.1383
0.5	0.0643	0.0861	0.1080	0.1297	0.1512	0.1724	0.1935	0.2142
0.6	0 0931	0.1245	0.1554	0.1859	0.2158	0.2451	0.2736	0.3015
0.7	0.1271	0.1691	0.2101	0.2499	0.2884	0.3256	0.3613	0.3955
0.8	0.1657	0.2191	0.2705	0.3196	0.3663	0.4105	0.4522	0.4915
0.9	0.2084	0.2736	0.3352	0.3928	0.4465	0.4962	0.5421	0.5844
1.0	0.2547	0.3315	0.4024	0.4672	0.5261	0.5794	0.6274	0.6704
1.1	0.3037	0.3914	0.4703	0.5405	0.6026	0.6572	0.7049	0.7465
1.2	0.3548	0.4522	0.5372	0.6107	0.6737	0.7272	0.7726	0.8109
1.3	0.4071	0.5125	0.6016	0.6761	0.7377	0.7883	0.8297	0.8634
1.4	0.4597	0.5712	0.6622	0.7354	0.7937	0.8398	0.8760	0.9044
1.5	0.5119	0.6274	0.7179	0.7878	0.8412	0.8818	0.9123	0.9352
1.6	0.5629	0.6801	0.7680	0.8330	0.8805	0.9150	0.9397	0.9574
1.7	0.6121	0.7286	0.8121	0.8710	0.9121	0.9404	0.9598	0.9730
1.8	0.6588	0.7726	0.8503	0.9023	0.9367	0.9593	0.9739	0.9834
1.9	0.7027	0.8119	0.8826	0.9274	0.9555	0.9729	0.9836	0.9901
2.0	0.7432	0.8463	0.9093	0.9471	0.9694	0.9824	0.9900	0.9943
2.1	0.7803	0.8760	0.9311	0.9622	0.9794	0.9889	0.9940	0.9968
2.2	0.8138	0.9013	0.9485	0.9735	0.9865	0.9932	0.9966	0.9983
2.3	0.8437	0.9224	0.9621	0.9818	0.9913	0.9959	0.9981	0.9991
2.4	0.8700	0.9397	0.9726	0.9877	0.9945	0.9976	0.9990	0.9995
2.5	0.8929	0.9538	0.9805	0.9919	0.9967	0.9986	0.9994	0.9998
2.6	0.9127	0.9651	0.9863	0.9947	0.9980	0.9992	0.9997	0.9999
2.7	0.9294	0.9739	0.9906	0.9966	0.9988	0.9996	0.9999	
2.8	0.9435	0.9808	0.9936	0.9979	0.9993	0.9998		
2.9	0.9552	0.9860	0.9957	0.9987	0.9996	0.9999		
3.0	0.9649	0.9899	0.9972	0.9992	0.9998			
3.1	0.9727	0.9929	0.9982	0.9995	0.9999			
3.2	0.9790	0.9950	0.9989	0.9997				
3.3	0.9840	0.9966	0.9993	0.9999				
3.4	0.9879	0.9977	0.9996					
3.5	0.9910	0.9984	0.9997					
3.6	0.9933	0.9990	0.9998					
3.7	0.9951	0.9993	0.9999					
3.8	0.9964	0.9996						
3.9	0.9974	0.9997						
4.0	0.9982	0.9998						

Tab. 6. Truncated distribution function of the distance between seedling no. 1 and its nearest neighbour.

