RIGIDITY OF CORONA ALGEBRAS

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Abstract

In this thesis we use techniques from set theory and model theory to study the isomorphisms between certain classes of C*-algebras. In particular we look at the isomorphisms between corona algebras of the form $\prod \mathbb{M}_{k(n)}(\mathbb{C})/\bigoplus \mathbb{M}_{k(n)}(\mathbb{C})$ for sequences of natural numbers $\{k(n) : n \in \mathbb{N}\}$. We will show that the question “whether any isomorphism between these C*-algebras is trivial", is independent from the usual axioms of set theory (ZFC). We extend the classical Feferman-Vaught theorem to reduced products of metric structures. This implies that the reduced powers of elementarily equivalent structures are elementarily equivalent. We also use this to find examples of corona algebras of the form $\prod \mathbb{M}_{k(n)}(\mathbb{C})/\bigoplus \mathbb{M}_{k(n)}(\mathbb{C})$ which are non-trivially isomorphic under the Continuum Hypothesis. This gives the first example of genuinely non-commutative structures with this property.

In chapter 6 we show that SAW^*-algebras are not isomorphic to ν-tensor products of two infinite dimensional C*-algebras, for any C*-norm ν. This answers a question of S. Wassermann who asked whether the Calkin algebra has this property.

To my parents

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1 Introduction

This dissertation is focused on some of the applications of logic to operator algebras. The connection between logic and operator algebras in the past decade has been overwhelmingly fruitful. A number of long-standing problems in the theory of C*-algebras were solved by using set-theoretic methods, and solutions to some of them were even shown to be independent from ZFC. I will focus on some of the set-theoretic and model-theoretic properties of the isomorphisms between certain classes of corona algebras. The reader may refer to [18] for an overview of the recent developments of the applications of logic to operator algebras.

The main objective of this thesis is to study the rigidity of certain coronas of C*-algebras. In general for quotient structures X/I, Y/J and Φ a homomorphism between them, a representation of Φ is a map $\Phi_* : X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{\Phi_*} & Y \\ \downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_J} \\ X/I & \xrightarrow{\Phi} & Y/J \end{array}$$

commutes, where π_I and π_J denote the respective quotient maps. Note that since representation is not required to satisfy any algebraic properties, its existence follows from the axiom of choice. If Φ has a representation which is a homomorphism itself we say Φ is *trivial*. The question whether automorphisms between some quotient structures are trivial, sometimes is called the *rigidity* question and has been studied for various structures (see for example [51], [15], [37], [36], [49], [48] and [17]). It turns out that for many structures the answers to these questions highly depend on the set-theoretic axioms. Theses results belong to a line of results starting with Shelah's ground-breaking construction of a model of set theory in which all automorphisms of the quotient Boolean algebra $P(\mathbb{N})/Fin$ are trivial ([50]).

The rigidity question for quotients of Boolean algebras, more specifically $P(\mathbb{N})$, was the first to consider in order to answer the following basic question: How does a change of the ideal $\mathcal{I}$ of $P(\mathbb{N})$ effect the change of its quotient $P(\mathbb{N})/\mathcal{I}$? This turns out to be a quite subtle question and for many non-trivial cases the answer to this question is independent from ZFC. Motivated by a question of Brown-Douglas-Fillmore [4, 1.6(ii)] the rigidity question for the category of C*-algebras has been studied for various corona algebras. For a separable Hilbert space H let $\mathcal{C}(H)$ denote the Calkin algebra over H. An automorphism Φ of the Calkin algebra is said to be *inner* if it is implemented by a partial isometry between cofinite dimensional subspaces of H, i.e., there exists $u \in \mathcal{B}(H)$ such that $\pi(u)$ is a unitary in $\mathcal{C}(H)$

and $\Phi(\pi(a)) = \pi(u^*au)$ for all $a \in \mathcal{B}(H)$, where $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ is the natural quotient map. Assuming the Continuum Hypothesis, N.C. Phillips and N. Weaver [48] constructed $2^{\aleph_1}$ many automorphism of the Calkin algebra over a separable Hilbert space. Since there are only continuum many inner automorphisms of the Calkin algebra, this implies that there are many outer automorphisms. On the other hand I. Farah [17] showed that under the *Todorcevic's Axiom* (the open coloring axiom) all automorphisms of the Calkin algebra over a separable Hilbert space are inner.

A very brief introduction to C*-algebras and their coronas is given in section 1.1. The interested reader is referred to [3] or [45] for an extensive treatment of these algebras. In chapter 3 I study the rigidity question for the isomorphisms between the coronas of direct sums of full matrix algebras, $\bigoplus \mathbb{M}_{k(n)}(\mathbb{C})$. I will prove that it is consistent with ZFC that all such isomorphisms are trivial ([32]) in the strongest possible sense. This generalizes the result of Shelah ([50]), in its dual form, about automorphisms of $P(\mathbb{N})/Fin$, which corresponds to the case where the automorphisms are restricted to centers of these algebras.

The forcing extension that we use is a countable support iteration of proper forcings of the form $\mathbb{P}_{\mathcal{I}}$ for a σ-ideal $\mathcal{I}$ on $\mathbb{N}$. These forcings are well-studied by J. Zapletal in [58] and [59]. It connects the practice of proper forcing introduced by Shelah [50] with the study of various σ-ideals on Polish spaces from the point of view of abstract analysis, descriptive set theory and measure theory. Many forcings encountered in practice can be represented as $\mathbb{P}_{\mathcal{I}}$ for a suitable σ-ideal $\mathcal{I}$ on a Polish space. In chapter 2 I give a brief introduction to the theory of forcing with ideals and their countable support iterations, and provide the necessary information about these forcings in what comes next in chapter 3.

In the last few years the model theory for operator algebras has been developed and specialized from the model theory for metric structures. This has proved to be very fruitful as many properties of C*-algebras and tracial von Neumann algebras have equivalent model theoretic reformulations (see [22], [19], ...). For instance, given a sequence of C*-algebras $\{\mathcal{A}_n : n \in \mathbb{N}\}$, the *asymptotic sequence algebra* $\ell_\infty(\mathcal{A}_n)/c_0(\mathcal{A}_n)$ is the reduced product over the Fréchet ideal and is an important example of corona algebras, which has many interesting model-theoretic properties. The most interesting model-theoretic property of the asymptotic sequence algebras is the fact that they are countably saturated ([24]). Hence under the Continuum Hypothesis the question whether two such reduced products are isomorphic reduces to the weaker question of whether they are elementarily equivalent. In general it seems that even though particular reduced products of some metric structures have been studied in analysis, unlike classical first order logic, the model theory of reduced products of metric structures has not been studied until very recently in [40] and [24].

In chapter 5 I prove a metric version of the Feferman-Vaught theorem (Theorem 5.1.3) for reduced products of metric structures, which just like the classical Feferman-Vaught theorem, implies the preservation of the elementary equivalence relation by arbitrary direct products, ultraproducts and reduced products of metric structures. The metric Feferman-Vaught theorem (see [30]) has number of interesting consequences. In particular I use this theorem to solve an outstanding problem on coronas of C*-algebras (§5.3). Namely, I prove the existence of two separable C*-algebras of the form $\bigoplus_i \mathbb{M}_{k(i)}(\mathbb{C})$ such that the assertion that their coronas are isomorphic is independent from ZFC, which gives the first example of genuinely non-commutative coronas of separable C*-algebras with this property.

It is well known that C*-algebras can be viewed as non-commutative topological spaces and the correspondence $X \leftrightarrow C(X)$ is a contravariant category equivalence between the category of compact Hausdorff spaces and continuous maps and the category of commutative unital C*-algebras and unital *-homomorphisms. Each property of a locally compact Hausdorff space can be reformulated in terms of the function algebra $C_0(X)$, so it usually make sense to ask about these properties for non-commutative C*-algebras. SAW^*-algebras were introduced by G.K. Pedersen ([46]) as non-commutative analogues of sub-Stonean spaces (also known as F-spaces, see [33]) in topology. In [47] and [46] some of the properties of sub-Stonean spaces are generalized to SAW^*-algebras. It is proved ([46]) that the corona algebra of any σ-unital C*-algebra is a SAW^*-algebra. In particular for a separable Hilbert space the Calkin algebra is a SAW^*-algebra. In [19] I. Farah and B. Hart noticed that some of the nice properties of SAW^*-algebras, like Countable Riesz Separation Property (CRISP), Kasparov's Technical Theorem (KTT), ..., follows from their somewhat saturated nature. They introduced *countably degree-1 saturated* C*-algebras and showed that the class of all countably degree-1 saturated C*-algebras contains all coronas of σ-unital C*-algebra, ultrapowers of C*-algebras and the relative commutants of separable subalgebras of a countably degree-1 saturated C*-algebra, and they are all SAW^*-algebras. The countably degree-1 saturation is a weakening of the 'fully' countably saturation and it is shown that the Calkin algebra (which is countably degree-1 saturated) is not fully countably saturated ([19]). Therefore it makes sense to ask how much of the properties of fully countably saturated algebras are passed on to countable degree-1 saturated C*-algebras or even SAW^*-algebras.

Chapter 6 of this thesis is devoted to show that SAW^*-algebras (hence the Calkin algebra, the reduced products over the Fréchet ideal and ultraproducts) are essentially non-factorizable ([31]). Meaning that they can not be written as $\mathcal{B} \otimes_\nu \mathcal{C}$ where both $\mathcal{B}$ and $\mathcal{C}$ are infinite dimensional, for any C*-algebra norm ν. This gives a positive answer to a question asked by S. Wassermann and also implies that the ultrapowers of C*-algebras and relative commutants of separable subalgebras of a countably degree-1 saturated C*-algebra are also essentially non-factorizable.

1.1 C*-algebras and their coronas

In one of his papers on Hilbert space theory (1929), John von Neumann defined a ring of operators M (nowadays called a *von Neumann algebra*) as a self-adjoint subalgebra of the algebra of bounded linear operators on a Hilbert space which is closed in the *weak operator* topology. A sequence $\{a_n\}$ of bounded operators weakly converges to a when $\langle a_n\xi|\eta\rangle \to \langle a\xi|\eta\rangle$ for all $\xi, \eta \in H$. Von Neumann's primary motivation in studying rings of operators came from quantum mechanics. In 1925 Heisenberg discovered a model of quantum mechanics, which at the time was called "matrix mechanics". Schrödinger was led to a seemingly different formulation of the theory, which he called "wave mechanics". Modern day formalism of quantum mechanics was developed by Dirac and von Neumann. They realized that to each quantum system, one can associate a separable Hilbert space over the field of complex numbers $\mathbb{C}$. In quantum mechanics observables such as position, momentum, angular momentum and spin are represented by self-adjoint operators on a Hilbert space. The possible states of an observable in a quantum mechanical system are represented by unit vectors (called state vectors) in the associated Hilbert space. In quantum theory a sequence of observables a_n converges to the observable a if the expectation value $\langle a_n\xi|\xi\rangle$ of a_n in the state ξ converges to $\langle a\xi|\xi\rangle$, for each $\xi \in H$

(given that $\|\xi\| = 1$). Thereby von Neumann introduced and studied weakly-closed rings of operators (von Neumann algebras) as models for quantum mechanical systems. In 1931, he proved a famous theorem - now called the Stone-von Neumann theorem - which explained that Heisenberg's matrix mechanics and Schrödinger's wave mechanics are just two different representations of the same theory. The theory of von Neumann algebras was developed in a series of papers by Murray and von Neumann in the 1930's and 1940's. In his book *Mathematische Grundlagen der Quantenmechanik* (1932), von Neumann formulated the abstract concept of a Hilbert space, developed the spectral theory of bounded as well as unbounded normal operators on a Hilbert space.

For a complex Hilbert space H, let $\mathcal{B}(H)$ denote the Banach algebra of bounded linear operators on H equipped with the operation $*$ of taking the adjoint. For $T \in \mathcal{B}(H)$ define the norm of T by

$$\|T\| = \sup\{\|T\xi\| : \|\xi\| \leq 1\}.$$

In addition to von Neumann algebras, the mathematical formalism of quantum mechanics and quantum field theory led to study of "norm-closed" self-adjoint subalgebras of $\mathcal{B}(H)$ as suitable models for an "algebra of observables". These algebras are called *concrete C*-algebras*. In 1943, the work of Israel Gelfand and Mark Naimark [29] yielded an abstract characterisation of C*-algebras without any reference to operators on a Hilbert space. Based on their work, Irving Segal (1947) introduced

C*-algebras in its present form, and what is now called the Gelfand-Naimark-Segal (GNS) construction, connecting states to representations. An *abstract* C*-algebra $\mathcal{A}$ is a Banach *-algebra over the field of complex numbers which satisfies the *C*-identity*

$$\|xx^*\| = \|x\|^2$$

for all $x \in \mathcal{A}$.

The Gelfand-Naimark theorem states that an arbitrary abstract C*-algebra $\mathcal{A}$ is isometrically *-isomorphic to a subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. As with von Neumann algebras, there is a fruitful interaction between C*-algebras and quantum physics. For example, certain quantum field theories allow many different “vacuum states", which are closely related to representations of C*-algebras. This connection can be made precise using the GNS construction. In fact what physicists usually call an “algebra of observables" is a C*-algebra. The theory of C*-algebras turned out to be interesting both for intrinsic reasons (structure and representation theory of C*-algebras), as well as because of its connections with a number of other fields of mathematics. Many mathematical structures such as group actions, groupoids, foliations, and complex domains can be analysed through an appropriate C*-algebra. Theory of C*-algebras also has extensive applications in dynamical systems and non-commutative geometry.

Notably, all algebraic isomorphisms between C*-algebras are isometries. The *strong operator topology* is the initial topology on $\mathcal{B}(H)$ induced by the family of seminorms $a \to \|a\xi\|$ for all $\xi \in H$. We will always assume H is separable unless specifically stated. It is not hard to see that for every $M < \infty$ the strong operator topology on $\mathcal{B}(H)_{\leq M} = \{a \in \mathcal{B}(H) : \|a\| \leq M\}$ is a Polish space.

For a locally compact Hausdorff space X, the algebra

$$C_0(X) = \{f : X \to \mathbb{C} : f \text{ is continuous and vanishes at infinity}\}$$

is a C*-algebra with the involution $f^* = \bar{f}$, pointwise multiplication and the sup norm. Here "vanishes at infinity" means that f continuously extends to the one point compactification $X \cup \{\infty\}$ of X such that the extension vanishes at ∞. If X is compact we write $C(X) = C_0(X)$. By the Gelfand-Naimark duality (see e.g., [3]) every commutative C*-algebra is isometrically *-isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space X. One aspect of non-commutative topology is to view a general C*-algebra as a "non-commutative $C_0(X)$". Each property concerning a locally compact Hausdorff space X can be formulated in terms of the function algebra $C_0(X)$ and will usually make sense for any non-commutative C*-algebra. Instead of this translation one may look directly for the objects. In a non-commutative C*-algebra $\mathcal{A}$ the open and closed projections in the enveloping von Neumann algebra $\mathcal{A}''$ of $\mathcal{A}$ replace the open and closed sets in the commutative case (see [46]). The open projections are in a bijective correspondence with the hereditary C*-subalgebras of $\mathcal{A}$ (of the form $\mathcal{L} \cap \mathcal{L}^*$ for some closed left ideal $\mathcal{L}$ of $\mathcal{A}$), see [45, 1.5.2, 3.10.7, 3.11.9].

Let $\mathcal{A}$ be a C*-algebra. The set of all *positive* elements of $\mathcal{A}$, elements of the form aa^* for $a \in \mathcal{A}$, is denoted by $\mathcal{A}_+$ and we use $\mathcal{A}_{\leq 1}$ to denote the (closed) unit ball of $\mathcal{A}$. An approximate unit for $\mathcal{A}$ is an increasing net (h_λ) of positive elements of $\mathcal{A}$ of norm ≤ 1, indexed by a directed set Λ, such that $h_\lambda x \to x$ for all $x \in A$. A C*-algebra is σ-unital if it contains a countable approximate unit. Every separable C*-algebra is σ-unital, but there are non-separable σ-unital C*-algebras. For example, $C_0(X)$ is σ-unital if and only if X is σ-compact.

A (two-sided and closed) ideal of $\mathcal{A}$ is called *essential* if it has a non-trivial intersection with any non-zero ideal of $\mathcal{A}$. For a non-unital C*-algebra $\mathcal{A}$ there are various ways in which $\mathcal{A}$ can be embedded as an ideal in a unital C*-algebra. If $A = C_0(X)$ is commutative this corresponds to the ways in which the locally compact Hausdorff space X can be embedded as an open set in a compact Hausdorff space Y. The minimal way to do so, is to take the one-point compactification of X and the maximal way is the Čech-Stone compactification βX of X. The C*-analogue of the Čech-Stone compactification is called the *multiplier algebra* of $\mathcal{A}$. The multiplier algebra $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ is the unital C*-algebra containing $\mathcal{A}$ as an essential ideal, which is universal in the sense that whenever $\mathcal{A}$ is an ideal of a unital C*-algebra $\mathcal{D}$, the identity map on $\mathcal{A}$ extends uniquely to a *-homomorphism from $\mathcal{D}$ into $\mathcal{M}(\mathcal{A})$. There are several ways of constructing $\mathcal{M}(\mathcal{A})$ (cf. [3, II.7.3]).

A traditional way is to take a faithful nondegenerate representation ρ of $\mathcal{A}$ on a Hilbert space H, and consider $\mathcal{M}(\mathcal{A})$ as the idealizer of $\rho(\mathcal{A})$ in $B(H)$,

$$\mathcal{M}(\mathcal{A}) \cong \{m \in B(H) \ : \ \ \forall a \in \mathcal{A} \quad m\rho(a) \text{ and } \rho(a)m \in \rho[\mathcal{A}]\}.$$

The *strict topology* on $\mathcal{M}(\mathcal{A})$ is the initial topology induced by the seminorms $x \to \|ax\|$ and $x \to \|xa\|$ for $a \in \mathcal{A}$, i.e., $x_i \to x$ strictly in $\mathcal{M}(\mathcal{A})$ if and only if $ax_i \to ax$ and $x_i a \to xa$ in norm for all $a \in \mathcal{A}$. In fact, $\mathcal{M}(\mathcal{A})$ is the strict completion of $\mathcal{A}$. The quotient C*-algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is called the *corona* of $\mathcal{A}$ and is denoted by $\mathcal{C}(\mathcal{A})$.

Examples of corona algebras. (i) If $\mathcal{A}$ is unital, then $\mathcal{M}(\mathcal{A}) = \mathcal{A}$ and the strict topology is the norm topology. Therefore the corona of $\mathcal{A}$ is trivial.

(ii) If $\mathcal{A} = \mathcal{K}(H)$, the closed ideal of all compact operators on a Hilbert space H, then $\mathcal{M}(\mathcal{A}) = \mathcal{B}(H)$. The corona of $\mathcal{A}$ is the Calkin algebra $\mathcal{C}(H)$.

(iii) If X is a locally compact Hausdorff space, then $\mathcal{M}(C_0(X)) \cong C(\beta X)$, which is isomorphic to the C*-algebra $C_b(X)$ of bounded continuous complex-valued functions on X. The corona of $C_0(X)$ is isomorphic to the C*-algebra $C_b(X)/C_0(X) \cong C(X^*)$, where X^* is the Čech-Stone remainder $\beta X \setminus X$ of X.

(iv) Suppose $\mathcal{A}_n$ is a sequence of unital C*-algebras and let

$$\prod_n^{\infty} \mathcal{A}_n = \{(x_n) : \quad x_n \in \mathcal{A}_n \ \text{ and } \ \sup_n \|x_n\| < \infty\}$$
$$\bigoplus_n^{\infty} \mathcal{A}_n = \{(x_n) \in \prod_n \mathcal{A}_n : \|x_n\| \to 0\}.$$

If $\mathcal{A} = \bigoplus_n \mathcal{A}_n$ then $\mathcal{M}(\mathcal{A}) = \prod_n \mathcal{A}_n$. The corona of $\mathcal{A}$, $\prod_n \mathcal{A}_n / \bigoplus_n \mathcal{A}_n$, is usually called the *reduced product* of the sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ (over the Fréchet ideal).

(v) If $\mathcal{A}$ is a C*-algebra and X is a locally compact Hausdorff space, let

$$C_b(X, \mathcal{A}) = \{f : X \to \mathcal{A} : \quad f \text{ is a norm continuous and bounded function}\}$$

$$C_0(X, \mathcal{A}) = \{f : X \to \mathcal{A} : \quad f \text{ is continuous and vanishes at } \infty\},$$

then the $\mathcal{M}(C_0(X, \mathcal{A})) \cong C_b(X, \mathcal{A})$, which is isomorphic to the C*-algebra of strictly continuous functions from βX to $\mathcal{M}(\mathcal{A})$. Then $\mathcal{C}(C_0(X, \mathcal{A})) \cong C_b(X, \mathcal{A})/C_0(X, \mathcal{A}) \cong \mathcal{C}(X^*, \mathcal{C}(\mathcal{A}))$.

Note that if $X = \mathbb{N}$ then $C_b(X, \mathcal{A})/C_0(X, \mathcal{A}) = \prod_n \mathcal{A} / \bigoplus_n \mathcal{A}$.

For a locally compact Hausdorff space X there is a bijective correspondence between autohomeomorphisms of the Čech-Stone remainder X^* of X and automorphisms of the quotient C*-algebra $C_b(X)/C_0(X) \cong C(X^*)$. By the Gelfand-Naimark duality this reduces the study of the automorphisms of the corona of commutative C*-algebras to the study of autohomeomorphisms of the Čech-Stone remainders (also called corona) of locally compact Hausdorff spaces.

1.2 Analytic ideals

We denote the set of natural numbers by $\mathbb{N}$ (or sometimes ω) and $P(\mathbb{N})$ (or $P(\omega)$) is the Boolean algebra of all subsets of the natural numbers ordered with inclusion.

An ideal $\mathcal{I}$ on $\mathbb{N}$ is an ideal of the Boolean algebra $P(\mathbb{N})$ and the quotient Boolean algebra is denoted by $P(\mathbb{N})/\mathcal{I}$. By identifying sets with their characteristic functions we equip $P(\mathbb{N})$ with the compact metric topology taken from $\{0,1\}^{\mathbb{N}}$. Thus we can speak of Borel, or analytic ideals on $\mathbb{N}$. By Fin we denote the ideal of all finite subsets of $\mathbb{N}$, so called the *Fréchet ideal*. In order to avoid trivial considerations, all non-empty ideals that we consider include Fin and are not all of $P(\mathbb{N})$, i.e., *proper*. Sets $A, B \subseteq \mathbb{N}$ are *almost disjoint* if $A \cap B \in Fin$ and A is *almost included* in B ($A \subseteq^* B$) if $A \setminus B \in Fin$. For a map f we shall use $f[X]$ to denote the image of the set X under the mapping f. Also by $[m, n) \subset \mathbb{N}$ we mean the set $\{m, m+1, \ldots, n-1\}$.

For an ideal $\mathcal{I}$ on $\mathbb{N}$, a set $A \subseteq \mathbb{N}$ is $\mathcal{I}$-positive if $A \notin \mathcal{I}$. A restriction of $\mathcal{I}$ to an $\mathcal{I}$-positive set A, $\mathcal{I} \restriction_A$, is an ideal on A defined by

$$\mathcal{I} \restriction_A = \mathcal{I} \cap P(A).$$

For two ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$, the *Fubini product*, $\mathcal{I} \times \mathcal{J}$, of $\mathcal{I}$ and $\mathcal{J}$ is the ideal on $\mathbb{N} \times \mathbb{N}$ defined by

$$A \in \mathcal{I} \times \mathcal{J} \quad \leftrightarrow \quad \{i : A_i \notin \mathcal{J}\} \in \mathcal{I},$$

where $A_i = \{j \in \mathbb{N} : (i, j) \in A\}$ is the vertical section of A at i.

Definition 1.2.1. *Ideals $\mathcal{I}$ and $\mathcal{J}$ are Rudin-Keisler isomorphic, $\mathcal{I} \approx_{RK} \mathcal{J}$, if there are $A \in \mathcal{I}$, $B \in \mathcal{J}$, and a bijection $h : \mathbb{N} \setminus A \to \mathbb{N} \setminus B$ such that for all $X \subseteq \mathbb{N} \setminus A$*

we have

$$X \in \mathcal{I} \quad \leftrightarrow \quad h[X] \in \mathcal{J}.$$

It is not difficult to see that in this situation the map $[X]_{\mathcal{I}} \to [h[X]]_{\mathcal{J}}$ is an isomorphism between $P(\mathbb{N})/\mathcal{I}$ and $P(\mathbb{N})/\mathcal{J}$ ([11, Lemma 1.2]).

An ideal $\mathcal{I}$ is a P-ideal if for every sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{I}$ there is a single set A_∞ in $\mathcal{I}$ such that $A_n \subseteq^* A_\infty$ for all n, i.e., $\mathcal{I}$ is a P-ideal if the partial ordering $\langle \mathcal{I}, \subseteq^* \rangle$ is σ-directed.

Definition 1.2.2. *A map $\mu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a submeasure supported by $\mathbb{N}$ if for $A, B \subseteq \mathbb{N}$*

$$\mu(\emptyset) = 0$$

$$\mu(A) \leq \mu(A \cup B) \leq \mu(A) + \mu(B).$$

It is lower semicontinuous if for all $A \subseteq \mathbb{N}$ we have

$$\mu(A) = \lim_{n \to \infty} \mu(A \cap [1, n]).$$

For a lower semicontinuous submeasure μ let

$$Exh(\mu) = \{A \subseteq \mathbb{N} \ : \ \lim_{n} \mu(A \setminus [1, n]) = 0\},$$

$$Fin(\mu) = \{A \subseteq \mathbb{N} \ : \ \mu(A) < \infty\}.$$

It is not hard to see that $Exh(\mu)$ is an $F_{\sigma\delta}$ P-ideal on $\mathbb{N}$ (see [11]), and the following theorem shows that all analytic P-ideals are of this form.

Theorem 1.2.3 (Mazur, Solecki)**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then*

(a) $\mathcal{I}$ is an F_σ-ideal if and only if $\mathcal{I} = Fin(\mu)$ for some lower semicontinuous submeasure μ.

(b) $\mathcal{I}$ is an analytic P-ideal if and only if $\mathcal{I} = Exh(\mu)$ for some lower semicontinuous submeasure μ.

(c) $\mathcal{I}$ is a F_σ P-ideal if and only if $\mathcal{I} = Exh(\mu) = Fin(\mu)$ for some lower semicontinuous submeasure μ.

For the proof of (a) see [42, Lemma 1.2] and for (b) and (c) see [52, Theorem 3.1].

Definition 1.2.4. *An ideal $\mathcal{I}$ is* layered *if there is $f : P(\mathbb{N}) \to [0, \infty]$ such that*

1. $f(A) \leq f(B)$ if $A \subseteq B$,

2. $\mathcal{I} = \{A : \ f(A) < \infty\}$,

3. $f(A) = \infty$ implies $f(A) = \sup_{B \subseteq A} f(B)$.

Layered ideals were introduced in [14], where in particular the following is proved.

Lemma 1.2.5. *[14, Proposition 6.6]*

1. Every F_σ ideal is layered.

2. If $\mathcal{J}$ is a layered ideal and $\mathcal{I}$ is an arbitrary ideal on $\mathbb{N}$, then $\mathcal{J} \times \mathcal{I}$ is layered.

Proof. By a theorem of K. Mazur ([42]) for every F_σ ideal $\mathcal{I}$ there is a lower semi-continuous submeasure μ such that

$$\mathcal{I} = Fin(\mu) = \{A \subseteq \mathbb{N} \ : \ \mu(A) < \infty\}$$

and $f = \mu$ satisfies all the conditions above. For (2) let $f_{\mathcal{J}}$ be a map witnessing that $\mathcal{J}$ is layered, and define f by

$$f(A) = f_{\mathcal{J}}(\{n : A_n \notin \mathcal{I}\})$$

for $A \subseteq \mathbb{N}^2$. It is not hard to see that f satisfies the conditions (1) - (3) stated above. □

1.3 Descriptive set theory

Descriptive set theory is the study of "definable sets" in a *Polish* (i.e., separable completely metrizable) spaces. In this theory, sets are classified in hierarchies according to the complexity of their definitions. Descriptive set theory has been one of the main areas of research in set theory and its concepts and results are being used in diverse fields of mathematics, such as mathematical logic, combinatorics, topology, real harmonic analysis, measure theory, operator algebras, etc. A standard reference for descriptive set theory is [39].

In this short section we review some of the basic concepts of descriptive set theory. As usual we identify each subset of the natural numbers with its characteristic function and turn $P(\mathbb{N})$ into a Polish space, by equipping it with the Cantor set topology. By *reals* we may refer to the elements of any of the sets $\mathbb{R}, 2^\omega, \omega^\omega$, or $P(\mathbb{N})$. Recall that a set of reals is *meager* (or it is of *first category*) if it can be covered by a countable union of nowhere dense sets, and a set of reals is *comeager* if its complement is meager.

For a finite $s \subseteq \mathbb{N}$ we use $[s]$ to denote the basic clopen subset $\{t \subseteq \mathbb{N} : s \subseteq t\}$ of $P(\mathbb{N})$. The following is a well-known characterization of (co)meager subsets of $P(\mathbb{N})$ and will be used extensively throughout this thesis.

Lemma 1.3.1 (Jalali-Naini, Talagrand)**.** *A subset $\mathcal{X}$ of $P(\mathbb{N})$ is comeager if and only if there is a sequence $0 = n_0 < n_1 < \dots$ of natural numbers and $s_i \subseteq [n_i, n_{i+1})$ for $i \geq 0$ such that for any $A \subseteq \mathbb{N}$ if $A \cap [n_i, n_{i+1}) \supseteq s_i$ for infinitely many i then $A \in \mathcal{X}$.*

Proof. Assume $\mathcal{X}$ is meager and it is covered by a countable union $\bigcup_{n=0}^{\infty} F_n$ of closed nowhere dense sets. We can assume $F_n \subset F_{n+1}$ for every n. Recursively find $0 = n_0 < n_1 < n_2 < \dots$ and $s_i \subseteq [n_i, n_{i+1})$ such that

$$[s_m] \cap F_m = \emptyset$$

for all m. This is possible because each F_i is nowhere dense. Then any set A which contains s_i for infinitely many i avoids all F_m's and hence belongs to the complement of $\mathcal{X}$. On the other hand if there are such sequences $\{n_i\}$ and $s_i \subseteq [n_i, n_{i+1})$, then for all i the set

$$U_i = \{A : \ s_n \subseteq A \text{ for some } n \geq i\}$$

is a dense open subset of $P(\mathbb{N})$. The dense G_δ set $\bigcap_{i=0}^{\infty} U_i$ is exactly the set of all subsets of $\mathbb{N}$ which include infinitely many of s_i's, therefore $\bigcap_{i=0}^{\infty} U_i \subseteq \mathcal{X}$. $\square$

This implies that for any meager subset $\mathcal{Y}$ of $P(\mathbb{N})$ there is a partition of $\mathbb{N}$ into finite intervals $[n_i, n_{i+1})$ such that for any infinite $y \subseteq \mathbb{N}$ the set $\bigsqcup_{i\in y}[n_i, n_{i+1})$ does not belong to $\mathcal{Y}$. We say such a partition *witnesses* the meagerness of $\mathcal{Y}$.

