MAXIMAL SATURATED LINEAR ORDERS

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Abstract

The goal of this dissertation is to prove two theorems related to a question posed by Felix Hausdorff in 1907,[1] regarding *pantachies*: maximal linearly ordered subsets of the space of real-valued sequences partially ordered by eventual domination.

In Chapter 1, some terminology is defined, and Hausdorff's question about pantachies is explored. Some related work by other mathematicians is examined, both preceding and following Hausdorff's paper. In Chapter 2, relevant definitions and results about forcing, gaps, and saturated linear orders are collected. Chapter 3 contains the complete proof of the first theorem, namely, the consistency of the existence of a saturated Hausdorff pantachie in a model where the continuum hypothesis (CH) fails. Finally, in Chapter 4, a different method is used to prove a stronger result, namely, the consistency of the existence of a saturated Hausdorff pantachie in a model of Martin's Axiom along with the negation of CH. The appendix mentions a few related open questions and some partial answers.

[1]See [Hau05], page 165, question (α): is there a pantachie without (ω_1, ω_1)-gaps?

This work is dedicated to my mother, the late Maria Cristina Aida Kibedi. Having her as my mother is one of the greatest blessings in my life.

In her name, I offer this work to the divine source of all.

Jai Sai Ram

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1 Introduction

This chapter introduces a few basic definitions, as well as some historical background and motivations for studying pantachies and maximal saturated linear orders. We focus on the motivations that are relevant to the two results presented in this thesis: Theorem 3.0.1 and Theorem 4.0.3. For a more detailed history the reader may wish to consult [Ste12].

1.1 A Brief History

The results of this thesis are based on a question posed in 1907 by Felix Hausdorff, who was analyzing Paul du Bois-Reymond's attempt to classify rates of convergence (in the context of mathematical analysis). Not only do the motivations of du Bois-Reymond find their roots in analysis, but this present work has some connections with analysis, in particular, with the infinitesimal controversy that arose in the development of calculus, especially in the sense that infinitesimals relate to saturation of linear orders.

In the 17th century, Gottfried Leibniz and Isaac Newton (independently) advanced the notion of infinitesimals, quantities which are like real numbers but infinitely small. More specifically, an infinitesimal is a positive number (greater than 0), but smaller than every (non-infinitesimal) positive real number; for example, an infinitesimal is smaller than each of the numbers in the sequence $\{1/2, 1/3, 1/4, \ldots\}$. Although there are no real numbers with this property, the notion of an infinitesimal was very useful as it enabled the development of many tools and applications in calculus.[2] Although they developed many useful tools using infinitesimals, neither Newton nor Leibniz were able to make the notion of infinitesimals mathematically precise. For this reason, the usage of infinitesimal quantities was criticized by other mathematicians at the time, including Michel Rolle and Bishop George Berkeley.

Augmenting the set of real numbers by the inclusion of infinitesimals[3] can be understood as an attempt to obtain a greater degree of saturation in the linear order of the real numbers. The reals enjoy the following density property: between any two real numbers, there is another real number. This implies that between two finite sets of reals A and B, where each member of A is less than each member of B, there will always be a real number c that lies above all the members of A and below

[2]For example, an instantaneous rate of change or the slope of a curve at a given point could be calculated with the use of infinitesimals, and the area of a region bounded by curves could be calculated by adding the areas of rectangles with infinitesimally small widths.

[3]More precisely, augmenting by expanding to the so-called set of hyper-reals, which includes all real numbers, infinitesimals, and any number that can be obtained with these by means of the usual operations on real numbers.

all the members of B. But, what happens if we allow at least one of the sets A or B to be infinite? For example, if A is the singleton set consisting of the number 0, and B is the infinite set consisting of the terms in the sequence $1/n$ (where n is any positive integer), then there is no real number c that lies between A and B—but an infinitesimal does fit into this slot. Nevertheless, in a sense, sequences that converge to 0 (such as the $1/n$ sequence) approximate the idea of an infinitesimal: given any positive (non-infinitesimal) real number greater than 0, all but finitely many terms of the sequence are less than that given real number.

Thanks to the contributions of Augustin-Louis Cauchy and Karl Theodor Wilhelm Weierstrass, calculus was formalized first without the formal need of infinitesimals, but rather with the use of sequences and limits of sequences, via the well-known (ϵ, δ)-definition of limits. In fact, Cauchy defined an important class of sequences that can be used to formally define the set of real numbers. These so-called Cauchy sequences are defined by the following condition: given any positve real distance, no matter how small, all but a finite number of elements of the sequence are less than that given distance from each other. The real numbers are the limits (or equivalence classes) of Cauchy sequences of rational numbers. In other words, every Cauchy sequence should converge to a point, and the set of real numbers are complete in the sense that every Cauchy sequence (even those whose elements range over the real numbers) *does* have a real limit. This is another sort of saturation property of the set of real numbers.

It was not until the 20th century that Abraham Robinson, making use of the so-called Hyper-real numbers, was able to provide the first rigorous backing to the intuition behind infinitesimals.The Hyper-reals can be developed using model theory, or, via an ultrapower construction on sequences of real numbers with an ultrafilter and its corresponding ideal.[4] With this approach, one obtains a mathematically precise set of numbers (the Hyper-reals) which includes all the reals and the infinitesimals as well; in other words, the full saturation property needed for infinitesimals is found in the Hyper-reals. However, there is still more to be said about saturation in linear orders.

Hausdorff, in 1907, was analyzing du Bois-Reymond's attempt to classify rates of convergence of positive real-valued functions. Du Bois-Reymond defined an ordering on these functions based on the limit of the quotient of two functions from this space, and he called this space of functions endowed with this ordering a *pantachie*. His intention was to use his ordering to determine a "cut point" separating convergence from divergence. One of the complications of du Bois Reymond's approach is that his ordering was not linear—there are many pairs of functions that are simply incomparable under the ordering. Hausdorff decided to pursue du Bois Reymond's work further, but with modifications. First, Hausdorff decided to re-

[4]The use of ideals on sequences of real numbers (such as the mod-finite ideal used in the present work) continues to be a major topic of research in the set-theoretic study of the reals.

strict consideration to functions with domain the natural numbers; in other words, sequences of reals (or positive reals). Moreover, Hausdorff used a slightly simpler ordering, namely, eventual domination (see Definition 2). Hausdorff knew that this ordering was also not a linear ordering, but that one could restrict to a subset that did form a linear order, and in fact, Hausdorff was interested in such linear orders that were *maximal*, in the sense that no more sequences could be added to the subset without destroying linearity. Hausdorff, borrowing du Bois-Reymond's terminology, called such a maximal linearly ordered subset of this space a *pantachie*.

Hausdorff was interested in the saturation properties of pantachies, which can be interpreted as an alternate approach to the infinitesimal controversy.[5] The saturation properties of interest to Hausdorff involved the consideration of higher orders of infinity. By Hausdorff's time, the notion of different sizes or *cardinalities* for infinite sets had already been defined by Georg Cantor. In particular, the natural numbers, or sequences enumerated by the natural numbers, are said to be countable, and these sets are provably of a smaller cardinality than sets which cannot be put into one-to-one correspondence with the natural numbers, such as the set of all real numbers. Sets which can be placed in a one-to-one correspondence with the set of real numbers are said to have the same cardinality as the reals, and that cardinality is called *continuum*. But is there an uncountable set with

[5] Pantachies are saturated for countable sets, which suffices to embed the reals and infinitesimals.

cardinality less than continuum, or is there not? The latter conjecture, namely, that continuum is the smallest uncountable cardinality, is known as the continuum hypothesis (CH). In Hausdorff's time, CH was seen as an open question. It wasn't until 1963 that Paul Cohen showed that CH is independent of the usual, commonly accepted axioms of set theory, known today as ZFC; there are models of the axioms of ZFC where CH holds, and models where it fails. Hausdorff was able to obtain saturation in his pantachies for up to countably infinite sets. More specifically, for any countable (i.e., finite or countably infinite) subsets A and B of a pantachie where each element of A is less than each element of B, there is some element c in the pantachie that lies above all members of A and below all members of B. However, what about further saturation properties? What if A or B are infinite sets that are not countable?

The best saturation property that could possibly be obtained would be saturation for all sets A and B of size less than continuum. Thus, under CH, a pantachie is already as saturated as possible. Hausdorff asked a question that he could not answer, about the saturation properties of a pantachie in the absence of CH. Hausdorff's question can be phrased as follows: are there pantachies that are saturated for all sets A and B of size up to the first uncountable cardinality (if CH fails)?[6] In

[6] Although this version uses modern language and might seem slightly more general than question (α) on page 165 of [Hau05], Hausdorff's own deductions culminating on page 141 of [Hau05] show that the two versions of the question are equivalent.

this thesis, we answer the latter question by producing a model of ZFC where CH fails (in fact, continuum in this model can be arbitrarily large, so that there can be any number of uncountable cardinalities less than continuum) and yet there is a pantachie that is as saturated as possible. We then extend this result to a model where CH still fails and another statement that is independent of the axioms of set theory holds, namely, Martin's Axiom (MA).[7]

1.2 More Background, Motivations, and Some Definitions

As mentioned in Section 1.1, the term *pantachie* was first introduced by Paul du Bois-Reymond, who was interested in classifying rates of convergence (in the context of mathematical analysis) into a hierarchy:

Definition 1. Let $G = {}^{\mathbb{R}}(0, \infty)$, the set of positive real functions, and let $f, g \in G$. Define

- $f < g$ iff $\lim_{x \to +\infty} f(x)/g(x) = 0$
- $f > g$ iff $\lim_{x \to +\infty} f(x)/g(x) = +\infty$
- $f \sim g$ iff $\lim_{x \to +\infty} f(x)/g(x) \in (0, +\infty)$

[7] Martin's Axiom implies that uncountable cardinalities less than continuum behave, in many ways, like the smallest (countable) infinite cardinality.

Du Bois-Reymond called $(G, <)$ a pantachie.[8] He considered a function f to represent a larger infinity than a function g if $f > g$, and he considered f and g to represent the same infinity if $f \sim g$. Du Bois-Reymond wanted to find, in this structure, a cut point separating convergence from divergence. One of the difficulties in pursuing this approach, is that there are incomparable pairs of functions under this ordering; in other words, there are pairs of functions f, g such that each of the statements $f < g$, $f > g$, and $f \sim g$ fail. Thus, this ordering does not produce a true hierarchy.

Felix Hausdorff continued this study, but with modifications. In his 1907 paper entitled *Investigations into Order Types*[9] (translated in [Hau05]), Hausdorff explores various alternate definitions of the term pantachie. By the end of the paper, he seems to have settled on a definition which makes use of eventual (strict) domination:

Definition 2. Fix $f, g \in {}^{\leq\omega}\mathbb{R}$, and suppose $|f| = |g| = \alpha$.

- $f \leq g$ (i.e., g dominates f) if $\forall k \in \alpha$, $f(k) \leq g(k)$;
- $f < g$ (i.e., g strictly dominates f) if $\forall k \in \alpha$, $f(k) < g(k)$.

[8] The origin of the term "pantachie", and the related adjective "pantachish", are from the Greek words for "everywhere." See page 180 in [Kan07].

[9] In German, *Untersuchungen über Ornungstypen*. This paper includes the fourth and fifth parts of a five part study Hausdorff began in his 1906 publication by the same title. The 1906 paper contains the first three parts of the study, however, only the 1907 paper deals with pantachies.

If $\alpha = \omega$, then

- $f \leq^* g$ (i.e., g eventually dominates f) if $\exists m \in \omega$ such that $\forall k > m$, $f(k) \leq g(k)$;
- $f <^* g$ (i.e., g eventually strictly dominates f) if $\exists m \in \omega$ such that $\forall k > m$, $f(k) < g(k)$.
- $f =^* g$, i.e., f is *eventually equal to* or *almost equal to* g, if $\exists m \in \omega$ such that $\forall k > m$, $f(k) = g(k)$.

Definition 3 (Hausdorff pantachie)**.** Consider the set of real-valued ω-sequences, ${}^{\omega}\mathbb{R}$, partially ordered by $<^*$, eventual strict domination. A *pantachie* is any subset of ${}^{\omega}\mathbb{R}$ that is a maximal linear order under $<^*$.[10]

Hausdorff was able to prove the existence of pantachies by simply using the well-ordering principle.[11] Analyzing the nature of pantachies led Hausdorff to consider their saturation properties, which in turn leads to the question upon which the results of this thesis are based. Before stating Hausdorff's question precisely, we collect some definitions:

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[10] Actually, Hausdorff alternates between sequences of reals and sequences of positive reals, even towards the end of his 1907 paper (see [Hau05]). For the purposes of this thesis, there is no need to restrict to $\mathbb{R}^+$, although the theorems can be easily transferred to this restricted space, and to other similar spaces.

[11] See page 138 of [Hau05].

Definition 4. Let $(P, <)$ be a partial order, and suppose $X \subseteq P$ forms a linear order with $<$. A *cut* in X is a pair (A, B) of subsets of X such that $A < B$ (i.e., $\forall a \in A$, $\forall b \in B$, $a < b$). A cut (A, B) is called a *partition cut* if $A \cup B = X$. The *size* of the cut (A, B) is $\max\{|A|, |B|\}$. Given $p \in P$, we say p fills the cut (A, B) iff $A < p < B$ (i.e., $A < \{p\} < B$).

The notion of a cut is closely related to that of a gap (or pregap).

Definition 5. Let $(P, <)$ be a partial order. A *pregap* in P is a pair of sequences $(\{a_\alpha : \alpha < \gamma\}, \{b_\beta : \beta < \delta\})$ from P, where γ, δ are ordinals and $\forall \alpha_0 < \alpha_1 \in \gamma$, $\forall \beta_0 < \beta_1 \in \delta$, $a_{\alpha_0} < a_{\alpha_1} < b_{\beta_1} < b_{\beta_0}$. (The pair of indexing ordinals (γ, δ) is said to be the *type* of this pregap, and the pregap is referred to as a (γ, δ)-pregap; if $\gamma = \delta$, the pregap is referred to as a γ-pregap.) A pregap is said to be *filled* if there is some $p \in P$ such that for any $\alpha \in \gamma$, and any $\beta \in \delta$, $a_\alpha < p < b_\beta$. A *gap* is a pregap that is *unfilled* (i.e., not filled).

Definition 6. Let κ be any cardinal. A linear order (or partial order) $(L, \prec)$ is κ-saturated if every cut (A, B) of size $< \kappa$ can be filled, i.e., $\exists r \in L$ such that $A \prec r \prec B$. If $|L| = \kappa$ and L is κ-saturated, we say L is a *saturated linear order of size* κ. Furthermore, in the context of Hausdorff's partial order $({}^\omega\mathbb{R}, <^*)$, a saturated linear order $L \subseteq {}^\omega\mathbb{R}$ of size $\mathfrak{c}$ will be called simply a saturated linear order.

Apart from eventual (strict, or non-strict) domination, there are various other related partial orderings which are studied in the literature.[12] For example:

Definition 7. Let $D \subseteq \mathbb{R}$; as usual, ${}^{\omega}D$ denotes the set of functions from ω into D.

1. *The divergence ordering,* $\prec$. Suppose D is unbounded. Let $f, g \in {}^{\omega}D$. Write $f \prec g$ iff $\lim_{n\to\infty} g(n) - f(n) = \infty$; i.e., g diverges from f.

2. *Eventual non-strict domination without eventual equality,* $\lneq^*$. Let $f, g \in {}^{\omega}D$. Write $f \lneq^* g$ iff $f \leq^* g$ and $\{n : f(n) < g(n)\}$ is infinite.

In the case of $({}^{\omega}2, \lneq^*)$, a more common implementation is $\mathcal{P}(\omega)/Fin$, or $\mathcal{P}(\omega)$ under inclusion mod finite, or *(strict) almost containment.*

Definition 8 (Strict almost containment)**.** Let $A, B \subseteq \omega$. Say A *is almost contained in* B, and write $A \subseteq^* B$, iff $A \setminus B$ is finite. Say A *is strictly almost contained in* B, and write $A \subset^* B$, iff $A \subseteq^* B$ but not $B \subseteq^* A$; i.e., $A \setminus B$ is finite but $B \setminus A$ is infinite.

Remark 1. For the actual forcing constructions in this thesis, we work with Hausdorff's partial order restricted to the rationals: $({}^{\omega}\mathbb{Q}, <^*)$. It would not be difficult to transfer the definitions and results to other related spaces, such as:

[12] Cf. [Far96] (about embedding posets into ${}^{\omega}\omega$), which focuses on eventual (non-strict) domination in ${}^{\omega}\omega$; cf. [Sch93] (about gaps in ${}^{\omega}\omega$), which compares various related partial orderings; and cf. [BLT08] (about bounds on the continuum based on work of Godel), which makes use of eventual (non-strict) domination in ${}^{\omega}(\mathbb{R}^+)$.

- $({}^{\omega}(\mathbb{Q}^+), <^*)$;

- $(\{f \in {}^{\omega}\omega : f \text{ is not bounded}\}, \prec)$;

- $(\{f \in {}^{\omega}\omega : f \text{ is not eventually } 0\}, \lneq^*)$; and

- $(\{A \subseteq \omega : A \text{ is infinite and co-infinite}\}, \subset^*)$.

Note that all of these posets are free from endpoints; in fact, each is ω_1-saturated (cf. Theorem 1 from [Sch93], page 443).

Returning to [Hau05], it is interesting to note that Hausdorff realized that a pantachie would always be ω_1-saturated.[13] Furthermore, he argued that under the continuum hypothesis, all pantachies contain (ω_1, ω_1)-gaps. (Cf. Theorems 2.1.4 and 2.1.5.) Of course, it was Hausdorff himself who first produced a construction (with no additional assumptions beyond the usual axioms of set theory, known today as ZFC) of an (ω_1, ω_1)-gap, published first in 1909, in the context of eventual domination, and then again in 1936, this time within $\mathcal{P}(\omega)$ under the inclusion mod finite ordering.[14]

Towards the end of his 1907 *Investigations into Order Types*, Hausdorff poses several questions that he cannot answer, the first of which is labelled (α):

[13] See Hausdorff's deductions, culminating in line (K). on page 141 of [Hau05]; see also the definition of *H-type* immediately preceding line (K). Of course, Hausdorff does not use the modern terminology.

[14] See [Hau05], pages 271-301 and 305-316.

Question 1. [Hausdorff's Question (α)] Is there a pantachie without (ω_1, ω_1)-gaps?[15]

Under CH, the answer to this question is "no", by Theorems 2.1.4 and 2.1.5. It turns out that Hausdorff's question is independent of ZFC, and so answering Hausdorff's question, in modern terms, amounts to showing that *consistently* there is a pantachie without (ω_1, ω_1)-gaps:

Theorem 1.2.1. *Con(ZFC + ¬CH + ∃ a pantachie with no (ω_1, ω_1)-gaps)*

In Chapter 3 of this thesis, based on the work of Richard Laver in [Lav79], a ccc forcing extension is presented where continuum can be essentially as large as desired, and there is a pantachie which contains no (κ, λ)-gaps for any $\kappa, \lambda < \mathfrak{c}$. This shows Con(ZFC + ¬CH + ∃ a maximal saturated linear order in $({}^\omega\mathbb{R}, <^*)$), and Theorem 1.2.1 is an immediate corollary, which in turn answers Hausdorff's Question (α), as described above.

In [Lav79], rather than the partial order $({}^\omega\mathbb{R}, <^*)$, Laver was considering the space of sequences of natural numbers, ${}^\omega\omega$, under eventual domination, and under divergence.[16] The existence of a saturated linear order of size continuum in $({}^\omega\omega, \prec)$ is independent of ZFC: since $({}^\omega\omega, \prec)$ embeds every linear order of size $\leq \omega_1$, under

[15]See the question labelled (α) on page 165 of [Hau05]. The precise phrasing of the question, as translated into English by J.M. Plotkin, is as follows: "Is there a pantachie without $\Omega\Omega^*$-gaps?" Of course, Ω is alternate notation for ω_1.

[16]Note that $\forall f, g \in {}^\omega\omega$, $f \prec g \Rightarrow f \lneq^* g$. So a linearly ordered subset in $({}^\omega\omega, \prec)$ is automatically a linearly ordered subset in $({}^\omega\omega, \lneq^*)$.

CH, we do have such a saturated linear order; on the other hand, Laver produces a model with a saturated linear order of size $\mathfrak{c}$ in $({}^{\omega}\omega, \prec)$ without CH. The latter model was constructed for the purpose of answering the following question:

Question 2. Without CH, which linear orders of size $\leq \mathfrak{c}$ are embeddable in $({}^{\omega}\omega, \lneq^{*})$?[17]

Laver's goal was to prove:

Proposition 1.2.2. *Con(¬CH + $({}^{\omega}\omega, \lneq^{*})$ embeds every linear order of size $\mathfrak{c}$)*[18]

The difficulty in trying to embed (inductively) linear orders of size ω_2 into ${}^{\omega}\omega$ (under the eventual domination ordering or the divergence ordering) is the possibility of creating a Hausdorff-type gap, i.e., an (ω_1, ω_1)-gap which cannot be filled in any extension preserving ω_1. Laver's approach, in order to circumvent this difficulty, was to use a finite-support ccc iterated forcing construction (of length κ,

[17]Robert M. Solovay raised this question in connection to his work with W. Hugh Woodin on homomorphisms of Banach Algebras; in particular, with regard to the question of automatic continuity: *Given X a compact space, let $C(X)$ denote the space of real-valued functions with domain X, continuous (with the* sup *norm). Suppose A is a commutative Banach algebra, and $H : C(X) \longrightarrow A$ is a homomorphism. Under what conditions is H automatically continuous?* It turns out that CH $\implies$ there is a discontinuous homomorphism, while PFA (the Proper Forcing Axiom) $\implies$ there are no discontinuous homomorphisms. The non-existence of a saturated linear order in ${}^{\omega}\omega$ under PFA is important here. The reader might wish to consult [TF95] and [DW87] for more details.

[18]A number of related facts were already known to Laver at that time: 2^{ω} can be arbitrarily large with ω_2 not embeddable in ${}^{\omega}\omega$; assuming Martin's Axiom (MA), ${}^{\omega}\omega$ embeds every well-ordering and converse well-ordering of size $\leq \mathfrak{c}$, and ${}^{\omega}\omega$ embeds every linear ordering of size $< \mathfrak{c}$. On the other hand, Kenneth Kunen had shown Con(MA + $\exists$ linear order of size $\mathfrak{c}$ not embeddable in ${}^{\omega}\omega$).

where $\kappa > \omega_1$ is a regular cardinal and $2^{<\kappa} = \kappa$), and generically add at each stage α a function f_α. By bookkeeping, all possible cuts of size $< \kappa$ are enumerated as $\{C_\alpha : \alpha < \kappa\}$, and f_α fills the cut C_α. The resulting set $\{f_\alpha : \alpha < \kappa\}$ is a saturated linear order of size $\kappa = 2^\omega$ in the final forcing extension.

Given that Laver obtained a saturated linear order (of size $\mathfrak{c} > \omega_1$) in $({}^\omega\omega, \prec)$, the following question arises:

Question 3. Can we obtain a *maximal* saturated linear order of size $\mathfrak{c} > \omega_1$ in $({}^\omega\omega, \prec)$, or rather, in $({}^\omega\mathbb{Q}, <^*)$?

Switching from Laver's partial order to Hausdorff's partial order (or rather, to the related partial order $({}^\omega\mathbb{Q}, <^*)$) is not difficult. Then, by careful bookkeeping and exploiting genericity, we show that Laver's method can produce such a maximal saturated linear order. In fact, we show that this saturated linear order is maximal in $({}^\omega\mathbb{R}, <^*)$, thus answering Question 1 (Hausdorff's Question (α)) — as described above, producing such a model shows that the existence of a Hausdorff pantachie with no ω_1-gaps is independent of ZFC.

To explore Hausdorff's question further, we consider whether it is possible to have a maximal saturated linear order of size $\mathfrak{c} > \omega_1$ in a model of Martin's Axiom[19] — in other words, a model containing a pantachie with no (ω_1, ω_1)-gaps, which in addition satisfies Martin's Axiom. Background and motivation for this extension is

[19]The statement of Martin's Axiom can be found in Chapter 2, Definition 24.

provided by work of James Baumgartner, Kenneth Kunen, and Hugh Woodin, as explained below.

In [Bau84], Baumgartner presents various applications of the Proper Forcing Axiom (PFA).[20] In particular, Baumgartner expands on work of Kunen about gap-filling partial orders,[21] and presents the following results, in the context of $(\mathcal{P}(\omega), \subset^*)$:

Theorem 1.2.3. *Assume PFA. If $\kappa \leq \lambda$ are regular infinite cardinals and (A, B) is a (κ, λ^*)-gap, then either $\kappa = \omega$ and $\lambda \geq \omega_2$ or $\kappa = \lambda = \omega_1$.*

In particular, PFA $\Rightarrow$ there are no $(\mathfrak{c}, \mathfrak{c}^*)$-gaps. This also leads to the following result:

Theorem 1.2.4. *PFA $\implies$ there is a linear ordering of size $\mathfrak{c}$ which is not embeddable in $(\mathcal{P}(\omega), \subset^*)$.*

Thus, under PFA, there are no $(\mathfrak{c}, \mathfrak{c}^*)$-gaps, and no saturated linear order of size continuum in $(\mathcal{P}(\omega), \subset^*)$ (nor in related partial orders). In fact, Kunen had already shown Con(MA + $\neg\exists(\mathfrak{c}, \mathfrak{c}^*)$-gap). On the other hand, another result of Kunen's is Con(MA + $\exists(\mathfrak{c}, \mathfrak{c}^*)$-gap). So, a natural question arises:

Question 4. Is there a model of MA where there is a saturated linear order (in $(\mathcal{P}(\omega), \subset^*)$, or any of the related partial orders)?

[20]The statement of the Proper Forcing Axiom can be found in Chapter 2, Definition 27.

[21]See Definition 28 for an explanation of the Kunen forcing notion for filling a cut (or gap).

In his doctoral thesis [Woo84], Woodin provides a positive answer to this question. The next question is:

Question 5. Is there a model of MA along with a *maximal* saturated linear order in $({}^{\omega}\mathbb{Q}, <^*)$?

We answer Question 5 affirmatively in Chapter 4. In fact, we produce a model of MA along with a saturated linear order that is maximal in the unrestricted Hausdorff partial order, $({}^{\omega}\mathbb{R}, <^*)$. This result extends our answer to Hausdorff's question (α), since the model produced satisfies Martin's Axiom and includes a Hausdorff pantachie with no ω_1-gaps.

Woodin's argument does not directly lend itself to obtaining this extended result, since Woodin's approach produces a saturated linear order which is necessarily *not* maximal. Woodin's approach involves an iterated forcing construction of length $\omega_2 \cdot \omega_2$. A partial suborder of $({}^{\omega}\omega, \lneq^*)$ is recursively constructed by generically filling all the gaps[22] that arise as the partial suborder emerges through the course of the recursive construction of the forcing.

In addition, through the course of the iterated forcing construction, Woodin recursively defines a function which embeds the saturated linear order of *eventually alternating* sequences[23] into the partial suborder of $({}^{\omega}\omega, \lneq^*)$.

[22]Woodin uses a variation of Kunen's gap-filling forcing notion. Cf. Definition 28.

[23]Call $s \in {}^{\omega_2}2$ *eventually alternating* if there exists $\gamma < \omega_2$ such that for each $\beta > \gamma$, $s(\beta) = 0$ iff

To be more precise, Woodin defines $T(\omega_2) = \{s \in {}^{\leq\omega_2}2 : |s| = \omega_2 \Rightarrow s$ is eventually alternating$\}$, and recursively defines an embedding $\varphi : T(\omega_2) \longrightarrow {}^\omega\omega \times {}^\omega\omega$ such that for all $s, t \in T(\omega_2)$:

1. $s <_{lex} t \Rightarrow g_s \lneq^* f_s \lneq^* g_t \lneq^* f_t$; and

2. $t \subseteq s \Rightarrow g_t \lneq^* g_s \lneq^* f_s \lneq^* f_t$.

Here, g_s denotes the first component of $\varphi(s)$ while f_s denotes the second component of $\varphi(s)$. The embedding of both the tree ordering and lexicographic ordering of $T(\omega_2)$ is important in Woodin's construction to ensure all appropriate gaps are filled through ccc forcing. The saturated linear order is embedded recursively by the first component of φ. In other words, the embedding is into the first copy of ${}^\omega\omega$; i.e., the image is a collection of "g-functions." So all the functions obtained from the second component of φ (i.e., the "f-functions") are not in the image of the saturated linear order. In other words, by the nature of Woodin's construction, the range of the embedding function excludes $\mathfrak{c}$-many elements that are linearly ordered with the image of the saturated linear order.[24]

Although Woodin's argument does not directly lend itself to obtaining maximality, it is by combining many of the clever ideas introduced by Woodin, Laver

β is even. It is not hard to see that as long as $2^{<\omega_2} = \omega_2$, which Woodin assumes, the collection of eventually alternating sequences is a saturated linear order of size ω_2 under the lexicographic ordering. Cf. Lemma 2.1.1.

[24]See pages 31 to 47 of [Woo84] for more details. Note that Woodin only considers the case where $\mathfrak{c} = \omega_2$ in the final forcing extension.

and Kunen in a strategic manner that we are able to construct the appropriate model.

2 Basic Definitions and Facts

In this chapter we collect some important general definitions, notation and results that are related to (and many are needed for) the work in Chapters 3 and 4. Some of our definitions here come from [Kun80]. For more definitions and related results, the reader may also wish to consult [Kun11], [JW96], and [JW97].

2.1 Gaps and Saturated Linear Orders

The definitions of cuts, partition cuts, pregaps, gaps and saturated linear orders can be found in Chapter 1 (see Definitions 4, 5 and 6).

Note that a pregap is automatically a cut. Conversely, it's easy to find a pregap that is equivalent to any given cut, in the following sense:

Definition 9. Suppose $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ are cuts in $({}^\omega\mathbb{Q}, <^*)$. Say $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ are equivalent, and write $(\mathcal{F}, \mathcal{G}) \cong (\mathcal{F}', \mathcal{G}')$, if $\mathcal{F}$, $\mathcal{F}'$ are mutually cofinal, and $\mathcal{G}$, $\mathcal{G}'$ are mutually coinitial; i.e., given any $f \in \mathcal{F}$, $g \in \mathcal{G}$ there is some $f' \in \mathcal{F}'$, $g' \in \mathcal{G}'$ such that $f <^* f'$, $g' <^* g$, and for each $f' \in \mathcal{F}'$, $g' \in \mathcal{G}'$ there is some

$f \in \mathcal{F}$, $g \in \mathcal{G}$ such that $f' <^* f$, $g <^* g'$.

Definition 10. Given a cut (L, R) in some linear order X, we say a pregap (F, G) in X *represents* the cut (L, R) if $(F, G) \cong (L, R)$.

Note that if two regular pregaps (i.e., two pregaps indexed by regular cardinals) represent the same cut, then the two pregaps must have the same type.

Definition 11. The type of a cut is defined to be the type of any regular pregap that represents it. If the type of a cut is (γ, γ), we refer to it as a γ-cut.

Remark 2. Note that two gaps are equivalent iff they represent the same partition cut.

Definition 12. Let $(X, <)$ be a linear order, let (A, B) and (C, D) be distinct partition cuts in X. We say that the cut (A, B) is *to the left of* the cut (C, D), and we write $(A, B) < (C, D)$, iff $A \subset C$. Note that this defines a linear ordering on the collection of partition cuts of X.

Definition 13. Let $\mathcal{L}$ be a linear order and let A be any subset of $\mathcal{L}$. Let $LeftPart_{\mathcal{L}}(A) = \{r \in \mathcal{L} : \exists a \in A, r \leq a\}$. (This is the downward closure of A within $\mathcal{L}$ that makes A into the left side of a partition cut in $\mathcal{L}$; the right side of the partition cut would be $\mathcal{L} \setminus LeftPart_{\mathcal{L}}(A)$.)

