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OPERATOR ALGEBRAS AND ABSTRACT CLASSIFICATION

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Abstract

This dissertation is dedicated to the study of operator spaces, operator algebras, and their automorphisms using methods from logic, particularly descriptive set theory and model theory. The material is divided into three main themes. The first one concerns the notion of Polish groupoids and functorial complexity. Such a study is motivated by the fact that the categories of Elliott-classifiable algebras, Elliott invariants, abelian separable C*-algebras, and arbitrary separable C*-algebras have the same complexity according to the usual notion of Borel complexity. The goal is to provide a functorial refinement of Borel complexity, able to capture the complexity of classifying the objects in a functorial way. Our main result is that functorial Borel complexity provides a finer distinction of the complexity of functorial classification problems.

The second main theme concerns the classification problem for automorphisms of C*-algebras from the perspective of Borel complexity theory. Our results show that, for any non-elementary simple separable C*-algebra, the problem of classifying its automorphisms up to unitary equivalence transcends countable structures. Furthermore we prove that in the unital case the relation of unitary equivalence obeys the following dichotomy: it is either smooth, when the algebra has continuous trace, or not classifiable by countable structures.

The last theme concerns applications of model theory to the study and construction of interesting operator spaces and operator systems. Specifically we show that the Gurarij operator space introduced by Oikhberg can be characterized as the Fraïssé limit of the class of finite-dimensional 1-exact operator spaces. This proves that the Gurarij operator space is unique, homogeneous, and universal among separable 1-exact operator spaces. Moreover we prove that, while being 1-exact, the Gurarij operator space does not embed into any exact C*-algebra. Furthermore the ternary ring of operators generated by the Gurarij operator space is canonical, and does not depend on the concrete representation chosen. We also construct the operator system analog of the Gurarij operator space, and prove that it has analogous properties.

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Introduction

One of the main revolution in physics in the 20th Century is with no doubt the introduction of quantum mechanics, aiming at explaining the physical phenomena at extremely small scale. In this framework the classical laws of physics are replaced by their quantized version, obtained—roughly speaking—by replacing functions with operators on the Hilbert space. The study of algebras of operators on the Hilbert space was initiated by Murray and von Neumann in the 1930s, aiming at providing rigorous mathematical foundation to quantum mechanics. Murray and von Neumann focused on the case of algebras that are closed in the weak operator topology (von Neumann algebras, in the modern terminology), which are the quantized analog of measure spaces. The topological side of this study was initiated by the abstract characterization due to Gelfand and Neumark of algebras of operators that are closed in the norm topology [48]. These can be regarded as the quantum analog of compact Hausdorff spaces. The fact that quantized spaces have no actual points correspond to the physical principle that it is meaningless to speak about points in the phase space of a quantum particle. Since the seminal results of Murray-von Neumann and Gelfand-Neumark, the study of operator algebras has expanded enormously, finding many new applications and connections to a variety of branches of mathematics and physics.

In the last decade many long-standing problems in operator algebras have been settled using methods originally developed within the field of mathematical logic. These problems include the existence of outer automorphisms of the Calkin algebra [32] or the existence of nonisomorphic ultrapowers or relative commutants of a given II_1 factors [35]. These breakthroughs have been the starting point for a new line of research in operator algebras that uses tools and ideas coming from logic and set theory. Generally speaking, such a line of research can be divided into three main areas: set theory and forcing, descriptive set theory and Borel complexity, and model theory and continuous logic.

Descriptive set theory is, broadly speaking, the study of definable subsets of Polish spaces. A generous notion of definability for subsets of Polish spaces and maps between them is being Borel-measurable. The idea of studying abstractly the existence of classifying maps that are definable in this sense was first investigated in [43]. Such a study led to what is now called Borel complexity theory, providing a general framework where the complexity of classification problems in mathematics can be measured and compared. Many strong tools have been developed, most notably Hjorth's theory of turbulence [59], to distinguish between the complexity of classification problems, and refute the possibility of a satisfactory classification by means of a certain kind of invariants. This is especially valuable for fields, such as the theory of C*-algebras, were classification plays a prominent role.

Originating from the seminal classification results of UHF and AF C*-algebras due to Glimm [50] and Elliott-Bratteli [27, 16], the Elliott classification program is an ambitious project aiming at a full classification of all simple, separable, nuclear, unital C*-algebras by their Elliott invariant. This is a functorial invariant consisting of K-theoretical information (K-groups), measure-theoretical information (trace simplex), and their interaction (canonical pairing between K_0 and traces). The possible scope of such a program has been recently limited by counterexamples due to Rørdam and Toms [122, 131, 132]. These examples have shown that additional regularity properties are needed to ensure Elliott-classifiability. Three such properties ($\mathcal{Z}$-absorption, finite nuclear dimension, and strict comparison) of very different nature (algebraic, topological, and cohomological) have been recently shown to be equivalent for a large class of simple, separable, nuclear, unital C*-algebras, partially confirming a conjecture due to Toms and Winter; see [15] and references therein. The classification program can then be recast for the class of *well behaved* (in any of the conjecturally equivalent acceptations above) simple, separable, unital, nuclear C*-algebras. The study of the Elliott classification program from the perspective of Borel complexity theory has been initiated in [38]. As observed in [34, §3], by combining results from [30, 99, 45, 125, 137] one can conclude that the following classes of objects have the same complexity:

- separable C*-algebras,
- well-behaved simple, unital, nuclear, separable C*-algebras,
- abelian C*-algebras,
- Elliott invariants.

However the functorial nature of the classification of C*-algebras is not taken into account in the analysis above. This observation motivated us to introduce the notion of functorial Borel complexity, aiming at capturing the complexity of classifying the object of a category in an explicit and functorial way. Such a notion can be seen as a refinement and generalization of the usual notion of Borel complexity, since it reduces to the latter one when the morphisms are ignored, and only the isomorphism relations are considered. For simplicity we restricted ourselves to the case of groupoids, i.e. categories where every arrow is invertible. Furthermore we considered groupoids endowed with a canonical Polish topology, making composition and inversion of arrows continuous (Polish groupoids). It turns out that most of the fundamental results about the descriptive set theory of Polish group actions can be generalized to the case of Polish groupoids, as we show in Chapter 1; see also [89]. As an application we prove that the notion of functorial Borel complexity provides a finer distinction between functorial classification problems than the usual notion of Borel complexity. Precisely, for any nontreeable countable Borel equivalence E relation there are Polish groupoids with G and H both having E as associated orbit equivalence relation, such that G and H have distinct functorial Borel complexity. Such a property in fact characterizes the nontreeable Borel equivalence relations among all the countable Borel equivalence relations: for groupoids with (essentially) treeable orbit equivalence relation Borel complexity and functorial Borel complexity coincide.

In Chapter 2 we apply the theory of Borel complexity, and primarily Hjorth's theory of turbulence, to the study of automorphisms of C*-algebras. A prominent role in the theory of C*-algebras, as a source of both examples and applications, is played by the theory of C*-dynamical systems and crossed products. In its simplest form, a C*-dynamical system is a pair (A, α) where A is a C*-algebra and α is an automorphism of A, i.e. a function $\alpha : A \to A$ preserving all the C*-algebra structure. This can be regarded as a quantized analog of a dynamical system, which is a homeomorphic transformation of a compact Hausdorff space. Two automorphisms α, α' of A are *conjugate* if α' equals $\beta \circ \alpha \circ \beta^{-1}$ for some automorphism β of A. This can be seen as a quantum analog of the usual notion of conjugacy of topological dynamical systems. In joint work with David Kerr and N. Christopher Phillips we have shown that the classification problem for automorphisms up to conjugacy is intractable for any Elliott-classifiable C*-algebra [76]. The main technical tool used in the proof is Hjorth's theory of turbulence [59], providing dynamical conditions on a Polish group action ensuring that the corresponding orbit equivalence relation is not classifiable with countable structures as invariants.

The above result suggests that other quantum analogs of conjugacy should be considered to obtain a satisfactory classification in the noncommutative case. Any unitary element u of a C*-algebra A induces an automorphisms of A defined by $x \mapsto uxu^*$. Automorphisms of this form—called inner—can be thought as "rotations" of the space on which A acts, and should be regarded as trivial. It is therefore natural, in classifying the automorphisms of A, to identify automorphisms that are unitarily equivalent, i.e. agree modulo an inner automorphism.

Our main result on the subject, presented in Chapter 2 and also contained in [91], shows that the classification problem of automorphisms up to unitary equivalence is intractable for any simple separable C*-algebra. Furthermore, we prove that the relation of unitary equivalence of automorphisms of separable unital C*-algebra obeys the following dichotomy: it is either smooth (when the algebra has continuous trace), or not classifiable by countable structures. Such a result is reminiscent of the analogous behaviour of the relation of unitary equivalence of representations of locally compact groups and C*-algebras [58, 33, 75].

Combining conjugacy and unitarily equivalence one obtains the notion of cocycle conjugacy. Precisely, two automorphisms are cocycle conjugate if one is unitarily equivalent to a conjugate of the other. Satisfactory classification results of automorphisms of $\mathcal{O}_2$ and other C*-algebras up to cocycle conjugacy have been obtained in [61, 62] under suitable freeness conditions. Little is known in general about the complexity of the relation of cocycle conjugacy. In joint work with Eusebio Gardella I have shown that, in the case of the Cuntz algebra $\mathcal{O}_2$, the relation of cocycle conjugacy is a complete analytic set [47].

The last topic that we consider in this thesis concerns applications of model theory to operator spaces and operator systems. The theory of operator spaces can be considered as a noncommutative generalization of Banach space theory. It provides the natural framework where important notions for C*-algebras such as nuclearity and exactness can be defined and studied. A noncommutative analog of the Gurarij Banach space has been introduced by Oikhberg in [104]. In Chapter 3 we prove that such a space can be realized as the Fraïssé limit of the class of finite-dimensional 1-exact operator spaces. As a consequence we deduce that the Gurarij operator space $\mathbb{NG}$ is unique, homogeneous, and universal for separable 1-exact operator spaces. These results are also presented in [90]. Moreover we prove that the ternary ring of operators $\mathcal{T}(\mathbb{NG})$ generated by $\mathbb{NG}$ does not depend on the concrete realization of $\mathbb{NG}$ as a subspace of $B(H)$, and is not exact. This implies that $\mathbb{NG}$ does not embed into any exact C*-algebra.

In Chapter 4 we consider the analogous construction in the operator systems category, yielding the Gurarij operator system $\mathbb{GS}$; see also [92]. The Gurarij operator system is unique, homogeneous, and universal for separable 1-exact operator systems. Furthermore the C*-algebra $C^*(\mathbb{GS})$ generated by $\mathbb{GS}$ does not depend from the concrete realization of $\mathbb{GS}$ as a unital subspace of $B(H)$, and it is not exact. This implies that $\mathbb{GS}$ does not unitally embed into any exact unital C*-algebra.

In joint work with Isaac Goldbring [51] we have shown that the first order theory of $\mathbb{NG}$ is the model-completion of the theory of operator spaces, it has a unique separable 1-exact model, and any two separable models of $\mathbb{NG}$ are n-isometric for every $n \in \mathbb{N}$. However $\mathbb{NG}$ is not separably categorical, and in fact it has a continuum of pairwise non completely isometric models, and no atomic model. The analogous properties for the Gurarij operator systems are also established.

Chapter 1

Polish groupoids and functorial Borel complexity

Classification of mathematical structures is one of the main components of modern mathematics. It is safe to say that most results in mathematics can be described as providing an explicit classification of a class of mathematical objects by a certain type of invariants.

In the last 25 years the notion of constructive classification has been given a rigorous formulation in the framework of invariant complexity theory. In this context a classification problem is regarded as an equivalence relation on a standard Borel space (virtually all classification problems in mathematics fit into this category). The concept of constructive classification is formalized by the notion of *Borel reduction*. A Borel reduction from an equivalence relation E on X to an equivalence relation E' on X' is a *Borel* function $f : X \to X'$ with the property that, for every $x, y \in X$,

$$xEy \text{ if and only if } f(x)E'f(y).$$

In other words f is a Borel assignment of complete invariants for E that are equivalence classes of E'. The existence of such a function can be interpreted as saying that classifying the objects of X' up to E' is at least as complicated as classifying the objects of X up to E. This offers a notion of comparison between the complexity of different classification problems.

Several natural equivalence relation can then be used as benchmarks to measure the complexity of classification problems. Perhaps the most obvious such a benchmark is the relation of equality $=_{\mathbb{R}}$ of real numbers. This gives origin to the basic dichotomy smooth vs. non-smooth: an equivalence relation is *smooth* if it is Borel reducible to $=_{\mathbb{R}}$. (The real numbers can here be replaced by any other uncountable standard Borel space.) Beyond smoothness the next fundamental benchmark is classifiability by countable structures. Here the test is Borel reducibility to the relation of isomorphism within some class of countable first order structures, such as (ordered) groups, rings, etc. Equivalently one can consider orbit equivalence relation associated with Borel actions of the Polish group S_∞ of permutations of $\mathbb{N}$. Replacing S_∞ with an arbitrary Polish group yields the notion of equivalence relation classifiable by orbits of a Polish group action.

This framework allows one to build a hierarchy between different classification problems. Many efforts have been dedicated to the attempt to draw a picture as complete as possible of classification problems in mathematics and their relative complexity. To this purpose powerful tools such as Hjorth's theory of turbulence [59] have been developed in order to disprove the existence of Borel reduction between given equivalence relations, and to distinguish between the complexity of different classification problems. This can be interpreted as a way to formally exclude the possibility of a full classification of a certain class of objects by means of a given type of invariants. For example the relation of isomorphism of simple separable C*-algebras has been shown to transcend countable structures in [38]; see also [125]. Similar results have been obtained for several other equivalence relations, such as affine homeomorphism of Choquet simplexes [38], conjugacy of unitary operators on the infinite dimensional separable Hilbert space [74], conjugacy of ergodic measure-preserving transformations of the Lebesgue space [41], conjugacy of homeomorphisms of the unit square [59], conjugacy of irreducible representations of non type I groups [58] or C*-algebras [33, 75], conjugacy and unitary equivalence of automorphisms of classifiable simple separable C*-algebras [76, 91], isometry of separable Banach spaces [99] and complete order isomorphism of separable operator systems. Furthermore the relations of isomorphism and Lipschitz isomorphisms of separable Banach spaces, topological isomorphism of (abelian) Polish groups, uniform homeomorphism of complete separable metric spaces [40], and the relation of completely bounded isomorphism of separable operator spaces [4] have been shown to be not classifiable by the orbits of a Polish group action (and in fact to have maximal complexity among analytic equivalence relations). An exhaustive introduction to invariant complexity theory can be found in [44].

Considering how helpful the theory of Borel complexity has been so far in giving us a clear understanding of the relative complexity of classification problems in mathematics, it seems natural to look at refinements to the notion of Borel reducibility, that can in some situations better capture the notion of explicit classification from the practice of mathematics. Such a line of research has been suggested in [34], where the results of the present chapter has been announced. This is the case for example when the classification problem under consideration concerns a category. In this case it is natural to ask to the classifying map to be *functorial*, and to assign invariants not only to the objects of the category, but also to the *morphisms*. This is precisely what happens in many explicit examples of classification results in mathematics. In fact in many such examples the consideration of invariants of morphisms is essential to the proof. This is particularly the case in the Elliott classification program of simple C*-algebras, starting from Elliott's seminal paper of AF algebras [27]. Motivated by similar considerations, Elliott has suggested in [29] an abstract approach to classification by functors. In this chapter we bring Elliott's theory of functorial classification within the framework of Borel complexity theory. For simplicity we consider only categories where every arrow is invertible, called groupoids. Such categories will be assumed to have a global Borel structure that is at least analytic, and makes the set of objects (identified with their identity arrows) a standard Borel space. In the particular case when between any two objects there is at most one arrow (principal groupoids) these are precisely the analytic equivalence relations. One can then consider the natural constructibility requirement for classifying functors, which is being Borel with respect to the given Borel structures. This gives rise to the notion of functorial Borel complexity, which in the particular case of principal groupoids is the usual notion of Borel complexity.

In this chapter we study such a notion of functorial Borel complexity for groupoids, focusing on the case of Polish groupoids. These are the groupoids where the Borel structure is induced by a topology that makes composition and inversion of arrows continuous and open, and has a basis of open sets which are Polish in the relative topology. These include all Polish groups, groupoids associated with Polish group actions, and locally compact groupoids [106, Definition 2.2.2]. The latter ones include the holonomy groupoids of foliations and the tangent groupoids of manifolds [106, Chapter 2], the groupoids of row-finite directed graphs [84], the localization groupoids of actions of countable inverse semigroups [106, Chapter 4]. The main results of the present chapter assert that, for Polish groupoids with essentially countable equivalence relation, the existence of a Borel reducibility between the groupoids is equivalent to the Borel reducibility of the corresponding orbit equivalence relations. On the other hand for every countable equivalence relation E that is not treeable there are two Polish groupoid with orbit equivalence relation E that have distinct functorial Borel complexity; see Section 1.7. This shows that Borel reducibility of groupoids provides a finer notion of complexity than the usual Borel reducibility of equivalence relations. Having a finer notion of complexity is valuable, because it allows one to further distinguish between the complexity of problems that, in the usual framework, turn out to have the same complexity. An example of this phenomenon occurs in the classification problem for C*-algebras, where it turns out [38, 30, 125] that classifying arbitrary separable C*-algebras is as difficult as classifying the restricted class of C*-algebras that are considered to be well behaved (precisely the amenable simple C*-algebras, or even more restrictively the simple C*-algebras that can be obtained as direct limits of interval algebras).

In order to prove the above mentioned characterization of essentially treeable equivalence relations we will generalize some fundamental results of the theory of actions of Polish groups to actions of Polish groupoids, answering a question of Ramsay from [120]. These include the Becker-Kechris results on Polishability of Borel G-spaces [7, Chapter 5], existence of universal Borel G-spaces [7, Section 2.6], and characterization of Borel G-spaces with Borel orbit equivalence relation [7, Chapter 7]. The fundamental technique employed is a generalization of the *Vaught transform* [134] from actions of Polish groups to actions of Polish groupoids.

This chapter is organized as follows: In Section 1.1 we recall some background notions, introduce the notation to be used in the rest, and state the basic properties of the Vaught transform for actions of Polish groupoids. In Section 1.2 we generalize the local version of Effros' theorem from Polish group actions to actions of Polish groupoids, and infer the Glimm-Effros dichotomy for Polish groupoids and Borel reducibility, refining results from [119]. Section 1.3 contains the proof of the Polishability result for Borel G-spaces, showing that any Borel G-space is isomorphic to a Polish G-space, where G is a Polish groupoid. A characterization for Borel G-spaces with Borel orbit equivalence relation is obtained as a consequence in Section 1.4. Section 1.5 contains the construction of a universal Borel G-space for a given Polish groupoid G, generalizing [7, Section 2.6]. Section 1.6 considers countable Borel groupoids, i.e. analytic groupoids with only countably many arrows with a given source. It is shown that every such a groupoid has a Polish groupoid structure compatible with its Borel structure. In particular all results about Polish groupoids apply to countable Borel groupoids. Finally in Section 1.7 the above mentioned characterization of essentially treeable equivalence relations in terms of Borel reducibility is proved.

1.1 Descriptive set theory and Polish groupoids

1.1.1 Descriptive set theory

A *Polish space* is a separable and completely metrizable topological space. Equivalently a topological space is Polish if it is T_1, regular, second countable, and *strong Choquet* [73, Theorem 8.18]. A subspace of a Polish space is Polish with respect to the subspace topology if and only if it is a G_δ [73, Theorem 3.11].

A *standard Borel space* is a space endowed with a σ-algebra which is the σ-algebra of Borel sets with respect to some Polish topology. An *analytic space* is a space endowed with a countably generated σ-algebra which is the image of a standard Borel space under a Borel function. A subset of a standard Borel space is analytic if it is an analytic space with the relative standard Borel structure. A subset of a standard Borel space is *co-analytic* if its complement is analytic. It is well known that for a subset of a standard Borel space it is equivalent being Borel and being both analytic and co-analytic [73, Theorem 14.7]. If X, Y are standard Borel space and A is a subset of $X \times Y$, then for $x \in X$ the section

$$\{y \in Y : (x, y) \in A\}$$

is denoted by A_x. The *projection* of A onto the first coordinate is

$$\{x \in X : A_x \neq \varnothing\},$$

while the *co-projection* of A is

$$\{x \in X : A_x = Y\}.$$

The projection of an analytic set is analytic, while the co-projection of a co-analytic set is co-analytic.

If X is a Polish space, then the space of closed subsets of X is denoted by $F(X)$. The *Effros Borel structure* on $F(X)$ is the σ-algebra generated by the sets

$$\{F \in F(X) : F \cap U \neq \varnothing\}$$

for $U \subset X$ open. This makes $F(X)$ a standard Borel space [73, Section 12.C].

Recall that a subset A of a Polish space X has the *Baire property* if there is an open subset U of X such that the symmetric difference $A \triangle U$ is meager [73, Definition 8.21]. It follows from [73, Corollary 29.14] that any analytic subset of X has the Baire property.

A topological space X is a *Baire space* if every nonempty open subset of X is not meager. Every completely metrizable topological space is a Baire space; see [73, Theorem 8.4].

If X, Y are standard Borel spaces, then we say that Y is *fibred* over X if there is a Borel surjection $p : Y \to X$. If $x \in X$, then the inverse image of x under p is called the x-fiber of

Y and denoted by Y_x. If Y_0, Y_1 are fibred over X, then the fibred product

$$Y_0 * Y_1 = \{(y_0, y_1) : p_0(y_0) = p_1(y_1)\}$$

is naturally fibred over X. Similarly if $(Y_n)_{n\in\mathbb{N}}$ is a sequence of Borel spaces fibred over X we define

$$\mathop{\ast}_{n\in\mathbb{N}} Y_n = \{(y_n)_{n\in\mathbb{N}} : p(y_n) = p(y_m) \text{ for } n, m \in \mathbb{N}\}$$

which is again fibred over X. A *Borel fibred map* from Y_0 to Y_1 is a Borel function $\varphi : Y_0 \to Y_1$ which sends fibers to fibers, i.e. $p_1 \circ \varphi = p_0$.

If E is an equivalence relation on a standard Borel space X, then a subset T of X is a *transversal* for E if it intersects any class of E in exactly one point. A *selector* for E is a Borel function $\sigma : X \to X$ such that $\sigma(x)Ex$ for every $x \in X$ and $\sigma(x) = \sigma(y)$ whenever xEy.

1.1.2 Locally Polish spaces

Definition 1.1.1. A *locally Polish space* is a topological space with a countable basis of open sets which are Polish spaces in the relative topology.

By [73, Theorem 8.18] a locally Polish space is T_1, second countable, and strong Choquet. Moreover it is a Polish space if and only if it is regular. It follows from [73, Lemma 3.11] that a G_δ subspace of a locally Polish space is locally Polish.

Suppose that X is a locally Polish space. Denote by $F(X)$ the space of closed subsets of X. The Effros Borel structure on $F(X)$ is the σ-algebra generated by the sets of the form

$$\{F : F \cap U \neq \varnothing\}$$

for $U \subset X$ open. It has been proved by Anush Tserunyan that the Effros Borel structure on $F(X)$ is standard.

One can deduce from this that the Borel σ-algebra of X is standard. In fact the function

$$\begin{aligned} X &\to F(X) \\ x &\mapsto \{x\} \end{aligned}$$

is clearly a Borel isomorphism onto the set $F_1(X)$ of closed subsets of $F(X)$ containing exactly one element. It is therefore enough to show that $F_1(X)$ is a Borel subset of $F(X)$. Fix a countable basis $\mathcal{A}$ of open Polish subsets of X. Suppose also that for every $U \in \mathcal{A}$ it is fixed a compatible complete metric d_U on U. Observe that $F_1(X)$ contains precisely the closed subsets F of X such that $F \cap U \neq \varnothing$ for some $U \in \mathcal{A}$ and for every $U \in \mathcal{A}$ such that $F \cap U \neq \varnothing$ and for every $n \in \mathbb{N}$ there is $W \in \mathcal{A}$ such that $cl(W) \subset U$, $diam_U\left(cl\left(\overline{W}\right)\right) < 2^{-n}$ and $F \cap (X \backslash cl(W)) = \varnothing$, where $diam_U\left(cl\left(\overline{W}\right)\right)$ is the diameter of $\overline{W}$ with respect to the metric d_U. This shows that $F_1(X)$ is a Borel subset of X.

1.1.3 The Effros fibred space

Suppose that Z is a locally Polish space, X is a Polish space, and $p : Z \to X$ is a continuous open surjection. For $x \in X$ denote by Z_x the inverse image of x under p. Define $F^*(Z)$ to be the space of *nonempty* subsets of Z endowed with the Effros Borel structure. Define $F^*(Z, X)$ to be the Borel subset of closed subsets of Z contained in Z_x for some $x \in X$. The Borel function from $F^*(Z, X)$ onto X assigning to an element F of $F^*(Z, X)$ the unique $x \in X$ such that $F \subset Z_x$ endows $F^*(Z, X)$ with the structure of fibred Borel space. The obvious embedding of $F^*(Z_x)$ into $F^*(Z, X)$ is a Borel isomorphism onto the x-fiber of $F^*(Z, X)$.

Consider for every $x \in X$ a copy $\varnothing_x$ of the empty set. Define $F(Z, X)$ to be the disjoint union of $F^*(Z, X)$ with $\{\varnothing_x : x \in X\}$, which is again a standard Borel space fibred over X in the obvious way. Moreover the x-fiber of $F(Z, X)$ is now naturally isomorphic to the space $F(Z)$ of (possibly empty) subsets of Z_x. We will call $F(Z, X)$ the (standard) *Effros fibred space* of the fibration $p : Z \to X$.

1.1.4 Analytic and Borel groupoids

A *groupoid* G is a small category where every arrow is invertible. The set of objects of G is denoted by G^0. We will regard G^0 as a subset of G, by identifying an object with its identity arrow. Denote by G^2 the (closed) set of pairs of composable arrows

$$G^2 = \{(\gamma, \rho) : s(\gamma) = r(\rho)\}.$$

If A, B are subsets of G, then AB stands for the set

$$\left\{\gamma\rho : (\gamma, \rho) \in (A \times B) \cap G^2\right\}.$$

In particular if $Y \subset X$ then

$$YB = \{\gamma \in B : r(\gamma) \in Y\}$$

and

$$BY = \{\gamma \in B : s(\gamma) \in Y\}.$$

We write xB for $\{x\} B = r^{-1}[\{x\}] \cap B$ and Bx for $B\{x\} = s^{-1}[\{x\}] \cap B$. If A is a set of objects, then the *restriction* $G_{|A}$ of G to A (this is called "contraction" in [95, 118]) is the groupoid

$$\{\gamma \in G : s(\gamma) \in A, r(\gamma) \in A\}$$

with set of objects A and operations inherited from G.

To every groupoid G one can associate the *orbit equivalence relation* E_G on G^0 defined by $(x, y) \in E_G$ if and only if there is $\gamma \in G$ such that $s(\gamma) = x$ and $r(\gamma) = y$. The function

$$\begin{array}{rcl} G & \to & E_G \\ \gamma & \mapsto & (r(\gamma), s(\gamma)) \end{array}$$

is a continuous surjection. We say that a groupoid is *principal* when such a map is injective. Thus a principal groupoid is just an equivalence relation on its set of objects. Conversely any equivalence relation can be regarded as a principal groupoid.

The notion of *functor* between groupoids is the usual notion from category theory. Thus a functor from G to H is a function from G to H such that, for every $\gamma \in G$ and $(\rho_0, \rho_1) \in G^2$ the following holds:

- $F(s(\gamma)) = s(F(\gamma))$;
- $F(r(\gamma)) = r(F(\gamma))$;
- $F(\gamma^{-1}) = F(\gamma)^{-1}$;
- $F(\rho_0\rho_1) = F(\rho_0)F(\rho_1)$.

When E and E' are principal groupoids, then functors from E to E' are in 1:1 correspondence with *homomorphisms* from E to E' in the sense of [44, Definition 10.1.3].

Definition 1.1.2. An *analytic groupoid* is a groupoid endowed with an analytic Borel structure making composition and inversion of arrows Borel and such that the set of objects and, for every object x, the set of elements with source x, are standard Borel spaces with respect to the induced Borel structure. A *(standard) Borel groupoid* is a groupoid endowed with a standard Borel structure making composition and inversion of arrows Borel, and such that the set of objects is a Borel subset.

It is immediate to verify that principal analytic groupoids are precisely analytic equivalence relations on standard Borel spaces. Similarly principal Borel groupoids are precisely the Borel equivalence relations on standard Borel spaces. A functor between analytic groupoids is Borel if it is Borel as a function with respect to the given Borel structures.

1.1.5 Polish groupoids and Polish groupoid actions

Definition 1.1.3. A *topological groupoid* is a groupoid endowed with a topology making composition and inversion of arrows continuous.

It is not difficult to see that for a topological groupoid the following conditions are equivalent:

1. Composition of arrows is open;
2. The source map is open;
3. The range map is open.

(See [121, Exercise I.1.8].)

Definition 1.1.4. A *Polish groupoid* is a groupoid endowed with a locally Polish topology such that

1. composition and inversion of arrows are continuous and open,
2. the set G^0 of objects is a Polish space with the subspace topology,
3. for every $x \in G^0$ the sets Gx and xG are Polish spaces with the subspace topology.

Polish groupoids have been introduced in [119] with the extra assumption that the topology be regular or, equivalently, globally Polish. It is nonetheless noticed in [119, page 362] that one can safely dispense of this additional assumption, without invalidating the results proved therein.

Suppose that G is a Polish groupoid, and X is a Polish space. A continuous action of G on X is given by a continuous function $p : X \to G^0$ called *anchor map* together with a continuous function $(g, x) \mapsto gx$ from

$$G \ltimes X = \{(\gamma, x) : p(x) = s(\gamma)\}$$

to X such that, for all $\gamma, \rho \in G$ and $x \in X$

1. $\gamma(\rho x) = (\gamma\rho)x$,

2. $p(\gamma x) = r(\gamma)$, and

3. $p(x)x = x$.

In such a case we say that X is a Polish G-space. Similarly if X is a standard Borel space, then a Borel action of G on X is given by a Borel map $p : X \to G^0$ together with a Borel map

$$\begin{aligned} G \ltimes X &\to X \\ (\gamma, x) &\to \gamma x \end{aligned}$$

satisfying the same conditions as above. In this case X will be called a Borel G-space.

Clearly any Polish groupoid acts continuously on its space of objects G^0 by setting $p(x) = x$ and $(\gamma, x) \mapsto r(\gamma)$. This will be called the *standard* action of G on G^0.

Most of the usual notions for actions of groups, such as orbits, or invariant sets, can be generalized in the obvious way to actions of groupoids. If X is a G-space, and $x \in X$, then its orbit $\{\gamma x : s(\gamma) = p(x)\}$ is denoted by $[x]$. The orbit equivalence relation E_G^X on X is defined by $xE_G^X y$ iff $[x] = [y]$. If A is a subset of X, then its saturation

$$\{\gamma a : a \in A,\, \gamma \in Gp(a)\}$$

is denoted by $[A]$. An action is called *free* if $\gamma x = \rho x$ implies $\gamma = \rho$ for any $x \in X$ and $\gamma, \rho \in Gp(x)$.

Suppose that G is a Polish groupoid, and X is a Borel G-space. If $x, y \in G^0$ are in the same orbit define the stabilizer

$$G_x = \{\gamma \in G : s(\gamma) = p(x) \text{ and } \gamma x = x\}$$

of x, and

$$G_{x,y} = \{\gamma \in G : s(\gamma) = p(x) \text{ and } \gamma x = y\}.$$

Observe that by [73, Theorem 9.17] G_x is a closed subgroup of $p(x)Gp(x)$. Therefore $G_{x,y}$ is also closed, since $G_{x,y} = G_{x,x}\rho$ for any ρ such that $s(\rho) = p(x)$ and $\rho x = y$.

Suppose that X and Y are Borel G-spaces with anchor maps p_X and p_Y. A Borel fibred map from X to Y is a Borel function $\varphi : X \to Y$ such that $p_Y \circ \varphi = p_X$. A Borel fibred map from X to Y is *G-equivariant* if

$$\varphi(\gamma x) = \gamma\varphi(x)$$

for $x \in X$ and $\gamma \in Gp(x)$. A *Borel G-embedding* from X to Y is an injective G-equivariant Borel fibred map from X to Y. Finally a *Borel G-isomorphism* from X to Y is a Borel G-embedding which is also onto.

1.1.6 Some examples of Borel groupoids

In this subsection we show how several natural categories of interest can be endowed (after a suitable parametrization) with the structure of Borel groupoid.

Let us first consider the category of *complete separable metric spaces*, having surjective isometries as morphisms. This can be endowed with the structure of Borel groupoid in the following way. Denote by $\mathbb{U}$ the Urysohn universal metric space. (A survey about $\mathbb{U}$ and its remarkable properties can be found in [100].) Let $F(\mathbb{U})$ be the Borel space of closed subsets of $\mathbb{U}$ endowed with the Effros Borel structure. By universality of the Urysohn space, $F(\mathbb{U})$ contains an isometric copy of any separable metric space. Moreover any surjective isometry between closed subsets of $\mathbb{U}$ can be identified with its graph, which is a closed subset of $\mathbb{U} \times \mathbb{U}$. The set **CMS** of such graphs is easily seen to be a Borel subset of $F(\mathbb{U})$. Moreover a standard computation shows that composition and inversion of arrows are Borel functions in **CMS**. This shows that **CMS** is a Borel groupoid that can be seen as a parametrization of the category of metric spaces with surjective isometries as arrows.

More generally one can look at the category of *separable $\mathcal{L}$-structures* in some signature $\mathcal{L}$ of continuous logic. (A complete introduction to continuous logic is [9].) One can identify any $\mathcal{L}$-structure with an $\mathcal{L}$-structure having as support a closed subset of $\mathbb{U}$. In such case the interpretation of a function symbol f can be seen as a closed subset of $\mathbb{U}^{|f|+1}$ where $|f|$ denotes the arity of f. The interpretation of a relation symbol B can be seen as a closed subset of $\mathbb{U}^{|R|} \times \mathbb{R}$ where again $|R|$ denotes the arity of B. (Here distances and relations are allowed to attain value in the whole real line.) The set $\mathrm{Mod}(\mathcal{L})$ of such structures can be verified to be a Borel subset of

$$F(\mathbb{U}) \times \prod_f F(\mathbb{U}) \times \prod_B F(\mathbb{U})$$

where f and B range over the function and relation symbols of $\mathcal{L}$. Similar parametrizations of the space of $\mathcal{L}$-structures can be found in [30] and [10]. As before the space $\mathrm{Mod}(\mathcal{L})$ of isomorphisms between $\mathcal{L}$-structures (identified with their graph) is a Borel subset of $F(\mathbb{U})$, and composition and inversion of arrows are Borel maps. Thus one can regard $\mathrm{Mod}(\mathcal{L})$ as the Borel groupoid of $\mathcal{L}$-structures. In the particular case when one considers *discrete* structures then one can replace the Urysohn space with $\mathbb{N}$.