Let X be a Polish space. A subset of X has the *property of Baire* (or it is Baire-measurable) if its symmetric difference with some open set is meager. It is easy to see that every Baire measurable function is continuous on a G_δ set. *Borel subsets* of X are those sets which can be obtained from the basic open sets by repeated applications of countable union, countable intersection and taking complements. This class of sets is closed under continuous preimages and continuous one-to-one images, but not under arbitrary continuous images. The class of continuous images of Borel sets is called *analytic sets*, denoted by Σ^1_1, and their complements, *coanalytic sets*, is denoted by Π^1_1. Every analytic set $A \subset X$ is a projection of a closed subset C of $X \times \omega^\omega$. Analytic sets (as well as coanalytic sets) share some of the regularity properties of Borel sets such as property of Baire and measurability with respect to Borel measures.

For a *tree* $T \subseteq \omega^{<\omega}$ the set

$$[T] = \{t \in \omega^\omega : \forall n \in \omega \; t \restriction_n \in T\}$$

is the set of all *braches* of T. For every analytic set $A \subset \omega^\omega$ there is a tree $T \subset (\omega \times \omega)^{<\omega}$ which projects into A, i.e., $a \in A$ if and only if there exists $b \in \omega^\omega$ and $(a, b) \in [T]$.

Uniformization theorems from descriptive set theory usually play a crucial roll in the rigidity questions. The following is a well-known *uniformization* theorem ([39, Theorem 18.1]).

Theorem 1.3.2 (Jankov, von Neumann)**.** *If X and Y are polish spaces and $A \subseteq X \times Y$ is analytic, then there is a Baire measurable function $f : X \to Y$ such that for all $x \in X$, if $(x, y) \in A$ for some y then $(x, f(x)) \in A$.*

A function f as above *uniformizes* A. The following well-known absoluteness theorems will also be used throughout this thesis.

Theorem 1.3.3. *(Analytic absoluteness) Suppose that M is a transitive model of ZFC, $\bar{x} \in M \cap \omega^\omega$ is a sequence of parameters, and ϕ is a Σ^1_1-formula with free variables. Then $\phi(\bar{x})$ holds if and only if $M \models \phi(\bar{x})$.*

Theorem 1.3.4. *(Shoenfield absoluteness) Suppose that M is a transitive model of*

ZFC containing all countable ordinals, $\bar{x} \in M \cap \omega^\omega$ *is a sequence of parameters, and* ϕ *is a* Σ^1_2*-formula with free variables. Then* $\phi(\bar{x})$ *holds if and only if* $M \models \phi(\bar{x})$*.*

2 Groupwise Silver forcing and forcings of the form $\mathbb{P}_{\mathcal{I}}$

2.1 Forcing with ideals

Some standard reference books for the forcing terminology are [35], [1] and [50]. For the convenience of the reader we start with reviewing some of the general facts about forcing with ideals. These forcings are well-studied by J. Zapletal in [58] and [59].

Suppose X is a Polish space and $\mathcal{I}$ is a σ-ideal on X. We will always assume that X is uncountable and $\mathcal{I}$ contains all singletons. The symbol $\mathbb{P}_{\mathcal{I}} = B(X)/\mathcal{I}$ denotes the partial order of all $\mathcal{I}$-positive Borel subsets of X ordered by inclusion. Throughout this section M will denote a countable elementary submodel of some large structure (typically H_θ for some large ordinal θ). For a poset $\mathbb{P}$, an M-generic filter $G \subset \mathbb{P}$ is a filter on $\mathbb{P} \cap M$ which intersects every dense subset of $\mathbb{P}$ which belongs to M. The generic extension of M by G is denoted by $M[G]$.

Proposition 2.1.1. *Suppose G is an M-generic filter. The poset $\mathbb{P}_{\mathcal{I}}$ adds an element $\dot{r}_{gen}$ of the Polish space X such that for every Borel set $B \subseteq X$, coded in the ground model, $B \in G$ if and only if $\dot{r}_{gen} \in B$.*

Proof. The closed sets contained in the generic filter form a collection closed under intersection which contains sets of arbitrarily small diameter. A completeness argument shows that such a collection has a nonempty intersection containing a single point, and $\dot{r}_{gen}$ is a name for the single point in the intersection. By induction on the complexity of the Borel set B prove that $B \Vdash \dot{r}_{gen} \in \check{B}$. This follows from the definition of $\dot{r}_{gen}$ for closed sets. Suppose $B = \bigcup_n C_n$ and each set C_n forces that $\dot{r}_{gen} \in \check{C}_n$. Since $\mathcal{I}$ is a σ-ideal, whenever $D \subset B$ is a Borel $\mathcal{I}$-positive set then for some n, $D \cap C_n$ is $\mathcal{I}$-positive and $D \cap C_n \Vdash \dot{r}_{gen} \in \check{C}_n$. Therefore by the genericity $B \Vdash \dot{r}_{gen} \in \check{B}$. Now assume $B = \bigcap_n C_n$ such that C_n forces that $\dot{r}_{gen} \in \check{C}_n$. Since $B \subseteq C_n$ for every n we have $B \Vdash \dot{r}_{gen} \in \check{C}_n$ and hence $B \Vdash \dot{r}_{gen} \in \check{B}$. On the other hand it is easy to see that $C \Vdash \dot{r}_{gen} \in \check{B}$ if and only if $C \setminus B \in \mathcal{I}$. □

Definition 2.1.2. *A point $x \in X$ is called M-generic if the collection $\{B \in \mathbb{P}_{\mathcal{I}} \cap M : x \in B\}$ is an M-generic filter.*

Proposition 2.1.3. *Suppose that $\mathcal{I}$ is a σ-ideal on a Polish space X. The following are equivalent.*

1. *The forcing $\mathbb{P}_{\mathcal{I}}$ is proper.*

2. *For every countable elementary submodel M of a large enough structure and every condition $B \in M \cap \mathbb{P}_{\mathcal{I}}$ the set $\{x \in B : x \text{ is } M\text{-generic}\}$ is Borel $\mathcal{I}$-positive.*

Proof. Recall that a forcing notion $\mathbb{P}$ is proper if for every countable elementary submodel M of a large enough structure containing the poset $\mathbb{P}$ and for every condition $p \in \mathbb{P} \cap M$ there exists a *master condition* q below p, i.e., q forces that $\dot{G} \cap \check{M}$ is an M-generic filter, where $\dot{G}$ is the canonical name for the generic filter. Note that for every $B \in M \cap \mathbb{P}_{\mathcal{I}}$ the set $C = \{x \in B : x \text{ is } M\text{-generic}\}$ is Borel, since it is equal to $B \cap \bigcap\{\bigcup O \cap M : O \in M \text{ is an open dense subset of } \mathbb{P}_{\mathcal{I}}\}$. If the set C is $\mathcal{I}$-positive, then it is a condition in the poset $\mathbb{P}_{\mathcal{I}}$, therefore by Proposition 2.1.1, C forces that $\dot{r}_{gen} \in \dot{C}$. Since the statement "$x$ is M-generic" is Π^1_1, by analytic absoluteness C forces $\dot{r}_{gen}$ is $\check{M}$-generic as required.

For the other direction assume $\mathbb{P}_{\mathcal{I}}$ is proper and suppose that there are $M \preceq H_\theta$ for large enough θ containing $\mathbb{P}_{\mathcal{I}}$, and $B \in \mathbb{P}_{\mathcal{I}} \cap M$ such that $C = \{x \in B : x \text{ is } M\text{-generic}\}$ belongs $\mathcal{I}$. Therefore $\mathbb{P}_{\mathcal{I}} \Vdash \dot{r}_{gen} \notin \dot{C}$, by analytic absoluteness $B \Vdash \dot{r}_{gen}$ is not an $\check{M}$-generic condition, so there is no master condition extending B. $\square$

The simplest property of the proper forcings of the form $\mathbb{P}_{\mathcal{I}}$ is that every real in the extension is the image of the canonical generic real under some ground model coded Borel function.

Lemma 2.1.4. *Let $\mathcal{I}$ be a σ-ideal on reals. Suppose $\mathbb{P}_\mathcal{I}$ is proper and $\dot{x}$ is a $\mathbb{P}_\mathcal{I}$-name for a real. Then for every $B \in \mathbb{P}_\mathcal{I}$ there is a Borel $\mathcal{I}$-positive set $C \subseteq B$ and a ground model coded Borel function $f : C \to 2^\omega$ such that $C \Vdash f(\dot{r}_{gen}) = \dot{x}$.*

Proof. Suppose $\dot{x}$ is a $\mathbb{P}_\mathcal{I}$-name for an element of 2^ω and $B \in \mathbb{P}_\mathcal{I}$. Let M be a countable elementary submodel of a large enough structure containing $\mathbb{P}_\mathcal{I}$ and all the necessary information. Let $C \subseteq B$ be the set of all M-generic reals in B. Define a ground model function $f : C \to 2^\omega$ as follows: for each $n \in \omega$ and $i \in \{0, 1\}$ let

$$\mathcal{D}_n^i = \{D \in \mathbb{P}_\mathcal{I} \cap M : D \Vdash \dot{x}(\check{n}) = \check{i}\}$$

and clearly $\mathcal{D}_n = \mathcal{D}_n^0 \sqcup \mathcal{D}_n^1$ is dense in $\mathbb{P}_\mathcal{I}$. For $r \in C$ since r is M-generic the map f defined by $f(r)(n) = i$ if and only if there is $B \in \mathcal{D}_n^i$ such that $r \in B$, is well-defined and Borel, and $C \Vdash f(\dot{r}_{gen}) = \dot{x}$. $\square$

Lemma 2.1.5 (Uniformization)**.** *Suppose I is a σ-ideal on reals and $\mathbb{P}_\mathcal{I}$ is proper. If B is a Borel $\mathcal{I}$-positive set and $A \subseteq \mathbb{R} \times \mathbb{R}$ is an analytic relation such that for every real $r \in B$ there is a real s such that $(r, s) \in A$, then there is an $\mathcal{I}$-positive Borel set $C \subseteq B$ and a Borel function $f : C \to \mathbb{R}$ such that for every $r \in C$ we have $(r, f(r)) \in A$.*

Proof. Note that by Shoenfield absoluteness $B \Vdash \exists \dot{s} \in \mathbb{R} \ \ (\dot{r}_{gen}, \dot{s}) \in \dot{A}$. Let M be a large enough countable elementary model, by Lemma 2.1.4 find Borel $\mathcal{I}$-positive set $C \subseteq B$ and a ground model Borel function $f : C \to \mathbb{R}$ such that C forces

$\check{f}(\dot{r}_{gen}) = \dot{s}$. Now for every $r \in C$ by the analytic absoluteness for the model $M[r]$ we have $(r, f(r)) \in A$. □

Definition 2.1.6. *Let $\mathcal{I}$ be a σ-ideal on a Polish space X. The forcing notion $\mathbb{P}_\mathcal{I}$ is said to have continuous reading of names if for every $B \in \mathbb{P}_\mathcal{I}$ and a $\mathbb{P}_\mathcal{I}$-name $\dot{x}$ for an element of $X^{V[G]}$, there is a Borel $\mathcal{I}$-positive set $C \subseteq B$ and a ground model coded continuous function $f : C \to X$ such that $C \Vdash \check{f}(\dot{r}_{gen}) = \dot{x}$.*

Recall that a forcing notion $\mathbb{P}$ is ω^ω-*bounding* if for every $p \in \mathbb{P}$ and a $\mathbb{P}$-name for a function $\dot{f} : \omega \to \omega$ there are $q \leq p$ and $g \in \omega^\omega \cap V$ such that $q \Vdash \dot{f}(\check{n}) \leq \check{g}(\check{n}) \ \ \forall n$.

Lemma 2.1.7. *Let $\mathcal{I}$ be a σ-ideal on a Polish space X and $\mathbb{P}_\mathcal{I}$ is a proper forcing. Then following are equivalent.*

1. *$\mathbb{P}_\mathcal{I}$ is ω^ω-bounding.*

2. *Compact sets are dense in $\mathbb{P}_\mathcal{I}$ and $\mathbb{P}_\mathcal{I}$ has continuous reading of names.*

Proof. It is enough to prove the case that $\mathcal{I}$ is an ideal on the Baire space ω^ω. Since for any other underlying uncountable Polish space we can choose a Borel bijection $f : \omega^\omega \to X$ and work with the ideal $\mathcal{J} = \{A \subset \omega^\omega : f[A] \in \mathcal{I}\}$. Note that $\mathbb{P}_\mathcal{I}$ and $\mathbb{P}_\mathcal{J}$ are isomorphic via the map $f : A \to f[A]$, since the forward Borel one-to-one images of Borel sets are Borel. For every $\mathcal{I}$-positive Borel set $B \subseteq X$ there is a $\mathcal{J}$-positive and compact subset $C \subseteq f^{-1}[B]$ such that $f \restriction C$ is continuous. Now clearly $f[C]$ is an $\mathcal{I}$-positive and compact.

To see (2) implies (1) let $B \in \mathbb{P}_{\mathcal{I}}$ and $\dot{x}$ be a $\mathbb{P}_{\mathcal{I}}$-name for an element of ω^ω. By our assumption there are a Borel $\mathcal{I}$-positive set $C \subseteq B$ and a continuous function $g : C \to \omega^\omega$ such that $C \Vdash \check{f}(\dot{r}_{gen}) = \dot{x}$. We can assume C is compact and since $f[C]$ is also compact and therefore bounded in ω^ω, $\mathbb{P}_{\mathcal{I}}$ is ω^ω bounding.

For the converse first we show that for every $\mathcal{I}$-positive Borel set B we can find B_1 an $\mathcal{I}$-positive G_δ subset of B_0. Since B is analytic find a tree $T \subset (\omega \times \omega)^{<\omega}$ which projects into B. The set $B \Vdash \dot{r}_{gen} \in \dot{B}$, hence by analytic absoluteness there is a $\mathbb{P}_{\mathcal{I}}$-name $\dot{a}$ for an element of ω^ω such that $B \Vdash (\dot{r}_{gen}, \dot{a}) \in [T]$. Since the forcing $\mathbb{P}_{\mathcal{I}}$ is ω^ω-bounding there is a condition $C \subseteq B$ and a function $g \in \omega^\omega \cap V$ such that $C \Vdash \dot{a}(\check{n}) \leq \check{g}(\check{n}) \ \ \forall n$. Define $B_1 = \{r \in B : \text{ there is } f \leq g \text{ such that } (r, f) \in [T]\}$. This set is a closed subset of B since

$$B_1 = \bigcap_{n \in \omega} \bigcup_{s \leq g} \{[r] : (r, s) \in T \cap (\omega \times \omega)^n\}$$

and C forces that $\dot{r}_{gen} \in B_1$, therefore it is $\mathcal{I}$-positive.

Now we find a compact $\mathcal{I}$-positive subset $C \subseteq B_1$. The poset $\mathbb{P}_{\mathcal{I}}$ is ω^ω-bounding, therefore there is a condition $C \subseteq B_1$ and a function $g \in \omega^\omega \cap V$ such that $C \Vdash \dot{r}_{gen} \leq \check{g}$ pointwise. The set $B_2 = B_1 \cap \{r \in \omega^\omega : r \leq g \text{ pointwise}\}$ is compact and $\mathcal{I}$-positive since C forces $\dot{r}_{gen}$ is in B_2.

For the continuous reading of names by Lemma 2.1.4 it is enough to show that for every Borel $\mathcal{I}$-positive set B and every Borel function $f : B \to \omega^\omega$, there is a Borel positive subset $C \subseteq B$ such that $f \restriction C$ is continuous. Suppose B and f as above are given. For every $n \in \omega$ let A_n be a maximal antichain consisting of compact sets deciding the value of $\dot{f}(\dot{r}_{gen})$. Let M be a countable elementary submodel of a large enough structure containing these antichains and B. Let $D \subseteq B$ be a compact $\mathcal{I}$ positive consisting of only M-generic elements. Note that D is compatible with only countably many elements of each A_n, namely with those with belong to M. Since $\mathcal{P}_{\mathcal{I}}$ is ω^ω-bounding we can find an $\mathcal{I}$-positive Borel set $E \subseteq D$ such that for every n there are only finitely many elements of A_n compatible with E. Let $X_n \subset A_n$ be the finite set consisting of these elements and let $C = E \cap \bigcap_n \bigcup X_n$. The set C is clearly compact and $\mathcal{I}$-positive and for every basic open set $O_{n,m} = \{g \in \omega^\omega : g(n) = m\}$. The f-preimage

$$f^{-1}(O_{n,m}) = C \setminus \bigcup \{F \in X_n : \ F \Vdash \dot{f}(\dot{r}_{gen})(\check{n}) \neq m\}.$$

is relatively open in C.

$\square$

Note that it is essential in above that the conditions are compact sets since the Cohen forcing is proper and of the form $\mathbb{P}_{\mathcal{I}}$, but it is not ω^ω-bounding, yet it does have the continuous reading of names.

2.2 The countable support iteration

In Zapletal's theory the countable support iteration of forcings of the form $\mathbb{P}_{\mathcal{I}}$ has been studied for reasonably definable ideals called *iterable* [58, Definition 3.1.1]. We will avoid repeating the definition of iterable ideals here, for technical reasons. It is enough to know that all the ideals that ever came up in practice within the realm of definable proper real forcings are iterable. For an ordinal κ, let $\mathbb{P}_\kappa$ be the countable support iteration of the posets $\mathbb{P}_{\mathcal{I}_\alpha}$ of length κ for σ-ideals $\mathcal{I}_\alpha,\ \alpha < \kappa$. Let $\dot{r}_{gen}$ be the canonical $\mathbb{P}_\kappa$-name for the generic sequence of reals.

Definition 2.2.1. *Let α be a countable ordinal and $\{\mathcal{I}_\xi\}_{\xi<\alpha}$ be a sequence of σ-ideals on the reals. An $\{\mathcal{I}_\xi\}_{\xi<\alpha}$-perfect set is a set $B \subseteq \mathbb{R}^\alpha$ such that*

- *for every ordinal $\beta \in \alpha$ and every sequence $\bar{s} \in B \restriction \beta$ the set $\{r \in \mathbb{R} : \bar{s}^\frown r \in B \restriction_{\beta+1}\}$ is $\mathcal{I}_\beta$-positive*
- *for every increasing sequence β_n, $n \in \omega$ of ordinals below α and every inclusion increasing sequence $\bar{s}_n \in B \restriction_{\beta_n}$, we have $\bigcup_n \bar{s}_n \in B \restriction_{\bigcup_n \beta_n}$.*

Define the poset $\mathbb{L}_\kappa$ to consist of all Borel $\{\mathcal{I}_\xi\}_{\xi\in X}$-perfect (identify X with its ordertype) subsets B of $\mathbb{R}^X$ for all countable subsets X of κ. The set X is called the *domain* of B, $dom(B)$. The ordering is defined by $C \leq B$ if and only if $dom(B) \subseteq dom(C)$ and for every $\bar{s} \in C$, $\bar{s} \restriction_{dom(B)} \in B$. For a generic filter G, let

$\dot{s}_{gen}$ be the $\mathbb{L}_\kappa$-name for a κ-sequence of reals defined by $\dot{s}_{gen}(\alpha)(n) = m$ if and only if the set $B_{\alpha,n,m} = \{\langle \alpha, r\rangle : r \in \mathbb{R},\ r(n) = m\} \subset \mathbb{R}^{\{\alpha\}}$ belongs to G.

Lemma 2.2.2. *The posets $\mathbb{P}_\kappa$ and $\mathbb{L}_\kappa$ are forcing isometric.*

Proof. To see that the following map $\pi : \mathbb{L}_\kappa \to \mathbb{P}_\kappa$ is a dense embedding refer to [58, Corollary 3.1.6]. Suppose X is a countable subset of κ and $B \subseteq \mathbb{R}^X$ is Borel $\{I_\xi\}_{\xi \in X}$-perfect set, let $\pi(B)$ be the canonical condition $p \in \mathbb{P}_\kappa$ such that $dom(p) = dom(B) = X$ and for every $\alpha \in X$ we have $p \restriction_\alpha \Vdash p(\alpha) = \{r \in \mathbb{R} : (\bar{s}_{gen} \restriction_{dom(p)\cap\alpha})^\frown r \in \dot{B} \restriction_{\alpha+1}\}$. Note that perfectness of B implies that each $p(\alpha)$ is $\mathcal{I}_\alpha$-positive. $\square$

The following definition, due to J. Zapletal, is a generalization of the classical *Fubini product* of two ideals, and it computes the ideals that arise in the iteration process in the theory of the countable support iteration.

Definition 2.2.3. *For a countable ordinal α and σ-ideals $\{\mathcal{I}_\xi : \xi \in \alpha\}$ on the reals, the Fubini product, $\prod_{\xi\in\alpha} \mathcal{I}_\xi$, is the ideal on $\mathbb{R}^\alpha$ defined as the collection of all sets $A \subseteq \mathbb{R}^\alpha$ for which the player I has a winning strategy in the game $G(A)$ as follows: At stage $\beta \in \alpha$ player I plays a set $B_\beta \in \mathcal{I}_\beta$ and player II produces a real $r_\beta \in \mathbb{R} \setminus B_\beta$. Player II wins the game $G(A)$ if the sequence $\{r_\beta : \beta \in \alpha\}$ belongs to the set A.*

It is easy to see that $\prod_{\xi\in\alpha} \mathcal{I}_\xi$ is a σ-ideal on $\mathbb{R}^\alpha$ since player I can always combine countably many of his winning strategies into one. If $\beta \in \alpha$ are countable ordinals then $\prod_{\xi\in\alpha} \mathcal{I}_\xi$ is naturally isomorphic to the Fubini product of $\prod_{\xi\in\beta} \mathcal{I}_\xi$ and $\prod_{\beta\leq\xi<\alpha} \mathcal{I}_\xi$, that is a set $C \subset \mathbb{R}^\alpha$ if and only if the set

$$\{\bar{s} \in \mathbb{R}^\beta : \{\bar{t} \in \mathbb{R}^{\alpha-\beta} : \bar{s}^\frown\bar{t}\} \notin \prod_{\beta\leq\xi<\alpha} \mathcal{I}_\xi\}$$

is $\prod_{\xi\in\beta} \mathcal{I}_\xi$. It is not difficult to see that for every set $A \in \mathbb{R}^\alpha$, player II has a winning strategy in the game $G(A)$ if and only if the set A has a $\prod_{\xi\in\alpha} \mathcal{I}_\xi$-perfect subset. In the presence of large cardinals the game $G(A)$ is always determined for iterable ideals. However without large cardinals we need some additional definability assumptions on the ideals to guarantee that the game $G(A)$ is determined, see [58] section 3.3.

Recall that for Polish spaces X, Y and $A \subseteq X \times Y$, for any $x \in X$ the vertical section of A at x is the set $A_x = \{y \in Y : (x, y) \in A\}$.

Definition 2.2.4. *A σ-ideal $\mathcal{I}$ on a polish space X is Π^1_1 on Σ^1_1 if for every Σ^1_1 set $B \subseteq 2^{\mathbb{N}} \times X$ the set $\{x \in 2^{\mathbb{N}} : B_x \in \mathcal{I}\}$ is Π^1_1.*

The following is due to J. Zapletal and V. Kanovei.

Theorem 2.2.5. *Suppose α is a countable ordinal and $\{\mathcal{I}_\xi : \xi < \alpha\}$ is a sequence of Π^1_1 on Σ^1_1 σ-ideals on the reals. The Borel $\{\mathcal{I}_\xi\}_{\xi\in\alpha}$-perfect sets are dense in the poset $\mathcal{B}(\mathbb{R}^\alpha)/\prod_{\xi\in\alpha} \mathcal{I}_\xi$.*

Proof. The proof is similar to [58, Lemma 3.3.1], where it is stated for the case $\mathcal{I}_\xi = \mathcal{I}$ for all $\xi < \alpha$. $\square$

Corollary 2.2.6. *Suppose α is a countable ordinal and $\{\mathcal{I}_\xi : \xi < \alpha\}$ is a sequence of Π^1_1 on Σ^1_1 σ-ideals on the reals. The poset $\mathcal{B}(\mathbb{R}^\alpha)/\prod_{\xi\in\alpha}\mathcal{I}_\xi$ is forcing equivalent to the countable support iteration of the ground model forcings $\mathbb{P}_{\mathcal{I}_\xi}$, $\xi \leq \alpha$ of length α.*

A forcing notion $\mathbb{P}$ is called to be *Suslin* if its underlying set is an analytic set of reals and both $\leq$ and $\perp$ are analytic relations. Some commonly used ideals fail to be Π^1_1 on Σ^1_1, e.g. a σ-ideal $\mathcal{I}$ for which the forcing $\mathbb{P}_\mathcal{I}$ is proper and adds a dominating real is not Π^1_1 on Σ^1_1. However the σ-ideals corresponding to the Silver forcing, random forcing and many other natural forcings are Π^1_1 on Σ^1_1 . In fact for a σ-ideal $\mathcal{I}$ on $\mathbb{R}$ if the poset $\mathbb{P}_\mathcal{I}$ consists of compact sets and is Suslin, proper with continuous reading of names, then $\mathcal{I}$ is Π^1_1 on Σ^1_1 (see [58], Appendix C).

We will occasionally use the following property of countable support forcing iterations, which was defined in [23], to prove our main theorem.

Definition 2.2.7. *A countable support forcing iteration $\mathbb{P}_\kappa = \{\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa\}$ such that each $\dot{\mathbb{Q}}_\eta$ is a $\mathbb{P}_\eta$-name for a ground-model forcing notion which adds a generic real $\dot{g}_\eta$, has continuous reading of names if for every $\mathbb{P}_\kappa$-name $\dot{x}$ for a new real the set of conditions p such that there exist a countable $S \subset \kappa$, a compact*

$K \subset \mathbb{R}^S$, and a continuous $h : K \to \mathbb{R}$ such that

$$p \Vdash \langle \dot{g}_\xi : \ \xi \in S \rangle \in K \text{ and } \dot{x} = h(\langle \dot{g}_\xi : \ \xi \in S \rangle)$$

is dense.

Proposition 2.2.8. *If $\mathbb{P}_\kappa = \{\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa\}$ is a countable support iteration of ground model forcings such that each $\dot{\mathbb{Q}}_\eta$ is a $\mathbb{P}_\eta$-name for a partial order $\mathbb{P}_\mathcal{I}$ which consists of compact sets as conditions and is Suslin, proper and ω^ω-bounding, then $\mathbb{P}_\kappa$ has the continuous reading of names.*

Proof. Suppose $\mathbb{Q}_\xi = \mathbb{P}_{\mathcal{I}_\xi}$ and $p \in \mathbb{P}_\kappa$ forces that $\dot{x}$ a $\mathbb{P}_\kappa$-name for a real. Let S be the support of p and $\mathcal{I} = \prod_{\xi \in S} \mathcal{I}_\xi$. By our assumptions each $\mathcal{I}_\xi$ is Π^1_1 on Σ^1_1, and therefore $\mathbb{P}_\kappa$ is forcing equivalent to $\mathbb{P}_\mathcal{I} = \mathcal{B}(\mathbb{R}^\alpha)/\mathcal{I}$. We can assume that $\dot{x}$ is a $\mathbb{P}_\mathcal{I}$-name and since $\mathbb{P}_\mathcal{I}$ is proper, ω^ω-bounding and contains only compact sets by the Zapletal's lemma (Lemma 2.1.7) there are $q \leq p$ and a continuous function $h : q \mapsto \mathbb{R}$ such that $q \Vdash h(\langle \dot{g}_\xi : \ \xi \in S \rangle) = \dot{x}$. □

The forcing used in chapter 3 is a countable support iteration of the forcings of the form $\mathbb{P}_\mathcal{I}$. Let $\mathbb{P}_\kappa = \{\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa\}$ be such a forcing of length κ. In Lemma 2.2.10 we show that assuming MA, any Σ^1_2 set in the generic extension by $\mathbb{P}_\kappa$ can be uniformized by a Baire measurable map in the ground model. In order to prove this we first need the following lemma. It is proved in [59] but we include the proof here for the convenience of the reader.

Lemma 2.2.9. *Suppose $\mathcal{I}$ is a σ-ideal on a Polish space X such that $\mathbb{P}_{\mathcal{I}}$ is proper. Let Y be a Polish space and $p \in \mathbb{P}_{\mathcal{I}}$ forces that $\dot{B}$ is a Borel subset of Y. Then there is a Borel $\mathcal{I}$-positive condition $q \leq p$ and a ground model coded Borel set $D \subseteq q \times Y$ such that $q \Vdash \dot{D}_{\dot{r}_{gen}} = \dot{B}$.*

Proof. The proof is carried out by induction on the Borel rank of $\dot{B}$. Since the forcing $\mathbb{P}_{\mathcal{I}}$ preserves $\aleph_1$ by possibly strengthening the condition p we may assume that the Borel rank of $\dot{B}$ is forced to be $\leq \alpha$ for a fixed countable ordinal α. Let $\mathcal{M}$ be a countable elementary submodel of a large enough structure.

Assume $\dot{B}$ is forced to be a closed set. Fix a countable base $\mathcal{O}$ for the topology of the space Y. Since $\mathbb{P}_{\mathcal{I}}$ is proper we can find [58] a Borel I-positive set $q \leq p$ (in fact q is the set of all $\mathcal{M}$-generic reals in p) and a ground model Borel function $f : q \mapsto \mathcal{P}(\mathcal{O})$ such that $q \Vdash \check{f}(\dot{r}_{gen}) = \{O \in \mathcal{O} : \dot{B} \cap O = \emptyset\}$. Define $D = \{(x, y) \in q \times Y : y \notin \bigcup f(x)\}$. It is easy to check that D is the required Borel set. The proof for open sets is similar.

Now suppose p forces that $\dot{B} = \bigcup_n \dot{B}_n$ where $\dot{B}_n$'s are sets of lower Borel rank. Let $q = \{x \in p : x \text{ is } \mathcal{M}\text{-generic}\}$. Using the inductive assumption for each $n \in \mathbb{N}$ find a maximal antichain $A(n) \subset \mathbb{P}_{\mathcal{I}}$ below p, such that for every condition $s \in A(n)$ there is a Borel set $D(s, n) \subset s \times Y$ such that $s \Vdash \dot{D}(s, n)_{\dot{r}_{gen}} = \dot{B}(n)$. For every $n \in \mathbb{N}$ let $D(n) = \bigcup\{D(s, n) : s \in \mathcal{M} \cap A(n)\} \cap q \times Y \subset q \times Y$. The condition q forces that the generic real $\dot{r}_{gen}$ belongs to exactly one condition in the antichain $\mathcal{M} \cap A(n)$

for every n. Therefore $\dot{B}(n) = \bigcup\{\check{D}(s,n) : s \in \mathcal{M} \cap A(n)\}_{\dot{r}_{gen}} = \dot{D}(n)_{\dot{r}_{gen}}$. Now the set $D = \bigcup_n D(n)$ is clearly a Borel subset of $q \times Y$ and q forces that $\dot{B} = \dot{D}_{\dot{r}_{gen}}$.

The countable intersection case is a similar argument. $\square$

The following lemma can be ignored in proving Theorem 3.2.2 since it immediately follows from the large cardinal assumption. Nevertheless it implies that in order to get local triviality of isomorphisms or even *-homomorphisms of FDD-algebras, corollary 3.2.3, no large cardinal assumption is necessary.