As suggested in Definition 13, a cut can always be extended to a partition cut. To see this, let (A, B) be a cut in a linear order $\mathcal{L}$. Let $A' = LeftPart_{\mathcal{L}}(A)$, let

$B' = \mathcal{L} \setminus LeftPart_{\mathcal{L}}(A)$. Note that $A \subseteq A'$, $B \subseteq B'$, and (A', B') is a partition cut in $\mathcal{L}$.

We now prove three important facts about saturated linear orders.

Fact 1. Every saturated linear order of size κ contains an isomorphic copy of each linear order of size $\leq \kappa$.

Proof. Let $(S, <)$ be a saturated linear order of size κ, and let $(X, <)$ be any linear order of size $\lambda \leq \kappa$. Fix an arbitrary enumeration $X = \{x_\alpha : \alpha < \lambda\}$, and for each $\alpha < \lambda$, let $X_\alpha = \{x_\beta : \beta \leq \alpha\}$.

We define the sequence $(f_\alpha)_{\alpha<\lambda}$ by recursion such that $\forall \alpha < \lambda$:

1. $\forall \beta < \alpha$, $f_\beta \subseteq f_\alpha$, and

2. $f_\alpha : X_\alpha \longrightarrow S$ is order-preserving

Fix $\beta \in \lambda$ and let $\bar{f}_\beta = \bigcup_{\eta<\beta} f_\eta$. Let $A = \{x_\alpha : x_\alpha < x_\beta, \alpha < \beta\}$, and let $B = \{x_\alpha : x_\alpha > x_\beta, \alpha < \beta\}$. Let $A' = \bar{f}_\beta[A]$, and let $B' = \bar{f}_\beta[B]$. Then (A', B') is a cut in S of size $< \lambda \leq \kappa$. Since S is a saturated linear order of size κ, there is a $y \in S$ such that $A' < y < B'$. Let $f_\beta = \bar{f}_\beta \cup \{(x_\beta, y)\}$.

Having defined the sequence of functions $(f_\alpha)_{\alpha<\lambda}$, let $f = \bigcup_{\alpha<\lambda} f_\alpha$. So $f : X \longrightarrow S$ is an embedding. □

Fact 2. Fix κ any cardinal, and suppose L, R are both saturated linear orders of size κ. Then L and R are order-isomorphic.

Proof. Fix enumerations $L = \{l_\alpha : \alpha < \kappa\}$ and $R = \{r_\alpha : \alpha < \kappa\}$. By simultaneous recursion, the sequences of functions $(f_\alpha)_{\alpha<\kappa}$, $(g_\alpha)_{\alpha<\kappa}$ and the sequences of sets $(X_\alpha)_{\alpha<\kappa}$ and $(Y_\alpha)_{\alpha<\kappa}$ are defined such that for each $\alpha < \kappa$:

1. $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ and $g_\alpha : Y_\alpha \longrightarrow X_\alpha$ are order-preserving;

2. $\forall \beta < \alpha$, $X_\beta \subseteq X_\alpha \subseteq L$, $Y_\beta \subseteq Y_\alpha \subseteq R$, $f_\beta \subseteq f_\alpha$, and $g_\beta \subseteq g_\alpha$;

3. $f_\alpha^{-1} = g_\alpha$.

To begin, let $X_0 = \{l_0\}$, and let $f_0(l_0) = r_0$. Let $Y_0 = \{r_0\}$, and let $g_0(r_0) = l_0$.

Next, suppose f_α, g_α, X_α, and Y_α have been defined for all $\alpha < \beta$, where $\beta < \kappa$ is a limit ordinal. Let $f_\beta = \bigcup_{\alpha<\beta} f_\alpha$, let $g_\beta = \bigcup_{\alpha<\beta} g_\alpha$, let $X_\beta = \bigcup_{\alpha<\beta} X_\alpha$, and let $Y_\beta = \bigcup_{\alpha<\beta} Y_\alpha$.

Finally, fix $\beta < \kappa$, and suppose f_α, g_α, X_α, and Y_α have been defined for all $\alpha \leq \beta$. Let $\bar{X}_{\beta+1} = X_\beta \cup \{l_\gamma\}$, where $\gamma = \min\{\bar{\gamma} < \kappa : l_{\bar{\gamma}} \notin X_\beta\}$. Let $A = \{l \in X_\beta : l < l_\gamma\}$, and let $B = \{l \in X_\beta : l > l_\gamma\}$. Let $A' = f_\beta[A]$, and let $B' = f_\beta[B]$. Find $\bar{r} \in R$ such that $A' < \bar{r} < B'$. Let $\bar{Y}_{\beta+1} = Y_\beta \cup \{\bar{r}\}$. Let $\bar{f}_{\beta+1} = f_\beta \cup \{(l_\gamma, \bar{r})\}$, and let $\bar{g}_{\beta+1} = g_\beta \cup \{(\bar{r}, l_\gamma)\}$. Let $\delta = \min\{\bar{\delta} < \kappa : r_{\bar{\delta}} \in R \setminus \bar{Y}_{\beta+1}\}$. Let $Y_{\beta+1} = \bar{Y}_{\beta+1} \cup \{r_\delta\}$. Let $C = \{r \in \bar{Y}_{\beta+1} : r < r_\delta\}$, and let $D = \{r \in \bar{Y}_{\beta+1} : r > r_\delta\}$. Let $C' = \bar{g}_{\beta+1}[C]$, and let $D' = \bar{g}_{\beta+1}[D]$. Find $\bar{l} \in L$ such that $C' < \bar{l} < D'$. Let $X_{\beta+1} = \bar{X}_{\beta+1} \cup \{\bar{l}\}$. Let $f_{\beta+1} = \bar{f}_{\beta+1} \cup \{(\bar{l}, r_\delta)\}$, and let $g_{\beta+1} = \bar{g}_{\beta+1} \cup \{(r_\delta, \bar{l})\}$.

Having completed the recursive definitions, let $f = \bigcup_{\alpha<\kappa} f_\alpha$. It is straightforward to see that $f : L \longrightarrow R$ is an order-isomorphism. □

Fact 3. Let κ be an infinite cardinal. There exists a saturated linear order of size κ iff κ is regular and $2^{<\kappa} = \kappa$.

Proof. The proof of this fact follows from the three lemmas below. □

Lemma 2.1.1. *Let κ be an infinite cardinal. If κ is regular and $2^{<\kappa} = \kappa$, then there exists a saturated linear order of size κ.*

Proof. Fix any set A of size κ, and an enumeration $A = \{a_\alpha : \alpha < \kappa\}$. Since κ is regular, $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$, and so:

$$|[A]^{<\kappa} \times [A]^{<\kappa}| = \kappa^{<\kappa} = (2^{<\kappa})^{<\kappa} = 2^{<\kappa} = \kappa$$

It follows that the set $\mathcal{C} = \{(C, D) : C, D \subseteq A,\ |C|, |D| < \kappa, \text{ and } C \cap D = \emptyset\}$ is of size κ. Fix an enumeration $\{(C_\alpha, D_\alpha) : \alpha < \kappa\}$ of $\mathcal{C}$ such that each pair from $\mathcal{C}$ occurs cofinally often in the enumeration.

For each $\beta < \kappa$, let $A_\beta = \{a_\alpha : \alpha < \beta\}$. The increasing sequence of orderings $(\prec_\alpha)_{\alpha<\kappa}$ is defined by recursion such that, for each $\alpha < \kappa$:

1. $(A_\alpha, \prec_\alpha)$ is a linear order, and

2. $\forall \beta < \alpha$, $\prec_\beta \subseteq \prec_\alpha$.

Fix $\alpha < \kappa$. If α is a limit ordinal, let $\prec_\alpha = \bigcup_{\beta<\alpha} \prec_\beta$. If $\alpha = \beta+1$, look at the pair (C_β, D_β). If (C_β, D_β) is not a cut in $(A_\beta, \prec_\beta)$, extend $\prec_\beta$ to $\prec_\alpha$ by letting $A_\beta \prec_\alpha a_\beta$. Otherwise, if (C_β, D_β) is a cut in $(A_\beta, \prec_\beta)$, let $C'_\beta = LeftPart_{(A_\beta,\prec_\beta)}(C_\beta)$ and let $D'_\beta = A_\beta \setminus LeftPart_{(A_\beta,\prec_\beta)}(C_\beta)$. Extend $\prec_\beta$ to $\prec_\alpha$ such that $C'_\beta \prec_\alpha a_\beta \prec_\alpha D'_\beta$.

Let $\prec = \bigcup_{\alpha<\kappa} \prec_\alpha$. To see that $(A, \prec)$ is a saturated linear order of size κ, fix any cut (E^*, F^*) in A of size $< \kappa$. Let $E = \{\alpha < \kappa : a_\alpha \in E^*\}$, and let $F = \{\alpha < \kappa : a_\alpha \in F^*\}$. Since $|E|, |F| < \kappa$ and κ is regular, $E \cup F$ is bounded in κ. Find $\alpha > \sup(E \cup F)$ such that $(C_\alpha, D_\alpha) = (E^*, F^*)$. So (C_α, D_α) is a cut in A_α, and a_α fills this cut. □

Lemma 2.1.2. *Let κ be an infinite cardinal. If there exists a saturated linear order of size κ, then κ is regular.*

Proof. Suppose $(X, \prec)$ is a saturated linear order of size κ, and suppose instead that $cf(\kappa) = \lambda < \kappa$. Let $X = \{x_\delta : \delta < \kappa\}$.

Fix a strictly increasing sequence $(\beta_\alpha)_{\alpha<\lambda}$ cofinal in κ. Define, for each $\alpha < \lambda$, $X_{\beta_\alpha} = \{x_\delta : \delta < \beta_\alpha\}$; note that

$$X = \bigcup_{\alpha<\lambda} X_{\beta_\alpha}$$

Define a sequence $(y_\alpha)_{\alpha<\lambda}$ by recursion: for each $\alpha < \lambda$, choose y_α filling the cut $(X_{\beta_\alpha} \cup \{y_\delta : \delta < \alpha\}, \emptyset)$.

Next, fix $y^* \in X$ filling the cut $(\{y_\alpha : \alpha < \lambda\}, \emptyset)$. Find $\alpha < \lambda$ such that $y^* \in X_{\beta_\alpha}$; but $X_{\beta_\alpha} \prec y_\alpha \prec y^*$, a contradiction. □

Lemma 2.1.3. *Let κ be an infinite cardinal. If there exists a saturated linear order of size κ, then $2^{<\kappa} = \kappa$.*

Proof. Suppose R is a saturated linear order of size κ, and suppose instead $2^{<\kappa} > \kappa$. (Note that by Lemma 2.1.2, κ is regular.) Let

$$\lambda = \min\{\lambda' < \kappa : 2^{\lambda'} > \kappa\}$$

Let $X = {}^{<\lambda}2$, ordered by $\prec$, where $\prec$ is the lexicographic ordering:

Definition 14. $\forall s, t \in {}^{\leq\lambda}2$, define $s \prec t$ iff either

1. $dom(s) < dom(t)$ and $\forall \alpha \in dom(s)$, $s(\alpha) = t(\alpha)$; or
2. $\exists \alpha \in dom(s) \cap dom(t)$ such that $s(\alpha) \neq t(\alpha)$, and $s(\bar{\alpha}) < t(\bar{\alpha})$, where $\bar{\alpha} = \min\{\alpha : s(\alpha) \neq t(\alpha)\}$.

Let $A = \{f \in {}^{\lambda}2 : f$ is not eventually constant$\}$. Note that $|A| = 2^\lambda > \kappa$.

Definition 15. For each $f \in A$, define:

- $l(f) = \{s \in X : dom(s) = \alpha + 1, (\forall \beta < \alpha,\ s(\beta) = f(\beta))$, and $s(\alpha) < f(\alpha)\}$, and

- $r(f) = \{s \in X : dom(s) = \alpha + 1,\ (\forall \beta < \alpha,\ s(\beta) = f(\beta)),\text{ and } s(\alpha) > f(\alpha)\}$.

Note that for each $f \in A$, $(l(f), r(f))$ is a cut of size λ in $(X, \prec)$.

Definition 16. Given two cuts $(A^0, A^1), (B^0, B^1)$ in a linear order $(L, <)$, we say $(A^0, A^1), (B^0, B^1)$ are compatible, and we write $(A^0, A^1) \not\perp (B^0, B^1)$, iff $(A^0 \cup B^0, A^1 \cup B^1)$ is a cut in $(L, <)$. We say $(A^0, A^1), (B^0, B^1)$ are incompatible and we write $(A^0, A^1) \perp (B^0, B^1)$ iff the two cuts are not compatible (i.e., iff $\exists x \in A^0 \cup B^0, y \in A^1 \cup B^1)$ such that $y < x$).

Claim 1. *For each $f \neq g \in A$, $(l(f), r(f)) \perp (l(g), r(g))$.*

Proof. Fix $f \prec g \in A$. Let $\bar{\alpha} = \min\{\alpha : f(\alpha) \neq g(\alpha)\}$; so $f(\bar{\alpha}) = 0$, $g(\bar{\alpha}) = 1$. Define $\bar{t} \in {}^{\bar{\alpha}+1}2$ by $\bar{t} = g \restriction (\bar{\alpha} + 1)$. So $\bar{t} \in r(f)$. Let $\bar{\gamma} = \min\{\gamma > \bar{\alpha} : g(\gamma) = 1\}$. Define $\bar{s} \in {}^{\bar{\gamma}+1}2$ by:

$$\bar{s}(\beta) = \begin{cases} g(\beta), & \text{if } \beta < \bar{\gamma}, \\ 0 & \text{if } \beta = \bar{\gamma}. \end{cases}$$

So $\bar{s} \in l(g)$. Since $\bar{s} \restriction (\bar{\alpha} + 1) = \bar{t}$, $\bar{t} < \bar{s}$. □

So each element $f \in A$ determines a distinct cut in X; in other words, there are $|A| = 2^{\lambda} > \kappa$ many incompatible cuts of size $< \kappa$ in X. But $|X| = 2^{<\lambda} \leq \kappa$, by choice of λ. Thus, since R is a saturated linear order of size κ, by Fact 1, there must be a copy X' of X in R. But then there are $> \kappa$-many distinct cuts of size

$< \kappa$ in $X' \subseteq R$, each of which must be filled by a distinct element of R, of which there are only κ-many. □

This section concludes with two theorems relating saturated linear orders and $\mathfrak{c}$.

Theorem 2.1.4. *Under CH, every pantachie is a saturated linear order.*

Proof. Let L be a maximal linear order in $({}^\omega\mathbb{R}, <^*)$. Suppose that (A, B) is an unfilled cut in L of size $< \mathfrak{c} = \omega_1$. Note that any such cut is fillable in $({}^\omega\mathbb{R}, <^*)$ (cf. Theorem 1 from [Sch93], page 443), so suppose $b \in {}^\omega\mathbb{R}$ fills (A, B). Since (A, B) is unfilled in L, $b \notin L$. But $(L \cup \{b\}, <^*)$ is a linear order, contrary to the maximality of L. □

Theorem 2.1.5. *Every saturated linear order contains a* $(\mathfrak{c}, \mathfrak{c})$*-gap.*

Proof. Let $(X, <)$ be a saturated linear order of size $\mathfrak{c}$. Fix an enumeration $X = \{x_\alpha : \alpha < \mathfrak{c}\}$. The sequences $A = \{a_\alpha : \alpha < \mathfrak{c}\}$ and $B = \{b_\alpha : \alpha < \mathfrak{c}\}$ are defined by recursion, along with the auxiliary sets $\bar{A}_\alpha$ and $\bar{B}_\alpha$ for each $\alpha < \mathfrak{c}$.

To begin, choose any $a_0 \in X$. If $x_0 \leq a_0$, let $\bar{A}_0 = \{a_0, x_0\}$ and let $B_0 = \emptyset$. Otherwise, let $\bar{A}_0 = \{a_0\}$ and let $B_0 = \{x_0\}$. Choose b_0 filling the cut $(\bar{A}_0, B_0)$, and let $\bar{B}_0 = B_0 \cup \{b_0\}$.

Next, suppose a_α, b_α, $\bar{A}_\alpha$, $\bar{B}_\alpha$ have been defined. If $\exists a \in \bar{A}_\alpha$ such that $x_{\alpha+1} \leq a$, let $A_{\alpha+1} = \bar{A}_\alpha \cup \{x_{\alpha+1}\}$ and let $B_{\alpha+1} = \bar{B}_\alpha$. Otherwise, let $A_{\alpha+1} = \bar{A}_\alpha$, and let

$B_{\alpha+1} = \bar{B}_\alpha \cup \{x_{\alpha+1}\}$. Choose $a_{\alpha+1}$ filling $(A_{\alpha+1}, B_{\alpha+1})$, and let $\bar{A}_{\alpha+1} = A_{\alpha+1} \cup \{a_{\alpha+1}\}$; choose $b_{\alpha+1}$ filling $(\bar{A}_{\alpha+1}, B_{\alpha+1})$, and let $\bar{B}_{\alpha+1} = B_{\alpha+1} \cup \{b_{\alpha+1}\}$.

Finally, suppose δ is a limit ordinal and a_α, b_α, $\bar{A}_\alpha$, $\bar{B}_\alpha$ have been defined for all $\alpha < \delta$. Let $A_\delta = \bigcup_{\alpha<\delta} \bar{A}_\alpha \cup \{x_\alpha : \alpha \leq \delta, \exists\beta < \delta \text{ such that } x_\alpha \leq a_\beta\}$, and let $B_\delta = \bigcup_{\alpha<\delta} \bar{B}_\alpha \cup \{x_\alpha : \alpha \leq \delta, \forall\beta < \delta, a_\beta < x_\alpha\}$. Choose a_δ filling the cut (A_δ, B_δ), and let $\bar{A}_\delta = A_\delta \cup \{a_\delta\}$; choose b_δ filling the cut $(\bar{A}_\delta, B_\delta)$, and let $\bar{B}_\delta = B_\delta \cup \{b_\delta\}$.

Clearly, (A, B) is a pregap of type $(\mathfrak{c}, \mathfrak{c})$. To see that (A, B) is a gap in $(X, <)$, suppose instead $x_\beta \in X$ fills (A, B). But then x_β fills $(\bar{A}_\beta, \bar{B}_\beta)$; so either $x_\beta < a_{\beta+1}$ or $b_{\beta+1} < x_\beta$, each of which contradicts the fact that x_β fills (A, B). □

2.2 Forcing

It is assumed that the reader is familiar with forcing, so we have not included the definitions of basic forcing terminology in this section, such as the notion of $\mathbb{P}$-*names* in general (where $\mathbb{P}$ is a partial order). However, for the reader's convenience, we do record certain definitions that are used in the subsequent chapters, including *nice names*, $\mathbb{P}$-*names for partial orders*, and other notions relating to iterated forcing. These definitions come from [Kun80], and the reader is referred there for more background.

Throughout this section, M denotes a countable transitive model (ctm) for ZFC.

Definition 17 (Nice Names)**.** Given a $\mathbb{P}$-name σ, a *nice name* for a subset of σ is a $\mathbb{P}$-name τ of the form $\bigcup\{\{\pi\} \times A_\pi : \pi \in dom(\sigma)\}$, where each A_π is an antichain in $\mathbb{P}$.

Definition 18 (Name for a partially ordered set)**.** Given $\mathbb{P}$ a p.o. (partially ordered set) in M, a $\mathbb{P}$-*name for a p.o. (in M)* is a triple of $\mathbb{P}$-names $\langle \pi, \leq_\pi, \mathbf{1}_\pi \rangle \in M$ such that $\mathbf{1}_\pi \in \operatorname{dom}(\pi)$ and

$$\mathbf{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} (\mathbf{1}_\pi \in \pi) \wedge (\leq_\pi \text{ is a partial order on } \pi \text{ with largest element } \mathbf{1}_\pi).$$

We'll often write just π for $\langle \pi, \leq_\pi, \mathbf{1}_\pi \rangle$.

Definition 19. If $\mathbb{P}$ is a p.o. in M and π is a $\mathbb{P}$-name for a p.o. (in M), then $\mathbb{P} * \pi$ is the p.o. with base set

$$\{(p, \tau) : p \in \mathbb{P} \wedge \tau \in \operatorname{dom}(\pi) \text{ and } p \Vdash \tau \in \pi\},$$

with ordering defined by

$$(p, \tau) \leq (q, \sigma) \Leftrightarrow p \leq_{\mathbb{P}} q \text{ and } p \Vdash \tau \leq_\pi \sigma.$$

Note that $\mathbf{1}_{\mathbb{P}*\pi} = (\mathbf{1}_{\mathbb{P}}, \mathbf{1}_\pi)$. Define $i : \mathbb{P} \longrightarrow \mathbb{P} * \pi$ by $i(p) = (p, \mathbf{1}_\pi)$.[25]

Definition 20. In the above notation, if G is $\mathbb{P}$-generic over M and $H \subseteq \pi_G$, then $G * H = \{(p, \tau) \in \mathbb{P} * \pi : p \in G \wedge \tau_G \in H\}$.

[25]Note that the notation $\dot{\mathbb{Q}}$ is often used for a $\mathbb{P}$-name for a partial order, rather than π.

Remark 3. If H above is π_G-generic over $M[G]$, then $G * H$ is $\mathbb{P} * \pi$-generic over M.

Finally, we come to Kunen's definition of a (finite support) iterated forcing construction.

Definition 21 (Iterated Forcing Construction)**.** Suppose α is any ordinal. An α-*stage iterated forcing construction* (with finite supports), or an α-IFC for short, is an object (in M) of the form:

$$(\{\mathbb{P}_\xi : \xi \leq \alpha\}, \{\pi_\xi : \xi < \alpha\})$$

which satisfies the following:

- For each $\xi \leq \alpha$, $\mathbb{P}_\xi$ is a p.o. (in M), and for each $\xi < \alpha$, π_ξ is a $\mathbb{P}_\xi$-name for a p.o. (in M).
- Elements p of $\mathbb{P}_\xi$ are sequences $\langle \rho_\mu : \mu < \xi \rangle$ of length ξ such that each $\rho_\mu \in \operatorname{dom}(\pi_\mu)$. We also write $p(\mu) = \rho_\mu$.
- If $p \in \mathbb{P}_\eta$ and $\xi < \eta$, then $p \upharpoonright \xi \in \mathbb{P}_\xi$.

For each $p \in \mathbb{P}_\xi$, define $\operatorname{supp}(p) = \{\mu < \xi : p(\mu) \neq \mathbf{1}_{\pi_\mu}\}$. The following conditions must also be satisfied:

1. *Basis.* $\mathbb{P}_0 = \{\emptyset\}$.
2. *Successors.*

- If $p = \langle \rho_\mu : \mu \leq \xi \rangle$, then $p \in \mathbb{P}_{\xi+1}$ iff $p \restriction \xi \in \mathbb{P}_\xi$, $\rho_\xi \in \operatorname{dom}(\pi_\xi)$, and $p \restriction \xi \Vdash \rho_\xi \in \pi_\xi$.
- If $p, p' \in \mathbb{P}_{\xi+1}$, then $p \leq p'$ iff $p \restriction \xi \leq p' \restriction \xi$ and $p \restriction \xi \Vdash p(\xi) \leq p'(\xi)$.

3. *Limits.*

- If $\eta \leq \alpha$ is a limit ordinal and $p = \langle \rho_\mu : \mu < \eta \rangle$, then $p \in \mathbb{P}_\eta$ iff $\forall \xi < \eta$, $p \restriction \xi \in \mathbb{P}_\xi$, and $\operatorname{supp}(p)$ is finite.
- If $p, p' \in \mathbb{P}_\eta$, then $p \leq p'$ iff $\forall \xi < \eta$, $p \restriction \xi \leq p' \restriction \xi$.

Remark 4. Note that for each ξ, once π_ξ is defined, $\mathbb{P}_{\xi+1}$ is determined, and $\mathbb{P}_{\xi+1}$ is isomorphic to $\mathbb{P}_\xi * \pi_\xi$. For η a limit ordinal (including the case $\eta = \alpha$), $\mathbb{P}_\eta$ is determined uniquely once the π_ξ (for $\xi < \eta$) are defined.

Definition 22. In the notation of Definition 21, let $\xi < \eta \leq \alpha$. Define $i_{\xi\eta} : \mathbb{P}_\xi \to \mathbb{P}_\eta$ by letting $i_{\xi\eta}(p) = p'$, where for each $\beta \in \eta$, $p'(\beta) = p(\beta)$ if $\beta < \xi$ and $p'(\beta) = \mathbf{1}_{\pi_\beta}$ otherwise. Given a $\mathbb{P}_\eta$-generic over M filter G, let $G_{\xi\eta} = i_{\xi\eta}^{-1}(G)$. Sometimes we write simply G_ξ if the context permits, or the more informal $G \cap \mathbb{P}_\xi$.

Definition 23. Given p.o.'s $\mathbb{P}$ and $\mathbb{Q}$ and a function $i : \mathbb{P} \longrightarrow \mathbb{Q}$, define, by recursion over $\mathbb{P}$-names τ,

$$i_*(\tau) = \{\langle i_*(\sigma), i(p) \rangle : \langle \sigma, p \rangle \in \tau\}$$

Remark 5. Referring to Definition 22, we have the following:

- For each $\tau \in M^{\mathbb{P}_\xi}$, $\tau[i_{\xi\eta}^{-1}(G)] = i_*(\tau)[G]$.

- Given a formula $\varphi(x_1, \ldots, x_n)$ which is absolute for transitive models of ZFC,

$$p \Vdash_{\mathbb{P}_\xi} \text{“}\varphi(\tau_1, \ldots, \tau_n)\text{” iff } i(p) \Vdash_{\mathbb{P}_\eta} \text{“}\varphi(i_*(\tau_1), \ldots, i_*(\tau_n)).\text{”}$$

For the reader's convenience, we record the statement of two important forcing axioms.

Definition 24 (Martin's Axiom (MA))**.** Fix a cardinal κ. MA(κ) is the statement: Whenever $(\mathbb{P}, \leq)$ is a non-empty ccc partial order and $\mathcal{D}$ is a family of $\leq \kappa$ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ such that $\forall D \in \mathcal{D}$, $G \cap D \neq \emptyset$. MA is the statement $\forall \kappa < 2^\omega$, MA($\kappa$). MA[$\sigma$-centered] (respectively, MA[σ-linked]) is the statement MA except with “ccc” replaced by “σ-centered” (respectively, “σ-linked”).

Definition 25 (Closed, Unbounded, Stationary)**.** Fix A any set, and λ a regular cardinal. Fix $C \subseteq [A]^{<\lambda}$. C is said to be *unbounded* in $[A]^{<\lambda}$ if $(\forall x \in [A]^{<\lambda})(\exists y \in C)(x \subseteq y)$. C is said to be *closed* in $[A]^{<\lambda}$ if for every increasing sequence $(x_\alpha)_{\alpha<\mu}$ of length $\mu < \lambda$ from C, $\bigcup\{x_\alpha : \alpha < \mu\} \in C$. A set $S \subseteq [A]^{<\lambda}$ is called *stationary* in $[A]^{<\lambda}$ if S meets all closed unbounded sets in $[A]^{<\lambda}$.

Definition 26 (Proper Partial Orders)**.** A partial order $\mathbb{P}$ is *proper* if for every uncountable set X, and for every stationary set $S \subseteq [X]^{<\omega_1}$, S is still stationary in the forcing extension by $\mathbb{P}$.

Definition 27 (Proper Forcing Axiom (PFA))**.** PFA is the statement: Whenever $(\mathbb{P}, \leq)$ is a non-empty proper partial order and $\mathcal{D}$ is a family of $\leq \omega_1$ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ such that $\forall D \in \mathcal{D}$, $G \cap D \neq \emptyset$.

3 A Saturated Pantachie and $\neg$CH

This chapter features an iterated forcing construction similar to [Lav79] to obtain a saturated linear order in Hausdorff's partial order restricted to the rationals, $({}^{\omega}\mathbb{Q}, <^*)$, with $\mathfrak{c} > \omega_1$. We then show that in this forcing extension, the saturated linear order is in fact maximal in Hausdorff's (unrestricted) partial order, $({}^{\omega}\mathbb{R}, <^*)$, and thus is a Hausdorff pantachie with no ω_1-gaps. This answers Hausdorff's question (α) from 1907 (see page 165 from [Hau05]), as described in Chapter 1. In other words, we prove the following:

Theorem 3.0.1. *Con(ZFC + $\exists$ a maximal saturated linear order of size $\mathfrak{c} \geq \omega_2$ in $({}^{\omega}\mathbb{R}, <^*)$).*

3.1 The Kunen Partial Order

Definition 28. Fix γ an ordinal, and for each $\alpha < \gamma$, let $h_\alpha \in {}^{\omega}\mathbb{Q}$ such that $\mathcal{L}_\gamma = \{h_\alpha : \alpha < \gamma\}$ is linearly ordered by $<^*$. Suppose further that (A^*, B^*) is a partition cut in $\mathcal{L}_\gamma$, and let $A = \{\alpha : h_\alpha \in A^*\}$, $B = \{\alpha : h_\alpha \in B^*\}$. Define:

- $K(A, B) = \{(L, R, s) \mid L \in [A]^{<\omega}, R \in [B]^{<\omega}, s \in {}^{<\omega}\mathbb{Q}, \forall k \geq |s|, \forall \alpha \in L, \beta \in R, h_\alpha(k) < h_\beta(k)\}$.[26]

When A and B are clear from the context, we write simply K rather than $K(A, B)$. Define a partial ordering on K by $(L', R', s') \leq (L, R, s)$ iff

(i) $L \subseteq L', R \subseteq R', s \subseteq s'$; and

(ii) $\forall k \in |s'| \setminus |s|$, $\forall \alpha \in L, \beta \in R$, $h_\alpha(k) < s'(k) < h_\beta(k)$.[27]

Under this ordering, call K the Kunen forcing for filling the cut (A^*, B^*).

Fact 4. In the notation of Definition 28, K is a partial order.

Proof. It's easy to show reflexivity and symmetry, so we turn to transitivity. Fix $(L'', R'', s'') \leq (L', R', s') \leq (L, R, s)$. Fix $k \in |s''| \setminus |s|$, $\alpha \in L$ and $\beta \in R$. If $k < |s'|$, then $h_\alpha(k) < s''(k) < h_\beta(k)$ since $(L', R', s') \leq (L, R, s)$ and $s''(k) = s'(k)$. If $k \geq |s''|$, then $h_\alpha(k) < s''(k) < h_\beta(k)$ since $(L'', R'', s'') \leq (L', R', s')$, $L \subseteq L'$ and $R \subseteq R'$. □

Let (A^*, B^*) be a partition cut, and let $K = K(A^*, B^*)$. Forcing with K adds a function which fills the cut. More precisely:

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[26]It would be easy to define instead $K(A^*, B^*)$ and thus have no need for the indexing. However, for our purposes there will normally be such an indexing, which is convenient especially in the forcing constructions. Nevertheless, we occasionally write $K(A^*, B^*)$ instead of $K(A, B)$.

[27]More precisely, we should say $\forall k \in |s'| \setminus |s|$: $\forall \alpha \in L$, $h_\alpha(k) < s'(k)$ and $\forall \beta \in R$, $s'(k) < h_\beta(k)$; this version correctly covers the case when one of L or R is empty.