x (metres)	$F_{11}(x \mid R_{11} > 0.8)$ at various numbers of seedlings/ha							
	1,500	2,000	2,500	3,000	3,500	4,000	4,500	5,000
0.9	0.051	0.070	0.089	0.108	0.127	0.145	0.164	0.183
1.0	0.107	0.144	0.181	0.217	0.252	0.287	0.320	0.352
1.1	0.165	0.221	0.274	0.325	0.373	0.418	0.461	0.501
1.2	0.227	0.298	0.366	0.428	0.485	0.537	0.585	0.628
1.3	0.289	0.376	0.454	0.524	0.586	0.641	0.689	0.731
1.4	0.352	0.451	0.537	0.611	0.674	0.728	0.774	0.812
1.5	0.415	0.523	0.613	0.688	0.750	0.799	0.840	0.873
1.6	0.476	0.590	0.682	0.755	0.811	0.856	0.890	0.916
1.7	0.535	0.652	0.742	0.810	0.861	0.899	0.927	0.947
1.8	0.591	0.709	0.795	0.856	0.900	0.931	0.952	0.967
1.9	0.644	0.759	0.839	0.893	0.930	0.954	0.970	0.981
2.0	0.692	0.803	0.876	0.922	0.952	0.970	0.982	0.989
2.1	0.737	0.841	0.906	0.944	0.968	0.981	0.989	0.994
2.2	0.777	0.874	0.929	0.961	0.979	0.988	0.994	0.997
2.3	0.813	0.901	0.948	0.973	0.986	0.993	0.996	0.998
2.4	0.844	0.923	0.962	0.982	0.991	0.996	0.998	0.999
2.5	0.872	0.941	0.973	0.988	0.995	0.998	0.999	
2.6	0.895	0.955	0.981	0.992	0.997	0.999		
2.7	0.915	0.967	0.987	0.995	0.998			
2.8	0.932	0.975	0.991	0.997	0.999			
2.9	0.946	0.982	0.994	0.998				
3.0	0.958	0.987	0.996	0.999				
3.1	0.967	0.991	0.998					
3.2	0.975	0.994	0.998					
3.3	0.981	0.996	0.999					
3.4	0.986	0.997						
3.5	0.989	0.998						
3.6	0.992	0.999						
3.7	0.994							
3.8	0.996							
3.9	0.997							
4.0	0.998							

Tab. 7. Mean distances (metres) at various numbers of seedlings/ha.

Number of seedlings/ha	Square lattice			Randomly distributed seedlings				
	$E(R_1)$	$E(R_2)$	$E(R_3)$	$E(R_1)$	$E(R_2)$	$E(R_3)$	$E(R_{11})$	$E(R_{11}\|R_{11} > 0.8)$
1,500	0.988	1.81	2.34	1.29	1.94	2.42	1.54	1.74
1,600	0.956	1.78	2.27	1.25	1.88	2.34	1.49	*1.70*
1,700	0.928	1.70	2.20	1.21	1.82	2.27	1.45	*1.66*
1,800	0.902	1.65	2.14	1.18	1.77	2.21	1.41	*1.62*
1,900	0.878	1.60	2.08	1.15	1.72	2.15	1.37	*1.59*
2,000	0.855	1.56	2.03	1.12	1.68	2.10	1.33	1.56
2,100	0.835	1.53	1.98	1.09	1.64	2.05	1.30	*1.53*
2,200	0.816	1.49	1.94	1.07	1.60	2.00	1.27	*1.50*
2,300	0.798	1.46	1.89	1.04	1.56	1.95	1.24	*1.48*
2,400	0.781	1.43	1.85	1.02	1.53	1.91	1.22	*1.46*
2,500	0.765	1.40	1.82	1.00	1.50	1.88	1.19	1.44
2,600	0.750	1.37	1.78	0.981	1.47	1.84	1.17	*1.42*
2,700	0.736	1.35	1.75	0.962	1.44	1.80	1.15	*1.40*
2,800	0.723	1.32	1.72	0.945	1.42	1.77	1.13	*1.38*
2,900	0.710	1.30	1.67	0.928	1.39	1.74	1.11	*1.36*
3,000	0.699	1.28	1.66	0.913	1.37	1.71	1.09	1.35
3,100	0.687	1.26	1.63	0.898	1.35	1.68	1.07	*1.34*
3,200	0.676	1.24	1.61	0.884	1.33	1.66	1.05	*1.32*
3,300	0.666	1.22	1.58	0.870	1.31	1.63	1.04	*1.31*
3,400	0.656	1.20	1.56	0.857	1.29	1.61	1.02	*1.30*
3,500	0.647	1.18	1.54	0.845	1.27	1.58	1.01	1.29
3,600	0.638	1.17	1.51	0.833	1.25	1.56	0.994	*1.28*
3,700	0.629	1.15	1.49	0.822	1.23	1.54	0.980	*1.27*
3,800	0.621	1.13	1.47	0.811	1.22	1.52	0.967	*1.26*
3,900	0.613	1.12	1.45	0.801	1.20	1.50	0.955	*1.25*
4,000	0.605	1.11	1.44	0.791	1.19	1.48	0.943	1.24
4,100	0.598	1.09	1.42	0.781	1.17	1.46	0.931	*1.23*
4,200	0.590	1.08	1.40	0.772	1.16	1.45	0.921	*1.22*
4,300	0.583	1.07	1.38	0.762	1.14	1.43	0.909	*1.21*
4,400	0.577	1.05	1.37	0.754	1.13	1.41	0.899	*1.20*
4,500	0.570	1.04	1.35	0.745	1.12	1.40	0.889	1.20
4,600	0.564	1.03	1.34	0.737	1.11	1.38	0.879	*1.19*
4,700	0.558	1.02	1.32	0.729	1.09	1.37	0.870	*1.18*
4,800	0.552	1.01	1.31	0.722	1.08	1.35	0.861	*1.17*
4,900	0.547	0.999	1.30	0.714	1.07	1.34	0.852	*1.17*
5,000	0.541	0.989	1.28	0.707	1.06	1.33	0.843	1.16