As a particular case of separable structures in a given signature one can consider *separable C*-algebras*. (The book [11] is a complete reference for the theory of operator algebras.) The complexity of the classification problem for separable C*-algebras has recently attracted considerable interest; see [38, 37, 30, 125]. Particularly important classes for the classification program are *nuclear* and *exact* C*-algebras; see [11, Section IV.3]. Separable exact C*-algebras are precisely the closed self-adjoint subalgebras of the Cuntz algebra $\mathcal{O}_2$ [78]. Thus the Borel groupoid $\mathbf{C^*Exact}$ of closed subalgebras of $\mathcal{O}_2$—where a *-isomorphism between closed subalgebras is identified with its graph—can be regarded as a parametrization for the category of exact C*-algebras having *-isomorphisms as arrows. The category of (simple, unital) nuclear C*-algebras can be regarded as the restriction of $\mathbf{C^*Exact}$ to the Borel set of (simple, unital) self-adjoint subalgebras of $\mathcal{O}_2$; see [38, Section 7].

We now look at the category of *Polish groups* with continuous group isomorphisms as arrows. Denote by $\mathrm{Iso}(\mathbb{U})$ the group of isometries of the Urysohn space endowed with the topology of pointwise convergence. Recall that $\mathrm{Iso}(\mathbb{U})$ is a universal Polish group [133], i.e. it contains any other Polish group as closed subgroup. The space $SG\left(\mathrm{Iso}(\mathbb{U})\right)$ of closed subgroups of $\mathrm{Iso}(\mathbb{U})$ endowed with the Effros Borel structure can be regarded as the standard Borel space of Polish groups. Moreover a continuous isomorphism between closed subgroups of $\mathrm{Iso}(\mathbb{U})$ can be identified with its graph, which is a closed subgroup of $\mathrm{Iso}(\mathbb{U})\times\mathrm{Iso}(\mathbb{U})$. It is not difficult to check that the set $\mathbf{PG}$ of such closed subgroups of $\mathrm{Iso}(\mathbb{U})\times\mathrm{Iso}(\mathbb{U})$ is a Borel subset of the space $SG\left(\mathrm{Iso}(\mathbb{U})\times\mathrm{Iso}(\mathbb{U})\right)$ of closed subgroups of $\mathrm{Iso}(\mathbb{U})\times\mathrm{Iso}(\mathbb{U})$ endowed with the Effros Borel structure. (Fix a countable neighborhood basis $\mathcal{N}$ of the identity in $\mathrm{Iso}(\mathbb{U})$, and observe that a closed subgroup H of $\mathrm{Iso}(\mathbb{U})\times\mathrm{Iso}(\mathbb{U})$ is in $\mathbf{PG}$ if and only if $\forall U\in\mathcal{N}$ $\exists V\in\mathcal{N}$ such that $H\cap(U\times(\mathrm{Iso}(\mathbb{U})\backslash cl(V)))=\varnothing$ and $H\cap((\mathrm{Iso}(\mathbb{U})\backslash cl(V))\times U)=\varnothing$.) Moreover a standard calculation shows that composition and inversion of arrows are Borel functions in $\mathbf{PG}$. (For composition of arrows, observe that if as before $\mathcal{N}$ is a countable basis of neighborhoods of the identity in $\mathrm{Iso}(\mathbb{U})$, D is a dense subset of $\mathrm{Iso}(\mathbb{U})$, A and B are open subsets of $\mathrm{Iso}(\mathbb{U})$, and $\varphi,\psi\in\mathbf{PG}$, then $(\varphi\circ\psi)\cap(A\times B)\neq\varnothing$ if and only if there are $U,V\in\mathcal{N}$ and $g,h\in D$ with $cl(V)^2h\subset B$ and $Ug\subset A$ such that $\psi\cap\left(U^2\times(\mathrm{Iso}(\mathbb{U})\backslash cl(V))\right)=\varnothing$, $\varphi\cap(A\times Ug)\neq\varnothing$, and $\psi\cap(Ug\times Vh)\neq\varnothing$.) This shows that $\mathbf{PG}$ is a Borel groupoid that can be seen as a parametrization of the category of Polish groups with continuous isomorphisms as arrows.

A similar discussion applies to the category of separable Banach spaces with linear (not necessarily isometric) isomorphisms as arrows. In this case one considers a universal separable Banach space, such as $C[0,1]$. One then looks at the standard Borel space of closed subspaces of $C[0,1]$ as set of objects, and the set of closed subspaces of $C[0,1] \oplus C[0,1]$ that code a linear isomorphism between closed subspaces of $C[0,1]$ as set of arrows. The proof that these sets are Borel with respect to the Effros Borel structure is analogous to the case of Polish groups.

1.1.7 The action groupoid

Suppose that G is a Polish groupoid, and X is a Polish G-space. Consider the groupoid

$$G \ltimes X = \{(\gamma, x) \in G \times X : s(\gamma) = p(x)\},$$

where composition and inversion of arrows are defined by

$$(\rho, \gamma x)(\gamma, x) = (\rho\gamma, x)$$

and

$$(\gamma, x)^{-1} = \left(\gamma^{-1}, \gamma x\right).$$

The set of objects of $G \ltimes X$ is

$$G^0 \ltimes X = \left\{(a, x) \in G^0 \times X : p(x) = a\right\}.$$

Endow $G \ltimes X$ with the subspace topology from $G \times X$. Observe that the function

$$\begin{array}{rcl} X & \to & G^0 \ltimes X \\ x & \mapsto & (p(x), x) \end{array}$$

is a homeomorphism from X to the set of objects of $G \ltimes X$. We can therefore identify the latter with X. Under this identification the source of (γ, x) is x and the range is γx. We claim that $G \ltimes X$ is a Polish groupoid, called the *action groupoid* associated with the Polish G-space X. Clearly the topology is locally Polish, and composition and inversion of arrows are continuous. We need to show that the source map is open. Suppose that V is an open subset of G, U is an open subset of X, and W is the open subset

$$\{(\gamma, x) : \gamma \in V, x \in U\}$$

of $G \ltimes X$. Suppose that W is nonempty and pick $(\gamma_0, x_0) \in W$. Thus $x_0 \in U$ and $p(x_0) = s(\gamma_0) \in s[V]$. Therefore there is an open subset U_0 of U containing x_0 such that $p[U_0] \subset s[V]$. We claim now that U_0 is contained in the image of W under the source map. In fact if $x \in U_0$ then $p(x) = s(\gamma)$ for some $\gamma \in V$ and therefore x is the source of the arrow (γ, x) in W. This concludes the proof of the fact that $G \ltimes X$ is a Polish groupoid. To summarize we can state the following proposition.

Proposition 1.1.5. *Suppose that G is a Polish groupoid, and X is a Polish G-space. The action groupoid $G \ltimes X$ as defined above is a Polish groupoid. Moreover the map*

$$\begin{aligned} X &\rightarrow (G \ltimes X)^0 \\ x &\mapsto (p(x), x) \end{aligned}$$

is a homeomorphism such that, for every $x, x' \in X$,

$$x E_G^X x' \quad \textit{iff} \quad (p(x), x)\, E_{G \ltimes X} \left(p(x'), x'\right).$$

1.1.8 Functorial reducibility

Definition 1.1.6. Suppose that G and H are analytic groupoids. A *Borel reduction* from G to H is a Borel functor F from G to H such that $xGy \neq \varnothing$ whenever $F(x)HF(y) \neq \varnothing$.

Equivalently a Borel functor F from G to H is a Borel reduction from G to H when the function

$$\begin{aligned} G^0 &\rightarrow H^0 \\ x &\mapsto F(x) \end{aligned}$$

is a Borel reduction from E_G to E_H in the sense of [44, Definition 5.1.1].

Definition 1.1.7. Suppose that G and H are analytic groupoids. We say that G is *Borel reducible* to H—in formulas $G \leq_B H$—if there is a Borel reduction from G to H.

The notion of bireducibility is defined accordingly.

Definition 1.1.8. Suppose that G and H are analytic groupoids. We say that G is *Borel bireducible* to H—in formulas $G \sim_B H$—if G is Borel reducible to H and vice versa.

When E and E' are principal analytic groupoids, then the Borel reductions from E to E' are in 1:1 correspondence with Borel reductions from E to E' in the usual sense of Borel complexity theory; see [44, Definition 5.1.1]. In particular Definition 1.1.7 *generalizes* the notion of Borel reducibility from analytic equivalence relations to analytic groupoids.

Similarly as in the case of reducibility for equivalence relations, one can impose further requirements on the reduction map. If G and H are analytic groupoid, we say that G is injectively Borel reducible to H—in formulas $G \sqsubseteq_B H$ if there is an *injective* Borel reduction from G to H. When G and H are Polish groupoid, one can also insist that the reduction be continuous rather than Borel. One then obtains the notion of continuous reducibility $\leq_c$ and continuous injective reducibility $\sqsubseteq_c$.

Definition 1.1.7 provides a natural notion of comparison between analytic groupoids. This allows one to build a hierarchy of complexity of analytic groupoids, that includes the usual hierarchy of Borel equivalence relations. The *functorial Borel complexity* of an analytic groupoid will denote the position of the given groupoid in such a hierarchy.

1.1.9 Category preserving maps

According to [101, Definition A.2] a continuous map $f : X \to Y$ between Polish spaces is *category preserving* if for any comeager subset C of Y the inverse image $f^{-1}[C]$ of C under f is a comeager subset of X. It is not difficult to see that any continuous open map is category preserving [101, Proposition A.3].

Category-preserving maps satisfy a suitable version of the classical Kuratowski-Ulam theorem for coordinate projections. We will state the particular case of this result for continuous open maps in the following lemma, which is Theorem A.1 in [101].

Lemma 1.1.9. *Suppose that X is second countable space, Y is a Baire space, and $f : X \to Y$ is an open continuous map such that $f^{-1}\{y\}$ is a Baire space for every $y \in Y$. If $A \subset X$ has the Baire property, then the following statements are equivalent:*

1. *A is comeager;*
2. *$\forall^* y \in Y$, $A \cap f^{-1}\{y\}$ is comeager in $f^{-1}\{y\}$.*

1.1.10 Vaught transforms

Suppose in the following that G is a Polish groupoid,

$$\mathcal{A} = \{U_n : n \in \mathbb{N}\}$$

is a basis of Polish open subsets of G, and X is a Borel G-space.

Definition 1.1.10. For $A \subset X$ and $V \subset G$, define the *Vaught transforms*

$$A^{\triangle V} = \{x \in X : Vp(x) \neq \varnothing \text{ and } \exists^* \gamma \in Vp(x),\, \gamma x \in A\}$$

and

$$A^{*V} = \{x \in X : Vp(x) \neq \varnothing \text{ and } \forall^* \gamma \in Vp(x),\, \gamma x \in A\}.$$

In the particular case when G is a Polish group, and X is a Borel G-space, this definition coincide with the usual Vaught transform; cf. [44, Definition 3.2.2].

Lemma 1.1.11. *Assume that B and A_n for $n \in \mathbb{N}$ are subsets of X. If V is an open subset of G, then the following hold:*

1. *$B^{\triangle G}$ and B^{*G} are invariant subsets of X;*
2. *$\left(\bigcap_n A_n\right)^{*V} = \bigcap_n A_n^{*V}$;*
3. *$\left(\bigcup_n A_n\right)^{\triangle V} = \bigcup_n A_n^{\triangle V}$;*
4. *$p^{-1}\left[s\left[V\right]\right]$ is the disjoint union of $\left(X \setminus B\right)^{*V}$ and $B^{\triangle V}$;*
5. *If B is analytic, then*

$$B^{\triangle V} = \bigcup \left\{B^{*U} : V \supset U \in \mathcal{A}\right\} \quad \textit{and} \quad B^{*V} = \bigcap \left\{B^{\triangle U} : V \supset U \in \mathcal{A}\right\}.$$

Lemma 1.1.11 is elementary and can be proved similarly as [44, Proposition 3.2.5].

Lemma 1.1.12. *Suppose that $B \subset X$ is analytic, and $U \subset G$ is open. If $x \in X$ and $\gamma \in Gp(x)$, then the following statements are equivalent:*

1. *$\gamma x \in B^{\triangle U}$;*
2. *$x \in B^{*V}$ for some $V \in \mathcal{A}$ such that $V\gamma^{-1} \subset Ur(\gamma)$;*
3. *$x \in B^{\triangle V}$ for some $V \in \mathcal{A}$ such that $V\gamma^{-1} \subset Ur(\gamma)$;*
4. *there are $V, W \in \mathcal{A}$ such that $VW^{-1} \subset U$, $\gamma \in W$, and $x \in B^{\triangle V}$.*

(1)⇒(2) By hypothesis $Ur(\gamma) \neq \varnothing$ and $\exists^* \rho \in Ur(\gamma)$ such that $\rho\gamma x \in B$. Therefore $U\gamma \neq \varnothing$ and $\exists^* \rho \in U\gamma$ such that $\rho x \in B$. Since B is analytic and the action is Borel, the set

$$\{\rho \in U\gamma : \rho x \in B\}$$

is analytic and in particular it has the Baire property. It follows that there is $V \in \mathcal{A}$ such that $Vp(x) \neq \varnothing$, $Vp(x) \subset U\gamma$, and $\forall^* \rho \in V$, $\rho x \in B$. Observe that $V\gamma^{-1} \subset Ur(\gamma)$.

(2)⇒(3) Obvious.

(3)⇒(1) Observe that $\varnothing \neq Vp(x) \subset U\gamma$. Thus $U\gamma \neq \varnothing$ and $\exists^* \rho \in U\gamma$ such that $\rho x \in B$. Thus $Up(\gamma z) \neq \varnothing$ and $\exists^* \rho \in Up(\gamma x)$, $\rho\gamma x \in B$. This shows that $\gamma x \in B^{\triangle U}$.

(2)⇒(4) Pick $v \in Vp(x)$ and observe that $v\gamma^{-1} \in Ur(\gamma)$. Therefore there are $W, V_0 \in \mathcal{A}$ such that $v \in V_0 \subset V$, $\gamma \in W$, and $V_0 W^{-1} \subset U$. Moreover since $x \in B^{*V}$ and $V_0 \subset V$ we have that $x \in B^{*V_0}$.

(4)⇒(2) Obvious.

If A is a subset of $G \ltimes X$ and $x \in X$, then A_x denotes the x-fiber

$$\{\gamma \in G : (\gamma, x) \in A\}$$

of A. The following lemma is inspired by the Montgomery-Novikov theorem; see [73, Theorem 16.1].

Lemma 1.1.13. *If A is a Borel subset of $G \ltimes X$ and $V \subset G$ is open, then*

$$\{x \in X : Vp(x) \neq \varnothing \text{ and } A_x \text{ is nonmeager in } Vp(x)\}$$

is Borel. The same conclusion holds if one replaces "nonmeager" with "comeager" or "meager".

Proof. Define $\mathcal{E}$ to be the class subsets of subsets A of $G \ltimes X$ such that

$$\{x \in X : Vp(x) \neq \varnothing \text{ and } A_x \text{ is nonmeager in } Vp(x)\}$$

is Borel for every nonempty open subset V of G. We claim that:

1. $\mathcal{E}$ contains the sets of the form

$$U \ltimes B = \{(\rho, x) \in G \ltimes X : x \in B,\ \rho \in U\}$$

for $B \subset X$ Borel and $U \subset G$ open;

2. $\mathcal{E}$ is closed by taking countable unions;

3. $\mathcal{E}$ is closed by taking complements.

In fact:

1. If $A = U \ltimes B$ where $B \subset X$ is Borel and $U \subset G$ is open then for every nonempty open set V

$$\begin{aligned} & \{x \in X : Vp(x) \neq \varnothing \text{ and } A_x \text{ is nonmeager in } Vp(x)\} \\ = \quad & B \cap p^{-1}\left[s\left[U \cap V\right]\right]; \end{aligned}$$

   is Borel.

2. If $A = \bigcup_n A_n$ then for every nonempty open set

$$\begin{aligned} & \{x \in X : Vp(x) \neq \varnothing \text{ and } A_x \text{ is nonmeager in } Vp(x)\} \\ = \quad & \bigcup_{n\in\mathbb{N}} \{x \in X : Vp(x) \neq \varnothing \text{ and } \left(A_n\right)_x \text{ is nonmeager in } Vp(x)\}; \end{aligned}$$

3. If $A \subset G \ltimes X$ then for every nonempty open set V

$$\begin{aligned} & \{x \in X : Vp(x) \neq \varnothing \text{ and } \left((G \ltimes X) \setminus A\right)_x \text{ is nonmeager in } V\} \\ = \quad & \{x \in X : Vp(x) \neq \varnothing \text{ and } A_x \text{ is not comeager in } V\} \\ = \quad & \bigcup_{U_n \subset V} \{x \in X : U_n p(x) \neq \varnothing \text{ and } A_x \text{ is meager in } U_n\} \\ = \quad & \bigcup_{U_n \subset V} \left(p^{-1} s\left[U_n\right] \setminus \{x \in X : U_n p(x) \neq \varnothing \text{ and } A_x \text{ is nonmeager in } U_n\}\right). \end{aligned}$$

A similar argument shows that the same conclusion holds after replacing "nonmeager" with "meager" or "comeager". □

Lemma 1.1.14. *If $A \subset X$ is Borel and $V \subset G$ is open then $A^{\triangle V}$ and A^{*V} are Borel.*

Proof. Consider the subset

$$\widetilde{A} = \{(\rho, x) \in G \ltimes X : \rho x \in A\}$$

and observe that $\widetilde{A}$ is a Borel subset of $G \ltimes X$ such that

$$A^{\triangle V} = \left\{ x \in X : Vp(x) \neq \varnothing \text{ and } \widetilde{A}_x \text{ is nonmeger in } Vp(x) \right\}$$

and

$$A^{*V} = \left\{ x \in X : Vp(x) \neq \varnothing \text{ and } \widetilde{A}_x \text{ is comeager in } Vp(x) \right\}.$$

The conclusion now follows from Lemma 1.1.13. □

Lemma 1.1.15. *Assume that X is a Polish G-space. If $B \subset X$, $U \subset G$ is open, and $\alpha \in \omega_1$, then the following hold:*

1. *If B is open, then $B^{\triangle U}$ is open;*
2. *If B is $\mathbf{\Sigma}^0_\alpha$, then $B^{\triangle U}$ is $\mathbf{\Sigma}^0_\alpha$ relatively to $p^{-1}s[U]$;*
3. *If B is $\mathbf{\Pi}^0_\alpha$, then B^{*U} is $\mathbf{\Pi}^0_\alpha$ relatively to $p^{-1}s[U]$.*

Proof. The proof is analogous to the corresponding one for group actions; see Theorem 3.2.9 of [44]. Suppose that B is open, and pick $x \in B^{\triangle U}$. Thus $Up(x) \neq \varnothing$ and $\exists^* \rho \in Up(x)$ such that $\rho x \in B$. Pick $U_0 \subset U$ open such that $x \in B^{*U_0}$ and $\rho \in U_0 p(x)$ such that $\rho x \in B$. Since B is open and the action is continuous there are open subsets W and V containing x and ρ such that $V \subset U_0$, $VW \subset B$, and $p[W] \subset s[V]$. We claim that $W \subset B^{\triangle U}$. In fact if $w \in W$ then $Vp(w) \neq \varnothing$. Moreover since $VW \subset B$, $\exists^* \rho \in Vp(w)$ such that $\rho w \in B$. This concludes the proof that $B^{\triangle U}$ is open. The other statements follow via (2),(3), and (4) of Lemma 1.1.11. □

Using the Vaught transform it is easy to see that, if X is a Borel G-space, then the orbit equivalence relation E^X_G is idealistic. (This is well known when G is a Polish group; cf. [44, Proposition 5.4.10].) Recall that an equivalence relation E on a standard Borel space X is *idealistic* if there is a map $[x]_E \mapsto I_{[x]_E}$ assigning to each equivalence class $[x]_E$ of E and ideal $I_{[x]_E}$ of subsets of $[x]_E$ such that $[x]_E \notin I_{[x]_E}$, and for every Borel subset A of $X \times X$ the set A_I defined by $x \in A_I$ iff $\{y \in [x]_E : (x, y) \in A\} \in I_{[x]_E}$ is Borel; see [44, Definition 5.4.9].

Proposition 1.1.16. *If X is a Borel G-space, then the orbit equivalence relation E^X_G is idealistic.*

Proof. Pick $x \in X$ and denote by C the orbit of x. Define the ideal I_C of subsets of C by $S \in I_C$ iff $\forall^* \rho \in Gp(x)$, $r(\rho) \notin S$. Observe that this does not depend from the choice of x.

In fact suppose that $y \in C$ and hence $y = \gamma x$ for some $\gamma \in Gp(x)$. Assume moreover that $S \subset C$ is such that $\forall^* \rho \in Gp(x)$, $r(\rho) \notin S$. Consider the homeomorphism Φ from $Gp(x)$ to $Gp(y)$ given by $\rho \mapsto \rho\gamma$. It is apparent that

$$\Phi\left[\{\rho \in Gp(x) : r(\rho) \notin S\}\right] = \{\rho \in Gp(y) : r(\rho) \notin S\}.$$

This shows that $\forall^* \rho \in Gp(y)$, $r(\rho) \notin S$, and hence the definition of I_C is does not depend from the choice of $x \in C$. Clearly $C \notin I_C$ since $Gp(x)$ is a Baire space. It is not difficult to verify that I_C is a σ-ideal. Suppose that $A \subset X \times X$ is Borel, and consider the set A_I defined by $x \in A_I$ iff $\{y \in [x] : (x, y) \in A\} \in I_{[x]}$. Observe that $x \in A_I$ iff $\forall^* \rho \in Gp(x)$, $(x, r(\rho)) \notin A$. Consider

$$X * X = \{(x, y) \in X \times X : p(x) = p(y)\}$$

and the action of G on $X * X$ defined by $p(x, y) = p(x) = p(y)$ and $\gamma(x, y) = (x, \gamma y)$ for $\gamma \in Gp(x, y)$. Observe that $x \in A_I$ if and only if

$$(x, x) \in ((X * X) \setminus A)^{*G}.$$

This shows that A_I is Borel by Lemma 1.1.14. $\square$

Let us denote as customary by E_1 the tail equivalence relation for sequences in $[0, 1]$. If E is an idealistic Borel equivalence relation, then E_1 is not Borel reducible to E by [71, Theorem 4.1]. Therefore we obtain from Proposition 1.1.16 the following corollary:

Corollary 1.1.17. *If X is a Borel G-space with Borel orbit equivalence relations, then the orbit equivalence relation E_1 is not Borel reducible to E_G^X.*

Corollary 1.1.17 holds more generally for arbitrary Borel G-spaces, with not necessarily Borel orbit equivalence relations. This was shown by the present author in collaboration with Samuel Coskey, George Elliott, and Ilijas Farah by adapting the proof of [59, Chapter 8].

An equivalence relation E on a standard Borel space E is *smooth* if it is Borel reducible to the relation of equality in some Polish space [44, Definition 5.4.1]. By [44, Theorem 5.4.11] an equivalence relation has a Borel selector precisely when it is smooth and idealistic. Therefore the following corollary follows immediately from Proposition 1.1.16.

Corollary 1.1.18. *If X is a Polish G-space such that E_G^X is smooth, then E_G^X has a Borel selector.*

Corollary 1.1.19. *If G and H are Polish groupoids such that E_G and E_H are smooth, then $G \leq_B H$ if and only if $E_G \leq E_H$.*

1.1.11 Borel orbits

We now observe that, if G is a Polish groupoid, then the orbits of any Polish G-space are Borel.

Proposition 1.1.20. *If G is a Polish groupoid, and X is a Polish G-space, then the orbit equivalence relation E_G^X is analytic and has Borel classes.*

Proof. By Proposition 1.1.5 we can consider without loss of generality the case of the standard action of G on its set of objects G^0. Fix $x \in G^0$ and consider the right action of xGx on Gx by composition. Observe that xGx is a Polish group, and Gx is a right Polish xGx-space with closed orbits. Therefore by [44, Proposition 3.4.6] the corresponding orbit equivalence relation E_{xGx}^{Gx} has a Borel transversal T. The orbit $[x]$ is the image of T under the range map r. Since r is 1:1 on T, it follows that $[x]$ is Borel by [73, Theorem 15.1]. Observe now that the orbit equivalence relation E_G is the image of the standard Borel space G under the Borel function $\gamma \mapsto (r(\gamma), s(\gamma))$. This shows that E_G is analytic. $\square$

Similarly as in the case of Polish group actions, a uniform bound on the complexity of the orbits in the Borel hierarchy entails Borelness of the orbit equivalence relation.

Theorem 1.1.21. *Suppose that G is Polish groupoid, and X is a Polish G-space. The orbit equivalence relation E_G^X is Borel if and only if there is $\alpha \in \omega_1$ such that every orbit is $\mathbf{\Pi}^0_\alpha$*

Proof. One direction is obvious. For the other one consider for $\alpha \in \omega_1$ the relation E_α of X defined by

$$(x, y) \in E_\alpha$$

iff for every G-invariant $\mathbf{\Pi}^0_\alpha$ set $W \subset X$ we have that $x \in W$ iff $y \in W$. If every orbit is $\mathbf{\Pi}^0_\alpha$ then $E_G^X = E_\alpha$. It is thus enough to prove that E_α is co-analytic for every $\alpha \in \omega_1$. Consider a universal $\mathbf{\Pi}^0_\alpha$ subset U of $\mathbb{N}^{\mathbb{N}} \times X$. Define the action of G on $\mathbb{N}^{\mathbb{N}} \times X$ by setting $p\,(a, b) = p(b)$ and $\gamma\,(a, b) = (a, \gamma b)$. Define now

$$T = U^{*G}$$

and observe that T is $\mathbf{\Pi}^0_\alpha$ since U is $\mathbf{\Pi}^0_\alpha$. Denote by T_a the section

$$\{b \in X : (a, b) \in T\}$$

for $a \in \mathbb{N}^{\mathbb{N}}$. We have that

$$\begin{aligned} b \in T_a &\Leftrightarrow (a,b) \in T \\ &\Leftrightarrow \forall^* \gamma \in Gp(b),\ (a, \gamma b) \in U \\ &\Leftrightarrow \forall^* \gamma \in Gp(b), \gamma b \in U_a \\ &\Leftrightarrow b \in (U_a)^{*G}. \end{aligned}$$

This shows that T_a is a G-invariant $\mathbf{\Pi}^0_\alpha$ subset of X for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. Conversely if A is a G-invariant $\mathbf{\Pi}^0_\alpha$ set then $A = U_a$ for some $a \in \mathbb{N}^{\mathbb{N}}$ and hence

$$A = A^{*G} = (U_a)^{*G} = T_a.$$

This shows that $\left\{T_a : a \in \mathbb{N}^{\mathbb{N}}\right\}$ is the collection of all invariant $\mathbf{\Pi}^0_\alpha$ sets. It follows that $(x, y) \in E_\alpha$ iff $\forall a \in \mathbb{N}^{\mathbb{N}}$, $(a, x) \in T$. Therefore E_α is co-analytic. □

Theorem 1.1.21 was proved for Polish group actions in [126, Sections 3.6 and 3.7].

1.2 Effros' theorem and the Glimm-Effros dichotomy

1.2.1 Effros' theorem

Lemma 1.2.1. *Suppose that G is a Polish groupoid. Consider the standard action of G on G^0, and the corresponding Vaught transform. If $A \subset G^0$ is meager, then $A^{\triangle G}$ is meager.*

Proof. The source map $r : G \to G^0$ is open and, in particular, category preserving; see Subsection 1.1.9. Thus $r^{-1}[A]$ is a meager subset of G. Therefore, since the source map s is also open, by Lemma 1.1.9 the set of $x \in X$ such that $Gx \cap r^{-1}[A]$ is meager. This set is by definition $A^{\triangle G}$. □

Theorem 1.2.2. *Suppose that G is a Polish groupoid, X is a Polish G-space, and $x \in X$. Denote by $[x]$ the orbit of x. The following statements are equivalent:*

1. $[x]$ is a G_δ subset of X;

2. $[x]$ is a Baire space;

3. $[x]$ is nonmeager in itself.

Proof. By Proposition 1.1.5 we can assume without loss of generality that $X = G^0$ and $G \curvearrowright G^0$ is the standard action. The only nontrivial implication is $3 \Rightarrow 1$. After replacing G with the restriction of G to the closure of $[x]$, we can assume that $[x]$ is dense in G^0 and hence nonmeager in G^0. By Proposition 1.1.20, $[x]$ is a Borel subset of G^0 and in particular it has the Baire property. Therefore by [73, Proposition 8.23] the orbit $[x]$ is the union of a meager set M and a G_δ set U. One can conclude that $[x] = U^{*G}$ arguing as in [126, Proposition 4.4]. Clearly $[x]$ is the union of U^{*G} and $M^{\triangle G}$. By Lemma 1.2.1, $M^{\triangle G}$ is meager and hence, since $[x]$ is nonmeager, $M^{\triangle G} = \varnothing$. Therefore $[x] = U^{*G}$ is G_δ by Lemma 1.1.15. □

Theorem 2.1 of [119] asserts that it is equivalent for the conditions in Theorem 1.2.2 to hold for all points of X.

Suppose that G is a Polish groupoid, X is a Polish G-space, and $x \in G^0$. The fiber $Gp(x)$ is a Polish space, and the stabilizer G_x of x is a Polish group acting from the right by composition on $Gp(x)$. One can then consider the quotient space $Gp(x)/G_x$ and the quotient map $\pi_x : Gp(x) \to Gp(x)/G_x$, which is clearly continuous and open. When $G \curvearrowright G^0$ is the standard action of G on its set of objects and $x \in G^0$, then the stabilizer G_x is just xGx.

It is not difficult to see that the proof of [119, Theorem 3.2] can be adapted to the context where G is a not necessarily regular Polish groupoid, as observed in [119, page 362]. The following lemma can then be obtained as an immediate consequence.

Lemma 1.2.3 (Ramsay)**.** *Suppose that G is a Polish groupoid, and $x \in G^0$. If the orbit $[x]$ of x is G_δ, then the map $\phi_x : Gx/xGx \to [x]$ defined by $\phi_x(\pi(\gamma)) = r(\gamma)$ is a homeomorphism.*

Corollary 1.2.4. *Suppose that G is a Polish groupoid, X is a Polish G-space, and $x \in X$. If the orbit $[x]$ of x is G_δ, then the map $\phi_x : Gp(x)/G_x \to [x]$ defined by $\phi_x(\pi(\gamma)) = \gamma x$ is a homeomorphism.*

Proof. Consider the action groupoid $G \ltimes X$, and let us identify X with the space of objects of $G \ltimes X$ as in Proposition 1.1.5. Consider the map ψ defined by

$$\begin{array}{rcl} Gp(x) & \to & (G \ltimes X)\,x \\ \gamma & \mapsto & (\gamma, x). \end{array}$$

Observe that ψ is a continuous map with continuous inverse

$$\begin{array}{rcl}(G \ltimes X)\, x & \to & Gp(x) \\ (\gamma, x) & \mapsto & \gamma.\end{array}$$

Moreover the image of G_x under ψ is precisely $x\,(G \ltimes X)\,x$. The proof is then concluded by invoking Lemma 1.2.3. $\square$

1.2.2 A Polish topology on quotient spaces

Suppose in this subsection that G is a Polish groupoid, which is moreover regular. Equivalently the topology of G is (globally) Polish. The following lemma is proved in [119, page 362].

Lemma 1.2.5 (Ramsay)**.** *Suppose that G is a regular Polish groupoid. If U is an open subset of G containing the set of objects G^0, then there is an open subset V of G containing the set of objects G^0 such that $VV \subset U$.*

Fix $x \in G^0$. If V is a neighborhood of G^0 in G, define the set

$$A_{V,x} = \left\{(\rho, \gamma) \in Gx \times Gx : \rho\gamma^{-1} \in V\right\}.$$

Observe that, if $\gamma \in Gx$, then the collection of open subsets of the form $V\gamma$, where V is an open neighborhood of $r(\gamma)$ in G, is a basis of neighborhoods of γ in Gx. It follows from this observation and Lemma 1.2.5 that the collection

$$\mathcal{U}_x = \left\{A_{V,x} : V \text{ is a neighborhood of } G^0 \text{ in } G\right\}$$

generates a uniformity compatible with the topology of Gx.

Suppose now that H is a closed subgroup of xGx, and consider the right action of H on Gx by translation. Denote by π the quotient map $Gx \to Gx/H$. Observe that π is continuous and open. If V is a neighborhood of G^0 in G define

$$A_{V,x,H} = \left\{(\pi(\gamma), \pi(\rho)) \in Gx/H \times Gx/H : \rho h\gamma^{-1} \in V \text{ for some } h \in xGx\right\}.$$

As before the collection

$$\mathcal{U}_{x,H} = \left\{A_{V,x,H} : V \text{ is a neighborhood of } G^0 \text{ in } G\right\}$$

generates a uniformity compatible with the topology of Gx/H.

Proposition 1.2.6. *The quotient Gx/H is a Polish space.*

Proof. The topology on Gx/H is induced by a countably generated uniformity, and hence it is metrizable. Since the quotient map $\pi : Gx \to Gx/H$ is continuous and open, it follows from [44, Theorem 2.2.9] that Gx/H is Polish. $\square$

Proposition 1.2.7. *Suppose that G is a regular Polish groupoid, and $x \in G^0$. Denote by π the quotient map*

$$\pi : Gx \to Gx/xGx.$$

The following statements are equivalent:

1. *The orbit $[x]$ of x is a G_δ subset of G^0;*
2. *The map $\phi_x : Gx/xGx \to [x]$ defined by $\phi_x(\pi(\gamma)) = r(\gamma)$ is a homeomorphism.*

Proof. The quotient space Gx/xGx is Polish by Proposition 1.2.6. Therefore if ϕ_x is a homeomorphism, then $[x]$ is Polish, and hence a G_δ subset of G^0 by [73, Theorem 3.11]. The converse implication follows from Lemma 1.2.3. $\square$

1.2.3 G_δ orbits

Lemma 1.2.8. *Suppose that G is a Polish groupoid, and $(U_n)_{n\in\mathbb{N}}$ is an enumeration of a basis of nonempty open subsets of G^0. If G has a dense orbit, then every element of $\bigcap_n [U_n]$ has dense orbit.*

The proof of Lemma 1.2.8 is immediate. Recall that $[U_n]$ denotes the G-saturation $r\left[s^{-1}[U_n]\right]$ of U_n.