Fact 5. Suppose M is any countable transitive model for ZFC. Working in M, fix $\bar{\gamma}$ an ordinal, and for each $\alpha < \bar{\gamma}$, let $h_\alpha \in {}^\omega\mathbb{Q}$ such that $\mathcal{L}_{\bar{\gamma}} = \{h_\alpha : \alpha < \bar{\gamma}\}$ is linearly ordered by $<^*$. Suppose further that (A^*, B^*) is a partition cut in $\mathcal{L}_{\bar{\gamma}}$, and let $A = \{\alpha : h_\alpha \in A^*\}$, $B = \{\alpha : h_\alpha \in B^*\}$. Let $K = K(A, B)$, let G be any K-generic over M filter, and let $h_{\bar{\gamma}} = h_G = \bigcup\{s_p : p \in G\}$. Then

(a) $h_{\bar{\gamma}} : \omega \longrightarrow \mathbb{Q}$

(b) $A^* <^* h_{\bar{\gamma}} <^* B^*$.

Proof. For each $n \in \omega$, let $D_n = \{(L, R, s) \in K : n \in |s|\}$; for each $\beta \in A$, let $E_\beta = \{(L, R, s) \in K : \beta \in L\}$, and for each $\beta \in B$, let $J_\beta = \{(L, R, s) \in K : \beta \in R\}$. The following claim will aid in showing that these sets are dense.

Claim 2. *$\forall n \in \omega$, $\forall p = (L, R, s) \in K$, if $n \geq |s|$ then $\exists s' \supseteq s$ such that $|s'| = n$ and $p' = (L, R, s') \leq p$.*

Proof. Fix n and $p = (L, R, s)$ such that $n \geq |s|$. Define $s' \in {}^n\omega$ by

$$s'(k) = \begin{cases} s(k), & \text{if } k < |s| \\ \frac{\max\{h_\gamma(k):\gamma\in L\}+\min\{h_\delta(k):\delta\in R\}}{2} & \text{otherwise.} \end{cases}$$

It's easy to see that $p' = (L, R, s') \leq p$. □

To see that D_n is dense (for each n) is now immediate: fix n, and fix $p = (L, R, s) \in \mathbb{Q}$. If $n \in \text{dom}(s)$, then $p \in D_n$; otherwise, use Claim 2 to get $p' =$

$(L, R, s') \leq p$ with $|s'| = n + 1$; in particular, $n \in \text{dom}(s')$ so $p' \in D_n$.

Next, fix $\beta \in A$. To see that E_β is dense, fix $p = (L, R, s) \in K$, and let $L' = L \cup \{\beta\}$. Find $n > |s|$ s.t. $\forall k \geq n$, $\forall \gamma \in L'$, $\forall \delta \in R$, $h_\gamma(k) < h_\delta(k)$. Claim 2 gives us $p' = (L, R, s') \leq p$ such that $|s'| = n + 1$. So $(L', R, s') \in E_\beta$ and $(L', R, s') \leq p' \leq p$.

Finally, fix $\delta \in B$. To see that J_δ is dense, fix $p = (L, R, s) \in K$, and let $R' = R \cup \{\delta\}$. Find $n > |s|$ s.t. $\forall k \geq n$, $\forall \gamma \in L$, $\forall \delta \in R'$, $h_\gamma(k) < h_\delta(k)$. Using Claim 2 yet again, we get $p' = (L, R, s') \leq p$ such that $|s'| = n + 1$. So $(L, R', s') \in J_\delta$ and $(L, R', s') \leq p' \leq p$.

For part (a) now, fix any $n \in \omega$ and find $p = (L, R, s) \in G \cap D_n$. So $n \in \text{dom}(s) \subseteq \text{dom}(h_{\bar{\gamma}})$.

For (b), first fix $\beta \in A$. Find $p = (L, R, s) \in G \cap E_\beta$; so $\beta \in L$. Let $n_0 = |s|$. Fix $k \geq n_0$, and fix $q \in G \cap D_k$. Find $r \in G$ s.t. $r \leq p, q$. So $h_{\bar{\gamma}}(k) = s_r(k) > h_\beta(k)$.

Next, fix $\delta \in B$ and find $p \in G \cap J_\delta$; so $\delta \in R_p$. Let $n_0 = |s_p|$. Fix $k \geq n_0$, and fix $q \in G \cap D_k$. Find $r \in G$ s.t. $r \leq p, q$. So $h_{\bar{\gamma}}(k) = s_r(k) < h_\delta(k)$.

□

3.2 An Enumeration of Potential Cut-Names

Fix a countable transitive model M for ZFC, and in M, fix κ, a regular uncountable cardinal such that $2^{<\kappa} = \kappa$. In the next section, we will be defining a finite-support iterated forcing construction of length κ that will yield the p.o. $\mathbb{P} = \mathbb{P}_\kappa$ needed for the proof of Theorem 3.0.1. In the course of this iteration, we will be defining, for each $\alpha < \kappa$, a name for a cut (A^*_α, B^*_α) that will be filled, and we want to make sure that every possible cut in the saturated linear order of the forcing extension is filled cofinally often through the course of the iterated forcing construction.

We begin with some notation.

Notation. Given any ordinal α, let $X_\alpha = [\alpha]^{<\omega} \times [\alpha]^{<\omega} \times {}^{<\omega}\mathbb{Q}$. Note that for each α, $|X_\alpha| = \max\{|\alpha|, \omega\}$.

Definition 29. For each $\alpha \leq \kappa$, and each $p \in \Pi_{\beta<\alpha} X_\beta$, define the *support* of p as $\mathrm{supp}(p) = \{\beta < \alpha : p(\beta) \neq (\emptyset, \emptyset, \emptyset)\}$.

Definition 30. For $\alpha \leq \kappa$, let $P_\alpha = \{p \in \Pi_{\beta<\alpha} X_\beta : \mathrm{supp}(p) \text{ is finite}\}$, and let $P = P_\kappa$.

Definition 31. Let $\mathcal{K} = \{\{(\alpha, p) : \alpha \in \lambda, p \in A_\alpha\} : \lambda < \kappa$, and $(A_\alpha)_{\alpha<\lambda}$ is a λ-sequence of elements from $[P]^{\leq\omega}\}$.

In other words, $S \in \mathcal{K}$ iff there is some $\lambda < \kappa$ and some λ-sequence $(A_\alpha)_{\alpha<\lambda}$ of countable subsets of P such that $S = \{(\alpha, p) : \alpha \in \lambda, p \in A_\alpha\}$. As will be seen in Section 3.3, the set $\mathcal{K}$ serves as a set of "potential nice names for cuts".

Given any $S \in \mathcal{K}$, we want to be able to recover the associated ordinal λ, and for each $\alpha < \lambda$, the countable set A_α.

Definition 32. The *rank* of S is defined as $rank(S) = \bigcup \mathrm{dom}(S)$. Letting $\lambda = rank(S)$, define, for any $\alpha < \lambda$, $A_\alpha(S) = \bigcup\{p : (\alpha, p) \in S\}$. Note that $\lambda < \kappa$, and $\langle A_\alpha(S)\rangle_{\alpha<\lambda} \in {}^\lambda([P]^{\leq\omega})$.

We would also like to find the maximal ordinal that occurs in the support of any element from any of the countable sets A_α.

Definition 33. Define $\mathcal{A}(S) = \bigcup_{\alpha<\lambda} A_\alpha(S)$, and let $top(S) = \sup\{\max(\mathrm{supp}(p)) : p \in \mathcal{A}(S)\}$. Note that $top(S) < \kappa$.

It isn't too hard to see that $|P| = \kappa$ (cf. Lemma 3.8.1); hence, by our assumptions on κ, $|[P]^{\leq\omega}| = \kappa^{\leq\omega} \leq \kappa^{<\kappa} = (2^{<\kappa})^{<\kappa} = 2^{<\kappa} = \kappa$. This calculation is the main ingredient needed to see that $|\mathcal{K}| = \kappa$.

Fact 6. There is an enumeration $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$ of $\mathcal{K}$ where each member of $\mathcal{K}$ is enumerated cofinally often, and for each $\alpha < \kappa$, either $C_\alpha = \emptyset$ or $top(C_\alpha), rank(C_\alpha) < \alpha$.

Proof. To see that such an enumeration is possible, begin with a bijection $g : \kappa \to \mathcal{K} \times \kappa$. Define $h : \kappa \to \mathcal{K}$ by setting, for each $\alpha < \kappa$,

$$h(\alpha) = \begin{cases} S & \text{if } g(\alpha) = (S, \delta) \text{ for some (unique) } \delta \text{, and} \\ & rank(S), top(S) < \alpha \\ \emptyset & \text{otherwise} \end{cases}$$

Claim 3. *Each $S \in \mathcal{K}$ is mapped to cofinally often by h; i.e., $|h^{-1}(S)| = \kappa$.*

Proof. Fix $S \in \mathcal{K}$. Choose $\beta < \kappa$ such that each of $rank(S)$ and $top(S)$ are less than β. For each $\delta \in \kappa$, let $\alpha_\delta = g^{-1}(S, \delta)$. Since $\{\alpha_\delta : \delta \in \kappa\}$ is of size κ, the sets $Z = \{\alpha_\delta > \beta : \delta \in \kappa\}$ and $Y = \{\delta \in \kappa : \alpha_\delta > \beta\}$ are each of size κ. Note that for each $\delta \in Y$, $g(\alpha_\delta) = (S, \delta)$ and $rank(S), top(S) < \beta < \alpha_\delta$; it follows that for each $\alpha_\delta \in Z$, $h(\alpha_\delta) = S$. □

For each $\alpha < \kappa$, let $C_\alpha = h(\alpha)$. □

Fix an enumeration $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$ of $\mathcal{K}$ as in Fact 6. This enumeration $\mathcal{C}$ will be used throughout this chapter.

3.3 The Iterated Forcing Construction

The desired partial order is obtained through a κ-length finite-support iterated forcing over M, using the somewhat standard notation $(\{\mathbb{P}_\alpha : \alpha \leq \kappa\}, \{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\})$. By simultaneous recursion, and making use of the canonical names $\dot{G}_\alpha$ for the generics on the $\mathbb{P}_\alpha$, and the associated names $H_\alpha = \{t \in \dot{\mathbb{Q}}_\alpha : (\exists p \in \dot{G}_\alpha)\ p * t \in \dot{G}_{\alpha+1}\}$ for the generics on the $\dot{\mathbb{Q}}_\alpha$, we define $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$, $\{\mathcal{L}_\alpha : \alpha < \kappa\}$, $\{(A_\alpha, B_\alpha) : \alpha < \kappa\}$, and $\{h_\alpha : \alpha < \kappa\}$ such that for all $\alpha < \kappa$, $\mathcal{L}_\alpha = \{h_\beta : \beta < \alpha\}$ is a linear order in $({}^{\omega}\mathbb{Q}, <^*)$.

For $\alpha < \kappa$, look first at C_α, the α^{th} member of the enumeration $\mathcal{C}$ of $\mathcal{K}$ from the previous section (see Fact 6). Either $C_\alpha = \emptyset$ or $rank(C_\alpha), top(C_\alpha) < \alpha$. In the latter case, we can define, for each $p \in \mathcal{A}(C_\alpha)$, a corresponding $p' \in \mathbb{P}_\alpha$ by setting, for each $\beta < \alpha$, $p'(\beta)$ equal to the check name for $p(\beta)$, i.e., $p'(\beta) = (p(\beta))\check{}$.

Define $A'_\alpha = \{(\check{\beta}, p') : (\beta, p) \in C_\alpha\}$.

If $C_\alpha = \emptyset$ or if A'_α does not name a subset of α, let (A_α, B_α) name the pair of index sets $(\emptyset, \alpha)$ for a trivial partition cut. On the other hand, if A'_α does name a subset of α, let $(A'_\alpha)^*$ be the name for the corresponding subset of $\mathcal{L}_\alpha$ (i.e., the subset indexed by A'_α), let A_α be the index set for $LeftPart_{\mathcal{L}_\alpha}(A'_\alpha)^*$, and let B_α be the index set for $\mathcal{L}_\alpha \setminus LeftPart_{\mathcal{L}_\alpha}(A'_\alpha)^*$.

Let $\dot{\mathbb{Q}}_\alpha = K(A_\alpha, B_\alpha)$, the Kunen forcing for filling the cut (A^*_α, B^*_α).

Let $h_\alpha = \bigcup_{p \in H_\alpha} s_p$, the generic function that fills this cut.

Having completed the definition of the forcing construction, we let $\mathbb{P} = \mathbb{P}_\kappa$. Also, fix now and for the rest of this chapter a filter $G \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over M.

We next prove three lemmas that will show that $\mathbb{P}$ has the ccc. These three lemmas, and their proofs, are adapted from [Lav79].

3.4 The Uniformity Lemma

We begin with some definitions:

Remark 6. Fix $\alpha < \kappa$, fix $p \in \mathbb{P}_\alpha$. Recall that the *support* of p is $\text{supp}(p) = \{\beta < \alpha : p(\beta) \neq \mathbf{1}_{\dot{\mathbb{Q}}_\beta}\}$. Recall that $\mathbf{1}_{\dot{\mathbb{Q}}_\beta} = (\emptyset, \emptyset, \emptyset)\check{}$.

Definition 34. Fix $\alpha \leq \kappa$, and fix $p \in \mathbb{P}_\alpha$. If for each $\beta \in \text{supp}(p)$, there is a triple $(L_{p,\beta}, R_{p,\beta}, s_{p,\beta}) \in X_\beta \cap M$ such that $p(\beta) = (\check{L}_{p,\beta}, \check{R}_{p,\beta}, \check{s}_{p,\beta})$, then p is called a *determined* condition. We sometimes omit the subscript p or β when the context permits.

Definition 35. Fix $\alpha \leq \kappa$, fix $p \in \mathbb{P}_\alpha$. We say p has *closed support* if $\forall \beta \in \text{supp}(p)$, $L_\beta \cup R_\beta \subseteq \text{supp}(p)$.

Definition 36. Fix $\alpha \leq \kappa$, fix $p \in \mathbb{P}_\alpha$. We say p is *uniform* if $\exists l_p$ such that for all $\beta \in \text{supp}(p)$, $|s_\beta| = l_p$.

Definition 37. Fix $1 \leq \gamma \leq \kappa$. Let $U_\gamma = \{p \in \mathbb{P}_\gamma : p$ is determined, uniform and of closed support$\}$. Let $U_{\gamma,n} = \{p \in U_\gamma : l_p \geq n\}$.

Lemma 3.4.1. *For each $1 \leq \gamma \leq \kappa$ and each $n \in \omega$, $U_{\gamma,n}$ is dense in $\mathbb{P}_\gamma$.*

Proof. The proof proceeds by induction on γ. The basis and limit cases are straightforward, so we turn to the successor case. Fix $1 \leq \alpha < \kappa$, and suppose $U_{\alpha,n}$ is dense in $\mathbb{P}_\alpha$, for all $n \in \omega$. Fix a particular $n \in \omega$, and let $\gamma = \alpha + 1$. We will show that $U_{\gamma,n}$ is dense in $\mathbb{P}_\gamma$. Fix $p \in \mathbb{P}_\gamma$, and let $q_0 = p \restriction \alpha$. Choose $(L_\alpha, R_\alpha, s_\alpha) \in M$ and $q_1 \leq q_0$ such that $q_1 \Vdash_{\mathbb{P}_\alpha}$ "$p(\alpha) = (\check{L}_\alpha, \check{R}_\alpha, \check{s}_\alpha)$". Choose $q_2 \leq q_1$ such that

$L_\alpha \cup R_\alpha \subseteq \operatorname{supp}(q_2)$; for example, define $q_2(\beta)$ to be equal to $q_1(\beta)$ except when $\beta \in (L_\alpha \cup R_\alpha) \setminus \operatorname{supp}(q_1)$, in which case define $q_2(\beta) = (\emptyset, \emptyset, \check{(0, 108)})$.

Now, since $U_{\alpha,n}$ is dense in $\mathbb{P}_\alpha$ by induction hypothesis, find $q_3 \leq q_2$ such that $q_3 \in U_{\alpha,n}$. In particular, we have $L_\alpha \cup R_\alpha \subseteq \operatorname{supp}(q_2) \subseteq \operatorname{supp}(q_3)$. Also, $l_{q_3} \geq n$.

Let $\bar{n} \geq \max(l_{q_3}, |s_\alpha|) \geq n$, and find $q_4 \leq q_3$ such that $q_4 \in U_{\alpha,\bar{n}}$; in particular, $l_{q_4} \geq \bar{n}$.

Claim 4. *There is an s in $M \cap {}^{l_{q_4}}\mathbb{Q}$ such that*

$$q_4{}^\frown(\check{L}_\alpha, \check{R}_\alpha, \check{s}) \leq q_4{}^\frown(\check{L}_\alpha, \check{R}_\alpha, \check{s}_\alpha).$$

Proof. Since $|s_\alpha| \leq \bar{n} \leq l_{q_4}$, we can extend s_α (using Claim 2) to $s \in {}^{l_{q_4}}\mathbb{Q} \cap M$ so that $q_4 \Vdash_{\mathbb{P}_\alpha} (\check{L}_\alpha, \check{R}_\alpha, \check{s}) \leq (\check{L}_\alpha, \check{R}_\alpha, \check{s}_\alpha)$. So we have $q_4{}^\frown(\check{L}_\alpha, \check{R}_\alpha, \check{s}) \leq q_4{}^\frown(\check{L}_\alpha, \check{R}_\alpha, \check{s}_\alpha)$, and $|s| = l_{q_4}$. □

Let $p' = q_4{}^\frown(\check{L}_\alpha, \check{R}_\alpha, \check{s})$. Clearly, p' is a determined condition. Furthermore, since $q_4 \leq q_1$ and $q_1 \Vdash_{\mathbb{P}_\alpha}$ "$p(\alpha) = (\check{L}_\alpha, \check{R}_\alpha, \check{s}_\alpha)$", it follows that $q_4{}^\frown(\check{L}_\alpha, \check{R}_\alpha, \check{s}_\alpha) \leq p$, and hence $p' \leq p$. Since $L_\alpha \cup R_\alpha \subseteq \operatorname{supp}(q_3) \subseteq \operatorname{supp}(p')$, p' is of closed support. Moreover, since $|s| = l_{q_4}$, it follows that for each $\beta \in \operatorname{supp}(p')$, $|s_{p',\beta}| = l_{q_4} \geq \bar{n} \geq n$. Thus, $p' \in U_{\gamma,n}$, and $U_{\gamma,n}$ is dense. □

3.5 The Star-Extension Lemma

Definition 38. Given $1 \leq \gamma \leq \kappa$, and $p, q \in U_\gamma$, we say p *star-extends* q, and we write $p \leq^* q$, iff $p \leq q$, $\text{supp}(p) = \text{supp}(q)$, and for each $\beta \in \text{supp}(p) = \text{supp}(q)$, $s_{\beta,p} = s_{\beta,q}$.

Remark 7. Thus, p above is obtained from q by expanding some of the L_β and R_β sets to include more members from $\text{supp}(q)$.

Lemma 3.5.1. *Fix $1 \leq \gamma \leq \kappa$, and fix $p \in U_\gamma$. Given $\beta, \delta \in$ supp(p) such that $\beta < \delta$, we have*

- $p \Vdash_{\mathbb{P}_\gamma}$ *"$\dot{h}_\beta <^* \dot{h}_\delta$" $\rightarrow \exists r \in U_\gamma$ such that $r \leq^* p$ and $\beta \in L_{r,\delta}$.*
- $p \Vdash_{\mathbb{P}_\gamma}$ *"$\dot{h}_\delta <^* \dot{h}_\beta$" $\rightarrow \exists r \in U_\gamma$ such that $r \leq^* p$ and $\beta \in R_{r,\delta}$.*

Proof. Since the arguments for the basis and limit cases are quite straightforward, we turn to the successor case. Fix α such that $0 < \alpha < \kappa$, and let $\gamma = \alpha + 1$. Suppose we have, for each $q \in U_\alpha$,

- I.H.: $\forall \beta < \delta \in \text{supp}(q)$, $q \Vdash_{\mathbb{P}_\alpha}$ "$\dot{h}_\beta <^* \dot{h}_\delta$" $\rightarrow \exists r \leq^* q$ s.t. $\beta \in L_{r,\delta}$, and $q \Vdash_{\mathbb{P}_\alpha}$ "$\dot{h}_\delta <^* \dot{h}_\beta$" $\rightarrow \exists r \leq^* q$ s.t. $\beta \in R_{r,\delta}$.

Fix $\beta < \delta \in \text{supp}(p)$. We suppose $p \Vdash_{\mathbb{P}_\gamma} \dot{h}_\beta <^* \dot{h}_\delta$, and then show that $\exists r \leq^* p$ such that $\beta \in L_{r,\delta}$. (The other case is similar, by symmetry.) We might as well assume $\delta = \alpha$, since the case $\delta < \alpha$ follows directly from the induction hypothesis.

Let $q_0 = p \restriction \alpha$. Now, since $p \Vdash_{\mathbb{P}_\gamma} \dot{h}_\beta <^* \dot{h}_\alpha$, we have $q_0 \Vdash_{\mathbb{P}_\alpha} \dot{h}_\beta \in A_\alpha$, and so for every $\sigma \in R_\alpha$, $q_0 \Vdash_{\mathbb{P}_\alpha} \dot{h}_\beta <^* \dot{h}_\sigma$. (Note that β, $\sigma \in \text{supp}(q_0)$). Applying I.H. $|R_\alpha|$ times successively, we obtain $q_1 \leq^* q_0$ such that for all $\sigma \in R_\alpha$, we have $\beta \in L_{q_1,\sigma}$ if $\beta < \sigma$, and $\sigma \in R_{q_1,\beta}$ if $\sigma < \beta$.

Let $r = q_2 \frown (L_\alpha \cup \{\beta\}, R_\alpha, s_\alpha)\check{}$. It is now easy to see that this r satisfies the induction hypothesis.

$\square$

3.6 The Uniform Extension Lemma

We now come to the final lemma needed to prove that $\mathbb{P}$ is ccc.

Lemma 3.6.1. *Fix $1 \leq \gamma \leq \kappa$, and fix $p, q \in U_\gamma$ such that $l_p = l_q$. Suppose that for all $\beta \in$ supp$(p) \cap$ supp(q), $s_{p,\beta} = s_{q,\beta}$. Then $\exists r \in U_\gamma$ such that $r \leq p, q$ and $l_r = l_p = l_q$.*

Proof. The proof proceeds by induction on γ. Since the basis and limit cases are quite straightforward, we turn to the successor case. So, fix $1 \leq \alpha < \kappa$, and let $\gamma = \alpha + 1$. Fix $p, q \in U_\gamma$ such that $l_p = l_q$, and suppose that for all $\beta \in \text{supp}(p) \cap \text{supp}(q)$, $s_{p,\beta} = s_{q,\beta}$. Now, apply the I.H. to $p \restriction \alpha$, $q \restriction \alpha$ to obtain $r_1 \in U_\alpha$ such that $r_1 \leq p \restriction \alpha, q \restriction \alpha$, and $l_{r_1} = l_p = l_q$.

Case 1. Suppose $\alpha \notin \text{supp}(p) \cap \text{supp}(q)$. Let

$$r_1(\alpha) = \begin{cases} p(\alpha), & \text{if } \alpha \notin \text{ supp}(q), \\ \\ q(\alpha) & \text{otherwise.} \end{cases}$$

Let $r = r_1 ^\frown r_1(\alpha)$. Then $r \in U_\gamma$, $r \leq p, q$ and $l_r = l_p = l_q$, as required.

Case 2. Suppose $\alpha \in \text{supp}(p) \cap \text{supp}(q)$. Let $L = L_{p,\alpha} \cup L_{q,\alpha}$, let $R = R_{p,\alpha} \cup R_{q,\alpha}$, and let $s = s_{p,\alpha} = s_{q,\alpha}$.

Claim 5. $r_1 \Vdash_{\mathbb{P}_\alpha} \forall \beta \in L, \forall \delta \in R, \dot{h}_\beta <^* \dot{h}_\delta$.

Proof. Note that $p \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha}$ "$(\forall \beta \in L_{p,\alpha}) \dot{h}_\beta \in A_\alpha$", and $p \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha}$ "$(\forall \delta \in R_{p,\alpha}) \dot{h}_\delta \in B_\alpha$". Similarly, $q \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha}$ "$(\forall \beta \in L_{q,\alpha}) \dot{h}_\beta \in A_\alpha$", and $q \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha}$ "$(\forall \delta \in R_{q,\alpha}) \dot{h}_\delta \in B_\alpha$". The proof of the claim is now completed by the fact that $r_1 \leq p \upharpoonright \alpha, q \upharpoonright \alpha$. □

Claim 6. $\exists r_2 \leq^* r_1$ *such that for all* $(\beta, \delta) \in L \times R$, $\beta \in L_{r_2,\delta}$ *or* $\delta \in R_{r_2,\beta}$.

Proof. Fix a pair $(\beta, \delta) \in L \times R$. Note first that $\beta, \delta \in \text{supp}(r_1)$. Also, $r_1 \Vdash_{\mathbb{P}_\alpha}$ "$\dot{h}_\beta <^* \dot{h}_\delta$" by the previous claim. By Lemma 3.5.1, we know that if $\beta < \delta$, then $\exists \bar{r} \leq^* r_1$ such that $\beta \in L_{\bar{r},\delta}$, and if $\delta < \beta$, then $\exists \bar{r} \leq^* r_1$ such that $\delta \in R_{\bar{r},\beta}$. So, applying Lemma 3.5.1 successively, we get $r_2 \leq^* r_1$ such that $\beta \in L_{r_2,\delta}$ (if $\beta < \delta$) or $\delta \in R_{r_2,\beta}$ (if $\delta < \beta$). □

Let $r = r_2 ^\frown (L, R, s)$. It is now easy to see that $r \in U_\gamma$, $r \leq p, q$ and $l_r = l_p = l_q$.

□

3.7 $\mathbb{P}$ has the ccc

Lemma 3.7.1. *For each $1 \leq \gamma \leq \kappa$, $\mathbb{P}_\gamma$ has the ccc.*

Proof. Fix $1 \leq \gamma \leq \kappa$, and suppose instead that $P_1 = \{p_\alpha : \alpha < \omega_1\} \subseteq \mathbb{P}_\gamma$ is an antichain. By Lemma 3.4.1, for each $p_\alpha \in P_1$ we can find $q_\alpha \leq p_\alpha$ such that $q_\alpha \in U_\gamma$. Since P_1 is an uncountable antichain, $Q_1 = \{q_\alpha : \alpha < \omega_1\}$ is too.

Now, there are only countably many possibilities $l \in \omega$ for the "l_{q_α}"-value. So there must be some $l \in \omega$ such that $Q_2 = \{q_\alpha \in Q_1 : l_{q_\alpha} = l\}$ is uncountable. Define $S = \{\text{supp}(q_\alpha) : q_\alpha \in Q_2\}$; so S is an uncountable family of finite sets. Applying the Δ-system lemma, let $S' \subseteq S$ be an uncountable Δ-system with root A, and let $Q_3 = \{q_\alpha \in Q_2 : \text{supp}(q_\alpha) \in S'\}$.

For each $q_\alpha \in Q_3$, there is a mapping $\beta \longmapsto s_{q_\alpha,\beta}$ for each $\beta \in A$. But since $|s_{q_\alpha,\beta}|$ is always l, there are only countably many choices for each $s_{q_\alpha,\beta}$, and so only countably many choices for the finite sequence $\langle s_{q_\alpha,\beta}\rangle_{\beta\in A}$. So there must be some finite sequence $\langle \bar{s}_\beta\rangle_{\beta\in A}$ such that

$$Q_4 = \{q_\alpha \in Q_3 : \langle s_{q_\alpha,\beta}\rangle_{\beta\in A} = \langle \bar{s}_\beta\rangle_{\beta\in A}\}$$

is uncountable. Now, fix $p, q \in Q_4 \subseteq U_\gamma$. We have $l_p = l_q = l$, and for each $\beta \in \text{supp}(p) \cap \text{supp}(q) = A$, $s_{p,\beta} = s_{q,\beta} = \bar{s}_\beta$. So Lemma 3.6.1 applies, yielding $r \in U_\gamma$ such that $r \leq p, q$. But p, q are supposed to be incompatible, since Q_4 is an antichain. $\square$

3.8 $2^\omega = \kappa$ in $M[G]$

We would like to show that $2^\omega = \kappa$ in $M[G]$. First, the following lemma shows that $|\mathbb{P}_\kappa| = \kappa$.

Lemma 3.8.1. *For each $\alpha \leq \kappa$, $|\mathbb{P}_\alpha| \leq \max\{|\alpha|, \omega\}$.*

Proof. (By induction on α.)

Basis. ($\alpha = 0$). By definition, $\mathbb{P}_0 = \{0\}$.

Successor. Suppose $\alpha = \beta + 1$, and suppose $|\mathbb{P}_\beta| \leq \max\{|\beta|, \omega\}$. Note that $\mathbb{P}_\alpha \subseteq \mathbb{P}_\beta \times$ dom$(\mathbb{Q}_\alpha)$. But $|\text{dom}(\mathbb{Q}_\alpha)| \leq |X_\alpha| = \max\{|\alpha|, \omega\}$.

Limit. Suppose α is a limit ordinal, and $\forall \beta < \alpha$, $|\mathbb{P}_\beta| \leq \max\{|\beta|, \omega\}$. For each $B \in [\alpha]^{<\omega}$, define $C_B = \{p \in \mathbb{P}_\alpha : \text{supp}(p) = B\}$. Since $|[\alpha]^{<\omega}| = \alpha$, $\{C_B : B \in [\alpha]^{<\omega}\}$ is a partition of $\mathbb{P}_\alpha$ into $|\alpha|$-many classes. It suffices to show:

Claim 7. For each $B \in [\alpha]^{<\omega}$, $|C_B| \leq |\alpha|$.

Proof. Let $\lambda = |\max B|$. So $\lambda \leq |\alpha|$. Let $n = |B|$. So $|C_B| \leq |\Pi_{\gamma \in B} X_\gamma| \leq \lambda^n \leq |\alpha|$. □

□

Lemma 3.8.2. $2^\omega = \kappa$ *in* $M[G]$.

Proof. First, note that in $M[G]$, $\mathcal{L}_\kappa \subseteq {}^\omega\mathbb{Q}$, and $|\mathcal{L}_\kappa| = \kappa$, so $2^\omega \geq \kappa$ in $M[G]$. Next, let $A = \{Y \in M[G] : (Y \subseteq \omega)^{M[G]}\}$. In $M[G]$, $2^\omega = |A|$. Now, every subset of ω in

$M[G]$ gets represented by a nice-name[28] (for a subset of $\check{\omega}$), and each such nice-name corresponds to an ω-sequence of antichains from $\mathbb{P}$. Since $|\mathbb{P}| = \kappa$, and since each antichain is countable (Lemma 3.7.1), there are at most $|[\kappa]^{\leq\omega}| = \kappa^\omega$ antichains, and so at most $(\kappa^\omega)^\omega = \kappa^\omega \leq \kappa^{<\kappa} = (2^{<\kappa})^{<\kappa} = 2^{<\kappa} = \kappa$ such nice-names. So $2^\omega = |A| \leq \kappa$ in $M[G]$. □

3.9 $\mathcal{L}_\kappa$ is Saturated in $M[G]$

The following lemma implies that $\mathcal{L}_\kappa$ is a saturated linear order of size continuum in the forcing extension.

Lemma 3.9.1. *Working in $M[G]$, fix $B^1, B^2 \in [\mathcal{L}_\kappa]^{<\kappa}$ such that $B^1 <^* B^2$. Then $\exists \alpha < \kappa$ such that $B^1 <^* h_\alpha <^* B^2$.*

Proof. Let $\gamma < \kappa$ be the minimal ordinal such that $B^1, B^2 \subseteq \mathcal{L}_\gamma$. Note that we have $B^1, B^2 \in M[G]$, $B^1, B^2 \subseteq \mathcal{L}_\kappa$, and $|B^1|, |B^2| \leq \kappa$. Let $\bar{B}^1 = \{\eta : h_\eta \in B_1\}$, $\bar{B}^2 = \{\eta : h_\eta \in B_2\}$. Note that $\bar{B}^1, \bar{B}^2 \subseteq \gamma$.