Tab. 8. Standard deviations of the number of seedlings of a circular sample plot; square lattice.

Number of seedlings/ha	σ: (number of seedlings / sample plot)		σ: (number of seedlings/ha)	
	5 sq. m. plot	10 sq. m. plot	5 sq. m. plot	10 sq. m. plot
1,000	0.500	0.426	1,000	426
1,200	0.490	0.476	980	476
1,400	0.458	0.503	917	503
1,600	0.404	0.497	808	497
1,800	0.403	0.569	806	569
2,000	0.426	0.653	851	653
2,200	0.453	0.720	905	720
2,400	0.476	0.764	953	764
2,600	0.494	0.784	987	784
2,800	0.503	0.776	1,006	776
3,000	0.502	0.737	1,004	737
3,200	0.497	0.666	994	666
3,400	0.528	0.622	1,055	622
3,600	0.569	0.572	1,138	572
3,800	0.612	0.494	1,224	494
4,000	0.653	0.404	1,306	404

Tab. 9. Standard deviations of the number of seedlings of a circular sample plot; randomly distributed seedlings.

Number of seedlings/ha	σ: (number of seedlings/sample plot)		σ: (number of seedlings/ha)	
	5 sq. m. plot	10 sq. m. plot	5 sq. m. plot	10 sq. m. plot
1,000	0.7071	1.0000	1,414	1,000
1,200	0.7746	1.0954	1,549	1,095
1,400	0.8367	1.1832	1,673	1,183
1,600	0.8944	1.2649	1,789	1,265
1,800	0.9487	1.3416	1,897	1,342
2,000	1.0000	1.4142	2,000	1,414
2,200	1.0488	1.4832	2,098	1,483
2,400	1.0954	1.5492	2,191	1,549
2,600	1.1402	1.6125	2,280	1,612
2,800	1.1832	1.6733	2,366	1,673
3,000	1.2247	1.7321	2,449	1,732
3,200	1.2649	1.7889	2,530	1,789
3,400	1.3038	1.8439	2,608	1,844
3,600	1.3416	1.8973	2,683	1,897
3,800	1.3784	1.9494	2,757	1,949
4,000	1.4142	2.0000	2,828	2,000
4,200	1.4491	2.0494	2,898	2,049
4,400	1.4832	2.0976	2,966	2,098
4,600	1.5166	2.1448	3,033	2,145
4,800	1.5492	2.1909	3,098	2,191

Tab. 10. Mean distances, means of squares of distances as well as of reciprocals of squares of distances; sample survey made by the Boxholm Company Ltd.