Lemma 1.2.9. *Suppose that G is a Polish groupoid. Define the equivalence relation $\overline{E}$ on G^0 by $(x, y) \in \overline{E}$ iff the orbits of x and y have the same closure. The equivalence relation $\overline{E}$ is G_δ and contains E_G.*

Proof. Suppose that $(U_n)_{n\in\mathbb{N}}$ is an enumeration of a countable open basis of G^0. We have that $(x, y) \in \overline{E}$ if and only if $\forall n \in \mathbb{N}$, $x \in [U_n]$ iff $y \in [U_n]$. It follows that $\overline{E}$ is G_δ. $\square$

Lemma 1.2.10. *Suppose that G is a Polish groupoid such that every orbit of G is G_δ. If $x, y \in G^0$ are such that $[x] \neq [y]$ and $[y] \cap \overline{[x]} \neq \varnothing$ then $\overline{[y]} \cap [x] = \varnothing$. Equivalently the quotient space G^0/E_G is T_0*

Proof. After replacing G with the restriction of G to $\overline{[x]}$ we can assume that $\overline{[y]} \subset \overline{[x]} = X$. Denote by $(U_n)_{n\in\mathbb{N}}$ an enumeration of a basis of nonempty open subsets of G^0. By Lemma 1.2.8, every element of $\bigcap_n [U_n]$ has dense orbit. Since $[x] \cap [y] = \varnothing$, $[y]$ is not dense in X (otherwise it would be comeager and it would intersect $[x]$). It follows that, for some $n \in \mathbb{N}$, $y \notin [U_n]$ and hence $[y] \cap U_n = \varnothing$. This shows that $\overline{[y]} \subset X \backslash U_n$. On the other hand U_n is invariant dense open and $[x]$ is comeager, hence $[x] \subset U_n$. This shows that $\overline{[y]} \cap [x] = \varnothing$. □

Lemma 1.2.11. *Suppose that G is a Polish groupoid, and X is a Polish G-space. If every orbit is G_δ, then E_G^X is smooth.*

Proof. By Proposition 1.1.5 we can assume that $X = G^0$ and $G \curvearrowright G^0$ is the standard action. Observe that if $x, y \in G^0$, then $[x] = [y]$ if and only if $[x]$ and $[y]$ have the same closure. This shows that the map $x \mapsto \overline{[x]}$ from G^0 to the space $F(G^0)$ of closed subsets of x endowed with the Effros Borel structure is a reduction from E_G^X to equality in G^0. It remains to observe that such a map is Borel. In fact if U is an open subset of G^0, then

$$\left\{x \in G^0 : \overline{[x]} \cap U \neq \varnothing\right\} = [U] = r\left[s^{-1}[U]\right]$$

is open. □

Proposition 1.2.12. *Suppose that G is a Polish groupoid, and X is a Polish G-space. The following statements are equivalent:*

1. *Every orbit is G_δ;*
2. *The orbit equivalence relation E_G^X is G_δ;*
3. *The quotient space X/E_G^X is T_0.*
4. *The quotient topology generates the quotient Borel structure*

Proof. In view of Proposition 1.1.5 we can assume without loss of generality that $X = G^0$ and $G \curvearrowright G^0$ is the standard action.

(1)⇒(2) Consider the equivalence relation $\overline{E}$ defined as in 1.2.9. Suppose that $x, y \in X$ are such that $(x, y) \in \overline{E}$. It follows that $[x]$ and $[y]$ are both dense subsets of $Y = \overline{[x]} = \overline{[y]}$. Since both the orbit of x and y are G_δ, $[x]$ and $[y]$ are comeager subsets of Y. It follows that they are not disjoint, and hence $[x] = [y]$. This shows that $E_G = \overline{F}$ and in particular E_G is G_δ.

(2)⇒(1) Obvious.

(2)⇒(3) Follows from Lemma 1.2.10.

(3)⇒(1) Since the quotient map $\pi : X \to X/E_G$ is continuous and open, X/E_G has a countable basis $\{U_n : n \in \mathbb{N}\}$. If $x \in X$ then

$$[x] = \bigcap \left\{ \pi^{-1} [U_n] : n \in \mathbb{N},\, \pi(x) \in U_n \right\}.$$

This shows that $[x]$ is G_δ.

(3)⇒(4) The Borel structure generated by the quotient topology is separating and countably generated. By [94, Theorem 4.2] it must coincide with the quotient Borel structure.

(4)⇒(3) Observe that the orbits are Borel. Therefore the quotient Borel structure is separating and hence the quotient topology separates points, i.e. it is T_0.

□

The equivalence of the conditions in Proposition 1.2.12 has been proved in [119, Theorem 2.1] under the additional assumption that the orbit equivalence relation is F_σ.

1.2.4 The Glimm-Effros dichotomy

Denote by E_0 the orbit equivalence relation on $2^{\mathbb{N}}$ defined by $(x, y) \in E_0$ iff $x(n) = y(n)$ for all but finitely many $n \in \mathbb{N}$. Observe that E_0 can be regarded as the (principal) Polish groupoid associated with the free action of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ on $\prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ by translation. The proof of the following result is contained in [119, Section 4]. An exposition of the proof in the case of Polish group actions can be found in [44, Theorem 6.2.1].

Proposition 1.2.13. *Suppose that G is a Polish groupoid. If E_G is dense and meager in $G^0 \times G^0$, then $E_0 \sqsubseteq_c G$.*

Recall that $E_0 \sqsubseteq_c G$ means that there is an injective continuous functor $F : E_0 \to G$ such that the restriction of F to the set of objects is a Borel reduction from E_0 to E_G; see Subsection 1.1.8. One can then obtain the following consequences:

Proposition 1.2.14. *Suppose that G is a Polish groupoid. If G has no G_δ orbits, then $E_0 \sqsubseteq_c G$.*

Proof. After replacing G with the restriction of G to a class of the equivalence relation $\overline{E}$ defined as in Lemma 1.2.9, we can assume that every orbit is dense. By Theorem 1.2.2 every orbit is meager. It follows from Lemma 1.1.9 that E_G is meager. One can now apply Proposition 1.2.13. □

Theorem 1.2.15. *Suppose that G is a Polish groupoid. If every G_δ orbit is F_σ, then either E_G is G_δ or $E_0 \sqsubseteq_c G$.*

Proof. Suppose that $E_0 \not\sqsubseteq_c G$. In particular for every G_δ subspace Y of G^0, $E_0 \not\sqsubseteq_c G_{|Y}$. Denote by $\overline{E}$ the equivalence relation defined as in Lemma 1.2.9. If $y \in X$ define $Y = [y]_{\overline{E}}$ and observe that Y is an E_G-invariant G_δ subset of G^0. Moreover every E_G-orbit contained in Y is dense in Y. Since E_0 is not continuously reducible to G, by Proposition 1.2.14 there is $z \in Y$ such that $[z]_{E_G}$ is a dense G_δ subset of Y. In particular $[z]_{E_G}$ is G_δ subset of G^0. Therefore by assumption also $Y \setminus [z]_G$ is G_δ. Since every orbit of Y is dense, $Y \setminus [z]_{E_G}$ must be empty and $[z]_{E_G} = Y = [y]_{\overline{E}}$ is G_δ. This shows that every orbit of G is G_δ and hence E_G is G_δ by Proposition 1.2.12. □

Corollary 1.2.16. *Suppose that G and H are Polish groupoids such that every G_δ orbit is F_σ. If E_G and E_H are Borel reducible to E_0, then $G \leq H$ if and only if $E_G \leq E_H$.*

Proof. Suppose that $E_G \leq E_H$. If E_H is smooth then the conclusion follows from Corollary 1.1.19. If E_G is not smooth then $G \sim_B H \sim_B E_0$ by Theorem 1.2.15; see Definition 1.1.8. □

We can combine Proposition 1.2.12 with Theorem 1.2.15 to get the following result. It was obtained in [119] under the additional assumption that the orbit equivalence relation E_G is F_σ. The notion of nonatomic and ergodic Borel measure with respect to an equivalence relation can be found in [44, Definition 6.1.5].

Theorem 1.2.17. *Suppose that G is a Polish groupoid such that every G_δ orbit is F_σ. The following statements are equivalent:*

1. *There is an orbit which is not G_δ;*
2. *$E_0 \sqsubseteq_c G$;*
3. *$E_0 \leq_B E_G$;*
4. *There is an E_G-nonatomic E_G-ergodic Borel probability measure on G^0;*
5. *E_G is not smooth;*

6. *E_G is not G_δ;*

7. *Some orbit is not open in its closure.*

Proof. The implication (1)⇒(2) follows from Theorem 1.2.15. The implication (2)⇒(3) is obvious. For (3)⇒(4), observe that if $f : 2^{\mathbb{N}} \to X$ is a Borel reduction from E_0 to E_G, μ is the product measure on $2^{\mathbb{N}}$, and ν is the push-forward of μ under f, then ν is an E_G-nonatomic and E_G-ergodic Borel probability measure on G^0. The implication (4)⇒(5) follows from [44, Proposition 6.1.6]. By Lemma 1.2.11 (5) implies 6). The implication (6)⇒(1) is contained in Proposition 1.2.12. Since a set that is open in its closure is G_δ, the implication (1)⇒(7) is obvious. Let us show that (7)⇒(1). Suppose that every orbit is G_δ, and fix $x \in G^0$. After replacing G with its restriction to the closure of the orbit of x, we can assume that x has dense orbit. Therefore $[x]$ is a dense G_δ in x. Since $[x]$ is by assumption also F_σ, $[x] = \bigcup_n F_n$ where the F_n's are closed in X. Being $[x]$ nonmeager in X, there is an open subset U of X contained in F_n for some $n \in \mathbb{N}$. Hence $[x] = [U]$ is open. □

1.3 Better topologies

1.3.1 Polishability of Borel G-spaces

Theorem 1.3.1. *Suppose that G is a Polish groupoid. Every Borel G-space is Borel G-isomorphic to a Polish G-space. Equivalently if X is a Borel G-space, then there is a Polish topology compatible with the Borel structure of G that makes the action of G on X continuous.*

Theorem 1.3.1 answers a question of Ramsay from [120]. The rest of this subsection is dedicated to the proof of Theorem 1.3.1. The analogous statement for actions of Polish groups is proved in a similar way in [7, Theorem 5.2.1]. Fix a countable basis $\mathcal{A}$ of Polish open subsets of G. Suppose that X is a Polish G-space. We want to define a topology t on X such that

1. t is Polish,

2. the action $G \curvearrowright (X, t)$ is continuous, i.e. the anchor map $p : X \to G^0$ is continuous and

$$\begin{array}{rcl} G \ltimes X & \to & X \\ (\gamma, x) & \mapsto & \gamma x \end{array}$$

is continuous,

3. t generates the Borel structure of X.

By Lemma 1.1.14 and [7, 5.1.3 and 5.1.4] there exists a countable Boolean algebra $\mathcal{B}$ of Borel subsets of X satisfying the following conditions:

- For all $B \in \mathcal{B}$ and $U \in \mathcal{A}$, $B^{\triangle U} \in \mathcal{B}$;
- The topology t' generated by the basis $\mathcal{B}$ is Polish.

Observe that the identity function from X with its original Borel structure to (X, t') is Borel measurable, and hence a Borel isomorphism by [73, Theorem 15.1]. It follows that t' generates the Borel structure of X. Define $\mathcal{S}$ to be the set

$$\left\{ B^{\triangle U} : B \in \mathcal{B}, U \in \mathcal{A} \right\},$$

and t to be the topology on X having $\mathcal{S}$ as subbasis.

Claim. The action $G \curvearrowright (X, t)$ is continuous.

Proof. If $V \in \mathcal{A}$ then
$$p^{-1}\left[s\left[V\right]\right] = X^{\triangle V} \in \mathcal{S}.$$

This shows that $p : X \to G^0$ is t-continuous. Let us now show that the map $G \ltimes X \to X$ is t-continuous. Suppose that $B \in \mathcal{B}$, $U \in \mathcal{A}$, and $(\gamma_0, x_0) \in G \ltimes X$ is such that $\gamma_0 x_0 \in B^{\triangle U}$. By Lemma 1.1.12 there are $W, V \in \mathcal{A}$ such that $VW^{-1} \subset U$, $x_0 \in B^{\triangle V}$, and $\gamma_0 \in W$. We claim that $\gamma x \in B^{\triangle U}$ for every $x \in B^{\triangle V}$ and $\gamma \in W$. Fix $x \in B^{\triangle V}$ and $\gamma \in W$ and observe that $V\gamma^{-1} \subset VW^{-1} \subset U$ and hence it follows from Lemma 1.1.12 that $\gamma x \in B^{\triangle U}$. This shows that the action is continuous. $\square$

Claim. The space (X, t) is T_1.

Proof. Pick distinct points x, y of X. If $p(x) \neq p(y)$ then there are disjoint $V, W \in \mathcal{A}$ such that $p(x) \in V$ and $p(y) \in W$. Thus $p^{-1}[V]$ and $p^{-1}[W]$ are open sets separating x and y. Suppose that $p(x) = p(y)$. Consider the function $f : Gp(x) \to X \times X$ defined by

$$f(\gamma) = (\gamma x, \gamma y).$$

Observe that f is Borel when $X \times X$ is endowed with the $t' \times t'$ topology. By [73, Theorem 8.38] there is a dense G_δ subset Q of Gx such that the restriction of f to Q is $(t' \times t')$-continuous. Let $\gamma_0 \in Q$. Since $\mathcal{B}$ is a basis for the Polish topology t' on X there are disjoint elements B, C of $\mathcal{B}$ such that $\gamma_0 x \in B$ and $\gamma_0 y \in C$. Since f is $(t' \times t')$-continuous on Q there is $U \in \mathcal{A}$ such that $Up(x) \neq \varnothing$ and

$$f\left[Up(x) \cap Q\right] \subset B \times C.$$

Thus $\forall^* \gamma \in Up(x)$, $\gamma x \in B$ and $\gamma y \in C$. This shows that

$$x \in B^{\triangle U},\ y \in C^{\triangle U},\ y \notin B^{\triangle U}, \text{ and } x \notin C^{\triangle U}.$$

This concludes the proof that (X, t) is T_1. □

Claim. The space (X, t) is regular.

Proof. Suppose that $B \in \mathcal{B}$ and $U \in \mathcal{A}$. Pick $x_0 \in B^{\triangle U}$. It is enough to show that there is a t-open subset N of $B^{\triangle U}$ containing x_0 such that the t-closure of N is contained in $B^{\triangle V}$. Since $x_0 \in B^{\triangle U}$ by Lemma 1.1.12 there are $W_1, V_1 \in \mathcal{A}$ such that $V_1 W_1^{-1} \cup V_1 \subset U$, $p(x_0) \in W_1$, and $x_0 \in B^{\triangle V_1}$. Since $x_0 \in B^{\triangle V_1}$ again by Lemma 1.1.12 there are $V_2, W_2 \in \mathcal{A}$ such that $V_2 W_2^{-1} \subset V_1$, $p(x_0) \in W_2$, and $x_0 \in B^{\triangle V_2}$. Define $W \in \mathcal{A}$ such that

$$p(x_0) \in W \subset W_1^{-1} \cap W_2.$$

Consider

$$N = B^{\triangle V_2} \cap p^{-1} s\left[W\right]$$

and observe that N is a t-open subset of X containing x_0. We claim that the closure of N is contained in $B^{\triangle U}$. Define $F = \left(B^{\triangle V_2}\right)^{*W}$ and observe that F is relatively closed in $p^{-1} s\left[W\right]$ by Lemma 1.1.11(4). We claim that

$$N \subset F \subset B^{\triangle U}.$$

Suppose that $x \in N$. If $\gamma \in Wp(x)$ we have that

$$V_2 \gamma^{-1} \subset V_2 W^{-1} \subset V_2 W_2^{-1} \subset V_1.$$

Therefore $\gamma x \in B^{\triangle V_1}$. Being this true for every $\gamma \in Wp(x)$, $x \in \left(B^{\triangle V_1}\right)^{*W} = F$. Suppose now that $x \in F$ and pick $\gamma \in Wp(x)$ such that $\gamma x \in B^{\triangle V_1}$. We thus have

$$V_1 \gamma \subset V_1 W \subset V_1 W_1^{-1} \subset U$$

which implies by Lemma 1.1.12 that $x = \gamma^{-1}(\gamma x) \in B^{\triangle U}$. This concludes the proof that $N \subset F \subset B^{\triangle U}$. We will now show that the t-closure of N is contained in $B^{\triangle U}$. It is enough to show that if $x \notin B^{\triangle U}$ then there is a t-open neighborhood of x disjoint from N. This is clear if $p(x) \notin s[W]$. Suppose now that $p(x) \in s[W]$. Since F is relatively closed in $p^{-1}s[W]$ and

$$N \subset F \subset B^{\triangle V} \cap p^{-1}s[W]$$

we have that $p^{-1}s[W] \backslash F$ is an open subset of X containing x and disjoint from N. This concludes the proof that the closure of N is contained in $B^{\triangle V}$. We have thus found an open neighborhood N of x whose closure is contained in $B^{\triangle V}$. This concludes the proof that (X, t) is regular. □

Claim. The space (X, t) is strong Choquet.

Proof. Define $\mathcal{C}$ to be the (countable) set of nonempty finite intersections of elements of $\mathcal{S}$ and observe that $\mathcal{C}$ is a basis for (X, t). Fix a well ordering E of the countable set $\mathcal{C} \times \mathcal{B} \times \mathcal{A}$. Let d' be a complete metric on X compatible with the Polish topology t'. We want to define a strategy for Player II in the strong Choquet game; see [73, Section 8.D]. Suppose that Player I plays t-open sets N_i for $i \in \mathbb{N}$ and $x_i \in N_i$. At the i-th turn Player II will choose an element (M_i, B_i, U_i) of $\mathcal{C} \times \mathcal{B} \times \mathcal{A}$ in such a way that the following properties hold:

1. $x_i \in M_i$;
2. The t-closure of M_i is contained in N_i;
3. The closure of U_{i+1} in U_0 is contained in U_i;
4. The t'-closure of B_{i+1} is contained in B_i;
5. The d'-diameter of B_i is less than 2^{-i};
6. The d_{U_0}-diameter of U_i is less than 2^{-i} for $i \geq 1$, where d_{U_0} is a compatible complete metric on U_0;
7. $M_i \subset B_i^{\triangle U_i}$.

Player II strategy is the following: At the i-th turn pick the E-least tuple (M_i, B_i, U_i) in $\mathcal{C} \times \mathcal{B} \times \mathcal{A}$ satisfying properties (1)–(7). We need to show that the set of such tuples is nonempty. Observe that $x_i \in N_i \subset M_{i-1} \subset B_{i-1}^{\triangle U_{i-1}}$. Thus $U_{i-1}p(x_i) \neq \varnothing$ and $\exists^* \gamma \in U_{i-1}p(x_i)$ such that $\gamma x_i \in B_{i-1}$. Since $\mathcal{B}$ is a basis for (X, t') and $\mathcal{A}$ is a basis for G we can find B_i and U_i such that

- (3)–(6) hold.
- $U_i p(x_i) \neq \varnothing$, and
- $\exists^* \gamma \in U_i p(x_i)$ such that $\gamma x_i \in B_i$.

Consider $M = B_i^{\triangle U_i} \cap N_i$ and observe that M is a t-open set containing x_i. Since (X, t) is regular there is $M_i \in \mathcal{C}$ such that $x_i \in M_i$ and the closure of M_i is contained in $N_i \cap B_i^{\triangle U_i}$. This ensures that (1),(2),(7) are satisfied. We now show that this gives a winning strategy for Player II. For every $i \in \mathbb{N}$ we have that $x_i \in M_i \subset B_i^{\triangle U_i}$ and hence there is $\gamma_i \in U_i p(x_i)$ such that $\gamma_i x_i = y_i \in B_i$. Define γ to be the limit of the sequence $(\gamma_i)_{i\in\mathbb{N}}$ in U_0 and y to be the t'-limit of the sequence $(y_i)_{i\in\mathbb{N}}$ in Y. Observe that

$$p(y) = \lim_i p(y_i) = \lim_i r(\gamma_i) = r(\gamma).$$

Define $x = \gamma^{-1} y \in X$ and observe that x is the t-limit of the sequence $(x_i)_{i\in\mathbb{N}}$. Fix $i \in \mathbb{N}$. For $j > i$ we have that $x_j \in N_j \subset M_i$ and hence x is contained in the t-closure of M_i, which is in turn contained in N_i. This shows that $x \in \bigcap_{i\in\mathbb{N}} N_i$, concluding the proof that Player II has a winning strategy in the strong Choquet game in (X, t). □

The proof of Theorem 1.3.1 is finished recalling that a regular T_1 strong Choquet space is Polish [73, Theorem 8.18].

1.3.2 Finer topologies for Polish G-spaces

Theorem 1.3.2. *Suppose that G is a Polish groupoid, and (X, τ) is a Polish G-space. Assume that $V \subset G$ is an open Polish subset, $P \subset X$ is $\mathbf{\Sigma}^0_\alpha$ for some $\alpha \in \omega_1$, and $Q = P^{\triangle V}$. There is a topology t on X such that:*

1. *t is a Polish topology;*
2. *t is finer that τ;*
3. *Q is t-open,*
4. *The action of G on (X, t) is continuous;*
5. *t has a countable basis $\mathcal{B}$ such that for every $B \in \mathcal{B}$ there is $n \in \mathbb{N}$ such that B is $\mathbf{\Sigma}^0_{\alpha+n}$ with respect to τ.*

The analogous statement for actions of Polish groups is proved in a similar way in [7, Theorem 5.1.8]. Let $\mathcal{A}$ be a countable basis of Polish open subsets for G containing V and let $\mathcal{D}$ be a countable basis for (X, τ). By Lemma 1.1.15 and [7, 5.1.3, 5.1.4] there is a countable Boolean algebra $\mathcal{B}$ of subsets of X satisfying the following:

1. For $B \in \mathcal{B}$ and $U \in \mathcal{A}$, $B^{\triangle V} \in \mathcal{A}$;
2. $P \in \mathcal{B}$;
3. $\mathcal{D} \subset \mathcal{B}$;
4. $\mathcal{B}$ is a basis for a Polish topology t';
5. For every $B \in \mathcal{B}$, there is $n \in \mathbb{N}$ such that B is $\mathbf{\Sigma}_{\alpha+n}$ with respect to τ.

Define

$$\mathcal{S} = \left\{ B^{\triangle V} : B \in \mathcal{B}, V \in \mathcal{A} \right\},$$

and

$$\mathcal{S}^* = \mathcal{S} \cup \mathcal{D}.$$

Consider the topology t on X having $\mathcal{S}^*$ as a subbasis. We claim that t is a Polish topology finer that τ and coarser that t' making the action continuous. Clearly t is finer that τ and in particular $p : (X, t) \to G^0$ is continuous. The proof that the action is continuous and that t is a Polish topology is analogous to the proof of Theorem 1.3.1. The following corollary can be obtained from Theorem 1.3.2 together with [7, Subsection 5.1.3].

Corollary 1.3.3. *Suppose that G is a Polish groupoid, and (X, τ) is a Polish G-space. If $\mathcal{J}$ is a countable collection of G-invariant Borel subsets of X, then there is a Polish topology t on X finer than τ and making the action continuous such that all elements of $\mathcal{J}$ are t-clopen.*

1.4 Borel orbit equivalence relations

1.4.1 A Borel selector for cosets

Suppose that G is a Polish groupoid. Denote by $F(G)$ the standard Borel space of closed subsets of G endowed with the Effros Borel structure. A similar proof as [73, Theorem 12.13] shows that there is a Borel function

$$\sigma : F(G) \backslash \{\varnothing\} \to G$$

such that $\sigma(A) \in A$ for every nonempty closed subset A of G. Denote by $S(G)$ the Borel space of closed subgroupoids of G. This is the Borel subset of $F(G)$ containing the closed subsets H of G such that for $\gamma, \rho \in H$, $\gamma^{-1} \in H$ and if $r(\gamma) = s(\rho)$ then $\rho\gamma \in H$. If $H \in S(G)$ denote by $\sim_H$ the equivalence relation on G defined by $\gamma \sim_H \rho$ iff $\gamma = \rho h$ for some $h \in H$ or, equivalently, $\gamma H = \rho H$.

Proposition 1.4.1. *The relation $\sim$ on $G \times S(G)$ defined by $(\gamma, H) \sim (\gamma', H')$ iff $H = H'$ and $\gamma H = \gamma' H'$ has a Borel transversal T.*

Proof. Define the map f from $S(G) \times G$ to $F(G)$ by $f(\gamma, H) = \gamma H$. We claim that f is Borel. Let us show that if U is an open subset of G then the set

$$A_U = \{(\gamma, H) \in G \times S(G) : \gamma H \cap U \neq \varnothing\}$$

is Borel. Since the set

$$\left\{(\rho, \gamma, H) \in G \times G \times S(G) : \gamma^{-1}\rho \in H \text{ and } \rho \in U\right\}$$

is Borel, its projection A_U on the last two coordinates is analytic. We want to show that A_U is co-analytic. Fix a countable basis of Polish open sets $\{U_n : n \in \mathbb{N}\}$ for G. Observe that $(H, \gamma) \in A_U$ if and only if there is $n \in \mathbb{N}$ such that $\gamma U_n \subset U$ and $U_n \cap H \neq \varnothing$. It is now enough to show that $\{\gamma \in G : \gamma U_n \subset U\}$ is co-analytic. This follows from the fact that

$$\{(\gamma, \rho) \in G \times U_n : \text{either } r(\rho) \neq s(\gamma) \text{ or } r(\rho) = s(\gamma) \text{ and } \gamma\rho \in U\}$$

is a Borel set and it co-projection on the first coordinate is $\{\gamma \in G : U_n\gamma \subset U\}$. If now $\sigma : F(G) \backslash \{\varnothing\} \to G$ is a Borel map such that $\sigma(A) \in A$ for every nonempty closed subset A of G, define $g(\gamma, H) = ((\sigma \circ f)(\gamma, H), H)$. Observe that g is a Borel selector for $\sim$. Therefore the set

$$T = \{(\gamma, H) : g(\gamma, H) = (\gamma, H)\}$$

is a Borel transversal for $\sim$. □

Corollary 1.4.2. *If G is a Polish groupoid, and X is a Borel G-space, then the orbits are Borel subsets of X.*

Proof. Observe that the stabilizer G_x is a closed subgroup of $p(x)Gp(x)$ by [73, Theorem 9.17]. Consider a Borel transversal T_x for the equivalence relation $\sim_{G_x}$. The function $\gamma \mapsto \gamma x$ from $T \cap Gx$ to X is a 1:1 Borel function from T_x onto the orbit of x. This shows that the orbit of x is Borel by [73, Theorem 15.1]. □

1.4.2 Borel orbit equivalence relations

Suppose that G is a Polish groupoid, and X is a Polish G-space. If $x \in X$ then Lemma 1.1.14 and [7, 5.1.3 and 5.1.4] show that there is a sequence $(B_{x,n})_{n\in\mathbb{N}}$ of Borel subsets of X such that $[x] = B_{x,0}$ and

$$\mathcal{B}(x) = \{B_{x,n} : n \in \mathbb{N}\}$$

is a Boolean algebra that is a basis for a topology $t(x)$ on X making the action continuous, and such that $B^{\triangle U} \in \mathcal{B}(x)$ whenever $B \in \mathcal{B}(x)$ and $U \in \mathcal{A}$. It is implicit in the proof of Lemma 1.1.14 and [7, 5.1.3, 5.1.4] that, under the additional assumption that the orbit equivalence relation E_G^X is Borel, the dependence of the sequence $(\mathcal{B}_{x,n})_{n\in\mathbb{N}}$ from x is Borel, i.e. the relation

$$\mathcal{B}(y, x, n) \Leftrightarrow y \in \mathcal{B}_{x,n}$$

is Borel. This concludes the proof of the following lemma; see also [7, Lemma 7.1.3].

Lemma 1.4.3. *Suppose that G is a Polish groupoid, and X is a Polish G-space. Assume that $\mathcal{A}$ is a countable basis of Polish open subsets of G. If the orbit equivalence relation E_G^X is Borel, then there is a Borel subset $\mathcal{B}$ of $X \times X \times \mathbb{N}$ such that, letting*

$$B_{x,n} = \{y : (y, x, n) \in \mathcal{B}\}$$

and

$$\mathcal{B}(x) = \{B_{x,n} : n \in \mathbb{N}\},$$

for every $x \in X$ the following hold:

1. $[x] = B_{x,0}$*;*
2. $B^{\triangle U} \in \mathcal{B}(x)$ *for every* $B \in \mathcal{B}(x)$*, and* $U \in \mathcal{A}$*;*
3. $\mathcal{B}(x)$ *is a Boolean algebra;*
4. $\mathcal{B}(x)$ *is a basis for a Polish topology* $t(x)$ *on* X *making* X *a Polish* G*-space.*

The following result provides a characterization of the Borel G-spaces with Borel orbit equivalence relation. The analogous result for Polish group actions is [7, Theorem 7.1.2].

Theorem 1.4.4. *Suppose that G is a Polish groupoid, and X is a Borel G-space. The following statements are equivalent*

1. *The function*

$$\begin{aligned} X &\to F(G) \\ x &\mapsto G_x \end{aligned}$$

is Borel;

2. *The function*

$$\begin{aligned} X \times X &\to F(G) \\ (x, y) &\mapsto G_{x,y} \end{aligned}$$

is Borel;

3. *The orbit equivalence relation E_G^X is Borel.*

Recall that, for $x, y \in G^0$, G_x denotes the (closed) stabilizer

$$\{\gamma \in Gp(x) : \gamma x = x\}$$

and while $G_{x,y}$ is the set

$$\{\gamma \in Gp(x) : \gamma x = y\} ;$$

see Subsection 1.1.5.

Proof. By Theorem 1.3.1 we can assume without loss of generality that X is a Polish G-space. Fix a countable basis $\mathcal{A} = \{U_n : n \in \mathbb{N}\}$ of nonempty Polish open subsets of G. Denote also by $T \subset G \times S(G)$ a Borel transversal for the relation $(\gamma, H) \sim (\gamma', H')$ iff $H = H'$ and $\gamma H = \gamma H'$ as in Proposition 1.4.1.

(1)⇒(2) Fix a nonempty open subset U of G. It is enough to show that the set

$$\{(x, y) \in X \times X : U \cap G_{x,y} \neq \varnothing\}$$

is co-analytic. Observe that $G_{x,y} \cap U \neq \varnothing$ if and only if there is a unique $\gamma \in G$ such that $s(\gamma) = p(x)$, $(\gamma, G_x) \in T$, and $\gamma G_x \cap U \neq \varnothing$. Moreover $\gamma G_x \cap U \neq \varnothing$ if and only if there is $n \in \mathbb{N}$ such that $\gamma U_n \subset U$ and $U_n \cap G_x \neq \varnothing$. Fix $n \in \mathbb{N}$ and recall that by the proof of Proposition 1.4.1 $\{\gamma \in G : \gamma U_n \subset U\}$ is co-analytic. This concludes the proof that

$$\{(x, y) \in X \times X : U \cap G_{x,y} \neq \varnothing\}$$

is co-analytic.

(2)⇒(1) Obvious.

(1)⇒(3) Observe that $(x, y) \in E_G$ if and only if there is a unique $\gamma \in T$ such that $(\gamma, G_x) \in T$ and $r(\gamma) = y$.

(3)⇒(1) Suppose that $\mathcal{B}$, $\mathcal{B}(x)$, $t(x)$, and $B_{x,n}$ for $x \in X$ and $n \in \mathbb{N}$ are defined as in Lemma 1.4.3. Observe that the orbit $[x] = B_{x,0}$ is open in $t(x)$. It follows from Lemma 1.2.3 that the map

$$\begin{array}{rcl} Gp(x)\,/G_x & \to & [x] \\ \gamma G_x & \mapsto & \gamma x \end{array}$$

is a $t(x)$-homeomorphism. We want to show that for every $U \in \mathcal{A}$ the set

$$\{x \in X : G_x \cap U \neq \varnothing\}$$

is Borel. It is enough to show that for every basic nonempty U, V the set

$$\{x \in X : UG_x \cap V \neq \varnothing\}$$

is co-analytic. We claim that $UG_x \cap V \neq \varnothing$ iff $\exists U_m \subset U$ such that $\forall B \in \mathcal{B}(x)$, $x \in B^{\triangle U_m}$ implies $x \in B^{\triangle V}$. In fact suppose that $UG_x \cap V \neq \varnothing$ and pick m such that $U_m \subset U$ and $U_m\gamma \subset V$ for some $\gamma \in G_x$. If $x \in B^{\triangle U_m}$ then $x = \gamma^{-1}x \in B^{\triangle U_m \gamma}$ and hence $x \in B^{\triangle V}$ by Lemma 1.1.12. Conversely suppose that $UG_x \cap V = \varnothing$ and hence

$$\{\gamma x : \gamma \in U\} \cap \{\gamma x : \gamma \in V\} = \varnothing.$$

Fix $m \in \mathbb{N}$. Since the map

$$\begin{array}{rcl} Gp(x)\,/G_x & \to & [x] \\ \gamma G_x & \mapsto & \gamma x \end{array}$$

is a $t(x)$-homeomorphism, the set $\{\gamma x : \gamma \in U_m\}$ is open in $[x]$. Thus there is $B \in \mathcal{B}(x)$ such that

$$x \in B \subset \{\gamma x : \gamma \in U_m\}.$$

Moreover

$$\{\gamma \in Gp(x) : \gamma x \in B\}$$

is an open subset of $Gp(x)$. Therefore there is $k \in \mathbb{N}$ such that $U_k \subset U_m$ and

$$U_k p(x) \subset \{\gamma \in Gp(x) : \gamma x \in B\}.$$

In particular $x \in B^{\triangle U_k}$ but $x \notin B^{\triangle V}$.

$\square$

1.5 Universal actions

Suppose that G is a Polish groupoid. The space G is fibred over the space of objects G^0 via the source map $r : G \to G^0$. One can then consider the corresponding Effros fibred space $F(G, G^0)$ of closed subsets of G contained in Gx for some $x \in G^0$; see Subsection 1.1.3. Recall that $F(G, G^0)$ is a standard Borel space fibred over G^0 via the Borel map assigning x to a closed nonempty subset F of xG. Moreover $F(G, G^0)$ has naturally the structure of Borel G-space given by the map

$$(\gamma, F) \mapsto \gamma F$$

for $F \subset s(\gamma)G$, where

$$\gamma F = \{\gamma\rho : \rho \in F\}.$$

Similarly the fibred product

$$\mathop{\ast}_{n\in\mathbb{N}} F(G, G^0) = \left\{(F_n)_{n\in\mathbb{N}} \in F\left(G, G\right)^{\mathbb{N}} : \exists x \in G^0 \ \forall n \in \mathbb{N},\ F_n \subset xG\right\}$$

is naturally a Borel G-space with respect to the coordinate-wise action of G

$$\gamma(F_n)_{n\in\mathbb{N}} = (\gamma F_n)_{n\in\mathbb{N}}.$$

We want to show that the Borel G-space $\ast_{n\in\mathbb{N}} F(G, G^0)$ is a universal Borel G-space. This means that if X is any Borel G-space, then there is a Borel G-embedding $\varphi : X \to \ast_{n\in\mathbb{N}} F(G, G^0)$; see Subsection 1.1.5.

The following lemma is well known. A proof is included for convenience of the reader.