Focusing on B^1, note that there is a nice name (for a subset of $\check{\gamma}$) $\tau_1 \in M^{\mathbb{P}}$ such that $\tau_1[G] = \bar{B}^1$. So τ_1 is of the form

$$\{(\check{\eta}, p) : \eta < \gamma, p \in A_\eta\}$$

where for each $\eta < \gamma$, A_η is an antichain in $\mathbb{P}$.

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[28]See Definition 17.

Since $\mathbb{P}$ is ccc, every antichain is countable, and so we can say

$$\tau_1 = \{(\check{\eta}, p) : \eta < \gamma, p \in A^1_\eta\}$$

where $\langle A^1_\eta \rangle_{\eta<\gamma} \in {}^\gamma([\mathbb{P}]^{\leq\omega})$.

For each $p \in \bigcup_{\eta<\gamma} A^1_\eta$, define a corresponding $\bar{p} \in P_\gamma$ by setting, for each $\beta < \gamma$, $\bar{p}(\beta) = (L, R, s)$ where $p(\beta) = (L, R, s)\check{}$. Let $K^1 = \{(\eta, \bar{p}) : (\check{\eta}, p) \in \tau_1\}$. So $K^1 \in \mathcal{K}$, which means K^1 occurs cofinally often in the enumeration $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$. Find $\alpha > \gamma = rank(K^1)$ such that $K^1 = C_\alpha$.

Now, for each $\bar{p}$, define (as in section 3.3) p' in $\mathbb{P}_\gamma$ by setting, for each $\beta < \gamma$, $p'(\beta) = [\bar{p}(\beta)]\check{}$. Note that for each $p \in \bigcup_{\eta<\gamma} A^1_\eta$, $p' = p$.

So $A'_\alpha = \{(\check{\beta}, p') : p \in C_\alpha\}$ (see section 3.3, page 42) and A'_α names a subset of α. So, as in section 3.3, A'_α names a subset of α, and so $(A'_\alpha)^*$ is the name for the corresponding subset of $\mathcal{L}_\alpha$ (i.e., the subset indexed by A'_α). Furthermore, A_α is the index set for $LeftPart_{\mathcal{L}_\alpha}(A'_\alpha)^*$, and B_α is the index set for $\mathcal{L}_\alpha \setminus LeftPart_{\mathcal{L}_\alpha}(A'_\alpha)^*$. Since $\bar{B}^1 \subseteq A_\alpha$ and $\bar{B}^2 \subseteq B_\alpha$, it follows that $B^1 <^* h_\alpha <^* B^2$.

3.10 $\mathcal{L}_\kappa$ is a Maximal Linear Order in $M[G]$

The following lemma will be needed to show that $\mathcal{L} = \mathcal{L}_\kappa$ is a maximal linear order in $({}^\omega\mathbb{R}, <^*)$. For the present chapter, the set E in the lemma could simply be α; however, in the following chapter, the lemma will be applied with a different set for

E.

Lemma 3.10.1. *Suppose $\mathcal{N}$ is any countable transitive model of ZFC, and in $\mathcal{N}$, α is an ordinal, $E \subseteq \alpha$ is cofinal in α, and $\mathcal{L}_\alpha = \{h_\beta : \beta \in E\} \subseteq {}^\omega\mathbb{Q}$ such that $\mathcal{L}_\alpha$ is linearly ordered by $<^*$. Fix (A^*, B^*), a partition cut in $\mathcal{L}_\alpha$, and let $A = \{\beta \in E : h_\beta \in A^*\}$, $B = \{\beta \in E : h_\beta \in B^*\}$. Let $K = K(A, B)$. Fix H, a K-generic filter over $\mathcal{N}$, and let $h_\alpha = \bigcup\{s_p : p \in H\}$. Note that, in $\mathcal{N}[H]$, $A^* <^* h_\alpha <^* B^*$. Suppose there is some $b : \omega \longrightarrow \mathbb{R}$ in $\mathcal{N}$ such that $A^* <^* b <^* B^*$. Then b and h_α are incomparable under $\leq^*$.*

Proof. Let $\dot{h}_\alpha$ be a name for h_α (i.e., $\dot{h}_\alpha[H] = h_\alpha$). For each $n \in \omega$, let $D_n = \{p \in K : \exists k > n, p \Vdash \dot{h}_\alpha(k) < b(k)\}$, and let $E_n = \{p \in K : \exists k > n, p \Vdash \dot{h}_\alpha(k) > b(k)\}$.

Claim 8. *$\forall n \in \omega$, D_n and E_n are dense.*

Proof. Fix $n \in \omega$, and fix $r \in K$; say $r = (L, R, s)$. Find $\bar{k} > \max\{n, |s|\}$ such that $\forall k \geq \bar{k}$, $\forall \gamma \in L, \delta \in R$, $h_\gamma(k) < b(k) < h_\delta(k)$. Find $b_1, b_2 \in \mathbb{Q}$ such that $\max\{h_\gamma(k) : \gamma \in L\} < b_1 < b(k) < b_2 < \min\{h_\delta(k) : \delta \in R\}$.

Define $\bar{s}, \bar{t} : \bar{k} + 1 \longrightarrow \mathbb{Q}$ by

$$\bar{s}(k) = \begin{cases} s(k) & \text{if } k < |s|, \\ (\max\{h_\gamma(k) : \gamma \in L\} + \min\{h_\delta(k) : \delta \in R\})/2 & \text{if } |s| \leq k < \bar{k}, \text{ and} \\ b_1 & \text{if } k = \bar{k}. \end{cases}$$

$$\bar{t}(k) = \begin{cases} s(k) & \text{if } k < |s|, \\ (\max\{h_\gamma(k) : \gamma \in L\} + \min\{h_\delta(k) : \delta \in R\})/2 & \text{if } |s| \leq k < \bar{k}, \text{ and} \\ b_2 & \text{if } k = \bar{k}. \end{cases}$$

Now, let $p = (L, R, \bar{s}) \in D_n$, and let $q = (L, R, \bar{t}) \in E_n$. Since $p \leq r$ and $q \leq r$, the sets D_n and E_n are dense. □

Having established the claim, fix $n \in \omega$, and find $p \in H \cap D_n$, $q \in H \cap E_n$. Find $k_1 > n$ such that $p \Vdash$ "$\dot{h}_\alpha(k_1) < b(k_1)$", and $k_2 > n$ such that $q \Vdash$ "$\dot{h}_\alpha(k_2) > b(k_2)$". So in $\mathcal{N}[H]$, both "$h_\alpha \leq^* b$" and "$b \leq^* h_\alpha$" are false; i.e., h_α and b are incomparable under $\leq^*$. □

Theorem 3.10.2. *In $M[G]$, $(\mathcal{L}, <^*)$ is a maximal linearly ordered subspace of $({}^\omega\mathbb{R}, <^*)$.*

Proof. (For the present chapter, let $EL = \kappa$; note that in the next chapter the proof below will be quoted except that EL will have a different definition.) Suppose instead $\exists b \in M[G]$ such that $b : \omega \longrightarrow \mathbb{R}$, $b \notin \mathcal{L}$, and $(\mathcal{L} \cup \{b\}, <^*)$ is a linear order. Since b can be identified with a subset of $\omega \times \omega$, there is an $\eta < \kappa$ such that $b \in M[G_\eta]$ (see Lemma 5.14 from Chapter VIII of [Kun80], page 276).

Working in $M[G_\eta]$, let

$$\bar{C} = \{h_\gamma : \gamma \in EL \cap \eta, h_\gamma <^* b\}$$

$$\bar{D} = \{h_\gamma : \gamma \in EL \cap \eta, b <^* h_\gamma\}$$

So $(\bar{C}, \bar{D})$ is a partition cut in $\mathcal{L}_\eta$. Let $C = \{\gamma : h_\gamma \in \bar{C}\}$, and $D = \{\gamma : h_\gamma \in \bar{D}\}$. Note that $\sup(C), \sup(D) \leq \eta$. Let $\mathcal{B} = \{\beta \geq \eta : \beta \in EL, \bar{C} \subseteq A^*_\beta, \text{ and } \bar{D} \subseteq B^*_\beta\}$.[29] Note that $\beta \in \mathcal{B}$ whenever $\beta \geq \eta$ and C_β names the set C; since there are κ-many such β's, in particular, $\mathcal{B} \neq \emptyset$. Let $\bar{\beta} = \min \mathcal{B}$.

Claim 9. *(i) $\bar{C}$ is cofinal in $A^*_{\bar{\beta}}$; i.e., $\forall a \in A^*_{\bar{\beta}}$, $\exists c \in \bar{C}$ such that $a <^* c$ or $c = a$; and (ii) $\bar{D}$ is coinitial in $B^*_{\bar{\beta}}$, i.e., $\forall a \in B^*_{\bar{\beta}}$, $\exists d \in \bar{D}$ such that $d <^* a$ or $d = a$.*

Proof. For (ii), suppose instead $\exists h_\alpha \in B^*_{\bar{\beta}}$ such that $h_\alpha <^* \bar{D}$. Note that $\alpha < \bar{\beta}$ and $\alpha \in EL$. Since $h_\alpha \in B^*_{\bar{\beta}}$, $A^*_{\bar{\beta}} <^* h_\alpha$. Thus, since $\bar{C} \subseteq A^*_{\bar{\beta}}$, $\bar{C} <^* h_\alpha$. In particular, h_α is neither in $\bar{C}$ nor in $\bar{D}$, which means h_α is not in $\mathcal{L}_\eta$. So $\alpha \geq \eta$, and since (by definition) $A^*_\alpha <^* h_\alpha <^* B^*_\alpha$, it must be that $\bar{C} \subseteq A^*_\alpha$ and $\bar{D} \subseteq B^*_\alpha$. Thus, $\alpha \in \mathcal{B}$, and $\alpha < \bar{\beta}$, contrary to the minimality of $\bar{\beta}$. The argument for (i) is very similar. $\square$

Now, by definition, $\bar{C} <^* b <^* \bar{D}$. Thus, by the above claim, $A^*_{\bar{\beta}} <^* b <^* B^*_{\bar{\beta}}$. But, by definition, $A^*_{\bar{\beta}} <^* h_{\bar{\beta}} <^* B^*_{\bar{\beta}}$. We now can apply Lemma 3.10.1 (where $\alpha = \bar{\beta}$ and $E = EL \cap \bar{\beta}$) to conclude that b and $h_{\bar{\beta}}$ are incomparable (under $\leq^*$) in $M[G_{\bar{\beta}+1}]$, and therefore in $M[G]$. This contradicts the assumed linearity of $(\mathcal{L}_\kappa \cup \{b\}, <^*)$. $\square$

[29] Recall that for each $\beta \in EL$, (A^*_β, B^*_β) is the cut in $\mathcal{L}_\beta$, while $A_\beta = \{\gamma : h_\gamma \in A^*_\beta\}$, $B_\beta = \{\gamma : h_\gamma \in B^*_\beta\}$ are the corresponding index sets; moreover, C_β is the β^{th} element in the enumeration $\mathcal{C} = \{C_\beta : \beta \in EL\}$ of potential names for subsets of κ of size $< \kappa$.

Remark 8. It is important for the above argument that for each $\alpha < \kappa$, $|\mathcal{L}_\alpha| < \kappa$, to ensure that the cut $(\bar{C}, \bar{D})$ is of size $< \kappa$. This remark will be especially relevant in the next chapter.

□

4 A Saturated Pantachie and MA

The goal of this chapter is to show the consistency of Martin's Axiom together with the existence of a saturated pantachie, in other words, the existence of a maximal saturated linear order of size $\mathfrak{c} \geq \omega_2$ in the partial order $({}^{\omega}\mathbb{R}, <^*)$ (i.e., functions from ω into the reals, ordered by eventual strict domination). In short, the goal of this chapter is to prove the following:

Theorem 4.0.3. *Con(ZFC + MA + $\exists$ a maximal saturated linear order of size $\mathfrak{c} \geq \omega_2$ in $({}^{\omega}\mathbb{R}, <^*)$).*

We work with the restricted partial order, $({}^{\omega}\mathbb{Q}, <^*)$ (i.e., functions from ω into the rationals, ordered by eventual strict domination), and eventually show that the saturated linear order constructed is a maximal linearly ordered subspace not merely of $({}^{\omega}\mathbb{Q}, <^*)$, but also of $({}^{\omega}\mathbb{R}, <^*)$.

4.1 Introduction and Strategy Outline

The strategy for proving Theorem 4.0.3 involves a finite support iterated forcing construction (within M, a countable transivite model for ZFC) of length κ, where κ is a regular cardinal such that $\kappa \geq \omega_2$ and $2^{<\kappa} = \kappa$. First, partition κ into the following classes:

- EL = EASY-LIMITS = $\{\beta < \kappa : \beta \in LIM,\, cf(\beta) \neq \omega_1\}$;
- HL = HARD-LIMITS = $\{\beta < \kappa : \beta \in LIM,\, cf(\beta) = \omega_1\}$
- ES = EASY-SUCCESSORS = $\{\beta < \kappa : \beta = \alpha + 1,\, \alpha \notin HL\}$;
- HS = HARD-SUCCESSORS = $\{\beta < \kappa : \beta = \alpha + 1,\, \alpha \in HL\}$

A subset $\mathcal{L}$ of ${}^{\omega}\mathbb{Q}$ will be constructed recursively, and shown to be a maximal saturated linear order of size $\kappa = \mathfrak{c}$ in the final forcing extension. First, using the class EL of easy limit ordinals as index set, fix an enumeration of cuts in $\mathcal{L}$ (or more precisely, an enumeration of candidates for names of cuts in $\mathcal{L}$) such that each cut (i.e., candidate) is enumerated cofinally often.[30] Similarly, fix a cofinally-often enumeration of all candidates for ccc-partial order names, using the class ES of easy successor ordinals as index set.

[30]Cf. section 3.2 from Chapter 3

The iterated forcing and $\mathcal{L}$ are defined by a simultaneous recursion of length κ. Through the course of the recursion, for each $\alpha \in EL$, a function h_α is defined, and for any $\beta \leq \kappa$, let $\mathcal{L}_\beta = \{h_\alpha : \alpha < \beta, \alpha \in EL\}$. (In the end we will set $\mathcal{L} = \mathcal{L}_\kappa$.) In a bit more detail:

1. When $\alpha \in EL$, force with the Kunen partial order for filling the cut in $\mathcal{L}_\alpha$ from the enumeration. (Note that it is only at these stages that we add functions into the emerging linear order; namely, we add the function h_α that fills the cut.)

2. For $\alpha \in ES$, force with the ccc partial order given by the ccc-p.o. enumeration. (This ensures that MA will hold in the final forcing extension.)

3. If $\alpha \in HL$, force with the (finite-support) product of the Kunen partial orders for filling each partition cut in L_α. (Note however that no function is added into the emerging linear order at this stage! We only force with these partial orders to render the appropriate gaps ccc-fillable[31] in later stages.)

4. Finally, for $\alpha \in HS$, force with the (finite-support) product of all Aronszajn-tree specializers. (It will be shown that forcing with this product renders it impossible to add new ω_1-gaps of exisiting elements in the linear order via any future ccc forcing.)

[31]See Definition 39.

After defining the forcing and $\mathcal{L} = \mathcal{L}_\kappa$, we show that the forcing is ccc. It then follows by standard methods that in the forcing extension, $\mathfrak{c} = \kappa$ and MA holds. Finally, we show that $\mathcal{L}$ is a maximal saturated linear order of size $\mathfrak{c}$ in $({}^\omega\mathbb{R}, <^*)$.

4.2 Special Gaps and Strong Gaps

The important notion of a "strong" or *ccc-indestructible* gap can now be introduced. This machinery is due to Kunen; similar versions of these definitions and results can be found in [Woo84]. In the definitions and results that follow, the statement "Let $(\mathcal{F}, \mathcal{G})$ be an ω_1-(pre)gap" is short for "Suppose $(\{f_\alpha : \alpha < \omega_1\}, \{g_\alpha : \alpha < \omega_1\})$ is an ω_1-(pre)gap in $(\mathbb{Q}, <^*)$, and let $(\mathcal{F}, \mathcal{G}) = (\{f_\alpha : \alpha < \omega_1\}, \{g_\alpha : \alpha < \omega_1\})$".

Definition 39. Let $(\mathcal{F}, \mathcal{G})$ be an ω_1-gap. $(\mathcal{F}, \mathcal{G})$ is called a *strong* gap or a *ccc-indestructible* gap if it cannot be filled in any ccc forcing extension. On the other hand, an ω_1-gap which is not strong (i.e., it *can* be filled by some ccc forcing extension) is called *weak*, or *ccc-fillable*.

Definition 40. An ω_1-pregap $(\mathcal{F}, \mathcal{G})$ is said to be *special* if there is some $k < \omega$ such that the following two conditions hold:

(i) $\forall \alpha < \omega_1$, $\forall n > k$, $f_\alpha(n) < g_\alpha(n)$; and

(ii) $\forall \alpha \neq \beta < \omega_1$, $\exists n > k$ such that $f_\alpha(n) \geq g_\beta(n)$ or $f_\beta(n) \geq g_\alpha(n)$.

Lemma 4.2.1. *For any ω_1-pregap $(\mathcal{F}, \mathcal{G})$, if $(\mathcal{F}, \mathcal{G})$ is special then $(\mathcal{F}, \mathcal{G})$ is a gap.*

Proof. Suppose instead that for some $h : \omega \longrightarrow \mathbb{Q}$, $\mathcal{F} <^* h <^* \mathcal{G}$. Find k such that (i) and (ii) from Definition 40 hold. Define $\pi : \omega_1 \longrightarrow \omega$ by $\pi(\alpha) = \min\{m \geq k : \forall n \geq m, f_\alpha(n) < h(n) < g_\alpha(n)\}$. Suppose for some $m \in \omega$, $\pi(\alpha) = m$ (for all $\alpha < \omega_1$). Also, suppose there is some $g : m \longrightarrow \mathbb{Q}$ such that for all $\alpha < \omega_1$, $g_\alpha \restriction m = g$.

Now property (ii) is contradicted for any pair $\alpha \neq \beta < \omega_1$. To see this, fix $n \geq k$. If $k \leq n < m$, then, using property (i), $f_\alpha(n) < g_\alpha(n) = g(n) = g_\beta(n)$, and $f_\beta(n) < g_\beta(n) = g(n) = g_\alpha(n)$. On the other hand, if $n \geq m$, then $f_\alpha(n) < h(n) < g_\beta(n)$ and $f_\beta(n) < h(n) < g_\alpha(n)$. □

Corollary 4.2.2. *A special gap is ccc-indestructible.*

Proof. Let $(\mathcal{F}, \mathcal{G})$ be special and let P be a ccc forcing notion. Find k such that (i) and (ii) from Definition 40 hold. In V^P, $(\mathcal{F}, \mathcal{G})$ is an $(\omega_1)^V$-pregap satisfying properties (i) and (ii) from Definition 40 for k, by absoluteness. Since P is ccc, $(\omega_1)^{V^P} = (\omega_1)^V$, so by Lemma 4.2.1 in V^P, $(\mathcal{F}, \mathcal{G})$ is a gap in V^P. □

Remark 9. Note that the above proof actually shows that a special gap remains unfilled not only in a ccc forcing extension, but in any forcing extension that preserves ω_1.

Lemma 4.2.3. *Let $(\mathcal{F}, \mathcal{G})$ be an ω_1-gap. If $K = K(\mathcal{F}, \mathcal{G})$[32] is not ccc, then $(\mathcal{F}, \mathcal{G})$*

[32]The notation $K(\mathcal{F}, \mathcal{G})$ is explained in Definition 28.

is equivalent to a special gap.

Proof. Fix $A = \{p_\gamma \mid \gamma < \omega_1\}$ an uncountable antichain in K, and suppose for each γ, $p_\gamma = (L_\gamma, R_\gamma, s_\gamma)$. By extending the p_γ if necessary, suppose that for all $\gamma < \omega_1$, $L_\gamma = R_\gamma$. Also, suppose without loss of generality that there is some $k \in \omega$ and $s : k \longrightarrow \mathbb{Q}$ such that for all $\gamma < \omega_1$, $s_\gamma = s$. Furthermore, suppose the set $\{L_\gamma : \gamma < \omega_1\}$ forms a Δ-system with root Σ such that $\forall \gamma < \gamma' < \omega_1$, $L_\gamma \setminus \Sigma < L_{\gamma'} \setminus \Sigma$.

Claim 10. *For each $\gamma \neq \gamma' \in \omega_1$, there is some $n \geq k$ such that* $\max\{f_\alpha(n) : \alpha \in L_\gamma\} \geq \min\{g_\alpha(n) : \alpha \in R_{\gamma'}\}$ *or* $\max\{f_\alpha(n) : \alpha \in L_{\gamma'}\} \geq \min\{g_\alpha(n) : \alpha \in R_\gamma\}$.

Proof. Suppose instead that there is a pair $\gamma \neq \gamma' \in \omega_1$ such that for each $n \geq k$, $\max\{f_\alpha(n) : \alpha \in L_\gamma\} < \min\{g_\alpha(n) : \alpha \in R_{\gamma'}\}$ and $\max\{f_\alpha(n) : \alpha \in L_{\gamma'}\} < \min\{g_\alpha(n) : \alpha \in R_\gamma\}$. Then $q = (L_\gamma \cup L_{\gamma'}, R_\gamma \cup R_{\gamma'}, s)$ is a condition extending both p_γ and $p_{\gamma'}$, which are supposed to be incompatible. $\square$

Next, for each $\gamma < \omega_1$, define $F_\gamma, G_\gamma : \omega \longrightarrow \mathbb{Q}$ by

$$F_\gamma(n) = \begin{cases} 0 & \text{if } n < k, \\ \max\{f_\alpha(n) : \alpha \in L_\gamma\} & \text{if } n \geq k. \end{cases}$$

$$G_\gamma(n) = \begin{cases} 1 & \text{if } n < k, \\ \min\{g_\alpha(n) : \alpha \in R_\gamma\} & \text{if } n \geq k. \end{cases}$$

Let $\mathcal{F}' = \{F_\gamma \mid \gamma < \omega_1\}$, and $\mathcal{G}' = \{G_\gamma \mid \gamma < \omega_1\}$. For each $\gamma < \omega_1$, let $\pi(\gamma) = \max L_\gamma = \max R_\gamma$. So $F_\gamma =^* f_{\pi(\gamma)}$ and $G_\gamma =^* g_{\pi(\gamma)}$. Thus, $(\mathcal{F}, \mathcal{G}) \cong (\mathcal{F}', \mathcal{G}')$.

To see that $(\mathcal{F}', \mathcal{G}')$ satisfies property (i) from Definition 40 for k, first fix $\gamma < \omega_1$ and $n \geq k$. For some $\beta \in L_\gamma$ and some $\delta \in R_\gamma$, $F_\gamma(n) = f_\beta(n)$ and $G_\gamma(n) = g_\delta(n)$. So $F_\gamma(n) < G_\gamma(n)$. Next, to see that $(\mathcal{F}', \mathcal{G}')$ satisfies property (ii) from Definition 40 for k, fix $\gamma \neq \gamma' < \omega_1$. By Claim 10, $\exists n \geq k$ such that $F_\gamma(k) \geq G_{\gamma'}(k)$ or $F_{\gamma'}(k) \geq G_\gamma(k)$. So $(\mathcal{F}', \mathcal{G}')$ is special. □

Corollary 4.2.4. *Let $(\mathcal{F}, \mathcal{G})$ be an ω_1-gap. $(\mathcal{F}, \mathcal{G})$ is ccc-indestructible iff $(\mathcal{F}, \mathcal{G})$ is equivalent to a special gap.*

Proof. Suppose first that $(\mathcal{F}, \mathcal{G})$ is equivalent to $(\mathcal{F}', \mathcal{G}')$, a special gap. By Corollary 4.2.2, $(\mathcal{F}', \mathcal{G}')$ is ccc-indestructible. But then, any ccc forcing for splitting $(\mathcal{F}, \mathcal{G})$ would also split $(\mathcal{F}', \mathcal{G}')$, since the gaps are equivalent. Thus, $(\mathcal{F}, \mathcal{G})$ is ccc-indestructible.

Next, suppose that $(\mathcal{F}, \mathcal{G})$ is ccc-indestructible. Since $K = K(\mathcal{F}, \mathcal{G})$ splits the gap, K cannot be ccc. Thus, by Lemma 4.2.3, $(\mathcal{F}, \mathcal{G})$ is equivalent to a special gap. □

Lemma 4.2.5. *Suppose $(\mathcal{F}, \mathcal{G})$ is an ω_1-gap. Then $K = K(\mathcal{F}, \mathcal{G})$ is ccc iff $(\mathcal{F}, \mathcal{G})$ is ccc-fillable (i.e., weak).*

Proof. Suppose first K is ccc. Since K fills the gap $(\mathcal{F}, \mathcal{G})$, the gap is ccc-fillable.

Next, suppose K is not ccc. Then, using Lemma 4.2.3, $(\mathcal{F}, \mathcal{G})$ is equivalent to some special gap $(\mathcal{F}', \mathcal{G}')$. By Corollary 4.2.2, $(\mathcal{F}', \mathcal{G}')$ is a ccc-indestructible gap. So then $(\mathcal{F}, \mathcal{G})$ must be ccc-indestructible too. $\square$

The following formulation will be useful.

Lemma 4.2.6. *Suppose $(\mathcal{F}, \mathcal{G})$ is a ccc-indestructible ω_1-gap, and $\bar{k} \in \omega$. Then there is some $k^* \geq \bar{k}$ and an uncountable subset $S \subseteq \omega_1$ such that*

- *$\forall \alpha \in S$, $\forall n \geq k^*$, $f_\alpha(n) < g_\alpha(n)$; and*
- *$\forall \alpha \neq \beta \in S$, $\exists n \geq k^*$ such that $f_\alpha(n) \geq g_\beta(n)$ or $f_\beta(n) \geq g_\alpha(n)$.*

Proof. Since $(\mathcal{F}, \mathcal{G})$ is ccc-indestructible, in particular, $K(\mathcal{F}, \mathcal{G})$ is not ccc. Let $A = \{p_\gamma : \gamma < \omega_1\}$ be an uncountable antichain where for each γ, $p_\gamma = (L_\gamma, R_\gamma, s_\gamma)$, $|s_\gamma| \geq \bar{k}$, and $L_\gamma = R_\gamma$. Find $k^* \geq \bar{k}$ and $s : k^* \longrightarrow \mathbb{Q}$ such that $S_1 = \{\gamma < \omega_1 : s_\gamma = s\}$ is uncountable. Find $\Sigma \in [\omega_1]^{<\omega}$ and $S_2 \in [S_1]^{\omega_1}$ such that $\{L_\gamma : \gamma \in S_2\}$ is a Δ-system with root Σ and $\forall \gamma < \gamma' \in S_2$, $L_\gamma \setminus \Sigma < L_{\gamma'} \setminus \Sigma$.

Claim 11. *For each $\gamma \neq \gamma' \in S_2$, there is some $n \geq k^*$ such that $\max\{f_\alpha(n) : \alpha \in L_\gamma\} \geq \min\{g_\alpha(n) : \alpha \in R_{\gamma'}\}$ or $\max\{f_\alpha(n) : \alpha \in L_{\gamma'}\} \geq \min\{g_\alpha(n) : \alpha \in R_\gamma\}$.*

Proof. Exactly the same as the proof of Claim 10 from Lemma 4.2.3. $\square$

Define F_γ and G_γ just as in the proof of Lemma 4.2.3. Define $\pi : S_2 \longrightarrow \omega_1$ by $\pi(\gamma) = \max L_\gamma = \max R_\gamma$. Define $e : S_2 \longrightarrow \omega$ by $e(\gamma) = \min\{m \geq k^* : \forall n \geq m,$

$F_\gamma(n) = f_{\pi(\gamma)}(n)$ and $G_\gamma(n) = g_{\pi(\gamma)}(n)\}$.

Find $m \geq k^*$, $f : m \longrightarrow \mathbb{Q}$, $g : m \longrightarrow \mathbb{Q}$, and $S_3 \in [S_2]^{\omega_1}$ such that $\forall \gamma \in S_3$,

$$e(\gamma) = m,\ f_{\pi(\gamma)} \restriction m = f, \text{ and } g_{\pi(\gamma)} \restriction m = g$$

Let $S = \{\pi(\gamma) : \gamma \in S_3\}$. Fix $n \geq k^*$ and $\gamma \in S_3$. Since $\pi(\gamma) \in L_\gamma = R_\gamma$, $f_{\pi(\gamma)}(n) < g_{\pi(\gamma)}(n)$. Next, fix $\gamma, \gamma' \in S_3$. By Claim 11, find $n \geq k^*$ such that $\max\{f_\alpha(n) : \alpha \in L_\gamma\} \geq \min\{g_\alpha(n) : \alpha \in R_{\gamma'}\}$ or $\max\{f_\alpha(n) : \alpha \in L_{\gamma'}\} \geq \min\{g_\alpha(n) : \alpha \in R_\gamma\}$. By the choice of m, it must be that $n \geq m$, which means that $f_{\pi(\gamma)}(n) \geq g_{\pi(\gamma')}(n)$ or $f_{\pi(\gamma')}(n) \geq g_{\pi(\gamma)}(n)$. Thus, the set S is as required. $\square$

We next prove a product version of Lemma 4.2.6 which will be used in the proof of Theorem 4.5.1.

Lemma 4.2.7. *Let $\{(\mathcal{F}^i, \mathcal{G}^i) : i < N\}$ be a collection of ω_1-gaps where for each $i < N$, $\mathcal{F}^i = \{f^i_\alpha : \alpha < \omega_1\}$ and $\mathcal{G}^i = \{g^i_\alpha : \alpha < \omega_1\}$. Suppose that for some $k' \in \omega$,*

$$\forall \gamma < \omega_1, \forall i < N, \forall k \geq k', f^i_\gamma(k) < g^i_\gamma(k) \tag{4.1}$$

and

$$\forall S \in [\omega_1]^{\omega_1}, \exists \alpha \neq \beta \in S, \forall i < N, \forall k \geq k', f^i_\alpha(k) < g^i_\beta(k) \text{ and } f^i_\beta(k) < g^i_\alpha(k). \tag{4.2}$$

Then $\Pi\{K(\mathcal{F}^i, \mathcal{G}^i) : i < N\}$ is ccc.

Proof. Our first task is to find a single gap, $(\mathcal{F}, \mathcal{G})$, which encodes the N gaps, $\{(\mathcal{F}^i, \mathcal{G}^i) : i < N\}$. To this end, a bijection between $\omega \times N$ and ω will be needed, and we find it preferable to use a concrete bijection obtained by partitioning ω into the N-modular classes.

Definition 41. Given $k \in \omega$, let $div(k)$ and $rem(k)$ be the unique finite ordinals such that $rem(k) < N$ and $k = div(k) \cdot N + rem(k)$.

Definition 42. Define $\pi : ({}^{\omega}\mathbb{Q})^N \longrightarrow ({}^{\omega}\mathbb{Q})$ by $\pi(f^0, \ldots, f^{N-1}) = f$, where f is defined by $f(k) = f^i(m)$, where $i = rem(k)$ and $m = div(k)$.

Remark 10. Note that π is a bijection, and $\pi^{-1}(f) = (f^0, \ldots, f^{N-1})$, where for each $i < N$ and $m \in \omega$, $f^i(m) = f(m \cdot N + i)$.

Definition 43. For each $\gamma < \omega_1$, define $f_\gamma = \pi(f^0_\gamma, \ldots, f^{N-1}_\gamma)$, and $g_\gamma = \pi(g^0_\gamma, \ldots, g^{N-1}_\gamma)$. Let $\mathcal{F} = \{f_\gamma : \gamma < \omega_1\}$, and $\mathcal{G} = \{g_\gamma : \gamma < \omega_1\}$.