Field no.	Number of sample points	Mean distance (metres)			Mean of squares of distances (sq.m.)			Mean of recipro-cals of squares of distances to seed-ling no. 3	Median distance of seed-ling no. 3
		$\bar{r}_1$	$\bar{r}_2$	$\bar{r}_3$	$\bar{w}_1$	$\bar{w}_2$	$\bar{w}_3$		
21 a	48	0.768	1.262	1.777	0.8738	1.9083	3.7034	0.44102	1.57
21 b	53	0.744	1.216	1.552	0.8064	1.7238	2.7701	0.54944	1.43
26	29	0.820	1.323	1.827	0.9978	2.0901	3.9045	0.47344	1.67
27	38	0.746	1.151	1.456	0.7630	1.4745	2.2564	0.54624	1.38
27 a	24	1.037	1.626	2.080	1.5208	3.2157	5.0385	0.36923	2.05
27 b	21	0.641	1.186	1.577	0.4652	1.5445	2.6415	0.46370	1.47
38	40	0.765	1.252	1.559	0.7185	1.7940	2.7203	0.52685	1.38
45	55	0.807	1.292	1.593	1.1006	2.2188	3.1408	0.48133	1.59
57	37	0.771	1.278	1.680	0.7980	1.8284	3.0591	0.44081	1.67
62 a	50	0.610	1.015	1.367	0.4853	1.1346	2.0158	0.67397	1.37
62 b	66	0.749	1.265	1.567	0.6666	1.7629	2.6322	0.48323	1.54
66 a	46	0.903	1.310	1.666	1.2179	2.0740	3.1468	0.49821	1.48
66 b	60	0.747	1.286	1.675	0.9310	2.3434	3.8622	0.53003	1.43
84 a	34	1.148	1.995	2.549	1.7638	5.2420	8.1815	0.28836	2.55
84 b	41	0.851	1.327	1.807	0.9826	1.9835	3.9797	0.43005	1.60
85	45	0.674	1.264	1.582	0.5777	1.7442	2.6645	0.47379	1.48
89	66	0.929	1.440	1.826	1.1572	2.3584	3.7218	0.39066	1.71
95	43	0.885	1.490	1.906	1.0027	2.5185	4.0983	0.38911	1.61
114	35	0.741	1.185	1.646	0.6767	1.5156	3.1864	0.50172	1.44
175	36	1.497	2.174	2.626	5.5307	9.2762	11.7377	0.33687	1.89
176	28	0.843	1.568	2.234	1.1164	2.9058	5.8065	0.34286	1.90
177	40	1.035	1.518	1.905	1.5478	2.9347	4.4005	0.42926	1.56
180 a	37	0.709	1.209	1.759	0.6644	1.8953	4.4118	0.60516	1.38
180 b	31	0.910	1.257	1.743	1.2869	2.0788	3.6989	0.48554	1.50
180 c	47	0.716	1.186	1.450	0.8871	1.7722	2.4674	0.59725	1.32
181	50	0.648	1.048	1.350	0.5252	1.1664	1.9049	0.64709	1.34
183	37	0.940	1.348	1.679	1.2790	2.1565	3.4412	0.53553	1.47
214	42	0.705	1.126	1.453	0.5793	1.3383	2.2527	0.58122	1.42
237	46	1.161	1.779	2.220	2.0517	3.8842	5.7847	0.30453	1.89
246	66	0.884	1.675	2.132	1.0877	3.4507	5.5833	0.33336	2.13

Tab. 11. Numbers of seedlings/ha according to differnt estimates; sample survey made by the Boxholm Company Ltd.