Lemma 2.2.10. *Assume MA holds in the ground model and $\mathbb{P}_\kappa$ is a countable support iteration of length κ of proper forcings of the form $\mathbb{P}_{\mathcal{I}}$ with compact conditions. If $\dot{C}$ is a $\mathbb{P}_\kappa$-name for a Σ^1_2 subset of $\mathbb{R} \times \mathbb{R}$ in the extension such that for every $\dot{x} \in \mathbb{R}$ the vertical section $\dot{C}_{\dot{x}}$ is non-empty, then there are $q \in \mathbb{P}_\kappa$ and a Baire-measurable map $h : \mathbb{R} \mapsto \mathbb{R}$ such that for every $\mathbb{P}_\kappa$-name $\dot{x}$ for a real*

$$q \Vdash (\dot{x}, \check{h}(\dot{x})) \in \dot{C}.$$

Proof. Let $\mathcal{I}_\xi$ be the σ-ideal associated with $\dot{\mathbb{Q}}_\xi$. Since Σ^1_2 sets are projections of Π^1_1 sets and MA implies that all Σ^1_2 sets have the property of Baire, it is enough to uniformize Π^1_1 sets. Assume some $p \in \mathbb{P}_\kappa$ forces that $\dot{C}$ is a Π^1_1 subset of $\mathbb{R} \times \mathbb{R}$. There is a $\mathbb{P}_\kappa$-name $\dot{B}$ for a Borel subset of $\mathbb{R}^3$ such that $p \Vdash \mathbb{R}^2 - pr_{\{1,2\}}(\dot{B}) = \dot{C}$ where $pr_{\{1,2\}}$ is the projection on the first and second coordinates of $\mathbb{R}^3$. Let the countable set $S \subset \kappa$ denote the support of p and

$$\mathbb{P}_S = \{\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \in S, \eta \in S\}$$

and let $\mathcal{I}^S = \prod_{\xi \in S} \mathcal{I}_\xi$. Since these forcings are proper Suslin and ω^ω-bounding [23, Lemma 4.3] we have $p \Vdash_{\mathbb{P}_S} \mathbb{R}^2 - pr_{\{1,2\}}(\dot{B}) = \dot{C}$.

Let α be the order-type of S. By forcing equivalence of $\mathbb{P}_S$ and $\mathbb{P}_{\mathcal{I}^S} = \mathcal{B}(\mathbb{R}^S)/\mathcal{I}^S$ and for simplicity assume $p \in \mathbb{P}_{\mathcal{I}^S}$. Since $\mathbb{P}_{\mathcal{I}^S}$ is proper, by Lemma 2.2.9, there is a ground model Borel set $D \subseteq \mathbb{R}^\alpha \times \mathbb{R}^3$ and $q \leq p$ such that $q \Vdash \dot{B} = \dot{D}_{\dot{r}_{gen}}$ where $\dot{r}_{gen}$ is the canonical $\mathbb{P}_{\mathcal{I}^S}$-name for the generic real in $\mathbb{R}^\alpha$. Therefore

$$q \Vdash \mathbb{R}^2 - pr_{\{\alpha+1,\alpha+2\}}(\dot{D}_{\dot{r}_{gen}}) = \dot{C}. \tag{2.1}$$

Now since the set $E = \mathbb{R}^{\alpha+2} - pr_{\{1,\ldots,\alpha+2\}}(D)$ is Π^1_1, by Kondô's uniformization theorem, E has a Π^1_1 and hence a Baire-measurable uniformization $g : pr_{\{1,\ldots,\alpha+1\}}(E) \mapsto \mathbb{R}$.

Let $\mathcal{M}$ be an elementary submodel of some large enough structure containing $\mathcal{I}^S$ and $\mathbb{P}_\kappa$, and also let $t = \{x \in q : x \text{ is } \mathcal{M}\text{-generic}\}$. Since $\mathbb{P}_{\mathcal{I}^S}$ is proper, t is a condition in $\mathbb{P}_{\mathcal{I}^S}$. Fix $x \in t$ and note that since the sections of $\dot{C}$ are non-empty, for every $y \in \mathbb{R}$ we have

$$[pr_{\{\alpha+1,\alpha+2\}}(\dot{D}_x)]_y = \dot{C}_y \neq \emptyset$$

Therefore $t \times \mathbb{R} \subseteq dom(g)$. For every $x \in t$ and $y \in \mathbb{R}$ we have $(x, y, g(x, y)) \in E$.

Define the function $h : \mathbb{R} \mapsto \mathbb{R}$ by

$$h(y) = g(\dot{r}_{gen}, y)$$

By above and (1) we have $t \Vdash (\dot{y}, \check{h}(\dot{y})) \in \dot{C}$. $\square$

2.3 Groupwise Silver forcing and capturing partitions of $\mathbb{N}$

The (groupwise) Silver forcing is defined similar to the *infinitely equal forcing* EE [1, §7.4.C]. Let $\vec{I} = (I_n)$ be a partition of $\mathbb{N}$ into non-empty finite intervals and G_n be a finite set for each $n \in \mathbb{N}$. We denote the the set of the *reals* by $\mathbb{R} = \prod_n G_n$ endowed with the product topology. For each n define $F_n^{\vec{I}} = \prod_{i \in I_n} G_i$ and let $F^{\vec{I}} = \prod_{n \in \mathbb{N}} F_n^{\vec{I}}$. Moreover for any $X \subseteq \mathbb{N}$ let $F_X^{\vec{I}} = \prod_{n \in X} F_n^{\vec{I}}$. For a fixed partition $\vec{I}$ we usually drop the superscript $\vec{I}$.

Fix a partition $\vec{I} = (I_n)$ of the natural numbers into finite intervals. Define the groupwise Silver forcing $\mathbb{S}_F$ associated with F, to be the following forcing notion: A condition $p \in \mathbb{S}_F$ is a function from $M \subseteq \mathbb{N}$ into $\bigcup_{n=0}^{\infty} F_n$, such that $\mathbb{N} \setminus M$ is infinite and $p(n) \in F_n$. A condition p is stronger than q if p extends q. Each condition p can be identified with $[p]$, the set of all its extensions to $\mathbb{N}$, as a compact subset of F. For a generic G, $f = \bigcup\{p : p \in G\}$ is the generic real.

Theorem 2.3.1. *$\mathbb{S}_F$ is a proper and ω^ω-bounding forcing.*

Proof. Let $\mathcal{M} \prec H_\theta$ for large enough θ, be a countable transitive model of ZFC containing $\vec{I}$ and $\mathbb{S}_F$. Suppose $\{A_n : n \in \mathbb{N}\}$ is the set of all maximal antichains in $\mathcal{M}$ and $q \in \mathbb{S}_F$ is given. First we claim that there exists $p \in \mathbb{S}_F$ such that for every n the set $\{q \in A_n : q \textit{ is compatible with } p\}$ is finite. To see this let $p \leq_n q$ if and only if $q \subset p$ and the first n elements that are not in the domain of q are not in the domain of p. We build a fusion sequence $p_0 \geq_0 p_1 \geq_1 \ldots p_n \geq_n p_{n+1} \geq_{n+1} \ldots$ recursively. For the given q let $p_0 = q$ and suppose p_n is chosen. Let $B = \{k_1 \ldots k_n\}$ be the set of first n elements of $\mathbb{N} \backslash dom(p_n)$ ordered increasingly Since A_n is a maximal antichain, p_n is compatible with some $s \in A_n$. Let $p_{n+1} = p_n \cup s \restriction_{(k_n, \infty)}$. Note that p_{n+1} is compatible with only finitely many elements of A_n. Let $p = \bigcup_n p_n$ be the fusion of the above sequence. For every n the set $C_n = \{q \in A_n : q \textit{ is compatible with } p\}$ is finite and predense below p for every n. Therefore $A_n \cap \mathcal{M}$ contains C_n and is predense below p.

To see $\mathbb{S}_F$ is ω^ω-bounding assume $\dot{f}$ is a $\mathbb{S}_F$-name such that $q \Vdash \dot{f} : \mathbb{N} \to \mathbb{N}$. As above we build a fusion sequence $q = p_0 \geq_0 p_1 \geq_1 \ldots p_n \geq_n p_{n+1} \geq_{n+1} \ldots$. Assume p_n is chosen and let B be defined as above and $\{r_j : j < k\}$ be the list of all functions $r : B \to \bigcup_{i \in B} F_i$ such that $r(k_i) \in F_{k_i}$ for all $i \in B$. Successively find $p_n = p_n^0 \geq_n p_n^1 \geq_n \cdots \geq_n p_n^{k-1}$ such that:

$$p_n^j \cup r_j \Vdash \dot{f}(n) = \check{a}_n^j.$$

Let $p_{n+1} = \bigcup p_n^j$ and $D_n = \{a_n^j : j < k\}$. Now the fusion of this sequence p forces that for every n we have $f(n) \in D_n$. Define a ground model map $g : \mathbb{N} \to \mathbb{N}$

by $g(n)$ to be the largest element of D_n. Therefore p forces that $g(n) \geq f(n)$ for all n.

$\square$

The following property is the main property of $\mathbb{S}_F$ which allows us to prove that in the forcing extension used in the following chapter, the graph of isomorphisms between certain corona algebras satisfy some local definability assumptions.

Definition 2.3.2. *We say a forcing notion $\mathbb{P}$ captures $F^{\vec{I}}$ if there exists a $\mathbb{P}$-name for a real $\dot{x}$ such that for every $p \in \mathbb{P}$ there is an infinite $M \subseteq \mathbb{N}$ such that for every $a \in F^{\vec{I}}_M$ there is $q_a \leq p$ such that $q_a \Vdash \dot{x} \restriction_M = \check{a}$.*

Lemma 2.3.3. *For any partition $\vec{I}$ of $\mathbb{N}$ into finite intervals, $\mathbb{S}_{F^{\vec{I}}}$ captures $F^{\vec{I}}$.*

Proof. Suppose $\dot{x}$ is the canonical $\mathbb{S}_{F^{\vec{I}}}$-name for the generic real and $p \in \mathbb{S}_{F^{\vec{I}}}$ is given. Let M be an infinite subset of $\mathbb{N} \setminus dom(p)$ such that $\mathbb{N} \setminus (M \cup dom(p))$ is also infinite. For every $a \in F^{\vec{I}}_M$ let $q_a = p \cup a$. Since $\mathbb{N} \setminus (M \cup dom(p))$ is infinite, q_a is a condition in $\mathbb{S}_{F^{\vec{I}}}$ and $q_a \Vdash \dot{x} \restriction_M = \check{a}$. $\square$

The following lemma shows that $\mathbb{S}_F$ has continuous reading of names. To see this it is enough to notice that the groupwise Silver forcing can be viewed as a forcing with Borel $\mathcal{I}$-positive sets, $\mathbb{P}_{\mathcal{I}}$, for a σ-ideal $\mathcal{I}$, where $\mathcal{I}$ is the σ-ideal σ-generated by partial functions with cofinite domains. Since groupwise Silver forcing (as well as the random forcing, which will be used in the next chapter) is proper and conditions are compact sets, by the Zapletal's lemma (Lemma 2.1.7) the continuous reading of names is equivalent to the forcing being ω^ω-bounding. Nevertheless here we give a direct proof of the continuous reading of names for $\mathbb{S}_F$.

Lemma 2.3.4. *For every $\mathbb{S}_F$-name for a real $\dot{x}$ and $q \in \mathbb{S}_F$ there are $p \leq q$ and a continuous function $f : p \to \mathbb{R}$ such that p forces $f(\dot{r}_{gen}) = \dot{x}$, where $\dot{r}_{gen}$ is the canonical name for the generic real.*

Proof. Assume q forces that $\dot{x}$ is a $\mathbb{S}_F$-name for a subset real. By identifying each condition with the corresponding compact set we can find a fusion sequence $\{p_s : s \in \bigcup_n F_{[0,n)}\}$ such that for each $s \in F_{[0,n)}$ (here s just would be used as an index) $p_s \Vdash \dot{x} \restriction_{[0,n)} = u_s$ for some $u_s \in F_{[0,n)}$. Let $p = \bigcap_{n\in\mathbb{N}} \bigcup_{s \in F_{[0,n)}} p_s$ be the fusion. For each $y \in p$ let $b \in F$ be the branch such that $y \in p_{b\restriction_{[0,n)}}$ for each n. Define $f(y) \restriction_{[0,n)} = u_{b\restriction_{[0,n)}}$. The map f is continuous and $y \in p_{b\restriction_{[0,n)}}$ implies $f(y) \in [u_s]$ and hence $d(f(y), \dot{x}) < 2^{-n}$. Therefore $p \Vdash f(\dot{r}_{gen}) = \dot{x}$. □

3 Rigidity of reduced products of matrix algebras

3.1 The rigidity question

The rigidity question for quotients of Boolean algebras, more specifically $P(\mathbb{N})$, was considered in order to answer the following basic question: How does a change of the ideal $\mathcal{I}$ of $P(\mathbb{N})$ effect the change of its quotient $P(\mathbb{N})/\mathcal{I}$? For example the Continuum Hypothesis, via a standard back-and-forth argument, makes many of these quotients isomorphic and therefore trivializing the problem (see [38]). Under the Continuum Hypothesis the Boolean algebra $P(\mathbb{N})/Fin$ has $2^{2^{\aleph_0}}$ automorphisms ([49]). On the other hand in a ground breaking result S. Shelah ([50, §4]) proved that it is consistent that every automorphism Φ of $P(\mathbb{N})/Fin$ is trivial in a very strong sense, i.e., there are finite subsets $a, b \subset \mathbb{N}$ and a bijection $h : \mathbb{N} \setminus a \to \mathbb{N} \setminus b$ such that for every $x \subseteq \mathbb{N} \setminus a$, $\Phi(\pi(x)) = \pi(h[x])$, where π is the natural quotient map. Clearly there are only $2^{\aleph_0}$ such automorphisms. Shelah's construction uses the oracle chain condition and is quite involved. Subsequently Shelah and Steprāns ([51]) showed that the same conclusion follows from the *Proper Forcing Axiom* (PFA) and Veličković ([54]) proved the same result assuming the *Open Coloring Axiom* and the *Martin's axiom* (OCA+ $MA_{\aleph_1}$). In [15] I. Farah conjectured that the PFA implies that all isomorphisms between two quotient algebras of the form $P(\mathbb{N})/\mathcal{I}$, for a Borel ideal $\mathcal{I}$, are trivial. In a more recent work ([23]), Farah and Shelah proved that assuming the existence of a measurable cardinal, it is consistent with ZFC that for any Borel ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$, every isomorphism form $P(\mathbb{N})/\mathcal{I}$ into $P(\mathbb{N})/\mathcal{J}$ has a continuous representation.

Using the Stone duality one can reformulate the rigidity results for Boolean algebras in the category of topological spaces. In general if X and Y are Polish spaces, a continuous map $\Phi : X^* \to Y^*$ is trivial if there are a compact subset K of X and a continuous map $f : X \setminus K \to Y$ such that $\Phi = \beta f \restriction_{X^*}$, where $\beta f : \beta X \to \beta Y$ is the unique continuous extension of f. In particular under CH there are non-trivial autohomeomorphisms of the Čech-Stone remainder of the natural numbers $\mathbb{N}^*$, and PFA (or OCA+ MA) implies that all such autohomeomorphisms are trivial. By a classical result of Parovičenko the Continuum Hypothesis implies that all Čech-Stone remainders of locally compact, zero dimensional, non-compact Polish spaces are homeomorphic. Hence for example Čech-Stone remainders of ordinals ω and ω^2 are homeomorphic, assuming CH. However, in [53, Theorem

2.2.1] J. van Mill proved that ω^* and $(\omega^2)^*$ are not homeomorphic assuming that all autohomeomorphisms ω^* are trivial. Later in [8] A. Dow and K.P. Hart proved that under the Open Coloring Axiom the only Čech-Stone remainder of a locally compact, σ-compact and non-compact space which is a surjective image of ω^* is ω^* itself. The key part of their proof is the fact that $(\omega^2)^*$ is not a surjective image of ω^*. Recently Farah and Shelah ([24]) showed that PFA also implies that every autohomeomorphism of the Čech-Stone remainder, $[0,1)^*$, of $[0,1)$ is trivial.

By the Gelfand-Naimark transform (see [3], §II.2.2) each commutative C*-algebra is isometrically isomorphic to $C_0(X)$ for a locally compact Hausdorff space X. The correspondence $X \leftrightarrow C(X)$ is a contravariant category equivalence between the category of compact Hausdorff spaces and continuous maps and the category of commutative unital C*-algebras and unital *-homomorphisms. Hence each property of a locally compact Hausdorff X space can be reformulated in terms of the function algebra $C_0(X)$. Under this duality the rigidity question has an equivalent reformulation in the category of commutative C*-algebras.

Recall that for a non-unital C*-algebra $\mathcal{A}$ the corona of $\mathcal{A}$ is the non-commutative analogue of the Čech-Stone reminder of a non-compact locally compact topological space (see section 1.1). Motivated by a question of Brown-Douglas-Fillmore [4, 1.6(ii)] the rigidity question for the category of C*-algebras has been studied for various corona algebras.

Let $\tilde{\mathcal{A}}$ denote the unitization of the C*-algebra $\mathcal{A}$.

Definition 3.1.1. *Assume $\mathcal{A}$ is a C*-algebras. Each unitary $u \in \mathcal{M}(\mathcal{A})$ defines an automorphism Ad_u of $\mathcal{A}$ given by $a \to uau^*$ and such an automorphism is called a multiplier inner automorphism. An automorphism Ad_u with u a unitary element in $\tilde{\mathcal{A}}$, is called inner.*

For a separable Hilbert space H we use $\mathcal{C}(H)$ to denote the Calkin algebra over H. The following well-known theorems show that the rigidity question for the Calkin algebra is independent from ZFC.

Theorem 3.1.2 (C. Phillips, N. Weaver [48])**.** *Assuming the Continuum Hypothesis the Calkin algebra has an outer automorphism.*

Theorem 3.1.3 (I. Farah [17])**.** *Under the Open Coloring Axiom (OCA) all automorphisms of the Calkin algebra are inner.*

Later Farah showed that PFA implies that all automorphisms of the Calkin algebra over any Hilbert space are inner ([16]).

The following conjectures by S. Coskey and I. Farah ([7]) generalize their commutative counterparts.

Conjecture 1: The Continuum Hypothesis implies that the corona of every separable, non-unital C*-algebra has nontrivial automorphisms.

Conjecture 2: Forcing axioms imply that the corona of every separable, non-unital C*-algebra has only trivial automorphisms.

In conjecture 2 the notion of triviality refers to a weaker notion than the one used in this thesis, and it assures that automorphisms are *definable* in ZFC in a strong sense. Let us call it *weakly trivial* to avoid confusion in the future references.

Definition 3.1.4 (Coskey, Farah)**.** *An automorphism Φ of the corona of a separable C*-algebra $\mathcal{A}$ is weakly trivial if the set*

$$\Gamma_\Phi = \{(a, b) \in \mathcal{M}(\mathcal{A})^2 : \Phi(\pi(a)) = \pi(b)\}$$

is Borel, where $\mathcal{M}(\mathcal{A})$ is endowed with the strict operator topology.

In [7] it has been proved that assuming CH every σ-unital C*-algebra which is either simple or stable has non-trivial automorphisms. On the other hand OCA + MA implies that all automorphisms of reduced products of UHF-algebras are weakly trivial ([43]).

In this chapter I employ some of the techniques used in [23] and [17] in order to generalize the main result of Farah-Shelah ([23]) to isomorphisms between corona algebras of the form $\prod_{n=0}^{\infty} \mathbb{M}_{k(n)}(\mathbb{C}) / \bigoplus \mathbb{M}_{k(n)}(\mathbb{C})$, where $M_n(\mathbb{C})$ is the C*-algebra of all $n \times n$ matrices over the field of complex numbers. I will show that the result of Farah-Shelah follows from the main theorem of this chapter.

Definition 3.1.5. *Assume $\mathcal{A}_n$ is a sequence of unital C*-algebras and $\mathcal{I}$ is an ideal* *on* $\mathbb{N}$. *Consider the C*-algebras*

$$\begin{aligned}
\prod_n \mathcal{A}_n &= \{(a_n) : \quad \forall n \ a_n \in \mathcal{A}_n \ \textit{and} \ \sup_n \|a_n\| < \infty\}, \\
\bigoplus_{\mathcal{I}} \mathcal{A}_n &= \{(a_n) \in \prod_n \mathcal{A}_n : \quad \limsup_{n \to \mathcal{I}} \|a_n\| = 0\}
\end{aligned}$$

with their usual norms. The expression $\limsup_{n\to\mathcal{I}} \|a_n\| = 0$ *means that for every* $\epsilon > 0$ *the set* $\{n \in \mathbb{N} : \|a_n\| \geq \epsilon\} \in \mathcal{I}$. *Clearly* $\bigoplus_{\mathcal{I}} \mathcal{A}_n$ *is a closed ideal of* $\prod_n \mathcal{A}_n$ *and the quotient* $\prod_n \mathcal{A}_n / \bigoplus_{\mathcal{I}} \mathcal{A}_n$ *is usually called the* reduced product *of* $\{\mathcal{A}_n\}$ *over the ideal* $\mathcal{I}$.

If $\mathcal{I} = Fin$ as usual we drop the subscript and write $\prod_n \mathcal{A}_n / \bigoplus \mathcal{A}_n$ instead of $\prod_n \mathcal{A}_n / \bigoplus_{Fin} \mathcal{A}_n$. We will use $\pi_{\mathcal{I}}$ to denote the natural quotient from $\prod_n \mathcal{A}_n$ onto $\prod_n \mathcal{A}_n / \bigoplus_{\mathcal{I}} \mathcal{A}_n$.

In [17] the corona algebras of the form $\prod_n \mathbb{M}_{k(n)}(\mathbb{C}) / \bigoplus_n \mathbb{M}_{k(n)}(\mathbb{C})$ play a crucial role in proving that “OCA implies all automorphisms of the Calkin algebra are inner". In the class of C*-algebras $\prod_n \mathbb{M}_{k(n)}(\mathbb{C})$ can be considered as a good counterpart of $P(\mathbb{N})$ in set theory and as it will be clear from next section, trivial automorphisms of the corona of these algebras give rise to trivial automorphisms of the boolean algebra $P(\mathbb{N})/Fin$.

Assume ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$ are Rudin-Keisler isomorphic (Definition 1.2.1) via a bijection $\sigma : \mathbb{N} \setminus A \to \mathbb{N} \setminus B$ for $A \in \mathcal{I}$ and $B \in \mathcal{J}$. If $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_n\}$ are sequences of C*-algebras such that there are isomorphisms $\varphi_n : \mathcal{A}_n \cong \mathcal{B}_{\sigma(n)}$ for every $n \in \mathbb{N} \setminus A$, then there is an obvious (and trivial) isomorphism Φ between algebras $\prod_n \mathcal{A}_n / \bigoplus_{\mathcal{I}} \mathcal{A}_n$ and $\prod_n \mathcal{B}_n / \bigoplus_{\mathcal{J}} \mathcal{B}_n$. Namely define Φ by

$$\Phi(\pi_{\mathcal{I}}((a_n))) = \pi_{\mathcal{J}}(\varphi_n(a_n)).$$

Let us call such an isomorphism *strongly trivial.*

In the rest of this chapter we investigate the isomorphisms between these quotient algebras. We will show that it is impossible to construct non-strongly trivial isomorphisms between reduced products of the sequence $\{\mathbb{M}_{k(n)}(\mathbb{C}) : n \in \mathbb{N}\}$ associated with analytic P-ideals on $\mathbb{N}$, without appealing to some additional set-theoretic axioms. This is a consequence of our main result (Theorem 3.2.2) which implies the following corollary.

Corollary 3.1.6. *It is relatively consistent with ZFC that for all analytic P-ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$ all isomorphisms between $\prod_n \mathbb{M}_n(\mathbb{C}) / \bigoplus_{\mathcal{I}} \mathbb{M}_n(\mathbb{C})$ and $\prod_n \mathbb{M}_n(\mathbb{C}) / \bigoplus_{\mathcal{J}} \mathbb{M}_n(\mathbb{C})$ are strongly trivial. In particular all automorphisms of the corona $\prod_n \mathbb{M}_n(\mathbb{C}) / \bigoplus_n \mathbb{M}_n(\mathbb{C})$ are strongly trivial and inner.*

We follow [17] and use the terminology 'FDD-algebras' (Finite Dimensional Decomposition) for spatial representations of $\prod_n \mathbb{M}_{k(n)}(\mathbb{C})$ on separable Hilbert spaces, but throughout this chapter we usually identify FDD-algebras with $\prod_n \mathbb{M}_{k(n)}(\mathbb{C})$ for some sequence of natural numbers $\{k(n)\}$.

3.2 FDD-algebras and closed ideals associated with Borel ideals

Recall that for a separable infinite dimensional Hilbert space H, $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H.

Definition 3.2.1. *Fix a separable infinite dimensional Hilbert space H with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let $\vec{E} = (E_n)$ be a partition of $\mathbb{N}$ into finite intervals, i.e., a finite set of consecutive natural numbers, and $\mathcal{D}[\vec{E}]$ denote the C^*-algebra of all operators in $\mathcal{B}(H)$ such that the subspace spanned by $\{e_i : i \in E_n\}$ is invariant. These algebras are called FDD-algebras.*

Clearly $\mathcal{D}[\vec{E}]$ is isomorphic to $\prod_{n=0}^{\infty} \mathbb{M}_{|E_n|}(\mathbb{C})$. The unit ball of $\mathcal{D}[\vec{E}]$ is a Polish space when equipped with the strong operator topology and this allows us to use tools from descriptive set theory in this context.

For $M \subseteq \mathbb{N}$ let $P^{\vec{E}}_M$ be the projection on the closed span of $\bigcup_{n\in M}\{e_i : i \in \vec{E}_n\}$ and $\mathcal{D}_M[\vec{E}]$ be the closed ideal $P^{\vec{E}}_M \mathcal{D}_M[\vec{E}] P^{\vec{E}}_M = P^{\vec{E}}_M \mathcal{D}[\vec{E}]$. For a fixed $\vec{E}$ we often drop the superscript and write P_M and P_n instead of $P^{\vec{E}}_M$ and $P^{\vec{E}}_{\{n\}}$.

For a Borel ideal $\mathcal{J}$ on $\mathbb{N}$, the subspace $\mathcal{D}^{\mathcal{J}}[\vec{E}] = \overline{\bigcup_{X\in\mathcal{J}} \mathcal{D}_X[\vec{E}]}$ is a closed ideal of $\mathcal{D}[\vec{E}]$. Equivalently

$$\mathcal{D}^{\mathcal{J}}[\vec{E}] = \{(a_n) \in \mathcal{D}[\vec{E}] : \limsup_{n\to\mathcal{J}} \|a_n\| = 0\}.$$

Let $\mathcal{C}^{\mathcal{J}}[\vec{E}] = \mathcal{D}[\vec{E}]/\mathcal{D}^{\mathcal{J}}[\vec{E}]$ and $\pi_{\mathcal{J}}$ be the natural quotient map. For operators a and b in $\mathcal{D}[\vec{E}]$ we usually write $a =^{\mathcal{J}} b$ instead of $a - b \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$.

The following theorem is the main result of this chapter.

Theorem 3.2.2. *Assume there is a measurable cardinal. There is a forcing extension in which for partitions $\vec{E}$ and $\vec{F}$ of the natural numbers into finite intervals, and for $\mathcal{I}$ and $\mathcal{J}$ Borel ideals on the natural numbers, the following are true.*

1. *Any automorphism $\Phi : \mathcal{C}^{\mathcal{J}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{E}]$ has a (strongly) continuous representation.*

2. *Any isomorphism $\Phi : \mathcal{C}^{\mathcal{I}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{F}]$ has a continuous representation.*

   If $\mathcal{I}$ and $\mathcal{J}$ are analytic P-ideals then

3. *Any automorphism $\Phi : \mathcal{C}^{\mathcal{J}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{E}]$ has a *-homomorphism representation.*

4. *Any isomorphism $\Phi : \mathcal{C}^{\mathcal{I}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{F}]$ has a *-homomorphism representation.*

The following corollary follows from the proof of Theorem 3.2.2 and does not require any large cardinal assumption. See §3.4 for definition of local triviality.

Corollary 3.2.3. *There is a forcing extension in which if $\mathcal{I}$ and $\mathcal{J}$ are (P)-ideals on $\mathbb{N}$, any *-homomorphism $\Phi : \mathcal{C}^{\mathcal{I}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{F}]$ has a locally (*-homomorphism) continuous representation.*

In order to avoid making notations more complicated we only prove Theorem 3.2.2 for automorphisms and it is easy to see that the same proof works for isomorphisms.

In our forcing extension every such isomorphism has a simple description as it turns out that these isomorphisms are implemented by isometries between "co-small" subspaces. For partitions $\vec{E} = (E_n)$ and $\vec{F} = (F_n)$ of $\mathbb{N}$ into finite intervals in the following proposition let $\mathcal{D}[\vec{E}]$ and $\mathcal{D}[\vec{F}]$ be the FDD-algebras associated with $\vec{E}$ and $\vec{F}$ with respect to fixed orthonormal basis $\{e_n : n \in \mathbb{N}\}$ and $\{f_n : n \in \mathbb{N}\}$ for Hilbert spaces H and K respectively. Also let

$$\begin{aligned} H_n &= span\{e_i : i \in E_n\} \qquad P_n = Proj(H_n) \\ K_n &= span\{f_i : i \in F_n\} \qquad Q_n = Proj(K_n). \end{aligned}$$

Proposition 3.2.4. *Assume there is a measurable cardinal. There is a forcing extension in which the following holds. Assume $\mathcal{I}$, $\mathcal{J}$ are analytic P-ideals on $\mathbb{N}$ and $\vec{E} = (E_n)$, $\vec{F} = (F_n)$ are partitions of $\mathbb{N}$ into finite intervals. Then there is an isomorphism $\Phi : \mathcal{C}^{\mathcal{I}}[\vec{E}] \mapsto \mathcal{C}^{\mathcal{J}}[\vec{F}]$ if and only if*

1. *$\mathcal{I}$ and $\mathcal{J}$ are Rudin-Keisler isomorphic, i.e., there are sets $B \in \mathcal{I}$ and $C \in \mathcal{J}$ and a bijection $\sigma : \mathbb{N} \setminus B \mapsto \mathbb{N} \setminus C$ such that $X \in \mathcal{I}$ if and only if $\sigma[X] \in \mathcal{J}$, and*

2. *$|E_n| = |F_{\sigma(n)}|$ for every $n \in \mathbb{N} \setminus B$.*

Moreover, for every $n \in \mathbb{N} \setminus B$ there is a linear isometry $u_n : H_n \mapsto K_{\sigma(n)}$ such that if $u = \sum_{n \in \mathbb{N} \setminus B} u_n$, then the map $a \mapsto uau^*$ is a representation of Φ.