Claim 12. *$(\mathcal{F}, \mathcal{G})$ is a pregap.*

Proof. Fix $\beta < \gamma < \omega_1$. Fix $j > N$ such that $\forall i < N, \forall k \geq j$, $f^i_\beta(k) < f^i_\gamma(k) < g^i_\gamma(k) < g^i_\beta(k)$. Now, fix $k \geq j \cdot (N+1)$, and suppose $k = m \cdot N + i$, where $i < N$. So $m \geq j$, and so $f^i_\beta(m) < f^i_\gamma(m) < g^i_\gamma(m) < g^i_\beta(m)$. Hence, $f_\beta(k) < f_\gamma(k) < g_\gamma(k) < g_\beta(k)$. □

Claim 13. *(a) If $(\mathcal{F}, \mathcal{G})$ is filled then $\forall i < N$, $(\mathcal{F}^i, \mathcal{G}^i)$ is filled.*

(b) $(\mathcal{F}, \mathcal{G})$ is a gap.

Proof. Suppose that $(\mathcal{F}, \mathcal{G})$ is filled; say $\mathcal{F} <^* h <^* \mathcal{G}$. Fix $i < N$, and define $h^i : \omega \longrightarrow \mathbb{Q}$ by $h^i(m) = h(m \cdot N + i)$. Fix j such that $\forall k \geq j$, $f_\gamma(k) < h(k) < g_\gamma(k)$. Now, fix $m \geq j$ and let $k = m \cdot N + i$. Since $m \geq j$, $k \geq j$. Thus, $f^i_\gamma(m) = f_\gamma(k) < h(k) = h^i(m) < g_\gamma(k) = g^i_\gamma(m)$. So h^i fills $(\mathcal{F}^i, \mathcal{G}^i)$; since the latter is actually a gap (i.e., unfilled), it must be that $(\mathcal{F}, \mathcal{G})$ is also a gap. □

Remark 11. Although not needed for the proof of Lemma 4.2.7, it's interesting to note (as demonstrated by the above proof of Claim 13) that if there is even just one $i < N$ such that $(\mathcal{F}^i, \mathcal{G}^i)$ is a gap, then $(\mathcal{F}, \mathcal{G})$ will be a gap as well.

Notation. Let $K = K(\mathcal{F}, \mathcal{G})$, and let $K' = \Pi\{K(\mathcal{F}^i, \mathcal{G}^i) : i < N\}$.

Claim 14. *If K is ccc, then K' is ccc.*

Proof. Suppose K is ccc. In V^K, since $(\mathcal{F}, \mathcal{G})$ is filled, each $(\mathcal{F}^i, \mathcal{G}^i)$ is also filled (by Claim 13). Since K is ccc, $(\omega_1)^V = (\omega_1)^{V^K}$, and so each $(\mathcal{F}^i, \mathcal{G}^i)$ is a filled ω_1-pregap; hence, each $K(\mathcal{F}^i, \mathcal{G}^i)$ is σ-centered in V^K (see Lemma 4.2.8 below). So K' is σ-centered in V^K, since a product of σ-centered forcing notions is σ-centered. Now, suppose towards a contradiction that $A \subseteq K'$ is an antichain of size ω_1 in V. Then A is still an ω_1-antichain in V^K, whence K' is both ccc and not ccc in V^K. □

Finally, we suppose, toward a contradiction, that $K' = \Pi\{K(\mathcal{F}^i, \mathcal{G}^i) : i < N\}$ is not ccc. Thus, by Claim 14, $K = K(\mathcal{F}, \mathcal{G})$ is not ccc. So, by Lemma 4.2.5, $(\mathcal{F}, \mathcal{G})$ is ccc-indestructible. Thus, we may apply Lemma 4.2.6, using $\bar{k} = k' \cdot N + N$. So we obtain a set $X \in [\omega_1]^{\omega_1}$ and an integer $k^* \geq \bar{k}$ such that for each $\gamma \neq \eta \in X$, there is some $k \geq k^*$ such that $f_\gamma(k) \geq g_\eta(k)$ or $f_\eta(k) \geq g_\gamma(k)$.

Claim 15. *$\exists S \in [\omega_1]^{\omega_1}$, $\forall \gamma \neq \eta \in S$, $\exists i < N$ and $m \geq k'$ such that $f^i_\gamma(m) \geq g^i_\eta(m)$ or $f^i_\eta(m) \geq g^i_\gamma(m)$.*

Proof. Let $S = X$. Fix any $\gamma \neq \eta \in X$. So, there is some $k \geq k^* \geq \bar{k}$ such that either $f_\gamma(k) \geq g_\eta(k)$, or $f_\eta(k) \geq g_\gamma(k)$. Suppose first that $f_\gamma(k) \geq g_\eta(k)$. Say $k = m \cdot N + i$. So $f^i_\gamma(m) = f_\gamma(k) \geq g_\eta(k) = g^i_\eta(m)$. Note that $m \geq k'$, since $k \geq k^* \geq \bar{k} = k' \cdot N + N$. The case $f_\eta(k) \geq g_\gamma(k)$ is similar. □

Note that Claim 15 contradicts (4.2) from page 64.

□

Lemma 4.2.8. *Let $(\mathcal{F}, \mathcal{G})$ be an ω_1-pregap in $({}^{<\omega}\mathbb{Q}, <^*)$, let $h : \omega \longrightarrow \mathbb{Q}$, and suppose $\mathcal{F} <^* h <^* \mathcal{G}$. Then $K = K(\mathcal{F}, \mathcal{G})$ is σ-centered.*

Proof. For each $s \in {}^{<\omega}\mathbb{Q}$, $k \in \omega$, and $B, C \in [{}^k\mathbb{Q}]^{<\omega}$, let $K(s, k, B, C) = \{(L, R, s) \in K : k > |s|, \forall \gamma \in L, \forall \delta \in R, \forall n \geq k, f_\gamma(n) < h(n) < g_\delta(n), B = \{f_\gamma \restriction k : \gamma \in L\}, C = \{g_\delta \restriction k : \delta \in R\}\}$.

Claim 16. $K = \bigcup\{K(s, k, B, C) : s \in {}^\omega\mathbb{Q},\ k \in \omega,\ B, C \in [{}^k\mathbb{Q}]^{<\omega}\}$.

Proof. Fix $(L, R, s) \in K$. Since $L <^* h <^* R$, find $k > |s|$ such that $\forall \gamma \in L$, $\forall \delta \in R$, $\forall n > k$, $f_\gamma(n) < h(n) < g_\delta(n)$. Let $B = \{f_\gamma \upharpoonright k : \gamma \in L\}$, $C = \{g_\delta \upharpoonright k : \delta \in R\}$. Then $(L, R, s) \in K(s, k, B, C)$. □

Claim 17. *For each $s \in \mathbb{Q}^{<\omega}$, $k \in \omega$, and $B, C \in [{}^k\mathbb{Q}]^{<\omega}$, the set $K(s, k, B, C)$ is centered.*

Proof. Fix such s, k, B and C. Let $\{(L^i, R^i, s) : i < m\}$ be a finite subset of $K(s, k, B, C)$. Let $L = \bigcup_{i<m} L^i$, and $R = \bigcup_{i<m} R^i$. Then for each $i < m$, $(L, R, s) \leq (L^i, R^i, s)$. To see this, we need only show that $(L, R, s) \in K$. So, fix $\gamma^i \in L^i$, $\delta^r \in R^r$, and $n > |s|$. If $n > k$, then $f_{\gamma^i}(n) < h(n) < g_{\delta^r}(n)$, by definition of $K(s, k, B, C)$. So suppose that $n < k$. Note that $\{f_\gamma \upharpoonright k : \gamma \in L^i\} = B = \{f_\gamma \upharpoonright k : \gamma \in L^r\}$. So $\exists \gamma^r \in L^r$ such that $f_{\gamma^i} \upharpoonright k = f_{\gamma^r} \upharpoonright k$. So $f_{\gamma^i}(n) = f_{\gamma^r}(n) < g_{\delta^r}(n)$, since $(L^r, R^r, s) \in K$. □

So K is a countable union of centered sets.

□

Definition 44. A partial order P is pre-caliber $\aleph_1$ (or has $\aleph_1$ as a pre-caliber) if for every uncountable subset A of P, there is an uncountable $B \subseteq A$ that is a centered family in P (i.e., given any finite subset $F \subseteq B$, there is some $p \in P$ such that p extends every element of F).

Remark 12. It is easy to see that a σ-centered partial order is pre-caliber $\aleph_1$.

Lemma 4.2.9. *Let $(\mathcal{F}, \mathcal{G})$ be a (κ, λ)-gap in $({}^{\omega}\mathbb{Q}, <^*)$. If $cf(\kappa) \neq \omega_1$ or $cf(\lambda) \neq \omega_1$, then $K(\mathcal{F}, \mathcal{G})$ is pre-caliber $\aleph_1$.*

Proof. Write $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$, $\mathcal{G} = \{g_\alpha : \alpha < \lambda\}$, and $K = K(\mathcal{F}, \mathcal{G})$. It suffices to consider the case $cf(\kappa) \neq \omega_1$.

CASE 1 Suppose $cf(\kappa) = \omega$. Let $\{\xi_j : j < \omega\}$ be a strictly increasing sequence of ordinals from κ converging to κ. For each $j \in \omega$, let $A_j = \{(L, R, s) \in K : L \subseteq \xi_j\}$. Note that $K = \bigcup_{j \in \omega} A_j$.

For each $j \in \omega$, $\bar{s} \in \mathbb{Q}^{<\omega}$, $k \in \omega$, and $B, C \in [{}^{k}\mathbb{Q}]^{<\omega}$, let $A_j(\bar{s}, k, B, C) = \{(L, R, s) \in A_j : s = \bar{s},\ k > |s|,\ \forall \gamma \in L,\ \forall \delta \in R,\ \forall n \geq k,\ f_\gamma(n) < f_{\xi_j}(n) < g_\delta(n),\ B = \{f_\gamma \restriction k : \gamma \in L\},\ C = \{g_\delta \restriction k : \delta \in R\}\}$.

Claim 18. *For each $j < \omega$, $A_j = \bigcup \{A_j(\bar{s}, k, B, C) : \bar{s} \in {}^{\omega}\mathbb{Q},\ k \in \omega,\ B, C \in [{}^{k}\mathbb{Q}]^{<\omega}\}$.*

Proof. Similar to the proof of Claim 16, except now f_{ξ_j} plays the role of h. □

Claim 19. *For each $j \in \omega$, $s \in \mathbb{Q}^{<\omega}$, $k \in \omega$, and $B, C \in [{}^{k}\mathbb{Q}]^{<\omega}$, the set $A_j(s, k, B, C)$ is centered.*

Proof. Similar to the proof of Claim 17, except now f_{ξ_j} plays the role of h. □

So $K = \bigcup_{j \in \omega} A_j$ is a countable union of centered sets. Being σ-centered, K is pre-caliber $\aleph_1$.

CASE 2 Suppose $cf(\kappa) = \rho > \omega_1$. For each $A \in [K]^{<\rho}$, find $\xi_A < \kappa$ such that $\forall (L, R, s) \in A$, $L \subseteq \xi_A$. Fix A and ξ_A. We need to show A has an uncountable centered subset, but we will show something stronger, namely, that A is σ-centered.

For each $s \in \mathbb{Q}^{<\omega}$, $k \in \omega$, and $B, C \in [{}^k\mathbb{Q}]^{<\omega}$, let $A(\bar{s}, k, B, C) = \{(L, R, s) \in A : s = \bar{s},\ k > |s|,\ \forall \gamma \in L,\ \forall \delta \in R,\ \forall n \geq k,\ f_\gamma(n) < f_{\xi_A}(n) < g_\delta(n),\ B = \{f_\gamma \restriction k : \gamma \in L\},\ C = \{g_\delta \restriction k : \delta \in R\}\}$.

Claim 20. $A = \bigcup\{A(\bar{s}, k, B, C) : \bar{s} \in {}^\omega\mathbb{Q},\ k \in \omega,\ B, C \in [{}^k\mathbb{Q}]^{<\omega}\}$.

Proof. Just like the proof of Claim 16, except now f_{ξ_A} plays the role of h. □

Claim 21. *For each $s \in \mathbb{Q}^{<\omega}$, $k \in \omega$, and $B, C \in [{}^k\mathbb{Q}]^{<\omega}$, the set $A(s, k, B, C)$ is centered.*

Proof. Just like the proof of Claim 17, except now f_{ξ_A} plays the role of h. □

So A is a countable union of centered sets.

CASE 3 Suppose $cf(\kappa) < \omega$. For $cf(\kappa) = 0$, any finite set of conditions of the form $\{(\emptyset, R^i, s) : i < m\}$ will have lower bound $(\emptyset, \bigcup_{i<n} R^i, s)$. So suppose $cf(\kappa) = 1$. Let $\xi = \sup(\kappa) < \kappa$, and let $h = f_\xi$.

For each $s \in \mathbb{Q}^{<\omega}$, $k \in \omega$, and $B, C \in [{}^k\mathbb{Q}]^{<\omega}$, let $K(s, k, B, C) = \{(L, R, s) \in A : k > |s|,\ \forall \gamma \in L \setminus h,\ \forall \delta \in R,\ \forall n \geq k,\ f_\gamma(n) < h(n) < g_\delta(n),\ B = \{f_\gamma \restriction k : \gamma \in L\},\ C = \{g_\delta \restriction k : \delta \in R\}\}$.

Claim 22. $K = \bigcup\{K(s,k,B,C) : s \in {}^{\omega}\mathbb{Q},\ k \in \omega,\ B, C \in [{}^{k}\mathbb{Q}]^{<\omega}\}$.

Proof. Like the proof of Claim 16. □

Claim 23. *For each $s \in \mathbb{Q}^{<\omega}$, $k \in \omega$, and $B, C \in [{}^{k}\mathbb{Q}]^{<\omega}$, the set $K(s,k,B,C)$ is centered.*

Proof. Similar to the proof of Claim 17. □

So K is a countable union of centered sets. Once again, being σ-centered, K is pre-caliber $\aleph_1$.

□

We will need a few standard facts about pre-caliber $\aleph_1$, namely, that pre-caliber $\aleph_1$ posets are ccc, that the product of a pre-caliber $\aleph_1$ poset with a ccc partial order is ccc, and that the pre-caliber $\aleph_1$ property is preserved under finte products.

Lemma 4.2.10. *Let P and Q be partial orders.*

1. *If P is pre-caliber $\aleph_1$, then P is ccc.*

2. *If P and Q are pre-caliber $\aleph_1$, then $P \times Q$ is pre-caliber $\aleph_1$.*

3. *If P is pre-caliber $\aleph_1$ and Q is ccc, then $P \times Q$ is ccc.*

Proof. See [Kun11], page 182. □

4.3 The Suslin Tree Lemmas

Before describing in detail the actual forcing construction needed to prove Theorem 4.0.3, a few more lemmas will be needed. These results, and some of the proofs, can also be found in [Woo84].

Lemma 4.3.1. *Fix M, a countable transitive model for ZFC, and in M, suppose $(L, <)$ is an uncountable linear order, and P is a ccc forcing notion. If forcing with P creates a new ω_1-gap in L, then there is a Suslin tree T in M such that forcing with P shoots a branch through T.*

Proof. Before proceeding, some definitions will be needed. First, a subset $I \subseteq L$ will be called an interval if $\forall a, c \in I$, $\forall b \in L$, $a < b < c \Rightarrow b \in I$. It will be useful to identify gaps in L with the corresponding upper half interval in L:

Definition 45. Given a gap $(\mathcal{A}, \mathcal{B})$ in L, let $UHI(\mathcal{A}, \mathcal{B})$ (respectively, $LHI(\mathcal{A}, \mathcal{B})$) denote the upper-half interval (respectively, lower-half interval) corresponding to $(\mathcal{A}, \mathcal{B})$. I.e., $UHI(\mathcal{A}, \mathcal{B}) = \{x \in L : \forall a \in \mathcal{A}, a < x\}$, and $LHI(\mathcal{A}, \mathcal{B}) = \{x \in L : \forall b \in \mathcal{B}, x < b\}$.

Definition 46. Given a gap $(\mathcal{A}, \mathcal{B})$ in L, and an interval J from L, say $(\mathcal{A}, \mathcal{B})$ lies in J if both $LHI(\mathcal{A}, \mathcal{B}) \cap J$ and $UHI(\mathcal{A}, \mathcal{B}) \cap J$ are non-empty. Note that if $(\mathcal{A}, \mathcal{B})$ is an ω_1-gap and $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$, $\mathcal{B} = \{b_\alpha : \alpha < \omega_1\}$, then $(\mathcal{A}, \mathcal{B})$ lies in J iff $\exists \bar{\eta} < \omega_1$ such that "$(\mathcal{A}, \mathcal{B})$ beyond $\bar{\eta}$ is in J"; i.e., $\forall \eta > \bar{\eta}$, $a_\eta, b_\eta \in J$.

Definition 47. Given an interval $I \subseteq L$ and some $y \in I$, let $I^0(y) = \{x \in I : x \leq y\}$ and $I^1(y) = \{x \in I : x > y\}$.

Finally, fix a P-name τ such that $\mathbf{1} \Vdash$ "τ is an ω_1-gap in L", and such that $\forall x \in \mathcal{P}(L) \cap M$, $\forall p \in P$, $p \nVdash$ "$UHI(\tau) = \check{x}$".

Claim 24. *Suppose I is an interval from L such that $\exists p \in P$, $p \Vdash$ "τ lies in $\check{I}$". Then there is some $x_I \in I$ and conditions $p_0, p_1 \in P$ such that $p_0 \Vdash$ "τ lies in $I^0(\check{x}_I)$", and $p_1 \Vdash$ "τ lies in $I^1(\check{x}_I)$".*

Proof. Fix $(\dot{A}, \dot{B})$ such that $\mathbf{1} \Vdash$ "$\tau = (\dot{A}, \dot{B})$". Fix I and p such that $p \Vdash$ "τ lies in I". Suppose first that for each $y \in I$, $p \Vdash$ "τ lies in $I^0(y)$", or $p \Vdash$ "τ lies in $I^1(y)$". Let $I^1 = \{y \in L : p \Vdash \tau \text{ lies in } I^0(y)\}$. Note that I^1 is in M, and $p \Vdash$ "$UHI(\tau) = I^1$", contrary to hypothesis. To see more clearly that $p \Vdash$ "$UHI(\tau) = I^1$", fix G, a generic filter for P over M, such that $p \in G$. Let $A = \dot{A}[G]$, and let $B = \dot{B}[G]$. We need to show that $UHI(A, B) = I^1$. Fix first $y \in UHI(A, B)$. So for each $a \in A$, $a < y$. Either $y \in I$, or $y > I$. In the latter case, since $p \Vdash$ "τ lies in I", it follows that $p \Vdash$ "τ lies in $I^0(y)$". On the other hand, if $y \in I$, suppose instead that $p \Vdash$ "τ lies in $I^1(y)$". So in $M[G]$, (A, B) lies in $I^1(y)$. But then $\exists a \in A$ such that $a > y$, contrary to the assumption that $y \in UHI(A, B) = \{x \in L : \forall a \in A, a < x\}$. Next, fix $y \in I^1$. So $y \in L$ and $p \Vdash$ "τ lies in $I^0(y)$". So, in $M[G]$, (A, B) lies in $I^0(y)$. Then, given $a \in A$, it must be that $a < y$. So $y \in UHI(A, B)$.

So, there must be some x_I in I such that $p \not\Vdash$ "τ lies in $I^0(x_I)$", and $p \not\Vdash$ "τ lies in $I^1(x_I)$". So $\exists p_1 \leq p$ such that $p_1 \Vdash$ "τ does not lie in $I^0(y)$", and hence $p_1 \Vdash$ "τ lies in $I^1(y)$"; similarly, $\exists p_0 \leq p$ such that $p_0 \Vdash$ "τ lies in $I^0(y)$". □

Now, recursively define $\{S_\alpha : \alpha < \omega_1\}$ such that for each $\alpha < \omega_1$:

1. $S_\alpha \subseteq \mathcal{P}(L)$ is a non-empty collection of intervals of L;

2. $\forall I \neq J \in S_\alpha$, $I \cap J = \emptyset$;

3. $\mathbf{1} \Vdash$ "$(\exists I \in S_\alpha)\tau$ lies in I";

4. $\forall I \in S_\alpha$, $\exists p \in P$ such that $p \Vdash$ "τ lies in I";

5. $\forall \beta < \alpha$, $\forall I \in S_\beta$, $\exists I_0, I_1 \in S_\alpha$ such that $I_0 \cap I_1 = \emptyset$ and $I_0 \cup I_1 \subseteq I$;

6. $\forall \beta < \alpha$, $\forall I \in S_\alpha$, $\exists J \in S_\beta$ such that $I \subseteq J$.

To begin, let $S_0 = \{L\}$. Next, fix $\beta < \omega_1$, and suppose that for all $\delta < \beta$, S_δ has been defined subject to the conditions. If $\beta = \alpha + 1$, find x_I for each $I \in S_\alpha$ as per Claim 24. Let $S_\beta = \{I^0(x_I) : I \in S_\alpha\} \cup \{I^1(x_I) : I \in S_\alpha\}$.

If $\beta \in LIM$, first let $X_\beta = \{x \in L : \forall \alpha < \beta, \exists I \in S_\alpha \text{ such that } x \in I\}$. Define an equivalence relation $\sim$ on X_β by $x \sim y$ iff $\forall \alpha < \beta$, $\forall I \in S_\alpha$, $x \in I \Leftrightarrow y \in I$. Let $S = \{[x] : x \in X_\beta\}$, the collection of equivalence classes for $\sim$ on X_β.

Claim 25. *S is a collection of intervals of L.*

Proof. Fix $[x] \in S$. Suppose $y, z \in [x]$ such that $y < z$, and fix $w \in L$ such that $y < w < z$. To see that $w \sim x$, fix $\alpha < \beta$, $I \in S_\alpha$, and suppose first $x \in I$. Then $y, z \in I$, and so $w \in I$ since I is an interval. So suppose next that $w \in I$. Since $x \in X_\beta$, there is some $I' \in S_\alpha$ such that $x \in I'$. So $y, z \in I'$, and hence $w \in I'$. Since $w \in I \cap I'$, it must be that $I = I'$, and so $x \in I$. □

Now, for each $\alpha < \beta$, for each $I \in S_\alpha$, find $p_I \in P$ such that $p_I \Vdash \tau$ lies in I. Extend each $\{p_I\}$ to an antichain A_I such that $\forall p \in A_I$, $p \Vdash \tau$ lies in I, and A_I is maximal with respect to this property. For each $\alpha < \beta$, let $\mathbb{A}_\alpha = \bigcup\{A_I : I \in S_\alpha\}$.

Claim 26. *For each $\alpha < \beta$, $\mathbb{A}_\alpha$ is a maximal antichain in P.*

Proof. Fix $p, q \in \mathbb{A}_\alpha$, and suppose $p \in A_I$, $q \in A_J$ (where $I, J \in S_\alpha$). If $I = J$, then $p \perp q$ since A_I is an antichain; if $I \neq J$, then $I \cap J = \emptyset$, so $p \perp q$. Thus, $\mathbb{A}_\alpha$ is an antichain. For maximality, suppose instead there is some $r \in P$ such that $\forall p \in \mathbb{A}_\alpha$, $r \perp p$. Since $\mathbf{1} \Vdash$ "$(\exists I \in S_\alpha)\tau$ lies in I", find $I \in S_\alpha$ and $q \leq r$ such that $q \Vdash \tau$ lies in I. So $\exists p \in A_I$ such that $p \not\perp q$. But then $r \not\perp p$, a contradiction. □

Claim 27. *For each $G \subseteq P$ generic over M, there is a $p_G \in G$ such that $p_G \Vdash$ $(\exists I \in S)\tau$ lies in I.*

Proof. Fix such a G, and for each $\alpha < \beta$, let p_α denote the unique condition such that $p_\alpha \in G \cap \mathbb{A}_\alpha$. Let I_α denote the unique $I \in S_\alpha$ such that $p_\alpha \Vdash \tau$ lies in $\dot{I}_\alpha$ (where $\dot{I}_\alpha$ names the interval I_α, for each $\alpha < \beta$). In $M[G]$, suppose $\tau[G] = (\mathcal{A}, \mathcal{B})$, where $\mathcal{A} = \{a_\eta : \eta < \omega_1\}$, $\mathcal{B} = \{b_\eta : \eta < \omega_1\}$. Since β is countable and $(\mathcal{A}, \mathcal{B})$ is an ω_1-gap, there must be some $\bar{\eta}$ such that $\forall \alpha < \beta$, $(\mathcal{A}, \mathcal{B})$ beyond $\bar{\eta}$ is in I_α. Thus, $I^* = \bigcap_{\alpha<\beta} I_\alpha$ is an (uncountable) interval of L such that $(\mathcal{A}, \mathcal{B})$ lies in I^*. Note that for each $x \in I^*$, $I^* = [x] \in S$. So in $M[G]$, $(\exists I \in S)\tau[G]$ lies in I. The forcing lemma now completes the proof. $\square$

Claim 27 can be rephrased as follows:

Claim 28. *$\mathbb{1} \Vdash$ "$(\exists I \in S)$ τ lies in I".*

Let $S' = \{I \in S : \exists p \in P,\ p \Vdash \text{"}\tau \text{ lies in } I\text{"}\}$. Note that by Claim 27, S' is non-empty. For each $I \in S'$, find x_I as per Claim 24, and let $S_\beta = \{I^0(x_I) : I \in S'\} \cup \{I^1(x_I) : I \in S'\}$.

Having completed the construction, let $T = \bigcup\{S_\alpha : \alpha < \omega_1\}$.

Claim 29. *$(T, \supseteq)$ is an ω_1-Suslin tree in M.*

Proof. Clearly T is a tree of height ω_1 with levels the S_α's. To see that T has no uncountable levels, or more in general, no uncountable antichains, suppose instead that $A = \{I_\alpha : \alpha < \omega_1\}$ is an antichain in T. For each $\alpha < \omega_1$, choose $p_\alpha \in P$ such that $p_\alpha \Vdash \tau$ lies in I_α. But then $\{p_\alpha : \alpha < \omega_1\}$ is an uncountable antichain in P, contrary to the assumption that P is ccc.

The fact that (in M) every chain in T is countable, follows immediately now by a well-known result on trees. To be explicit, suppose instead that $B \subseteq T$ is an

ω_1-branch of T in M. For each $\alpha \in \omega_1$, let $B(\alpha)$ denote the node of T on the α^{th} level of T; i.e., $\{B(\alpha)\} = B \cap S_\alpha$, and pick $I_\alpha \in S_{\alpha+1} \setminus B$ such that $I_\alpha \subseteq B(\alpha)$. Then $\{I_\alpha : \alpha < \omega_1\}$ is an uncountable antichain, contrary to the preceding paragraph. $\square$

Finally, note that τ determines an ω_1-branch through T in M^P. More explicitly, fixing $G \subseteq P$ generic over M, the set $\{I \in T : \exists p \in G$ such that $p \Vdash \tau$ lies in $I\}$ is a branch through T in $M[G]$. $\square$

Definition 48. Given any tree T, define $S(T) = \{p \in Fn(T, \omega) \mid \forall n \in \omega, p^{-1}\{n\}$ is an antichain in $T\}$, ordered by reverse inclusion.[33] Call these forcing notions the tree-specializers.

Remark 13. Note that forcing with $S(T)$ makes T into a special tree — see Lemma 4.3.4 for a proof. Furthermore, Baumgartner has shown that $S(T)$ is ccc whenever T is an Aronszajn tree.

Lemma 4.3.2. *Given any Aronszajn tree T, $S(T)$ is ccc.*[34]

Proof. Suppose instead that $A \subseteq S(T)$ is an uncountable antichain. Suppose that $\{dom(p) : p \in A\}$ is a Δ-system with root R and tails X_p, such that for all $p \in A$, $|X_p| = n$ (for some fixed $n \in \omega$). For each $p \in A$, write $X_p = \{t_i^p : i < n\}$. Suppose further that $\exists \bar{p} : R \longrightarrow \omega$ such that for each $p \in A$, $p \restriction R = \bar{p}$.

[33] Given sets I, J, $Fn(I, J)$ denotes the set of finite partial functions from I into J.

[34] Cf. [BMR70].

Claim 30. *For each* $p \neq q \in A$, $\exists k, l < n$ *such that* t^p_k, t^q_l *are comparable and* $p(t^p_k) = q(t^q_l)$.

Proof. If not, then $p \cup q \in S(T)$, and then p, q would be compatible. □

Fix $\mathcal{U}$, a non-principal ultrafilter on A (i.e., on $(\mathcal{P}(A), \subseteq)$) such that all co-countable subsets of A are in $\mathcal{U}$.

Claim 31. $\forall p \in A$, $\exists k = k_p, l = l_p < n$ *such that* $A_{p,k,l} = \{q \in A : p(t^p_k) = q(t^q_l)$, *and* t^p_k, t^q_l *are comparable*$\} \in \mathcal{U}$.

Proof. Fix $p \in A$. Note that $A \setminus \{p\} = \bigcup\{A_{p,k,l} : k, l < n\}$.[35] But $A \setminus \{p\} \in \mathcal{U}$, since $\mathcal{U}$ is non-principal. Since $\{A_{p,k,l} : k, l < n\}$ is a finite collection, there must be some $k, l < n$ such that $A_{p,k,l} \in \mathcal{U}$. □

For each $p \in A$, choose $k_p < n$ and $l_p < n$ as per Claim 31. Suppose $\exists k, l < n$ such that $\forall p \in A$, $k_p = k$ and $l_p = l$.

Claim 32. $\{t^p_k : p \in A\}$ *is a chain in* T.

Proof. Fix $p \neq q \in A$, and let $A' = A_{p,k,l} \cap A_{q,k,l} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter containing all co-countable sets, A' cannot be countable. Furthermore, note that

[35] To see that $A \setminus \{p\} = \bigcup\{A_{p,k,l} : k, l < n\}$, first fix $q \in A \setminus \{p\}$. By Claim 30, there are $k, l < n$ such that $q \in A_{p,k,l}$. For the other direction, given any $k, l < n$, note that $p \notin A_{p,k,l}$, by definition of $S(T)$.

for all $u \in A'$, t_k^p, t_l^u are comparable, and t_k^q, t_l^u are comparable. Since T is an ω_1-tree, there are only countably many nodes that lie below t_k^p (in the tree ordering). Since A' is uncountable, there must be some $u \in A'$ such that $t_k^p \leq t_l^u$. But then t_k^p and t_l^q are comparable. □

Since A is uncountable, $\{t_k^p : p \in A\}$ is an uncountable branch in T, contrary to the assumption that T is Aronszajn.

□

Definition 49. Let $\mathcal{T} = \{(T, \leq) : T \subseteq \omega_1, (T, \leq)$ is an Aronszajn tree$\}$, and let $\mathbb{S} = \Pi\{S(T) : T \in \mathcal{T}\}$, the finite support product of these tree specializers.

Corollary 4.3.3. $\mathbb{S}$ *is ccc.*

Proof. It suffices to show that given any finite collection $\{T_i : i < k\}$ of Aronszajn trees, the product $\Pi_{i<k}S(T_i)$ is ccc. To begin, let T denote the disjoint union of these finitely many trees. Note that $S(T)$ is order isomorphic to $\Pi_{i<k}S(T_i)$; the isomorphism is given by:

$$\varphi : S(T) \longrightarrow \Pi_{i<k}S(T_i);\ \varphi(p) = (p \restriction T_0, \ldots, p \restriction T_{k-1})$$

But since each T_i is an Aronszajn tree, T is also an Aronszajn tree. Thus, by Lemma 4.3.2, S(T) is ccc, and so $\Pi_{i<k}S(T_i)$ is ccc.