Field no.	Area, hectare	Number of sample plots	Number of seedlings/ha according to sample plot survey			Number of seedlings/ha according to the assumption of square lattice and				Number of seedlings/ha according to the assumption of randomly distributed seedlings, using						
			mean $\bar{x}$	standard error of sample plot	standard error of sample plot mean	$\bar{r}_1$	$\bar{r}_2$	$\bar{r}_3$	z_3^*	$\bar{r}_1$	$\bar{r}_2$	$\bar{r}_3$	z_3^*	max. lik. method and $\bar{w}_2$	max. lik. method and $\bar{w}_3$	max. lik. method and $\bar{h}_3$
21 a	1.4	48	3,500	1,800	260	2,500	3,050	2,600	3,400	4,250	3,550	2,800	3,450	3,350	2,600	2,800
21 b	0.6	53	4,000	1,750	240	2,650	3,300	3,400	4,100	4,500	3,800	3,650	4,150	3,700	3,450	3,500
26	1.5	29	2,950	1,800	340	2,200	2,800	2,450	3,000	3,700	3,200	2,650	3,050	3,050	2,450	3,000
27	—	38	4,200	1,650	270	2,650	3,700	3,900	4,400	4,500	4,250	4,150	4,450	4,300	4,250	3,500
27 a	0.2	24	2,350	1,750	360	1,350	1,850	1,900	2,000	2,300	2,150	2,050	2,050	2,000	1,900	2,350
27 b	0.3	21	3,300	1,250	280	3,550	3,500	3,300	3,900	6,100	4,000	3,550	3,950	4,100	3,600	2,950
38	1.1	40	3,450	1,700	270	2,500	3,100	3,400	4,400	4,250	3,600	3,600	4,450	3,550	3,500	3,350
45	2.6	55	3,500	1,100	140	2,250	2,950	3,250	3,300	3,850	3,350	3,450	3,350	2,850	3,050	3,050
57	3.2	37	3,400	1,500	240	2,450	3,000	2,900	3,000	4,200	3,450	3,100	3,050	3,500	3,100	2,800
62 a	1.0	50	4,700	1,650	230	3,950	4,750	4,400	4,450	6,700	5,450	4,700	4,550	5,600	4,750	4,300
62 b	0.7	66	3,700	1,250	150	2,600	3,050	3,350	3,550	4,450	3,500	3,600	3,600	3,600	3,650	3,100
66 a	1.0	47	3,300	2,000	290	1,800	2,850	2,950	3,800	3,050	3,300	3,150	3,900	3,050	3,050	3,150
66 b	3.3	60	3,700	1,600	210	2,600	2,950	2,950	4,100	4,500	3,400	3,150	4,150	2,700	2,450	3,350
84 a	1.7	34	2,200	1,650	280	1,100	1,250	1,250	1,300	1,900	1,400	1,350	1,300	1,200	1,150	1,850
84 b	0.8	41	3,150	1,500	230	2,000	2,800	2,550	3,250	3,450	3,200	2,700	3,350	3,200	2,400	2,750
85	1.5	45	3,300	1,350	200	3,200	3,050	3,300	3,800	5,500	3,500	3,500	3,900	3,650	3,600	3,000
89	2.7	66	2,800	1,550	190	1,700	2,350	2,450	2,850	2,900	2,700	2,650	2,900	2,700	2,550	2,500
95	1.1	43	2,900	1,600	250	1,850	2,200	2,250	3,250	3,200	2,550	2,400	3,300	2,550	2,350	2,500
114	2.0	35	3,650	1,350	220	2,650	3,500	3,050	4,050	4,550	4,000	3,250	4,100	4,200	3,000	3,200
175	2.4	36	2,350	1,800	300	650	1,050	1,200	2,350	1,100	1,200	1,250	2,400	700	800	2,150
176	1.3	29	2,600	1,650	300	2,050	2,000	1,650	2,300	3,500	2,300	1,750	2,350	2,200	1,650	2,200
177	3.0	40	3,050	1,900	300	1,350	2,100	2,250	3,450	2,350	2,450	2,400	3,500	2,150	2,150	2,750
180 a	0.9	37	4,050	2,050	340	2,900	3,350	2,650	4,400	4,950	3,850	2,850	4,450	3,350	2,150	3,850
180 b	2.3	31	3,600	2,100	380	1,750	3,100	2,700	3,700	3,000	3,550	2,900	3,800	3,050	2,600	3,100
180 c	0.6	47	4,200	1,800	260	2,850	3,500	3,900	4,800	4,900	4,000	4,200	4,900	3,600	3,850	3,800
181	0.6	50	4,750	1,550	220	3,500	4,450	4,550	4,650	5,950	5,100	4,800	4,750	5,450	5,000	4,100
183	1.5	37	3,700	2,050	340	1,650	2,700	2,950	3,900	2,850	3,100	3,100	3,950	2,950	2,750	3,400
214	1.8	42	3,650	1,500	240	2,950	3,850	3,900	4,150	5,050	4,450	4,150	4,200	4,750	4,250	3,700
237	—	46	2,100	1,550	230	1,100	1,550	1,650	2,350	1,850	1,800	1,800	2,400	1,650	1,650	1,950
246	3.4	66	2,650	1,500	190	1,850	1,750	1,800	1,850	3,200	2,000	1,950	1,900	1,850	1,700	2,100

Tab. 12. Confidence intervals for the number of seedlings/ha; sample survey made by the Boxholm Company Ltd.

Field no.	95 % confidence interval for the number of seedlings/ha according to				Field no.	95 % confidence interval for the number of seedlings/ha according to			
	median distance of seedling no. 3		sample plot survey			median distance of seedling no. 3		sample plot survey	
21 a	2,750	4,250	3,000	4,000	85	3,050	4,800	2,900	3,700
21 b	3,350	5,100	3,500	4,500	89	2,400	3,500	2,400	3,200
26	2,250	4,000	2,250	3,650	95	2,550	4,100	2,400	3,400
27	3,450	5,650	3,650	4,750	114	3,100	5,250	3,200	4,100
27 a	1,450	2,700	1,650	3,050	175	1,800	3,050	1,750	2,950
27 b	2,750	5,350	2,750	3,850	176	1,750	3,100	2,000	3,200
38	3,450	5,600	2,900	4,000	177	2,700	4,400	2,450	3,650
45	2,700	4,100	3,200	3,800	180 a	3,450	5,650	3,350	4,750
57	2,350	3,850	2,900	3,900	180 b	2,800	4,900	2,850	4,350
62 a	3,600	5,550	4,250	5,150	180 c	3,850	6,050	3,700	4,700
62 b	2,950	4,300	3,400	4,000	181	3,850	5,800	4,300	5,200
66 a	3,050	4,800	2,700	3,900	183	3,000	5,000	3,000	4,400
66 b	3,400	5,000	3,300	4,100	214	3,300	5,250	3,150	4,150
84 a	1,000	1,650	1,650	2,750	237	1,900	2,950	1,650	2,550
84 b	2,600	4,150	2,700	3,600	246	1,550	2,250	2,250	3,050