Proof. The inverse direction of the first statement is trivial. To prove the forward direction assume $\Phi : \mathcal{C}^{\mathcal{I}}[\vec{E}] \mapsto \mathcal{C}^{\mathcal{J}}[\vec{F}]$ is an isomorphism. Using Theorem 3.2.2 there is a forcing extension in which there is a *-homomorphism $\Psi : \mathcal{D}[\vec{E}] \mapsto \mathcal{D}[\vec{F}]$ which is a representation of Φ. For every n we have $\Psi(P_n)(K) \subseteq Q_m(K)$ for some m. It is easy to see that since Φ is an isomorphism there are $B \in \mathcal{I}$, $C \in \mathcal{J}$ and a bijection $\sigma : \mathbb{N} \setminus B \mapsto \mathbb{N} \setminus C$ such that for every $n \in \mathbb{N} \setminus B$ we have $\Psi(P_n)(K) = Q_{\sigma(n)}(K)$. The map σ witnesses that $\mathcal{I}$ and $\mathcal{J}$ are Rudin-Keisler isomorphic. Moreover, for every one-dimensional projection $P \in \mathcal{B}(H_n)$ the image, $\Psi(P)$, is also a one-dimensional projection in $B(K_{\sigma(n)})$. In particular $|E_n| = |F_{\sigma(n)}|$.

Now for every $n \in \mathbb{N} \setminus B$ assume $E_n = [k_n, k_{n+1}]$ and define a unitary $a \in B(H_n)$ by

$$a(e_{k_i}) = \begin{cases} e_{k_i+1} & k_n \leq i < k_{n+1} \\ e_{k_n} & i = k_{n+1}. \end{cases}$$

Fix $\xi_0 \in K_{\sigma(n)}$. Let $b = \Psi(a)$ and $\xi_j = b^j(\xi_0)$ (b^j is the j-th power of b) for each $0 \leq j < |E_n|$. Then $\{\xi_j : 0 \leq j < |E_n|\}$ forms a basis for $K_{\sigma(n)}$ and $e_{k_j} \mapsto \xi_j$ defines an isometry u_n as required.

Now $u = \bigoplus_{n \in \mathbb{N} \setminus B} u_n$ is an isometry from $\bigoplus_{n \in \mathbb{N} \setminus B} H_n$ to $\bigoplus_{n \in \mathbb{N} \setminus C} K_n$ such that $\Psi(a) - uau^* \in \mathcal{D}^{\mathcal{J}}(\vec{F})$ for all $a \in \mathcal{D}[\vec{E}]$. □

As we mentioned in the introduction the result of Farah and Shelah ([23]) can be obtained from Theorem 3.2.2.

Corollary 3.2.5. *If there is measurable cardinal, there is a forcing extension in which every isomorphism between quotient Boolean algebras $P(\mathbb{N})/\mathcal{I}$ and $P(\mathbb{N})/\mathcal{J}$ over Borel ideals has a continuous representation.*

Proof. Let $E_n = \{n\}$. Then $\mathcal{D}[\vec{E}] \cong \ell_\infty$ is the standard atomic masa (maximal abelian subalgebra) of $B(H)$ and for

$$\hat{\mathcal{J}} = \{(\alpha_n) \in \ell_\infty : \limsup_{n \to \mathcal{J}} \alpha_n = 0\}$$

clearly $\mathcal{C}^{\mathcal{J}}[\vec{E}] = \ell_\infty / \hat{\mathcal{J}} \cong C(st(P(\mathbb{N})/\mathcal{J}))$ where $st(P(\mathbb{N})/\mathcal{J})$ is the Stone space of $P(\mathbb{N})/\mathcal{J}$. The duality between categories implies that every isomorphism Φ between $P(\mathbb{N})/\mathcal{I}$ and $P(\mathbb{N})/\mathcal{J}$ corresponds to an isomorphism $\tilde{\Phi}$ between $C(st(P(\mathbb{N})/\mathcal{I}))$ and $C(st(P(\mathbb{N})/\mathcal{J}))$. The continuous map witnessing the topological triviality of $\tilde{\Phi}$ corresponds to a continuous map witnessing the topological triviality of Φ. □

For any partition $\vec{E}$ let $Z(\mathcal{C}^{\mathcal{J}}[\vec{E}])$ denote the center of $\mathcal{C}^{\mathcal{J}}[\vec{E}]$ and $U(n)$ be the compact group of all unitary $n \times n$ matrices equipped with the bi-invariant normalized Haar measure μ. More generally the following are true.

Lemma 3.2.6. *For any ideal $\mathcal{J}$*

$$Z(\mathcal{C}^{\mathcal{J}}[\vec{E}]) = \frac{Z(\mathcal{D}[\vec{E}])}{\mathcal{D}^{\mathcal{J}}[\vec{E}] \cap Z(\mathcal{D}[\vec{E}])}.$$

Proof. Clearly we have $Z(\mathcal{D}[\vec{E}])/(\mathcal{D}^{\mathcal{J}}[\vec{E}] \cap Z(\mathcal{D}[\vec{E}])) \subseteq Z(\mathcal{C}^{\mathcal{J}}[\vec{E}])$. For the other direction it is enough to show that for every $a + \mathcal{D}^{\mathcal{J}}[\vec{E}] \in Z(\mathcal{C}^{\mathcal{J}}[\vec{E}])$ there exists a $a' \in Z(\mathcal{D}[\vec{E}])$ such that $a - a' \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$, in other words every element of $Z(\mathcal{C}^{\mathcal{J}}[\vec{E}])$ can be lifted to an element of $Z(\mathcal{D}[\vec{E}])$. Let $a = (a_n)$ be such that each a_n belongs to $M_{|E_n|}(\mathbb{C})$ and $a + \mathcal{D}^{\mathcal{J}}[\vec{E}] \in Z(\mathcal{C}^{\mathcal{J}}[\vec{E})]$. For every n let

$$a'_n = \int_{u \in U(|E_n|)} u a_n u^* d\mu$$

and since μ is bi-invariant, for every unitary $u \in M_{|E_n|}(\mathbb{C})$ we have $u a'_n u^* = a'_n$. If $a' = (a'_n)$ then $a' \in Z(\mathcal{D}[\vec{E}])$ and $a - a' \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$. □

Proposition 3.2.7. *$Z(\mathcal{C}^{\mathcal{J}}[\vec{E}]) \cong C(st(P(\mathbb{N})/\mathcal{J}))$.*

Proof. Clearly we have $Z(\mathcal{D}[\vec{E}]) \cong \ell_\infty$ and $\mathcal{D}^{\mathcal{J}}[\vec{E}] \cap Z(\mathcal{D}[\vec{E}]) \cong \hat{\mathcal{J}}$. Therefore by Lemma 3.2.6 we have $Z(\mathcal{C}^{\mathcal{J}}[\vec{E}]) \cong \ell_\infty/\hat{\mathcal{J}} \cong C(st(P(\mathbb{N})/\mathcal{J}))$. □

3.3 Topologically trivial automorphisms of analytic P-ideal quotients of FDD-algebras

In this section we study the automorphisms of quotients of FDD-algebras over ideals associated with analytic P-ideals with Baire-measurable representations. We will show that if an automorphism $\Phi : \mathcal{C}^{\mathcal{J}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{E}]$ is topologically trivial (i.e., has a Baire-measurable representation), then it must be trivial. Our result resembles the fact that for an analytic P-ideal $\mathcal{J}$ any automorphism of $P(\mathbb{N})/\mathcal{J}$ with a Baire-measurable representation has an asymptotically additive representation (see [11], §1.5).

For the rest of this section let $\mathcal{J} = Exh(\mu)$ be an analytic P-ideal on $\mathbb{N}$ for a lower semicontinuous submeasure μ, containing all finite sets ($Fin \subseteq \mathcal{J}$). For each $a \in \mathcal{D}[\vec{E}]$ define $supp(a) \subseteq \mathbb{N}$ by

$$supp(a) = \{n \in \mathbb{N} : P_n a \neq 0\}$$

and in order to make notations simpler let $\hat{\mu} : \mathcal{D}[\vec{E}] \to [0, \infty]$ be $\hat{\mu}(a) = \mu(supp(a))$.

Definition 3.3.1 (Approximate *-homomorphism)**.** *Assume A and B are unital C^*-algebras. A map $\Psi : A \to B$ is an ϵ-approximate unital *-homomorphism if for every a and b in $A_{\leq 1}$ the following hold:*

1. $\| \Psi(ab) - \Psi(a)\Psi(b) \| \leq \epsilon$
2. $\| \Psi(a + b) - \Psi(a) - \Psi(b) \| \leq \epsilon$
3. $\| \Psi(a^*) - \Psi(a)^* \| \leq \epsilon$
4. $|\|\Psi(a)\| - \|a\|| \leq \epsilon$
5. $\| \Psi(I) - I \| \leq \epsilon$

*We say Ψ is δ-approximated by a unital *-homomorphism Λ if $\| \Psi(a) - \Lambda(a) \| \leq \delta$ for all $a \in A_{\leq 1}$.*

Next lemma is an Ulam-stability type result for finite-dimensional C*-algebras which will be required in the proof of Lemma 3.3.4. To see a proof look at [17, Theorem 5.1].

Lemma 3.3.2 (I. Farah)**.** *There is a universal constant $K < \infty$ such that for every ϵ small enough , $\mathcal{A}$ and $\mathcal{B}$ finite-dimensional C*-algebras, every Borel-measurable ϵ-approximate unital *-homomorphism $\Psi : \mathcal{A} \to \mathcal{B}$ can be $K\epsilon$-approximated by a unital *-homomorphism.*

To see a proof of the following refer to [17, Theorem 5.8].

Lemma 3.3.3. *If $0 < \epsilon < 1/8$ then in every C*-algebra A the following holds. For every $a \in A$ satisfying $\|a - a^2\| \leq \epsilon$ and $\|a - a^*\| \leq \epsilon$, there is a projection $P \in A$ such that $\|P - a\| \leq 4\epsilon$.*

Assume $\Phi : \mathcal{C}^{\mathcal{J}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{E}]$ is an automorphism and $\mathcal{D}[\vec{E}]$ is equipped with the strong operator topology. Recall that if $M \subseteq \mathbb{N}$ then P_M denotes the projection on the closed span of $\bigcup_{n\in M}\{e_i : i \in \vec{E}_n\}$. For each n fix a finite set of operators G_n which is 2^{-n}-dense (in norm) in the unit ball of $\mathcal{D}_{\{n\}}[\vec{E}] \cong \mathbb{M}_{|E_n|}(\mathbb{C})$. Let $F = \prod_{n=1}^{\infty} G_n$ and $F_M = P_M F$ for any $M \subseteq \mathbb{N}$.

Lemma 3.3.4. *If an automorphism $\Phi : \mathcal{C}^{\mathcal{J}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{E}]$ has a Baire-measurable representation Φ_*, then it has a *-homomorphism representation.*

Proof. First we show that Φ has a (strongly) continuous representation on F and then we construct a *-homomorphism representation on $\mathcal{D}[\vec{E}]$ by using a similar argument used in chapter 6 of [17].

The first part is a well-known fact (see [11]). To see this let G be a dense G_δ set such that the restriction of Φ_* is continuous on G and $G = \bigcap_{i=1}^{\infty} U_i$ where U_i are dense open sets in F. Assume $U_{i+1} \subseteq U_i$ for each i. Recursively choose $1 = n_1 \leq n_2 \leq \ldots$ and $s_i \in F_{[n_i,n_{i+1})}$ such that for every $a \in F$ if $P_{[n_i,n_{i+1})}a = s_i$ then $a \in U_i$. Now let

$$t_0 = \sum_i s_{2i} \qquad t_1 = \sum_i s_{2i+1}$$

Let $Q_0 = \sum_{i\ even} P_{[n_i,n_{i+1})}$ and $Q_1 = \sum_{i\ odd} P_{[n_i,n_{i+1})}$. Define Ψ on F by

$$\Psi(a) = \Psi_0(a) + \Psi_1(a)$$

where

$$\Psi_0(a) = \Phi_*(Q_0 a + t_1) - \Phi_*(t_0)$$

$$\Psi_1(a) = \Phi_*(Q_1 a + t_0) - \Phi_*(t_1).$$

It is easy to see that Ψ is a continuous representation of Φ on F. By possibly replacing Ψ with the map $a \to \Psi(a)\Psi(I)^*$ we can assume Ψ is unital.

In order to find a *-homomorphism representation of Φ, first we find a representation of Φ which is *stabilized* by a sequence $\{u_n\}$ of orthogonal elements of F in the sense to be made clear below.

Claim 1. For all n and $\epsilon > 0$ there are $k > n$ and $u \in F_{[n,k)}$ such that for every a and b in F satisfying $P_{[n,\infty)}a = P_{[n,\infty)}b$ and $P_{[n,k)}a = P_{[n,k)}b = u$, there exists $c \in \mathcal{D}[\vec{E}]$ such that $\|\Psi(a) - \Psi(b) - c\| < \epsilon$ and $\hat{\mu}(P_{[k,\infty)}c) < \epsilon$.

Proof. Suppose claim fails for n and $\epsilon > 0$. Recursively build sequences m_i, u_i, s_i and t_i for $i \in \mathbb{N}$ as follows

(a) $n = m_0 < m_1 < m_2 < \dots,$

(b) $u_i \in F_{[m_i, m_{i+1})}$,

(c) s_i and t_i are elements of $F_{[0,n)}$,

(d) for every $c \in \mathcal{D}[\vec{E}]$ if $\|\Psi(s_i + u_i) - \Psi(t_i + u_i) - c\| < \epsilon$ then $\hat{\mu}(P_{[i,\infty)}c) \geq \epsilon$.

This can be easily done by our assumption. Since $F_{[0,n)}$ is finite let $\langle s, t\rangle$ be a pair $\langle s_i, t_i\rangle$ which appears infinitely often. Note that Ψ is a representation of an automorphism, therefore we can find k large enough and $d, h \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$ such that for every $j \in \mathbb{N}$

$$\begin{aligned}
\|\Psi(s + \sum_i u_i) - \Psi(s + u_j) - \Psi(\sum_{i \neq j} u_i) - d\| &< \epsilon/3 \\
\|\Psi(t + \sum_i u_i) - \Psi(t + u_j) - \Psi(\sum_{i \neq j} u_i) - h\| &< \epsilon/3,
\end{aligned}$$

and

$$\hat{\mu}(P_{[k,\infty)}d) \leq \epsilon/3, \qquad \hat{\mu}(P_{[i,\infty)}h) \leq \epsilon/3. \tag{3.1}$$

Both d and h can be chosen to be $\Psi(0)$. Also fix a $c \in \mathcal{D}[\vec{E}]$ such that

$$\|\Psi(s + \sum_i u_i) - \Psi(t + \sum_i u_i) - c\| < \epsilon/3. \tag{3.2}$$

We will see that with these assumptions no such c could belong to $\mathcal{D}^{\mathcal{J}}[\vec{E}]$. For infinitely many $j \geq k$ we have

$$\begin{aligned}
\|\Psi(s + u_j) \;&-\; \Psi(t + u_j) - (d + h + c)\| \\
&\leq \|\Psi(s + \sum_i u_i) - \Psi(s + u_j) - \Psi(\sum_{i \neq j} u_i) - d\| \\
&+ \|\Psi(t + \sum_i u_i) - \Psi(t + u_j) - \Psi(\sum_{i \neq j} u_i) - h\| \\
&+ \|\Psi(s + \sum_i u_i) - \Psi(t + \sum_i u_i) - c\| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

Hence by condition (d) we have $\hat{\mu}(P_{[j,\infty)}(d + h + c)) \geq \epsilon$ and

$$\hat{\mu}(P_{[j,\infty)}d) + \hat{\mu}(P_{[j,\infty)}h) + \hat{\mu}(P_{[j,\infty)}c) \geq \hat{\mu}(P_{[j,\infty)}(d + h + c)) \geq \epsilon.$$

Therefore by (2) we have $\hat{\mu}(P_{[j,\infty)}c) \geq \epsilon$ for infinitely many $j \geq k$. Since c was arbitrary this implies that for any c satisfying (3) we have $\lim_{i\to\infty} \hat{\mu}(P_{[i,\infty)}c) > \epsilon$. Hence $\Psi(s+\sum_i u_i)-\Psi(t+\sum_i u_i)$ does not belong to $\mathcal{D}^{\mathcal{J}}[\vec{E}]$. This is a contradiction since $(s + \sum_i u_i) - (t + \sum_i u_i)$ is a compact operator and therefore $\Psi(s + \sum_i u_i) - \Psi(t + \sum_i u_i) \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$. □

We build two increasing sequences of natural numbers (n_i) and (k_i) such that

$n_i < k_i < n_{i+1}$ for every i and so called “stabilizers" $u_i \in F_{[n_i,n_{i+1})}$ such that for all $a, b \in F$ which $P_{[n_i,n_{i+1})}a = P_{[n_i,n_{i+1})}b = u_i$ the following holds:

1. If $P_{[n_{i+1},\infty)}a = P_{[n_{i+1},\infty)}b$ then there exists $c \in \mathcal{D}[\vec{E}]$ such that $\|[\Psi(a) - \Psi(b)]P_{[k_i,\infty)} - c\| < 2^{-n_i}$ and $\hat{\mu}(P_{[k_i,\infty)}c) < 2^{-n_i}$.

2. If $P_{[0,n_i)}a = P_{[0,n_i)}b$ then $\|[\Psi(a) - \Psi(b)]P_{[k_i,\infty)}\| \leq 2^{-n_i}$.

Assume n_i, k_{i-1} and u_{i-1} have been chosen. By the claim above we can find k_i and $u_i^0 \in F_{[n_i,k_i)}$ such that (1) holds. Now since Ψ is strongly continuous we can find $n_{i+1} \geq k_i$ and $u_i \in F_{[n_i,n_{i+1})}$ extending u_i^0 such that (2) holds.

Let $J_i = [n_i, n_{i+1})$ and $\nu_i = \mathcal{D}_{J_i}[\vec{E}]$. Then $\mathcal{D}[\vec{E}] = \prod \nu_i$ and for $b \in \mathcal{D}[\vec{E}]$ we have $b = \sum_j b_j$ where $b_j \in \nu_j$. Note that F_{J_i} is finite and 2^{-n_i+1}-dense in ν_i. Fix a linear ordering of F_{J_i} and define $\sigma_i : \nu_i \longrightarrow F_{J_i}$ by letting $\sigma_i(b)$ to be the least element of F_{J_i} which is in the 2^{-n_i+1}- neighbourhood of b. For $b \in \mathcal{D}[\vec{E}]_{\leq 1}$ let $b_{even} = \sum \sigma_{2i}(b_{2i})$ and $b_{odd} = \sum \sigma_{2i+1}(b_{2i+1})$. Both of these elements belong to F and $b - b_{even} - b_{odd}$ is compact.

Define $\Lambda_{2i+1} : \nu_{2i+1} \longrightarrow \mathcal{D}[\vec{E}]$ by

$$\Lambda_{2i+1}(a) = \Psi(u_{even} + \sigma_{2i+1}(a)) - \Psi(u_{even}).$$

Since Ψ is continuous and σ_i is Borel-measurable, Λ_{2i+1} is Borel-measurable. Let $Q_i = P_{[k_{i-1},k_{i+1})}$ with $k_{-1} = 0$. Note that if $\mid i - j \mid > 1$ then Q_i and Q_j are orthogonal.

Let $\Lambda\ :\ \prod_{i=0}^{\infty}\nu_{2i+1}\longrightarrow\mathcal{D}[\vec{E}]$ be defined by

$$\Lambda(b)=\Psi(u_{even}+b_{odd})-\Psi(u_{even}).$$

Since $b-b_{odd}$ is compact we have $\Psi(b)-\Lambda(b)\in\mathcal{D}^{\mathcal{J}}[\vec{E}]$. Therefore Λ is a representation of Φ on $\prod_{i=0}^{\infty}\nu_{2i+1}$.

Claim 2. For $b=\sum_j b_{2j+1}\in\prod_{j=0}^{\infty}\nu_{2j+1}$, the operator $\Psi(b)-\sum_{i=0}^{\infty}Q_{2i+1}\Lambda_{2i+1}(b_{2i+1})$ belongs to $\mathcal{D}^{\mathcal{J}}[\vec{E}]$.

Since Λ is a representation of Φ on $\prod_{i=0}^{\infty}\nu_{2i+1}$, there exists $c\in\mathcal{D}^{\mathcal{J}}[\vec{E}]$ such that for every large enough l, $\|[\Psi(b)+\Lambda(b)]Q_{2l+1}-c\|<2^{-n_{2l}}$ and $\hat{\mu}(P_{[k_{2l},\infty)}c)<2^{-n_{2l}}$. Let $b^l=\sum_{j=l}^{\infty}\sigma_{2j+1}(b_{2j+1})$ and apply (1) to b^l and b_{odd} implies that there exists $c'\in\mathcal{D}[\vec{E}]$ such that $\|[\Psi(u_{even}+b^l)-\Psi(u_{even}-b_{odd})]Q_{2l+1}-c'\|<2^{-n_{2l}}$ and $\hat{\mu}(P_{[k_{2l},\infty)}c')<2^{-n_{2l}}$. Therefore

$$\begin{aligned}
&\|Q_{2l+1}[\Psi(b)-\sum_i Q_{2i+1}\Lambda_{2i+1}(b_{2i+1})]-(c+c')\|\\
&\leq\|Q_{2l+1}[\Psi(b)-\Lambda(b)]-c\|+\|Q_{2l+1}[\Lambda(b)-\sum_i Q_{2i+1}\Lambda_{2i+1}(b_{2i+1})]-c'\|\\
&\leq 2^{-n_{2l}}+\|Q_{2l+1}[\Lambda(b)-\Psi(u_{even}+b^l)-\Psi(u_{even})]-c'\|\\
&+\|Q_{2l+1}[\Psi(u_{even}+b^l)-\Psi(u_{even})-\Lambda_{2l+1}(b_{2l+1})]\|\qquad\text{[Apply (2)]}\\
&\leq 3.2^{-n_{2l}}.
\end{aligned}$$

Now for $d=P_{[k_{2l},\infty)}(c+c')+[\sum_{n=0}^{l}Q_{2n+1}\Lambda_{2n+1}(b_{2n+1})-P_{[0,k_{2l})}\Psi(b)]$ and any large enough l we have

$$\|\Psi(b) - \sum_i Q_{2i+1}\Lambda_{2i+1}(b_{2i+1}) - d\| \leq \sum_{j=l}^{\infty} 2^{-n_{2j}}$$

and $\hat{\mu}(P_{[k_{2l},\infty)}d) < 2.2^{-n_{2l}}$. This completes the proof of the claim 2.

Now let $\Lambda'_{2i+1} : \nu_{2i+1} \to Q_{2i+1}\mathcal{D}[\vec{E}]$ be defined as

$$\Lambda'_{2i+1}(b) = Q_{2i+1}\Lambda_{2i+1}(b).$$

Let $c_{2i+1} = \Lambda'_{2i+1}(I_{2i+1})$, where I_{2i+1} is the unit of ν_{2i+1}, and $\delta_i = max\{\|c_{2i+1}^2 - c_{2i+1}\|, \|c_{2i+1}^* - c_{2i+1}\|\}$. We show that $\limsup_i \delta_i = 0$. Assume not; find $\delta > 0$ and an infinite set $M \subset 2\mathbb{N}+1$ such that for all $i \in M$ we have $max\{\|c_i^2 - c_i\|, \|c_i^* - c_i\|\} > \delta$. Let $c = \sum_{i\in M} c_i$, by our previous claim if $P = \sum_{i\in M} Q_i$ then $\Psi(P) - c$ is compact. Therefore $c - c^2$ and $c - c^*$ are compact. Since c_i's are orthogonal we have $c^2 = \sum_{i\in M} c_i^2$ and $c^* = \sum_{i\in M} c_i^*$. Thus for large enough $i \in M$ we have $\|c_i - c_i^2\| = \|Q_i(c - c^2)\| \leq \delta$ and $\|c_i - c_i^*\| = \|Q_i(c - c^*)\| \leq \delta$, which is a contradiction.

Applying lemma 3.3.3 to c_{2i+1} for large enough i we get projections $S_{2i+1} \leq Q_{2i+1}$ such that $\limsup_{i\to\infty} \|S_{2i+1} - \Lambda'_{2i+1}(I_{2i+1})\| = 0$. Let

$$\Lambda''_i(a) = S_{2i+1}\Lambda'_{2i+1}(a)S_{2i+1}$$

for $a \in \nu_{2i+1}$. Now by re-enumerating indices we can assume Λ''_i is ϵ-approximate unital *-homomorphism, for small enough ϵ. Then $\Lambda''(a) = \sum_i \Lambda''_i(a)$ is a representation of Φ on $\prod_i \nu_{2i+1}$. Let

$$\begin{aligned}
\delta_i^0 &= \sup_{a,b\in\nu_{2i+1}\leq 1}\{\|\Lambda_i''(ab)-\Lambda_i''(a)\Lambda_i''(b)\|\}\\
\delta_i^1 &= \sup_{a,b\in\nu_{2i+1}\leq 1}\{\|\Lambda_i''(a+b)-\Lambda_i''(a)-\Lambda_i''(b)\|\}\\
\delta_i^2 &= \sup_{a\in\nu_{2i+1}\leq 1}\{\|\Lambda_i''(a^*)-\Lambda_i''(a)^*\|\}\\
\delta_i^3 &= \sup_{a\in\nu_{2i+1}\leq 1}\{\|\Lambda_i''(a)\|-\|a\|\}.
\end{aligned} \tag{3.3}$$

We claim that $\lim_i max_{0\leq k\leq 3}\delta_i^k = 0$. We only show $\lim_i \delta_i^0 = 0$ since the others are similar. Take a and b in $\sum_i \nu_{2i+1}$ such that $P_{J_i}a = a_i$ and $P_{J_i}b = b_i$ for all i. Since $\Psi(ab)-\Psi(a)\Psi(b)$ is compact, by claim 2 so is $\Lambda''(ab)-\Lambda''(a)\Lambda''(b)$, which implies $\lim \delta_i^0 = 0$. Let $\delta_j = max_{0\leq i\leq 3}\{\delta_j^i\}$. Each Λ_j'' is a Borel measurable δ_j-approximate *-homomorphism. Therefore by lemma 3.3.2 for any large enough j we can find a *-homomorphism Θ_j defined on ν_{2j+1} which is $K\delta_j$-approximation of Λ_j''. Define $\Theta : \sum_i \nu_{2i+1} \longrightarrow \mathcal{D}[\vec{E}]$ by $\Theta = \sum \Theta_i$. Since $\lim_j \delta_j = 0$, Θ is a representation of Φ on $\sum_{i>n} \nu_{2i+1}$. Hence Θ can be extended to a *-homomorphism representation of Φ on $\sum_i \nu_{2i+1}$. By repeating the same argument for even intervals instead of odd intervals, one can get a *-homomorphism representation of Φ on $\sum_i \nu_{2i}$. Now by combining these two representation we get the desired representation of Φ.

□

3.4 Topologically trivial automorphisms

This section is devoted to find local Baire-measurable representations of Φ. For this section it is enough to assume $\mathcal{J}$ is a Borel ideal on natural numbers containing all finite sets and we also assume that all elements of the FDD-algebra are taken from the unit ball. We say an automorphism $\Phi : \mathcal{C}^{\mathcal{J}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{E}]$ is trivial if it has a representation which is *-homomorphism and that it is Δ^1_2 if the set $\{(a, b) : \Phi(\pi_{\mathcal{J}}(a)) = \pi_{\mathcal{J}}(b)\}$ is Δ^1_2.

Fix a partition $\vec{I} = (I_n)$ of natural numbers into finite intervals and for each n fix a finite set G_n of operators which is 2^{-n}-dense (in norm) in the unit ball of $\mathcal{D}_{\{n\}}[\vec{E}] \cong \mathbb{M}_{|E_n|}(\mathbb{C})$. As before let

$$F_n = \prod_{i \in I_n} G_i, \qquad\qquad F = \prod_{n \in \mathbb{N}} F_n$$

and for $M \subset \mathbb{N}$ let

$$F_M = \prod_{n \in M} F_n.$$

Note that each F_n is 2^{k-1}-dense in $\mathcal{D}_{I_n}[\vec{E}]$ where k is the smallest element of I_n. Since each G_n is finite the product topology and the strong operator topology coincide on F. For any $M \subseteq \mathbb{N}$ let $\hat{P}_M = P_{\cup_{n \in M} I_n}$.

Lemma 3.4.1. *If a forcing notion $\mathbb{P}$ captures F, then there is a $\mathbb{P}$-name $\dot{x}$ for a real such that for every $p \in \mathbb{P}$ there is an infinite $M \subset \mathbb{N}$ such that for every $a \in \mathcal{D}_{\cup_{n \in M} I_n}[\vec{E}]$ there is $q_a \leq p$ such that $q_a \Vdash \hat{P}_M \dot{x} =^{\mathcal{J}} \check{a}$.*

Proof. Since the ideal $\mathcal{J}$ contains all finite sets and the sequence $\{F_n\}$ is eventually dense in $\mathcal{D}_{\cup_{n\in M} I_n}[\vec{E}]$ the proof follows from Definition 2.3.2. $\square$

Let $\mathcal{C}_M[\vec{E}] = \mathcal{D}_M[\vec{E}]/\mathcal{D}_M[\vec{E}] \cap \mathcal{D}^{\mathcal{J}}\vec{E}]$ and define the following ideals on $\mathbb{N}$,

$$Triv^0_\Phi \;= \{M \subset \mathbb{N} : \; \Phi \restriction \mathcal{C}_M[\vec{E}] \text{ has a strongly continuous representation}\}$$

$$Triv^1_\Phi \;= \{M \subset \mathbb{N} : \; \Phi \restriction \mathcal{C}_M[\vec{E}] \; is \; \Delta^1_2\}.$$

We say that Φ is locally topologically trivial if $Triv^0_\Phi$ is non-meager and it is locally Δ^1_2 if $Triv^1_\Phi$ is non-meager.

The following lemma is well-known and is proved in [23, lemma 4.5], where $\mathbb{P}$ is countable support iteration of some creature forcings and the random forcing. Since groupwise Silver forcings as well as random forcing are also Suslin proper, ω^ω-bounding and have continuous reading of names the same proof works for $\mathbb{P} = \{\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \; : \; \xi \leq \kappa, \eta < \kappa\}$, a countable support iteration of forcings such that each $\dot{\mathbb{Q}}_\eta$ is forced to be either some groupwise Silver forcing or the random forcing.

Lemma 3.4.2. *Assume $\mathbb{P} = \{\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \; : \; \xi \leq \kappa, \eta < \kappa\}$ is as above and $\dot{x}$ is a $\mathbb{P}$-name for a real. For $A \subseteq \mathbb{R}$ a Borel set and $g : \mathbb{R}^2 \to \mathbb{R}$ a Borel function, if $p \in \mathbb{P}$ is such that $\dot{x}$ is continuously read below p, then the set*

$$\{a : \; p \Vdash g(\check{a}, \dot{x}) \in A\}$$

is Δ^1_2.

Note that since $\mathbb{P}$ is ω^ω- bounding we can assume all partitions of $\mathbb{N}$ into finite intervals in the generic extension by $\mathbb{P}$ are ground model partitions. We will use the previous lemma to show that if all partitions are captured by some groupwise Silver forcings in stationary many steps of uncountable cofinality then any automorphism Φ is forced to be Δ^1_2 in the generic extension.