Lemma 1.5.1. *If X is a Polish space, $A \subset X$, and $E(A)$ is the set of $x \in X$ such that for every neighborhood V of x, $V \cap A$ is not meager, then $E(A)$ is closed in X. Moreover A*

has the Baire property iff $A \triangle E(A)$ is meager.

Proof. Clearly $E(A)$ is closed, and if $A \triangle E(A)$ is meager then A has the Baire property. Observe that if $A, B \subset X$ are such that $A \triangle B$ is meager, then $E(A) = E(B)$. If A has the Baire property, there is an open subset U of X such that $A \triangle U$ is meager. Thus $E(A) = E(U)$ is equal to the closure $\overline{U}$ of U. It follows that

$$A \triangle E(A) \subset (A \triangle U) \cup \left(\overline{U} \backslash U\right)$$

is meager. $\square$

Suppose that X is a Borel G-space. In view of Theorem 1.3.1 we can assume without loss of generality that X is in fact a Polish G-space. Fix a countable open basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ of nonempty open subsets of X. Assume further that $\mathcal{A}$ is a countable basis of Polish open subsets of G. Define for $n \in \mathbb{N}$ the fibred Borel map $\varphi_n : X \to F(G, G^0)$ by setting

$$\varphi_n(x) = \left(E\left(\{\gamma \in Gp(x) : \gamma x \in B_n\}\right)\right)^{-1}.$$

Define the Borel fibred map $\varphi : X \to \mathop{\ast}_{n\in\mathbb{N}} F(G, G^0)$ by $\varphi(x) = (\varphi_n(x))_{n\in\mathbb{N}}$.

Claim. φ is Borel measurable

It is enough to show that φ_n is Borel measurable for every $n \in \mathbb{N}$. Suppose that $V \in \mathcal{A}$. We want to show that the set of $x \in X$ such that

$$E\left(\{\gamma \in Gp(x) : \gamma x \in B_n\}\right) \cap V \neq \varnothing$$

is Borel. Observe that

$$E\left(\{\gamma \in Gp(x) : \gamma x \in B_n\}\right) \cap V \neq \varnothing$$

if and only if $\exists W \in \mathcal{A}$ such that $W \subset V$ and

$$\{\gamma \in Gp(x) : \gamma x \in B_n\}$$

is comeager in $Wp(x)$. The set of such elements x of X is Borel by Lemma 1.1.13.

Claim. φ is G-equivariant, i.e. $\varphi(\gamma x) = \gamma\varphi(x)$ for $(\gamma, x) \in G \ltimes X$

It is enough to show that $\varphi_n(\gamma x) = \gamma\varphi_n(x)$ for $n \in \mathbb{N}$. Observe that

$$\varphi_n(\gamma x) = \left(E\left(\{\rho \in Gr(\gamma) : \rho\gamma x \in B_n\}\right)\right)^{-1}$$

and

$$\varphi_n(x) = \left(E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\right)\right)^{-1}$$

We thus have to prove that

$$\left(E\left(\{\rho \in Gr(\gamma) : \rho\gamma x \in B_n\}\right)\right)^{-1} = \gamma\left(E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\right)\right)^{-1}$$

or equivalently

$$E\left(\{\rho \in Gr(\gamma) : \rho\gamma x \in B_n\}\right) = E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\right)\gamma^{-1}$$

Since $\tau \mapsto \tau\gamma^{-1}$ is a homeomorphism from $Gp(x)$ to $Gr(\gamma)$ we have that

$$\begin{aligned} E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\right)\gamma^{-1} &= E\left(\{\tau \in Gp(x) : \tau x \in B_n\}\gamma^{-1}\right) \\ &= E\left(\{\rho \in Gr(\gamma) : \rho\gamma x \in B_n\}\right) \end{aligned}$$

Claim. φ is injective

Assume that $x, y \in X$ are such that $\varphi(x) = \varphi(y)$. Thus $p(x) = p(y)$ and for every $n \in \mathbb{N}$

$$\{\gamma \in Gp(x) : \gamma x \in B_n\} \triangle \{\gamma \in Gp(y) : \gamma y \in B_n\}$$

is meager. Thus $\forall^*\gamma \in Gp(x)$, $\forall n \in \mathbb{N}$, $\gamma x \in B_n$ iff $\gamma y \in B_n$. Thus for some $\gamma \in Gp(x)$, $\gamma x \in B_n$ iff $\gamma y \in B_n$ for all $n \in \mathbb{N}$. This implies that $\gamma x = \gamma y$ and hence $x = y$.

1.6 Countable Borel groupoids

1.6.1 Actions of inverse semigroups on Polish spaces

An *inverse semigroup* is a semigroup T such that every $t \in T$ has a semigroup-theoretic inverse $t^* \in T$. This means that t^* is the unique element of T such that

$$tt^*t = t \text{ and } t^*tt^* = t^*.$$

If T is an inverse semigroup, then the set $E(T)$ of idempotent elements is a commutative subsemigroup of T, and hence a semilattice; see [106, Proposition 2.1.1]. In particular $E(T)$ has a natural order defined by

$$e \leq f \text{ iff } ef = fe = e.$$

Observe that for every $t \in T$ the elements tt^* and t^*t are idempotent.

Suppose that X is a Polish space. The semigroup $\mathcal{H}(X)$ of partial homeomorphisms between open subsets of X is clearly an inverse semigroup.

Definition 1.6.1. An action $\theta : T \curvearrowright X$ of a countable inverse semigroup T on the Polish space X is a semigroup homomorphism $\theta : t \mapsto \theta_t$ from T to $\mathcal{H}(X)$.

Observe that a semigroup homomorphism between inverse semigroups automatically preserves inverses; see [106, Proposition 2.1.1].

1.6.2 Étale Polish groupoids

Suppose that G is a Polish groupoid. A subset u of G is a bisection if the source and range maps restricted to u are injective. A bisection of G is open if it is an open subset of G. It is not difficult to verify that the following conditions are equivalent:

1. The source and range maps of G are local homeomorphisms from G to G^0;
2. Composition of arrows in G is a local homeomorphism from G^2 to G;
3. G has a countable basis of open bisections;
4. G has a countable inverse semigroup of open bisections that is basis for the topology of G;
5. G^0 is an open subset of G.

When these conditions are satisfied, G is called étale Polish groupoid. If G is an étale Polish groupoid, then in particular for every $x \in G^0$ the fiber Gx is a countable discrete subset of G.

1.6.3 The groupoid of germs

Suppose that $\theta : T \curvearrowright X$ is an action of a countable inverse semigroup on a Polish space. We want to associate to such an action an étale Polish groupoid $\mathcal{G}(\theta, T, X)$ that contains all the information about the action. This construction can be found in [31] in the case when X is locally compact.

If $e \in E(T)$ denote by D_e the domain of θ_e. Observe that the domain of θ_t is D_{t^*t} and the range of θ_t is D_{tt^*}. Define Ω to be the subset of $T \times X$ of pairs (u, x) such that $x \in D_{u^*u}$. Consider the equivalence relation $\sim$ on Ω defined by $(u, x) \sim (v, y)$ iff $x = y$ and for some

$e \in E(S)$, $ue = ve$ and $x \in D_e$. The equivalence class $[u, x]$ of (u, x) is called the *germ* of u at x. Observe that if e witnesses that $(u, x) \sim (v, x)$ then, after replacing e with u^*uv^*ve we can assume that $e \leq u^*u$ and $e \leq v^*v$. It can be verified as in [31, Proposition 4.7] that if (u, x) and (v, y) are in Ω and $x = \theta_v(y)$ then $(uv, y) \in \Omega$. Moreover the germ $[uv, y]$ of uv at y depends only on $[x, s]$ and $[y, t]$.

One can then define the groupoid $\mathcal{G}(\theta, T, X) = \Omega/\sim$ of germs of the action $S \curvearrowright X$ obtained by setting

- $\mathcal{G}(\theta, T, X)^2 = \{([u, x], [v, y]) : \theta_v(y) = x\}$,
- $[u, x]\,[v, y] = [uv, y]$, and
- $[u, x]^{-1} = [\theta_{u^*}, \theta_u(x)]$.

Observe that the map $x \mapsto [e, x]$ from X to G, where e is any element of $E(S)$ such that $x \in D_e$, is a well-defined bijection from X to the set of objects G^0 of G. Identifying X with G^0 we have that the source and range maps s and r are defined by

$$s\,[u, x] = x$$

and

$$r\,[u, x] = \theta_u(x).$$

We now define the topology of $\mathcal{G}(\theta, T, X)$. For $u \in T$ and $U \subset D_{u^*u}$ open define

$$\Theta\,(u, U) = \{[u, x] \in G : x \in U\}$$

It can be verified as in [31, Proposition 4.14, Proposition 4.15, Corollary 4.16, Proposition 4.17, and Proposition 4.18] that the following hold:

1. $\mathcal{G}(\theta, T, X)$ is an étale Polish groupoid;
2. the map $x \mapsto [e, x]$ where e is any element of $E(S)$ such that $x \in D_s$, is a homeomorphism from X onto the space of objects of $\mathcal{G}\,(\theta, T, x)$;
3. if $u \in T$ and $U \subset D_{u^*u}$ then $\Theta\,(u, U)$ is an open bisection of U, and the map $x \mapsto [u, x]$ is a homeomorphism from U onto $\theta\,(u, U)$;
4. if $\mathcal{A}$ is a basis for the topology of X, then the collection

$$\{\theta\,(u, A \cap D_{u^*u}) : u \in S, A \in \mathcal{A}\}$$

is a basis of open bisections for $\mathcal{G}(\theta, T, X)$.

1.6.4 Regularity of the groupoid of germs

The groupoid of germs $\mathcal{G}(\theta, T, X)$ for an action $\theta : T \curvearrowright X$ is in general not Hausdorff, even when X is locally compact. Here we isolate a condition that ensures that $\mathcal{G}(\theta, T, X)$ is regular.

Define the order $\leq$ on T by setting $u \leq v$ iff $u = vu^*u$. Observe that this extends the order of $E(T)$. Moreover if $u \leq v$ then

$$u^*u = v^*vu^*u^*v^*v = v^*vu^*u$$

and hence $u^*u \leq v^*v$. We say that T is a *semilattice* if it is a semilattice with respect to the order $\leq$ just defined, i.e. for every pair u, v of elements of T there is a largest element $u \wedge v$ below both u and v.

Proposition 1.6.2. *Suppose that T is a semilattice. If there is a subset C of T such that:*

1. *for every $u \in T$ and $x \in D_{u^*u}$ there is $c \in C$ such that $x \in D_{(u\wedge c)^*(u\wedge c)}$, and*
2. *for every distinct $c, d \in C$, $\Theta(c, D_{c^*c}) \cap \Theta(d, D_{d^*d}) = \varnothing$,*

then the groupoid of germs $\mathcal{G}(\theta, T, X)$ is regular.

Proof. Suppose that $[u, x]$ is an element of $\mathcal{G}(\theta, T, X)$, and W is an open neighborhood of $[u, x]$ in $\mathcal{G}(\theta, T, X)$. There are an open subset U of X contained in D_{u^*u} such that $[u, x] \in \Theta(u, U) \subset W$. Pick $c \in C$ such that $x \in D_{(u\wedge c)^*(u\wedge c)}$, and an open neighborhood V of x with $\overline{V}$ contained in $U \cap D_{(u\wedge c)^*(u\wedge c)}$. We claim that $\Theta(u \wedge c, V)$ is an open neighborhood of $[u, x]$ whose closure is contained in W. To show this it is enough to show that $\Theta\left(u \wedge c, \overline{V}\right)$ is closed in $\mathcal{G}(\theta, T, X)$. Pick $[v, y] \in \mathcal{G}(\theta, T, X) \backslash \Theta\left(u \wedge c, \overline{V}\right)$. If $y \notin \overline{V}$ then clearly there is an open neighborhood of $[t, y]$ disjoint from $\Theta\left(u \wedge c, \overline{V}\right)$. Suppose that $y \in \overline{V}$. Pick $d \in C$ such that $y \in D_{(u\wedge d)^*(u\wedge d)}$. In such case we have that

$$\Theta\left(u \wedge d, D_{(u\wedge d)^*(u\wedge d)}\right)$$

is an open neighborhood of y disjoint from $\Theta\left(u \wedge c, \overline{V}\right)$. This concludes the proof. $\square$

1.6.5 Étale groupoids as groupoids of germs

Suppose that G is an étale Polish groupoid, and Σ is a countable inverse semigroup of open bisections of G. One can define the standard action of Σ on G^0 by setting $D_e = e$ for every $e \in E(\Sigma)$, and $\theta_u : D_{u^*u} \to D_{uu^*}$ by

$$\theta_u(x) = r(ux),$$

where ux is the only element of u with source x. The same proof as [31, Proposition 5.4] shows the following fact:

Proposition 1.6.3. *Suppose that Σ is a countable inverse semigroup of open bisections of G such that $\bigcup \Sigma = G$ and for every $u, v \in \Sigma$, $u \cap v$ is the union of the elements of Σ contained in $u \cap v$. Consider the standard action $\theta : \Sigma \curvearrowright G^0$. The map from $\mathcal{G}(\theta, \Sigma, X)$ to G assigning to the germ $[u, x]$ of u at x the unique element of u with source x is well defined, and it is an isomorphism of étale Polish groupoids.*

In particular every étale Polish groupoid is isomorphic to the groupoid of germs of an action of an inverse semigroup on a Polish space.

1.6.6 Borel bisections

We will say that a (standard) Borel groupoid is countable if for every $x \in G^0$, the set $Gx = s^{-1}[\{x\}]$ is countable. Observe that the countable Borel equivalence relations are exactly the principal countable Borel groupoids.

Suppose that G is a countable Borel groupoid. Observe that the set $\mathcal{S}(G)$ of Borel bisections of G is an inverse semigroup. The idempotent semilattice $E(S)$ is the Boolean algebra of Borel subsets of G^0. The order $\leq$ on $\mathcal{S}(G)$ as in Subsection 1.6.4 is defined by $u \leq v$ iff $u \subset v$. Therefore (S) is a semilattice with $u \wedge v = u \cap v$.

Lemma 1.6.4. *Suppose that X, Z are standard Borel spaces and $s : Z \to X$ is a Borel countable-to-one surjection. There is a countable partition $(P_n)_{n\in\mathbb{N}}$ of Z into Borel subsets such that $s_{|P_n}$ is 1:1 for every $n \in \mathbb{N}$.*

Proof. It is enough to show that $Z = \bigcup_n P_n$, where P_n are Borel subsets of Z such that $s_{|P_n}$ is 1:1. After replacing Z with the disjoint union of Z and $X \times \mathbb{N}$, and setting $s(x, n) = x$ for $(x, n) \in X \times \mathbb{N}$, we can assume that for every $x \in X$ the inverse image $s^{-1}\{x\}$ is countably infinite. We want to define a Borel function $e : X \to Z^{\mathbb{N}}$ such that $\{e(x)_n : n \in \mathbb{N}\}$ is an enumeration of $s^{-1}\{x\}$ for every $x \in X$. Consider the Borel subset E of $X \times Z^{\mathbb{N}}$ defined by

$$\begin{aligned} (x,(e_n)) \in E \quad &\Leftrightarrow \quad (e_n) \text{ is a enumeration of } s^{-1}\{x\} \\ &\Leftrightarrow \quad s(e_n) = x \text{ and } \forall z \in s^{-1}\{x\}\ \exists n \text{ such that } z = e_n. \end{aligned}$$

(Recall that the image of a standard Borel space under a countable-to-one Borel function is Borel; see [73, Exercise 18.15].) We want to find a Borel uniformization of E. For each $x \in X$ endow $s^{-1}\{x\}$ with the discrete topology and $s^{-1}\{x\}^{\mathbb{N}}$ with the product topology. Observe that for $(e_n) \in s^{-1}\{x\}^{\mathbb{N}}$ we have that $(e_n) \in E_x$ iff $\forall z \in s^{-1}\{x\}\ \exists n \in \mathbb{N}$ such that $e_n = z$. Thus E_x is a dense G_δ subset of $s^{-1}\{x\}^{\mathbb{N}}$. Define the following σ-ideal $\mathcal{I}_x$ in $Z^{\mathbb{N}}$: $A \in \mathcal{I}_x$ iff $A \cap E_x$ is meager in $s^{-1}\{x\}^{\mathbb{N}}$. Thus $E_x \notin \mathcal{I}_x$. In order to conclude that E has a Borel uniformization, by [73, Theorem 18.6] it is enough to show that the assignment $x \mapsto \mathcal{I}_x$ is Borel-on-Borel as in [73, Definition 18.5]. Suppose that Y is a standard Borel space and $A \subset Y \times X \times Z^{\mathbb{N}}$. Consider the set

$$\begin{aligned} &\{(y,x) \in Y \times X : A_{y,x} \in \mathcal{I}_x\} \\ = \quad &\left\{(y,x) \in Y \times X : A_{y,x} \cap E_x \text{ is meager in } s^{-1}\{x\}^{\mathbb{N}}\right\} \end{aligned}$$

Clearly we can assume that $A \subset Y \times E$. If $e : \mathbb{N} \to s^{-1}\{x\}$ is a bijection, then e induces a homeomorphism $\pi_e : \mathbb{N}^{\mathbb{N}} \to s^{-1}\{x\}^{\mathbb{N}}$. Therefore for $(y,x) \in Y \times X$ we have that

$$\begin{aligned} A_{y,x} \cap E_x \text{ is meager in } s^{-1}\{x\}^{\mathbb{N}} \quad &\Leftrightarrow \quad \pi_e^{-1}\left[A_{y,x} \cap s^{-1}\{x\}^{\mathbb{N}}\right] \text{ is meager} \\ &\Leftrightarrow \quad \left\{w \in \mathbb{N}^{\mathbb{N}} : \pi_e(w) \in A_{y,x}\right\} \text{ is meager.} \end{aligned}$$

Consider the Borel subset Q of $Y \times X \times Z^{\mathbb{N}}$ defined by $(y,x,e) \in Q$ iff $(x,e) \in E$ and $\forall n,m \in \mathbb{N}$ if $n \neq m$ then $e_n \neq e_m$ and $\left\{w \in \mathbb{N}^{\mathbb{N}} : (y,x,e \circ w) \in A_{y,x}\right\}$ is meager. We have that

$$\begin{aligned} A_{y,x} \in \mathcal{I}_x \quad &\Leftrightarrow \quad \exists e \text{ such that } (y,x,e) \in Q \\ &\Leftrightarrow \quad \forall e \forall n \neq m \in \mathbb{N},\ (x,e) \in E, \text{ and } e_n \neq e_m \ \Rightarrow \ (z,x,e) \in Q. \end{aligned}$$

This shows that $\{(y,x) : A_{y,x} \in \mathcal{I}_x\}$ is both analytic and co-analytic, and hence Borel. $\square$

Proposition 1.6.5. *If G is a countable Borel groupoid, then there is a countable partition* *of G into Borel bisections. Moreover for every $n \in \mathbb{N}$ we have that*

$$\left\{x \in G^0 : |Gx| = n\right\}$$

is Borel.

Proof. The source map $s : G \to G^0$ satisfies the hypothesis of Lemma 1.6.4. Therefore one can find a countable partition $\mathcal{H}$ of G into Borel subsets such that the source map is 1:1 on every element of $\mathcal{H}$. Define

$$\mathcal{C} = \left\{u \cap v^{-1} : u, v \in \mathcal{H}\right\}$$

an observe that $\mathcal{C}$ is a countable collection of pairwise disjoint Borel bisections of G. Observe now that for every $u \in \mathcal{C}$,

$$\left\{x \in G^0 : \exists \gamma \in u,\, x = s(\gamma)\right\} = s\,[u] = u^{-1}u$$

is Borel being 1:1 image of a Borel set. Moreover $|Gx| = m$ iff $\exists u_0, \ldots, u_{m-1} \in \mathcal{C}$ pairwise distinct such that $x \in u_i u_i^{-1}$ for $i \in m$ and $\forall w \in \mathcal{C}$ if $x \in ww^{-1}$ then $w = u_i$ for some $i \in m$. □

Let us say that a Borel bisection u is *full* if $uu^{-1} = u^{-1}u = G^0$. It is clear from Proposition 1.6.5 that if G is a countable Borel groupoid, then there is a partition of G into full Borel bisections.

1.6.7 A Polish topology on countable Borel groupoids

In this subsection we observe that any countable Borel groupoid is Borel isomorphic to a regular zero-dimensional étale Polish groupoid. Suppose that G is a countable Borel groupoid. Pick a countable partition C of G into full Borel bisections and consider the smallest inverse subsemigroup of T with the property that $u \cap v \in T$ whenever $u, v \in T$. Observe that T is countable. By [73, Exercise 13.5] there is a zero-dimensional Polish topology τ^0 on G^0 generating the Borel structure on G^0 such that $u^{-1}u$ is clopen for every $u \in T$. Consider the standard action θ of T on $\left(G^0, \tau^0\right)$ and observe that it satisfies the condition of Proposition 1.6.2. Therefore the associated groupoid of germs $\mathcal{G}\left(\theta, T, G^0\right)$ is an étale zero-dimensional regular Polish groupoid. Arguing as in the proof of [31, Proposition 5.4] one can verify that the function ϕ from G to $\mathcal{G}\left(\theta, T, G^0\right)$ sending γ to $[c, s(\gamma)]$ where c is the only element of C such that $\gamma \in C$ is a well defined Borel isomorphism of countable Borel groupoids.

1.6.8 Treeable Borel groupoids

Suppose that G is a countable Borel groupoid. A *graphing* Q of G is a Borel subset Q of $G\backslash G^0$ such that $Q = Q^{-1}$ and $\bigcup_{n\in\mathbb{N}} Q^n = G$, where $Q^0 = G^0$. Suppose that Q is a graphing of G. Define $P^*(Q)$ to be the set of finite nonempty sequences $(\gamma_i)_{i\in n+1}$ in Q such that $r(\gamma_{i+1}) = s(\gamma_i)$ and $\gamma_{i+1} \neq \gamma_i^{-1}$ for $i \in n$. For $(\gamma_i)_{i\in n+1}$ in $P^*(Q)$ one can define

$$\prod_{i\in n+1} \gamma_i$$

to be the product $\gamma_n\gamma_{1n-2}\cdots\gamma_1\gamma_0$ in G. We say that Q is a *treeing* if for every $(\gamma_i)_{i\in n+1} \in P^*(Q)$,

$$\prod_{i\in n+1} \gamma_i \notin Q^0$$

or, equivalently, for every $\gamma \in G\backslash G^0$ there is exactly one element $(\gamma_i)_{i\in n+1}$ of $P^*(Q)$ such that $\prod_{i\in n+1} \gamma_i = \gamma$. A countable Borel groupoid is *treeable* when it admits a treeing [3, Section 8].

It is not difficult to verify that a principal countable Borel groupoid is treeable precisely when it is treeable as an equivalence relation. A countable group is treeable as groupoid if and only if it is a free group.

In the following if Q is a treeing of G we denote by $P(Q)$ the union of $P^*(Q)$ and $\{\varnothing\}$. In analogy with free groups, if $(\gamma_n, \ldots, \gamma_0) \in P(Q)$ we say that $\gamma_n \cdots \gamma_0$ is a *reduced word*, and that the length $l(\gamma_n \cdots \gamma_0)$ of $\gamma_n \cdots \gamma_0$ is $n + 1$.

Proposition 1.6.6. *Suppose that G is a countable Borel groupoid. If there is a Borel complete section A for E_G such that $G_{|A}$ is treeable, then G is treeable.*

Proof. Pick a Borel function $f : G^0 \to G$ such that $f(a) = a$ for $a \in A$, $s(f(x)) = x$ and $r(f(x)) \in A$ for $x \in G^0$. Suppose that Q_A is a treeing for $G_{|A}$. Observe that $Q_A \cup f\left[G^0 \backslash A\right]$ is a treeing for G. □

We want to show that Borel subgroupoids of treeable groupoid are treeable. A particular case of this statement is that a subgroup of a countable free group is free, which is the well known *Nielsen-Schreier theorem.* The strategy of our proof will be a Borel version for groupoids of Schreier's proof of the Nielsen-Schreier theorem.

Suppose that G is a treeable groupoid with no elements of order 2, and H is a Borel subgroupoid of G. In the rest of the subsection we will show that H is treeable. Denote by $\sim_H$ the equivalence relation $\gamma \sim_H \rho$ iff $\gamma H = \rho H$. Suppose that Q is a treeing for

G. Since G has no elements of order 2 we can write $Q = Q^+ \cup Q^-$ where Q^+ and Q^- are disjoint and $Q^+ = (Q^-)^{-1}$. A Borel transversal U for $\sim_H$ is *Schreier* if $\gamma_n \cdots \gamma_0 \in T$ implies $\gamma_k \cdots \gamma_0 \in T$ for $k \in n$. We want to show that there is a Schreier Borel transversal for H.

Suppose that $(V_n)_{n\in\mathbb{N}}$ is a partition of $G \backslash G^0$ into full Borel bisections. If $\gamma_n \cdots \gamma_0$ and $\gamma'_m \cdots \gamma'_0$ are reduced words with $r(\gamma_n) = r(\gamma'_m) = x$, set

$$\gamma_n \cdots \gamma_0 <_x \gamma'_m \cdots \gamma'_0$$

iff $n < m$, or $n = m$ and for some $k \in n$, $\gamma_i = \gamma'_i$ for $i \in k$ and for some $N \in \mathbb{N}$, $\gamma_k \in V_N$ while $\gamma'_k \notin V_n$ for any $n \leq N$. Define also

$$x <_x \gamma_n \cdots \gamma_0.$$

Observe that $<_x$ is a Borel order of xG with minimum x, and the function $x \mapsto <_x$ is Borel. Define now for $\gamma \in G$, $\overline{\gamma}$ to be the $<_{r(\gamma)}$-least element of γH. Thus $\overline{\gamma} \in \gamma H$ and hence $\overline{\gamma}^{-1}\gamma \in H$. Consider $U = \left\{ \overline{\gamma}^{-1}\gamma : \gamma \in G \right\}$ and observe that, since x is the $<_x$-minimum element of xG, $U \cap H \subset H^0$. Arguing as in [64, Section 2.3] one can show that U is a Schreier transversal for $\sim_H$. Define then

$$A = \left\{ \overline{\gamma u}^{-1} \gamma u : u \in U, \, \gamma \in Q \right\} \subset H.$$

The same proof as Lemma 3 in [64, Section 3.3] shows that $\bigcup_{n\in\mathbb{N}} A^n = H$. Define now

$$B = \left\{ \overline{\gamma u}^{-1} \gamma u : u \in U, \, \gamma \in Q^+, \text{ and } \gamma u \notin U \right\}.$$

The same proof as Lemma 4 in [64, Section 3.4] shows that

$$B^{-1} = \left\{ \overline{\gamma u}^{-1} \gamma u : u \in U, \, \gamma \in Q^-, \text{ and } \gamma u \notin U \right\},$$

and $A \backslash H^0$ is the disjoint union of B and B^{-1}. Finally one can show that $A \backslash H^0$ is a treeing for G as in [64, Section 3.6]. The proof is the same as the proof of Theorem 1 in [64, Section 3.6]. The fundamental lemma is the following:

Lemma 1.6.7. *Suppose that $b = \overline{u\gamma}^{-1} u\gamma \in A \backslash H^0$ and $b' = \overline{v\rho}^{-1} v\rho \in A \backslash H^0$. The product $\rho v \overline{\gamma u}^{-1} \gamma$ is equal to a reduced word $\rho w \gamma$ for some $w \in G$, unless $v = \overline{\gamma u}$ and $\rho = \gamma^{-1}$, in which case*

$$u = \overline{\gamma^{-1} \overline{\gamma u}} = \overline{\rho v}$$

and

$$b' = b^{-1}.$$

The proof of Lemma 1.6.7 is analogous to the proof of Lemma 5 in [64, Section 3.5].

1.7 Functorial Borel complexity and treeable equivalence relations

1.7.1 The lifting property

Definition 1.7.1. Suppose that G is a Polish groupoid. We say that G has the *lifting property* if the following holds: For any Polish groupoid H such that E_H is Borel, and any Borel function $f : G^0 \to H^0$ such that $f(x) E_H f(x')$ whenever $x E_G x'$, there is a Borel functor $F : G \to H$ that extends f.

Remark 1.7.2. If E_G has the lifting property (as a principal groupoid), then G has the lifting property.

Proposition 1.7.3. *A treeable countable Borel groupoid with no elements of order* 2 *has the lifting property.*

Proof. Suppose that G is a treeable countable Borel groupoid with no elements of order 2, H is a Polish groupoid such that E_H is Borel, and $f : G^0 \to H^0$ is a Borel function such that $f(x) E_G f(x')$ whenever $x E_G x'$. Suppose that Q is a treeing for G. Write $Q = Q^+ \cup Q^-$ where $Q^+ = (Q^-)^{-1}$ and Q^+ and Q^- are disjoint. Since E_H is Borel, then map $(x, y) \mapsto xHy$ from E_G to $F(H) \setminus \{\varnothing\}$ is Borel by Theorem 1.4.4. Fix a Borel map $\sigma : F(H) \setminus \{\varnothing\} \to H$ such that $\sigma(A) \in A$ for every $A \in F(H) \setminus \{\varnothing\}$. Define

- $F(x) = f(x)$ for $x \in G^0$,
- $F(\gamma) = \sigma\left(f(r(\gamma)) H f(s(\gamma))\right)$ for $\gamma \in Q^+$,
- $F(\gamma) = F\left(\gamma^{-1}\right)^{-1}$ for $\gamma \in (Q^+)^{-1}$, and
- $F(\gamma_n \cdots \gamma_0) = F(\gamma_n) \cdots F(\gamma_0)$ if $\gamma_n \cdots \gamma_0 \in G \backslash G^0$ is a reduced word.

It is immediate to check that F is a Borel functor such that $F_{|G^0} = f$. $\square$

Proposition 1.7.4. *If G is a Polish groupoid and $A \subset G^0$ is a Borel complete section for E_G such that $G_{|A}$ has the lifting property and there is a Borel map $\phi : G^0 \to G$ such that $s(\phi(x)) = x$ and $r(\phi(x)) \in A$ for every $x \in G^0$, then G has the lifting property.*

Proof. Without loss of generality we can assume that $\phi(x) = x$ for $x \in A$. Define $y(x) = r(\phi(x))$ for $x \in G^0$. Suppose that $f : G^0 \to H^0$ is a Borel function such that $f(x) E_H f(x')$ whenever $x E_G x$. Since $G_{|A}$ ha the lifting property there is a Borel functor $F : G_{|A} \to H$ such that $F_{|A} = f_{|A}$. Define $h(x) = \sigma(f(y(x)) H f(x))$. Define now for $\rho \in G$ such that $s(\rho) = x$ and $r(\rho) = y$

$$F(\rho) = h(y)^{-1} F\left(\phi(y)\rho\phi(x)^{-1}\right) h(x)$$

and observe that F is a Borel functor such that $F_{|G^0} = f$. □

Theorem 1.7.5. *Suppose that G is a Polish groupoid. If E_G is essentially treeable, then E_G has the lifting property.*

Proof. Observe that the assignment $[x]_{E_G} \mapsto I_{[x]_{E_G}}$,where

$$A \in I_{[x]_{E_G}} \Leftrightarrow \{\gamma \in xG : s(\gamma) \in A\} \text{ is meager}$$

is a Borel ccc assignment of σ-ideals in the sense of [72, page 285]; see Subsection 1.1.10. It follows from [72, Theorem 1.5] together with the fact that E_G is essentially treeable that there is a countable Borel subset A of G^0 meeting every orbit in a countable nonempty set. Thus $(E_G)_{|A}$ is treeable equivalence relation. In particular by Proposition 1.7.3 the equivalence relation $(E_G)_{|A}$ has the lifting property. Therefore $G_{|A}$ has the lifting property. Since $(E_G)_{|A}$ is countable one can find a Borel map $p : X \to A$ such that $(x, p(x)) \in E_G$ for every $x \in X$ and $p(x) = x$ for $x \in A$. It follows from Proposition 1.7.4 that E_G has the lifting property. □

Corollary 1.7.6. *Suppose that G and H are Polish groupoids. If E_G is essentially treeable, and E_H is Borel, then $G \leq_B H$ if and only if $E_G \leq E_H$.*

Proposition 1.7.7. *Suppose that G is a Polish groupoid. If E_G is essentially countable, then there is an invariant dense G_δ set $C \subset G^0$ such that $(E_G)_{|C}$ is essentially hyperfinite.*

Proof. By [60, Theorem 6.2] there is a comeager and invariant subset C_0 of G^0 such that $(E_G)_{|C_0}$ is essentially hyperfinite. Pick a dense G_δ subset C_1 of C_0 and then define

$$C = \{x \in X : \forall^* \gamma \in Gx, \, \gamma x \in C_1\}.$$

The properties of the Vaught transform together with Lemma 1.1.9 imply that C is an invariant dense G_δ set contained in C_0. In particular $(E_G)_{|C}$ is essentially hyperfinite. □

Corollary 1.7.8. *Suppose that G is a Polish groupoid such that E_G is essentially countable. There is an invariant dense G_δ subset C of G^0 with the following property: For any Polish groupoid H such that $E_G \leq_B E_H$ and E_H is Borel, $G_{|C} \leq_B H$.*

1.7.2 The cocycle property

Definition 1.7.9. An analytic groupoid G has the *cocycle property* if there is a Borel functor $F : E_G \to G$ such that $F(x,x) = x$ for every $x \in G^0$.

It is immediate to verify that a Polish group action $G \curvearrowright X$ has the cocycle property as defined in [60] if and only if the action groupoid $G \ltimes X$ has the cocycle property as in Definition 1.7.9. The proof of the following proposition is essentially the same of the proof of the implication (ii)⇒(iii) in [63, Theorem 3.7], and it is presented for convenience of the reader.