□

Lemma 4.3.4. *Fix M a countable transitive model for ZFC, and suppose T is a tree in M. Fix $G \subseteq S(T)$, generic over M. In $M[G]$, T is a special tree.*

Proof. For each $t \in T$, let $D_t = \{p \in S(T) : t \in \mathrm{dom}(p)\}$. To see that each D_t is dense, fix $t \in T$ and $q \in S(T)$ such that $t \notin \mathrm{dom}(q)$. Choose $m \in \omega \setminus \mathrm{ran}(q)$, and let $p = q \cup \{(t, m)\}$. So $p^{-1}\{m\} = \{t\}$, an antichain in T. Thus, $p \in D_t$ and $p \leq q$.

Define $F = \bigcup G$, and note that $F : T \longrightarrow \omega$.

Claim 33. *For each $n \in \omega$, $F^{-1}\{n\}$ is an antichain in T.*

Proof. Fix $n \in \omega$, and suppose $s, t \in F^{-1}\{n\}$. Fix $p \in G$ such that $s \in \mathrm{dom}(p)$, fix $q \in G$ such that $t \in \mathrm{dom}(q)$, and find $r \in G$ extending p and q. Since $s, t \in \mathrm{dom}(r)$, $r(s) = F(s) = n = F(t) = r(t)$ and s, t are incomparable. □

Since $T = \bigcup_{n\in\omega} F^{-1}\{n\}$, T is a special tree. □

Remark 14. Forcing with $\mathbb{S} = \Pi\{S(T) : T \in \mathcal{T}\}$ specializes all Aronszajn trees of the ground model.

Lemma 4.3.5. *Let V be a countable transitive model for ZFC, and let $(L, <)$ be a linear order in V. Suppose P is (forced to be) a ccc partial order in $V^{\mathbb{S}}$. Then P does not introduce any new ω_1-gaps in L.*

Proof. Suppose instead that P did introduce a new ω_1-gap in L. Note that since $\mathbb{S}$ is ccc and $\mathbb{S} \Vdash P$ is ccc, the iteration $\mathbb{S} * P$ is ccc. Thus, by Lemma 4.3.1, there must be some Suslin tree T in V such that in $V^{\mathbb{S}*P}$, there is an uncountable branch in T. But T is special in $V^{\mathbb{S}}$, and hence in $V^{\mathbb{S}*P}$. So T is both special and contains an uncountable branch in $V^{\mathbb{S}*P}$, which is impossible. □

Lemma 4.3.6. *Fix M a countable transitive model for ZFC, T a Suslin tree in M, P a ccc forcing notion in M, and G a P-generic filter over M. Suppose in $M[G]$, B is a branch through T. Then in $M[G]$, P is not ccc.*

Proof. Let τ be a P-name for B; i.e., $\tau[G] = B$.

Claim 34. *Given $t \in T$ and $p_t \in P$ such that $p_t \Vdash t \in \tau$, there are nodes $t_0, t_1 \in T$, and conditions $p_{t_0}, p_{t_1} \in P$ such that*

- *$t \leq t_0, t_1$, and t_0, t_1 are incomparable;*
- *$p_{t_0}, p_{t_1} \leq p_t$; $p_{t_0} \Vdash t_0 \in \tau$; and $p_{t_1} \Vdash t_1 \in \tau$.*

Proof. Suppose instead there is some $t \in T$ and $p_t \Vdash t \in \tau$ for which there are no nodes t_0, t_1 and conditions p_0, p_1 satisfying the stated properties. But then p_t decides τ; i.e., for each β such that $ht(t) < \beta < \omega_1$, there is some $r(\beta) \in T \cap M$ such that $p_t \Vdash$ "$r(\beta)$ is the β^{th} element of the branch τ". Let $B' = \{t' \in T : p_t \Vdash t' \in \tau\}$. Then B' is a branch through T that lies in M, a contradiction. □

Now, working in $M[G]$, let $t(\beta)$ be the β^{th} element of the branch B; i.e., the node from the branch on the β^{th} level of T. By the forcing lemma, for each $\beta < \omega_1$, there is some $p_\beta \in G$ such that $p_\beta \Vdash t(\beta) \in \tau$. For each $\beta < \omega_1$, Claim 34 provides a $q_\beta \leq p_\beta$ and an $r(\beta) \geq t(\beta)$ such that $r(\beta) \notin B$, but $q_\beta \Vdash r(\beta) \in \tau$. Thus, in $M[G]$, $\{q_\beta : \beta < \omega_1\}$ is an uncountable antichain in P. □

The following definition is from [Woo84]. Some of the results which follow appear without proof in [Woo84].

Definition 50. A partial order P shall be called *nice* iff P is ccc (in V, the ground model), P is ccc in V^P, and P is ccc in any ccc forcing extension of V^P.

Corollary 4.3.7. *Forcing with nice partial orders cannot add ω_1-branches to a ground model Suslin tree.*

Proof. Immediate by Lemma 4.3.6. □

Corollary 4.3.8. *Let V be a countable transitive model for ZFC, and let $(L, <)$ be a linear order in V. Suppose P is a nice partial order in V. Then P does not introduce any new ω_1-gaps in L.*

Proof. Suppose P did add a new ω_1-gap to L. Then by Lemma 4.3.1, P adds a branch to a Suslin tree of V, contradicting Corollary 4.3.7. □

Lemma 4.3.9. *Let $(\mathcal{F}, \mathcal{G})$ be a pregap, and let $K = K(\mathcal{F}, \mathcal{G})$. K is nice iff K is ccc.*

Proof. Suppose K is ccc. Then forcing with K fills the pregap, so in V^K the pregap is filled, and so K is σ-centered in V^K by Lemma 4.2.8. In any further ccc extension, the pregap is of course still filled, so K remains σ-centered. □

Lemma 4.3.10. *Fix $n \in \omega$, and for each $i < n$ suppose $(\mathcal{F}^i, \mathcal{G}^i)$ are pregaps. For each $i < n$, suppose $K^i = K(\mathcal{F}^i, \mathcal{G}^i)$. Then ΠK^i is nice iff ΠK^i is ccc.*

Proof. Suppose ΠK^i is ccc. In $V^{\Pi K^i}$, all the pregaps are filled, so each is σ-centered. Thus, ΠK^i is σ-centered in $V^{\Pi K^i}$ since the product of σ-centered partial orders is easily seen to be σ-centered. In any further ccc extension, all the pregaps remain filled, so ΠK^i remains σ-centered. □

Lemma 4.3.11. *Fix $\alpha \in \mathbb{ON}$, and for each $i < \alpha$ suppose $(\mathcal{F}^i, \mathcal{G}^i)$ are pregaps. For each $i < \alpha$, suppose $K^i = K(\mathcal{F}^i, \mathcal{G}^i)$. Then $\Pi_{i<\alpha} K^i$ is nice iff $\Pi_{i<\alpha} K^i$ is ccc.*

Proof. Suppose $\Pi_{i<\alpha} K^i$ is ccc. In $V^{\Pi_{i<\alpha} K^i}$, all the pregaps are filled, so each is σ-centered. To see that the finite support product is ccc in $V^{\Pi_{i<\alpha} K^i}$, we'll show that any finite product is σ-centered. Fix $n \in \omega$, and look at (without loss of generality) $\{K^i : i < n\}$. To see that $\Pi_{i<n} K^i$ is σ-centered in $V^{\Pi_{i<\alpha} K^i}$, note once again that the product of finitely many σ-centered partial orders is σ-centered. In any further ccc extension, all the pregaps remain filled, so any finite product of K^i's remains σ-centered. □

Fact 7. (a) For any Aronszajn tree T, $S(T)$ is nice. Furthermore, (b) The finite-support product $\mathbb{S}$ is a nice partial order as well.

Proof. For (a), note first that $S(T)$ is ccc if T is Aronszajn, by Lemma 4.3.2. In $V^{S(T)}$, T is still Aronszajn — T cannot have a branch in this extension since it is special. This remains true in any further ccc extension. For (b), by Lemma 4.3.3, $\mathbb{S}$ is ccc. But in $V^{\mathbb{S}}$, all the relevant trees have been specialized and remain Aronszajn, so $\mathbb{S}$ is ccc in $V^{\mathbb{S}}$, and in any further ccc extension. □

4.4 The Construction

The strategy for the proof of Theorem 4.0.3 has already been outlined in Section 4.1, but here we make the details of the iterated forcing construction more explicit. Fix M a countable transitive model for ZFC, and in M, fix κ a regular cardinal such that $\kappa \geq \omega_2$ and $2^{<\kappa} = \kappa$. Recall (from Section 4.1) our partition of κ into the following classes:

- EL = EASY-LIMITS = $\{\beta < \kappa : \beta \in LIM, cf(\beta) \neq \omega_1\}$;
- HL = HARD-LIMITS = $\{\beta < \kappa : \beta \in LIM, cf(\beta) = \omega_1\}$
- ES = EASY-SUCCESSORS = $\{\beta < \kappa : \beta = \alpha + 1, \alpha \notin HL\}$;
- HS = HARD-SUCCESSORS = $\{\beta < \kappa : \beta = \alpha + 1, \alpha \in HL\}$

Using the class EL of easy limit ordinals as index set, fix an enumeration $\mathcal{C} = \{C_\alpha : \alpha \in EL\}$ of (potential nice names for elements of) $[EL]^{<\kappa}$, and suppose that each member of $[EL]^{<\kappa}$ is enumerated cofinally often.[36] $\mathcal{C}$ serves as our enumeration of *candidates for cuts*. Similarly, using the class ES of easy successor ordinals as index set, fix a cofinally-often enumeration of all candidates for ccc-partial order names: $\mathcal{R} = \{\dot{\mathbb{R}}_\alpha : \alpha \in ES\}$.

The finite support iterated forcing construction of length κ will be defined using the somewhat standard notation $(\{\mathbb{P}_\alpha : \alpha \leq \kappa\}, \{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\})$. By simultaneous recursion, and making use of the canonical names $\dot{G}_\alpha$ for the generics on the $\mathbb{P}_\alpha$, and the associated names $H_\alpha = \{t \in \dot{\mathbb{Q}}_\alpha : (\exists p \in \dot{G}_\alpha)\ p * t \in \dot{G}_{\alpha+1}\}$ for the generics on the $\dot{\mathbb{Q}}_\alpha$, we define $\{\dot{\mathbb{Q}}_\alpha : \alpha < \kappa\}$, $\{\mathcal{L}_\alpha : \alpha < \kappa\}$, $\{(A_\alpha, B_\alpha) : \alpha \in EL\}$, and $\{h_\alpha : \alpha \in EL\}$ such that for all $\alpha < \kappa$, $\mathcal{L}_\alpha = \{h_\beta : \beta \in \alpha \cap EL\}$ is a linear order in $({}^\omega\mathbb{Q}, <^*)$. For $\alpha < \kappa$:

- If $\alpha \in EL$, look at C_α. If C_α names a subset of $\alpha \cap EL$, let $A_\alpha = C_\alpha \cup \{\beta \in (\alpha \cap EL) : \exists \gamma \in C_\alpha,\ h_\beta <^* h_\gamma\}$, and let $B_\alpha = (\alpha \cap EL) \setminus A_\alpha$. Otherwise, simply let $(A_\alpha, B_\alpha) = (\alpha \cap EL, \emptyset)$, the pair of index sets for a trivial partition cut. Also, let $A^*_\alpha = \{h_\beta : \beta \in A_\alpha\}$, and let $B^*_\alpha = \{h_\beta : \beta \in B_\alpha\}$. Now, let $\dot{\mathbb{Q}}_\alpha = K(A_\alpha, B_\alpha)$, the Kunen partial order for filling the cut (A^*_α, B^*_α), and let $h_\alpha = \bigcup_{p \in H_\alpha} s_p$, the generic function that fills this cut.

[36]See Chapter 3, section 3.2.

- Suppose next $\alpha \in ES$. If $\dot{\mathbb{R}}_\alpha$ names a valid ccc partial order, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{R}}_\alpha$; otherwise, let $\dot{\mathbb{Q}}_\alpha$ name the trivial (singleton) partial order.

- If $\alpha \in HL$, first let $\mathcal{X}_\alpha = \{(A^\beta_\alpha, B^\beta_\alpha) : \beta < \kappa\}$ enumerate all partition cuts in $\mathcal{L}_\alpha$. Let $\dot{\mathbb{Q}}_\alpha = \Pi\{K(A^\beta_\alpha, B^\beta_\alpha) : \beta < \kappa\}$, the (finite-support) product of the Kunen partial orders for filling each such cut.[37]

- Finally, for $\alpha \in HS$, we force with the (finite-support) product of all Aronszajn tree specializers. More explicitly, as in the previous section, let $\mathcal{T} = \{(T, \leq) : T \subseteq \omega_1 \text{ and } (T, \leq) \text{ is an Aronszajn tree}\}$. Let $\dot{\mathbb{Q}}_\alpha = \Pi\{S(T) : T \in \mathcal{T}\}$, the finite support product of these tree specializers. Lemma 4.3.5 shows that forcing with this product renders it impossible to add new ω_1-gaps of exisiting elements in the linear order via any future ccc forcing.

Let $\mathbb{P} = \mathbb{P}_\kappa$, and $\mathcal{L} = \mathcal{L}_\kappa$. Fix G a $\mathbb{P}$-generic filter over M, and for each $\alpha < \kappa$, let $G_\alpha = G \cap \mathbb{P}_\alpha$. It must be shown that $\mathbb{P}$ is ccc. It then follows by standard methods that $|\mathbb{P}| = \kappa$ (cf. Lemma 3.8.1), and that in $M[G]$, $\mathfrak{c} = \kappa$ (cf. Lemma 3.8.2), and MA holds (see [Kun80], Chapter VIII, section 6, starting on page 278). Finally, we'll show that $\mathcal{L}$ is a maximal saturated linear order of size $\mathfrak{c}$ in $({}^\omega\mathbb{R}, <^*)$. This

[37]Note that no function is added into the emerging linear order at this stage. We only force with these partial orders to render the corresponding cuts ccc-fillable in later stages. If the functions added at this state of the forcing were included into the emerging linear order, then $|\mathcal{L}_\alpha|$ would be $> \kappa$ (for cofinally many α). It is necessary to ensure $|\mathcal{L}_\alpha|$ remains $< \kappa$ for each α in order to prove maximality of the linear order in the final forcing extension. See Theorem 4.6.4 and Remark 17.

work will be done in the following two sections. The current section now concludes with a definition that will be needed for the proof that $\mathbb{P}$ is ccc.

Definition 51. Fix $\theta < \kappa$. Let (C, D) be a partition cut (or a gap which represents a partition cut) in $\mathcal{L}_\theta = \{h_\alpha : \alpha \in EL \cap \theta\}$. If there is some $\lambda < \theta$ such that $\{h_\alpha \in C : \alpha \in EL \cap \lambda\}$ is cofinal in C, then we say the left tower is bounded in θ; if there is no such $\lambda < \theta$ the left tower is said to be unbounded in θ. Similarly, if there is some $\lambda < \theta$ such that $\{h_\alpha \in D : \alpha \in EL \cap \lambda\}$ is coinitial in D, then we say the right tower is bounded in θ; otherwise the right tower is unbounded in θ. If both left and right towers are bounded, the gap is said to be *left/right-bounded in* θ, or just *bounded in* θ for short; if the left tower is bounded but the right tower is unbounded, the gap is said to be *left-bounded in* θ; if the right tower is bounded but the left is unbounded, the gap is called *right-bounded in* θ; if both the left and right towers are unbounded, the gap is called *left/right-unbounded in* θ, or just *unbounded in* θ for short. We suppress mention of θ when the context permits.

4.5 A Product of Gap-fillers

The theorem (Theorem 4.5.1) proved in this section says that under the right conditions, a product of Kunen gap-filling notions for ω_1-gaps that arise in the course of our iterated forcing construction (as described in the previous section) is ccc. More precisely, fix $\theta < \kappa$ such that $cf(\theta) = \omega_1$. Suppose $\{(A^i, B^i) : i < N\}$ is a collection of unfilled ω_1-partition cuts in the linear order $\mathcal{L}_\theta$, each of which arises at stage θ in the finite-support iterated forcing construction. If the iterated forcing construction is ccc up to stage θ, then the product forcing $\Pi\{K(A^i, B^i) : i < N\}$ is ccc.

Theorem 4.5.1 will be needed for the proof that our forcing notion $\mathbb{P}$ is ccc (cf. Theorem 4.6.2). Before stating the theorem, we list the following general assumptions that will be needed:[38]

Assumption 1. Suppose M is a countable transitive model for ZFC. (We work in M, except where otherwise indicated.)

1. $\theta \in \mathbb{ON}$, $cf(\theta) = \omega_1$.
2. $(\{\mathbb{P}_\beta : \beta \leq \theta\}, \{\dot{\mathbb{Q}}_\beta : \beta < \theta\})$ is a ccc finite-support iterated forcing construction.
3. $\mathbb{L} \subseteq LIM$ is a cofinal subset of θ, and $\forall \alpha \in \mathbb{L}$, $\mathbb{P}_{\alpha+1} \Vdash h_\alpha \in {}^\omega\mathbb{Q}$.
4. For each $\alpha \leq \theta$, $\mathbb{P}_\alpha \Vdash$ "$\mathcal{L}_\alpha = \{h_\beta : \beta \in \mathbb{L} \cap \alpha\}$ is a linear order (under $<^*$)."
5. For each $\alpha \in \mathbb{L}$, there is a partition cut (A_α, B_α) of $\mathcal{L}_\alpha$ such that $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = K(A_\alpha, B_\alpha)$", and h_α is the generic function added by $K(A_\alpha, B_\alpha)$.

[38]We have endeavoured to make this section somewhat self-contained by explicitly stating the assumptions that are needed. The use of the less restrictive $\mathbb{L}$ rather than the set EL (as well as the lack of mention of the sets ES and HS) reflects some generalization that is possible, and highlights which features from the partition of κ in Section 4.4 are relevant to the proof of Theorem 4.5.1 and which are not.

6. For each $i < N$, (A^i, B^i) is an unfilled ω_1-partition cut in $\mathcal{L} = \mathcal{L}_\theta$ that is either left-bounded, right-bounded, or left/right-unbounded in θ. (See Definition 51. Note that this assumption will be used in the proof of Claim 43.)

7. $HL(\theta) = \{\alpha < \theta : cf(\alpha) = \omega_1\}$ is disjoint from $\mathbb{L}$, and for each $\alpha \in HL(\theta)$, $\dot{\mathbb{Q}}_\alpha = \Pi_{i \in J} K(\mathcal{E}^i, \mathcal{H}^i)$, where J is an index set and all the partition cuts (see Definition 4) of $\mathcal{L}_\alpha$ occur in the enumeration $\{(\mathcal{E}^i, \mathcal{H}^i) : i \in J\}$.

The following notation will be useful.

Definition 52. For each $i < N$, let $\theta_{A^i} = \min\{\epsilon \leq \theta : A^i \cap \mathcal{L}_\epsilon \text{ is cofinal in } A^i\}$, and let $\theta_{B^i} = \min\{\epsilon \leq \theta : B^i \cap \mathcal{L}_\epsilon \text{ is coinitial in } B^i\}$. Note that if (A^i, B^i) is left-bounded then $\theta_{A^i} < \theta$ while $\theta_{B^i} = \theta$; if (A^i, B^i) is right-bounded then $\theta_{A^i} = \theta$ while $\theta_{B^i} < \theta$; if (A^i, B^i) is unbounded then $\theta_{A^i} = \theta_{B^i} = \theta$. Now, for each $i < N$, if (A^i, B^i) is left-bounded or right-bounded, let $\epsilon^i = \min\{\theta_{A^i}, \theta_{B^i}\}$; otherwise, let $\epsilon^i = 0$. Let $\epsilon = \max\{\epsilon^i : i < N\}$.

Theorem 4.5.1. $\mathbb{P}_\theta \Vdash \Pi\{K(A^i, B^i) : i < N\}$ *is ccc.*

Proof. Suppose instead that for some $\bar{q}_0 \in \mathbb{P}_\theta$, $\bar{q}_0 \Vdash$ "$\Pi\{K(A^i, B^i) : i < N\}$ is not ccc". Obtain (for each $i < N$) $\mathbb{P}_\theta$-names for strictly increasing sequences in θ, $(\dot{\mu}^i_\beta)_{\beta<\omega_1}$ and $(\dot{\lambda}^i_\beta)_{\beta<\omega_1}$, such that the following condition holds:

Condition 1. $\bar{q}_0 \Vdash$ "$\forall i < N$, $(\dot{\mu}^i_\beta)_{\beta<\omega_1}$ is strictly increasing and cofinal in θ_{A^i},

$(\dot\lambda^i_\beta)_{\beta<\omega_1}$ is strictly increasing and cofinal in θ_{B^i}, $\{h_{\dot\mu^i_\beta} : \beta < \omega_1\}$ is strictly increasing and cofinal in A^i, and $\{h_{\dot\lambda^i_\beta} : \beta < \omega_1\}$ is strictly decreasing and coinitial in B^i".

Find $k' \in \omega$, a name S_0 for an uncountable subset of ω_1, and a condition $\bar q_1 \leq \bar q_0$ such that:

Condition 2. $\bar q_1 \Vdash$ "$\forall \beta \in S_0$, $\forall i < N$, $\forall n \geq k'$, $h_{\dot\mu^i_\beta}(n) < h_{\dot\lambda^i_\beta}(n)$".

Let $F^i = \{h_{\dot\mu^i_\beta} : \beta \in S_0\}$, and let $G^i = \{h_{\dot\lambda^i_\beta} : \beta \in S_0\}$; let $K^i = K(F^i, G^i)$, and let $\tilde K^i = K(A^i, B^i)$; let $K = \Pi_{i<N} K^i$; and let $\tilde K = \Pi_{i<N} \tilde K^i$. Note that we are assuming $\bar q_1 \leq \bar q_0 \Vdash$ "$\tilde K$ is not ccc".

Claim 35. $\bar q_1 \Vdash$ "*K is not ccc*".

Proof. Fix a filter G, generic for $\mathbb{P}_\theta$ over M such that $\bar q_1 \in G$. Suppose instead K is ccc in $M[G]$. Let H be a filter generic for K over $M[G]$. In $M[G][H]$, each gap (F^i, G^i) is filled, so each cut (A^i, B^i) is filled. Thus, each $\tilde K^i$ is σ-centered in $M[G][H]$, and so $\tilde K$ is σ-centered in $M[G][H]$.

Since $\bar q_1 \in G$, $\tilde K$ is not ccc in $M[G]$. Let $A \subseteq \tilde K$ be an ω_1-antichain in V. But then A is still an ω_1-antichain in $M[G][H]$, since we are assuming K is ccc in $M[G]$. But then $\tilde K$ is both ccc and not ccc in $M[G][H]$. □

Claim 36. *There is a $\mathbb{P}_\theta$-name $\dot S$ for a subset of ω_1 and a condition $\bar q_2 \leq \bar q_1$ such* *that*

$$\bar{q}_2 \Vdash \forall \alpha \neq \beta \in \dot{S},\, \exists i < N,\, \exists n \geq k',\, h_{\dot{\mu}^i_\alpha}(n) \geq h_{\dot{\lambda}^i_\beta}(n) \text{ or } h_{\dot{\mu}^i_\beta}(n) \geq h_{\dot{\lambda}^i_\alpha}(n)$$

Proof. Fix a filter G generic for $\mathbb{P}_\theta$ over M such that $\bar{q}_1 \in G$. In $M[G]$, apply Lemma 4.2.7 (in contrapositive form) to conclude that there exists an uncountable subset S of ω_1 such that:

$$\forall \alpha \neq \beta \in S,\, \exists i < N,\, \exists n \geq k',\, h_{\dot{\mu}^i_\alpha}(n) \geq h_{\dot{\lambda}^i_\beta}(n) \text{ or } h_{\dot{\mu}^i_\beta}(n) \geq h_{\dot{\lambda}^i_\alpha}(n)$$

Find a $\mathbb{P}_\theta$-name $\dot{S}$ and a condition $r \in G$ such that

$$r \Vdash \forall \alpha \neq \beta \in \dot{S},\, \exists i < N,\, \exists n \geq k',\, h_{\dot{\mu}^i_\alpha}(n) \geq h_{\dot{\lambda}^i_\beta}(n) \text{ or } h_{\dot{\mu}^i_\beta}(n) \geq h_{\dot{\lambda}^i_\alpha}(n)$$

Since r and $\bar{q}_1$ are in G, they are compatible, and so there is some condition $\bar{q}_2$ extending both r and $\bar{q}_1$. □

We shall exhibit a contradiction by finding ordinals η, ξ and a condition $p \leq \bar{q}_2 \in \mathbb{P}_\theta$ which forces $\eta, \xi \in \dot{S}$ and $\forall i < N, \forall n \geq k', h_{\dot{\mu}^i_\eta}(n) < h_{\dot{\lambda}^i_\xi}(n)$ and $h_{\dot{\mu}^i_\xi}(n) < h_{\dot{\lambda}^i_\eta}(n)$.

Remark 15. Since there must be some $q' \leq \bar{q}_2 \in \mathbb{P}_\theta$ which decides the ordering[39] of the N cuts from the collection $\{(A^i, B^i) : i < N\}$, without loss of generality suppose that $\forall i < j < N$, $q' \Vdash (A^i, B^i) < (A^j, B^j)$. Recursively, for each $i < N-1$, find $\sigma_i < \theta$ and q_i extending q' and all q_j for $j < i$ such that $q_i \Vdash h_{\sigma_i} \in B^i \cap A^{i+1}$. Moreover, let $\bar{\sigma} = \max\{\sigma_i : i < N-1\}$, and let $\bar{q}_3 = q_{N-2}$.

[39]See Definition 12 on page 21 for an explanation of the ordering on cuts.

Let $S = \{\alpha < \omega_1 : \exists p \leq \bar{q}_3,\, p \Vdash \check{\alpha} \in \dot{S}\}$.

Now, for each $\alpha \in S$, for each $i < N$, find μ^i_α and λ^i_α in $M \cap \theta$ as well as $p_\alpha \leq \bar{q}_3$ such that for each $i < N$, $\mu^i_\alpha, \lambda^i_\alpha \in supp(p_\alpha)$, $p_\alpha \Vdash \check{\alpha} \in \dot{S}$, and $p_\alpha \Vdash \forall i < N$, $\dot{\mu}^i_\alpha = \check{\mu}^i_\alpha$, $\dot{\lambda}^i_\alpha = \check{\lambda}^i_\alpha$.

For each $\alpha \in S$, for each $i < N$, find $m[\alpha, i]$ such that $\forall j \leq i$, $p_\alpha \Vdash$ "$\forall n \geq m[\alpha, i]$, $h_{\mu^j_\alpha}(n) < h_{\sigma_i}(n)$" and $\forall j > i$, $p_\alpha \Vdash$ "$\forall n \geq m[\alpha, i]$, $h_{\lambda^j_\alpha}(n) > h_{\sigma_i}(n)$".[40]

For every $\alpha \in S$, choose $m[\alpha] \geq \max\{m[\alpha, i] : i < N\} + k'$. So we have:

$$\forall \alpha \in S, \forall i < N, \forall j \leq i, p_\alpha \Vdash \forall n \geq m[\alpha], h_{\mu^j_\alpha}(n) < h_{\sigma_i}(n) \tag{4.3}$$

and

$$\forall \alpha \in S, \forall i < N, \forall j < i, p_\alpha \Vdash \forall n \geq m[\alpha], h_{\sigma_i}(n) < h_{\lambda^j_\alpha}(n) \tag{4.4}$$

Definition 53. Fix $\xi \in S$, fix $i < N$. Define $S^L_{\xi,i} : \omega \longrightarrow \mathcal{P}(\mathbb{L})$ by recursion: first, let $S^L_{\xi,i}(0) = \{\mu^i_\xi\}$. Having defined $S^L_{\xi,i}(n)$, let $S^L_{\xi,i}(n+1) = \bigcup\{L^\xi_\beta : \beta \in S^L_{\xi,i}(n)\}$. Let $L^{\xi,i} = \bigcup_{n\in\omega} S^L_{\xi,i}(n)$. Note that for some $n^* \in \omega$, $\forall j \geq n^*$, $S^L_{\xi,i}(j) = \emptyset$, so $L^{\xi,i}$ is actually a finite union of finite sets. Similarly, define $S^R_{\xi,i} : \omega \longrightarrow \mathcal{P}(\mathbb{L})$ by recursion: first, let $S^R_{\xi,i}(0) = \{\lambda_\xi\}$. Having defined $S^R_{\xi,i}(n)$, let $S^R_{\xi,i}(n+1) = \bigcup\{R^\xi_\beta : \beta \in S^R_{\xi,i}(n)\}$. Let $R^{\xi,i} = \bigcup_{n\in\omega} S^R_{\xi,i}(n)$, a finite set since for some n^*, $\forall j \geq n^*$, $S^R_{\xi,i}(j) = \emptyset$.

Claim 37. *For each $i < N$, for each $\xi \in S$, for each $\alpha \in \mathbb{L}$:*

[40] See Remark 15 for the definition of the h_{σ_i} functions. Furthermore, while it may be necessary to extend p_α to p'_α, we simply suppress the prime notation.

(a) $\forall \gamma < \theta$, *if* $\gamma \in L^{\xi}_{\alpha}$ *and* $\alpha \in L^{\xi,i}$, *then* $\gamma \in L^{\xi,i}$; *and*

(b) $\forall \delta < \theta$, *if* $\delta \in R^{\xi}_{\alpha}$ *and* $\alpha \in R^{\xi,i}$, *then* $\delta \in R^{\xi,i}$.

Proof. We show only (a), since (b) is very similar. Since $\alpha \in L^{\xi,i} = \bigcup_{n\in\omega} S^{L}_{\xi,i}(n)$, find n such that $\alpha \in S^{L}_{\xi,i}(n)$. But then, since $\gamma \in L^{\xi}_{\alpha}$, it follows that $\gamma \in S^{L}_{\xi,i}(n+1)$, by Definition 53. So $\gamma \in L^{\xi,i}$, as claimed. □

Now, for each pair $i < j < N$ and each $\alpha \in S$, find $m(\alpha, i, j) < \omega$ such that

$$\forall n \geq m(\alpha, i, j), \forall \gamma \in L^{\alpha,i}, \forall \delta \in R^{\alpha,j}, p_\alpha \Vdash h_\gamma(n) < h_\delta(n) \tag{4.5}$$

Let $m(\alpha) = \max\{m(\alpha, i, j) : i < j < N\}$.

Assumption 2. Without loss of generality, suppose:

1. Each p_α is determined and of closed support over $\mathbb{L}$. In other words, for each $\alpha \in S$, for each $\beta \in supp(p_\alpha) \cap \mathbb{L}$, $p_\alpha(\beta) = (\check{L}^{\alpha}_{\beta}, \check{R}^{\alpha}_{\beta}, \check{s}^{\alpha}_{\beta})$ (i.e., determined), and $L^{\alpha}_{\beta} \cup R^{\alpha}_{\beta} \subseteq supp(p_\alpha) \cap \mathbb{L}$ (i.e., closed support). (Cf. Section 3.4.)

2. Each p_α is uniform; more specifically, there is some k_α greater than both $m[\alpha]$ and $m(\alpha)$ such that $\forall \beta \in supp(p_\alpha) \cap \mathbb{L}$, $|s^{\alpha}_{\beta}| = k_\alpha$. (Cf. Section 3.4.)

Find an uncountable subset Z_0 of S and some $k \in \omega$ such that $\forall \alpha \in Z_0$, $k_\alpha = k > k'$.