Lemma 3.4.3. *Assume $\mathbb{P}$ is a countable support iteration forcing notion as above such that for every partition $\vec{I}$ of $\mathbb{N}$ into finite intervals the set*

$$\{\xi < \mathfrak{c}^+ : \ \Vdash_{\mathbb{P}_\xi} \dot{\mathbb{Q}}_\xi \textit{ captures } F^{\vec{I}} \textit{ and } cf(\xi) \geq \aleph_1\}$$

is stationary. Then every automorphism $\Phi : \mathcal{C}^{\mathcal{J}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{E}]$ is forced to be locally Δ^1_2.

Proof. Let $\dot{\Phi}$ be a $\mathbb{P}$-name for an automorphism in the generic extension as above and $\dot{\Phi}_*$ be an arbitrary representation of $\dot{\Phi}$. Let $G \subset \mathbb{P}$ be a generic filter.

Assume $Triv^1_{int_G(\dot{\Phi})}$ is meager in $V[G]$ with a witnessing partition $\vec{I} = (I_n)$, i.e. for every infinite $A \subset \mathbb{N}$ the set $\bigcup_{n\in A} I_n$ is not in $Triv^1_{int_G(\dot{\Phi})}$. Since our forcings have cardinality $< \mathfrak{c}^+$, the set of all $\xi < \mathfrak{c}^+$ of uncountable cofinality such that $int_{G\restriction\xi}(\vec{I})$ witnesses $Triv^1_{int_{G\restriction\xi}(\dot{\Phi}\restriction\xi)}$ is meager in $V[G \restriction \xi]$ includes a club C (cf. [23]) relative to $\{\xi < \mathfrak{c}^+ : cf(\xi) > \aleph_0\}$.

By our assumption there is a stationary set S of ordinals of uncountable cofinalities such that for all $\xi \in S$ we have $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}_\xi$ adds a real $\dot{x}$ which captures $F^{\vec{I}}$".

Fix $\eta \in S \cap C$. Let $\dot{y}$ be a $\mathbb{P}_{[\eta,\mathfrak{c}^+]}$-name such that

$$\Phi(\pi_{\mathcal{J}}(\dot{x})) = \pi_{\mathcal{J}}(\dot{y}).$$

Note that $\dot{x}$ is the generic real added by $\mathbb{Q}_\eta$ and since $\mathbb{P}$ has the continuous reading of names for any $p \in \mathbb{P}_{[\eta,\mathfrak{c}^+]}$ there are $q \leq p$, a countable set S containing η, a compact set $K \subseteq \mathbb{R}^S$ and a continuous map $h : K \mapsto \mathbb{R}$ such that q forces that $\check{h}(\langle \dot{q}_\xi : \xi \in S \rangle) = \dot{y}$. Since $\dot{\mathbb{Q}}_\eta$ captures $F^{\vec{I}}$ there is an infinite $A \subset \mathbb{N}$ such that if $M = \bigcup_{n \in A} I_n$ for every $a \in \mathcal{D}_M(\vec{E})$ there is $q_a < q$ such that $q_a \Vdash \check{P}_M \dot{x} =^{\mathcal{J}} \check{a}$ and therefore $\Phi_*(\check{P}_M)\dot{y} =^{\mathcal{J}} \Phi_*(\check{a})$. For every $a \in \mathcal{D}_M(\vec{E})$ we have

$$\Phi_*(a) =^{\mathcal{J}} b \iff q_a \Vdash b =^{\mathcal{J}} \Phi_*(\check{P}_M)\check{h}(\langle \dot{q}_\xi : \xi \in S \rangle)$$

so Lemma 3.4.2 implies that this set is Δ^1_2. Therefore M is in $Triv^1_{int_{G\restriction\eta}(\dot{\Phi}\restriction\eta)}$, which contradicts the assumption that $\vec{I}$ witnesses the meagerness of $Triv^1_{int_{G\restriction\eta}(\dot{\Phi}\restriction\eta)}$. $\square$

The following lemmas is very similar to [23, lemma 4.9].

Lemma 3.4.4. *Suppose f and g are functions such that each of them is a representation of a *-homomorphism from $\mathcal{C}^{\mathcal{J}}[\vec{E}]$ into $\mathcal{C}^{\mathcal{J}}[\vec{E}]$. Assume*

$$\Delta_{f,g,\mathcal{J}} = \{a \in F \;:\; f(a) \neq^{\mathcal{J}} g(a)\}$$

is null. Then $\Delta_{f,g,\mathcal{J}}$ is empty.

Proof. By inner regularity of the Haar measure we can find a compact set $K \subset F$ disjoint from $\Delta_{f,g,\mathcal{J}}$ of measure $> 1/2$. Fix any $a \in F$. Since the set $K + a$ also has measure $> 1/2$, we can find $b \in K$ such that $b + a$ is also in K. Now we have

$$f(a) =^{\mathcal{J}} f(a+b) - f(b) =^{\mathcal{J}} g(a+b) - g(b) =^{\mathcal{J}} g(a)$$

□

Corollary 3.4.5. *Suppose f and g are continuous functions such that each of them is a representations of a *-homomorphism from $\mathcal{D}[\vec{E}]$ into $\mathcal{C}(A)$ and the random forcing $\mathcal{R}$ forces that $f(\dot{x}) =^{\mathcal{J}} g(\dot{x})$, where $\dot{x}$ is the canonical name for the random real. Then $f(a) =^{\mathcal{J}} g(a)$ for every $a \in \mathcal{D}[\vec{E}]$.*

Proof. Let $\Delta_{f,g,\mathcal{J}}$ be as defined in previous lemma. If $\Delta_{f,g,\mathcal{J}}$ is null by Lemma 3.4.4 we are done. Assume $\Delta_{f,g,\mathcal{J}}$ has positive measure and M is a countable model of ZFC containing codes for f, g and $\mathcal{J}$, since $\dot{x}$ is the random real, $\dot{x} \in \Delta_{f,g,\mathcal{J}}$ and therefore $f(\dot{x}) \neq^{\mathcal{J}} g(\dot{x})$ in the generic extension. But our assumption $f(\dot{x}) =^{\mathcal{J}} g(\dot{x})$ is a Δ^1_1 statement so it is true in V. Which is a contradiction. □

Recall that for $M \subset \mathbb{N}$, P_M is the projection on the closed span of $\bigcup_{n \in M}\{e_i : i \in \vec{E}_n\}$.

Lemma 3.4.6. *Suppose $\mathcal{J}$ is a Borel ideal on $\mathbb{N}$. If $a \in \mathcal{D}[\vec{E}] \setminus \mathcal{D}^{\mathcal{J}}[\vec{E}]$ and $\mathcal{L}$ is a non-meager ideal on $\mathbb{N}$, then there exists $M \in \mathcal{L}$ such that $P_M a \notin \mathcal{D}^{\mathcal{J}}[\vec{E}]$.*

Proof. Since a does not belong to $\mathcal{D}^{\mathcal{J}}[\vec{E}]$ there is $\epsilon > 0$ such that

$$A = \{n \in \mathbb{N} : \|a_n\| > \epsilon\} \notin \mathcal{J}.$$

Since $A \cap \mathcal{J}$ is a proper Borel ideal on A there are disjoint finite sets I_n such that $\bigcup_{n\in\mathbb{N}} I_n = A$ and for every infinite $X \subseteq \mathbb{N}$ the set $\bigcup_{n\in X} I_n \notin A \cap \mathcal{J}$. Let $\vec{J} = (J_n)$ be a partition of $\mathbb{N}$ such that $J_n \cap A = I_n$ for every n. Since $\mathcal{L}$ is a non-meager ideal there exists an infinite $X \subseteq \mathbb{N}$ such that $\bigcup_{n\in X} J_n \in \mathcal{L}$. For $M = \bigcup_{n\in X} J_n$ we have $\bigcup_{n\in X} I_n \subseteq supp(P_M a) \notin \mathcal{J}$ and clearly $\|a_n\| \geq \epsilon$ for every $n \in \bigcup_{n\in X} I_n$. Hence $P_M a \notin \mathcal{D}^{\mathcal{J}}[\vec{E}]$.

□

Next lemma shows that every locally topologically trivial automorphism in the extension is forced to have a “simple" definition.

Lemma 3.4.7. *Assume $\mathbb{P} = \{\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \mathfrak{c}^+, \eta < \mathfrak{c}^+\}$ be as above where $\dot{\mathbb{Q}}_0$ is the poset for the random forcing and assume $\dot{\Phi}$ is a $\mathbb{P}$-name for an automorphism which extends a locally topologically trivial ground model automorphism $\Phi : \mathcal{C}^{\mathcal{J}}[\vec{E}] \to \mathcal{C}^{\mathcal{J}}[\vec{E}]$ such that $int_G \dot{\Phi}$ is itself locally topologically trivial with the same local continuous maps witnessing local triviality of Φ, then there exists a $q \in \mathbb{P}$ such that $q \Vdash \{(a,b) : \Phi(\pi_{\mathcal{J}}(a)) = \pi_{\mathcal{J}}(b)\}$ is Π^1_2.*

Proof. Let $\dot{g}_\xi$ be the canonical $\mathbb{Q}_\xi$-name for the generic real added by $\mathbb{Q}_\xi$ and let $\dot{y}$ be a $\mathbb{P}$-name such that $\dot{\Phi}(\pi_{\mathcal{J}}(\dot{g}_0)) = \pi_{\mathcal{J}}(\dot{y})$. Note that $\dot{g}_0$ is the canonical $\mathbb{Q}_0$-name for the random real. Since $\mathbb{P}$ has continuous reading of names, we can find a condition p with countable support S containing 0, a compact set $K \subset F^S$ and a continuous function $h : K \to F$ such that $p \Vdash h(\langle \dot{g}_\xi : \ \xi \in S \rangle) = \dot{y}$.

Let $\mathcal{Z}$ be the set of all pairs (M, N, f) such that

1. $M, N \subseteq \mathbb{N}$.
2. $f : \mathcal{D}_M[\vec{E}] \to \mathcal{D}_N[\vec{E}]$ is a continuous representation of a *-homomorphism from $\mathcal{C}_M^{\mathcal{J}}[\vec{E}]$ into $\mathcal{C}_N^{\mathcal{J}}[\vec{E}]$.
3. $f(P_M) =^{\mathcal{J}} P_N$.
4. $f(a) \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$ if and only if $a \in \mathcal{D}^{\mathcal{J}}[\vec{E}] \cap \mathcal{D}_M(\vec{E})$.
5. $p \Vdash f(\check{P}_M \dot{g}_0) =^{\mathcal{J}} \check{P}_N \dot{y}$.

It is not hard to see that conditions (1),(2),(3), and (4) are Π_1^1 and therefore by (co)analytic absoluteness still hold in the generic extension. Moreover by Lemma 3.4.2 condition (5) is Δ_2^1. Therefore $\mathcal{Z}$ is Δ_2^1. The set

$$\Gamma = \{M \ : \ (M, N, f) \in \mathcal{Z} \text{ for some } N \text{ and } f\}$$

is an ideal on $\mathbb{N}$ and $Triv_\Phi^0 \subseteq \Gamma$. Since Φ is locally topologically trivial, Γ is non-meager. For any $M \in \Gamma$ let f_M be such that $(M, N, f_M) \in \mathcal{Z}$ for some $N \subseteq \mathbb{N}$. Let Φ_* be an arbitrary representation of the extension of Φ in the forcing extension.

Claim 1: For all $M \in \Gamma$ we have $f_M(a) =^{\mathcal{J}} \Phi_*(a)$ for every a in $\mathcal{D}_M(\vec{E})$.

This clearly holds for any finite M. Assume $M \in \Gamma$ is infinite. By our assumption p forces that

$$f_M(P_M \dot{g}_0) =^{\mathcal{J}} P_N \Phi_*(\dot{g}_0).$$

Now by Corollary 3.4.5, since $P_M \dot{g}_0$ is the random real with respect to $\prod_{n \in M} G_n$, for every a in $\mathcal{D}_M(\vec{E})$

$$f_M(a) =^{\mathcal{J}} P_N \Phi_*(a).$$

Let $d = (I - P_N)\Phi_*(a)$. It's enough to show that $d \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$. Let $c = \Phi_*^{-1}(d)$ and note that $(I - P_M)c \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$ since

$$\Phi_*((I - P_M)c) =^{\mathcal{J}} \Phi_*(I - P_M)\Phi_*(a)(I - P_N) =^{\mathcal{J}} 0.$$

On the other hand we have

$$f_M(P_M c) =^{\mathcal{J}} P_N \Phi_*(P_M c) =^{\mathcal{J}} 0.$$

By assumption (4) we have $P_M c \in \mathcal{D}^{\mathcal{J}}[\vec{E}]$. This implies c and hence d belong to $\mathcal{D}^{\mathcal{J}}[\vec{E}]$.

As a consequence of claim (1) if $M \in \Gamma$ then f_M witnesses that $M \in Triv^0_\Phi$ and therefore $\Gamma = Triv^0_\Phi$.

Claim 2: The following holds in the generic extension:

$$\{(a, b) \;:\; \Phi(\pi_{\mathcal{J}}(a)) = \pi_{\mathcal{J}}(b)\} = \{(a, b) \;:\; (\forall (M, N, f) \in \mathcal{Z})\ f(P_M a) =^{\mathcal{J}} P_N b\}.$$

Suppose $\Phi(\pi_{\mathcal{J}}(a)) = \pi_{\mathcal{J}}(b)$. Again let Φ_* be an arbitrary representation of the extension of Φ in the forcing extension. For any $(M, N, f) \in \mathcal{Z}$ by claim (1) we have $f(P_M a) =^{\mathcal{J}} \Phi_*(P_M a) =^{\mathcal{J}} P_N b$.

To see the other direction take (a, b) such that $\Phi(\pi_{\mathcal{J}}(a)) \neq \pi_{\mathcal{J}}(b)$. Since Φ is an automorphism we can find a $\mathcal{D}^{\mathcal{J}}[\vec{E}]$-positive element c such that $\Phi_*(c) =^{\mathcal{J}} \Phi_*(a) - b$. Since Γ is a non-meager ideal by Lemma 3.4.6 we can find an infinite $M \in \Gamma$ such that $P_M c$ is $\mathcal{D}^{\mathcal{J}}[\vec{E}]$-positive. Now for $(M, N, f_M) \in \mathcal{Z}$ we have

$$f_M(P_M a) - P_N b =^{\mathcal{J}} \Phi_*(P_M a) - \Phi_*(P_M) b =^{\mathcal{J}} \Phi_*(P_M)(\Phi_*(a) - b) =^{\mathcal{J}} \Phi_*(P_M c)$$

and therefore (a, b) does not belong to the left hand side of the equation.

This completes the proof since the right hand side of the equation is Π^1_2. □

3.5 Proof of the main theorem

Proof of Theorem 3.2.2. Start with a countable model of ZFC+MA and consider the countable support iteration $\mathbb{P} = \{\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \ : \ \xi \leq \mathfrak{c}^+, \eta < \mathfrak{c}^+\}$ of forcings of the form $\mathbb{S}_{F\vec{I}}$ and the random forcing such that

1. For every partition $\vec{I}$ of $\mathbb{N}$ into finite intervals the set $\{\xi : \mathbb{Q}_\xi$ is $\mathbb{S}_{F\vec{I}}$ and $cf(\xi) > \aleph_0\}$ is stationary.

2. The set $\{\xi : \mathbb{Q}_\xi$ is the random forcing and $cf(\xi) > \aleph_0\}$ is also a stationary set.

Let G be a generic filter on $\mathbb{P}$. Fix a $\mathbb{P}$-name $\dot{\mathcal{J}}$ for a Borel ideal on $\mathbb{N}$ and a $\mathbb{P}$-name $\dot{\Phi}$ for an automorphism of $\mathcal{C}^{\mathcal{J}}[\vec{E}]$ in the extension. Since every partition is captured in stationary many steps of uncountable cofinalities, by Lemma 3.4.3 $\dot{\Phi}$ is forced to be a $\mathbb{P}$-name for a locally Δ^1_2 automorphism. Each $\mathbb{P}_\xi$ is proper, hence no reals are added at stages of uncountable cofnality. For every η with uncountable cofinality $H(\aleph_1)^{V[G\restriction\eta]}$ is the direct limit of $H(\aleph_1)^{V[G\restriction\xi]}$ for $\xi < \eta$. By a basic model theory fact there is a club C relative to $\{\xi < \mathfrak{c}^+ : cf(\xi) \geq \aleph_1\}$ such that for every $\xi \in C$ and $\dot{A}$ a $\mathbb{P}$-name for a set of reals we have

$$(H(\aleph_1), int_{G\restriction\xi}(\dot{A} \restriction \xi))^{V[G\restriction\xi]} \preceq (H(\aleph_1), int_G(\dot{A}))^{V[G]}.$$

Therefore for every $\xi \in C$, $\dot{\Phi} \restriction \xi$ is a $\mathbb{P}_\xi$-name for a locally Δ^1_2 automorphism and $cf(\xi) > \aleph_0$. Fix such a ξ and by condition 2 assume $\dot{\mathbb{Q}}_\xi$ is the name for the random forcing. By **MA** in the ground model and applying lemma 2.2.10 locally we can find Baire-measurable and hence continuous representations of $\dot{\Phi}$ in V. Therefore $\dot{\Phi}$ is a $\mathbb{P}_{[\xi,\mathfrak{c}^+]}$-name for a locally topologically trivial automorphism which its local triviality is witnessed by ground model continuous maps. Therefore lemma 3.4.7 implies that $int_G(\dot{\Phi})$ is forced to be Π^1_2 in $V[G]$. Since our assumption that there is a measurable cardinal implies that Π^1_2 sets have Π^1_2-uniformizations and all Π^1_2 sets have the property of Baire, the automorphism $int_G\dot{\Phi}$ has a Baire-measurable and hence a continuous representation. If $\mathcal{J}$ is a Borel P-ideal by lemma 3.3.4 we can get a representation of $int_G\dot{\Phi}$ which is a *-homomorphism.

□

The following corollary is essentially proved in [23] where the authors show the consistency of having all automorphisms of $P(\mathbb{N})/\mathcal{I}$ trivial for a Borel ideal $\mathcal{I}$ while the Calkin algebra has an outer automorphism.

Corollary 3.5.1. *It is relatively consistent with ZFC that all automorphisms of $\mathcal{C}^{\mathcal{J}}[\vec{E}]$ are (trivial) topologically trivial for a Borel (P-)ideal $\mathcal{J}$ and every partition $\vec{E}$ of natural numbers into finite intervals while the Calkin algebra has an outer automorphism.*

Proof. Since $\mathbb{P}$ is a countable support iteration of proper ω^ω-bounding forcings, it is proper and ω^ω-bounding [50, § xVI.2.8(D)]. Hence the dominating number $\mathfrak{d} = \aleph_1$. This and the weak continuum hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ imply that the Calkin algebra has an outer automorphism (see [17], the paragraph after the proof of Theorem 1.1). In order to get $2^{\aleph_0} < 2^{\aleph_1}$ start with a model of CH and force with the poset consisting of all countable partial functions $f : \aleph_3 \times \aleph_1 \to \{0,1\}$ ordered by the reverse inclusion to add $\aleph_3$ so-called Cohen subsets of $\aleph_1$. This will increase $2^{\aleph_1}$ to $\aleph_3$ while preserving CH. Now force with $\mathbb{P}$ the iteration of length $\aleph_2$ as above to make all automorphisms of $\mathcal{C}^{\mathcal{J}}[\vec{E}]$ trivial. A simple Δ-system argument shows that $\mathbb{P}$ is $\aleph_2$-cc and hence it preserves $2^{\aleph_1}$. □

The forcing $\mathbb{P}$ used in this chapter in fact can be written as a countable support iteration of the random forcing and a single groupwise Silver forcing in the way described in the proof of Theorem 3.2.2. To see this notice that if two partitions of natural numbers $\vec{I}$ and $\vec{J}$ are such that $\vec{J}$ is coarser than $\vec{I}$, then $\mathbb{S}_{F^{\vec{J}}}$ captures $F^{\vec{I}}$. Let $\vec{J} = (J_n)$ be such that $|J_n| = n$. It is enough to show that for every $\vec{I}$ there exists a condition p in $\mathbb{S}_{F^{\vec{J}}}$ such that the partial order $\{q \in \mathbb{S}_{F^{\vec{J}}} : q \leq p\}$ is forcing equivalent to $\mathbb{S}_{F^{\vec{I}}}$. By the remark above we can assume $|I_n| = k_n$ is increasing. Let p be such that $dom(p) = \mathbb{N} \setminus \{k_1, k_2, \dots\}$. Clearly any such p is a condition in $\mathbb{S}_{F^{\vec{J}}}$ since $|J_n| = n$. Now it is not hard to check that $\{q \in \mathbb{S}_{F^{\vec{J}}} : q \leq p\}$ and $\mathbb{S}_{F^{\vec{I}}}$ are forcing equivalent.

Note that the results of this chapter and [23] can not be immediately modified to work for the category of compact metric groups; for example in ZFC the quotient group $\prod \mathbb{Z}/2\mathbb{Z}/\bigoplus \mathbb{Z}/2\mathbb{Z}$ has $2^{\mathfrak{c}}$ automorphisms and therefore it has nontrivial automorphisms ([12, Proposition 9]).

4 Model theory and non-trivial isomorphisms

The classical first-order logic deals with the axiomatizable classes of discrete structures. However, it does not work very well for metric structures. In [2] the authors introduced a variant of the classical model theory which is suitable for studying metric structures. A metric structure, in the sense of [2], is a many-sorted structure in which each sort is a complete metric space of finite diameter. A slightly modified version of the this logic was introduced in [21] which does not require the structures to be bounded, and it is more suited for applications. As in [21] our structures are not assumed to be bounded. Additionally, a structure may consist of some distinguished elements (constants) as well as some maps (of several variables) between sorts (sorted functions) and maps from sorts to bounded subsets of $\mathbb{R}$ (sorted predicates), and these maps are all required to be uniformly continuous functions. To each metric structure $\mathcal{M}$ one can associate a *language* $\mathcal{L}$, i.e., a set of predicates, functions and constant symbols, and associates to each predicate and function symbol its arity. Each predicate and function symbol is equipped with a *modulus of uniform continuity* which is also part of the language. When the predicate, function, and constant symbols of $\mathcal{L}$ interpreted exactly to be the corresponding predicates, functions, and distinguished elements of which $\mathcal{M}$ consists, then we say $\mathcal{M}$ is an $\mathcal{L}$-structure. The reader my refer to [2] or [21] for the precise definitions and the syntax.

In this logic *terms* are formed by composing function symbols and variables and they are interpreted in the usual manner in structures. Formulas are, as usual, defined by induction.

(i) If R is a relation and $\tau_1, \ldots, \tau_n$ are terms then $R(\tau_1, \ldots, \tau_n)$ is a (atomic) formula.

(ii) If $\varphi_1, \ldots, \varphi_n$ are in formulas and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then $f(\varphi_1, \ldots \varphi_n)$ is a formula.

(iii) If φ is a formula and x is a variable of φ then both $\sup_{x \in D} \varphi$ and $\inf_{x \in D} \varphi$ are formulas.

In (iii) D is a *domain of quantification* for the sort of x. More precisely, assume $\mathcal{S}$ is the set of all sorts of $\mathcal{L}$, for each sort $S \in \mathcal{S}$ a set of domains $\mathcal{D}_S$ is associated. Elements of $\mathcal{D}_S$ are meant to be domains of quantifications when interpreted in $\mathcal{L}$-structures; for an $\mathcal{L}$-structure $\mathcal{A} = (A_S : S \in \mathcal{S})$, a sort $S \in \mathcal{S}$ and $D \in \mathcal{D}_S$ the interpretation of D in $\mathcal{A}$, denoted by $D^{\mathcal{A}}$, is a complete bounded subset of A_S. The collection $\{D^{\mathcal{A}} : D \in \mathcal{D}_S\}$ covers A_S.

Formulas are interpreted in the obvious manner in structures. The boundedness of domains of quantifications is essential to guarantee that the supremums and infimums exist when interpreted. In particular for every $\mathcal{L}$-formula φ, the set of all evaluations of φ in $\mathcal{L}$-structures is a bounded subset of the real numbers. A (closed) *condition* is an expression of the form $\varphi(\bar{x}) \leq r$ for a formula $\varphi(\bar{x})$ and a real number r. We consider conditions over a model $\mathcal{A}$, in which φ is allowed to have elements from $\mathcal{A}$ as parameters.

Formulas with no free variables are called *sentences*; every variable appears in the scope of a supremum or infimum. Formulas which are constructed only using (i) and (ii) are called *quantifier-free* formulas.

Assume $\mathcal{A}$ is an $\mathcal{L}$-structure. Define the *theory* of $\mathcal{A}$ to be

$$Th(\mathcal{A}) = \{\varphi : \varphi \text{ is an } \mathcal{L}\text{-sentence and } \ \varphi^{\mathcal{A}} = 0\}.$$

Two metric structures $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent*, $\mathcal{A} \equiv \mathcal{B}$, if $Th(\mathcal{A}) = Th(\mathcal{B})$.

The evaluation of each formula is not assumed to be a positive number. However, since the ranges of formulas are bounded in all interpretations and evaluations are linear functionals, by composing with linear functions, we can deal with only $[0, 1]$-valued formulas, as we will do in Chapter 5. In general, $\mathcal{L}$-formulas which have positive evaluations in every $\mathcal{L}$-structure are called *positive*. The *universal* *theory* $Th_\forall(\mathcal{A})$ of $\mathcal{A}$ is the subset of $Th(\mathcal{A})$ consisting of sentences of the form $\sup_{x_1} \dots \sup_{x_n} \varphi$ where φ is a positive quantifier-free formula.

A category $\mathcal{C}$ is *axiomatizable* if there is a language $\mathcal{L}$, theory T in $\mathcal{L}$, and a collection of conditions Σ such that $\mathcal{C}$ is equivalent to the category of models of T with morphisms given by maps that preserve Σ. The category of all C*-algebras (as well as tracial von Neumann algebras) is axiomatizable ([21]) as two-sorted structures; one sort for the algebra itself and the other for a copy of $\mathbb{C}$. Domains of quantifications for each sort are closed balls with radius n for each $n \in \mathbb{N}$. In the language of C*-algebras terms are *-polynomial in non-commuting variables $x_1, \dots, x_n$ for some n. An atomic formula is an expression of the form $\|t\|$ where t is a term.

For a metric structure $\mathcal{A}$ we usually abbreviate a tuple $(a_1, \dots, a_n)$ of elements of $\mathcal{A}$ by $\bar{a}$, when there is no confusion about the length and sort of the tuple. In most interesting cases all entries of the tuple will belong to a single sort, such as the unit ball of the C*-algebra under the consideration, and we shall suppress discussion of sorts by assuming all variables are of the same sort.

4.1 Reduced products of metric structures

Let's recall some definitions and basic theorems regarding reduced products of metric structures from [24] and [40]. Fix a language $\mathcal{L}$ in logic of metric structures.

Throughout this and next chapter $\mathcal{L}$ can be many-sorted, but in order to avoid distracting notations we often surpass the discussion about the sorts. We let $\mathcal{D}$ be the set of domains of the quantification in $\mathcal{L}$. For an $\mathcal{L}$-structure $\mathcal{A}$ and $D \in \mathcal{D}$, recall that we write $D^{\mathcal{A}}$ to denote the interpretation of D in $\mathcal{A}$. For instance if $\mathcal{A}$ is a C*-algebra, $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ and $D_n^{\mathcal{A}}$ is the closed ball of $\mathcal{A}$ of radius n (notice that even though C*-algebras are two-sorted structures, in order to make notations simpler we treat them as one-sorted structures).

Assume $\{(\mathcal{A}_\gamma, d_\gamma),\ \gamma \in \Omega\}$ is a family of metric $\mathcal{L}$-structures indexed by a set Ω. Consider the direct product

$$\prod_\Omega \mathcal{A}_\gamma = \{\langle a(\gamma)\rangle : \exists D \in \mathcal{D} \text{ such that } a(\gamma) \in D^{\mathcal{A}_\gamma} \ \ \forall \gamma \in \Omega\}.$$

Let $\mathcal{I}$ be an ideal on Ω. Define a map $d_\mathcal{I}$ on $\prod_\Omega \mathcal{A}_\gamma$ by

$$d_\mathcal{I}(x,y) = \limsup_{i \to \mathcal{I}} d_\gamma(x(\gamma), y(\gamma)) = \inf_{S \in \mathcal{I}} \sup_{\gamma \notin S} d_\gamma(x(\gamma), y(\gamma))$$

where $x = \langle x(\gamma) : \gamma \in \Omega\rangle$ and $y = \langle y(\gamma) : \gamma \in \Omega\rangle$. The map $d_\mathcal{I}$ defines a pseudo-metric metric on $\prod_\Omega \mathcal{A}_\gamma$. For $x, y \in \prod_\Omega \mathcal{A}_\gamma$ define $x \sim_\mathcal{I} y$ to mean $d_\mathcal{I}(x,y) = 0$. Then $\sim_\mathcal{I}$ is an equivalence relation and the quotient

$$\prod_\mathcal{I} \mathcal{A}_\gamma = (\prod_\Omega \mathcal{A}_\gamma)/ \sim_\mathcal{I}$$

with the induced metric $d_\mathcal{I}$ is a complete bounded metric space. We will use $\pi_\mathcal{I}$ to denote the natural quotient map from $\prod_\Omega \mathcal{A}_\gamma$ onto $\prod_\mathcal{I} \mathcal{A}_\gamma$. For a tuple $\bar{a} =$

$(a_1, \dots, a_k)$ of elements of $\prod_\Omega \mathcal{A}_\gamma$ we write $\pi_\mathcal{I}(\bar{a})$ for $(\pi_\mathcal{I}(a_1), \dots, \pi_\mathcal{I}(a_k))$ and by $\bar{a}(\gamma)$ we denote the corresponding tuple $(a_1(\gamma), \dots, a_k(\gamma))$ of elements of $\mathcal{A}_\gamma$.

Let R be a predicate symbol in $\mathcal{L}$ and $\bar{a}$ be a tuple of elements of $\prod_\Omega \mathcal{A}_\gamma$ of appropriate size define

$$R(\pi_\mathcal{I}(\bar{a})) = \limsup_{\mathcal{I}} R(\bar{a}(\gamma)).$$

If f is a function symbol in $\mathcal{L}$ for an appropriate $\bar{a}$ define

$$f(\pi_\mathcal{I}(\bar{a})) = \pi_\mathcal{I}(\langle f(\bar{a}(\gamma))\rangle),$$

and if $c \in \mathcal{L}$ is a constant symbol let

$$c^{\prod_\mathcal{I} \mathcal{A}_\gamma} = \pi_\mathcal{I}(\langle c^{\mathcal{A}_\gamma}\rangle).$$

The quotient $\prod_\mathcal{I} \mathcal{A}_\gamma$ is called the *reduced product* of the family $\{(\mathcal{A}_\gamma, d_\gamma) : \gamma \in \Omega\}$ over the ideal $\mathcal{I}$. Note that if $\mathcal{I}$ is a maximal (prime) ideal, then $\prod_\mathcal{I} \mathcal{A}_\gamma$ is the ultraproduct of the family $\{\mathcal{A}_\gamma, \quad \gamma \in \Omega\}$ over the ultrafilter $\mathcal{U}$ consisting of the complements of the elements of $\mathcal{I}$, usually denoted by $\prod_\mathcal{U} \mathcal{A}_\gamma$ or $(\prod_\Omega \mathcal{A}_\gamma)_\mathcal{U}$ or $\prod_\Omega \mathcal{A}_\gamma/\mathcal{U}$. Also, in the case when $\mathcal{L}$ includes a distinguished constant symbol for 0 (e.g., language of C*-algebras) the reduced product of $\mathcal{L}$-structures $\{\mathcal{A}_\gamma, \quad \gamma \in \Omega\}$ over $\mathcal{I}$ is the quotient of $\prod_\Omega \mathcal{A}_\gamma$ over its closed ideal $\bigoplus_\mathcal{I} \mathcal{A}_\gamma$ defined by

$$\bigoplus_\mathcal{I} \mathcal{A}_\gamma = \{a \in \prod_\Omega \mathcal{A}_\gamma : \quad d_\mathcal{I}(a, 0^{\prod_\Omega \mathcal{A}_\gamma}) = 0\},$$

and usually denoted by $\prod_\Omega \mathcal{A}_\gamma / \bigoplus_\mathcal{I} \mathcal{A}_\gamma$ (see [24]).