Proposition 1.7.10. *Suppose that G is a countable Borel groupoid, and X a Borel G-space. If $G \ltimes X$ has the cocycle property, then there is a free Borel G-space Y such that $E_G^Y \sim_B E_G^X$. Moreover if G is treeable then E_G^X is treeable.*

Proof. Since $G \ltimes X$ has the cocycle property there is a Borel functor

$$F : E_G^X \to G$$

such that $s(F(x,y)) = p(y)$ and $F(x,y)y = x$. Consider the equivalence relation $\sim$ on $G \ltimes X$ defined by $(\gamma, x) \sim (\rho, y)$ iff $(x,y) \in E_G^X$ and $\gamma F(x,y) = \rho$. Clearly $\sim$ is Borel. We now show that it has a Borel selector. Observe that the range H of F is a Borel subgroupoid of G (since F is countable to one). By Proposition 1.4.1 there is a Borel selector $t : G \to G$ for the equivalence relation $\gamma \sim_H \gamma'$ iff $\gamma H = \gamma' H$. Observe that if $(\gamma, x) \sim (\rho, y)$ then $\gamma H = \rho H$ and hence $t(\gamma) = t(\rho)$. Moreover there is a unique element x_0 of X such that $(t(\gamma), x_0) \sim (\gamma, x)$. Define $S(\gamma, x) = (t(\gamma), x_0)$ and observe that S is a Borel selector for the equivalence relation $\sim$. Define Y to be the quotient of $G \ltimes X$ by $\sim$. Define now the Borel action of G on Y by $p[\gamma, x] = r(\gamma)$ and $\rho[\gamma, x] = [\rho\gamma, x]$ for $\rho \in Gr(\gamma)$. It is easy to verify that such an action is free, and $[\gamma, x]\, E_G^Y\, [\rho, y]$ iff $x E_G^X y$. Let us now observe that $E_G^X \sim_B E_G^Y$. If $q : X \to G$ is a Borel map such that $s(q(x)) = p(x)$ for every $x \in X$, then the map $x \mapsto [q(x), x]$ is a Borel reduction from E_G^X to E_G^Y. Conversely the map $[\gamma, x] \mapsto x^*$ where $[t(\gamma), x^*] = S(\gamma, x)$ is a Borel reduction from E_G^Y to E_G^X. Suppose finally that G is treeable with treeing Q. We want to show that E_G^X is treeable. Since $E_G^X \sim_B E_G^Y$, it is enough to show that E_G^Y is treeable. Fix an equivalence class $[[\gamma, x]]_F$ of E_G^Y. Observe that the map from $[[\gamma, x]]_{E_G^Y}$ to $Gp(x)$ defined by $[\rho, y] \mapsto \rho F(y, x)$ is bijective. One can then consider the treeing

$$\{[\rho, y] \in Y : \rho F(y, x) \in Q\}$$

for E_G^Y. $\square$

Lemma 1.7.11 can be proved similarly as Proposition 1.7.4.

Lemma 1.7.11. *Suppose that G is a countable Borel groupoid action, and $A \subset G^0$ is a Borel complete section for E_G. If there is a Borel function $\psi : (E_G)_{|A} \to G$ such that $s(\psi(x, y)) = y$ and $r(\psi(x, y)) = x$, then G has the cocycle property.*

Lemma 1.7.12. *Suppose that G is a countable Borel groupoid. If $E_G \leq_B G$, then there is an invariant Borel subset Y of G^0 such that $G_{|Y}$ has the cocycle property and $(E_G)_{|Y} \sim_B E_G$*

Proof. Since $E_G \leq_B G$ there is a Borel functor $F : E_G \to G$. Define $A \subset G^0$ to be image of X under F. Since $F_{|X}$ is countable-to-one, A is a Borel subset of G^0; see [73, Theorem 18.10]. By [73, Exercise 18.14] there is a Borel function $g : A \to X$ such that $(f \circ g)(y) = y$ for every $y \in A$. Define now Y to be the union of the orbits of G that meet A. Clearly $E_G \sim_B E_{G_{|Y}}$. The map

$$\begin{array}{rcl} (E_G)_{|A} & \to & G \\ (x, y) & \mapsto & F(g(x), g(y)) \end{array}$$

together with Lemma 1.7.11 imply that $G_{|Y}$ has the cocycle property. $\square$

1.7.3 Free actions of treeable groupoids

We want to show that, if G is a treeable groupoid, and $G \curvearrowright X$ is a free Borel action of G, then the associated orbit equivalence relation is treeable. This will follow from a more general result about $\mathcal{L}$-structured equivalence relations.

Suppose that $\mathcal{L} = \{R_n : n \in \mathbb{N}\}$ is a countable relational language in first order logic, where R_n has arity $k_n \in \mathbb{N}$. Suppose that E is a countable Borel equivalence relation on a standard Borel space X. According to [63, Definition 2.17] the equivalence relation E is $\mathcal{L}$-structured if there are Borel relations $R_n^E \subset X^{k_n}$ such that, for any $k \in \mathbb{N}$ and $x_1, \ldots, x_{k_n} \in X$, $x_1, \ldots, x_{k_n}$ belong to the same E-class whenever $(x_1, \ldots, x_{k_n}) \in R_n^E$. In particular every E-class $[x]$ is the universe of an $\mathcal{L}$-structure

$$\left\langle [x], \left([x]^{k_n} \cap R_n^E\right)_{n \in \mathbb{N}} \right\rangle.$$

Similarly, if X is a standard Borel space, then *standard Borel bundle $\mathcal{A}$ of countable $\mathcal{L}$-structures* over X is a standard Borel space $\mathcal{A}$ fibred over X with countable fibers $(A_x)_{x\in X}$, endowed with Borel subsets $R_n^{\mathcal{A}} \subset \mathcal{A}^{k_n}$ such that, for any $k \in \mathbb{N}$ and $a_1, \ldots, a_{k_n} \in \mathcal{A}$, $a_1, \ldots, a_{k_n}$ belong to the same fiber over X whenever $(a_1, \ldots, a_{k_n}) \in R_n^{\mathcal{A}}$. Suppose that G is a Polish groupoid, and $\mathcal{A}$ is a standard Borel of $\mathcal{L}$-structures over G^0. A Borel action $G \curvearrowright \mathcal{A}$ is a Borel map $(\gamma, a) \mapsto \gamma a$ defined for $(\gamma, a) \in \mathcal{A} \times G$ such that $a \in A_{s(\gamma)}$, and $(a_1, \ldots, a_{k_n}) \in R_n^{\mathcal{A}}$ if and only if $(\gamma a_1, \ldots, \gamma a_{k_n}) \in R_n^{\mathcal{A}}$ for every $n \in \mathbb{N}$, $\gamma \in G$, and $a_1, \ldots, a_{k_n} \in A_{s(\gamma)}$. The proof of the following theorem is very similar to the argument at the beginning of Section 3.2 in [63], and it is reproduced for convenience of the reader.

Theorem 1.7.13. *Suppose that G is a Polish groupoid such that there is a standard Borel bundle $\mathcal{A}$ of countable $\mathcal{L}$-structures and a Borel action $G \curvearrowright \mathcal{A}$ such that the corresponding orbit equivalence relation $E_G^{\mathcal{A}}$ is smooth, and for every $a \in \mathcal{A}$ the stabilizer $G_a = \{\gamma \in Gp_{\mathcal{A}}(a) : \gamma a = a\}$ is a compact subset of G. If X is a standard Borel space, and $G \curvearrowright X$ is a free Borel action of G on X, then there is an $\mathcal{L}$-structured countable Borel equivalence relation E such that $E \sim_B E_G^X$, and moreover every class of E is isomorphic to some fiber of $\mathcal{A}$.*

Proof. By Corollary 1.1.18 there is a Borel selector t for $E_G^{\mathcal{A}}$. Moreover by Theorem 1.3.1 we can assume without loss of generality that the action $G \curvearrowright X$ is continuous. Define

$$\mathcal{A} * X = \left\{(x, a) \in X \times A : a \in A_{p(x)}\right\}$$

and the action $G \curvearrowright (X * A)$ by $\gamma\,(x, a) = (\gamma x, \gamma a)$. Observe that such an action is free and in particular the associated orbit equivalence relation $\sim$ is Borel. We now show that $\sim$ has a Borel selector. Suppose that $(x, a) \in X * A$. Observe that if $(x, a) \sim (x, b)$ then $a E_G^{A} b$ and hence $t(a) = t\,(b)$. Therefore $t(a)$ depends only on the $\sim$-class $[x, a]$ of (x, a). Observe that

$$G_{t(a)}x = \{y \in X : (y, t(a)) \in [x, a]\}$$

is a compact subset of X. Denote by $\sigma : F(X) \backslash \{\varnothing\} \to X$ a Borel function such that $\sigma\,(A) \in A$ for every $A \in F(X) \backslash \{\varnothing\}$. Define $S\,(x, a) = \left(\sigma\left(G_{t(a)}x\right), t(a)\right)$, and observe that S is a Borel selector for $\sim$. Define the standard Borel space $Y = (X * A) / \sim$ and the countable Borel equivalence relation E on Y by $[x, a]\,E\,[y, b]$ iff $x E_G^X y$. We now define for every E-class $C = [[x, a]]_E$ an $\mathcal{L}$-structure on C. Fix $n \in \mathbb{N}$ and suppose that $[\gamma_i x, a_i]$ for $i \in k_n$ are elements of C, where $\gamma_i \in Gp(x)$ for $i \in k_n$. Set $([\gamma_i x, a_i])_{i\in k_n} \in R_n^C$ iff $\left(\gamma_i^{-1} a_i\right)_{i\in k_n} \in R_n^{\mathcal{A}}$. Using the fact that G acts by $\mathcal{L}$-isomorphisms one can verify that this does not depend on the choice of $[x, a] \in C$. Define now $([x_i, a_i])_{i \in k_n} \in R_n^E$ if and only if $[x_0, a_0]\, E\, [x_1, a_1]\, E \cdots E\, [x_{k_n - 1}, a_{k_n - 1}]$ and $([x_i, a_i])_{i \in k_n} \in R_n^C$ where $C = [[x, a]]_E$. This defines Borel relations R_n^E on E that make E $\mathcal{L}$-structured. Moreover the Borel map $f : C \to \mathcal{A}$ defined by $f\,[\gamma x, a] = \gamma^{-1} a$, where C is the class of $[x, a]$, shows that the $\mathcal{L}$-structure

$$\left\langle C, \left(R_n \cap C^{k_n}\right)_{n \in \mathbb{N}} \right\rangle$$

is isomorphic to $A_{p(x)}$. Finally we observe now that $E \sim E_G^X$. If $q : G^0 \to \mathcal{A}$ such that $q(x)$ belongs to the fiber A_x for every $x \in G^0$, then the map $x \mapsto [x, q\,(p(x))]$ is a Borel reduction from E_G^X to E. Conversely the map $[x, a] \mapsto x^*$ where $[x^*, a^*] = S\,([x, a])$ witnesses that E is Borel reducible to E_G^X. □

Let us now consider the particular case of Theorem 1.7.13 when $\mathcal{L}$ is the language with a single binary relation. Assume further that G is a treeable Borel groupoid. A standard Borel bundle of trees over X is a standard Borel bundle $(A_x)_{x \in X}$ of countable $\mathcal{L}$-structures such that A_x is a tree for every $x \in X$. A treeing of G defines on G a structure of standard Borel bundle of trees over G. Moreover the action of G on itself by left translation is compatible with such a bundle of trees structure, and has a smooth orbit equivalence relation. Therefore by Theorem 1.7.13 E is treeable. This concludes the proof of the following corollary.

Corollary 1.7.14. *If G is a countable treeable groupoid, and $G \curvearrowright X$ is a free Borel action, then the orbit equivalence relation E_G^X is treeable.*

1.7.4 Characterizing treeable equivalence relations

Denote by $\mathbb{F}_\infty$ the free countable group on infinitely many generators. The following result subsumes [63, Theorem 3.7].

Theorem 1.7.15. *Suppose that E is a countable Borel equivalence relation on a standard Borel space X. The following statements are equivalent:*

1. *E is treeable;*
2. *E has the lifting property;*
3. *For every countable Borel groupoid G and Borel action $G \curvearrowright X$ such that $E_G^X = E$, the groupoid $G \ltimes X$ has the cocycle property;*
4. *For every Borel action $\mathbb{F}_\infty \curvearrowright X$ such that $E_{\mathbb{F}_\infty}^X = E$, $E_{\mathbb{F}_\infty}^X \leq_B \mathbb{F}_\infty \ltimes X$;*

5. *For every countable Borel groupoid G and Borel action $G \curvearrowright X$ such that $E_G^X = E$, there is a free Borel action $G \curvearrowright Y$ such that $E_G^Y \sim E_G^X$;*

6. *For every countable Borel groupoid G and Borel action $G \curvearrowright X$ such that $E \subset E_G^X$ there is a free Borel action $G \curvearrowright Y$ such that $E \sqsubseteq_B E_G^Y$.*

Proof. **(1)⇒(2)** It follows from Proposition 1.7.3.

(2)⇒(3) It follows form the fact that if E_G^X has the lifting property, then $G \ltimes X$ has the cocycle property.

(3)⇒(4) Obvious.

(4)⇒(1) It follows from Lemma 1.7.12 and Proposition 1.7.10.

(3)⇒(5) This follows from Proposition 1.7.10.

(5)⇒(1) Consider an action $\mathbb{F}_\infty \curvearrowright X$ such that $E = E_{\mathbb{F}_\infty}^X$ and then apply (5) and Corollary 1.7.14.

(2)⇒(6) Since E has the lifting property, there is a Borel function $F : E \to G$ such that $s(F(x,y)) = p(y)$ and $F(x,y)y = x$ for every $(x,y) \in E$. Consider on $G \ltimes X$ the equivalence relation $(\gamma, x) \sim (\rho, y)$ iff xEy and $\rho = \gamma F(x,y)$. Proceeding as in the proof of Proposition 1.7.10 one can show that $\sim$ has a Borel selector. Thus the quotient Y of $G \ltimes X$ by $\sim$ is standard. Define the Borel action $G \curvearrowright Y$ by $p[\gamma, x] = r(\gamma)$ and $\rho[\gamma, x] = [\rho\gamma, x]$. As in the proof of Proposition 1.7.10 one can show that such an action is free. Moreover the map $x \mapsto [p(x), x]$ is an injective Borel reduction from E to E_G^Y.

(6)⇒(1) It follows from Corollary 1.7.14 together with the fact that a subrelation of a treeable equivalence relation is treeable [63, Proposition 3.3]; see also Subsection 1.6.8. □

The following corollary is a direct consequence of the implication (4)⇒(1) in Theorem 1.7.15.

Corollary 1.7.16. *For any nontreeable equivalence relation E there are countable Borel groupoids G and H that have E as orbit equivalence relation such that G is not Borel reducible to H. Moreover one can take $G = E$ and H to be the action groupoid of an action of $\mathbb{F}_\infty$.*

Corollary 1.7.16 can be interpreted as asserting that functorial Borel complexity provides a finer distinction between the complexity of classification problems in mathematics than the traditional notion of Borel complexity for equivalence relations

Chapter 2

Unitary equivalence of automorphisms of C*-algebras

If A is a separable C*-algebra, the group $\mathrm{Aut}(A)$ of automorphisms of A is a Polish group with respect to the topology of pointwise norm convergence. An automorphism of A is called (multiplier) *inner* if it is induced by the action by conjugation of a unitary element of the multiplier algebra $M(A)$ of A. Inner automorphisms form a Borel normal subgroup $\mathrm{Inn}(A)$ of the group of automorphisms of A. The relation of *unitary equivalence* of automorphisms of A is the coset equivalence relation on $\mathrm{Aut}(A)$ determined by $\mathrm{Inn}(A)$. (The reader can find more background on C*-algebras in Section 2.1.) The main result presented in this chapter asserts that if A does not have continuous trace, then it is not possible to effectively classify the automorphisms of A up to unitary equivalence using countable structures as invariants; in particular this rules out classification by K-theoretic invariants. (The K-theoretic invariants of C*-algebras were shown to be computable by a Borel function in [37, Theorem 3.3]. Even though [37, Theorem 3.3] does not consider the K-theory of *-homomorphisms, it is not difficult to verify that the proof can be adapted to show that the computation of K-theory of *-homomorphisms is given by a *Borel functor*. The main ingredient of the proof is the fact that one can enumerate in a Borel fashion dense sequences of projections and of unitary elements of the algebra and of all its amplifications [38, Lemma 3.13].) In the course of the proof of the main result we will show that the existence of an outer derivation on a C*-algebra A is equivalent to a seemingly stronger statement, that we will refer to as Property AEP (see Definition 2.3.4), implying in particular the existence of an outer derivable automorphism of A.

In the last decade a number of natural equivalence relations arising in different areas of mathematics have been shown to be not classifiable by countable structures. For example the type of invariants that appear in the spectral theorem for normal operators transcend countable structures by a result of Kechris and Sofronidis [74, Theorem 1]. The theory of turbulence, developed by Greg Hjorth in the second half of the 1990s, plays a key role in the proof of this and of many other analogous results.

Turbulence is a dynamic condition on a continuous action of a Polish group on a Polish space, implying that the associated orbit equivalence relation is not classifiable by countable structures. Many nonclassifiability results were established directly or indirectly using this criterion. For instance Hjorth showed in [59, Section 4.3] that the orbit equivalence relation of a turbulent Polish group action is Borel reducible to the relation of homeomorphism of compact spaces, which in turn is reducible to the relation of isomorphism of separable simple nuclear unital C*-algebras by a result of Farah-Toms-Törnquist [38, Corollary 5.2]. As a consequence these equivalence relations are not classifiable by countable structures.

In this chapter, we use Hjorth's theory of turbulence to prove the following theorem. (See Definition 2.1.1 for the notion of continuous trace C*-algebra.)

Theorem 2.0.17. *If A is a separable C*-algebra that does not have continuous trace, then the automorphisms of A are not classifiable by countable structures up to unitary equivalence.*

Theorem 2.0.17 strengthens [111, Theorem 3.1], where the automorphisms of A are shown to be not concretely classifiable under the same assumptions on the C*-algebra A. We will in fact show that the same conclusion holds even if one only considers the subgroup consisting of approximately inner automorphisms of A, i.e. pointwise limits of inner automorphisms.

A particular implication of Theorem 2.0.17 is that it is not possible to classify the automorphisms of any separable C*-algebra that does not have continuous trace up to unitary equivalence by Borel-computable K-theoretic invariants. This should be compared with the classification results of (sufficiently outer) automorphisms up to other natural equivalence relations, such as outer conjugacy; see [103, Section 3]. Nakamura showed in [103, Theorem 9] that aperiodic automorphisms of Kirchberg algebras are classified by their KK-classes up to outer conjugacy. Theorem 1.4 of [81] asserts that there is only one outer conjugacy class of uniformly aperiodic automorphisms of UHF algebras. These results were more recently generalized and expanded to classification of actions of $\mathbb{Z}^2$ and $\mathbb{Z}^n$ up to outer conjugacy or cocycle conjugacy (see [97], [96], [68], and [98]).

Phillips and Raeburn obtained in [113] a cohomological classification of automorphisms of a C*-algebra with continuous trace up to unitary equivalence. Such classification implies that if A has continuous trace and the spectrum of A is homotopy equivalent to a compact space, then the normal subgroup $\mathrm{Inn}(A)$ of inner automorphisms is closed in $\mathrm{Aut}(A)$; see [116, Theorem 0.8]. In particular (cf. [11, Corollary II.6.5.7]) this conclusion holds when A is unital and has continuous trace. It follows from a standard result in descriptive set theory –see [44, Exercise 6.4.4]– that the automorphisms of A are concretely classifiable up to unitary equivalence if and only if $\mathrm{Inn}(A)$ is a closed subgroup of $\mathrm{Aut}(A)$. Theorem 0.8 of [116] and Theorem 2.0.17 therefore imply the following dichotomy result:

Theorem 2.0.18. *If A is a separable unital C*-algebra, then the following statements are equivalent:*

1. *the automorphisms of A are concretely classifiable up to unitary equivalence;*
2. *the automorphisms of A are classifiable by countable structures up to unitary equivalence;*
3. *A has continuous trace.*

More generally the same result holds if A is a separable C*-algebra with (not necessarily Hausdorff) compact spectrum. Without this hypothesis the implication $3 \Rightarrow 1$ of Theorem 2.0.18 does not hold, as pointed out in [116, Remark 0.9]. We do not know if the implication $3 \Rightarrow 2$ holds for a not necessarily unital C*-algebra A.

In particular Theorem 2.0.18 offers another characterization of unital C*-algebras that have continuous trace, in addition to the classical Fell-Dixmier spectral condition (see [39], [21]) or the reformulation in terms of central sequences by Akemann and Pedersen; see [2, Theorem 2.4].

The dichotomy in the Borel complexity of the relation of unitary equivalence of automorphisms of a unital C*-algebra expressed by Theorem 2.0.18 should be compared with the analogous phenomenon concerning the relation of unitary equivalence of irreducible representations of a C*-algebra A. It is a classical result of Glimm from [49] that such a relation is smooth if and only if A is Type I. It was proved in [75] and, independently, in [33] that the irreducible representations of a C*-algebra that is not Type I are in fact not classifiable by countable structures up to unitary equivalence.

The strategy of the proof of Theorem 2.0.17, summarized in Figure 1, is the following: We first introduce in Definition 2.3.4 and 2.3.9 properties AEP and AEP$^+$, named after Akemann, Elliott, and Pedersen since they can be found *in nuce* in their works [2] and [28]. (The main result of [28] is a characterization of C*-algebras with only inner derivations as

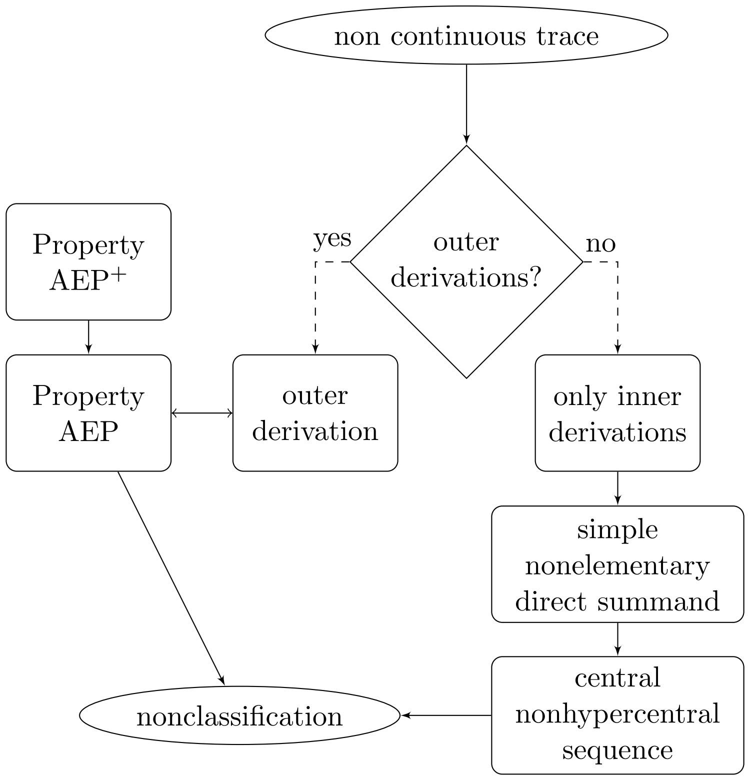

Figure 1:

direct sum of simple C*-algebras and C*-algebras with no nontrivial central sequence [28, Theorem 1]. Theorem 2.4 of [2] shows that a C*-algebra is does not have any nontrivial central sequence if and only if it has continuous trace.) We then show in Proposition 2.3.11 that Property AEP$^+$ is stronger than Property AEP; moreover by Theorem 2.3.5 Property AEP is equivalent to the existence of an outer derivation, and by Lemma 2.3.3 it implies that the conclusion of Theorem 2.0.17 holds.

This concludes the proof under the assumption that the C*-algebra A has an outer derivation. We then assume that A does not have continuous trace and has only inner derivations. Using the already mentioned characterization of C*-algebras with only inner derivations from [28, Theorem 1] and the characterization of continuous trace C*-algebras in terms of central sequences given in [2, Theorem 2.4], we infer that in this case A has a simple nonelementary direct summand. We then deduce in Proposition 2.4.5 that A contains a central sequence that is not strict-hypercentral. (A similar result was proved by Phillips in the unital case, cf. [112, Theorem 3.6].) The proof is finished by proving that the existence of a central sequence that is not strict-hypercentral implies that the conclusion of Theorem

2.0.17 holds. This is done in Proposition 2.4.6.

This chapter is organized as follows: Section 2.1 contains some background on C*-algebras and introduces the notations used in the rest of the chapter; Section 2.2 infers from Hjorth's theory of turbulence a criterion of nonclassifiability by countable structures (Criterion 2.2.3), to be applied in the proof of Theorem 2.0.17; Section 2.3 establishes Theorem 2.0.17 in the case of C*-algebras with outer derivations, while Section 2.4 deals with the case of C*-algebras with only inner derivations; Section 2.5 present a dichotomy result for derivations analogous to Theorem 2.0.18 (Theorem 2.5.1).

2.1 Some background notions on C*-algebras

The *multiplier algebra* $M(A)$ of a C*-algebra A is the largest unital C*-algebra containing A as an essential ideal; see [11, II.7.3]. It can be regarded as the noncommutative analog of the Stone-Čech compactification of a locally compact Hausdorff space. The strict topology on $M(A)$ is the locally convex vector space topology on $M(A)$ generated by the seminorms $x \mapsto \|ax\| + \|xa\|$ for $a \in A$ [11, II.7.3.11]. A positive contraction b_0 of A is *strictly positive* if

$$ab_0^{\frac{1}{n}} \to a$$

for every $a \in A$ [11, II.4.2.1]. If b_0 is any strictly positive contraction in A, then the strict topology on $M(A)$ can be equivalently defined as the locally convex vector space topology on A generated by the single seminorm $x \mapsto \|b_0x\| + \|xb_0\|$. The multiplier algebra of a separable C*-algebra A is not norm separable (unless A is unital, in which case $M(A)$ coincides with A). Nonetheless the *strict topology* of $M(A)$ is Polish and induces a Polish group structure on the group $U(A)$ of unitary elements of $M(A)$. If u is a unitary multiplier of A, i.e. an element of $U(A)$, then one can define as before the automorphism $\mathrm{Ad}(u)$ of A. An automorphism of A is called *inner* if it is of the form $\mathrm{Ad}(u)$ for some unitary multiplier u, and *outer* otherwise. Inner automorphisms of a separable C*-algebra A form a Borel normal subgroup of $\mathrm{Aut}(A)$; see [111, Proposition 2.4] . Two automorphisms α and β of A are called *unitarily equivalent* if $\alpha \circ \beta^{-1}$ is inner or, equivalently,

$$\alpha(x) = \beta(uxu^*)$$

for some unitary multiplier u and every $x \in A$. This defines a Borel equivalence relation on $\mathrm{Aut}(A)$.

A *representation* of A on a Hilbert space H is a $*$-homomorphism from A to the C*-

algebra $B(H)$ of bounded linear operators on H; see [11, Definition II.6.1.1]. Two representations π ,π' of A on Hilbert spaces H,H' are *unitarily equivalent* if there is a surjective linear isometry $U : H \to H'$ such that

$$U\pi(a) = \pi'(a)U$$

for every $a \in A$. A representation π of A on a Hilbert space H is called *irreducible* if there is no nontrivial closed subspace of H which is $\pi(a)$-invariant for every $a \in A$. The *spectrum* $\hat{A}$ of a separable C*-algebra A is the space of unitary equivalence classes of irreducible representations of A on a separable Hilbert space [109, Section 4.1]. This is canonically endowed with the *hull-kernel topology*, which is the topology having as open basis the collection of sets of the form

$$\mathcal{O}_I = \left\{[\pi] \in \hat{A} : I \nsubseteq \mathrm{Ker}(\pi)\right\}$$

for some closed ideal I of A. In general this topology has very poor separation properties, and can even fail to be T_0. A closed ideal of A is *primitive* if it is the kernel of an irreducible representation of A. A C*-algebra A is called *primitive* if $\{0\}$ is a primitive ideal in A, i.e. A has a faithful irreducible representation. The *primitive spectrum* $\check{A}$ of A is the space of primitive ideals of A endowed with the quotient topology from the canonical surjection

$$\begin{array}{ccc} \hat{A} & \to & \check{A} \\ [\pi] & \mapsto & \mathrm{Ker}(\pi). \end{array}$$

An element x of a C*-algebra A is *abelian* if the closure of x^*Ax in A is a commutative subalgebra.

Definition 2.1.1. A separable C*-algebra A has *continuous trace* if it is generated by abelian elements, and the spectrum $\hat{A}$ endowed with the hull-kernel topology is a Hausdorff space.

Equivalent reformulations of the notion of continuous trace C*-algebras can be found in [11, Definition IV.1.4.12 and Proposition IV.1.4.19] . The class of C*-algebras that do not have continuous trace is fairly large, and in particular includes all C*-algebras that are not Type I. (A C*-algebra A is Type I if every nonzero quotient of A contains a nonzero abelian element. Several equivalent characterizations of Type I C*-algebras are listed in [11, IV.1.5.1].) More information about C*-algebras with continuous trace can be found in the monograph [117].

In the rest of the chapter, *we assume all C*-algebras to be norm separable*, apart from multiplier algebras and enveloping von Neumann algebras. If A is a C*-algebra, then the *universal representation* π_u of A is the direct sum of all cyclic representations of A associated with states of A [109, 3.7.6, 3.7.8]. The *enveloping von Neumann algebra* of A is the closure of $\pi_u[A]$ in the strong operator topology. It is a well known theorem (see [109, Proposition 3.7.8]) that the enveloping von Neumann algebra of A is isometrically isomorphic –as a Banach space– to the second dual of A. We will therefore denote in the following by A^{**} the enveloping von Neumann algebra of A. The *σ-weak topology* on A^{**} coincides with the weak* topology of A^{**} regarded as the dual Banach space of A^*. The algebra A can be identified with a σ-weakly dense subalgebra of A^{**}. Moreover by [109, 3.12.3] we can identify the multiplier algebra $M(A)$ of A with the idealizer of A inside A^{**}, i.e. the algebra of elements x such that $xa \in A$ and $ax \in A$ for every $a \in A$. Analogously, the *unitization* $\tilde{A}$ of A [11, II.1.2] is identified with the subalgebra of $M(A)$ generated by A and 1.

If x is a *normal* element of A, i.e. commuting with its adjoint, and f is a complex-valued continuous function defined on the spectrum of x, then $f(x)$ denotes the element of $\tilde{A}$ obtained from x and f using functional calculus (II.2 of [11] is a complete reference for the basic notions of spectral theory and continuous functional calculus in operator algebras). If x, y are element of a C*-algebra, then $[x, y]$ denotes their *commutator* $xy - yx$; moreover if S is a subset of a C*-algebra A, then $S' \cap A$ denotes the *relative commutant* of S in A; see [11, I.2.5.3]. The set $\mathbb{N}$ of natural numbers is supposed not to contain 0. Boldface letters $\mathbf{t}$ and $\mathbf{s}$ indicate sequences of real numbers whose n-th terms are t_n and s_n respectively. Analogously $\mathbf{x}$ stands for the sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a C*-algebra A.

2.2 Nonclassifiability criteria

Recall that a subset A of a Polish space X has the Baire property [73, Definition 8.21] if its symmetric difference with some open set is meager. A function between Polish spaces is Baire measurable [73, Definition 8.37] if the inverse image of any open set has the Baire property. Observe that, in particular, any Borel function is Baire measurable. Suppose that E and R are equivalence relations on Polish spaces X and Y respectively. We say that E is *generically R-ergodic* if, for every Baire measurable function $f : X \to Y$ such that $f(x)Rf(y)$ whenever xEy, there is a comeager subset C of X such that $f(x)Rf(y)$ for every $x, y \in C$ [44, Definition 10.1.4]. Observe that if E is generically R-ergodic and no equivalence class of E is comeager then, in particular, E is not Borel reducible to R.

One of the main tools in the study of Borel complexity of equivalence relations is Hjorth's theory of turbulence. A standard reference for this subject is [59]. Turbulence is a dynamical property of a continuous group action of a Polish group G on a Polish space X; see [59, Definition 3.13]. The main result about turbulent actions is the following result of Hjorth (Theorem 3.21 in [59]):

The orbit equivalence relation E_G^X associated with a turbulent action $G \curvearrowright X$ of a Polish group G on a Polish space X is generically $\simeq_{\mathcal{C}}$-ergodic for every class $\mathcal{C}$ of countable structures, where $\simeq_{\mathcal{C}}$ denotes the relation of isomorphism for elements of $\mathcal{C}$. Since (by definition of turbulence) E_G^X has meager equivalence classes, it is in particular not classifiable by countable structures.

This result is valuable because it allows one to obtain several nonclassification results. In order to apply such result it will be useful to first state and prove the following to easy lemmas:

Lemma 2.2.1. *Suppose that E, F, and R are equivalence relations on Polish spaces X, Y, and Z respectively, and that F is generically R-ergodic. If there is a comeager subset $\tilde{C}$ of Y and a Baire measurable function $f : \widetilde{C} \to X$ such that:*

- *$f(x)Ef(y)$ for any $x, y \in \widetilde{C}$ such that xFy;*
- *$f[C]$ is comeager in X for every comeager subset C of $\widetilde{C}$;*

then the relation E is generically R-ergodic as well.

Proof. Suppose that $g : X \to Z$ is a Baire measurable function such that $g(x)Rg(x')$ for any $x, x' \in X$ such that xEx'. The composition $g \circ f$ is a Baire measurable function from $\tilde{C}$ to Z such that $(f \circ g)(y)R(f \circ g)(y')$ for any $y, y' \in \tilde{C}$ such that yFy'. Since $\tilde{C}$ is comeager in Y, and F is generically R-ergodic, there is a comeager subset C of $\tilde{C}$ such that $(g \circ f)(y)R(g \circ f)(y')$ for every $y, y' \in C$. Therefore, $f[C]$ is a comeager subset of X such that $g(x)Rg(x')$ for every $x, x' \in f[C]$. $\square$

Observe that if f is continuous, open, and onto, then it will automatically satisfy the second condition of Lemma 2.2.1.

Lemma 2.2.2. *Suppose that E and F are equivalence relations on Polish spaces X and Y respectively, and F is generically $\simeq_{\mathcal{C}}$-ergodic for every class $\mathcal{C}$ of countable structures. If there is a Baire measurable function $f : Y \to X$ such that*

- *$f(x) E f(y)$ whenever xFy, and*

- *no preimage of an E-class is comeager,*

then the relation E is not classifiable by countable structures.

Proof. Suppose by contradiction that there is a class $\mathcal{C}$ of countable structures and a Borel reduction $g : X \to \mathcal{C}$ of E to $\simeq_{\mathcal{C}}$. The composition $g \circ f : Y \to \mathcal{C}$ is a Baire measurable function from Y to $\mathcal{C}$ such that $(g \circ f)(y) \simeq_{\mathcal{C}} (g \circ f)(y')$ for any $y, y' \in Y$ such that yFy'. Since F is generically $\simeq_{\mathcal{C}}$-ergodic, there is a comeager subset C of Y such that $(g \circ f)(y) \simeq_{\mathcal{C}} (g \circ f)(y')$ for every $y, y' \in C$. Therefore, being g a reduction of E to $\simeq_{\mathcal{C}}$, $f(y) E f(y')$ for every $y, y' \in C$. This contradicts our assumptions. $\square$

Consider $\mathbb{R}^{\mathbb{N}}$ as a Polish space with the product topology and ℓ^1 as a Polish group with its Banach space topology. The fact that the action of ℓ^1 on $\mathbb{R}^{\mathbb{N}}$ by translation is turbulent is a particular case of [59, Proposition 3.25]. It then follows by Hjorth's turbulence theorem that the associated orbit equivalence relation $E_{\mathbb{R}^{\mathbb{N}}}^{\ell^1}$ is generically $\simeq_{\mathcal{C}}$-ergodic for every class $\mathcal{C}$ of countable structures. It is not difficult to see that the function $f : (\mathbb{R} \setminus \{0\})^{\mathbb{N}} \to (0,1)^{\mathbb{N}}$ defined by

$$f(\mathbf{t}) = \left(\frac{|t_n|}{|t_n| + 1} \right)_{n \in \mathbb{N}}$$

satisfies both the first (being continuous, open, and onto) and the second condition of Lemma 2.2.1, where:

- F is the relation $E_{\mathbb{R}^{\mathbb{N}}}^{\ell^1}$ of equivalence modulo ℓ^1 of sequences of real numbers;

- E is the relation $E_{(0,1)^{\mathbb{N}}}^{\ell^1}$ of equivalence modulo ℓ^1 of sequences of real numbers between 0 and 1.