Tab. 13. Mean distances between seedling no. 1 and its nearest neighbour and numbers of seedlings/ha estimated thereof; sample survey made by the Boxholm Company Ltd.

Field no.	Mean distance (metres)	Number of seedlings/ha according to		Field no.	Mean distance (metres)	Number of seedlings/ha according to	
		mean distance	sample plot survey			mean distance	sample plot survey
21 a	1.26	3,800	3,500	85	1.31	3,300	3,300
21 b	1.24	4,000	4,000	89	1.40	2,700	2,800
26	1.27	3,700	2,950	95	1.55	2,000	2,900
27	1.28	3,600	4,200	114	1.25	3,900	3,650
27 a	1.49	2,200	2,350	175	1.61	1,800	2,350
27 b	1.39	2,800	3,300	176	1.45	2,400	2,600
38	1.29	3,500	3,450	177	1.34	3,100	3,050
45	1.53	2,100	3,500	180 a	1.18	4,700	4,050
57	1.36	3,000	3,400	180 b	1.16	5,000	3,600
62 a	1.13	5,400	4,700	180 c	1.23	4.100	4,200
62 b	1.33	3,200	3,700	181	1.13	5,400	4,750
66 a	1.27	3,700	3,300	183	1.22	4,200	3,700
66 b	1.36	3,000	3,700	214	1.27	3,700	3,650
84 a	1.62	1,800	2,200	237	1.48	2,300	2,100
84 b	1,37	2.900	3,150	246	1.71	1,600	2,650

Tab. 14. Distribution of the distance between sample point and seedling no. 1 in a group of five fields; sample survey made by the Boxholm Company Ltd.

Class (centimetre)	Sample frequencies		Theoretical frequencies (%) according to the assumption of	
	absolute	ratio (%)	square lattice	randomly distributed seedlings
0— 19	10	5.3	4.4	4.1
20— 39	25	13.2	12.9	11.8
40— 59	37	19.5	21.6	16.6
60— 79	36	18.9	30.8	18.0
80— 99	36	18.9	23.9	16.3
100—119	11	5.8	6.4	12.9
120—139	15	7.9	0.0	9.0
140—159	7	3.7		5.6
160—179	7	3.7		3.2
180—199	2	1.1		1.6
200—219	1	0.5		0.8
220—239	1	0.5		0.1
240—259				
260—279				
280—299	1	0.5		
300—319				
320—339				
340—359	1	0.5		

Tab. 15. Results from testing the programme for artificial sampling in the computer FACIT EDB (2,500 seedlings/ha).

	Theoretical value	Sample value at a number of sample points of			
		25	50	100	200
Mean number of seedlings of a 5 sq.m. sample plot....................	1.25	1.36	1.18	1.21	1.22
Variance of the number of seedlings of the sample plot	0.24	0.31	0.15	0.25	0.25
Mean number of seedlings of a 10 sq.m. sample plot....................	2.50	2.68	2.34	2.52	2.48
Variance of the number of seedlings of the sample plot	0.60	0.54	0.38	0.65	0.60
Mean distance to seedling no. 1 (metres)........................	0.77	0.83	0.70	0.77	0.74
Variance of the distance to seedling no 1	0.081	0.065	0.051	0.082	0.090
Mean distance to seedling no. 2 (metres)........................	1.40	1.33	1.41	1.40	1.41
Variance of the distance to seedling no. 2	0.043	0.029	0.034	0.036	0.047
Mean distance to seedling no. 3 (metres)........................	1.82	1.84	1.86	1.83	1.83
Variance of the distance to seedling no. 3	0.034	0.035	0.028	0.037	0.035

Distribution function of the distance to seedling no. 1

x (metres)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
$F_1(x)$ (%)	1	3	7	13	20	28	38	50	64	79	89	95	99	100
$\overset{*}{F}_1(x)$ (%); $n=200$	2	4	10	14	24	32	40	54	66	82	88	94	98	100

Tab. 16. Summary of the models for artificial sampling in the computer FACIT EDB.