Without loss of generality, suppose the collection $\mathcal{D} = \{supp(p_\xi) : \xi \in Z_0\}$ is a Δ-system with root Σ and tails T_ξ (for each $\xi \in Z_0$). Suppose all the tails of $\mathcal{D}$ are of the same finite size, $\bar{t}$, and suppose that for each $\xi \in Z_0$, $T_\xi = \{t^\xi_j : j < \bar{t}\}$.

Moreover, thinning down Z_0 if necessary, we can also make the following assumptions:

Assumption 3. Suppose that:

1. for each $\beta \in \Sigma \cap \mathbb{L}$, there is some s_β such that $\forall \eta \in Z_0$, $s^\eta_\beta = s_\beta$;
2. for each $i < N$: if (A^i, B^i) is left-unbounded, then $\exists l < \bar{t}$ such that for each $\xi \in Z_0$, $\mu^i_\xi = t^\xi_l$; and, if (A^i, B^i) is right-unbounded, then $\exists r < \bar{t}$ such that for each $\xi \in Z_0$, $\lambda^i_\xi = t^\xi_r$;
3. for each $i < N$, for each $\eta, \xi \in Z_0$, $h_{\mu^i_\eta} \restriction k = h_{\mu^i_\xi} \restriction k$ and $h_{\lambda^i_\eta} \restriction k = h_{\lambda^i_\xi} \restriction k$.

There are various possibilities for the structure of the Δ-system $\mathcal{D} = \{supp(p_\xi) : \xi \in Z_0\}$, as per the following definition:

Definition 54. Fix $\lambda \geq \omega_1$. Let $\mathcal{A}$ be a Δ-system on λ with root R and tails $\{A_\eta : \eta < \omega_1\}$. Define the *type of* $\mathcal{A}$ by *type*$(\mathcal{A}) = \min\{n \in \omega : \exists S \in [R]^n, \forall \eta < \omega_1, \forall \rho \in R, \exists s \in S$ such that $\forall \alpha \in A_\eta$, $\alpha < \rho$ iff $\alpha < s\}$.

Note that the type of a Δ-system corresponds to the number of "blocks" the root is divided into, where "blocks" are divided by elements of the tails (i.e., a block is a maximal subset of the root such that the interval between the mininimum and maximum of the subset contains no tail elements). Without loss of generality we suppose that:

Assumption 4. There are uncountably many tail elements between any two blocks of the root in the Δ-system $\mathcal{D} = \{supp(p_\xi) : \xi \in Z_0\}$.

Now, let $\Theta = \{\bar{\theta} < \theta : cf(\bar{\theta}) = \omega_1,\ \{T_\eta \cap \bar{\theta} : \eta \in Z_0\}$ is unbounded in $\bar{\theta}\}$. Note that since the tails of the Δ-system $\mathcal{D}$ are all of the same (finite) size $\bar{t}$, the set Θ is finite; to be specific, $|\Theta| \leq \bar{t}$. If Θ is not empty, let $\theta^* = \max \Theta$, otherwise, let $\theta^* = 0$.

Remark 16. Note that if the type of the Δ-system $\mathcal{D}$ is larger than 1, then $\theta^* > 0$. Moreover, it's possible that $\max \Sigma > \theta^*$, in which case $\Sigma \backslash \theta^*$ is the "top block" of the Δ-system. In any case, we can assume without loss of generality that $(\theta^*, \max \Sigma)$ is an interval containing no tail elements; i.e., $(\theta^*, \max \Sigma) \cap \bigcup \{T_\eta : \eta \in Z_0\} = \emptyset$. To see this, note first that if $\max \Sigma \leq \theta^*$, the interval $(\theta^*, \max \Sigma)$ is empty. On the other hand, if $\max \Sigma > \theta^*$, suppose instead there is some $\eta \in Z_0$ such that $(\theta^*, \max \Sigma) \cap T_\eta \neq \emptyset$: if there were countably many such η's in Z_0, we could remove those η's from Z_0; if there were uncountably many such η's in Z_0, this would contradict the maximality of θ^*. We restate the now justified aforementioned assumption for ease of reference.

Assumption 5. $(\theta^*, \max\Sigma)$ is an interval containing no tail elements; i.e., $(\theta^*, \max\Sigma) \cap \bigcup\{T_\eta : \eta \in Z_0\} = \emptyset$.

Claim 38. *Let P be ccc and let $Y \subseteq P$ be uncountable. Then there is some generic filter $G \subseteq P$ such that $Y \cap G$ is uncountable.*

Proof. First, consider the following general fact:

Fact 8. Given a ccc partial order P and $X \subseteq P$ uncountable, there is some $p \in P$ such that $\forall r \leq p$, $X(r) = \{x \in X : r \not\perp x\}$ is uncountable.

Proof. Fix P ccc, and suppose instead there is some uncountable $X \subseteq P$ such that $\forall p \in P$, there is a $\bar{p} = \bar{p}(X) \leq p$ such that $X(\bar{p})$ is countable. Note that if this is true of X, it is true of any subset of X.

Construct recursively sequences $(q_\alpha)_{\alpha<\omega_1}$ and $(X_\alpha)_{\alpha<\omega_1}$ such that for each $\beta < \omega_1$, $\{q_\alpha : \alpha \leq \beta\}$ is an antichain, and $(X_\alpha)_{\alpha<\omega_1}$ is a descending sequence of uncountable sets. To begin, let $X_0 = X$ and choose any q_0 in X_0. Having chosen X_α and $q_\alpha \in X_\alpha$, let $\bar{q}_\alpha = \bar{q}_\alpha(X_\alpha)$, let $X_{\alpha+1} = X \setminus X_\alpha(\bar{q}_\alpha)$, and choose $q_{\alpha+1} \in X_{\alpha+1}$. Note that $\{q_\beta : \beta \leq \alpha + 1\}$ is an antichain, by induction hypothesis and the fact that $q_{\alpha+1} \in X_{\alpha+1}$.

Finally, having chosen X_β and q_β for each $\beta < \alpha$ where $\alpha \in LIM$, let $X_\alpha = X \setminus \bigcup_{\beta<\alpha} X_\beta(\bar{q}_\beta)$ (which is uncountable since the latter union consists of countable sets), and let $q_\alpha \in X_\alpha$. Once again, note that $\{q_\beta : \beta \leq \alpha\}$ is an antichain.

So $\{q_\alpha : \alpha < \omega_1\}$ is an antichain, contradicting that P is ccc. $\square$

The proof of Claim 38 is now completed by Fact 9, below. $\square$

Fact 9. Fix any ccc poset P, any $X \subseteq P$ uncountable, and let p be as in Fact 8. Then $p \Vdash$ "$\check{X} \cap \dot{G}$ is uncountable", where $\dot{G}$ is the canonical name for a generic filter on P.

Proof. Suppose instead that there is some $r \leq p$ such that $r \Vdash$ "$\check{X} \cap \dot{G}$ is countable". Since $X(r)$ is uncountable, choose $x \in X(r)$ such that $r \Vdash$ "$\check{x} \notin \dot{G}$". Since $x \in X(r)$, we can find $p \leq r, x$. Since $p \leq r$, $p \Vdash \check{x} \notin \dot{G}$; but since $p \leq x$, $p \Vdash \check{x} \in \dot{G}$. $\square$

By Assumptions 1 part 2, $\mathbb{P}_{\theta^*+1}$ is ccc.

Claim 39. *There is a generic filter $G \subseteq \mathbb{P}_{\theta^*+1}$ such that the set $Z_1 = \{\alpha \in Z_0 : p_\alpha \restriction (\theta^* + 1) \in G\}$ is uncountable.*

Proof. Let $Y = \{p_\alpha \restriction (\theta^* + 1) : \alpha \in Z_0\}$. If Y is countable, then there must be some $\bar{p} \in \mathbb{P}_{\theta^*+1}$ such that the set $\tilde{Z} = \{\alpha \in Z_0 : p_\alpha \restriction (\theta^* + 1) = \bar{p}\}$ is uncountable. Choose a generic filter $G \subseteq \mathbb{P}_{\theta^*+1}$ such that $\bar{p} \in G$; then $\tilde{Z} \subseteq Z_1 = \{\alpha \in Z_0 : p_\alpha \restriction (\theta^* + 1) \in G\}$. On the other hand, if Y is uncountable, then by Claim 38, obtain a generic filter $G \subseteq \mathbb{P}_{\theta^*+1}$ such that $Y \cap G$ is uncountable. But then $Z_1 = \{\alpha \in Z_0 : p_\alpha \restriction (\theta^* + 1) \in Y \cap G\}$ is uncountable. $\square$

Now, fix G and Z_1 as in Claim 39.

For each $i < N$, let $\mathcal{E}^{i*} = \{h_\alpha : \alpha \in L^{\eta,i} \cap \theta^*, \eta \in Z_1\}$, $\mathcal{H}^{i*} = \{h_\alpha : \alpha \in R^{\eta,i} \cap \theta^*, \eta \in Z_1\}$.

Claim 40. *For each $i < N$, $(\mathcal{E}^{i*}, \mathcal{H}^{i*})$ is a cut in $\mathcal{L}_{\theta^*}$.*

Proof. Fix $i < N$. It must be shown that $\forall \eta, \xi \in Z_1$, $\forall \alpha \in L^{\eta,i} \cap \theta^*$, $\forall \beta \in R^{\xi,i} \cap \theta^*$, $h_\alpha <^* h_\beta$. So fix such η, ξ, α, β. Since $\alpha \in L^{\eta,i}$, $h_\alpha <^* h_{\mu^i_\eta}$ or $h_\alpha = h_{\mu^i_\eta}$ (see Definition 53). Similarly, $h_{\lambda^i_\xi} <^* h_\beta$ or $h_{\lambda^i_\xi} = h_\beta$. But $h_{\mu^i_\eta} <^* h_{\lambda^i_\xi}$, by Condition 2.[41] $\square$

Note that the cut $(\mathcal{E}^{i*}, \mathcal{H}^{i*})$ is definable in $M[G]$. To see this, note first that each p_η (for $\eta \in Z_1$) is a determined condition (see Assumption 2, part 1, on page 93), and so $L^{\eta,i}$, $R^{\eta,i}$ are sets in the ground model, M (see Definition 53); furthermore, $Z_1 = \{\alpha \in Z_0 : p_\alpha \restriction (\theta^* + 1) \in G\} \in M[G]$.

If the cut $(\mathcal{E}^{i*}, \mathcal{H}^{i*})$ is filled in $M[G]$, let $h^{i*} \in M[G]$ denote such a cut-filler. Note that if $\theta^* = 0$, then there will certainly be such a cut-filler h^{i*} since the cut $(\mathcal{E}^{i*}, \mathcal{H}^{i*})$ would be finite (in fact, empty). Otherwise, suppose there is no such h^{i*} in $M[G]$; in particular, $\theta^* > 0$ and $cf(\theta^*) = \omega_1$. So $\theta^* \in HL(\theta)$ (see Assumptions 1, part 7).

[41] Since Condition 2 involves forcing over $\mathbb{P}_\theta$, rather than $\mathbb{P}_{\theta^*}$, it would seem at first that we have only shown $\bar{q}_1 \Vdash$ "$h_\alpha <^* h_\beta$". But h_α and h_β are $\mathbb{P}_{\theta^*}$-names. To see that $\bar{q}_1 \restriction \theta^* \Vdash$ "$h_\alpha <^* h_\beta$", suppose instead that $\bar{q}_1 \restriction \theta^* \not\Vdash h_\alpha <^* h_\beta$. Then there must be some $p' \leq \bar{q}_1 \restriction \theta^*$ in $\mathbb{P}_{\theta^*}$ such that $p' \Vdash \neg(h_\alpha <^* h_\beta)$. Let $p'' = p' {}^\frown \bar{q}_1 \restriction [\theta^*, \theta)$. Then $p'' \leq \bar{q}_1$, but $p'' \Vdash$ "$\neg(h_\alpha <^* h_\beta)$" while $\bar{q}_1 \Vdash$ "$h_\alpha <^* h_\beta$", a contradiction. Thus, in $M[G]$, we really do have $h_\alpha <^* h_\beta$.

Working in $M[G]$, for each $\zeta \in Z_1$, note that $p_\zeta(\theta^*) \in \Pi_{j\in J} K(\mathcal{E}^j, \mathcal{H}^j)$, where J is an index set and all of the partition cuts of $\mathcal{L}_{\theta^*}$ occur in the enumeration $\{(\mathcal{E}^j, \mathcal{H}^j) : j \in J\}$. So, for each $i < N$, $\exists j(i) \in J$ such that $\mathcal{E}^{i*} \subseteq \mathcal{E}^{j(i)}$, and $\mathcal{H}^{i*} \subseteq \mathcal{H}^{j(i)}$. Let H_{θ^*} be the filter such that $G \cong G_{\theta^*} * H_{\theta^*}$, and let $H_{\theta^*}(j(i))$ denote the associated filter that is generic for $K(\mathcal{E}^{j(i)}, \mathcal{H}^{j(i)})$. Let h^{i*} be the generic function obtained from the latter filter; so $\mathcal{E}^{i*} <^* h^{i*} <^* \mathcal{H}^{i*}$.

Definition 55. Let $\bar{\theta} = \max(\Sigma \cup \{\epsilon, \bar{\sigma}, \theta^* + 1\}) + 1$.[42]

Claim 41. *There is a generic filter $\bar{G} \subseteq \mathbb{P}_{\bar{\theta}}$ such that the set $Z_2 = \{\alpha \in Z_1 : p_\alpha \restriction \bar{\theta} \in \bar{G}\}$ is uncountable.*

Proof. Apply the argument of Claim 39 within $M[G]$. □

Note that $\bar{G}$ is chosen so that it extends G, since $\bar{G}$ is obtained in $M[G]$.[43]

For each $i < N$, let $\bar{\mathcal{E}}^i = \{h_\alpha : \alpha \in L^{\eta,i} \cap \bar{\theta}, \eta \in Z_2\}$, $\bar{\mathcal{H}}^i = \{h_\alpha : \alpha \in R^{\eta,i} \cap \bar{\theta}, \eta \in Z_2\}$.

We work now in $M[\bar{G}]$. Note that for each $i < N$, in moving from $(\mathcal{E}^{i*}, \mathcal{H}^{i*})$ to $(\bar{\mathcal{E}}^i, \bar{\mathcal{H}}^i)$, only finitely many functions are added to either side of the cut—namely, the functions h_α for $\alpha \in (\Sigma \setminus \theta^*) \cap \mathbb{L}$.[44] It is possible that the function h^{i*} (which

[42] ϵ is defined in Definition 52, on page 89; $\bar{\sigma}$ is defined in Remark 15 on page 91.

[43] To be more precise, $\bar{G}$ (which is actually defined in $M[G]$) is a filter for the iteration partial order from P_θ up to $P_{\bar{\theta}}$. In M, let $\dot{H}$ be a name for $\bar{G}$. The filter required is $G * \dot{H}$, but for simplicity, we refer to the latter filter as $\bar{G}$.

[44] Because of closed support (see Assumption 2, part 1 on page 93) and Assumption 5 on page 95, the h_α's added must come from the root Σ, which is finite.

fills the cut $(\mathcal{E}^{i*}, \mathcal{H}^{i*}))$ already fills the cut $(\bar{\mathcal{E}}^i, \bar{\mathcal{H}}^i)$. If not, then there is some $\nu(i) \in \Sigma \cap \mathbb{L}$ $(\nu(i) > \theta^*)$ such that $h_{\nu(i)}$ splits $(\bar{\mathcal{E}}^i \setminus \{h_{\nu(i)}\}, \bar{\mathcal{H}}^i \setminus \{h_{\nu(i)}\})$. Let $\nu(i)$ be such an index if h^{i*} does not fill the cut $(\bar{\mathcal{E}}^i, \bar{\mathcal{H}}^i)$; otherwise, if h^{i*} *does* fill the cut, let $\nu(i) = -1$ and let $h_{-1} = h^{i*}$. So, in either case,

$$h_{\nu(i)} \text{ fills the cut } (\bar{\mathcal{E}}^i \setminus \{h_{\nu(i)}\}, \bar{\mathcal{H}}^i \setminus \{h_{\nu(i)}\}). \tag{4.6}$$

For each $i < N$, for each $\zeta \in Z_2$, $\exists m^i_\zeta, \forall n \geq m^i_\zeta, \forall \gamma \in (L^{\zeta,i} \setminus \{\nu(i)\}) \cap \bar{\theta}, \forall \delta \in (R^{\zeta,i} \setminus \{\nu(i)\}) \cap \bar{\theta}, h_\gamma(n) < h_{\nu(i)}(n) < h_\delta(n)$. So, obtain, for each $i < N$, $\bar{m}^i$ such that $Z_3 = \{\zeta \in Z_2 : m^i_\zeta = \bar{m}^i\}$ is uncountable. Choose $\bar{m}$ such that $\bar{m} \geq \max\{\bar{m}^i : i < N\} + k$. (Recall the definition of k immediately following Claim 37.)

Finally, for each $i < N$, find integers $\bar{l}^i, \bar{r}^i \in \omega$ and sets $\{f^{l,i} : l < \bar{l}^i\}, \{g^{r,i} : r < \bar{r}^i\}$ such that for an uncountable $Z_4 \subseteq Z_3$, for each $\zeta \in Z_4$, for each $i < N$,

$$L^{\zeta,i} \cap \bar{\theta} = \{\gamma^{\zeta,i}_l : l < \bar{l}^i\},\ R^{\zeta,i} \cap \bar{\theta} = \{\delta^{\zeta,i}_r : r < \bar{r}^i\} \tag{4.7}$$

and,

$$\forall l < \bar{l}^i,\ r < \bar{r}^i,\ h_{\gamma^{\zeta,i}_l} \restriction \bar{m} = f^{l,i} \text{ and } h_{\delta^{\zeta,i}_r} \restriction \bar{m} = g^{r,i}. \tag{4.8}$$

The following Lemma will be helpful in the proof of Claim 42.

Lemma 4.5.2. *Fix $i < N$. In $M[\bar{G}]$, $\forall \eta \in Z_4$, $\forall \gamma \in L^{\eta,i} \cap \bar{\theta}$, $\forall \delta \in R^{\eta,i} \cap \bar{\theta}$, $\forall j \geq k$, $h_\gamma(j) < h_\delta(j)$.*

Proof. Fix $\eta \in Z_4$, $\gamma \in L^{\eta,i} \cap \bar{\theta}$, $\delta \in R^{\eta,i} \cap \bar{\theta}$, $j \geq k$. (Recall the definition of k immediately following Claim 37.) The first task is to show that $p_\eta \Vdash h_\gamma(j) \leq$

$h_{\mu^i_\eta}(j)$, and the argument is by induction on $n \in \omega$, where $\gamma \in S^L_{\eta,i}(n)$. (Recall from Definition 53 that $L^{\eta,i} = \bigcup_{n\in\omega} S^L_{\eta,i}(n)$.) The case $n = 0$ is immediate since $S^L_{\eta,i}(0) = \{\mu^i_\eta\}$. So fix $n > 0$ and say $\gamma \in L^\eta_\beta$, where $\beta \in S^L_{\eta,i}(n-1)$. By the induction hypothesis, $p_\eta \Vdash h_\beta(j) \leq h_{\mu^i_\eta}(j)$. Since $\gamma \in L^\eta_\beta$ and $j \geq k$, it follows that $p_\eta \Vdash h_\gamma(j) < h_\beta(j)$; thus $p_\eta \Vdash h_\gamma(j) \leq h_{\mu^i_\eta}(j)$, as required. A similar induction argument shows that $p_\eta \Vdash h_{\lambda^i_\eta}(j) \leq h_\delta(j)$. Since $p_\eta \Vdash h_{\mu^i_\eta}(j) < h_{\lambda^i_\eta}(j)$ (see Condition 2, and note that $k > k'$), it follows that $p_\eta \Vdash h_\gamma(j) < h_\delta(j)$. Finally, since $h_\gamma, h_\delta \in M[\bar{G}]$, it must be the case that, in $M[\bar{G}]$, $h_\gamma(j) < h_\delta(j)$ (i.e., the inequality is no longer merely a forcing statement but actually true in $M[\bar{G}]$).[45]

□

Without loss of generality, we make the following assumption:

Assumption 6. Suppose for all $\zeta \in Z_4$: if (A^i, B^i) is not left-bounded, then $\mu^i_\zeta > \bar{\theta}$, and if (A^i, B^i) is not right-bounded, then $\lambda^i_\zeta > \bar{\theta}$.

Now choose $\eta < \xi \in Z_4$ with $\max(T_\eta \setminus \theta^*) < \min(T_\xi \setminus \theta^*)$. Note that $supp(p_\eta) \cap supp(p_\xi) \subseteq \bar{\theta}$. Since $p_\eta \restriction \bar{\theta}, p_\xi \restriction \bar{\theta} \in \bar{G}$, find $q \in \bar{G} \cap \mathbb{P}_{\bar{\theta}}$ such that $q \leq p_\eta \restriction \bar{\theta}, p_\xi \restriction \bar{\theta}$.

Definition 56. For each $\alpha \in \bigcup_{i<N} L^{\eta,i} \cup \bigcup_{i<N} R^{\eta,i}$, let $\bar{L}^\eta_\alpha = L^\eta_\alpha \cup \bigcup\{L^{\xi,i} \cap \bar{\theta} : i < N, \alpha \in R^{\eta,i}\}$, and let $\bar{R}^\eta_\alpha = R^\eta_\alpha \cup \bigcup\{R^{\xi,i} \cap \bar{\theta} : i < N, \alpha \in L^{\eta,i}\}$. For each

[45] Since $p_\eta \Vdash h_\gamma(j) < h_\delta(j)$, and h_γ, h_δ are $\mathbb{P}_{\bar{\theta}}$-names, it must be the case that $p_\eta \restriction \bar{\theta} \Vdash h_\gamma(j) < h_\delta(j)$. To see this more clearly, suppose instead that $p_\eta \restriction \bar{\theta} \nVdash h_\gamma(j) < h_\delta(j)$. Then there must some $p' \leq p_\eta \restriction \bar{\theta}$ in $\mathbb{P}_{\bar{\theta}}$ such that $p' \Vdash h_\gamma(j) \geq h_\delta(j)$. Let $p'' = p' {}^\frown p_\eta \restriction [\bar{\theta}, \theta)$. Then $p'' \leq p_\eta$, but $p'' \Vdash$ "$h_\gamma(j) \geq h_\delta(j)$" while $p_\eta \Vdash$ "$h_\gamma(j) < h_\delta(j)$".

$\alpha \in \bigcup_{i<N} L^{\xi,i} \cup \bigcup_{i<N} R^{\xi,i}$, let $\bar{L}^{\xi}_{\alpha} = L^{\xi}_{\alpha} \cup \{\mu^i_{\eta} : i < N, \alpha \in R^{\xi,i}\}$, and let $\bar{R}^{\xi}_{\alpha} = R^{\xi}_{\alpha} \cup \{\lambda^i_{\eta} : i < N, \alpha \in L^{\xi,i}\}$. Define $p : \theta \longrightarrow M$ as follows:

$$p(\alpha) = \begin{cases} q(\alpha) & \text{if } \alpha < \bar{\theta} \\ (\bar{L}^{\eta}_{\alpha}, \bar{R}^{\eta}_{\alpha}, s^{\eta}_{\alpha}) & \text{if } \alpha \geq \bar{\theta} \text{ and } \alpha \in \bigcup_{i<N} L^{\eta,i} \cup \bigcup_{i<N} R^{\eta,i} \\ (\bar{L}^{\xi}_{\alpha}, \bar{R}^{\xi}_{\alpha}, s^{\xi}_{\alpha}) & \text{if } \alpha \geq \bar{\theta} \text{ and } \alpha \in \bigcup_{i<N} L^{\xi,i} \cup \bigcup_{i<N} R^{\xi,i} \\ (p_{\eta} \cup p_{\xi})(\alpha) & \text{otherwise.} \end{cases}$$

Claim 42. *p is a condition; i.e., $p \in \mathbb{P}$.*

Proof. The argument proceeds by induction on $\alpha \in supp(p)$.

CASE 42.1 Suppose $\alpha \in \bigcup_{i<N} L^{\eta,i} \cup \bigcup_{i<N} R^{\eta,i}$. The goal is to show that $p \restriction \alpha \Vdash$ "$p(\alpha) \in \dot{\mathbb{Q}}_{\alpha}$", which means, in this case, $p \restriction \alpha \Vdash$ "$\forall n \geq k, \forall \gamma \in \bar{L}^{\eta}_{\alpha}, \forall \delta \in \bar{R}^{\eta}_{\alpha}, h_{\gamma}(n) < h_{\delta}(n)$." So we fix such n, γ, δ.

Note that either $\gamma \in L^{\eta}_{\alpha}$, or $\gamma \in L^{\xi,j} \cap \bar{\theta}$, where $\alpha \in R^{\eta,j}$, for some $j < N$. Also, either $\delta \in R^{\eta}_{\alpha}$, or $\delta \in R^{\xi,j} \cap \bar{\theta}$, where $\alpha \in L^{\eta,j}$, for some $j < N$. Furthermore, $\gamma < \bar{\theta}$ or $\gamma > \bar{\theta}$, and $\delta < \bar{\theta}$ or $\delta > \bar{\theta}$. We consider all the combinations that are possible.

CASE 42.1.a Suppose $\gamma \in L^{\eta}_{\alpha}$, and $\delta \in R^{\eta}_{\alpha}$. Regardless of whether δ or γ are less than or greater than $\bar{\theta}$, the argument is completed by the fact that $p \restriction \alpha \leq p_{\eta} \restriction \alpha$.

CASE 42.1.b Suppose $\gamma \in L^{\eta}_{\alpha}$, $\delta \in R^{\xi,j} \cap \bar{\theta}$, where $\alpha \in L^{\eta,j}$, and $\gamma < \bar{\theta}$. Since $\gamma \in L^{\eta}_{\alpha}$ and $\alpha \in L^{\eta,j}$, it follows that $\gamma \in L^{\eta,j}$ by Claim 37.

So $\gamma \in L^{\eta,j} \cap \bar{\theta}$ and $\delta \in R^{\xi,j} \cap \bar{\theta}$. So, if $n > \bar{m}$, then, by (4.6), $h_\gamma(n) \leq h_{\nu(j)}(n)$ (with equality only if $\gamma = \nu(j)$), and $h_{\nu(j)}(n) \leq h_\delta(n)$ (with equality only if $\delta = \nu(j)$). Since at least one of the preceding inequalities must be strict, $h_\gamma(n) < h_\delta(n)$. On the other hand, suppose $n < \bar{m}$. Note that since $\gamma \in L^{\eta,j} \cap \bar{\theta}$, (4.7) applies; say $\gamma = \gamma_l^{\eta,j}$. Similarly, say $\delta = \delta_r^{\xi,j}$. But $h_{\delta_r^{\xi,j}}(n) = h_{\delta_r^{\eta,j}}(n)$ by the fact that these functions are equal below $\bar{m}$ — see (4.8). So we may now apply Lemma 4.5.2 to conclude that $h_{\gamma_l^{\eta,j}}(n) < h_{\delta_r^{\eta,j}}(n)$ — note that both functions refer to a single $\eta \in Z_4$, as required by Lemma 4.5.2. Thus, $h_\gamma(n) = h_{\gamma_l^{\eta,j}}(n) < h_{\delta_r^{\eta,j}}(n) = h_{\delta_r^{\xi,j}}(n) = h_\delta(n)$, as needed.

CASE 42.1.c Suppose $\gamma \in L^\eta_\alpha$, $\delta \in R^{\xi,j} \cap \bar{\theta}$, where $\alpha \in L^{\eta,j}$, and $\gamma > \bar{\theta}$. Again, since $\gamma \in L^\eta_\alpha$ and $\alpha \in L^{\eta,j}$, it follows that $\gamma \in L^{\eta,j}$ by Claim 37. Thus, since $\gamma < \alpha$ and $\gamma \in L^{\eta,j}$, the induction hypothesis applies, and the second case of Definition 56 gives $p \restriction \gamma \Vdash \dot{h}_\gamma(n) < h_\delta(n)$.[46]

CASE 42.1.d Suppose $\gamma \in L^{\xi,j} \cap \bar{\theta}$, where $\alpha \in R^{\eta,j}$, $\delta \in R^\eta_\alpha$, and $\delta < \bar{\theta}$. Since $\delta \in R^\eta_\alpha$ and $\alpha \in R^{\eta,j}$, it follows that $\delta \in R^{\eta,j}$ (see Claim 37). Thus, the argument in the second paragraph of CASE 42.1.b applies.

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[46]The dot on $\dot{h}_\gamma$, and on other h-functions throughout, is intended to remind the reader that this function is the generic function added at that stage (in this case, stage γ) in the forcing. Moreover, to see in more detail how the second case of Definition 56 gives (by the induction hypothesis) $p \restriction \gamma \Vdash \dot{h}_\gamma(n) < h_\delta(n)$, note that the induction hypothesis yields $p \restriction \gamma \Vdash$ "$(\bar{L}^\eta_\gamma, \bar{R}^\eta_\gamma, s^\eta_\gamma) \in \dot{\mathbb{Q}}_\gamma$", which means, in particular, that $p \restriction \gamma \Vdash \dot{h}_\gamma(n) < h_\delta(n)$. (Note that $\delta \in R^{\xi,j} \cap \bar{\theta}$ and $\gamma \in L^{\eta,j}$, so $\delta \in \bar{R}^\eta_\gamma$.) A similar explanation applies to the other cases in the proof of Claim 42 where the Induction Hypothesis is used (and the relevant part of Definition 56 is referenced).

CASE 42.1.e Suppose $\gamma \in L^{\xi,j} \cap \bar{\theta}$, where $\alpha \in R^{\eta,j}$, $\delta \in R^{\eta}_{\alpha}$, and $\delta > \bar{\theta}$. Again, by Claim 37, since $\delta \in R^{\eta}_{\alpha}$ and $\alpha \in R^{\eta,j}$, it follows that $\delta \in R^{\eta,j}$. Thus, the I.H. applies (since $\delta < \alpha$ and $\delta \in R^{\eta,j}$), and so the second case of Definition 56 gives $p \upharpoonright \delta \Vdash h_{\gamma}(n) < \dot{h}_{\delta}(n)$. (Note that $\gamma \in L^{\xi,j} \cap \bar{\theta}$ and $\delta \in R^{\eta,j}$, so $\gamma \in \bar{L}^{\eta}_{\delta}$.)

CASE 42.1.f Suppose $\gamma \in L^{\xi,j} \cap \bar{\theta}$, for some $j < N$, where $\alpha \in R^{\eta,j}$, and $\delta \in R^{\xi,l} \cap \bar{\theta}$, for some $l < N$, where $\alpha \in L^{\eta,l}$. Note that $j < l$, since $\alpha \in L^{\eta,l} \cap R^{\eta,j}$.[47] Thus, by (4.5), $p_{\eta} \Vdash h_{\gamma}(n) < h_{\delta}(n)$.[48]

CASE 42.2 Suppose $\alpha \in \bigcup_{i<N} L^{\xi,i} \cup \bigcup_{i<N} R^{\xi,i}$. As in CASE 42.1, the goal is to show that $p \upharpoonright \alpha \Vdash$ "$p(\alpha) \in \dot{\mathbb{Q}}_{\alpha}$". In this case, we must show $p \upharpoonright \alpha \Vdash$ "$\forall n \geq k, \forall \gamma \in \bar{L}^{\xi}_{\alpha}, \forall \delta \in \bar{R}^{\xi}_{\alpha}, h_{\gamma}(n) < h_{\delta}(n)$." Fix such n, γ, δ.