It follows that the latter relation is generically $\simeq_{\mathcal{C}}$ -ergodic for every class $\mathcal{C}$ of countable structures. Considering the particular case of Lemma 2.2.2 when F is the relation $E_{(0,1)^{\mathbb{N}}}^{\ell^1}$ one obtains the following nonclassifiability criterion:

Criterion 2.2.3. If E is an equivalence relation on a Polish space X and there is a Baire measurable function $f : (0,1)^{\mathbb{N}} \to X$ such that:

- $f(\mathbf{x}) E f(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in (0,1)^{\mathbb{N}}$ such that $\mathbf{x} - \mathbf{y} \in \ell^1$;

- the preimage under f of any E-class is meager;

then the relation E is not classifiable by countable structures.

In order to apply Criterion 2.2.3 we will need the following fact about nonmeager subsets of $(0,1)^{\mathbb{N}}$:

Lemma 2.2.4. *If X is a nonmeager subset of $(0,1)^{\mathbb{N}}$, then there is an uncountable $Y \subset X$ such that, for every pair of distinct points $\mathbf{s}, \mathbf{t}$ of Y, $\|\mathbf{s}-\mathbf{t}\|_\infty \geq \frac{1}{4}$, where*

$$\|\mathbf{s}-\mathbf{t}\|_\infty = \sup_{n\in\mathbb{N}} |t_n - s_n| .$$

Proof. Define for every $\mathbf{s} \in (0,1)$,

$$K_{\mathbf{s}} = \left\{ \mathbf{t} \in (0,1)^{\mathbb{N}} \mid \|\mathbf{t}-\mathbf{s}\|_\infty \leq \frac{1}{4} \right\}.$$

Observe that $K_{\mathbf{s}}$ is a closed nowhere dense subset of $(0,1)^{\mathbb{N}}$. Consider the class $\mathcal{A}$ of subsets Y of X with the property that, for every $\mathbf{s}, \mathbf{t}$ in Y distinct, $\|\mathbf{s}-\mathbf{t}\| \geq \frac{1}{4}$. If $\mathcal{A}$ is partially ordered by inclusion, then it has some maximal element Y by Zorn's lemma. By maximality,

$$X \subset \bigcup_{\mathbf{t}\in Y} \left\{ \mathbf{s} \in (0,1)^{\mathbb{N}} \mid \|\mathbf{t}-\mathbf{s}\|_\infty \leq \frac{1}{4} \right\}.$$

Since the set X is nonmeager, Y is uncountable. $\square$

2.3 The case of algebras with outer derivations

The aim of this section is to show that if a C*-algebra A has an outer derivation, then the relation of unitary equivalence of approximately inner automorphisms of A is not classifiable by countable structures. In proving this fact we will also show that any such C*-algebra satisfies a seemingly stronger property, that we will refer to as Property AEP (see Definition 2.3.4).

A *derivation* of a C*-algebra A is a linear function

$$\delta : A \to A$$

satisfying the *derivation identity*:

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for $x, y \in A$. The derivation identity implies that δ is a bounded linear operator on A; see [109, Proposition 8.6.3]. The set $\Delta(A)$ of derivations of A is a closed subspace of the Banach space $B(A)$ of bounded linear operators on A. A derivation is called a *-derivation if it is a positive linear operator, i.e. it sends positive elements to positive elements. Any element a of the multiplier algebra of A defines a derivation $\mathrm{ad}(ia)$ of A, by

$$\mathrm{ad}(ia)(x) = [ia, x] \, .$$

This is a *-derivation if and only if a is self-adjoint. A derivation of this form is called *inner*, and *outer* otherwise. More generally, if a is an element of the enveloping von Neumann algebra of A that *derives* A, i.e. $ax - xa \in A$ for any $x \in A$, then one can define the (not necessarily inner) derivation $\mathrm{ad}(ia)$ of A. Since any derivation is linear combination of *-derivations (see [109, 8.6.2]), the existence of an outer derivation is equivalent to the existence of an outer *-derivation. The set $\Delta_0(A)$ of inner derivations of A is a Borel (not necessarily closed) subspace of $\Delta(A)$. The norm on $\Delta_0(A)$ defined by

$$\|\mathrm{ad}(ia)\|_{\Delta_0(A)} = \inf \left\{ \|a - z\| \mid z \in \mathcal{Z}(A) \right\} ,$$

where $\mathcal{Z}(A)$ denotes the *center* of A, makes $\Delta_0(A)$ a separable Banach space isometrically isomorphic to the quotient of A by $\mathcal{Z}(A)$. The inclusion of $\Delta_0(A)$ in $\Delta(A)$ is continuous, and the closure $\overline{\Delta_0(A)}$ of $\Delta_0(A)$ in $\Delta(A)$ is a closed separable subspace of $\Delta(A)$. If δ is a *-derivation then the exponential $\exp(\delta)$ of δ, regarded as an element of the Banach algebra $B(A)$ of bounded linear operators of A, is an automorphism of A. Automorphisms of this form are called derivable. If $\delta = \mathrm{ad}(ia)$ is inner then

$$\exp(\delta) = \mathrm{Ad}(\exp(ia))$$

is inner as well. Lemma 2.3.1 provides a partial converse to this statement. (The converse is in fact false in general; see [67, Example 6.1].) For more information on derivations and derivable automorphisms, the reader is referred to [109, Section 8.6].

Lemma 2.3.1. *Suppose that A is a primitive C*-algebra. If δ is a *-derivation of A with operator norm strictly smaller than 2π such that $\exp(\delta)$ is inner, then δ is inner.*

The lemma is proved in [67, Theorem 4.6 and Remark 4.7] under the additional assumption that A is unital. It is not difficult to check that the same proof works without change in the nonunital case.

Definition 2.3.2. Suppose that A is a C*-algebra, $(a_n)_{n\in\mathbb{N}}$ is a dense sequence in the unit ball of A , and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal positive contractions of A such that for every $n \in \mathbb{N}$ and $i \leq n$,

$$\|[x_n, a_i]\| \leq 2^{-n}.$$

Since the x_n's are pairwise orthogonal, if $\mathbf{t}$ is a sequence of real numbers of absolute value at most 1, then the series

$$\sum_{n\in\mathbb{N}} t_n x_n$$

converges in the strong operator topology to a self-adjoint element of A^{**}. Moreover, the sequence of inner automorphisms

$$\left(\mathrm{Ad}\left(\exp\left(i\sum_{k\leq n} t_k x_k\right)\right)\right)_{n\in\mathbb{N}}$$

of A converges –in view of (2.3.2)– to the approximately inner automorphism

$$\alpha_{\mathbf{t}} := \mathrm{Ad}\left(\exp\left(i\sum_{n\in\mathbb{N}} t_n x_n\right)\right).$$

The equivalence relation $E_{\mathbf{x}}$ on $(0,1)^{\mathbb{N}}$ is defined by

$$\mathbf{s}E_{\mathbf{x}}\mathbf{t} \quad \text{iff} \quad \alpha_{\mathbf{t}} \text{ and } \alpha_{\mathbf{s}} \text{ are unitarily equivalent.}$$

This equivalence relation is finer than the relation of ℓ^1-equivalence introduced in Section 2.2. In fact if $\mathbf{s}, \mathbf{t} \in (0,1)^{\mathbb{N}}$ and $\mathbf{s} - \mathbf{t} \in \ell^1$, then the series

$$\sum_{n\in\mathbb{N}} (t_n - s_n) x_n$$

converges in A. It is then easily verified that

$$u := \exp\left(i\sum_{n\in\mathbb{N}} (t_n - s_n)\, x_n\right)$$

is a unitary multiplier of A such that

$$\mathrm{Ad}(u) \circ \alpha_{\mathbf{s}} = \alpha_{\mathbf{t}}.$$

Therefore, if the equivalence classes of $E_{\mathbf{x}}$ are meager, the continuous function

$$\begin{aligned} (0,1)^{\mathbb{N}} &\to \mathrm{Aut}(A) \\ \mathbf{t} &\mapsto \alpha_{\mathbf{t}} \end{aligned}$$

satisfies the hypothesis of Criterion 2.2.3. This concludes the proof of the following lemma:

Lemma 2.3.3. *Suppose that A is a C^*-algebra. If for some sequence $\mathbf{x}$ of pairwise orthogonal positive contractions of A satisfying the commutation condition* (2.3.2) *the equivalence relation $E_{\mathbf{x}}$ has meager equivalence classes, then the approximately inner automorphisms of A are not classifiable by countable structures.*

Lemma 2.3.3 motivates the following definition.

Definition 2.3.4. A C*-algebra A has Property AEP if for every dense sequence $(a_n)_{n\in\mathbb{N}}$ in the unit ball of A there is a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of pairwise orthogonal positive contractions of A such that:

1. $\|[x_n, a_i]\| < 2^{-n}$ for $i \in \{1, 2, \ldots, n\}$;
2. the relation $E_{\mathbf{x}}$ as in Definition 2.3.2 has meager conjugacy classes.

It is clear that if a C*-algebra A has Property AEP, then A has an outer *-derivation. In fact, if $\mathbf{s}, \mathbf{t} \in (0,1)^{\mathbb{N}}$ are such that $\mathbf{s} \not\!\!E_{\mathbf{x}}\, \mathbf{t}$, then the self-adjoint element

$$a = \sum_{n\in\mathbb{N}} (t_n - s_n)\, x_n$$

of A^{**} derives A. The automorphism $\mathrm{Ad}(\exp(ia))$ is outer, and hence such is the *-derivation $\mathrm{ad}(ia)$. The rest of this section is devoted to prove that, conversely, if A has an outer derivation, then A has Property AEP.

Theorem 2.3.5. *If A is a C^*-algebra, the following statements are equivalent:*

1. *A has an outer derivation;*

2. A has Property AEP.

The following lemma shows that primitive nonsimple C*-algebras have Property AEP. The main ingredients of the proof are borrowed from [28, Lemma 6] and [2, Lemma 3.2].

Lemma 2.3.6. *If A is a primitive nonsimple infinite-dimensional C*-algebra, then it has Property AEP.*

Proof. Fix a faithful irreducible representation $\pi : A \to B(H)$. By [109, Theorem 3.7.7] π extends to a σ-weakly continuous representation $\pi^{**} : A^{**} \to B(H)$. Fix a dense sequence $(a_n)_{n\in\mathbb{N}}$ in the unit ball of A and a strictly positive contraction b_0 of A. (Recall that a positive contraction b_0 of A is strictly positive if

$$ab_0^{\frac{1}{n}} \to a$$

for every $a \in A$ [11, II.4.2.1].) As in the proof of [2, Lemma 3.2], one can define a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of pairwise orthogonal projections such that for some $\varepsilon > 0$ and every $k, n \in \mathbb{N}$ such that $k \leq n$,

- $\|x_n b_0\| > \varepsilon$;
- $\|[x_n, a_k]\| < 2^{-n}$.

Now suppose by contradiction that the equivalence relation $E_{\mathbf{x}}$ has a nonmeager equivalence class X. Thus for every $\mathbf{t}, \mathbf{s} \in X$ the automorphism

$$\alpha_{\mathbf{t},\mathbf{s}} = \mathrm{Ad}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right)\right)$$

is inner. Fix $\mathbf{s}, \mathbf{t} \in X$. Observe that $\alpha_{\mathbf{t},\mathbf{s}}$ is the exponential of the *-derivation

$$\delta_{\mathbf{t},\mathbf{s}} = \mathrm{ad}\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right).$$

By Lemma 2.3.1 the *-derivation $\delta_{\mathbf{t},\mathbf{s}}$ is inner. Thus, there is an element $z_{\mathbf{t},\mathbf{s}}$ of the center of the enveloping von Neumann algebra of A such that

$$\sum_{n\in\mathbb{N}}(t_n - s_n)x_n + z_{\mathbf{t},\mathbf{s}} \in M(A).$$

Recall that π has been extended to a σ-weakly continuous representation $\pi^{**}: A^{**} \to B(H)$ by [109, Theorem 3.7.7]. The image of a central element of A^{**} under π^{**} belongs to the relative commutant of $\pi[A]$ in $B(H)$, which consists only of scalar multiples of the identity by [11, II.6.1.8]. Thus,

$$\pi\left(\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right) \in \pi^{**}\left[M(A)\right]$$

Hence

$$\pi\left(b_0\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right) \in \pi[A].$$

By Lemma 2.2.4 one can find an uncountable subset Y of X such that any pair of distinct elements of Y has uniform distance at least $\frac{1}{4}$. Fix $\mathbf{s} \in Y$. For all $\mathbf{t}, \mathbf{t}' \in Y$, there is $m \in \mathbb{N}$ such that

$$\left|t_m - t'_m\right| \geq \frac{1}{4}.$$

Henceforth,

$$\begin{aligned}
&\left\|\pi\left(b_0\left(\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right)\right) - \pi\left(b_0\left(\sum_{n\in\mathbb{N}}(t'_n - s_n)x_n\right)\right)\right\| \\
= \quad & \left\|\pi\left(b_0\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n\right)\right\| \\
= \quad & \left\|b_0\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n\right\| \\
\geq \quad & \left\|b_0\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n x_m a_0\right\| \\
\geq \quad & \left|t_m - t'_m\right| \|(x_m b_0)^*(x_m b_0)\| \geq \frac{\varepsilon^2}{4}.
\end{aligned}$$

Since Y is uncountable this contradicts the separability of $\pi\,[A]$. $\square$

In order to prove Property AEP for all C*-algebra with outer *-derivations we need the fact that Property AEP is *liftable*. This means that if a C*-algebra A has a quotient with property AEP, then A has property AEP. (For an exhaustive introduction to liftable properties the reader is referred to Chapter 8 of [88].)

Lemma 2.3.7. *If $\pi : A \to B$ is a surjective *-homomorphism and B has Property AEP, then A has Property AEP.*

Proof. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a dense sequence in A. Thus, $(\pi(a_n))_{n\in\mathbb{N}}$ is a dense sequence in B. Pick a sequence $(y_n)_{n\in\mathbb{N}}$ in B obtained from $(\pi(a_n))_{n\in\mathbb{N}}$ as in the definition of Property AEP. By [88, Lemma 10.1.12], there is a sequence $(z_n)_{n\in\mathbb{N}}$ of pairwise orthogonal positive contractions of A such that $\pi(z_n) = y_n$ for every $n \in \mathbb{N}$. Fix an increasing quasicentral approximate unit of $\mathrm{Ker}(\pi)$ (cf. [11, Section II.4.3]), i.e. a sequence $(u_k)_{k\in\mathbb{N}}$ of elements of $\mathrm{Ker}(\pi)$ such that:

- $\lim_{k\to+\infty} \|u_k x - x\| = \lim_{k\to+\infty} \|x u_k - x\| = 0$ for every $x \in \mathrm{Ker}(\pi)$;
- $\lim_{k\to+\infty} \|[u_k, a]\| = 0$ for every $a \in A$.

For every $n, i \in \mathbb{N}$ such that $i \le n$, by [11, Proposition II.5.1.1],

$$\begin{aligned} \lim_{k\to+\infty} \left\| z_n^{\frac{1}{2}}(1-u_k)z_n^{\frac{1}{2}} a_i - a_i z_n^{\frac{1}{2}}(1-u_k) z_n^{\frac{1}{2}} \right\| &= \lim_{k\to+\infty} \|(1-u_k)(z_n a_i - a_i z_n)\| \\ &= \|y_n\, \pi(a_i) - \pi(a_i)\, y_n\| < 2^{-n}. \end{aligned}$$

Thus, there is $k_n \in \mathbb{N}$ such that, if

$$x_n = z_n^{\frac{1}{2}}(1-u_{k_n}) z_n^{\frac{1}{2}},$$

then

$$\|x_n a_i - a_i x_n\| < 2^{-n}$$

for every $i \le n$. Observe that $(x_n)_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal positive contractions of A. Moreover, if $E \subset (0,1)^{\mathbb{N}}$ is nonmeager, consider $\mathbf{s}, \mathbf{t} \in E$ such that the automorphism

$$\mathrm{Ad}\left(\exp\left(i \sum_{n\in\mathbb{N}} (t_n - s_n) y_n\right)\right)$$

of B is outer. We claim that the automorphism

$$\mathrm{Ad}\left(\exp\left(i \sum_{n\in\mathbb{N}} (t_n - s_n) x_n\right)\right)$$

of A is outer. Suppose that this is not the case. Thus, there is z in the center of the enveloping von Neumann algebra of A such that

$$\exp\left(i \sum_{n\in\mathbb{N}} (t_n - s_n) x_n\right) + z \in U(A).$$

Denoting by $\pi^{**}: A^{**} \to B^{**}$ the normal extension of π –see [11, III.5.2.10]– one has that

$$\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)y_n\right) + \pi^{**}(z) = \pi^{**}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n\right) + z\right) \in U(B)$$

by Theorem 4.2 of [1]. Since $\pi^{**}(z)$ belongs to the center of the enveloping von Neumann algebra of B,

$$\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)y_n\right) + \pi^{**}(z)$$

is a unitary multiplier of B that implements

$$\mathrm{Ad}\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)y_n\right)\right).$$

Hence, the latter automorphism of B is inner, contradicting the assumption. $\square$

Liftability of Property AEP allows one to easily bootstrap Property AEP from primitive nonsimple C*-algebras to C*-algebra whose primitive spectrum is not T_1.

Lemma 2.3.8. *If A is a C*-algebra whose primitive spectrum $\check{A}$ is not T_1, then A has Property AEP.*

Proof. Since $\check{A}$ is not T_1, by [109, 4.1.4] there is an irreducible representation π of A whose kernel is not a maximal ideal. This implies that the image of A under π is a nonsimple primitive C*-algebra. By Lemma 2.3.6 the latter C*-algebra has Property AEP. Therefore, being Property AEP liftable by Lemma 2.3.7, A has Property AEP. $\square$

In order to show that a C*-algebra A has Property AEP, it is sometimes easier to show that it has a stronger property that we will refer to as Property AEP$^+$. Property AEP$^+$ appears, without being explicitly defined, in the proofs of Lemma 17, Theorem 18, and the main Theorem of [28], as well as in the proofs of Lemma 3.5 and Lemma 3.6 of [2].

Recall that a bounded sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A is called central if for every $a \in A$,

$$\lim_{n\to+\infty} \|[x_n, a]\| = 0.$$

The beginning of Section 2.4 contains a discussion about the notion of central sequence, the related notion of hypercentral sequence, and their basic properties.

Definition 2.3.9. A C*-algebra A has Property AEP$^+$ if there is a sequence $(\pi_n)_{n\in\mathbb{N}}$ of irreducible representations of A such that, for some positive contraction b_0 of A and a central sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise orthogonal positive contractions of A:

- the sequence
$$(\pi_n((x_n-\lambda)b_0))_{n\in\mathbb{N}}$$
does not converge to 0 for any $\lambda\in\mathbb{C}$;
- $x_m\in\mathrm{Ker}(\pi_n)$ for every pair of distinct natural numbers n,m.

To prove that Property AEP$^+$ implies Property AEP we will need the following lemma:

Lemma 2.3.10. *Fix a strictly positive real number η. For every $\varepsilon>0$ there is $\delta>0$ such that for every C*-algebra A and every pair of positive contractions x,b of A such that $\|b\|\geq\eta$, if*
$$\|(\exp(ix)-\mu)b\|\leq\delta$$
for some $\mu\in\mathbb{C}$ then
$$\|(x-\lambda)b\|\leq\varepsilon$$
for some $\lambda\in\mathbb{C}$.

Proof. Fix $\varepsilon>0$. Let L be the principal branch of the logarithm. Since L is an analytic function on the open disc of radius 1 centered in 1, there is a polynomial

$$p(Z)=\rho_0+\rho_1Z+\ldots+\rho_nZ^n$$

such that
$$|p\left(z\right)-L\left(z\right)|\leq\frac{\varepsilon}{2}$$
for every $z\in\mathbb{C}$ such that $|z-1|\leq\exp\left(i\right)$. In particular for every $t\in[0,1]$

$$|p(\exp(it))-t|=|p\left(\exp\left(it\right)\right)-L\left(\exp\left(it\right)\right)|\leq\frac{\varepsilon}{2}.$$

If $\mu\in\mathbb{C}$ is such that $|\mu|\leq\frac{2}{\eta}$, define $p_\mu(Z)$ to be the polynomial in Z obtained from $p(Z)$ by replacing the indeterminate Z by $Z+\mu$. Observe that the j -th coefficient of $p_\mu(Z)$ is

$$\rho_j^\mu=\sum_{i=j}^n\rho_i\binom{i}{j}\mu^{j-i}$$

for $0 \leq j \leq n$. Finally define

$$C = \sum_{1 \leq j \leq i \leq n} |\rho_i| \binom{i}{j} \left(\frac{3}{\eta}\right)^{j-1} \left(\frac{2}{\eta}\right)^{j-i}$$

and

$$\delta = \min \left\{ \frac{\varepsilon}{2C}, 1 \right\}.$$

Suppose that A is a C*-algebra and $x, b \in A$ are positive contractions such that $\|b\| \geq \eta$ and, for some $\mu \in \mathbb{C}$,

$$\|(\exp(ix) - \mu)b\| \leq \delta.$$

Thus,

$$|\mu| \leq \frac{2}{\eta}.$$

Moreover

$$\begin{aligned}
\|(x - \rho_0^\mu)b\| &= \|(p(\exp(ix)) - \rho_0^\mu)b\| + \frac{\varepsilon}{2} \\
&= \left\| \left(\sum_{j=1}^{n} \rho_j^\mu (\exp(ix) - \mu)^j \right) b \right\| + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^{n} \left| \rho_j^\mu \right| \|\exp(ix) - \mu\|^{j-1} \delta + \frac{\varepsilon}{2} \\
&\leq \sum_{j=1}^{n} \sum_{i=j}^{n} |\rho_i| \binom{i}{j} \left(\frac{2}{\eta}\right)^{j-i} \left(\frac{3}{\eta}\right)^{j-1} \delta + \frac{\varepsilon}{2} \\
&\leq C\delta + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

This concludes the proof. ∎

We can now prove that Property AEP$^+$ implies property AEP.

Proposition 2.3.11. *If a C*-algebra A has Property AEP$^+$, then it has property AEP.*

Proof. Suppose that $(\pi_n)_{n \in \mathbb{N}}$ is a sequence of irreducible representations of A, b_0 is a positive contraction of A of norm 1, and $(x_n)_{n \in \mathbb{N}}$ is a sequence of orthogonal positive elements of A as in the definition of Property AEP$^+$. Fix a dense sequence $(a_n)_{n \in \mathbb{N}}$ in the unit ball of A. After passing to a subsequence of the sequence $(x_n)_{n \in \mathbb{N}}$, we can assume that for some

$\varepsilon > 0$, for every $\mu \in \mathbb{C}$ and every $n \in \mathbb{N}$,

$$\|\pi_n((x_n - \mu)b_0)\| \geq \varepsilon$$

and

$$\|[x_n, a_i]\| < 2^{-n}$$

for $i \leq n$. Thus, for every $\mu \in \mathbb{C}$, $n \in \mathbb{N}$, and $t \in \left(\frac{1}{4}, 1\right)$,

$$\|\pi_n((tx_n - \mu)b_0)\| \geq \frac{\varepsilon}{4}.$$

Observe that, in particular,

$$\|\pi_n(b_0)\| \geq \varepsilon$$

for every $n \in \mathbb{N}$. Consider $\delta > 0$ obtained from $\frac{\varepsilon}{8}$ as in Lemma 2.3.10 (where we set $\eta = \varepsilon$). We claim that for every $t \in \left(\frac{1}{4}, 1\right)$, $n \in \mathbb{N}$, and $\mu \in \mathbb{C}$,

$$\|\pi_n((\exp(itx_n) - \mu)b_0)\| \geq \delta.$$

In fact suppose by contradiction that there are $t \in \left(\frac{1}{4}, 1\right)$, $n \in \mathbb{N}$, and $\mu \in \mathbb{C}$ such that

$$\|(\exp(it\pi_n(x_n)) - \mu)\,\pi_n(b_0)\| = \|\pi_n((\exp(itx_n) - \mu)b_0)\| < \delta.$$

Thus by our choice of δ there is $\mu \in \mathbb{C}$ such that

$$\|\pi_n((itx_n - \mu)\,b_0)\| = \|(it\pi_n(x_n) - \mu)\pi_n(b_0)\| \leq \frac{\varepsilon}{8}.$$

Such inequality contradicts 2.3. This concludes the proof of the assertion that for every $t \in \left(\frac{1}{4}, 1\right)$, $n \in \mathbb{N}$, and $\mu \in \mathbb{C}$,

$$\|\pi_n((\exp(itx_n) - \mu)b_0)\| \geq \delta.$$

We now claim that the sequence $(x_n)_{n \in \mathbb{N}}$ witnesses the fact that A has property AEP. Assume by contradiction that there is a nonmeager subset X of $(0,1)^{\mathbb{N}}$ such that for every $\mathbf{s}, \mathbf{t} \in X$, the automorphism

$$\mathrm{Ad}\left(\exp\left(i\sum_{n \in \mathbb{N}}(t_n - s_n)x_n\right)\right)$$

of A is inner. If $\mathbf{s}, \mathbf{t} \in X$, then there is an element $z_{\mathbf{t},\mathbf{s}}$ in the center of the enveloping von Neumann algebra of A such that

$$\exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n + z_{\mathbf{t},\mathbf{s}}\right)$$

multiplies A. Hence

$$y_{\mathbf{t},\mathbf{s}} = \exp\left(i\sum_{n\in\mathbb{N}}(t_n - s_n)x_n + z_{\mathbf{t},\mathbf{s}}\right) b_0$$

is an element of A. By Lemma 2.2.4, one can find an uncountable subset Y of X such that, for any $\mathbf{t}, \mathbf{s} \in Y$, there is $m \in \mathbb{N}$ such that

$$|t_m - s_m| \geq \frac{1}{4}.$$

Fix $\mathbf{s} \in Y$ and observe that, for $\mathbf{t}, \mathbf{t'} \in Y$,

$$\pi_{n_0}(\exp(z_{\mathbf{t'},\mathbf{s}} - z_{\mathbf{t},\mathbf{s}})) = \mu 1$$

is a scalar multiple of the identity. Therefore

$$\begin{aligned}
& \left\|y_{\mathbf{t},\mathbf{s}} - y_{\mathbf{t'},\mathbf{s}}\right\| \\
= & \left\|\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n\right) - \exp(z_{\mathbf{t'},\mathbf{s}} - z_{\mathbf{t},\mathbf{s}})\right) a_0\right\| \\
\geq & \left\|\pi_{n_0}\left(\left(\exp\left(i\sum_{n\in\mathbb{N}}(t_n - t'_n)x_n\right) - \exp(z_{\mathbf{t'},\mathbf{s}} - z_{\mathbf{t},\mathbf{s}})\right) a_0\right)\right\| \\
= & \left\|\pi_{n_0}((\exp((t_{n_0} - t'_{n_0})x_n) - \mu)a_0)\right\| \\
\geq & \ \varepsilon.
\end{aligned}$$

This contradicts the separability of A. $\square$

The proofs of Lemma 2.3.12 and Lemma 2.3.13 are contained, respectively, in the proofs of Lemmas 3.6 and 3.7 of [2] and in the proof of the implication $(i) \Rightarrow (ii)$ at page 139 of [28].

Recall that a point x of a topological space X is called *separated* if, given any point y of X distinct from x, there are disjoint open neighborhoods of x and y.

Lemma 2.3.12. *Suppose that A is a C*-algebra whose primitive spectrum $\check{A}$ is T_1. Consider* *a sequence $(\xi_n)_{n\in\mathbb{N}}$ of separated points in $\check{A}$. Define F to be the set of limit points of the sequence $(\xi_n)_{n\in\mathbb{N}}$ and I to be the closed self-adjoint ideal of A corresponding to F. If either the quotient A/I does not have continuous trace, or the multiplier algebra of A/I has nontrivial center, then A has Property AEP$^+$.*

Lemma 2.3.13. *If A is a C*-algebra whose spectrum $\hat{A}$ is homeomorphic to the one-point compactification of a countable discrete space, then A has Property AEP$^+$.*

We can now prove the main result of this section that Property AEP as defined in 2.3.4 is equivalent to having an outer *-derivation.

Proof of Theorem 2.3.5. We have already pointed out that Property AEP implies the existence of an outer *-derivation. It remains only to show the converse. Suppose that A has an outer derivation. By [28, Lemma 16], either there is a quotient B of A whose spectrum $\hat{B}$ is homeomorphic to the one point compactification of a countable discrete space, or the primitive spectrum $\check{A}$ of A is not Hausdorff. In the first case, A has Property AEP by virtue of Lemma 2.3.13 and Lemma 2.3.7. Suppose that, instead, the primitive spectrum $\check{A}$ of A is not Hausdorff. If $\check{A}$ is not even T_1, the conclusion follows from Lemma 2.3.8. Suppose now that $\check{A}$ is T_1. Since $\check{A}$ is not Hausdorff, there are two points ρ_0, ρ_1 of $\check{A}$ that do not admit any disjoint open neighbourhoods. By [20, Proposition 1] the set of separated points is dense in $\check{A}$. Therefore can find a sequence $(\xi_n)_{n\in\mathbb{N}}$ of separated points of $\check{A}$ whose set F of limit points contains both ρ_0 and ρ_1. Define I to be the closed self-adjoint ideal I of A corresponding to the closed subset F of $\check{A}$. Since F contains at least two points, A/I is nonsimple. Consider now two cases: If A/I has continuous trace then by [2, Theorem 3.9] and [2, Lemma 3.1], the multiplier algebra of A/I has nontrivial center. Therefore A has Property AEP$^+$ by Lemma 2.3.12. On the other hand if A/I does not have continuous trace, then again A has Property AEP$^+$ by Lemma 2.3.12. In either case, it follows that A has Property AEP$^+$ and, in particular, Property AEP. □

2.4 The case of algebras with only inner derivations

In this section we will prove that, if a C*-algebra A with only inner derivations does not have continuous trace, then the relation of unitary equivalence of approximately inner automorphisms of A is not classifiable by countable structures. In proving this fact we will also show that any such C*-algebra contains a central sequence that is not strict-hypercentral.

If A is a C*-algebra, denote by A^∞ the quotient of the direct product $\prod_{n\in\mathbb{N}} A$ by the direct sum $\bigoplus_{n\in\mathbb{N}} A$; see [11, II.8.1.2]. Identifying as it is customary A with the algebra of elements of A^{∞} admitting constant representative sequence, denote by A_{∞} the relative commutant

$$A' \cap A^{\infty} = \{x \in A^{\infty} \mid \forall y \in A \ , \ [x, y] = 0\} \, .$$

Finally define

$$\mathrm{Ann}(A, A_{\infty}) = \{x \in A_{\infty} \mid \forall y \in A \ , \ xy = 0\}$$

to be the annihilator ideal of A in A_{∞}. Observe that, if A is unital, then $\mathrm{Ann}(A, A_{\infty})$ is the trivial ideal.

A **central sequence** in a C*-algebra A is a bounded sequence $(x_n)_{n \in \mathbb{N}}$ of elements of A that asymptotically commute with any element of A. This means that for any $a \in A$,

$$\lim_{n \to +\infty} \|[x_n, a]\| = 0.$$

Equivalently the image of $(x_n)_{n \in \mathbb{N}}$ in the quotient of $\prod_{n \in \mathbb{N}} A$ by $\bigoplus_{n \in \mathbb{N}} A$ belongs to A_{∞}. From the last characterization it is clear that if $(x_n)_{n \in \mathbb{N}}$ is a central sequence of normal elements A with spectra contained in some subset D of $\mathbb{C}$ and $f : D \to \mathbb{C}$ is a continuous function such that $f(0) = 0$, then the sequence $(f(x_n))_{n \in \mathbb{N}}$ is central. It is not difficult to verify that, if $(x_n)_{n \in \mathbb{N}}$ is a central sequence and $b \in M(A)$, then the sequence $([b, x_n])_{n \in \mathbb{N}}$ converges strictly to 0. (The strict topology on $M(A)$ has been defined in Section 2.1.)

Let us call a central sequence $(x_n)_{n \in \mathbb{N}}$ *norm-hypercentral* if it asymptotically commutes in the norm topology with any other central sequence. This amounts to say that for any other central sequence $(y_n)_{n \in \mathbb{N}}$

$$\lim_{n \to +\infty} \|[x_n, y_n]\| = 0.$$

Equivalently the image of $(x_n)_{n \in \mathbb{N}}$ in the quotient of $\prod_{n \in \mathbb{N}} A$ by $\bigoplus_{n \in \mathbb{N}} A$ belongs to the center of A_{∞}. For our purposes it will be more convenient to look at central sequences that asymptotically commute *in the strict topology* with any other central sequence. This motivates the following definition:

Definition 2.4.1. Suppose that A is a C*-algebra. A sequence $(x_n)_{n \in \mathbb{N}}$ of elements of A is **strict-hypercentral** if it is central and, for any other central sequence $(y_n)_{n \in \mathbb{N}}$, the sequence

$$([x_n, y_n])_{n \in \mathbb{N}}$$

converges to 0 in the strict topology.

Observe that a central sequence $(x_n)_{n \in \mathbb{N}}$ is strict-hypercentral if and only if the image of the element of A_∞ having $(x_n)_{n\in\mathbb{N}}$ as representative sequence in the quotient $A_\infty/\mathrm{Ann}(A, A_\infty)$ belongs to the center of $A_\infty/\mathrm{Ann}(A, A_\infty)$. It is clear from this characterization that, if $(x_n)_{n\in\mathbb{N}}$ is a strict-hypercentral sequence of normal elements of A with spectra contained in some subset D of $\mathbb{C}$ and $f : D \to \mathbb{C}$ is a complex-valued continuous function such that $f(0) = 0$, then the sequence $(f(x_n))_{n\in\mathbb{N}}$ is strict-hypercentral. When A is unital the ideal $\mathrm{Ann}(A, A_\infty)$ is trivial, and hence the notions of strict-hypercentral and norm-hypercentral sequence coincide. Therefore in the unital case a norm-hypercentral sequence will be simply called hypercentral.

The fact that a unital simple infinite-dimensional C*-algebra contains a central sequence that is not hypercentral is a particular case of [112, Theorem 3.6]. We will show here how one can generalize this fact to all simple nonelementary C*-algebras. The proof deeply relies on ideas from [112].