Proposition 4.1.1. *The metric space $\langle \prod_{\mathcal{I}} \mathcal{A}_\gamma, d_{\mathcal{I}} \rangle$ is a metric $\mathcal{L}$-structure.*

Proof. We only have to check that each function and predicate symbol has the same modulus of uniform continuity. we shall prove this only for a function symbol f of arity k. Let $\Delta : [0,1] \to [0,1]$ be the modulus of uniform continuity of f, i.e., for $\epsilon > 0$ and $\bar{x} = (x_1, \ldots, x_k)$, $\bar{y} = (y_1, \ldots, y_k)$ tuples in each $\mathcal{A}_\gamma$ we have

$$d_\gamma(\bar{x}, \bar{y}) < \Delta(\epsilon) \quad \to \quad d_\gamma(f(\bar{x}), f(\bar{y})) \leq \epsilon,$$

where $d_\gamma(\bar{x}, \bar{y}) < \Delta(\epsilon)$ means $d_\gamma(x_i, y_i) < \Delta(\epsilon)$ for ever $i = \{1, \ldots, k\}$.

Suppose $\bar{a}$ and $\bar{b}$ in $(\prod_\Omega \mathcal{A}_\gamma)^k$ are such that $d_{\mathcal{I}}(\pi_I(\bar{a}), \pi_{\mathcal{I}}(\bar{b})) < \Delta(\epsilon)$. Then by the definition of $d_{\mathcal{I}}$ there is an $\mathcal{I}$-positive set $S \subseteq \Omega$ such that for every $\gamma \in S$ we have $d_\gamma(\bar{a}(\gamma), \bar{b}(\gamma)) < \Delta(\epsilon)$, and therefore $d_\gamma(f(\bar{a}(\gamma)), f(\bar{b}(\gamma))) \leq \epsilon$. This implies that $d_{\mathcal{I}}(\pi_{\mathcal{I}}(f(\bar{a})), \pi_{\mathcal{I}}(f(\bar{y}))) \leq \epsilon$. □

Lemma 4.1.2. *Assume $\mathcal{I}$ is an ideal on Ω. If $\varphi(\bar{y})$ is an atomic $\mathcal{L}$-formula and $\bar{a}$ is a tuple of elements of $\prod_\Omega \mathcal{A}_\gamma$, then*

$$\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\prod_{\mathcal{I}} \mathcal{A}_\gamma} = \limsup_{\mathcal{I}} \varphi(\bar{a}(\gamma))^{\mathcal{A}_\gamma}.$$

Proof. This easily follows from the definition of $d_{\mathcal{I}}$ and the interpretation of atomic formulas. □

4.2 Saturated structures

Fix a language $\mathcal{L}$ in the logic for the metric structures with possibly multiple sorts and fix a tuple of variables $\bar{x}$ from a sequence of sorts $\bar{S}$. We can define a pseudo-metric on the formulas with free variables $\bar{x}$ by letting the distance between $\varphi(\bar{x})$ and $\psi(\bar{x})$ to be

$$d(\varphi(\bar{x}), \psi(\bar{x})) = \sup\{|\varphi^{\mathcal{M}}(\bar{a}) - \psi^{\mathcal{M}}(\bar{a})| : \mathcal{M} \text{ is an } \mathcal{L}\text{-structure and } \bar{a} \in \mathcal{M}\}.$$

This pseudo-metric induces the topology of the uniform convergence on the set of $\mathcal{L}$-formulas. Later we will see that if $\mathcal{L}$ is countable, then the set of $\mathcal{L}$-formulas is separable in this topology (Proposition 4.3.3).

For a topological space X we use $\chi(X)$ to denote the density character of X, which is the smallest cardinality of a dense subset. Assume $\mathcal{A}$ is an $\mathcal{L}$-structure with multiple sorts $(A_S : S \in \mathcal{S})$ and $X \subseteq \mathcal{A}$. Then by $\chi(X)$ we mean $\sum_{S \in \mathcal{S}} \chi(X \cap A_S)$.

Let $\mathcal{A}$ be an $\mathcal{L}$-structure. An n-*type* over $X \subseteq \mathcal{A}$ is a set of $\mathcal{L}$-conditions with in free variable $\bar{x} = (x_1, \ldots, x_n)$ from a sequence of sorts $\bar{S}$, and parameters from X. An n-type $\mathbf{t}(\bar{x})$ is *realized* in $\mathcal{A}$ if for some tuple $\bar{a}$ in $\mathcal{A}$, appropriate in both sorts and size, we have that $\varphi(\bar{a})^{\mathcal{A}} \leq r$ for all conditions $\varphi(\bar{x}) \leq r$ in $\mathbf{t}(\bar{x})$. A type is *consistent* (or finitely approximately realizable) if every one of its finite subsets can be realized up to an arbitrarily small $\epsilon > 0$.

Definition 4.2.1. *Suppose κ is an infinite cardinal. A model $\mathcal{A}$ is κ*-saturated *if every consistent type* $\boldsymbol{t}$ *over* $X \subseteq \mathcal{A}$ *with* $|X| < \kappa$*, is realized in* $\mathcal{A}$*. If* $\mathcal{A}$ *is* $\chi(X)$*-saturated then we say* $\mathcal{A}$ *is saturated.*

Instead of $\aleph_1$-saturated we usually say countably saturated. Saturated models have remarkable properties. Two saturated models of the same language and same character density are isomorphic if and only if they have the same theory (see any standard text on model theory, e.g., [6] or [41]).

Theorem 4.2.2. *Assume the Continuum Hypothesis. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two elementarily equivalent countably saturated metric structures of density character* $\aleph_1$*. Then* $\mathcal{A} \cong \mathcal{B}$*.*

The following recent result ([24, Theorem 1.5]) is of significant importance in studying the reduced products of metric structures and the isomorphisms between them.

Theorem 4.2.3 (Farah-Shelah)**.** *Assume* $\{\mathcal{A}_n : n \in \mathbb{N}\}$ *is a sequence of metric structures. The reduced product over the Fréchet ideal* $\prod \mathcal{A}_n / \bigoplus \mathcal{A}_n$ *is countably saturated.*

For a structure $\mathcal{A}$ and a subset $A \subseteq \mathcal{A}$ a map $f : A \to \mathcal{A}$ is called a *partial elementary map* if for every formula $\varphi(\bar{x})$ and $\bar{a} \in \mathcal{A}$ of the appropriate sort and size,

$$\mathcal{A} \models \varphi(\bar{a}) \quad \leftrightarrow \quad \mathcal{A} \models \varphi(f(\bar{a})).$$

In saturated models, partial elementary maps are just restrictions of automorphisms. The proof of the following theorem is due to Bradd Hart and it is included in [24].

Theorem 4.2.4. *Assume $\mathcal{A}$ is a metric structure with density character κ, and it is κ-saturated. Then $\mathcal{A}$ has 2^κ automorphisms.*

Proof. Fix a dense subset of $\mathcal{A}$, $\{a_\gamma : \gamma < \kappa\}$. Consider $2^{<\kappa} = \bigcup_{\gamma<\kappa} 2^\gamma$ and for $s \in 2^\gamma$ write $len(s) = \gamma$.

Recursively construct families f_s and A_s for $s \in 2^{<\kappa}$ with the following properties.

1. A_s is an elementary submodel of $\mathcal{A}$ of cardinality $< \kappa$ including $\{a_\gamma : \gamma < len(s)\}$.

2. $f_s : A_s \to \mathcal{A}$ is a partial elementary map.

3. If $s \sqsubseteq t$ then $A_s \subseteq A_t$ and $f_t \restriction_{A_s} = f_s$.

4. $f_{s^\frown 0} \neq f_{s^\frown 1}$.

The first three conditions can be easily assured. In order to get (4) suppose for $\gamma < \kappa$ and all $s \in 2^\gamma$ both A_s and f_s are chosen to satisfy the above conditions. Fix such a s and assume a_ξ has the least index ξ such that a_ξ does not belong to A_s

and let $\epsilon = dist(a_\xi, A_s)$. Assume

$$\mathbf{t}(x) = \{\varphi(x, \bar{a}) : \ \bar{a} \subset A_s \ , \ A \models \varphi(a_\xi, \bar{a})\}$$

is the type of a_ξ over A_s and let $\mathbf{s}(x, y)$ be the 2-type $\mathbf{t}(x) \cup \mathbf{t}(y) \cup \{d(x, y) \geq \epsilon\}$. Since A_s is an elementary submodel of $\mathcal{A}$ every finite subset of $\mathbf{s}(x, y)$ can be realized and therefore $\mathbf{S}(x, y)$ is consistent. By saturation of $\mathcal{A}$ we can find elements b_0 and b_1 in $\mathcal{A}$ realizing the type $\mathbf{s}(x, y)$. Since $\mathcal{A}$ is κ-saturated and $|A_s| < \kappa$, by κ-homogeneity of $\mathcal{A}$ (see [41, Proposition 4.3.3]) we can extend f_s to the partial elementary maps $f_{s^\frown 0} : A_s \cup b_0 \to \mathcal{A}$ and $f_{s^\frown 1} : A_s \cup b_1 \to \mathcal{A}$. Since these maps are partial elementary we have $\mathcal{A} \models d(f_{s^\frown 0}(b_0), f_{s^\frown 1}(b_1)) \geq \epsilon$. By a Löwenheim-Skolem argument we can choose elementary submodels $A_{s^\frown 0}, A_{s^\frown 1} \preceq \mathcal{A}$ which contain $A_s \cup \{a_\xi, b_0\}$ and $A_s \cup \{a_\xi, b_1\}$ respectively. By elementarity and saturatedness of $A_{s^\frown i}$ extend $f_{s^\frown i}$ to $A_{s^\frown i}$ for $i = 0, 1$.

Now for every $s \in 2^\kappa$ we have $\mathcal{A} = \bigcup_{\gamma < \kappa} A_{s \restriction \kappa}$ and let $f_s = \bigcup_{\gamma < \kappa} f_{s \restriction \kappa}$. Clearly $\{f_s : s \in 2^\kappa\}$ are distinct automorphisms of $\mathcal{A}$.

□

Corollary 4.2.5. *If each $\mathcal{A}_n$ is separable, then the Continuum Hypothesis implies that there are $2^{\aleph_1}$ automorphisms of $\prod A_n / \bigoplus A_n$. In particular, it has outer automorphisms.*

Proof. Since by Theorem 4.2.3 the reduced product $\prod A_n / \bigoplus A_n$ is countably saturated, this follows from Theorem 4.2.4. □

Corollary 4.2.6. *The assertion that all automorphisms of the corona algebra $\prod_n \mathbb{M}_{k(n)}(\mathbb{C})/\bigoplus_n \mathbb{M}_{k(n)}(\mathbb{C})$ are trivial, is independent from ZFC.*

Proof. This is a direct consequence of Corollary 3.1.6 and Theorem 4.2.5. □

It is known that the Continuum Hypothesis implies the existence of non-trivial autohomeomorphisms of the Čech-Stone remainder $[0,1)^*$ of $[0,1)$ (this is a result of Yu, see [34, §9]). This result is generalized in [24, Theorem 2.2] where the authors give a sufficient and necessary condition for countable saturation of $C_b(X, \mathcal{A})/C_0(X, \mathcal{A})$, for a locally compact Polish space X and a metric structure $\mathcal{A}$ such that each domain of $\mathcal{A}$ is compact and locally connected.

The class of operator algebras which are countably saturated is small, but in this setting there are useful weakenings of countable saturation that are satisfied by a variety of algebras.

Definition 4.2.7. *A C*-algebra $\mathcal{A}$ is said to be countably degree-1 saturated if every consistent and countable type consisting of degree-1 *-polynomials is realized in $\mathcal{A}$.*

This property was introduced by Farah and Hart in [19], where it was shown to imply a number of important consequences. Next theorem shows that the class of countably degree-1 saturated C*-algebras is actually quite large, in fact strictly larger than countably saturated ones, for example by the theorem below the Calkin algebra is countably degree-1 saturated but it is not even countably quantifier-free saturated (see [19]). Countable degree-1 saturation can serve to unify proofs about these algebras.

Theorem 4.2.8 (I. Farah, B. Hart)**.** *The following classes of C*-algebras are countably degree-1 saturated.*

1. *The corona of separable C*-algebras.*
2. *Any ultraproduct of a sequence of separable C*-algebras.*

Moreover, the relative commutant of a sparable subalgebras of any countably degree-1 saturated C-algebra is also countable degree-1 saturated.*

Further examples were found by Voiculescu ([56]). In [9] the authors gave a classification of the theories of abelian real rank zero C*-algebras in terms of the discrete first-order theories of Boolean algebras.

Theorem 4.2.9 (C. Eagle, A. Vignati)**.** *Let X be a compact 0-dimensional Hausdorff space without isolated points. Then the following are equivalent:*

- $C(X)$ *is countably degree-1 saturated,*
- $C(X)$ *is countably saturated,*
- $CL(X)$ *is countably saturated, where* $CL(X)$ *is the Boolean algebra of the clopen subset of* X.

It is worth noticing that it is not known whether countable degree-1 saturation suffices to construct non-trivial automorphisms.

4.3 Restricted connectives

We conclude this chapter with recalling some facts regarding the connectives in the model theorey from [2, Chapter 6], which will be used in chapter 5. Throughout the rest of the thesis we will assume formulas are $[0,1]$-valued. This is not always an assumption (e.g., formulas in the model theory for operator algebras) in the continuous model theory. In particular we assume that the connectives are continuous functions from $[0,1]^n$ to $[0,1]$ for some $n \geq 1$.

Definition 4.3.1. *A closed system of connectives is a family $\mathcal{F} = (F_n : n \geq 1)$ where each F_n is a set of connectives $f : [0,1]^n \to [0,1]$ satisfying the following conditions.*

(i) For each n, F_n contains the projection onto the j^{th} coordinate for each $j = 1, \dots, n$.

(ii) For each n and m, if $u \in F_n$, and $v_1, \dots, v_n \in F_m$, then the function $w : [0,1]^m \to [0,1]$ defined by $w(\bar{t}) = u(v_1(\bar{t}), \dots, v_n(\bar{t}))$ belongs to F_m.

Definition 4.3.2. *Given a closed system of connectives $\mathcal{F}$, the collection of $\mathcal{F}$-*restricted *formulas is defined by induction.*

1. *Atomic formulas are $\mathcal{F}$-restricted.*

2. *If $u \in F_n$ and $\varphi_1, \ldots, \varphi_n$ are $\mathcal{F}$-restricted formulas, then $u(\varphi_1, \ldots, \varphi_n)$ is also an $\mathcal{F}$-restricted formula.*

3. *If φ is an $\mathcal{F}$-restricted formula, so are $\sup_x \varphi$ and $\inf_x \varphi$.*

Define a binary function $\dot{-} : [0,1]^2 \to [0,1]$ by

$$x \dot{-} y = \begin{cases} x - y & x \geq y \\ 0 & \text{otherwise} \end{cases}$$

and let $\mathcal{F}_0 = (F_n : n \geq 1)$ be the closed system of connectives generated from $\{0, 1, x/2, \dot{-}\}$ by closing it under (i) and (ii) (where 0 and 1 are constant functions with one variable).

Proposition 4.3.3. *[2, Proposition 6.6] The set of all $\mathcal{F}_0$-restricted $\mathcal{L}$-formulas are uniformly dense in the set of all $\mathcal{L}$-formulas; that is, for any $\epsilon > 0$ and any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, there is an $\mathcal{F}_0$-restricted $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ such that for all $\mathcal{L}$-structures $\mathcal{A}$ we have*

$$|\varphi(a_1, \ldots, a_n)^{\mathcal{A}} - \psi(a_1, \ldots, a_n)^{\mathcal{A}}| < \epsilon$$

for all $a_1, \ldots, a_n \in \mathcal{A}$. In particular if $\mathcal{L}$ is countable, there is a countable set of $\mathcal{L}$-formulas which is uniformly dense in the set of all $\mathcal{L}$-formulas.

5 A metric Feferman-Vaught theorem

In the classical model theory S. Feferman and R.L. Vaught ([26] and [6, §6.3]) gave an *effective* (recursive) way to determine the satisfaction of formulas in the reduced products of models of the same language, over the ideal of all finite sets, Fin. They showed the preservation of the elementary equivalence relation $\equiv$ by arbitrary direct products and also by reduced products over Fin. Later Frayne, Morel and Scott ([27]) noticed that the results extend to arbitrary reduced products (see also [57]). The classical Feferman-Vaught theorem effectively determines the truth value of a formula φ in reduced products of discrete structures $\{\mathcal{A}_\gamma : \gamma \in \Omega\}$ over an ideal $\mathcal{I}$ on Ω, by the truth values of certain formulas in the models $\mathcal{A}_\gamma$ and in the Boolean algebra $P(\Omega)/\mathcal{I}$. The model theory of reduced products of metric structures has been recently studied in [40] and [24].

In this chapter we prove a metric version of the Feferman-Vaught theorem (Theorem 5.1.3) for reduced products of metric structures, which also implies the preservation of $\equiv$ by arbitrary direct products, ultraproducts and reduced products of metric structures. This answers a question stated in [40]. We also use this theorem to solve an outstanding problem on coronas of C*-algebras (§5.3).

In [24] Farah-Shelah showed that the reduced products of a sequence of metric structures $\{\mathcal{A}_n : n \in \mathbb{N}\}$ over layered ideals (see Definition 1.2.4) are countably saturated. Hence under the Continuum Hypothesis the question whether two such reduced products are isomorphic reduces to the weaker question of whether they are elementarily equivalent. More generally, if $\mathcal{L}$ is a countable language, a transfinite extension of Cantor's back-and-forth method shows that for any uncountable cardinal κ, any two κ-saturated $\mathcal{L}$-structures of the same density character $\leq \kappa$ are isomorphic if and only if they are elementarily equivalent (see e.g., [22] or [2]).

We say an ideal $\mathcal{I}$ on $\mathbb{N}$ is *atomless* if the Boolean algebra $P(\mathbb{N})/\mathcal{I}$ is atomless. The metric extension of the Feferman-Vaught theorem is used to prove the following theorem.

Theorem 5.0.1. *Suppose $\mathcal{A}$ is a metric $\mathcal{L}$-structure and ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$ are atomless, then the reduced powers of $\mathcal{A}$ over $\mathcal{I}$ and $\mathcal{J}$ are elementarily equivalent.*

Therefore in particular if $\mathcal{A}$ is a separable C*-algebra then under the Continuum Hypothesis such reduced powers of $\mathcal{A}$, if they are countably saturated, are all isomorphic to $\ell_\infty(\mathcal{A})/c_0(\mathcal{A})$.

For an ultrafilter $\mathcal{U}$ Łoś's theorem implies that a metric structure $\mathcal{A}$ is elementarily equivalent to its ultrapower $\mathcal{A}^{\mathcal{U}}$. Therefore Farah-Shelah's result shows that under the Continuum Hypothesis if $\mathcal{A}$ is a separable C*-algebra, $\ell_\infty(\mathcal{A})/c_0(\mathcal{A})$ is isomorphic to its ultrapower associated with any nonprincipal ultrafilter on $\mathbb{N}$ ([24, Corollary 4.1]). Theorem 5.0.1 can be used (§5.2) to show that under the Continuum Hypothesis any reduced power of an asymptotic sequence algebra $\ell_\infty(\mathcal{A})/c_0(\mathcal{A})$ over an atomless layered ideal is also isomorphic to $\ell_\infty(\mathcal{A})/c_0(\mathcal{A})$ itself.

In section 5.3 we show there are two reduced products (of matrix algebras) which are isomorphic under the Continuum Hypothesis but there are no 'trivial' isomorphisms between them. Commutative examples of such reduced products are well-known, for example under the Continuum Hypothesis $C(\beta\omega \setminus \omega) \cong C(\beta\omega^2 \setminus \omega^2)$ (note that $\ell_\infty/c_0 \cong C(\beta\omega \setminus \omega)$), since by a well-known result of Parovičenko under ([44]) the Continuum Hypothesis $\beta\omega \setminus \omega$ and $\beta\omega^2 \setminus \omega^2$ are homeomorphic. However under the *proper forcing axiom* they are not isomorphic (see [8] and [11, Chapter 4]). A naive way to obtain non-trivial isomorphisms, under the Continuum Hypothesis, between "non-commutative" coronas is by tensoring $C(\beta\omega \setminus \omega)$ and $C(\beta\omega^2 \setminus \omega^2)$ with a full matrix algebra. However, such non-trivial isomorphisms are just amplifications of the non-trivial isomorphisms between their corresponding commutative factors (see section 5.3 for details). It was asked by I. Farah to give examples of non-commutative reduced products of C*-algebras which are non-trivially isomorphic under the Continuum Hypothesis, for non-commutative reasons. As we showed in chapter 3, assuming there is a measurable cardinal, it is relatively consistent with ZFC that all isomorphisms between reduced products of matrix algebras over analytic P-ideals (e.g., the corona of $\bigoplus \mathbb{M}_{k(n)}$) are *trivial* (Theorem 3.2.2).

Theorem 5.0.2. *There is an increasing sequence of natural numbers $\{k_\infty(i) : i \in \mathbb{N}\}$ such that if $\{g(i)\}$ and $\{h(i)\}$ are two subsequences of $\{k_\infty(i)\}$, then under the Continuum Hypothesis, $\mathcal{M}_g = \prod_i \mathbb{M}_{g(i)}/\bigoplus \mathbb{M}_{g(i)}$ is isomorphic to $\mathcal{M}_h = \prod_i \mathbb{M}_{h(i)}/\bigoplus M_{h_(i)}$. Moreover if there is a measurable cardinal, the following are equivalent.*

1. *$\mathcal{M}_g$ and $\mathcal{M}_h$ are isomorphic in ZFC.*
2. *$\mathcal{M}_g$ and $\mathcal{M}_h$ are trivially isomorphic, i.e., $\{g(i) : i \in \mathbb{N}\}$ and $\{h(i) : i \in \mathbb{N}\}$ are equal modulo finite sets.*

Thus if $\{g(i) : i \in \mathbb{N}\}$ and $\{h(i) : i \in \mathbb{N}\}$ are almost disjoint, this gives an example of two genuinely non-commutative reduced products for which the question “whether or not they are isomorphic?", is independent from ZFC. We will also show (Theorem 5.3.3) that there is an abundance of different theories of reduced products of sequences of matrix algebras, by exhibiting $2^{\aleph_0}$ pairwise non-elementarily equivalent such reduced products.

5.1 An extension of Feferman-Vaught theorem for reduced products of metric structures

The evaluation of a non-atomic formula in reduced products turns out to be more complicated than the atomic case, see [40]. In this section we give an extension of Feferman-Vaught theorem to reduced products of metric structures, which just like its classical version, gives a powerful tool to prove elementary equivalence of reduced products.

Suppose $\{\mathcal{A}_\gamma : \gamma \in \Omega\}$ is a family of metric structures in a fixed language $\mathcal{L}$ and $\mathcal{I}$ is an ideal on Ω. For the purposes of this section let

$$\mathcal{A}_\Omega = \prod_\Omega \mathcal{A}_\gamma, \qquad \mathcal{A}_\mathcal{I} = \prod_\mathcal{I} \mathcal{A}_\gamma.$$

For an $\mathcal{L}$-formula $\varphi(\bar{x})$, a tuple $\bar{a}$ of elements of $\mathcal{A}$ and

$$X = \{\gamma \in \Omega : \phi(\bar{a}(\gamma))^{\mathcal{A}_\gamma} > r\}$$

for some $r \in \mathbb{R}$, we use $\tilde{X}$ to denote the set

$$\tilde{X} = \{\gamma \in \Omega : \phi(\bar{a}(\gamma))^{\mathcal{A}_\gamma} \geq r\}.$$

Definition 5.1.1. *For an $\mathcal{F}_0$-restricted $\mathcal{L}$-formula (see §4.3) $\varphi(x_1, \dots, x_l)$, we say φ is determined up to 2^{-n} by $(\sigma_0, \dots, \sigma_{2^n}; \psi_0, \dots, \psi_{m-1})$ if*

1. *Each σ_i is a formula in the language of Boolean algebras with at most $s = m2^n$*

many variables, which is monotonic, i.e.,

$$T_{BA} \vdash \forall y_1 \ldots, y_s, z_1, \ldots, z_s(\sigma_i(y_1, \ldots, y_s) \wedge \bigwedge_{i=1}^{s} y_i \leq z_i \\ \rightarrow \sigma_i(z_1, \ldots, z_s)).$$

(Here, T_{BA} denotes the theory of Boolean algebras.)

2. *Each $\psi_j(x_1, \ldots, x_l)$ is an $\mathcal{F}_0$-restricted $\mathcal{L}$-formula for $j = 0, \ldots, m-1$.*

3. *For any indexed set Ω, an ideal $\mathcal{I}$ on Ω, a family $\{\mathcal{A}_\gamma : \gamma \in \Omega\}$ of metric $\mathcal{L}$-structures and $a_1, \ldots, a_l \in \mathcal{A}_\Omega$ the following hold:*

   for every $\ell = 0, \ldots, 2^n$

$$P(\Omega)/\mathcal{I} \models \sigma_\ell([X_0^0]_\mathcal{I}, \ldots, [X_{2^n}^0]_\mathcal{I}, \ldots, [X_0^{m-1}]_\mathcal{I}, \ldots, [X_{2^n}^{m-1}]_\mathcal{I}) \\ \Longrightarrow \varphi(\pi_\mathcal{I}(\bar{a}))^{\mathcal{A}_\mathcal{I}} > \ell/2^n,$$

   and

$$\varphi(\pi_\mathcal{I}(\bar{a}))^{\mathcal{A}_\mathcal{I}} > \ell/2^n \\ \Longrightarrow P(\Omega)/\mathcal{I} \models \sigma_\ell([\tilde{X}_0^0]_\mathcal{I}, \ldots, [\tilde{X}_{2^n}^0]_\mathcal{I}, \ldots, [\tilde{X}_0^{m-1}]_\mathcal{I}, \ldots, [\tilde{X}_{2^n}^{m-1}]_\mathcal{I})$$

   where $X_i^j = \{\gamma \in \Omega : \psi_j(\bar{a}(\gamma))^{\mathcal{A}_\gamma} > i/2^n\}$ for each $j = 0, \ldots, m-1$ and $i = 0, \ldots, 2^n$.

Using Lemma 4.3.3 we can generalize this definition to all $\mathcal{L}$-formulas.

Definition 5.1.2. *We say an $\mathcal{L}$-formula φ is determined up to 2^{-n} if there is an $\mathcal{F}_0$-restricted $\mathcal{L}$-formula $\tilde{\varphi}$ which is uniformly within 2^{-n-1} of φ and $\tilde{\varphi}$ is determined up to 2^{-n} by some $(\sigma_0, \dots, \sigma_{2^n}; \psi_0, \dots, \psi_{m-1})$.*

Theorem 5.1.3. *Every formula is determined up to 2^{-n} for any given $n \in \mathbb{N}$.*

Proof. By Definition 5.1.2 and Lemma 4.3.3, without loss of generality, we can assume that formulas are $\mathcal{F}_0$-restricted. Assume φ is an atomic $\mathcal{L}$-formula and for each $i \leq 2^n$ define

$$\sigma_i(y_0, \dots, y_{2^n}) := y_i \neq 0.$$

We show that φ is determined up to 2^{-n} by $(\sigma_0, \dots, \sigma_{2^n}; \varphi)$. Conditions (1) and (2) of Definition 5.1.1 are clearly satisfied. For an indexed set Ω, an ideal $\mathcal{I}$ on Ω, a family $\{\mathcal{A}_\gamma : \gamma \in \Omega\}$ of metric $\mathcal{L}$-structures and $a_1, \dots, a_l \in \mathcal{A}_\Omega$ let

$$X_i = \{\gamma \in \Omega : \varphi(\bar{a}(\gamma))^{\mathcal{A}_\gamma} > i/2^n\},$$

since $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} = \limsup_{\mathcal{I}} \varphi(\bar{a}(\gamma))^{\mathcal{A}_\gamma}$ we have

$$\begin{aligned} P(\Omega)/\mathcal{I} \vDash \sigma_\ell([X_0]_{\mathcal{I}}, \dots, [X_{2^n}]_{\mathcal{I}}) \quad &\Longleftrightarrow \quad X_\ell \notin \mathcal{I} \\ &\Longleftrightarrow \quad \varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n. \end{aligned}$$

Since each $X_i^j \subseteq \tilde{X}_i^j$, by the monotonicity of σ_ℓ, $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n$ also implies that $P(\Omega)/\mathcal{I} \vDash \sigma_\ell([\tilde{X}_0]_{\mathcal{I}}, \dots, [\tilde{X}_{2^n}]_{\mathcal{I}})$. Thus φ is determined up to 2^{-n} by $(\sigma_0, \dots, \sigma_{2^n}; \varphi)$.

Assume $\varphi(\bar{x}) = f(\alpha(\bar{x}))$ where $f \in \{0, 1, x/2\}$ and α is some $\mathcal{L}$-formula determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$. The cases where $f \in \{0, 1\}$ are trivial; for example if $f = 0$ then $\varphi(\bar{x})$ is determined up to 2^{-n} by $(\tau_0, \ldots, \tau_{2^n}; 0)$ where $\tau_i := 1 \neq 1$ for each $i = 0, \ldots, 2^n$. If $f(x) = x/2$ then it is also straightforward to check that φ is determined up to 2^{-n-1} by

$$(\sigma_0, \ldots, \sigma_{2^n}, \tau_{2^n+1}, \ldots, \tau_{2^{n+1}}; f(\psi_0), \ldots, f(\psi_{m-1})),$$

where each τ_i is a false sentence (e.g, $1 \neq 1$).