Model no.	Original number of seedlings/ha	Number of seedlings/ha of the model	The seedlings are "killed"
1	2,500	1,750	30 % one by one
2	2,500	1,750	30 % one by one
3	2,500	1,744	15 % one by one, 15 % in groups of 3 in a row
4	2,500	1,747	15 % one by one, 15 % in groups of 5 crosswise
5	3,571	2,500	30 % one by one
6	3,571	2,500	30 % one by one
7	3,571	2,500	15 % one by one, 15 % in groups of 3 in a row
8	3,571	2,492	15 % one by one, 15 % in groups of 3 in a row
9	3,571	2,496	15 % one by one, 15 % in groups of 5 crosswise
10	3,571	2,488	15 % one by one, 15 % in groups of 5 crosswise

Tab. 17. Results from artificial sampling in the computer FACIT EDB; assumption no. 1 = square lattice, assumption no. 2 = randomly distributed seedlings.

Model no.	Number of samples between parentheses; number of sample points = 25	Mean number of seedlings/ha							
		according to sample plot survey and plot size		according to mean distance to					
				seedling no. 1 and assumption		seedling no. 2 and assumption		seedling no. 3 and assumption	
		5 sq. m.	10 sq. m.	1	2	1	2	1	2
	maximum	2,100	1,950	1,800	3,100	1,900	2,150	2,000	2,050
1	mean (15)	1,700	1,750	1,450	2,450	1,700	1,950	1,800	1,900
	minimum	1,200	1,550	1,050	1,800	1,550	1,800	1,600	1,700
		2,000	1,850	2,000	3,400	1,850	2,150	1,850	2,000
2	(10)	1,750	1,650	1,600	2,700	1,600	1,850	1,750	1,850
		1,200	1,250	1,150	1,900	1,350	1,550	1,600	1,700
		2,100	2,150	1,850	3,150	1,950	2,250	1,950	2,050
3	(15)	1,800	1,800	1,450	2,450	1,700	1,950	1,700	1,800
		1,200	1,400	1,050	1,750	1,500	1,550	1,500	1,600
		2,000	2,000	1,750	2,950	1,900	2,200	1,950	2,100
4	(15)	1,700	1,750	1,300	2,200	1,600	1,800	1,650	1,750
		1,300	1,250	1,050	1,800	1,400	1,600	1,400	1,500
		3,200	2,700	2,800	4,800	3,000	3,450	2,800	3,000
5	(15)	2,400	2,450	2,150	3,650	2,400	2,750	2,450	2,600
		1,850	1,950	1,450	2,450	1,800	2,050	2,000	2,150
		2,950	2,750	2,600	4,450	2,800	3,250	2,800	2,950
6	(10)	2,500	2,400	2,100	3,550	2,400	2,800	2,500	2,650
		2,000	1,900	1,750	2,950	2,100	2,450	2,000	2,150
		2,900	2,750	3,250	5,550	2,700	3,100	2,900	3,100
7	(15)	2,400	2,450	2,100	3,600	2,300	2,650	2,500	2,650
		1,750	2,100	1,550	2,650	1,900	2,200	2,050	2,200
		2,700	2,700	2,550	4,800	2,500	2,850	2,700	2,850
8	(10)	2,450	2,450	2,100	3,550	2,250	2,600	2,500	2,650
		2,000	2,200	1,250	2,150	2,100	2,400	2,150	2,300
		3,050	2,700	2,300	3,900	2,450	2,800	2,450	2,600
9	(15)	2,400	2,300	1,700	2,850	2,050	2,350	2,150	2,300
		1,500	1,950	1,150	1,900	1,550	1,800	1,750	1,850
		2,900	2,650	2,600	4,450	2,550	2,950	2,450	2,600
10	(10)	2,400	2,400	1,950	3,400	2,200	2,500	2,300	2,450
		1,900	2,150	1,250	2,150	1,800	2,100	2,100	2,200

Tab. 17. Continuation.