Lemma 2.4.2. *If $(x_n)_{n\in\mathbb{N}}$ is a strict-hypercentral sequence in A and α is an approximately inner automorphism of A, then $(\alpha(x_n) - x_n)_{n\in\mathbb{N}}$ converges strictly to 0.*

Proof. The same proof of Kaplansky's density theorem [109, Theorem 2.3.3] shows that the unit ball of A is strictly dense in the unit ball of $M(A)$; see [85, Proposition 1.4]. (The strict topology on the multiplier algebra of A has been defined in Section 2.1.) It follows that, if $\varepsilon > 0$ and a is an element of A, then there is a finite subset F of the unit ball of A, a positive real number δ, and a natural number n_0 such that, for every $n \geq n_0$ and every y in the unit ball $M(A)$ such that $\|[y, z]\| < \delta$ for every $z \in F$,

$$\max\{\|a(x_n y - y x_n)\|, \|(x_n y - y x_n)a\|\} \leq \varepsilon.$$

Consider the open neighbourhood

$$U = \{\alpha \in \mathrm{Aut}(A) \mid \|\alpha(x) - x\| < \delta \text{ for every } x \in F\}$$

of id_A in $\mathrm{Aut}(A)$. Observe that if $\beta \in U$ is inner, then for every $n \geq n_0$

$$\|(\beta(x_n) - x_n)a\| \leq \varepsilon$$

and

$$\|a(\beta(x_n) - x_n)\| \leq \varepsilon.$$

Approximating with inner automorphisms, one can see that the same is true if $\beta \in U$ is just approximately inner. Since α is approximately inner, there is a unitary multiplier u of

A and an approximately inner automorphism β of A in U such that

$$\alpha = \beta \circ \mathrm{Ad}(u).$$

Consider a natural number $n_1 \geq n_0$ such that, for $n \geq n_1$,

$$\left\| \beta^{-1}(a)[x_n, u] \right\| \leq \varepsilon$$

and

$$\left\| [x_n, u^*]\beta^{-1}(a) \right\| \leq \varepsilon$$

It follows that, if $n \geq n_1$,

$$\begin{aligned} \|a(\alpha(x_n) - x_n)\| &\leq \|a\beta(\mathrm{Ad}(u)\,(x_n) - x_n)\| + \|\beta(x_n) - x_n\| \\ &\leq \left\| \beta^{-1}(a)(ux_nu^* - x_n) \right\| + \varepsilon \\ &= \left\| \beta^{-1}(a)[x_n, u] \right\| + \varepsilon \\ &\leq 2\varepsilon \end{aligned}$$

and, analogously,

$$\|(\alpha(x_n) - x_n)a\| \leq 2\varepsilon.$$

Since ε was arbitrary, this concludes the proof of the fact that

$$(a(x_n) - x_n)_{n \in \mathbb{N}}$$

converges strictly to 0. $\square$

If α is an automorphism of a C*-algebra A, then α^{**} denotes the unique extension of α to a σ-weakly continuous automorphism of the enveloping von Neumann algebra A^{**} of A (defined as in [11, Proposition III.5.2.10]).

Lemma 2.4.3. *Suppose that A is a C*-algebra such that every central sequence in A is strict-hypercentral. If α is an approximately inner automorphism of A, then α^{**} fixes pointwise the center of A^{**}, i.e. $\alpha^{**}(z) = z$ for every central element of A^{**}.*

Proof. Observe that z derives A, since

$$za - az = 0 \in A$$

for every $a \in A$. Thus, by Lemma 1.1 of [2], there is a bounded net (z_λ) in A converging strongly to z such that, for every $a \in A$,

$$\lim_{\lambda} \|[z_\lambda - z, a]\| = 0 \ .$$

Recall that strong and σ-strong topology agree on bounded sets, and that the σ-strong topology is stronger than the σ-weak topology; see [11, Definition I.3.1.1]. Thus the net (z_λ) converges *a fortiori* σ-weakly to z. Since the σ-weak topology on A^{**} is the weak* topology on A^{**} regarded as the dual space of A^*, the unit ball of A^{**} is σ-weakly compact by Alaoglu's theorem [110, Theorem 2.5.2]. Moreover by Kaplanski's Density Theorem [109, Theorem 2.3.3] the unit ball of A is σ-weakly dense in the unit ball of A^{**}. As a consequence the unit ball of A^{**} is σ-weakly metrizable, and the same holds for balls of arbitrary radius Thus we can find a bounded *sequence* $(z_n)_{n\in\mathbb{N}}$ in A converging σ-weakly to z such that, for every $a \in A$,

$$\lim_{n\to+\infty} \|[z_n - z, a]\| = 0.$$

Since

$$[z_n - z, a] = [z_n, a]$$

for every $n \in \mathbb{N}$, $(z_n)_{n\in\mathbb{N}}$ is a central and hence strict-hypercentral sequence (every central sequence of A is strict-hypercentral by assumption). Since α^{**} is a σ-weakly continuous automorphism of A^{**} extending α, $(\alpha(z_n))_{n\in\mathbb{N}}$ converges σ-weakly to $\alpha^{**}(z)$. It follows from Lemma 2.4.3 and from the facts that α is approximately inner and the sequence $(z_n)_{n\in\mathbb{N}}$ is strict-hypercentral that the bounded sequence $(z_n - \alpha(z_n))_{n\in\mathbb{N}}$ converges strictly to 0. By [88, Lemma 1.3.1] and since weak and σ-weak topology agree on bounded sets, the sequence $(z_n - \alpha(z_n))_{n\in\mathbb{N}}$ converges σ-weakly to 0. Therefore $z = \alpha^{**}(z)$. $\square$

A C*-algebra is called *elementary* if it is *-isomorphic to the algebra of compact operators on some Hilbert space; see [11, Definition IV.1.2.1]. By Corollary 1 of Theorem 1.4.2 in [5] any elementary C*-algebra is simple. Moreover by Corollary 3 of Theorem 1.4.4 in [5] any automorphism of an elementary C*-algebra is inner; in particular the group $\mathrm{Inn}(A)$ of inner automorphisms of an elementary C*-algebra A is closed inside the group $\mathrm{Aut}(A)$ of all automorphisms. Conversely if the group of inner automorphisms of a simple C*-algebra A is closed, then A is elementary by [111, Theorem 3.1] together with [11, Corollary IV.1.2.6 and Proposition IV.1.4.19].

Recall that in this chapter all C*-algebras (apart from multiplier algebras and enveloping von Neumann algebras) are assumed to be norm separable. In particular separability of A

is assumed in Proposition 2.4.4; however we do not know if the separability assumption is necessary there. (This is also asked in [36, Question 4.3].)

Proposition 2.4.4. *If A is a simple C*-algebra such that every central sequence in A is strict-hypercentral, then A is elementary.*

Proof. It is enough to show that $\mathrm{Inn}(A)$ is closed in $\mathrm{Aut}(A)$ or, equivalently, that no outer automorphism is approximately inner. Fix an outer automorphism α of A. Since A is simple, by [80, Corollary 2.3], there is an irreducible representation π such that π and $\pi \circ \alpha$ are not unitarily equivalent; see Section 2.1. If z is the central cover of π in A^{**} (defined as in [109, 3.8.1]), then $\alpha^{**}(z)$ is the central cover of $\pi \circ \alpha$; moreover, being π and $\pi \circ \alpha$ not equivalent, $\alpha^{**}(z)$ is different from z by [109, Theorem 3.8.2]. Thus α^{**} does not fixes pointwise the center of A^{**} and, by Lemma 2.4.3, α is not approximately inner. $\square$

Proposition 2.4.4 shows that any simple nonelementary C*-algebra contains a central sequence that is not strict-hypercentral. It is clear that the same conclusion holds for any C*-algebra containing a simple nonelementary C*-algebra as a direct summand. By Theorem 3.9 of [2], this class of C*-algebras coincides with the class of C*-algebras that have only inner derivations and do not have continuous trace. This concludes the proof of the following proposition:

Proposition 2.4.5. *If A is a C*-algebra that does not have continuous trace and has only inner derivations, then A contains a central sequence that is not strict-hypercentral.*

In view of this result, in order to finish the proof of Theorem 2.0.17, it is enough to show that its conclusion holds for a C*-algebra A containing a central sequence that is not strict-hypercentral.

Proposition 2.4.6. *If A is a C*-algebra containing a central sequence that is not strict-hypercentral, then the approximately inner automorphisms of A are not classifiable by countable structures up to unitary equivalence.*

Proof. Fix a dense sequence $(a_n)_{n\in\mathbb{N}}$ in the unit ball of A. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a central sequence in A that is not strict-hypercentral. Thus there is a central sequence $(y_n)_{n\in\mathbb{N}}$ in A such that the sequence

$$([x_n, y_n])_{n\in\mathbb{N}}$$

does not converge strictly to 0. This implies that, for some positive contraction b in A, then the sequence

$$(b[x_n, y_n])_{n\in\mathbb{N}}$$

does not converge to 0 is norm. Without loss of generality we can assume that, for every $n \in \mathbb{N}$, x_n and y_n are positive contractions. Observe that the sequence $(\exp(itx_n) - 1)_{n\in\mathbb{N}}$ is not strict-hypercentral for any $t \in (0,1)$. After passing to subsequences, we can assume that for some strictly positive real number ε, for every $t \in (0,1)$, every $s \in \left(\frac{1}{2}, 1\right)$, and every $n, k \in \mathbb{N}$ such that $k \leq n$:

- $\|[a_k, \exp(itx_n)]\| < 2^{-n}$;
- $\|b[x_n, y_n]\| \geq \varepsilon$;
- $\|b[\exp(isx_n), y_n]\| \geq \varepsilon$.

Define $\eta = \frac{\varepsilon}{20}$. After passing to a further subsequence, we can assume that, for every $t \in (0,1)$ and every $n, k \in \mathbb{N}$ such that $k \leq n$:

- $\|[\exp(itx_k), y_n]\| < 2^{-n}\eta$;
- $\|[y_k, \exp(itx_n)]\| < 2^{-n}\eta$;
- $\|[\exp(itx_k), \exp(isx_n)]\| < 2^{-n}\eta$.

It is not difficult to verify that, if $\mathbf{t} \in (0,1)^{\mathbb{N}}$, then the sequence

$$\left(\mathrm{Ad}\left(\prod_{k=1}^{n} \exp(it_k x_k)\right)\right)_{n\in\mathbb{N}}$$

is Cauchy in $\mathrm{Aut}(A)$. Denoting by $f(\mathbf{t})$ its limit, one obtains a function

$$f : (0,1)^{\mathbb{N}} \to \overline{\mathrm{Inn}(A)}.$$

In the rest of the proof we will show that f satisfies the hypothesis of Criterion 2.2.3. Suppose that M is a Lipschitz constant for the function $t \mapsto \exp(it)$ on $[0,1]$. If $\mathbf{t}, \mathbf{s} \in (0,1)^{\mathbb{N}}$ and $n \in \mathbb{N}$ are such that $|t_k - s_k| < \varepsilon$ for $k \in \{1, 2, \ldots, n\}$, then it is easy to see that

$$\|f(\mathbf{t})(a_k) - f(\mathbf{s})(a_k)\| \leq 2^{-n+1} + \varepsilon M$$

for $k \leq n$. This shows that the function f is continuous, particularly, Baire measurable. Moreover, if $\mathbf{t}, \mathbf{s} \in (0,1)^{\mathbb{N}}$ are such that $\mathbf{s} - \mathbf{t} \in \ell^1$, then the sequence

$$(\exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1))_{n\in\mathbb{N}}$$

is Cauchy in $U(A)$, and hence has a limit $u \in U(A)$. It is now readily verified that

$$f(\mathbf{t}) = \mathrm{Ad}(u) \circ f(\mathbf{s})$$

and hence $f(\mathbf{t})$ and $f(\mathbf{s})$ are unitarily equivalent. Finally, suppose that C is a nonmeager subset of $(0,1)^{\mathbb{N}}$. Thus, there are $\mathbf{t}, \mathbf{s} \in C$ such that $|t_n - s_n| \in \left(\frac{1}{2}, 1\right)$ for infinitely many $n \in \mathbb{N}$. We claim that $f(\mathbf{t})$ and $f(\mathbf{s})$ are not unitarily equivalent. Suppose by contradiction that this is not the case. Thus there is $u \in U(A)$ such that

$$f(\mathbf{t}) = \mathrm{Ad}(u) \circ f(\mathbf{s}).$$

This implies that the sequence

$$(u\exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1))_{n\in\mathbb{N}}$$

in $U(A)$ is central, i.e. asymptotically commutes (in norm) with any element of A. Fix now any $n_0 \in \mathbb{N}$ such that $|t_{n_0} - s_{n_0}| \in \left(\frac{1}{2}, 1\right)$ and

$$\|b[y_n, u]\| < \eta$$

for $n \geq n_0$. Suppose that $n > n_0$. Using the fact that the elements $\exp(it_mx_m)$ and $\exp(it_kx_k)$ commute up to $\eta 2^{-m}$ for $k, m \in \mathbb{N}$, one can show that

$$by_{n_0}u\exp(it_1x_1) \cdots \exp(it_nx_n)\ \exp(-is_nx_n) \cdots \exp(-is_1x_1)$$

is at distance at most 5η from

$$\begin{aligned}&buy_{n_0}\exp(i(t_{n_0} - s_{n_0})x_{n_0})\exp(it_1x_1) \cdots \exp(\widehat{it_{n_0}x_{n_0}})\\&\cdots \mathrm{ex}(it_nx_n)\exp(-is_nx_n) \cdots \exp(\widehat{is_{n_0}x_{n_0}}) \cdots \exp(-is_1x_1),\end{aligned}$$

where $\exp(\widehat{it_{n_0}x_{n_0}})$ and $\exp(\widehat{is_{n_0}x_{n_0}})$ indicate omitted terms in the product. Similarly

$$bu\exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1)y_{n_0}$$

is at distance at most 5η from

$$\begin{aligned}&bu\exp\left(i\left(t_{n_0} - s_{n_0}\right)x_{n_0}\right)y_{n_0}\exp\left(it_1x_1\right) \cdots \exp\left(\widehat{it_{n_0}x_{n_0}}\right)\\&\cdots \exp\left(it_nx_n\right)\exp\left(-is_nx_n\right) \cdots \exp\left(\widehat{is_{n_0}x_{n_0}}\right) \ldots \exp\left(-is_1x_1\right).\end{aligned}$$

Thus, the norm of the commutator of

$$u\exp(it_1x_1)\cdots\exp(it_nx_n)\exp(-is_nx_n)\cdots\exp(-is_1x_1)$$

and y_0 is at least

$$\|b[\exp(i(t_{n_0}-s_{n_0})x_{n_0}),y_{n_0}]\|-10\eta\geq\varepsilon-10\eta\geq\frac{\varepsilon}{2}.$$

This contradicts the fact that the sequence

$$(u\exp(it_1x_1)\cdots\exp(it_nx_n)\exp(-is_nx_n)\cdots\exp\left(-is_1x_1\right))_{n\in\mathbb{N}}$$

is central and concludes the proof. □

2.5 A dichotomy for derivations

If A is a C*-algebra, then we denote as in Section 2.3 by $\Delta_0(A)$ the separable Banach space of inner derivations of A endowed with the norm $\|\cdot\|_{\Delta_0(A)}$ and by $\overline{\Delta_0(A)}$ the closure of $\Delta_0(A)$ inside the space $\Delta(A)$ of derivations of A endowed with the operator norm. Suppose that $E_{\Delta(A)}$ is the Borel equivalence relation on $\overline{\Delta_0(A)}$ defined by

$$\delta_0E_{\Delta(A)}\delta_1\quad\text{iff}\quad\delta_0-\delta_1\in\Delta_0(A).$$

Observe that $E_{\Delta(A)}$ is the orbit equivalence relation associated with the continuous action of the additive group of $\Delta_0(A)$ on $\overline{\Delta_0(A)}$ by translation.

Theorem 2.5.1. *If A is a unital C*-algebra, then the following statements are equivalent:*

1. *$\Delta_0(A)$ is closed in $\Delta(A)$;*
2. *$E_{\Delta(A)}$ is smooth;*
3. *$E_{\Delta(A)}$ is classifiable by countable structures;*
4. *A has continuous trace.*

The equivalence of 1 and 4 follows from [67, Theorem 5.3] together with the equivalence of 1 and 3 in Theorem 2.0.18. The implication 1 $\Rightarrow$ 2 follows from [44, Exercise 4.4]. Trivially 2 is stronger than 3. Finally observe that $\Delta_0(A)$ and $\overline{\Delta_0(A)}$ satisfy the hypothesis of [127, Lemma 2.1]. In fact, as discussed at the beginning of Section 2.4, $\Delta_0(A)$ endowed with the norm

$$\|\mathrm{ad}\,(ia)\|_{\Delta(A)} = \inf\left\{\|a - z\| \mid z \in A' \cap A\right\}$$

is a separable Banach space. Moreover the inclusion map of $\Delta_0(A)$ in $\overline{\Delta_0(A)} \subset \Delta(A)$ is bounded with norm at most 2. Thus, if $\Delta_0(A)$ is not closed in $\Delta(A)$, then the continuous action of the additive group $\Delta_0(A)$ on $\overline{\Delta_0(A)}$ by translation is turbulent. Hjorth's turbulence theorem recalled at the beginning of Section 2.2 concludes the proof of the implication $3 \Rightarrow 1$.

Chapter 3

The Gurarij operator space

The *Gurarij space* $\mathbb{G}$ is a Banach space first constructed by Gurarij in [53]. It has the following universal property: whenever $X \subset Y$ are finite-dimensional Banach spaces, $\phi : X \to \mathbb{G}$ is a linear isometry, and $\varepsilon > 0$, there is an injective linear map $\psi : Y \to \mathbb{G}$ extending ϕ such that $||\psi|| \, ||\psi^{-1}|| < 1 + \varepsilon$. The uniqueness of such an object was proved by Lusky [93]. A short proof was later provided by Kubis-Solecki [83]. The Gurarij space was first realized as a Fraïssé limit by Ben Yaacov in [8].

Fraïssé theory is a subject at the border between model theory and combinatorics originating from the seminar work of Fraïssé [42]. Broadly speaking, Fraïssé theory studies homogeneous structures and ways to construct them. In the discrete setting Fraïssé established in [42] a correspondence between countable homogeneous structures and what are now called Fraïssé classes. Let the *age* of a countable structure $\mathbb{S}$ be the collection of finitely generated substructures of $\mathbb{S}$. Any Fraïssé class is the age of a countable homogeneous structure. Conversely from any Fraïssé class one can build a countable homogeneous structure that has the given class as its age. Moreover such a structure is uniquely determined up to isomorphism by this property.

This correspondence has been recently generalized in [8] by Ben Yaacov from the purely discrete setting to the setting where metric structures are considered; see also [128]. The main results of discrete Fraïssé theory are recovered in this more general framework. In particular any Fraïssé class of metric structures is the age of a separable homogeneous structure, which is unique up to isometric isomorphism. An alternative categorical-theoretic approach to Fraïssé limits in the metric setting has been developed by Kubiś [82].

The Gurarij space is the limit of the Fraïssé class of finite-dimensional Banach spaces. This has been showed in [8] building on previous work of Henson. In particular this has yielded an alternative proof of the uniqueness of the Gurarij space up to isometric isomorphism. Other naturally occurring examples of Fraïssé limits are the Urysohn universal metric space [102], the hyperfinite II_1 factor and the infinite type UHF C*-algebras [22].

In this chapter we consider a noncommutative analog of the Gurarij space introduced by Oikhberg in [104] within the framework of operator spaces. Operator spaces can be regarded as noncommutative Banach spaces. In fact Banach spaces can be concretely defined as closed subspaces of $C(K)$ spaces, where K is a compact Hausdorff space. These are precisely the *abelian* unital C*-algebras. Replacing abelian C*-algebras with arbitrary C*-algebras or—equivalently—the algebra $B(H)$ of bounded linear operators on some Hilbert space H provides the notion of an *operator space*.

An operator space $X \subset B(H)$ is endowed with matricial norms on the algebraic tensor product $M_n \otimes X$ obtained by the inclusion $M_n \otimes X \subset M_n \otimes B(H) \cong B(H \oplus \cdots \oplus H)$. A linear operator ϕ between operator spaces is completely bounded with norm at most M if all its amplifications $id_{M_n} \otimes \phi$ are bounded with norm at most M. The notion of complete isometry is defined similarly. Operator spaces form then a category with completely bounded (or completely isometric) linear maps as morphisms. Any Banach space X has a canonical operator space structure induced by the inclusion $X \subset C(\mathrm{Ball}(X^*))$ where $\mathrm{Ball}(X^*)$ is the unit ball of the dual of X. However in this case the matricial norms do not provide any new information, and any linear map ϕ between Banach spaces is automatically completely bounded with same norm. For more general operator spaces it is far from being true that any bounded linear map is completely bounded. The matricial norms play in this case a crucial role.

According to [104] an operator space is *noncommutative Gurarij* if it satisfies the same universal property of the Gurarij Banach space, where finite-dimensional Banach spaces are replaced with arbitrary *finite-dimensional* 1*-exact operator spaces*, and the operator norm is replaced by the *completely bounded norm*. The restriction to 1-exact spaces is natural since a famous result of Junge and Pisier asserts that there is no separable operator space containing all the finite-dimensional operator spaces as subspaces [66, Theorem 2.3]; see also [115, Chapter 21]. Proposition 3.1 of [104] shows that any two noncommutative Gurarij are *approximately* completely isometrically isomorphic. Moreover [104, Theorem 1.1] shows that separable $\mathcal{OL}_{\infty,1+}$ spaces—as defined in [25, 65]—can be completely isometrically embedded in some noncommutative Gurarij space as completely contractively complemented subspaces. In view of [24, Theorem 4.7], Oikhberg's result implies that every separable 1-exact operator space can be completely isometrically embedded in some noncommutative Gurarij space.

The main result of this chapter is that the noncommutative Gurarij space can be realized as the limit of the Fraïssé class of finite-dimensional 1-exact operator spaces. We deduce as a consequence that the noncommutative Gurarij space—which we denote by $\mathbb{NG}$—is unique up to complete isometry, universal among separable 1-exact operator spaces, and moreover satisfies the following homogeneity property: for any finite-dimensional subspace $X \subset \mathbb{NG}$, any complete isometry $\phi : X \to \mathbb{NG}$, and any $\varepsilon > 0$ there is a surjective linear complete isometry $\psi : \mathbb{NG} \to \mathbb{NG}$ such that $\psi_{|X} - \phi$ has completely bounded norm at most ε.

This rest of this chapter is divided into three sections. Section 3.1 contains some background material on Fraïssé theory and operator spaces. We follow the presentation of Fraïssé theory for metric structures as introduced by Ben Yaacov in [8]. Similarly as [102] we adopt the slightly less general point of view—sufficient for our purposes—where one considers only structures where the interpretation of function and relation symbols are Lipschitz with a constant that does not depend on the structure. The material on operator spaces is standard and can be found for example in the monographs [115, 26, 107]. The topic of M_n-spaces is perhaps less well known and can be found in Lehrer's PhD thesis [86] as well as in [105, 104].

In Section 3.2 we show that the class of finite-dimensional M_n-spaces is a Fraïssé class. This can be seen as a first step towards proving that the class of finite-dimensional 1-exact operator spaces is a Fraïssé class. Any M_n-space can be canonically endowed with a compatible operator space structure. Therefore in principle it is possible to rephrase all the arguments and results in terms of operator spaces. Nonetheless we find it more convenient and enlightening to deal with M_n-space. This allows one to recognize and use the analogy with the Banach space case.

Finally Section 3.3 contains the proof of the main result, asserting that the class of finite-dimensional 1-exact operator spaces is a Fraïssé class. Its limit is then identified as the noncommutative Gurarij space.

3.1 Background material

3.1.1 Approximate isometries

Suppose that A, B are complete metric spaces. An *approximate isometry* from A to B is a map $\psi : A \times B \to [0, +\infty]$ satisfying the following:

$$\left|\psi\left(a,b\right)-\psi\left(a',b\right)\right| \leq d\left(a,a'\right) \leq \psi\left(a,b\right)+\psi\left(a',b\right)$$

and

$$\left|\psi\left(a,b\right)-\psi\left(a,b'\right)\right|\leq d\left(b,b'\right)\leq\psi\left(a,b\right)+\psi\left(a,b'\right)$$

for every $a, a' \in A$ and $b, b' \in B$. If ψ is an approximate isometry from A to B, then we write $\psi : A \rightsquigarrow B$. The set of all approximate isometries from A to B is denoted by $\mathrm{Apx}\left(A, B\right)$. This is a compact space endowed with the product topology from $[0, +\infty]^{A\times B}$. A *partial isometry* f from A to B is an isometry from a subset $\mathrm{dom}\left(f\right)$ of A to B.

Remark 3.1.1. Any partial isometry f will be identified with the approximate isometry ψ_f given by the distance function from the graph of f.

Explicitly ψ_f is defined by

$$\psi_f\left(a,b\right)=\inf_{z\in\mathrm{dom}(f)}\left(d\left(a,z\right)+d\left(f(z),b\right)\right).$$

If $\psi : A \rightsquigarrow B$ one can consider its *pseudo-inverse* $\psi^* : B \rightsquigarrow A$ defined by $\psi^*\left(b, a\right) = \psi\left(a, b\right)$. Moreover one can take *composition* of approximate isometries $\psi : A \rightsquigarrow B$ and $\phi : B \rightsquigarrow C$ by setting

$$\left(\phi\psi\right)\left(a,c\right)=\inf_{b\in B}\left(\psi\left(a,b\right)+\phi\left(b,c\right)\right).$$

These definitions are consistent with composition and inversion of partial isometries when regarded as approximate isometries.

If $A_0 \subset A$, $B_0 \subset B$, and $\psi : A \rightsquigarrow B$ then one can define the *restriction* $\psi_{|A_0\times B_0} = j^*\psi i : A_0 \rightsquigarrow B_0$ where i and j are the inclusion maps of A_0 into A and B_0 into B. Conversely if $\phi : A_0 \rightsquigarrow B_0$ then one can consider its *trivial extension* $j\phi i^* : A \rightsquigarrow B$. This allows one to regard $\mathrm{Apx}\left(A_0, B_0\right)$ as a subset of $\mathrm{Apx}\left(A, B\right)$ by identifying an approximate isometry with its trivial extension.

For approximate isometries $\phi, \psi : A \rightsquigarrow B$ we say that ϕ *refines* ψ and ψ *coarsens* ϕ—written $\phi \leq \psi$—if $\phi\left(a, b\right) \leq \psi\left(a, b\right)$ for every $a \in A$ and $b \in B$. The set of approximate isometries that refine ψ is denoted by $\mathrm{Apx}^{\leq\psi}\left(A, B\right)$. The interior of $\mathrm{Apx}^{\leq\psi}\left(A, B\right)$ is denote by $\mathrm{Apx}^{<\psi}\left(A, B\right)$. The *closure under coarsening* $\mathcal{A}^{\uparrow}$ of a set $\mathcal{A} \subset \mathrm{Apx}\left(A, B\right)$ is the collection of $\phi \in \mathrm{Apx}\left(A, B\right)$ that coarsen some element of $\mathcal{A}$.

3.1.2 Languages and structures

A *language* $\mathcal{L}$ is given by sets of predicate symbols and of function symbols. Every symbol has two natural numbers attached: its arity and its Lipschitz constant. An *$\mathcal{L}$-structure* $\mathfrak{A}$ is given by

- a complete metric space A,
- a c_B-Lipschitz function $B^{\mathfrak{A}} : A^{n_B} \to \mathbb{R}$ for every predicate symbol B, where c_B is the Lipschitz constant of B and n_B is the arity of B, and
- a c_f-Lipschitz function $f^{\mathfrak{A}} : A^{n_f} \to A$ for every function symbol f, where c_f is the Lipschitz constant of f and n_f is the arity of f.

Here and in the following we assume the power A^n to be endowed with the max metric $d(\bar{a}, \bar{b}) = \max_i d(a_i, b_i)$. An *embedding* of $\mathcal{L}$-structures $\phi : \mathfrak{A} \to \mathfrak{B}$ is a function that commutes with the interpretation of all the predicate and function symbols. An *isomorphism* is a surjective embedding. An *automorphism* of $\mathfrak{A}$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}$. If $\bar{a}$ is a finite tuple in $\mathfrak{A}$ then $\langle \bar{a} \rangle$ denotes the smallest substructure of $\mathfrak{A}$ containing $\bar{a}$. A *partial isomorphism* $\phi : \mathfrak{A} \dashrightarrow \mathfrak{B}$ is an embedding from $\langle \bar{a} \rangle$ to $\mathfrak{B}$ for some finite tuple $\bar{a}$ in $\mathfrak{A}$. An $\mathcal{L}$-structure $\mathfrak{A}$ is *finitely generated* if $\mathfrak{A} = \langle \bar{a} \rangle$ for some finite tuple $\bar{a}$ in $\mathfrak{A}$.

We will assume that the language $\mathcal{L}$ contains a distinguished binary predicate symbol to be interpreted as the metric. In particular this ensures that all the embeddings and (partial) isomorphisms are (partial) isometries. Therefore consistently with the convention from Remark 3.1.1 partial isomorphisms will be regarded as approximate isometries.

Definition 3.1.2. Suppose that $\mathcal{C}$ is a class of finitely-generated $\mathcal{L}$-structure. We say that $\mathcal{C}$ satisfies

- the *hereditary property* (HP) if $\langle \bar{a} \rangle \in \mathcal{C}$ for every $\mathfrak{A} \in \mathcal{C}$ and finite tuple $\bar{a} \in \mathfrak{A}$,
- the *joint embedding property* (JEP) if for any $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ there is $\mathfrak{C} \in \mathcal{C}$ and embeddings $\phi : \mathfrak{A} \to \mathfrak{C}$ and $\psi : \mathfrak{B} \to \mathfrak{C}$,
- the *near amalgamation property* (NAP) if, whenever $\mathfrak{A} \subset \mathfrak{B}_0$ and $\mathfrak{B}_1$ are elements of $\mathcal{C}$, $\phi : \mathfrak{A} \to \mathfrak{B}_1$ is an embedding, $\bar{a}$ is a finite tuple in $\mathfrak{A}$, and $\varepsilon > 0$, there exists $\mathfrak{C} \in \mathcal{C}$ and embeddings $\psi_0 : \mathfrak{B}_0 \to \mathfrak{C}$ and $\psi_1 : \mathfrak{B}_1 \to \mathfrak{C}$ such that
$$d\left(\psi_0(\bar{a}), \left(\psi_1 \circ \varphi\right)(\bar{a})\right) \leq \varepsilon.$$
- the *amalgamation property* (AP) if it satisfies (NAP) even when one takes $\varepsilon = 0$.

3.1.3 Fraïssé classes and limits

Suppose in the following that $\mathcal{C}$ is a class of finitely generated $\mathcal{L}$-structures satisfying (HP), (JEP), and (NAP).

Definition 3.1.3. A $\mathcal{C}$-structure is an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\langle \bar{a} \rangle \in \mathcal{C}$ for every finite tuple $\bar{a}$ in $\mathfrak{A}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{C}$-structures. Define $\mathrm{Apx}_{1,\mathcal{C}}(\mathfrak{A}, \mathfrak{B}) \subset \mathrm{Apx}(A, B)$ to be the set of all partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. Define $\mathrm{Apx}_{2,\mathcal{C}}(\mathfrak{A}, \mathfrak{B})$ to be the set of approximate isometries $\phi : A \rightsquigarrow B$ of the form

$$\phi = g^* f$$

where $f \in \mathrm{Apx}_{1,\mathcal{C}}(\mathfrak{A}, \mathfrak{C})$ and $g \in \mathrm{Apx}_{1,\mathcal{C}}(\mathfrak{B}, \mathfrak{C})$ for some $\mathcal{C}$-structure $\mathfrak{C}$. Finally set

$$\mathrm{Apx}_{\mathcal{C}}(\mathfrak{A}, \mathfrak{B}) = \overline{\mathrm{Apx}_{2,\mathcal{C}}(\mathfrak{A}, \mathfrak{B})^{\uparrow}}.$$

Elements of $\mathrm{Apx}_{\mathcal{C}}(\mathfrak{A}, \mathfrak{B})$ are called ($\mathcal{C}$-intrinsic) *approximate morphism.* A ($\mathcal{C}$-intrinsic) *strictly approximate morphism from* $\mathfrak{A}$ to $\mathfrak{B}$ is an approximate morphism ϕ such that the interior $\mathrm{Apx}_{\mathcal{C}}^{<\phi}(\mathfrak{A}, \mathfrak{B})$ of $\mathrm{Apx}_{\mathcal{C}}^{\leq\phi}(\mathfrak{A},\mathfrak{B})$ is nonempty. The set of strictly approximate morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is denoted by $\mathrm{Stx}_{\mathcal{C}}(\mathfrak{A}, \mathfrak{B})$.

Fix $k \in \mathbb{N}$ and denote by $\mathcal{C}(k)$ the set of pairs $(\bar{a}, \mathfrak{A})$ where $\mathfrak{A} \in \mathcal{C}$ and $\bar{a}$ is a finite tuple in $\mathfrak{A}$ such that $\mathfrak{A} = \langle \bar{a} \rangle$. Two such pairs $(\bar{a}, \mathfrak{A})$ and $(\bar{b}, \mathfrak{B})$ are identified if there is an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\phi(\bar{a}) = \bar{b}$. By abuse of notation we will denote $(\bar{a}, \mathfrak{A})$ simply by $\bar{a}$.

Definition 3.1.4. The Fraïssé metric $d_{\mathcal{C}}$ on $\mathcal{C}(k)$ is defined by

$$d_{\mathcal{C}}(\bar{a}, \bar{b}) = \inf_{\phi} \max_{i} \phi(a_i, b_i)$$

where ϕ ranges in $\mathrm{Apx}_{\mathcal{C}}(\langle \bar{a} \rangle, \langle \bar{b} \rangle)$ or, equivalently, in $\mathrm{Stx}_{\mathcal{C}}(\langle \bar{a} \rangle, \langle \bar{b} \rangle)$.

Such a metric can be equivalently described in terms of embeddings:

$$d_{\mathcal{C}}(\bar{a}, \bar{b}) = \inf_{f,g} d(f(\bar{a}), g(\bar{b}))$$

where f, g range over all the embeddings of $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ into a third structure $\mathfrak{C} \in \mathcal{C}$.

Definition 3.1.5. Suppose that $\mathcal{C}$ is a class of finitely-generated $\mathcal{L}$-structures satisfying (HP), (JEP), and (NAP) from Definition 3.1.2. We say that $\mathcal{C}$ is a *Fraïssé class* if the metric space $(\mathcal{C}(k), d_{\mathcal{C}})$ is complete and separable for every $k \in \mathbb{N}$.

Remark 3.1.6. In [8, Definition 2.12] a Fraïssé class is moreover required to satisfy the Continuity Property. Such a property is automatically satisfied in our more restrictive setting, where we assume that the interpretation of any symbol from $\mathcal{L}$ is a Lipschitz function with Lipschitz constant that does not depend from the structure.