Note that either $\gamma \in L^{\xi}_{\alpha}$, or $\gamma = \mu^{j}_{\eta}$, where $\alpha \in R^{\xi,j}$. Also, either $\delta \in R^{\xi}_{\alpha}$, or $\delta = \lambda^{j}_{\eta}$, for some $j < \eta$, where $\alpha \in L^{\xi,j}$. Furthermore, $\gamma < \bar{\theta}$ or $\gamma > \bar{\theta}$, and $\delta < \bar{\theta}$ or $\delta > \bar{\theta}$. We consider all the combinations that are possible.

CASE 42.2.a Suppose $\gamma \in L^{\xi}_{\alpha}$, and $\delta \in R^{\xi}_{\alpha}$. Regardless of whether δ or γ are less than or greater than $\bar{\theta}$, the argument is completed by the fact that $p \upharpoonright \alpha \leq p_{\xi} \upharpoonright \alpha$.

CASE 42.2.b Suppose $\gamma \in L^{\xi}_{\alpha}$, $\delta = \lambda^{j}_{\eta}$, for some $j < \eta$, where $\alpha \in L^{\xi,j}$, and

[47] To see more clearly that $j < l$, first note that $h_{\alpha} <^{*} h_{\mu^{l}_{\eta}}$ or $h_{\alpha} =^{*} h_{\mu^{l}_{\eta}}$, since $\alpha \in L^{\eta,l}$, and $h_{\lambda^{j}_{\eta}} <^{*} h_{\alpha}$ or $h_{\lambda^{j}_{\eta}} = h_{\alpha}$, since $\alpha \in R^{\eta,j}$ (see Definition 53). So h_{α} is on the left side of the l^{th} cut and on the right side of the j^{th} cut. So the j^{th} cut must be to the left of the l^{th} cut. By Remark 15, $j < l$.

[48] Since $p_{\eta} \Vdash h_{\gamma}(n) < h_{\delta}(n)$, it must be the case that $p_{\eta} \upharpoonright \bar{\theta} \Vdash h_{\gamma}(n) < h_{\delta}(n)$. See the argument in footnote 45 on page 101.

$\gamma > \bar{\theta}$. Note that $\delta \in R^{\eta,j}$ by Definition 53, and $\gamma \in L^{\xi,j}$ by Claim 37. Since $\gamma < \alpha$ and $\gamma \in L^{\xi,j}$, the I.H. applies, and so the third case of Definition 56 gives $p \restriction \gamma \Vdash \dot{h}_\gamma(n) < h_\delta(n)$. (Note that $\delta \in \bar{R}^\xi_\gamma$, since $\gamma \in L^{\xi,j}$. Note that this case applies regardless of whether $\delta < \bar{\theta}$ or $\delta > \bar{\theta}$.)

CASE 42.2.c Suppose $\gamma \in L^\xi_\alpha$, $\delta = \lambda^j_\eta$, for some $j < \eta$, where $\alpha \in L^{\xi,j}$, and $\gamma < \bar{\theta}$. If $\delta < \bar{\theta}$, then the argument in the second paragraph of CASE 42.1.b applies. If $\delta > \bar{\theta}$, then the I.H. applies, and the second case of Definition 56 gives $p \restriction \delta \Vdash h_\gamma(n) < \dot{h}_\delta(n)$. (Note that $\delta < \alpha$, $\delta \in R^{\eta,j}$, and $\gamma \in L^{\xi,j} \cap \bar{\theta}$, so $\gamma \in \bar{L}^\eta_\delta$.)

CASE 42.2.d Suppose $\gamma = \mu^j_\eta$, where $\alpha \in R^{\xi,j}$, and $\delta \in R^\xi_\alpha$. Since $\alpha \in R^{\xi,j}$, $\delta \in R^{\xi,j}$, by Claim 37. If $\delta > \bar{\theta}$, then the I.H. applies, and the third case of Definition 56 gives $p \restriction \delta \Vdash h_\gamma(n) < \dot{h}_\delta(n)$. (Note that $\gamma \in \bar{L}^\xi_\delta$, since $\delta \in R^{\xi,j}$.) If $\delta < \bar{\theta}$ and $\gamma < \bar{\theta}$, then the argument in the second paragraph of CASE 42.1.b applies. So suppose $\delta < \bar{\theta}$ and $\gamma > \bar{\theta}$. Since $\gamma \in L^{\eta,j}$, the I.H. applies, and so the second case of Definition 56 gives $p \restriction \gamma \Vdash \dot{h}_\gamma(n) < h_\delta(n)$. (Note that $\delta \in R^{\xi,j} \cap \bar{\theta}$, and $\gamma \in L^{\eta,j}$, so $\delta \in \bar{R}^\xi_\gamma$.)

CASE 42.2.e Suppose $\gamma = \mu^j_\eta$, where $\alpha \in R^{\xi,j}$, and $\delta = \lambda^l_\eta$, where $\alpha \in L^{\xi,l}$. (In this case it doesn't matter whether δ or γ are less than or greater than $\bar{\theta}$.) Note that $j < l$, since $\alpha \in L^{\xi,l} \cap R^{\xi,j}$.[49] By (4.3) and (4.4), $p_\eta \Vdash \dot{h}_{\mu^j_\eta}(n) < h_{\sigma_j}(n)$ and

[49] To see more clearly that $j < l$, first note that $h_\alpha <^* h_{\mu^l_\xi}$ or $h_\alpha =^* h_{\mu^l_\xi}$, since $\alpha \in L^{\xi,l}$, and $h_{\lambda^j_\xi} <^* h_\alpha$ or $h_{\lambda^j_\xi} =^* h_\alpha$, since $\alpha \in R^{\xi,j}$ (see Definition 53). So h_α is on the left side of the l^{th} cut and on the right side of the j^{th} cut. So the j^{th} cut must be to the left of the l^{th} cut. By Remark

$p_\eta \Vdash h_{\sigma_j}(n) < \dot{h}_{\lambda^l_\eta(n)}$. Since these ordinals are all less than α, it must be the case that $p_\eta \restriction \alpha$ forces these inequalities (see the argument in footnote 45 on page 101). The argument in this case is now complete since $p \restriction \alpha \leq p_\eta \restriction \alpha$.

□

Note that $p \leq p_\eta$, and $p \leq p_\xi$. In particular, $p \leq p_\eta \Vdash$ "$\eta \in \dot{S}$" and $p \leq p_\xi \Vdash$ "$\xi \in \dot{S}$" (see choice of the p_α following Remark 15).

Claim 43. *$p \Vdash \forall i < N$, $\forall n \geq k'$, $h_{\mu^i_\eta}(n) < h_{\lambda^i_\xi}(n)$ and $h_{\mu^i_\xi}(n) < h_{\lambda^i_\eta}(n)$.*

Proof. Fix $i < N$, and fix $n \geq k'$; we'll show $p \Vdash$ "$h_{\mu^i_\eta}(n) < h_{\lambda^i_\xi}(n)$ and $h_{\mu^i_\xi}(n) < h_{\lambda^i_\eta}(n)$". If $k' \leq n < k$, then by Assumption 3 (part 3), $h_{\mu^i_\eta}(n) = h_{\mu^i_\xi}(n) < h_{\lambda^i_\xi}(n)$ and $h_{\mu^i_\xi}(n) = h_{\mu^i_\eta}(n) < h_{\lambda^i_\eta}(n)$. Note that Condition 2 was also used here. So suppose $n > k$. By Assumption 1 (part 6), there are three cases to consider: (A^i, B^i) is left-bounded (but not right-bounded), (A^i, B^i) is right-bounded (but not left-bounded), and (A^i, B^i) is left/right unbounded.

CASE 43.1 Suppose first (A^i, B^i) is left-bounded, but not right-bounded. Since (A^i, B^i) is not right-bounded, $\lambda^i_\eta, \lambda^i_\xi > \bar{\theta}$, by Assumption 6. Now, noting that $\lambda^i_\xi \in R^{\xi,i}$ (see Definition 53), it follows that $p(\lambda^i_\xi) = (\bar{L}^\xi_\alpha, \bar{R}^\xi_\alpha, s^\xi_\alpha)$, by the third case of Definition 56. Thus, $p \restriction \lambda^i_\xi \Vdash h_{\mu^i_\eta}(n) < \dot{h}_{\lambda^i_\xi}(n)$. Moreover, since (A^i, B^i) is left-bounded, $\mu^i_\eta, \mu^i_\xi < \epsilon < \bar{\theta}$ (see Definition 55). Thus, since $\lambda^i_\eta \in R^{\eta,i}$, $p(\lambda^i_\eta) =$

15, $j < l$.

$(\bar{L}^\eta_\alpha, \bar{R}^\eta_\alpha, s^\eta_\alpha)$, by the second case of Definition 56. Thus, since $\mu^i_\xi \in L^{\xi,i} \cap \bar{\theta}$, $p \restriction \lambda^i_\eta \Vdash h_{\mu^i_\xi}(n) < \dot{h}_{\lambda^i_\eta}(n)$.

CASE 43.2 Suppose next that (A^i, B^i) is right-bounded, but not left-bounded. So $\lambda^i_\eta, \lambda^i_\xi < \epsilon < \bar{\theta}$ (by Definition 55), while $\mu^i_\eta, \mu^i_\xi > \bar{\theta}$ (by Assumption 6). Since $\mu^i_\eta \in L^{\eta,i}$, $p(\mu^i_\eta) = (\bar{L}^\eta_\alpha, \bar{R}^\eta_\alpha, s^\eta_\alpha)$, by the second case of Definition 56. Since $\lambda^i_\xi \in R^{\xi,i} \cap \bar{\theta}$, it follows that $p \restriction \mu^i_\eta \Vdash \dot{h}_{\mu^i_\eta}(n) < h_{\lambda^i_\xi}(n)$. Similarly, since $\mu^i_\xi \in L^{\xi,i}$, $p \restriction \mu^i_\xi \Vdash \dot{h}_{\mu^i_\xi}(n) < h_{\lambda^i_\eta}(n)$, by the third case of Definition 56.

CASE 43.3 Finally, suppose (A^i, B^i) is left/right unbounded. Since (A^i, B^i) is not right-bounded, just as in CASE 43.1, $\lambda^i_\eta, \lambda^i_\xi > \bar{\theta}$. Again, noting that $\lambda^i_\xi \in R^{\xi,i}$ (see Definition 53), it follows that $p(\lambda^i_\xi) = (\bar{L}^\xi_\alpha, \bar{R}^\xi_\alpha, s^\xi_\alpha)$, by the third case of Definition 56. Thus, $p \restriction \lambda^i_\xi \Vdash h_{\mu^i_\eta}(n) < \dot{h}_{\lambda^i_\xi}(n)$. Now, since (A^i, B^i) is not left-bounded, $\mu^i_\eta, \mu^i_\xi > \bar{\theta}$, by Assumption 6. In particular, since $\mu^i_\xi \in L^{\xi,i}$, it follows that $p \restriction \mu^i_\xi \Vdash \dot{h}_{\mu^i_\xi}(n) < h_{\lambda^i_\eta}(n)$, by the third case of Definition 56.

□

So $p \leq \bar{q}_2$, $p \Vdash$ "$\eta, \xi \in \dot{S}$", and $p \Vdash$ "$\forall i < N$, $\forall n \geq k'$, $h_{\mu^i_\eta}(n) < h_{\lambda^i_\xi}(n)$ and $h_{\mu^i_\xi}(n) < h_{\lambda^i_\eta}(n)$". This contradicts Claim 36.

□

4.6 The Linear Order is Maximal and Saturated

Finally, we prove Theorem 4.0.3. The first step is to show that $\mathbb{P}$ is ccc. We have already completed a crucial piece of the ccc proof by showing that a finite product of Kunen gap-fillers for ω_1-gaps is ccc, provided that the gaps arise at stages θ of cofinality ω_1, and that the gaps are not bounded in θ (cf. Theorem 4.5.1 in Section 4.5). The following lemma will also be needed for the proof that $\mathbb{P}$ is ccc.

Lemma 4.6.1. *Fix $\alpha < \kappa$, and suppose that for each $\delta < \alpha$, $\mathbb{P}_\delta \Vdash$ "$\dot{\mathbb{Q}}_\delta$ is ccc". Suppose $\mathbb{P}_\alpha$ forces (A, B) is an ω_1-cut in $\mathcal{L}_\alpha$ that is bounded in α, and suppose (A, B) is equivalent to $(C, D) \subseteq \mathcal{L}_\delta$, where $\delta \in \alpha \cap HL$ and the ω_1-cut (C, D) is not bounded in δ. Then (C, D) is in $M[G_\delta]$ and $\mathbb{P}_\alpha$ forces (A, B) is a filled cut.*

Proof. Suppose instead that (C, D) does not appear in $M[G_\delta]$. There are three possibilities:

(i) $\dot{\mathbb{Q}}_\delta$ adds the cut (C, D) to $\mathcal{L}_\delta$,

(ii) $\dot{\mathbb{Q}}_{\delta+1}$ adds the cut (C, D) to $\mathcal{L}_\delta$, or

(iii) The iteration from $\delta + 1$ to α adds the cut (C, D) to $\mathcal{L}_\delta$.

Suppose (i) holds. But $\dot{\mathbb{Q}}_\delta$ is the product of Kunen gap-fillers, which is ccc by hypothesis, and hence nice by Lemma 4.3.11; this contradicts Corollary 4.3.8. Suppose next that (ii) holds. Note that $\dot{\mathbb{Q}}_{\delta+1}$ is the finite support product of Aronszajn tree specializers. This forcing is nice, so again it cannot add a gap. (Alternatively, note that if $\dot{\mathbb{Q}}_{\delta+1}$ did add a gap to $\mathcal{L}_\delta$, it would shoot a branch through a Suslin tree of $M[G_\delta]$ by Lemma 4.3.1, which is impossible since all such Suslin trees are specialized by $\dot{\mathbb{Q}}_{\delta+1}$.) Finally, suppose (iii) holds. So the iteration from $\delta+1$ to α (which is ccc by hypothesis) added a new ω_1-gap to $\mathcal{L}_\delta$, contradicting Lemma 4.3.5.

Thus (C, D) is in $\mathcal{X}_\delta$, which means (C, D) gets filled before stage α, and so $\mathbb{P}_\alpha \Vdash (A, B)$ is a filled cut. □

Theorem 4.6.2. $\mathbb{P}$ *is ccc.*

Proof. Since $\mathbb{P}$ is a finite-support iterated forcing construction, it suffices to show by induction that $\forall \alpha < \kappa$, $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha$ is ccc.

First, suppose $\alpha \in EL$. If (A^*_α, B^*_α) is not equivalent to an ω_1-cut, then $K(A_\alpha, B_\alpha)$ is ccc by Lemma 4.2.9 and Lemma 4.2.10 part 1. If (A^*_α, B^*_α) is equivalent to an ω_1-cut, then there must be some $\delta \in \alpha \cap HL$ and an equivalent ω_1-cut $(C, D) \subseteq \mathcal{L}_\delta$ that is not bounded in δ (see Definition 51). By Lemma 4.6.1, (C, D) is in $M[G_\delta]$, which means that (C, D) is in $\mathcal{X}_\delta$, (C, D) gets filled before stage α, and $\mathbb{P}_\alpha \Vdash (A, B)$ is a filled cut. Thus, $\mathbb{P}_\alpha$ forces $K(A_\alpha, B_\alpha)$ is ccc (in fact, σ-centered, by Lemma 4.2.8).

If $\alpha \in ES$, then clearly $\dot{\mathbb{Q}}_\alpha$ is forced to be ccc. If $\alpha \in HS$, then $\dot{\mathbb{Q}}_\alpha$ is forced to be ccc by Corollary 4.3.3.

Finally, suppose that $\alpha \in HL$.

In order to show that $\Pi\{K(A^\beta_\alpha, B^\beta_\alpha) : \beta < \kappa\}$ is ccc, it suffices to show that any such finite product is ccc. So fix a finite set $\{(A^i, B^i) : i < \bar{N}\}$ from the collection of partition cuts $\{(A^\beta_\alpha, B^\beta_\alpha) : \beta < \kappa\}$, and without loss of generality, suppose for some $N \leq \bar{N}$, only the initial segment $\{(A^i, B^i) : i < N\}$ consists of ω_1-cuts that are unfilled. By Lemma 4.2.10 parts 1 and 2, the product $\Pi\{K(A^i, B^i) : N \leq i < \bar{N}\}$ is ccc. Thus, by Lemma 4.2.10 part 3, it suffices to show that $\Pi\{K(A^i, B^i) : i < N\}$ is ccc.

Note first that the cuts $\{(A^i, B^i) : i < N\}$ are not bounded in α (in the sense of definition 51). To see this, suppose instead (A, B) is an unfilled ω_1-cut from $\mathcal{X}_\alpha$ that is bounded in α. But then there must be some $\delta \in \alpha \cap HL$ and an equivalent ω_1-cut $(C, D) \subseteq \mathcal{L}_\delta$ such that (C, D) is not bounded in δ. By Lemma 4.6.1, (C, D) must be in $M[G_\delta]$, which means (C, D) is in $\mathcal{X}_\delta$, (C, D) gets filled before stage α, and so $\mathbb{P}_\alpha$ forces (A, B) is a filled cut. But we are assuming that (A, B) is unfilled.

Thus, all the statements from Assumptions 1 of Section 4.5 hold, where α plays the role of θ and EL plays the role of $\mathbb{L}$. So Theorem 4.5.1 from that section applies: $\Pi\{K(A^i, B^i) : i < N\}$ is ccc, as required.

□

Theorem 4.6.3. *$\mathcal{L}$ is a saturated linear order of size $\mathfrak{c}$ in $({}^\omega\mathbb{Q}, <^*)$.*

Proof. Clearly, in $M[G]$, $\mathcal{L}$ is a linear order in $({}^\omega\mathbb{Q}, <^*)$, of size $\kappa = \mathfrak{c}$. For saturation, fix any cut (A, B) of type (λ, δ) in $\mathcal{L}$, where $\lambda, \delta < \kappa$. Let $C^0 = \{\beta : h_\beta \in A\}$, and let $C^1 = \{\beta : h_\beta \in B\}$. Since $|C^0| < \kappa$, C^0 is named cofinally often in the enumeration $\mathcal{C}$. Choose $\alpha \in EL$ such that $\sup(C^0), \sup(C^1) < \alpha$ and C_α names C^0. So h_α fills the cut (A, B).[50] □

Theorem 4.6.4. *In $M[G]$, $(\mathcal{L}, <^*)$ is a maximal linearly ordered subspace of $({}^\omega\mathbb{R}, <^*)$.*

Proof. The argument is the same as that for the proof of Theorem 3.10.2, verbatim, except $EL \neq \kappa$; the set EL is as defined in this Chapter (see the beginning of section 4.1 or section 4.4).

□

Remark 17. As noted in Remark 8, it is important for the proof of this theorem that for each $\alpha < \kappa$, $|\mathcal{L}_\alpha| < \kappa$, to ensure that the cut $(\bar{C}, \bar{D})$ is of size $< \kappa$.

[50]See the proof of Lemma 3.9.1 for a more expanded version of this argument.

5 Appendix: Open Problems

In this appendix we mention some questions related to the subject of this thesis that remain open (at least to the knowledge of the present author). We also present some partial results. These questions concern the existence of saturated linear orders and maximal saturated linear orders in the various alternate partial orders. We focus on three partial orderings, namely, divergence, eventual domination without eventual equality, and almost inclusion; we consider each of the orderings over a set that renders the structures countably saturated (in particular, we are careful to omit endpoints).

Notation. Recall the definitions of $\prec$, $\lneq^*$ and $\subset^*$ from Chapter 1.

1. Let $\mathcal{F} = \{f \in {}^\omega\omega : f \text{ is not bounded}\}$, ordered by $\prec$;
2. let $\mathcal{T} = \{f \in {}^\omega\omega : f \text{ is not eventually } 0\}$, ordered by $\lneq^*$; and
3. let $\mathcal{B} = \{A \subseteq \omega : A \text{ is infinite and co-infinite}\}$, ordered by $\subset^*$.

In the context of any of these partial orders, a *saturated linear order* (or *SLO*) refers to a saturated linear order of size $\mathfrak{c}$.

The first open question is about whether it is possible to have a saturated linear order in some partial order, but no maximal saturated linear order.

Question 6. Does the existence of a SLO in $\mathcal{F}$, $\mathcal{T}$ or $\mathcal{B}$ imply the existence of a *maximal* SLO in the same partial order?

Obtaining maximality in cases where there is a SLO does not seem to be easy; for the arguments in Chapters 3 and 4, genericity played a crucial role. Nevertheless, we do not have an example of a model in which there is a saturated linear order, but no maximal one.

Another question of interest is whether the existence of a SLO in one partial order implies the existence of a SLO in another partial order. The following result is a positive answer to this question, at least for our three partial orders:

Theorem 5.0.5. *The following are equivalent:*

(i) $\exists$ *a SLO in* $\mathcal{F}$*,*

(ii) $\exists$ *a SLO in* $\mathcal{T}$*, and*

(iii) $\exists$ *a SLO in* $\mathcal{B}$*.*

Proof. (i) $\Rightarrow$ (ii): Suppose F is a SLO in $\mathcal{F}$. Then F is automatically a SLO in $\mathcal{T}$.

(ii) $\Rightarrow$ (iii): Let F be a SLO in $\mathcal{T}$. Define, for each $f \in F$, $A_f = \{(m,n) : n \leq f(m)\}$. Let $\mathcal{A}_F = \{A_f : f \in F\}$.

Fix a bijection $\varphi : \omega \times \omega \longrightarrow \omega$. Let $\Phi : \mathcal{P}(\omega \times \omega) \longrightarrow \mathcal{P}(\omega)$ be the lifting of φ; in other words, $\Phi(A) = \varphi[A] = \{\varphi(m,n) : (m,n) \in A\}$ for each $A \subset \omega \times \omega$. The following general fact will be useful.

Fact 10. Let X, Y be any infinite sets of the same cardinality, let $\psi : X \longrightarrow Y$ be a bijection, and let Ψ be the lifting of ψ. Then

(a) Ψ is a bijection, and

(b) Ψ preserves $\subseteq$, $\subseteq^*$, and $\subset^*$.[51]

Proof. Routine. $\square$

By Fact 10, Φ is a $\subset^*$-order isomorphism.

Now, define $\sigma : F \longrightarrow \mathcal{A}_F$ by $\sigma(f) = A_f$. It is straightforward to check that σ is an order isomorphism. Thus, $\mathcal{A}_F$ is a SLO in $(\mathcal{P}(\omega \times \omega), \subset^*)$. So the range of $\Phi \restriction \mathcal{A}_F$ is a SLO in $(\mathcal{P}(\omega), \subset^*)$, and therefore in $\mathcal{B}$.

(iii) $\Rightarrow$ (i): Let $\mathcal{A}$ be a SLO in $\mathcal{B}$. For each $A \subseteq \omega$, let $(f_A(n))_{n \in \omega}$ enumerate the elements of A in increasing order; in other words, if $A = \{a_n : n \in \omega\}$ is the increasing enumeration of A, then $f : \omega \longrightarrow \omega$ is defined by $f(n) = a_n$.

[51] Let S be any infinite set. For all $A, B \subseteq S$, define $A \subseteq^* B$ iff $A \setminus B$ is finite, and define $A \subset^* B$ iff $A \setminus B$ is finite but $B \setminus A$ is infinite.

Claim 44. *$\forall A, B \in \mathcal{A}$, $A \subset^* B$ iff $f_B \prec f_A$.*

Proof. Fix $A, B \in \mathcal{A}$. Suppose first $A \subset^* B$. Let $\bar{n} = \min\{n : A \setminus n \subseteq B\}$. Let $\bar{m} = \min\{m > \bar{n} : f_A(m) \geq f_B(m)\}$.

Now, fix $M \in \omega$. We wish to find M' such that for each $n > M'$, $f_A(n) - f_B(n) > M$. To do this, simply choose $M' > \bar{m}$ such that $|\{n : \bar{m} < n < M', n \in B \setminus A\}| > M + \bar{m}$. Then, for all $n > M'$, it must be that $f_A(n) - f_B(n) > M$.

Next, suppose that $f_B \prec f_A$. Clearly, $A \neq B$. Since $\mathcal{A}$ is a linear order, either $A \subset^* B$ or $B \subset^* A$. But the latter would imply, by the preceding argument, that $f_A \prec f_B$. So, it must be the case that $A \subset^* B$. □

Let $F = \{f_A : A \in \mathcal{A}\}$, and define $\varphi : \mathcal{A} \longrightarrow F$ by $\varphi(A) = f_A$. By Claim 44, φ is an order-reversing isomorphism. Thus, F is a SLO in $\mathcal{F}$.

□

Although the existence of a SLO in one of our partial orders implies the existence of a SLO in the other two, the situation for maximal saturated linear orders seems more complicated.

Question 7. Does the existence of a maximal SLO in one of $\mathcal{F}$, $\mathcal{T}$ or $\mathcal{B}$ imply the existence of a maximal SLO in the other two partial orders?

We do not know the answer to the latter question. However, we do have the following partial result:

Theorem 5.0.6. *If there is a maximal SLO in $\mathcal{F}$ and a maximal SLO in $\mathcal{A}$, then there is a maximal SLO in $\mathcal{T}$.*

Proof. Suppose $\mathcal{A}$ is a maximal SLO in $\mathcal{B}$ and F is a maximal SLO in $\mathcal{F}$. For each $X \subseteq \omega$, let 1_X denote the characteristic function of X as a subset of ω, and let $\bar{0} = 1_\emptyset$. Moreover, let $\bar{\mathcal{A}} = \mathcal{A} \cup \{\emptyset\}$, and let $\bar{F} = F \cup \{\bar{0}\}$. Define:

$$\tilde{F} = \{f + n + 1_A : f \in \bar{F}, n \in \mathbb{Z}, A \in \bar{\mathcal{A}}\} \setminus \{\bar{0}\}$$

where, if $n < 0$,

$$f(m) + n = \begin{cases} f(m) + n & \text{if } f(m) + n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Claim 45. *$(\tilde{F}, \lneq^*)$ is a linear order.*

Proof. Fix $f' \neq g' \in \tilde{F}$; say $f' = f + n + 1_A$, and $g' = g + m + 1_B$, where $f, g \in \bar{F}$, $m, n \in \mathbb{Z}$, and $A, B \in \bar{\mathcal{A}}$.

CASE 1: Suppose first $f \neq g$. Without loss of generality, suppose $f \prec g$. We need to show $f' \lneq^* g$, but we'll show something stronger, namely, that $f' \prec g'$. To this end, fix $M \in \omega$, and find n such that $\forall k \geq n$, $g(k) - f(k) > |m| + |n| + 2 + M$. Then, $\forall k \geq n$, $g'(k) - f'(k) = g(k) + m + 1_B - f(k) - n - 1_A > |m| + |n| + 2 + M + m - n + 1_B - 1_A > M$.

CASE 2: Suppose next that $f = g$, but $m \neq n$. Without loss of generality, say $n < m$. Since A is co-infinite, $f' \lneq^* g$.

CASE 3: Suppose finally that $f = g$ and $m = n$. So it must be the case that $A \neq B$. Since $\mathcal{A}$ is a linear order, either $A \subset^* B$ or $B \subset^* A$; without loss of generality, suppose $A \subset^* B$. But then $1_A \lneq^* 1_B$, and so $f' \lneq^* g'$. $\square$

Claim 46. *$(\tilde{F}, \lneq^*)$ is a saturated linear order.*

Proof. Let (C, D) be a cut of size $< \mathfrak{c}$ in $\tilde{F}$. Let $C' = \{f : \exists n \in \mathbb{Z}, \exists A \in \bar{\mathcal{A}}, \exists f' \in C$ such that $f' = f + n + 1_A\}$, and let $D' = \{f : \exists n \in \mathbb{Z}, \exists A \in \bar{\mathcal{A}}, \exists f' \in D$ such that $f' = f + n + 1_A\}$. If $C' \cap D' = \emptyset$, then (C', D') is a cut (of size $< \mathfrak{c}$) in $(F, \prec)$; but then any $h \in F$ which fills the cut (C', D') also fills the cut (C, D). So, suppose instead that $C' \cap D' \neq \emptyset$. It must be that $|C' \cap D'| = 1$, so let $C' \cap D' = \{f\}$. Define

$$\bar{m} = \sup\{m : \exists A \in \mathcal{A},\ f + m + 1_A \in C\}$$

and

$$\bar{n} = \inf\{m : \exists A \in \mathcal{A},\ f + m + 1_A \in D\}$$

Note that $\bar{m}$ and $\bar{n}$ are finite integers, and $\bar{m} \leq \bar{n}$, since $C \lneq^* D$. Define $\mathcal{A}_1 = \{A \in \mathcal{A} : f + \bar{m} + 1_A \in C\}$, and $\mathcal{A}_2 = \{A \in \mathcal{A} : f + \bar{n} + 1_A \in D\}$

Suppose first that $\bar{m} < \bar{n}$. Note that since $|C| < \kappa$, also $|\mathcal{A}_1| < \kappa$. So $(\mathcal{A}_1, \emptyset)$ is a cut of size $< \kappa$ in $\mathcal{A}$. Since $\mathcal{A}$ is a SLO, find $B \in \mathcal{A}$ filling this cut. Then $C \lneq^* f + \bar{m} + 1_B \lneq^* D$.

Next, suppose that $\bar{m} = \bar{n}$. Note that $(\mathcal{A}_1, \mathcal{A}_2)$ must be a cut in $\mathcal{A}$ since (C, D) is a cut in $\tilde{F}$. So find $B \in \mathcal{A}$ filling the cut $(\mathcal{A}_1, \mathcal{A}_2)$. But then, $C \lneq^* f + \bar{m} + 1_B \lneq^* D$. □

It now remains to show that $(\tilde{F}, \lneq^*)$ is maximal in $\mathcal{T}$. To this end, suppose instead there is a function $h \in \mathcal{T} \setminus \tilde{F}$ such that $(\tilde{F} \cup \{h\}, \lneq^*)$ is a linear order. Since $h \notin \tilde{F}$, it follows that $h \notin F$, since otherwise $h = h + 0 + 1_\emptyset$ would be in $\tilde{F}$. However, since F is a maximal SLO in $\mathcal{F}$, $(\mathcal{F} \cup \{h\}, \prec)$ is not a linear order. So there must be some $f \in \mathcal{F}$ such that f and h are incomparable under the divergence ordering. But since $f = f + 0 + 1_\emptyset \in \tilde{F}$, f and h must be comparable under $\lneq^*$, by hypothesis.

So either $f \lneq^* h$ or $h \lneq^* f$. We consider only the case $f \lneq^* h$, since the other case is similar.

Claim 47. *There is some $n > 0$ such that $h \lneq^* f + n$.*

Proof. Suppose instead that for each $n > 0$, $h \lneq^* f + n$ fails. Since, for each n, $f + n = f + n + 1_\emptyset \in \tilde{F}$ and $F \cup \{h\}$ is assumed to be a linear order under $\lneq^*$, it must be the case that for each $n > 0$, $f + n \lneq^* h$. Now, fix $N > 0$. Since $f + (N + 1) \lneq^* h$, find $\bar{m}$ such for each $m \geq \bar{m}$, $f(m) + (N + 1) \leq h(m)$. So, for each $m \geq \bar{m}$, $h(m) - f(m) > N$. This shows that $f \prec h$, but f and h are supposed to be incomparable under $\prec$. □

Let $\bar{n} = \min\{n > 0 : h \lneq^* f + n\}$. So $f + (\bar{n} - 1) \leq^* h \lneq^* f + \bar{n}$. So, there is some $B \subseteq \omega$ such that $h =^* f + (\bar{n} - 1) + 1_B$. Since $\mathcal{A}$ is maximal, there must be some $B' \in \bar{\mathcal{A}}$ such that B and B' are incomparable under $\subset^*$. But then h is incomparable to $f + (\bar{n} - 1) + 1_{B'} \in \tilde{F}$ under $\lneq^*$, contrary to the assumption that $(\tilde{F} \cup \{h\}, \lneq^*)$ is a linear order. $\square$

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