Let $\varphi(\bar{x}) = \alpha_1(\bar{x}) \dot{-} \alpha_2(\bar{x})$ where each α_t $(t \in \{1, 2\})$ is determined up to 2^{-n} by $(\sigma_0^t, \ldots, \sigma_{2^n}^t; \psi_0^t, \ldots, \psi_{m_t-1}^t)$. We claim that φ is determined up to 2^{-n} by $(\tau_0, \ldots, \tau_{2^n}; \psi_0^1, \ldots, \psi_{m_1-1}^1, 1-\psi_0^2, \ldots, 1-\psi_{m_2-1}^2)$ where the Boolean algebra formulas τ_k are defined by

$$\begin{aligned} \tau_k(x_0^0, \ldots, x_{2^n}^0, \ldots, x_0^{m_1-1} \quad , \ldots, \quad & x_{2^n}^{m_1-1}, z_0^0, \ldots, z_{2^n}^0, \ldots, z_0^{m_2-1}, \ldots, z_{2^n}^{m_2-1}) := \\ & \bigvee_{i_0=k}^{2^n} [\sigma_{i_0}^1(x_0^0, \ldots, x_{2^n}^0, \ldots, x_0^{m_1-1}, \ldots, x_{2^n}^{m_1-1}) \\ & \wedge \neg \sigma_{i_0-k}^2(-z_{2^n}^0, \ldots, -z_0^0, \ldots, -z_{2^n}^{m_2-1}, \ldots, -z_0^{m_2-1})], \end{aligned}$$

(here, $-z$ is the Boolean algebra complement of z). Conditions (1) and (2) in Definition 5.1.1 are clearly satisfied. For (3) let $\mathcal{A}_{\mathcal{I}}$ be a reduced product of $\mathcal{L}$-structures (indexed by Ω and over an ideal $\mathcal{I}$) and $a_1, \ldots, a_l \in \mathcal{A}_\Omega$. Let

$$X_i^j = \{\gamma \in \Omega : \psi_j^1(\bar{a}(\gamma))^{\mathcal{A}_\gamma} > i/2^n\} \qquad 0 \leq j \leq m_1 - 1,$$

$$Y_i^j = \{\gamma \in \Omega : \psi_j^2(\bar{a}(\gamma))^{\mathcal{A}_\gamma} > i/2^n\} \qquad 0 \le j \le m_2 - 1,$$

and

$$Z_i^j = \{\gamma \in \Omega : 1 - \psi_j^2(\bar{a}(\gamma))^{\mathcal{A}_\gamma} > i/2^n\} \qquad 0 \le j \le m_2 - 1.$$

Note that $\tilde{Y}^j_{2^n - i} = (Z_i^j)^c$ for each i and j. Assume

$$P(\Omega)/\mathcal{I} \vDash \tau_k([X_0^0]_{\mathcal{I}}, \ldots, [X_{2^n}^{m_1-1}]_{\mathcal{I}}, [Z_0^0]_{\mathcal{I}}, \ldots, [Z_{2^n}^{m_2-1}]_{\mathcal{I}}),$$

then for some $i_0 \ge k$,

$$\begin{aligned} P(\Omega)/\mathcal{I} \quad &\vDash \quad \sigma^1_{i_0}([X_0^0]_{\mathcal{I}}, \ldots, [X_{2^n}^{m_1-1}]_{\mathcal{I}}) \wedge \neg\sigma^2_{i_0-k}([(Z_{2^n}^0)^c]_{\mathcal{I}}, \ldots, [(Z_0^{m_2-1})^c]_{\mathcal{I}}) \\ &\implies \quad P(\Omega)/\mathcal{I} \vDash \sigma^1_{i_0}([X_0^0]_{\mathcal{I}}, \ldots, [X_{2^n}^{m_1-1}]_{\mathcal{I}}) \wedge \neg\sigma^2_{i_0-k}([\tilde{Y}_0^0]_{\mathcal{I}}, \ldots, [\tilde{Y}_{2^n}^{m_2-1}]_{\mathcal{I}}), \end{aligned}$$

and therefore

$$\alpha_1(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > i_0/2^n \text{ and } \alpha_2(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} \le (i_0 - k)/2^n.$$

Hence $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > k/2^n$. To prove the other direction assume $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > k/2^n$. For some $i_0 \ge k$,

$$\alpha_1(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > i_0/2^n \text{ and } \alpha_2(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} \le (i_0 - k)/2^n.$$

By the induction assumptions

$$P(\Omega)/\mathcal{I} \vDash \sigma^1_{i_0}([\tilde{X}_0^0]_{\mathcal{I}}, \ldots, [\tilde{X}_{2^n}^{m_1-1}]_{\mathcal{I}}) \wedge \neg\sigma^2_{i_0-k}([Y_0^0]_{\mathcal{I}}, \ldots, [Y_{2^n}^{m_1-1}]_{\mathcal{I}}),$$

and note that $(\tilde{Z}^j_{2^n-i})^c = Y^j_i$ for each i and j, which implies

$$
\begin{aligned}
P(\Omega)/\mathcal{I} \quad &\models \quad \sigma^1_{i_0}([\tilde{X}^0_0]_{\mathcal{I}}, \ldots, [\tilde{X}^{m_1-1}_{2^n}]_{\mathcal{I}}) \wedge \neg\sigma^2_{i_0-k}([(\tilde{Z}^0_{2^n})^c]_{\mathcal{I}}, \ldots, [(\tilde{Z}^{m_1-1}_0)^c]_{\mathcal{I}}), \\
&\Longrightarrow \quad P(\Omega)/\mathcal{I} \models \tau_k([\tilde{X}^0_0]_{\mathcal{I}}, \ldots, [\tilde{X}^{m_1-1}_{2^n}]_{\mathcal{I}}, [\tilde{Z}^0_0]_{\mathcal{I}}, \ldots, [\tilde{Z}^{m_2-1}_{2^n}]_{\mathcal{I}}).
\end{aligned}
$$

Therefore φ is determined up to 2^{-n} by $(\tau_0, \ldots, \tau_{2^n}; \psi^1_0, \ldots, \psi^1_{m_1-1}, 1-\psi^2_0, \ldots, 1-\psi^2_{m_2-1})$.

Assume $\varphi(\bar{x}) = \sup_z \psi(\bar{x}, z)$ where ψ is determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$. Let $d = 2^{n+m} - 1$ and $s_0, \ldots, s_{d-1}$ be an enumeration of non-empty elements of $\prod_{i=0}^{2^n} P(\{0, \ldots, m-1\})$, i.e, each $s_k = (s_k(0), \ldots, s_k(2^n))$ where $s_k(i) \subseteq \{0, \ldots, m-1\}$ for each i. Also assume that for each $0 \leq k \leq m-1$ we have $s_k = \{\{k\}, \emptyset, \ldots, \emptyset\}$. For any $k \in \{0, \ldots, d-1\}$ define an $\mathcal{L}$-formula θ_k by

$$\theta_k(\bar{x}) = \sup_z \min\{\psi_j(\bar{x}, z) : j \in \bigcup_{i=0}^{2^n} s_k(i)\}.$$

Note that if $0 \leq k \leq m-1$ then $\theta_k(\bar{x}) = \sup_z \psi_k(\bar{x}, z)$. For each $i \in \{0, \ldots, 2^n\}$ define a Boolean algebra formula τ_i by

$$
\begin{aligned}
\tau_i(y^0_0, \ldots, y^0_{2^n}, \ldots, y^{d-1}_0, \ldots, y^{d-1}_{2^n}) \quad = \quad & \exists z^0_0, \ldots, z^0_{2^n}, \ldots, z^{d-1}_0, \ldots, z^{d-1}_{2^n} \\
& [\bigwedge_{j=0}^{d-1} \bigwedge_{i=0}^{2^n} (z^j_i \leq y^j_i) \wedge \bigwedge_{i=0}^{2^n} \bigwedge_{\substack{s_k(t) \cup s_{k'}(t) = s_{k''}(t) \\ \forall t}} (z^k_i . z^{k'}_i = z^{k''}_i) \\
& \wedge \quad \sigma_i(z^0_0, \ldots, z^0_{2^n}, \ldots, z^{m-1}_0, \ldots, z^{m-1}_{2^n})].
\end{aligned}
$$

We claim that φ is determined up to 2^{-n} by $(\tau_0, \ldots, \tau_{2^n}; \theta_0, \ldots, \theta_{d-1})$. Again condition (1) is clearly satisfied. Condition (2) is also satisfied, since $\min\{x, y\} =$

$x \dot{-} (x \dot{-} y)$. For (3) assume a reduced product $\mathcal{A}_\mathcal{I}$ and $a_1, \ldots, a_l \in \mathcal{A}_\Omega$ are given.

First assume $\varphi(\pi_\mathcal{I}(\bar{a}))^{\mathcal{A}_\mathcal{I}} > \ell/2^n$ for some ℓ. Let

$$\delta = \frac{\min\{\varphi(\pi_\mathcal{I}(\bar{a}))^{\mathcal{A}_\mathcal{I}} - \ell/2^n, 1/2^n\}}{2}$$

and find $c = (c(\gamma))_{\gamma\in\Omega}$ such that

$$\psi(\pi_\mathcal{I}(\bar{a}, c))^{\mathcal{A}_\mathcal{I}} > \varphi(\pi_\mathcal{I}(\bar{a}))^{\mathcal{A}_\mathcal{I}} - \delta > \ell/2^n.$$

For each $i \leq 2^n$ and $k \leq d-1$ let

$$Y_i^k = \{\gamma \in \Omega : \theta_k(\bar{a}(\gamma))^{\mathcal{A}_\gamma} > i/2^n\},$$

and let

$$Z_i^k = \{\gamma \in \Omega : \min\{\psi_j(\bar{x}(\gamma), c(\gamma)) : j \in \bigcup_{t=0}^{2^n} s_k(t)\}^{\mathcal{A}_\gamma} > i/2^n\}.$$

From definition of θ_k it is clear that $\tilde{Z}_i^k \subseteq \tilde{Y}_i^k$, and

$$\bigwedge_{i=0}^{2^n} \bigwedge_{\substack{s_k(t)\cup s_{k'}(t)=s_{k''}(t)\\ \forall t}} (\tilde{Z}_i^k \cap \tilde{Z}_i^{k'} = \tilde{Z}_i^{k''}),$$

and by the inductive assumption

$$P(\Omega)/\mathcal{I} \models \sigma_\ell([\tilde{Z}_0^0]_\mathcal{I}, \ldots, [\tilde{Z}_{2^n}^0]_\mathcal{I}, \ldots, [\tilde{Z}_0^{m-1}]_\mathcal{I}, \ldots, [\tilde{Z}_{2^n}^{m-1}]_\mathcal{I}).$$

Hence $P(\Omega)/\mathcal{I} \models \tau_\ell([\tilde{Y}_0^0]_\mathcal{I}, \ldots, [\tilde{Y}_{2^n}^0]_\mathcal{I}, \ldots, [\tilde{Y}_0^{d-1}]_\mathcal{I}, \ldots, [\tilde{Y}_{2^n}^{d-1}]_\mathcal{I})$.

For the other direction let

$$Y_i^k = \{\gamma \in \Omega : \theta_k(\bar{a}(\gamma))^{\mathcal{A}_\gamma} > i/2^n\},$$

and suppose $P(\Omega)/\mathcal{I} \models \tau_\ell([Y_0^0]_\mathcal{I}, \ldots, [Y_{2^n}^0]_\mathcal{I}, \ldots, [Y_0^{d-1}]_\mathcal{I}, \ldots, [Y_{2^n}^{d-1}]_\mathcal{I})$. There are sets $Z_0^0, \ldots, Z_{2^n}^0, \ldots, Z_0^{d-1}, \ldots, Z_{2^n}^{d-1}$ such that the following hold.

$$[Z_i^k]_\mathcal{I} \subseteq [Y_i^k]_\mathcal{I} \qquad 0 \leq i \leq 2^n, 0 \leq k \leq d-1,$$

$$[Z_i^k]_\mathcal{I} \cap [Z_i^{k'}]_\mathcal{I} = [Z_i^{k''}]_\mathcal{I} \qquad \forall t \;\; s_k(t) \cup s_{k'}(t) = s_{k''}(t),$$

$$P(\Omega)/\mathcal{I} \models \sigma_\ell([Z_0^0]_\mathcal{I}, \ldots, [Z_{2^n}^0]_\mathcal{I}, \ldots, [Z_0^{m-1}]_\mathcal{I}, \ldots, [Z_{2^n}^{m-1}]_\mathcal{I}).$$

Since there are only finitely many conditions above, we can find a set $S \in \mathcal{I}$ such that if $D = \Omega \setminus S$ then

$$Z_i^k \cap D \subseteq Y_i^k \qquad 0 \leq i \leq 2^n, 0 \leq k \leq d-1, \tag{5.1}$$

$$Z_i^k \cap Z_i^{k'} \cap D = Z_i^{k''} \cap D \qquad \forall t \;\; s_k(t) \cup s_{k'}(t) = s_{k''}(t),$$

Fix $\gamma \in D$, and for each $i \in \{0, \ldots, 2^n\}$ let $u(i) = \{j \in \{0, \ldots, m-1\} : \gamma \in Z_i^j\}$. If $k \in \{0, \ldots, d-1\}$ be such that $s_k = (u(0), \ldots, u(2^n))$, then since $\gamma \in Z_i^j$ for all $j \in u(i)$, using (5.1) we have $\gamma \in Y_i^k$ (for all i) and hence

$$\theta_k(\bar{a}(\gamma))^{\mathcal{A}_\mathcal{I}} = \sup_z \min_{j \in \cup_{t=0}^{2^n} u(t)} \psi_j(\bar{a}(\gamma), z) > i/2^n.$$

Let

$$\delta = \frac{\min_{i,k}\{\theta_k(\bar{a}(\gamma))^{\mathcal{A}_\mathcal{I}} - i/2^n, 1/2^n\}}{2}.$$

We can pick $c(\gamma) \in A_\gamma$ such that for every i

$$\min_{j \in u(i)} \psi_j(\bar{a}(\gamma), c(\gamma)) > \theta_k(a(\gamma)) - \delta \geq i/2^n. \tag{5.2}$$

For $\gamma \notin D$ define $c(\gamma)$ arbitrarily and let $c = (c(\gamma))_{\gamma\in\Omega}$. For each $j \in \{0, \dots, m-1\}$ and $i \in \{0, \dots, 2^n\}$ let

$$X_i^j = \{\gamma \in \Omega : \psi_j(\bar{a}(\gamma), c(\gamma))^{\mathcal{A}_\gamma} > i/2^n\}.$$

Now (5.1) and (5.2) imply that $Z_i^j \cap D \subseteq X_i^j$ for all i and j. Therefore

$$P(\Omega)/\mathcal{I} \vDash \bigwedge_{j=0}^{m-1} \bigwedge_{i=0}^{2^n} [Z_i^j]_{\mathcal{I}} \leq [X_i^j]_{\mathcal{I}}.$$

Since $P(\Omega)/\mathcal{I} \vDash \sigma_\ell([Z_0^0]_{\mathcal{I}}, \dots, [Z_{2^n}^0]_{\mathcal{I}}, \dots, [Z_0^{m-1}]_{\mathcal{I}}, \dots, [Z_{2^n}^{m-1}]_{\mathcal{I}})$, by monotonicity of σ_ℓ we have

$$P(\Omega)/\mathcal{I} \vDash \sigma_\ell([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^0]_{\mathcal{I}}, \dots, [X_0^{m-1}]_{\mathcal{I}}, \dots, [X_{2^n}^{m-1}]_{\mathcal{I}}).$$

Therefore by the induction assumption we have

$$\psi(\pi_{\mathcal{I}}(\bar{a}, c))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n,$$

which implies that $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n$. $\square$

Let us give some interesting applications of Theorem 5.1.3. Assume $\{\mathcal{A}_\gamma : \gamma \in \Omega\}$ and $\{\mathcal{B}_\gamma : \gamma \in \Omega\}$ are families of metric $\mathcal{L}$-structures indexed by Ω and for an ideal $\mathcal{I}$ on Ω let $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{B}_{\mathcal{I}}$ denote the corresponding reduced products over $\mathcal{I}$. Next proposition shows that if each $\mathcal{A}_\gamma \equiv \mathcal{B}_\gamma$ for $\gamma \in \Omega$ then $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{B}_{\mathcal{I}}$ are also elementarily equivalent.

Proposition 5.1.4. *Reduced products, direct products and ultraproducts preserve elementary equivalence.*

Proof. We only need to show this for reduced products, since the others are special cases of reduced products. Let $\mathcal{A}_\mathcal{I}$ and $\mathcal{B}_\mathcal{I}$ be two reduced products over ideal $\mathcal{I}$ such that $\mathcal{A}_\gamma \equiv \mathcal{B}_\gamma$ for every $\gamma \in \Omega$. Let φ be an $\mathcal{F}_0$-restricted $\mathcal{L}$-sentence. For a given $n \in \mathbb{N}$ suppose φ is determined up to 2^{-n} by $(\sigma_0, \dots, \sigma_{2^n}; \psi_0, \dots, \psi_{m-1})$. For each $i \in \{0, \dots, 2^n\}$ and $j \in \{0, \dots, m-1\}$ let

$$X_i^j = \{\gamma \in \Omega : \psi_j^{\mathcal{A}_\gamma} > i/2^n\},$$

$$Y_i^j = \{\gamma \in \Omega : \psi_j^{\mathcal{B}_\gamma} > i/2^n\}.$$

By our assumption $X_i^j = Y_i^j$ for all i and j. Therefore

$$P(\Omega)/\mathcal{I} \vDash \sigma_i([X_0^0]_\mathcal{I}, \dots, [X_{2^n}^{m-1}]_\mathcal{I}) \leftrightarrow \sigma_i([Y_0^0]_\mathcal{I}, \dots, [Y_{2^n}^{m-1}]_\mathcal{I}),$$

and

$$P(\Omega)/\mathcal{I} \vDash \sigma_i([\tilde{X}_0^0]_\mathcal{I}, \dots, [\tilde{X}_{2^n}^{m-1}]_\mathcal{I}) \leftrightarrow \sigma_i([\tilde{Y}_0^0]_\mathcal{I}, \dots, [\tilde{Y}_{2^n}^{m-1}]_\mathcal{I}),$$

which implies that for each i,

$$\varphi^{\mathcal{A}_\mathcal{I}} > i/2^n \iff \varphi^{\mathcal{B}_\mathcal{I}} > i/2^n.$$

Since n was arbitrary this implies that $\varphi^{\mathcal{A}_\mathcal{I}} \geq r$ if and only if $\varphi^{\mathcal{B}_\mathcal{I}} \geq r$ for any real number r. Applying the same argument for $1 - \varphi$ instead of φ, we get $\varphi^{\mathcal{A}_\mathcal{I}} \leq r$

if and only if $\varphi^{\mathcal{B}_\mathcal{I}} \leq r$ for any real number r. Therefore $\varphi^{\mathcal{A}_\mathcal{I}} = \varphi^{\mathcal{B}_\mathcal{I}}$. Since $\mathcal{F}_0$-restricted form $\mathcal{L}$-sentences are uniformly dense in the set of all $\mathcal{L}$-sentences, we have $\mathcal{A}_\mathcal{I} \equiv \mathcal{B}_\mathcal{I}$. □

Theorem 5.1.5. *Assume $\mathcal{A}$ is an $\mathcal{L}$-structure and $\mathcal{I}$ and $\mathcal{J}$ are atomless ideals on Ω, then the reduced powers of $\mathcal{A}$ over $\mathcal{I}$ and $\mathcal{J}$ are elementarily equivalent.*

Proof. Let $\mathcal{A}_\mathcal{I}$ and $\mathcal{A}_\mathcal{J}$ denote the reduced powers of $\mathcal{A}$ ($\mathcal{A}_\gamma = \mathcal{A}$ for all $\gamma \in \Omega$) over $\mathcal{I}$ and $\mathcal{J}$, respectively. Let φ be an $\mathcal{L}$-sentence and for $n \geq 1$ find an $\mathcal{F}_0$-restricted $\mathcal{L}$-sentence $\tilde{\varphi}$ which is uniformly within 2^{-n} of φ and is determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$. Then

$$X_i^j = \{\gamma \in \Omega : \psi_j^{\mathcal{A}_\gamma} > i/2^n\}$$

is clearly either Ω or $\emptyset$, therefore $[X_i^j]_\mathcal{I} = [X_i^j]_\mathcal{J} = 0$ or 1. Since any two atomless Boolean algebras are elementarily equivalent, for every $i = 0, \ldots, 2^n$

$$P(\Omega)/\mathcal{I} \vDash \sigma_i([X_0^0]_\mathcal{I}, \ldots, [X_{2^n}^{m-1}]_\mathcal{I}) \iff P(\Omega)/\mathcal{J} \vDash \sigma_i([X_0^0]_\mathcal{J}, \ldots, [X_{2^n}^{m-1}]_\mathcal{J}).$$

Thus by Theorem 5.1.3 and the same argument as the proof of Proposition 5.1.4 we have $\varphi^{\mathcal{A}_\mathcal{I}} = \varphi^{\mathcal{B}_\mathcal{I}}$. □

5.2 Isomorphisms of reduced products under the Continuum Hypothesis

In C*-algebra context an important class of corona algebras is the reduced power of a C*-algebra $\mathcal{A}$ over the Fréchet ideal Fin. It is called the *asymptotic sequence algebra* of $\mathcal{A}$ and denoted by $\ell_\infty(\mathcal{A})/c_0(\mathcal{A})$. The C*-algebra $\mathcal{A}$ can be identified with its diagonal image in $\ell_\infty(\mathcal{A})/c_0(\mathcal{A})$. We will also use the same notation $\ell_\infty(\mathcal{A})/c_0(\mathcal{A})$ for the reduced power of an arbitrary metric structure $\mathcal{A}$ over Fin.

As mentioned in the introduction a result of Farah-Shelah shows that asymptotic sequence algebras are countably saturated and therefore if $\mathcal{A}$ is separable, assuming the Continuum Hypothesis, they have $2^{\aleph_1}$ automorphisms (Theorem 4.2.4), hence non-trivial ones. Furthermore, since saturated structures of the same density character which are elementarily equivalent are isomorphic, under the Continuum Hypothesis, $\ell_\infty(\mathcal{A})/c_0(\mathcal{A})$ for a separable $\mathcal{A}$, is isomorphic to its ultrapower associated with any nonprincipal ultrafilter on $\mathbb{N}$. We will show that this is also the case for the reduced powers of separable metric structures over a large family of ideals (Corollary 5.2.4). The following is a more general version of Theorem 4.2.3.

Theorem 5.2.1 (Farah-Shelah)**.** *Every reduced product $\prod_n \mathcal{A}_n / \bigoplus_{\mathcal{I}} \mathcal{A}_n$ is countably saturated if $\mathcal{I}$ is a layered ideal.*

Proof. See [24, Theorem 2.7]. □

Therefore an immediate consequence of Theorem 5.0.1 and Theorem 5.2.1 implies the following corollary.

Corollary 5.2.2. *Assume the Continuum Hypothesis. If $\mathcal{A}$ is a separable metric structure, $\mathcal{I}$ and $\mathcal{J}$ are atomless layered ideals, then the reduced powers $\mathcal{A}_\mathcal{I}$ and $\mathcal{A}_\mathcal{J}$ are isomorphic.*

In Corollary 5.2.4 we give an application of this result, but before we need the following lemma. Recall that for any $A \subseteq \mathbb{N} \times \mathbb{N}$ the *vertical section* of A at m is the set $A_m = \{n \in \mathbb{N} : (m, n) \in A\}$.

Lemma 5.2.3. *Suppose $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\mathbb{N}$ and $\mathcal{A}_\mathcal{I}$ is the reduced power of $\mathcal{A}$ over the ideal $\mathcal{I}$. Then*

$$\frac{\prod \mathcal{A}_\mathcal{I}}{\bigoplus_\mathcal{J} \mathcal{A}_\mathcal{I}} \cong \frac{\prod_{\mathbb{N}^2} \mathcal{A}}{\bigoplus_{\mathcal{J}\times\mathcal{I}} \mathcal{A}}.$$

Proof. Assume $\langle a_{n,m}\rangle$ is an element of $\prod_{\mathbb{N}^2} \mathcal{A}$. Define the map $\rho : \prod_{\mathbb{N}^2} \mathcal{A}/\bigoplus_{\mathcal{J}\times\mathcal{I}} \mathcal{A} \to \prod(\mathcal{A}_\mathcal{I})/\bigoplus_\mathcal{J}(\mathcal{A}_\mathcal{I})$ by

$$\rho(\pi_{\mathcal{J}\times\mathcal{I}}(\langle a_{m,n}\rangle)) = \pi_\mathcal{J}(\langle b_m\rangle),$$

where $b_m = \pi_\mathcal{I}(\langle a_{m,n}\rangle_n)$ for each $m \in \mathbb{N}$. In order to see this map is well-defined assume $\pi_{\mathcal{J}\times\mathcal{I}}(\langle a_{m,n}\rangle) = 0$. If $\pi_\mathcal{J}(\langle b_m\rangle) \neq 0$, then there is $\epsilon > 0$ such that for every $S \in \mathcal{J}$ we have

$$\sup_{m\notin S} \|b_m\|_{\mathcal{A}_\mathcal{I}} \geq \epsilon.$$

Since $\pi_{\mathcal{J}\times\mathcal{I}}(\langle a_{n,m}\rangle) = 0$, there is $X \in \mathcal{J}\times\mathcal{I}$ such that

$$\sup_{(m,n)\notin X} \|\langle a_{m,n}\rangle)\|_{\mathcal{A}} < \epsilon/4.$$

The set $S = \{m : X_m \notin \mathcal{I}\}$ belongs to $\mathcal{J}$ and hence $\sup_{m\notin S}\|b_m\|_{\mathcal{A}_\mathcal{I}} \geq \epsilon$. Pick $m_0 \notin S$ such that $\|b_{m_0}\|_{\mathcal{A}_\mathcal{I}} \geq \epsilon/2$ and then pick $n_0 \notin X_{m_0}$ such that $\|a_{m_0,n_0}\|_{\mathcal{A}} \geq \epsilon/4$, which is a contradiction. Therefore $\pi_{\mathcal{J}}(\langle b_m\rangle) = 0$.

To show the injectivity of ρ assume $\pi_{\mathcal{J}}(\langle b_m\rangle) = 0$. Therefore for every $\epsilon > 0$ there is $S \in \mathcal{J}$ such that $\|b_m\|_{\mathcal{A}_\mathcal{I}} \leq \epsilon$ for every $m \in \mathbb{N}\setminus S$. So for each $m \in \mathbb{N}\setminus S$ there is $X_m \in \mathcal{I}$ such that

$$\sup_{n\notin X_m} \|a_{(m,n)}\|_{\mathcal{A}} \leq 2\epsilon.$$

The set $X = (S\times\mathbb{N}) \cup \{(m,n) : n \in X_m\}$ belongs to the ideal $\mathcal{J}\times\mathcal{I}$ and

$$\sup_{(m,n)\notin X} \|a_{m,n}\|_{\mathcal{A}} \leq 2\epsilon.$$

Therefore $\pi_{\mathcal{J}\times\mathcal{I}}(\langle a_{n,m}\rangle) = 0$. It is easy to check that ρ is a surjective *-homomorphism.

□

The following corollary follows form Lemma 1.2.5 and Corollary 5.2.2.

Corollary 5.2.4. *Assume the Continuum Hypothesis. Suppose* $\mathfrak{A} = \ell_\infty(\mathcal{A})/c_0(\mathcal{A})$ *is the asymptotic sequence algebra of* $\mathcal{A}$ *and* $\mathcal{I}$ *is an atomless layered ideal on* $\mathbb{N}$*, then*

$$\frac{\prod \mathfrak{A}}{\bigoplus_\mathcal{I} \mathfrak{A}} \cong \mathfrak{A}.$$

5.3 Non-trivially isomorphic reduced products of matrix algebras.

In this section we use Theorem 5.1.3 in order to prove the existence of two reduced products (of matrix algebras) which are isomorphic under the Continuum Hypothesis, but not isomorphic in ZFC. Note that in the model theory for operator algebras, the ranges of formulas are bounded subsets of reals possibly different from $[0,1]$ (see for example [22]) . Nevertheless Definition 5.1.1 can be easily adjusted for any formula in the language of C*-algebras $\mathcal{L}$ and Theorem 5.1.3 can be proved similarly.

As mentioned in the introduction, commutative examples of such reduced products are well-known, for example by a classical result of Parovičenko, under the Continuum Hypothesis $(\ell_\infty(\mathbb{N})/c_0(\mathbb{N}) \cong) C(\beta\omega \setminus \omega) \cong C(\beta\omega^2 \setminus \omega^2)$, however under the proper forcing axiom they are not isomorphic, since there are no trivial isomorphisms between them (see [11, Chapter 4]). Other examples of non-trivial isomorphisms between (non-commutative) reduced products can be obtained by tensoring a matrix algebra with these commutative algebras. Recall that ([3]) for a locally compact Hausdorff topological space X and for any C*- algebra $\mathcal{A}$, $C_0(X, \mathcal{A})$ can be identified with $C_0(X) \otimes \mathcal{A}$, under the map $(f \otimes a)(x) = f(x)a$. Let M_n denote the algebra of all $n \times n$ matrices over the field of complex numbers. Assume

$\mathfrak{A} = \ell_\infty(M_2)/c_0(M_2)$ is the asymptotic sequence algebra of M_2, we have

$$\mathfrak{A} \cong C(\beta\omega \setminus \omega) \otimes M_2 \cong M_2(C(\beta\omega \setminus \omega)),$$

and Corollary 5.2.4 implies that

$$\frac{\ell_\infty(\mathfrak{A})}{c_0(\mathfrak{A})} \cong \frac{\prod_{\mathbb{N}^2} M_2}{\bigoplus_{Fin\times Fin} M_2} \equiv \mathfrak{A}.$$

Since $\prod_{\mathbb{N}^2} M_2/\bigoplus_{Fin\times Fin} M_2 \cong M_2(C(\beta\omega^2 \setminus \omega^2))$, for the same reason as the commutative case, under the Continuum Hypothesis $\mathfrak{A}$ and $\ell_\infty(\mathfrak{A})/c_0(\mathfrak{A})$ are non-trivially isomorphic.

Lemma 5.3.1. *There is an increasing sequence of natural numbers $\{k_\infty(i) : i \in \mathbb{N}\}$ such that for every $\mathcal{L}$-sentence ψ*

$$\lim_i \psi^{M_{k_\infty(i)}} = r_\psi$$

for some real number r_ψ.

Proof. Let $\psi_1, \psi_2, \ldots$ be an enumeration of all $\mathcal{F}_0$-restricted $\mathcal{L}$-sentences (or any countable uniformly dense sequence of $\mathcal{L}$-sentences). Starting with ψ_1, since the range of it is a bounded set, find a sequence $\{k_1(i)\}$ such that $\psi_1^{M_{k_1(i)}} \to r_{\psi_1}$ for some r_{ψ_1}. Similarly find a subsequence $\{k_2(i)\}$ of $\{k_1(i)\}$ such that $\psi_2^{M_{k_2(i)}} \to r_{\psi_2}$ for some r_{ψ_2}, and so on. If we let

$$k_\infty(i) = k_i(i) \qquad i \in \mathbb{N},$$

since $\mathcal{F}_0$-restricted $\mathcal{L}$-sentences are uniformly dense in the set of all $\mathcal{L}$-sentences, $\{k_\infty(i)\}$ has the required property. □

Proposition 5.3.2. *For any ideal $\mathcal{I}$ on $\mathbb{N}$ containing all finite sets, if $\{g(i)\}$ and $\{h(i)\}$ are two almost disjoint subsequences of $\{k_\infty(i)\}$, then*

$$\frac{\prod_i \mathbb{M}_{g(i)}}{\bigoplus_{\mathcal{I}} \mathbb{M}_{g(i)}} \equiv \frac{\prod_i \mathbb{M}_{h(i)}}{\bigoplus_{\mathcal{I}} M_{h_(i)}},$$

hence if $\mathcal{I}$ is a layered P-ideal, they are isomorphic under the Continuum Hypothesis, with no trivial isomorphisms between them.