Model no.	Number of samples between parentheses, number of sample points = 200	Mean number of seedlings/ha							
		according to sample plot survey and plot size		according to mean distance to					
				seedling no. 1 and assumption		seedling no. 2 and assumption		seedling no. 3 and assumption	
		5 sq. m.	10 sq. m.	1	2	1	2	1	2
	maximum	1,800	1,950	1,450	2,500	1,800	2,100	1,850	1,950
2	mean (5)	1,700	1,800	1,400	2,350	1,700	1,950	1,750	1,850
	minimum	1,550	1,750	1,250	2,100	1,650	1,900	1,700	1,800
		2,650	2,500	2,300	3,900	2,500	2,850	2,550	2,750
6	(5)	2,450	2,450	2,100	3,550	2,350	2,700	2,450	2,600
		2,300	2,350	1,900	3,250	2,250	2,550	2,400	2,550
		2,650	2,550	2,200	3,700	2,450	2,850	2,450	2,600
8	(5)	2,450	2,450	2,050	3,450	2,350	2,700	2,400	2,550
		2,400	2,350	1,900	3,250	2,300	2,650	2,350	2,500
		2,550	2,600	2,050	3,550	2,350	2,700	2,450	2,600
10	(5)	2,400	2,450	1,800	3,100	2,200	2,550	2,350	2,500
		2,150	2,250	1,600	2,700	2,050	2,350	2,300	2,450

Tab. 18. Sample distribution functions of the distance between sample point and seedling no. 1; artificial sampling in the computer FACIT EDB (1000 sample points).

x (metres)	$\overset{*}{F}_1(x)$, model no.			
	2	6	8	10
0.1	0.005	0.005	0.010	0.005
0.2	0.015	0.030	0.030	0.025
0.3	0.035	0.065	0.070	0.080
0.4	0.065	0.120	0.130	0.130
0.5	0.105	0.200	0.200	0.180
0.6	0.170	0.290	0.295	0.245
0.7	0.235	0.380	0.405	0.335
0.8	0.310	0.490	0.530	0.460
0.9	0.410	0.600	0.635	0.575
1.0	0.515	0.695	0.710	0.665
1.1	0.620	0.765	0.765	0.730
1.2	0.680	0.835	0.810	0.780
1.3	0.745	0.875	0.845	0.820
1.4	0.800	0.920	0.880	0.855
1.5	0.845	0.935	0.905	0.890
1.6	0.890	0.955	0.930	0.910
1.7	0.920	0.970	0.950	0.940
1.8	0.940	0.985	0.965	0.955
1.9	0.955	0.995	0.980	0.965
2.0	0.975		0.985	0.970
2.1	0.980		0.990	0.985
2.2	0.985		0.990	0.990
2.3	0.995		0.995	0.995

Tab. 19. Estimated numbers of circular open spaces; artificial sampling in the computer FACIT EDB (25 sample points).

Sample no.	Percentages of circular open spaces with a radius of 1.5 metres; sampling of model no.									
	1	2	3	4	5	6	7	8	9	10
1	20	12	16	24	8	0	8	20	12	8
2	16	16	12	12	4	12	0	4	20	8
3	16	24	12	16	4	0	16	4	8	16
4	16	16	20	12	12	12	8	8	8	12
5	16	24	16	8	8	4	4	4	4	4
6	12	4	4	24	8	0	12	8	4	16
7	8	20	20	12	4	4	8	24	16	8
8	20	28	16	24	4	4	8	8	16	8
9	12	16	24	20	8	8	12	4	12	8
10	8	16	20	32	12	8	4	8	16	4
11	20		4	20	4		20		20	
12	8		16	32	8		8		4	
13	16		16	16	0		16		16	
14	20		12	20	4		4		8	
15	12		12	28	4		4		24	
mean	15	18	15	20	6	5	9	9	12	9

Sample no.	Percentages of circular open spaces with a radius of 2.0 metres.; sampling of model no.									
	1	2	3	4	5	6	7	8	9	10
1	0	0	4	12	0	0	0	0	4	8
2	4	4	0	8	0	0	0	0	8	4
3	4	0	4	8	0	0	0	0	4	0
4	4	0	12	8	4	0	0	0	4	4
5	8	4	8	0	0	0	0	0	0	0
6	4	0	4	0	4	0	4	0	4	8
7	4	8	4	4	4	0	0	4	12	0
8	8	0	0	12	0	0	0	0	8	0
9	8	0	12	8	0	0	4	0	8	0
10	4	4	8	16	4	0	4	0	4	4
11	0		0	8	0		4		12	
12	0		12	8	0		0		4	
13	8		8	4	0		4		8	
14	4		8	12	0		0		8	
15	8		8	16	0		4		8	
mean	4	2	6	8	1	0	2	0	6	3