This together with Proposition 3.2.4 implies that these reduced products are not isomorphic in ZFC, and therefore Theorem 5.0.2 follows.

Proof. Let φ be an $\mathcal{L}$-sentence and for $n \geq 1$ find an $\mathcal{F}_0$-restricted $\mathcal{L}$-sentence $\tilde{\varphi}$ which is uniformly within 2^{-n} of φ and it is determined up to 2^{-n} by

$$(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1}).$$

Let

$$X_i^j = \{l \in \mathbb{N} : \psi_j^{M_{g(l)}} > i/2^n\}$$

and

$$Y_i^j = \{l \in \mathbb{N} : \psi_j^{M_{h(l)}} > i/2^n\}.$$

Since by Lemma 5.3.1 the sequence $\{\psi_j^{M_{k_\infty(l)}} : l \in \mathbb{N}\}$ is Cauchy for each $j \in \{0, \ldots, m-1\}$, and $\mathcal{I}$ contains all finite sets, we have $[X_i^j]_{\mathcal{I}} = [Y_i^j]_{\mathcal{I}}$. Hence Theorem

5.1.3 implies that $\tilde{\varphi}^{\mathcal{A}_\mathcal{I}} = \tilde{\varphi}^{\mathcal{B}_\mathcal{I}}$. By uniform density of $\mathcal{F}_0$-restricted $\mathcal{L}$-sentences, the result follows. □

The following theorem shows the abundance of different theories of reduced products of matrix algebras.

Theorem 5.3.3. *For any ideal $\mathcal{I}$, there are $2^{\aleph_0}$-many reduced products of matrix algebras over $\mathcal{I}$ which are pairwise non-elementarily equivalent.*

Proof. Let $E = \{p_1, p_2, \dots\} \subset \mathbb{N}$ be an increasing enumeration of prime numbers. Assume $\{A_\xi : \xi < 2^{\aleph_0}\}$ is an almost disjoint family of subsets of E. Let $A_\xi = \{n_1^\xi, n_2^\xi, \dots\}$ be an increasing enumeration of A_ξ. For each $\xi < 2^{\aleph_0}$ define a sequence $\langle k^\xi(n)\rangle$ of natural numbers by

$$\langle k^\xi(n)\rangle = \langle n_1^\xi, n_1^\xi n_2^\xi, n_1^\xi n_2^\xi n_3^\xi, \dots\rangle.$$

We will show that for any distinct $\xi, \eta < 2^{\aleph_0}$ the reduced products $\prod M_{k^\xi(n)}/\bigoplus_\mathcal{I} M_{k^\xi(n)}$ and $\prod M_{k^\eta(n)}/\bigoplus_\mathcal{I} M_{k^\eta(n)}$ are not elementarily equivalent.

Fix such ξ and η. Since A_ξ and A_η are almost disjoint, pick m such that $\{n_m^\xi, n_{m+1}^\xi, \dots\} \cap \{n_1^\eta, n_2^\eta, \dots\} = \emptyset$. Define a formula $\varphi(\bar{x}, \bar{y})$ by

$$\begin{aligned}\varphi(x_1, \dots, x_{n_m^\xi}, y_2, \dots, y_{n_m^\xi}) \;=\; & \sum_{i=1}^{n_m^\xi} (\|x_i^2 - x_i\| + \|x_i^* - x_i\|) + \|\sum_{i=1}^{n_m^\xi} x_i - 1\| \\ & + \sum_{i\neq j}^{n_m^\xi} \|x_i x_j\| + \sum_{i=2}^{n_m^\xi} (\|y_i y_i^* - x_1\| + \|y_i^* y_i - x_i\|).\end{aligned}$$

For a unital C*-algebra $\mathcal{A}$ and tuples $\bar{a}$ and $\bar{v}$ if $\varphi(\bar{a},\bar{v})^{\mathcal{A}} = 0$ then $a_1, a_2, \ldots, a_{n^\xi_m}$ are orthogonal pairwise Murray-von Neumann equivalent projections of $\mathcal{A}$, and therefore $M_{n^\xi_m}$ can be embedded into $\mathcal{A}$. Since for every $j \geq m$

$$n^\xi_m \mid n^\xi_1 n^\xi_2 \ldots n^\xi_j,$$

find a tuple of projections $\bar{a}(k^\xi(j))$ and a tuple of partial isometries $\bar{v}(k^\xi(j))$ in $M_{k^\xi(j)}$ such that $\varphi(\bar{a}(k^\xi(j)), \bar{v}(k^\xi(j)))^{M_{k^\xi(j)}} = 0$. The fact that $\mathcal{I}$ contains all the finite sets and Lemma 4.1.2 implies that

$$\varphi(\pi_{\mathcal{I}}(\bar{a}), \pi_{\mathcal{I}}(\bar{v}))^{\prod M_{k^\xi(n)} / \bigoplus_{\mathcal{I}} M_{k^\xi(n)}} = 0.$$

However since

$$n^\xi_m \nmid n^\eta_1 n^\eta_2 \ldots n^\eta_k$$

for every $k \in \mathbb{N}$, a similar argument shows that for every tuples $\bar{b}$ and $\bar{u}$ in $\prod M_{k^\eta(n)}$,

$$\varphi(\pi_{\mathcal{I}}(\bar{b}), \pi_{\mathcal{I}}(\bar{u}))^{\prod M_{k^\eta(n)} / \bigoplus_{\mathcal{I}} M_{k^\eta(n)}} \neq 0.$$

Hence $\prod M_{k^\xi(n)} / \bigoplus_{\mathcal{I}} M_{k^\xi(n)} \not\equiv \prod M_{k^\eta(n)} / \bigoplus_{\mathcal{I}} M_{k^\eta(n)}$. □

5.4 Further remarks and questions

For a locally compact Hausdorff topological space X and a metric structure $\mathcal{A}$ the *continuous reduced products* $C_b(X,\mathcal{A})/C_0(X,\mathcal{A})$ are studied as models for metric structures in [24], where in particular it has been shown that certain continuous reduced products, e.g., $C([0,1)^*)$, are *countably saturated.* In general $C_b(X, \mathcal{A})$ is a submodel of $\prod_{t \in X} \mathcal{A}$ and one may hope to use a similar approach as in section 4 in order to prove the following preservation (of $\equiv$) question.

Question (1). Assume $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent metric structures and X is a locally compact, non-compact Polish space. Are $C_b(X, \mathcal{A})/C_0(X, \mathcal{A})$ and $C_b(X, \mathcal{B})/C_0(X, \mathcal{B})$ elementarily equivalent?

Note that if X is a discrete space (e.g., $\mathbb{N}$) this follows from Proposition 5.1.4 since $C_b(X, \mathcal{A})/C_0(X, \mathcal{A}) \cong \prod_{t \in X} \mathcal{A}/\bigoplus \mathcal{A}$.

In [20] the authors showed the existence of two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv \mathcal{B}$, where $C([0,1]) \otimes \mathcal{A} \not\equiv C([0,1]) \otimes \mathcal{B}$, i.e., tensor products in the category of C*-algebras, do not preserve elementary equivalence.

The Cone and Suspension Algebras. Let $\mathcal{A}$ be a C*-algebra. The *cone* $C\mathcal{A} = C_0((0,1], \mathcal{A})$ and *suspension* $S\mathcal{A} = C_0((0,1), \mathcal{A})$ over $\mathcal{A}$ are the most important examples of contractible and subcontractible C*-algebras ([3]). Since $S\mathcal{A} \subset C\mathcal{A}$ and $C\mathcal{A}$ is homotopic to $\{0\}$ (contractible) by a well-known result of D. Voiculescu ([55]) both $C\mathcal{A}$ and $S\mathcal{A}$ are *quasidiagonal* C*-algebras. Every quasidiagonal C*-algebra embeds into a reduced product of full matrix algebras over the Fréchet ideal, $\prod \mathbb{M}_{k(n)}/\bigoplus \mathbb{M}_{k(n)}$, for some sequence $\{k(n)\}$ (such C*-algebras are called MF, see e.g., [5] and [3]). In general it is easy to check that if a metric structure $\mathcal{A}$ embeds into $\mathcal{B}$ then the universal theory of $\mathcal{A}$ (see §4), $Th_\forall(\mathcal{A})$, contains the universal theory of $\mathcal{B}$, $Th_\forall(\mathcal{A})$. Hence for any C*-algebra $\mathcal{A}$

$$C\mathcal{A} \hookrightarrow \prod \mathbb{M}_{k(n)} / \bigoplus \mathbb{M}_{k(n)}$$

for some $\{k(n)\}$, which implies that

$$Th_\forall(C\mathcal{A}) \supseteq Th_\forall(\prod \mathbb{M}_{k(n)} / \bigoplus \mathbb{M}_{k(n)}).$$

In general it is not clear "how the theory of $C_0(X, \mathcal{A})$ is related to the theory of $\mathcal{A}$".

Question (2). Assume $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent C*-algebras. For which locally compact, Hausdorff spaces, like X, $C_0(X) \otimes \mathcal{A} \equiv C_0(X) \otimes \mathcal{B}$ is true?

6 Countably degree-1 staturated and SAW*-algebras

6.1 Sub-Stonean spaces and SAW*-algebras

The class of SAW^*-algebras was introduced by G.K. Pedersen [46] as non-commutative analogues of sub-Stonean spaces (also known as F-spaces) in topology, which are the locally compact Hausdorff spaces in which disjoint σ-compact open subspaces have disjoint compact closure. In [46] Peredsen generalized some of the remarkable properties of sub-Stonean spaces to SAW^*-algebras.

Definition 6.1.1. *A C*-algebra $\mathcal{A}$ is a SAW^*-algebra if for every two orthogonal elements x and y in $\mathcal{A}_+$, there is an element e in $\mathcal{A}_+$ such that $ex = x$ and $ey = 0$.*

The following characterization of SAW^* algebras is very useful and justifies the above definition as the non-commutative analogues of sub-Stonean spaces.

Lemma 6.1.2. *For a C*-algebra $\mathcal{A}$ the following are equivalent.*

1. *$\mathcal{A}$ is a SAW^*-algebra.*

2. *Given any two orthogonal, hereditary σ-unital C*-subalgebras $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$, there is an e in $\mathcal{A}_+$ which is a unit for $\mathcal{B}$ and annihilates C.*

Proof. This immediately follows by noticing that a hereditary C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is σ-unital if and only if it is of the form $\mathcal{B} = \overline{x\mathcal{A}x}$ for some x in $\mathcal{A}_+$ (see [3]). □

The Gelfand transform of (2) implies the following.

Proposition 6.1.3. *A commutative C*-algebra $C_0(X)$ is a SAW^*-algebra if and only if X is a sub-Stonean space.*

In [47] and [46] some of the properties of sub-Stonean spaces are generalized to SAW^*-algebras. It is proved ([46]) that the corona algebra of any σ-unital C*-algebra is a SAW^*-algebra. In particular for a separable Hilbert space the Calkin algebra is a SAW^*-algebra.

The following is the generalization of a well-known fact that Čech-Stone remainder X^* of a locally compact, σ-compact and non-compact Hausdorff space X is a sub-Stonean space.

Theorem 6.1.4. *([46, Theorem 13]) For a σ-unital C*-algebra $\mathcal{A}$, its corona $\mathcal{C}(\mathcal{A})$ is a SAW^*-algebra.*

It was noticed in [19] the reason behind some of the nice properties of SAW^*-algebras, like Countable Riesz Separation Property (CRISP), Kasparov's Technical

Theorem (KTT), ..., is in their somewhat saturated nature. The inverse of the next theorem is not true since it is not hard to see that any von Neumann algebra is a SAW^*-algebra, but not necessarily countably degree-1 saturated (Definition 4.2.7).

Proposition 6.1.5. *Every countably degree-1 saturated C*-algebra $\mathcal{A}$ is a SAW^*-algebra.*

Proof. Assume $\mathcal{B}$ and $\mathcal{C}$ are two orthogonal σ-unital hereditary subalgebras of $\mathcal{A}$. Let b_n and c_n for $n \in \mathbb{N}$ be approximate units for $\mathcal{B}$ and $\mathcal{C}$, respectively. Define a 1-type $\mathbf{t}$ consisting of the following conditions, for all n.

(i) $b_n x = b_n$,

(ii) $c_n x = 0$,

(iii) $x = x^*$.

Every finite subset of $\mathbf{t}$ can be realized by b_n for large enough n, hence $\mathbf{t}$ is consistent. If an element a in $\mathcal{A}$ realizes $\mathbf{t}$, then $|a|$ is as required. $\square$

A SAW^*-algebra need not to be unital. However, every σ-unital SAW^*-algebra is unital, because a local unit for a strictly positive element has to be the unit. All SAW^*-algebras, and hence countably degree-1 saturated C*-algebras, tend to be large. In fact every separable SAW^* is finite-dimensional ([46, Corollary 2]).

To see the analogy in the commutative case ([33]) note that any first-countable sub-Stonean space is finite.

Theorem 6.1.6. *Let $\mathcal{I}$ be a closed ideal in a SAW^*-algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is a SAW^*-algebra.*

Proof. Let π be the canonical quotient map from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$, Let x and y be two positive elements of $\mathcal{A}$ such that $xy \in \mathcal{I}$. Set $x' = (x-y)_+$ and $y' = (x-y)_-$. Since $\pi(x)\pi(y) = \pi(xy) = 0$ we have $(\pi(x) - \pi(y))_+ = \pi(x)$ and $(\pi(x) - \pi(y))_- = \pi(y)$. So

$$\pi(x) = (\pi(x) - \pi(y))_+ = \pi((x-y)_+) = \pi(x') \tag{6.1}$$

similarly $\pi(y) = \pi(y')$. Since x' and y' are orthogonal positive elements of $\mathcal{A}$, there is $e' \in \mathcal{A}$ such that $e'x' = x$ and $e'y' = 0$. Now $e = \pi(e')$ is local unit for $\pi(x)$ and annihilates $\pi(y)$. □

It was shown by L. Ge ([28]), using free entropy, that if the group von Neumann algebra of $\mathbb{F}_2$, $L(\mathbb{F}_2)$, is written as the von Neumann tensor product of two von Neumann algebras M and N then either M or N has to be isomorphic to the algebra of $n \times n$ matrices $\mathbb{M}_n(\mathbb{C})$ for some n. For two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ the C*-algebra tensor product is not unique and for a C*-norm $\|.\|_\nu$ on the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ the completion is usually denoted by $\mathcal{A} \otimes_\nu \mathcal{B}$ (see [3]). A C*-algebra $\mathcal{A}$ is called *essentially non-factorizable* if it can not be written as $\mathcal{B} \otimes_\nu \mathcal{C}$

where both $\mathcal{B}$ and $\mathcal{C}$ are infinite dimensional for any C*-algebra norm ν.

Theorem 6.1.7 (Simon Wassermann)**.** *The reduced group C*-algebra of* $\mathbb{F}_2$, $C_r^*(\mathbb{F}_2)$, *is essentially non-factorazable. In fact if* $C_r^*(\mathbb{F}_2) = \mathcal{B} \otimes_\nu \mathcal{C}$, *for some C*-norm* ν *and infinite dimensional C*-algebra* $\mathcal{B}$ *then* $\mathcal{C} = \mathbb{M}_n(\mathbb{C})$ *with* $n = 1$.

It was asked by Wassermann whether the Calkin algebra is essentially non-factorizable. We prove that the answer to this question is positive, by showing that all SAW^*-algebras, of which the Calkin algebra is an example, are essentially non-factorizable.

In this chapter we will use another property of sub-Stonean spaces to show that SAW^*-algebras are essentially non-factorizable. Hence this will show that the reduced products (over layered ideals), ultrapowers of C*-algebras and relative commutants of separable subalgebras of this algebras are also essentially non-factorizable. A similar result for ultrapowers of type II_1-factors with respect to a free ultrafilter is proved in [10].

We don't require any knowledge about sub-Stonean spaces and it's enough to know that $\beta\mathbb{N}$, the Čech-Stone compactification of $\mathbb{N}$, is a sub-Stonean space.

6.2 SAW^*-algebras are essentially non-factorizable

It is well-known that if X and Y are two infinite sub-Stonean spaces then $X \times Y$ is not a sub-Stonean space in the product topology (see [33, Proposition 1.7]). In this section we shall prove a generalization of this fact for SAW^*-algebras, which in particular gives a positive answer to Wassermann's question.

We adopt standard notations from Ramsey theory and write $[\mathbb{N}]^2$ to denote the set of all $(m, n) \in \mathbb{N}^2$ such that $m < n$ and $\Delta^2\mathbb{N}$ to denote the diagonal of $\mathbb{N}^2$. For spaces X and Y a rectangle is a subset of $X \times Y$ of the form $A \times B$ for $A \subset X$ and $B \subset Y$. We say a map f on $A \times B$ depends only on the first coordinate if $f(x, y) = f(x, z)$ for every (x, y) and (x, z) in $A \times B$. In [11, lemma 5.1] Van Douwen proved that for any continuous map $f : \beta\mathbb{N}^2 \to \beta\mathbb{N}$ there is a clopen $U \subset \beta\mathbb{N}$ such that $f \restriction U^2$ depends on at most one coordinate and conjectured [11, conjecture 8.4] that there is a disjoint open cover of $\beta\mathbb{N}^2$ into such sets. In [4, theorem 3] I. Farah showed that for a sub-Stonean space Z, compact spaces X and Y, every continuous map $f : X \times Y \to Z$ is of a "*very simple*" form, which will be clear from Theorem 6.2.2 (in fact the theorem is proved for a larger class of spaces so called the $\beta\mathbb{N}$- spaces in the range and arbitrary powers of a compact space in the domain. However the theorem remains true if products of arbitrary compact spaces is replaced in the domain of the map). We sketch the proof of this theorem for the convenience of the reader. Before we need the following lemma.

Lemma 6.2.1. *Suppose X, Y and Z are arbitrary sets, $\rho : X \times Y \to Z$ a map, then exactly one of the following holds:*

1. *$X \times Y$ can be covered by finitely many mutually disjoint rectangles such that ρ depends on at most one coordinate on each of them.*

2. *There are sequences $x_i \in X$, $y_i \in Y$ such that for all i and all $j < k$ we have $\rho(x_i, y_i) \neq \rho(x_j, y_k)$.*

Moreover if X, Y and Z are topological spaces and ρ is a continuous map, we can assume that the rectangles in (1) *are clopen.*

Proof. For any map from X^2 into X this is an immediate consequence of [3, Theorem 3]. One can check the proof of this theorem to see that a small adjustment in definitions would give the same result for any map from $X \times Y$ into Z. To see the second part, note that the closures of this rectangles are still rectangles, and since ρ is continuous, it depends on at most one coordinate on each of this closures. By [4, Theorem 8.2] we can assume these rectangles are clopen. □

Theorem 6.2.2. *If ρ is a continuous map from $X \times Y$ into Z where X and Y are compact topological spaces and Z is a sub-Stonean space, then $X \times Y$ can be covered by finitely many mutually disjoint clopen rectangles such that ρ depends on at most one coordinate on each of them.*

Proof. We just need to show that the case (2) of Lemma 6.2.1 does not happen. Suppose $\{x_i\}$ and $\{y_i\}$ are sequences guaranteed by (2). Define the map $g : \mathbb{N}^2 \longrightarrow X \times Y$ by $g(m,n) = (x_m, y_n)$. Then g continuously extends to a map $\beta g : \beta\mathbb{N}^2 \longrightarrow X \times Y$ and the continuous map $h : \beta\mathbb{N}^2 \longrightarrow Z$ defined by $h = \rho \circ \beta g$ has the property that $h(l,l) \neq h(m,n)$ for all l and all m,n such that $m < n$. This contradicts Corollary 7.6 in [13] which states that if $h : \beta\mathbb{N}^2 \longrightarrow Z$ is a continuous map and Z is a sub-Stonean space, then the sets $h([\mathbb{N}]^2)$ and $h(\Delta^2\mathbb{N})$ have nonempty intersection. □

As a corollary of this, if X and Y are infinite, any such ρ is not injective. For C*-algebras, the product of non-commutative spaces corresponds to the tensor product of algebras. By the Gelfand transform we can restate Farah's theorem in terms of commutative C*-algebras.

Theorem 6.2.3. *Suppose $f : \mathcal{A} \to \mathcal{B} \otimes \mathcal{C}$ is a unital *-homomorphism, where $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are unital commutative C*-algebras and $\mathcal{A}$ is a SAW*-algebra. Then there are finitely many projections $p_1, \ldots p_s$ in $\mathcal{B}$ and $q_1, \ldots q_t$ projections in $\mathcal{C}$ such that $\sum_{i=1}^{s} p_i = 1_{\mathcal{B}}$ and $\sum_{i=1}^{t} q_i = 1_{\mathcal{C}}$ and for every $1 \leq i \leq s$ and $1 \leq j \leq t$ and either for every $a \in \mathcal{A}$ we have $(p_i \otimes q_j)f(a) \in (p_i\mathcal{B}p_i) \otimes q_j$ or for every $a \in \mathcal{A}$ we have $(p_i \otimes q_j)f(a) \in p_i \otimes (q_j\mathcal{C}q_j)$.*

Note that in particular every element in the image of f is a finite sum of elementary tensor products and if $\mathcal{A}$ is a commutative SAW^*-algebra with no projections (e.g. $A = C(X)$ where X is a connected sub-Stonean space like $\beta\mathbb{R} \setminus \mathbb{R}$) the image of f can be identified with a C*-subalgebra of $\mathcal{B}$ or $\mathcal{C}$.

Lemma 6.2.4. *If $\mathcal{B}$ is an infinite-dimensional, unital C*-algebra, we can find an orthogonal sequence $\{a_1, a_2, \dots\}$ in $\mathcal{B}$ such that $0 \leq a_i \leq 1_{\mathcal{B}}$ for all i and a sequence of states on $\mathcal{B}$, $\{\phi_n\}$, such that $\phi_n(a_n) = 1$ and $\phi_n(a_m) = 0$ if $m \neq n$.*

Proof. It is well-known that any maximal abelian subalgebra (MASA) of an infinite-dimensional C*-algebra $\mathcal{B}$ is also infinite-dimensional. If not then there are orthogonal 1-dimensional projections $\{p_1, p_2, \dots, p_n\}$ in the MASA such that $\sum_{i=1}^{n} p_i = 1_{\mathcal{B}}$. Since $\mathcal{B} = \sum_{i,j=1}^{n} p_i \mathcal{B} p_j$ and for each pair i, j, we have $p_i \mathcal{B} p_j$ is either $\{0\}$ or 1-dimensional, $\mathcal{B}$ is finite-dimensional. Fix such a MASA, and by the Gelfand-Naimark theorem identify it with $C(X)$ for some compact Hausdorff space X. We can also identify the set of pure states of $C(X)$ with X. Since X is an infinite normal space we can choose a discrete sequence of pure states $\{\phi_n\}$ in X and find a pairwise disjoint sequence $\{U_n\}$ of open neighbourhoods of $\{\phi_n\}$. By the Uryshon's lemma we get an orthogonal sequence $0 \leq a_n \leq 1_{\mathcal{B}}$ in $C(X)$ such that $\phi_n(a_n) = a_n(\phi_n) = 1$ and a_n vanishes outside of U_n. So $\phi_n(a_m) = a_m(\phi_n) = 0$ if $m \neq n$. Now by the Hahn-Banach extension theorem extend ϕ_n to a functional on $\mathcal{B}$ of norm 1. Since $\phi_n(1_{\mathcal{B}}) = 1$, this extension is a state. □

Note that if ϕ is a state on a C*-algebra $\mathcal{A}$ and $\phi(a) = 1$ for $0 \leq a \leq 1_{\mathcal{A}}$, as a consequence of the Cauchy-Schwartz inequality for states we have $\phi(b) = \phi(aba)$ for any $b \in \mathcal{A}$ (cf. [25, Lemma 4.8]).

Lemma 6.2.5. *âĂIJLet $\{\phi_n\}$ be a sequence of states on a SAW^*-algebra $\mathcal{A}$. If there exists a sequence $\{a_n\}$ of mutually orthogonal positive elements in $\mathcal{A}$ such that $\|a_n\| = \phi_n(a_n) = 1$ and $\phi_n(a_m) = 0$ if $m \neq n$ then the weak*-closure of $\{\phi_n\}$ is homeomorphic to $\beta\mathbb{N}$.*

Proof. Let D be a subset of $\mathbb{N}$. We show that $\overline{\{\phi_n : n \in D\}} \cap \overline{\{\phi_n : n \in D^c\}} = \emptyset$. Take $\psi \in \overline{\{\phi_n : n \in D\}}$. Let $a = \sum_{i\in D} 2^{-i}a_i$ and $b = \sum_{i\in D^c} 2^{-i}a_i$. Since $\mathcal{A}$ is a SAW^*-algebra, there exists a positive $e \in \mathcal{A}$ such that $ea = a$ and $eb = 0$. Then $ea_n = a_n$ for $n \in D$ and $ea_n = 0$ for every $n \in D^c$. For $n \in D$ we have $\phi_n(e) = \phi_n(ea_n) = \phi_n(a_n) = 1$ and for $n \in D^c$ we have $\phi_n(e) = \phi_n(ea_n) = \phi_n(0) = 0$. Hence $\psi(e) = 1$ and ψ is not in $\overline{\{\phi_n : n \in D^c\}}$.

Now let $F : \beta\mathbb{N} \longrightarrow \overline{\{\phi_n : n \in \mathbb{N}\}}$ be the continuous map such that $F(n) = \phi_n$. Let $\mathcal{U}$ and $\mathcal{V}$ be two distinct ultrafilters in $\beta\mathbb{N}$ and pick $X \subseteq \mathbb{N}$ such that $X \in \mathcal{U}$ but X is not in $\mathcal{V}$. Therefore

$$F(\mathcal{U}) \in F(\overline{X}) \subseteq \overline{F(X)} = \overline{\{\phi_n : n \in X\}}.$$

Similarly $F(\mathcal{V}) \in \overline{\{\phi_n : n \in X^c\}}$. So F is injective and clearly surjective. Since $\beta\mathbb{N}$ is compact and $\overline{\{\phi_n\}}$ is Hausdorff, it follows that F is a homeomorphism. □

Theorem 6.2.6. *Any SAW^* algebra is essentially non-factorizable.*

Proof. Let $\mathcal{A}$ be a SAW^*-algebra. Suppose that $\mathcal{A} \cong \mathcal{B} \otimes_\nu \mathcal{C}$ for the C*-completion of the algebraic tensor product $\mathcal{B} \odot \mathcal{C}$ of infinite dimensional C*-algebras $\mathcal{B}$ and $\mathcal{C}$ with respect to some C*-norm $\|.\|_\nu$. By Lemma 6.2.4 there are orthogonal sequences of positive contractions $\{b_n\} \subseteq \mathcal{B}$ and $\{c_n\} \subseteq \mathcal{C}$ and sequences $\{\phi_n\} \subseteq \mathcal{B}^*$ and $\{\psi_n\} \subseteq \mathcal{C}^*$ such that $\phi_n(b_n) = \psi_n(c_n) = 1$ and $\phi_n(b_m) = \psi_n(c_m) = 0$ for $m \neq n$. Identifying $\mathcal{A}$ with $\mathcal{B} \otimes_\nu \mathcal{C}$ and letting $a_{m,n} = b_m \otimes c_n$ and $\gamma_{m,n} = \phi_m \otimes \psi_n$, it is immediate that

$$\gamma_{m,n}(a_{m',n'}) = \begin{cases} 1 & (m,n) = (m',n') \\ 0 & (m,n) \neq (m',n') \end{cases}.$$

Let $X = \overline{\{\phi_n : n \in \mathbb{N}\}}^{w^*}$ and $Y = \overline{\{\psi_n : n \in \mathbb{N}\}}^{w^*}$. Then X and Y are compact subsets of $\mathcal{B}^*$ and $\mathcal{C}^*$, respectively, $\{\phi_n : n \in \mathbb{N}\} \times \{\psi_n : n \in \mathbb{N}\}$ is a dense subset of $\overline{\{\gamma_{m,n}\}}^{w^*}$ and by compactness $X \times Y$ is homeomorphic to $\overline{\{\gamma_{m,n}\}}^{w^*}$, which is homeomorphic to $\beta\mathbb{N}$ by Lemma 6.2.5. This contradicts the remark following the proof of Theorem 6.2.2. Hence there is no *-isomorphism $\mathcal{A} \cong \mathcal{B} \otimes_\nu \mathcal{C}$ with $\mathcal{B}$ and $\mathcal{C}$ infinite dimensional. $\square$

Corollary 6.2.7. *The Calkin algebra is essentially non-factorizable.*

Corollary 6.2.8. *For a sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of unital C*-algebras, the corona* *algebra* $\prod \mathcal{A}_n / \bigoplus \mathcal{A}_n$ *is essentially non-factorizable.*

Proof. This is a direct consequence of Theorem 6.1.4. $\square$

We don't know if Theorem 6.2.3 is true for non-commutative C*-algebras. But an analogous theorem for non-commutative C*-algebras would provide us with a strong tool to study the automorphisms between tensorial powers of the Calkin algebra or other SAW^*-algebras such as ultrapowers of C*-algebras. More precisely, in [13, Theorem 3] I. Farah proved that all continuous maps between powers of sub-Stonean spaces are of a very simple kind and he used this result to show that there is a dimension phenomena associated with these spaces (e.g., $\cdots \not\twoheadrightarrow (\mathbb{N}^*)^3 \not\twoheadrightarrow (\mathbb{N}^*)^2 \not\twoheadrightarrow \mathbb{N}^*$). The following conjectures are non-commutative generalizations of the Gelfand transform of these results.

Conjecture 6.2.9. *Suppose* $f : \mathcal{A} \to \mathcal{B} \otimes_{\min} \mathcal{C}$ *is a unital *-homomorphism, where* $\mathcal{A}, \mathcal{B}$ *and* $\mathcal{C}$ *are unital C*-algebras and* $\mathcal{A}$ *is a* SAW^**-algebra. Then there are finitely many projections* $p_1, \ldots p_s$ *in* $\mathcal{B}$ *and* $q_1, \ldots q_t$ *projections in* $\mathcal{C}$ *such that* $\sum_{i=1}^s p_i = 1_{\mathcal{B}}$ *and* $\sum_{i=1}^t q_i = 1_{\mathcal{C}}$ *and for every* $1 \leq i \leq s$ *and* $1 \leq j \leq t$ *and either for every* $a \in \mathcal{A}$ *we have* $(p_i \otimes q_j) f(a)(p_i \otimes q_j) \in (p_i \mathcal{B} p_i) \otimes q_j$ *or for every* $a \in \mathcal{A}$ *we have* $(p_i \otimes q_j) f(a)(p_i \otimes q_j) \in p_i \otimes (q_j \mathcal{C} q_j)$*.*

Conjecture 6.2.10. *Assume* $\mathcal{A}$ *is a* SAW^**-algebra. There are no surjective *-homomorphisms from the* n*-fold minimal tensor product* $\bigotimes_{\min}^n \mathcal{A}$ *onto* $\bigotimes_{\min}^m \mathcal{A}$ *when-*

ever $m \geq n$.

Note that these conjectures are true when $\mathcal{A}$ is a corona or an ultrapower of a commutative C*-algebra.

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