Definition 3.1.7. Suppose that $\mathcal{C}$ is a Fraïssé class. A limit of $\mathcal{C}$ is a separable $\mathcal{C}$-structure $\mathfrak{M}$ satisfying the following property: For every $\mathfrak{A} \in \mathcal{C}$, finite tuple $\bar{a}$ in $\mathfrak{A}$, embedding $\phi : \langle \bar{a} \rangle \to \mathfrak{M}$, and $\varepsilon > 0$ there is an embedding $\psi : \mathfrak{A} \to \mathfrak{M}$ such that $d\left(\psi(\bar{a}), \phi(\bar{a})\right) < \varepsilon$.

The definition given above is equivalent to [8, Definition 2.14] in view of [8, Corollary 2.20].

Definition 3.1.8. An $\mathcal{L}$-structure $\mathfrak{M}$ is *homogeneous* if for every finite tuple $\bar{a}$ in $\mathfrak{A}$, embedding $\phi : \langle \bar{a} \rangle \to \mathfrak{A}$, and $\varepsilon > 0$, there is an automorphism ψ of $\mathfrak{M}$ such that $d\left(\phi(\bar{a}), \psi(\bar{a})\right) < \varepsilon$.

The following theorem is a combination of the main results from [8].

Theorem 3.1.9 (Ben Yaacov)**.** *Suppose that $\mathcal{C}$ is a Fraïssé class. Then $\mathcal{C}$ has a limit $\mathfrak{M}$. If $\mathfrak{M}'$ is another limit of $\mathcal{C}$ then $\mathfrak{M}$ and $\mathfrak{M}'$ are isomorphic as $\mathcal{L}$-structures. Moreover $\mathfrak{M}$ is homogeneous and contains any separable $\mathcal{C}$-structure as a substructure.*

Remark 3.1.10. Suppose that $\mathcal{C}$ is a Fraïssé class. Assume that $\mathcal{A}$ is a class of separable $\mathcal{C}$-structure with the following properties:

- $\mathcal{A}$ is closed under isomorphism and countable direct limits, and
- every separable $\mathcal{C}$-structure embeds into an element of $\mathcal{A}$.

It is clear from the proof of [8, Lemma 2.17] that the Fraïssé limit $\mathfrak{M}$ of $\mathcal{C}$ belongs to $\mathcal{A}$.

3.1.4 Operator spaces

An operator space is a closed subspace of $B(H)$. Here and in the following we denote by M_n the algebra of $n \times n$ complex matrices. If X is a complex vector space, then we denote by $M_n \otimes X$ the algebraic tensor product. Observe that this can be canonically identified with the space $M_n(X)$ of $n \times n$ matrices with entries from X. If $X \subset B(H)$ is an operator space, then $M_n(X)$ is naturally endowed with a norm given by the inclusion $M_n(X) \subset B(H \oplus \cdots \oplus H)$. If $\phi : X \to Y$ is a linear map between operator spaces, then its n-th amplification is the linear map $id_{M_n} \otimes \phi : M_n \otimes X \to M_n \otimes Y$. Under the identification of $M_n \otimes X$ with $M_n(X)$ and $M_n \otimes Y$ with $M_n(Y)$, the map $id_{M_n} \otimes \phi$ is defined by

$$\left(id_{M_n} \otimes \phi\right)[x_{ij}] = \left[\phi\left(x_{ij}\right)\right].$$

We say that ϕ is completely bounded if

$$\sup_n \|id_{M_n} \otimes \phi\| < +\infty.$$

In such case we define its completely bounded norm

$$\|\phi\|_{cb} = \sup_n \|id_{M_n} \otimes \phi\|.$$

A linear map ϕ is a complete contraction if $id_{M_n} \otimes \phi$ is a contraction for every $n \in \mathbb{N}$. It is a complete isometry if $id_{M_n} \otimes \phi$ is an isometry for every $n \in \mathbb{N}$. Finally it is a completely isometric isomorphism if $id_{M_n} \otimes \phi$ is an isometric isomorphism for every $n \in \mathbb{N}$.

Operator spaces admit an abstract characterization due to Ruan [123]. A vector space X is matrix-normed if for every $n \in \mathbb{N}$ the space $M_n(X)$ is endowed with a norm such that whenever $\alpha \in M_{k,n}$, $x \in M_n(X)$, and $\beta \in M_{n,k}$

$$\|\alpha.x.\beta\|_k \leq \|\alpha\| \|x\| \|\beta\|$$

where $\alpha.x.\beta$ denotes the matrix product, and $\|\alpha\|, \|\beta\|$ are the norms of α and β regarded as operators between finite-dimensional Hilbert spaces. A matrix-normed vector space is L^∞-matrix-normed provided that

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\| = \max\{\|x\|, \|y\|\}$$

for $x \in M_n(X)$ and $y \in M_m(X)$. Every operator space $X \subset B(H)$ is canonically an L^∞-matrix-normed space. Ruan's theorem asserts that, conversely, any L^∞-matrix-normed space is completely isometrically isomorphic to an operator space [107, Theorem 13.4].

Equivalently one can think of an operator system X as a structure on $\mathcal{K}_0 \otimes X$; see [115, Section 2.2]. Suppose that H is the separable Hilbert space with a fixed orthonormal basis $(e_k)_{k\in\mathbb{N}}$. Let P_n be the orthogonal projection of span $\{e_1, \ldots, e_n\}$ for $n \in \mathbb{N}$. We can identify M_n with the subspace of $A \in B(H)$ such that $AP_n = P_nA = A$. The direct union $\mathcal{K}_0 = \bigcup_n M_n$ is a subspace of $B(H)$. We can identify $\bigcup_n M_n(X)$ with $\mathcal{K}_0[X] \cong \mathcal{K}_0 \otimes X$. Then $\mathcal{K}_0[X]$ is a complex vector space with a natural structure of $\mathcal{K}_0$-bimodule. In formulas if $\alpha, \beta \in \mathcal{K}_0$ and $x = \sum_i \gamma_i \otimes y_i \in \mathcal{K}_0[X]$, then

$$\alpha.x.\beta = \sum_i \alpha\gamma_i\beta \otimes y_i \in \mathcal{K}_0[X].$$

In this framework one can reformulate Ruan's axioms as follows; see [115, Page 35]. If $\alpha_i, \beta_i \in \mathcal{K}_0$ and $x_i \in \mathcal{K}_0[X]$ are finite sequences then

$$\left\| \sum_i \alpha_i . x_i . \beta_i \right\| \leq \left\| \sum_i \alpha_i^* \alpha_i \right\|^{\frac{1}{2}} \max_i \|x_i\| \left\| \sum_i \beta_i \beta_i^* \right\|^{\frac{1}{2}} .$$

The metric on $\mathcal{K}_0[X]$ is not complete. Nonetheless one can pass to the completion $\mathcal{K}\overline{\otimes}X$ and extend all the operations. (Here $\mathcal{K}$ is the closure of $\mathcal{K}_0$ inside $B(H)$, i.e. the ideal of compact operators.)

The abstract characterization of operator spaces mentioned above allows one to regard operator spaces as $\mathcal{L}_{OS}$-structures for a suitable language $\mathcal{L}_{OS}$. Denote by

$$\mathcal{K}_0(\mathbb{Q}(i)) = \bigcup_{n\in\mathbb{N}} M_n(\mathbb{Q}(i))$$

the space of matrices with coefficients in the field of Gauss rationals $\mathbb{Q}(i)$. Then $\mathcal{L}_{OS}$ contains, in addition to the special symbol d for the metric, a symbol $+$ for the addition in $\mathcal{K}\overline{\otimes}X$, a constant 0 for the zero vector in $\mathcal{K}\overline{\otimes}X$, function symbols $\sigma_{\alpha,\beta}$ for $\alpha, \beta \in \mathcal{K}_0(\mathbb{Q}(i))$ for the bimodule operation. The Lipschitz constant for the symbol $+$ is 2, while the Lipschitz constant of $\sigma_{\alpha,\beta}$ is $\|\alpha\| \|\beta\|$. An alternative description of operator spaces as metric structures—which fits in the framework of continuous logic [9, 35]—has been provided in [30, Section 3.3] and [52, Appendix B].

It is worth noting that the space X can be described as the set of $x \in \mathcal{K}\overline{\otimes}X$ such that $1.x = x$. Moreover a linear map $\phi : \mathcal{K}\overline{\otimes}X \to \mathcal{K}\overline{\otimes}Y$ that respects the $\mathcal{K}_0$-bimodule operations satisfies

$$\phi\left(\sum_i \gamma_i \otimes x_i\right) = \sum_i \gamma_i \otimes \phi(x)$$

and therefore is the amplification of a linear map from X to Y. Hence when operator spaces are regarded as $\mathcal{L}_{OS}$-structures, embeddings and isomorphisms as defined in Subsection 3.1.2 correspond, respectively, to completely isometric linear maps and completely isometric linear isomorphisms.

If X and Y are operator spaces, then the space $CB(X,Y)$ of completely bounded linear maps from X to Y is canonically endowed with an operator space structure obtained by identifying isometrically $M_n(CB(X,Y))$ with $CB(M_n(X), M_n(Y))$ with the completely bounded norm. Any linear functional ϕ on an operator system X is automatically completely bounded with $\|\phi\|_{cb} = \|\phi\|$. Therefore the dual space X^* of X can be regarded as the operator space $CB(X, \mathbb{C})$.

If X and Y are operator spaces, then their ∞-sum $X \oplus^\infty Y$ is the operator system supported on the algebraic direct sum $X \oplus Y$ endowed with norms

$$\|(x,y)\|_{M_n(X \oplus^\infty Y)} = \max\left\{ \|x\|_{M_n(X)}, \|y\|_{M_n(Y)} \right\}.$$

The ∞-sum of a sequence of operator spaces is defined analogously.

The 1-sum $X \oplus^1 Y$ is the operator system obtained by identifying $X \oplus Y$ with $(X^* \oplus^\infty Y^*)^*$. In formulas if $x, y \in M_n(X \oplus Y)$ then

$$\begin{aligned} \|(x,y)\|_{M_n(X \oplus^1 Y)} &= \|(x,y)\|_{CB(X^* \oplus Y^*, M_n)} \\ &= \sup \|(id_{M_n} \otimes \phi)(x) + (id_{M_n} \otimes \psi)(y)\|_{M_n \otimes M_k} \end{aligned}$$

where ϕ, ψ range over the unit balls of $CB(X^*, M_k)$ and $CB(Y^*, M_k)$ and k ranges in $\mathbb{N}$. Equivalently the norm on $X \oplus^1 Y$ can be described as

$$\|(x,y)\| = \sup_{u,v} \|(id_{M_n} \otimes u)(x) + (id_{M_n} \otimes v)(y)\|$$

where u, v range over all completely contractive maps from X and Y into $B(H)$; see [115, Section 2.6]. In analogous fashion one can define the 1-sum and the ∞-sum of a sequence of operator spaces.

We denote the sum

$$\overbrace{\mathbb{C} \oplus^1 \mathbb{C} \oplus^1 \cdots \oplus^1 \mathbb{C}}^{n \text{ times}}$$

by $\ell^1(n)$ and the sum

$$\overbrace{\mathbb{C} \oplus^\infty \mathbb{C} \oplus^\infty \cdots \oplus^\infty \mathbb{C}}^{n \text{ times}}$$

by $\ell^\infty(n)$. Moreover we denote by $\bar{e} = (e_i)$ the canonical basis of $\ell^1(n)$ and by $\bar{e}^*$ its dual basis of $\ell^\infty(n)$.

3.1.5 M_n-spaces

In this subsection we recall the definition and basic properties of M_n-spaces as defined in [86, Chapter I]. A *matricial n-norm* on a space X is a norm on $M_n(X)$ such that

$$\|\alpha.x.\beta\| \leq \|\alpha\| \|x\| \|\beta\|$$

for $\alpha, \beta \in M_n$ and $x \in M_n(X)$. Such a norm induces a norm on $M_k(X)$ for $k \leq n$ via the inclusion

$$x \mapsto \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

An L^∞-*matrix-n-norm* is a matricial n-norm satisfying moreover

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\| = \max \{\|x\|, \|y\|\}$$

for $k \leq n$, $x \in M_k(X)$, and $y \in M_{n-k}(Y)$.

Observe that M_n has a natural n-norm obtained by identifying $M_n(M_n)$ with $M_n \otimes M_n$ (spatial tensor product). More generally if K is a compact Hausdorff space then $C(K, M_n)$ is n-normed by identifying $M_n(C(K, M_n))$ with $C(K, M_n \otimes M_n)$. In particular $\ell^\infty(\mathbb{N}, M_n)$ has a natural n-norm obtained by the identification with $C(\beta\mathbb{N}, M_n)$.

If X, Y are n-normed spaces, then a linear map $\phi : X \to Y$ is *n-bounded* if $id_{M_n} \otimes \phi : M_n(X) \to M_n(Y)$ is bounded, and $\|\phi\|_n$ is by definition $\|id_{M_n} \otimes \phi\|$. The notions of n-contraction and n-isometry are defined similarly. Let $nB(X, Y)$ be the space of n-bounded linear functions from X to Y with norm $\|\cdot\|_n$. Identifying $M_n(X^*)$ with $nB(X, \mathbb{C})$ isometrically defines an L^∞-matrix-n-norm on the dual X^* of X. The same argument allows one to define an L^∞-matrix-norm on the second dual X^{**}.

An L^∞-matricially-n-normed space is called an *M_n-space* if it satisfies any of the following equivalent definitions—see [86, Théorème I.1.9]:

1. There is an n-isometry from X to $B(H)$;
2. The canonical inclusion of X into X^{**} is isometric;
3. $\|\sum_i \alpha_i x_i \beta_i\| \leq \|\sum_i \alpha_i \alpha_i^*\|^{\frac{1}{2}} \max_i \|x_i\| \, \|\sum_i \beta_i^* \beta_i\|^{\frac{1}{2}}$ for any $x_i \in M_n(X)$, $\alpha_i, \beta_i \in M_n$;
4. there is an n-isometry from X to $C(X, M_n)$ for some compact Hausdorff space K.

Clearly the case $n = 1$ gives the usual notion of Banach space. Characterization (3) allows one to show that M_n-spaces can be seen as structures in a suitable language $\mathcal{L}_{M_n}$. This is the same as the language for operator space described in Subsection 3.1.4 where one replaces $\mathcal{K}_0$ with M_n. When M_n-spaces are regarded as $\mathcal{L}_{M_n}$-spaces, embeddings and isomorphisms as defined in Subsection 3.1.2 correspond, respectively, to n-isometric linear maps and n-isometric linear isomorphisms.

The notions of quotient and subspace of an M_n-space can be defined analogously as in the case of operator spaces. Similarly the constructions of 1-sum and ∞-sum can be performed in this context. More details can be found in [86, Section I.2]. We will use the same notations for the 1-sum and ∞-sum of operator spaces and M_n-spaces. This will be clear from the context and no confusion should arise.

For later use we recall the following observation; see [86, Remarque I.1.5]. Suppose that X is a finite-dimensional M_n-space, $\bar{b}$ is a normalized basis of X, and $\overline{b}'$ is its dual basis. Assume that $\overline{b}'$ is also normalized. Then the n-norm on X admits the following expression:

$$\left\|\sum_i \alpha_i \otimes b_i\right\| = \sup\left\{\left\|\sum_i \alpha_i \otimes \beta_i\right\| : \beta_i \in M_n, \left\|\sum_i \alpha_i \otimes b_i^{'}\right\| \leq 1\right\}$$

for $\alpha_i \in M_n$. In particular if $\bar{e}$ is the canonical basis of $\ell^1(n)$ with dual basis $\bar{e}^{'}$ of $\ell^\infty(n)$, then we obtain

$$\left\|\sum_i \alpha_i \otimes e_i\right\| = \sup\left\{\left\|\sum_i \alpha_i \otimes \beta_i\right\| : \beta_i \in M_n, \|\beta_i\| \leq 1\right\}.$$

Similar expressions hold for the matrix norms in operator spaces; see [115].

In the following we will often use tacitly the fact that finite-dimensional M_n-spaces can be approximated by subspaces of finite ∞-sums of copies of M_n.

Lemma 3.1.11. *Suppose that X is a finite-dimensional M_n-space and $\varepsilon > 0$. Then there is $k \in \mathbb{N}$ and an injective linear n-contraction*

$$\phi : X \to \overbrace{M_n \oplus^\infty \cdots \oplus^\infty M_n}^{k \text{ times}}$$

such that $||\phi^{-1}||_n \leq 1 + \varepsilon$.

In Lemma 3.1.11 the map ϕ is not assumed to be injective. The expression $||\phi^{-1}||_n$ denotes the n-norm of ϕ^{-1} when regarded as a map from the range of ϕ to X. Similar conventions will be adopted in the rest of this dissertation. We conclude by recalling that the natural analog of the Hahn-Banach theorem holds for M_n-spaces. Such an analog asserts that M_n is an injective element in the category of M_n-spaces with n-contractive maps as morphisms; see [86, Proposition I.1.16].

3.2 The Fraïssé class of finite-dimensional M_n-spaces

The purpose of this section is to show that the class $\mathcal{M}_n$ of finite-dimensional M_n-spaces is a complete Fraïssé class as in Definition 3.1.5. This will allow us to consider the corresponding Fraïssé limit as in Theorem 3.1.9. The case $n = 1$ of these results recovers the already known fact that finite-dimensional Banach spaces form a complete Fraïssé class. This has been shown by Ben Yaacov [8, Section 3.3] building on previous works of Henson (unpublished) and Kubis-Solecki [83]. For Banach spaces the limit is the Gurarij Banach space, introduced by Gurarij in [53] and proved to be unique up to isometric isomorphism by Lusky in [93].

3.2.1 Amalgamation property

The properties (JEP) and (HP) as in Definition 3.1.2 are clear for $\mathcal{M}_n$. We now show that $\mathcal{M}_n$ has (AP). The proof is analogous to the one for Banach spaces, and consists in showing that the category of finite-dimensional M_n-spaces has pushouts; see [46, Lemma 2.1].

Lemma 3.2.1. *Suppose that $X_0 \subset X$ and Y are M_n-spaces, $\delta \geq 0$, and $f : X \to Y$ is a linear injective map such that $\|f\|_n \leq 1+\delta$ and $\left\|f^{-1}\right\|_n \leq 1+\delta$. Then there is an M_n-space Z and n-isometric linear maps $i : X \to Z$ and $j : Y \to Z$ such that $\|j \circ f - i\|_n \leq \delta$.*

Proof. Define δX_0 to be the M_n-space structure on X_0 given by the norm

$$\|[x_{ij}]\|_{M_n(\delta X)} = \delta \|[x_{ij}]\|_{M_n(X)} .$$

Let $\widehat{Z}$ be the 1-sum $X \oplus^1 Y \oplus^1 \delta X_0$, and Z be the quotient of $\widehat{Z}$ by the subspace

$$N = \left\{(-z, f(z), z) \in \widehat{Z} : z \in X_0\right\}.$$

Finally let $i : X \to Z$ and $j : Y \to Z$ the embeddings given by

$$x \mapsto (x, 0, 0) + N$$

and

$$y \mapsto (0, y, 0) + N.$$

We claim that i and j satisfy the desired conclusions. In fact it is clear that i and j are n-contractions such that $\|i \circ f - j\|_n \leq \delta$. We will now show that i is an n-isometry. The proof that j is an n-isometry is similar. Suppose that $x \in M_n(X)$ consider a linear

n-contraction $\phi : X \to M_n$ such that $\|\phi(x)\|_{M_n \otimes M_n} = \|x\|_{M_n(X)}$. Observe that

$$\frac{1}{1+\delta}\left(\phi \circ f^{-1}\right) : f[X] \to M_n$$

is an linear n-contraction and hence it extends to a linear n-contraction $\psi : f[X] \to M_n$. Similarly the map

$$\frac{\delta}{1+\delta}\phi : \delta X_0 \to M_n$$

is a linear n-contraction. Therefore we have that for every $z \in M_n(X_0)$

$$\begin{aligned}
& \|(x-z, f(z), z)\|_{M_n(X \oplus^1 Y \oplus^1 \delta X_0)} \\
\geq \quad & \left\| \phi\left(x-z\right) + \psi(z) + \frac{\delta}{1+\delta}\phi(z) \right\|_{M_n \otimes M_n} \\
= \quad & \left\| \phi(x) - \phi(z) + \frac{1}{1+\delta}\phi(z) + \frac{\delta}{1+\delta}\phi(z) \right\|_{M_n \otimes M_n} \\
= \quad & \|\phi(x)\|_{M_n \otimes M_n} = \|x\|_{M_n(X)} \,.
\end{aligned}$$

This concludes the proof that i is an n-isometry. □

In particular Lemma 3.2.1 for $\delta = 0$ shows that the class $\mathcal{M}_n$ has (AP).

3.2.2 The Fraïssé metric space

We fix now $k \in \mathbb{N}$ and consider the space $\mathcal{M}_n(k)$ of pairs $(\bar{a}, X)$ such that X is a k-dimensional M_n-space and $\bar{a}$ is a linear basis of X. Two such pairs $(\bar{a}, X)$ and $\left(\bar{b}, Y\right)$ are identified if there is an n-isometry ϕ from X to Y such that $\phi(\bar{a}) = \bar{b}$. For simplicity we will write an element $(\bar{a}, X)$ of $\mathcal{M}_n(k)$ simply by $\bar{a}$, and denote X by $\langle \bar{a} \rangle$. Our goal is to compute the Fraïssé metric in $\mathcal{M}_n(k)$ as in Definition 3.1.4. The following result gives an equivalent characterization of such a metric. The case $n = 1$ is a result of Henson (unpublished) that can be found in [8, Fact 3.2]. We denote by $\ell^1(k)$ the k-fold 1-sum of $\mathbb{C}$ by itself in the category of M_n-spaces with canonical basis $\bar{e}$. An explicit formula for the corresponding norm has been recalled at the end of Section 3.1.5.

Proposition 3.2.2. *Suppose that $\bar{a}, \bar{b} \in \mathcal{M}_n(k)$ and $M > 0$. The following statements are equivalent:*

1. $d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq M$;

2. *For every n-contractive $u : \langle \bar{a} \rangle \to M_n$ there is an n-contractive $v : \left\langle \bar{b} \right\rangle \to M_n$ such that the linear function $w : \ell^1(k) \to M_n$ defined by $w(e_i) = u(a_i) - v(b_i)$ has n-norm at most M, and vice versa.*

Proof. After normalizing we can assume that $M = 1$. We will denote $\langle \bar{a} \rangle$ by X and $\left\langle \bar{b} \right\rangle$ by Y.

(1)⇒(2) Suppose that $d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq 1$. Then there are n-isometries $\phi : \langle \bar{a} \rangle \to Z$ and $\psi : \left\langle \bar{b} \right\rangle \to Z$ for some M_n-space Z such that $\|\phi(a_i) - \psi(b_i)\| \leq 1$ for every $i \leq k$. Suppose that $u : X \to M_n$ is n-contractive. Consider the n-contractive map $u \circ \phi^{-1} : \phi[X] \to M_n$. By injectivity of M_n there is an n-contractive map $\eta : Z \to M_n$ extending $u \circ \phi^{-1}$. Define $v = \eta \circ \psi : Y \to M_n$ and observe that it is n-contractive. Define now $w : \ell^1(k) \to M_n$ by $w(e_i) = u(a_i) - v(b_i)$. We claim that w is n-contractive. In fact

$$\|\eta\left(\phi(e_i) - \psi(e_i)\right)\| \leq 1$$

for every $i \leq k$. Therefore if $\alpha_i \in M_n$

$$\begin{aligned} \left\| (id_{M_n} \otimes w) \left(\sum_i \alpha_i \otimes e_i \right) \right\| &= \left\| \sum_i \alpha_i \otimes \eta\left(\phi(e_i) - \psi(e_i)\right) \right\| \\ &\leq \left\| \sum_i \alpha_i \otimes e_i \right\|_{M_n(\ell^1(k))}. \end{aligned}$$

The vice versa is proved analogously.

(2)⇒(1) Conversely suppose that for every n-contractive $u : X \to M_n$ there is an n-contractive $v : Y \to M_n$ such that the linear function $w : \ell^1(k) \to M_n$ defined by $w(e_i) = u(a_i) - v(b_i)$ is n-contractive, and vice versa. Define $\widehat{Z}$ to be

$$X \oplus^1 Y \oplus^1 \ell^1(k).$$

Denote by N the closed subspace

$$\left\{ \left(-\sum_i \lambda_i a_i, \sum_i \lambda_i b_i, \sum_i \lambda_i e_i \right) : \lambda_i \in \mathbb{C} \right\}$$

of $\widehat{Z}$. Define Z to be the quotient of $\widehat{Z}$ by N. Let ϕ be the composition of the canonical inclusion of X into $\widehat{Z}$ with the quotient map from $\widehat{Z}$ to Z. Similarly define $\psi : Y \to Z$.

By the properties of 1-sums and quotients, ϕ and ψ are n-contractions. We claim that they are in fact n-isometries. We will only show that ϕ is an n-isometry, since the proof for ψ is entirely analogous. Suppose that $x \in M_n(X)$. Pick an n-contraction $u : X \to M_n$ such that

$$\|x\| = \left\|\left(id_{M_n} \otimes u\right)(x)\right\|.$$

By hypothesis there are n-contractions $v : Y \to M_n$ and $w : \ell^1(k) \to M_n$ such that $w(e_i) = u(e_i) - v(e_i)$. Therefore if $\alpha_i \in M_n$ then the norm of

$$\left(x - \sum_i \alpha_i \otimes a_i, \sum_i \alpha_i \otimes b_i, \sum_i \alpha_i \otimes e_i\right)$$

in $M_n(\widehat{Z})$ is bounded from below by the norm of

$$\begin{aligned} & \left(id_{M_n} \otimes u\right)\left(x - \sum_i \alpha_i \otimes a_i\right) \\ & + \left(id_{M_n} \otimes v\right)\left(\sum_i \alpha_i \otimes b_i\right) \\ & + \left(id_{M_n} \otimes w\right)\left(\sum_i \alpha_i \otimes e_i\right) \\ = \quad & \left(id_{M_n} \otimes u\right)(x) \end{aligned}$$

which equals $\|x\|$. Since this is true for every $\alpha_i \in M_n$, ϕ is an n-isometry. Similarly ψ is an n-isometry. The proof is concluded by observing that $\|\phi(a_i) - \psi(b_i)\| \leq 1$ for every $i \leq k$. $\square$

Corollary 3.2.3. *If $\bar{a}, \bar{b} \in \mathcal{M}_n(k)$ and $d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq M$ then for every $\alpha_i \in M$*

$$\left|\left\|\sum_i \alpha_i \otimes a_i\right\| - \left\|\sum_i \alpha_i \otimes b_i\right\|\right| \leq M \left\|\sum_i \alpha_i \otimes e_i\right\|$$

where $\bar{e}$ is the canonical basis of $\ell^1(k)$.

An *Auerbach system* in a Banach space is a basis $\bar{a}$ with dual basis $\bar{a}'$ such that $\|a_i\| = \left\|a_i'\right\| = 1$. By analogy we say that an element $\bar{a}$ of $\mathcal{M}_n(k)$ is N-*Auerbach* if $\|a_i\| \leq N$ and $\left\|a_i'\right\| \leq N$ for every $i \leq k$. Denote by $\mathcal{M}_n(k, N)$ the set of N-Auerbach $\bar{a} \in \mathcal{M}_n(k)$. It follows from Corollary 3.2.3 that the set $\mathcal{M}_n(k, N)$ is closed in $\mathcal{M}_n(k)$. It can be easily verified that if $\bar{a} \in \mathcal{M}_n(k, N)$ and $\alpha_i \in M_n$ then

$$\left\| \sum_i \alpha_i \otimes a_i \right\| \leq N \left\| \sum_i \alpha_i \otimes e_i \right\|$$

and

$$\left\| \sum_i \alpha_i \otimes e_i \right\| \leq kN \left\| \sum_i \alpha_i \otimes e_i \right\|$$

where $\bar{e}$ is the canonical basis of $\ell^1(k)$.

If $\bar{a}, \bar{b} \in \mathcal{M}_n(k)$, denote by $\iota_{\bar{a},\bar{b}}$ the linear isomorphism from $\langle \bar{a} \rangle$ to $\langle \bar{b} \rangle$ such that $\iota_{\bar{a},\bar{b}}(a_i) = b_i$ for $i \leq k$. Define the *n-bounded distance* $d_{nb}(\bar{a}, \bar{b})$ to be $||\iota_{\bar{a},\bar{b}}||_n \; ||\iota^{-1}_{\bar{a},\bar{b}}||_n$. (Observe that this is not an actual metric, but $\log(d_{nb})$ is a metric.)

In the following lemma we establish a precise relation between the n-bounded distance d_{nb} and the Fraïssé metric d_{nb} on $\mathcal{M}_n(k, N)$.

Proposition 3.2.4. *Suppose that $\bar{a}, \bar{b} \in \mathcal{M}_n(k, N)$. Then*

$$d_{nb}(\bar{a}, \bar{b}) \leq \left(1 + kN d_{\mathcal{M}_n}(\bar{a}, \bar{b})\right)^2$$

and

$$d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq d_{nb}(\bar{a}, \bar{b}) - 1.$$

Proof. Suppose that $d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq M$. Then by Corollary 3.2.3

$$\left| \left\| \sum_i \alpha_i \otimes a_i \right\| - \left\| \sum_i \alpha_i \otimes b_i \right\| \right| \leq M \left\| \sum_i \alpha_i \otimes e_i \right\|$$

for every $\alpha_i \in M_n$, where $\bar{e}$ is the canonical basis of $\ell^1(k)$. Since $\bar{a}, \bar{b}$ are N-Auerbach we have

$$\left\| \sum_i \alpha_i \otimes a_i \right\| \leq \left\| \sum_i \alpha_i \otimes b_i \right\| + M \left\| \sum_i \alpha_i \otimes e_i \right\| \leq (1 + kNM) \left\| \sum_i \alpha_i \otimes b_i \right\|$$

and similarly

$$\left\| \sum_i \alpha_i \otimes b_i \right\| \leq (1 + kNM) \left\| \sum_i \alpha_i \otimes b_i \right\|.$$

Therefore

$$d_{nb}(\bar{a}, \bar{b}) \leq (1 + kNM)^2.$$

The other inequality is an immediate consequence of Lemma 3.2.1. □

We can finally show that the space $(\mathcal{M}_n(k), d_{\mathcal{M}_n})$ is separable and complete. In view of Proposition 3.2.4 this can be proved by a standard argument; see for example [115, Theorem 21.1 and Remark 21.2]. A proof is included for the sake of completeness.

Proposition 3.2.5. *The space $(\mathcal{M}_n(k), d_{\mathcal{M}_n})$ is compact.*

Proof. Suppose that $\left(\bar{a}^{(m)}\right)_{m\in\mathbb{N}}$ is a sequence in $\mathcal{M}_n(k)$. If $\alpha_i \in M_n$ then

$$\left(\left\|\sum_i \alpha_i \otimes a_i^{(m)}\right\|\right)_{m\in\mathbb{N}}$$

is a bounded sequence of complex numbers. Therefore after passing to a subsequence we can assume that such a sequence converges for any choice of $\alpha_i \in M_n(\mathbb{Q}(i))$. This is easily seen to imply that in fact such a sequence convergence for any choice of $\alpha_i \in M_n$. Moreover the functions

$$(\alpha_1, \dots, \alpha_k) \mapsto \left\|\sum_i \alpha_i \otimes a_i^{(m)}\right\|$$

are equiuniformly continuous on the unit ball of M_n. Therefore, by the Ascoli-Arzelá theorem, after passing to a further subsequence we can assume that the convergence is uniform on the unit ball of M_n. We can now define an element $\bar{a}$ of $\mathcal{M}_n(k)$ by setting

$$\left\|\sum_i \alpha_i \otimes a_i\right\| = \lim_{m\to+\infty}\left\|\sum_i \alpha_i \otimes a_i^{(m)}\right\|.$$

The abstract characterization of M_n-spaces shows that $\bar{a}$ is indeed an element of $\mathcal{M}_n(k)$. By uniform convergence in the unit ball the sequence $\left(\bar{a}^{(m)}\right)_{m\in\mathbb{N}}$ is such that $d_{nb}(\bar{a}^{(m)}, \bar{a}) \to 1$. Therefore $d_{\mathcal{M}_n}(\bar{a}^{(m)}, \bar{a}) \to 0$ by Proposition 3.2.4. □

This concludes the proof that $\mathcal{M}_n$ is a complete Fraïssé class.

3.2.3 The Fraïssé limit

We have verified that the class $\mathcal{M}_n$ is a Fraïssé class in the sense of Definition 3.1.5. Therefore by Theorem 3.1.9 we can consider its Fraïssé limit. Observe that the $\mathcal{M}_n$-structures are precisely the M_n-spaces. We first provide a characterization for the Fraïssé limit of $\mathcal{M}_n$ similar in spirit to the universal property defining the Gurarij Banach space.

Proposition 3.2.6. *Suppose that Z is a separable M_n-space. The following statements are equivalent:*

1. *Z is the Fraïssé limit of the class $\mathcal{M}_n$;*

2. *If $X \subset Y$ are finite-dimensional M_n-spaces, $\psi : X \to Z$ is a linear n-isometry, and $\varepsilon > 0$, then there is a linear function $\phi : Y \to Z$ extending ϕ such that $||\phi||_n \, ||\phi^{-1}||_n < 1 + \varepsilon$.*

Proof. The proof is entirely analogous to the proof of [8, Theorem 3.3], and is presented here for convenience of the reader.

(1)⇒(2) Suppose that Z is the Fraïssé limit of the class $\mathcal{M}_n$. Suppose that $X \subset Y$ are finite-dimensional M_n-spaces, $\phi : X \to Z$ is a linear n-isometry, and $\varepsilon > 0$. Fix $\delta > 0$ small enough. Consider also a basis $(a_1, \ldots, a_k)$ of X and a basis $(b_1, \ldots, b_m)$ of Y such that $b_i = a_i$ for $i \leq k$. Since Z is by assumption the Fraïssé limit of the class $\mathcal{M}_n$, there is a linear n-isometry $\widehat{\phi} : Y \to Z$ such that $||\phi(a_i) - \widehat{\phi}(a_i)|| \leq \delta$ for every $i \leq k$. Define now $\psi : Y \to Z$ by setting $\psi(b_i) = \phi(a_i)$ for $i \leq k$ and $\psi(b_i) = \widehat{\phi}(a_i)$ for $k < i \leq m$. A routine calculation shows that, for δ small enough, ψ satisfies the desired inequality.

(2)⇒(1) Suppose now that Z satisfies condition (2). Consider $X \in \mathcal{M}_n(k)$, a finite l-tuple $\bar{b}$ in X, $\psi \in \mathrm{Stx}_{\mathcal{M}_n}(X, Z)$, and $\varepsilon > 0$. By [8, Lemma 2.16] in order to show that Z is the Fraise limit of $\mathcal{M}_n$ it is enough to find $\varphi \in \mathrm{Stx}_{\mathcal{M}_n}^{<\psi}(X, Z)$ with the following property: for every $i \leq l$ there is $y \in Z$ such that $\varphi(b_i, y) < \varepsilon$. By [8, Lemma 2.8(iii)] after enlarging X, and decreasing ε we can assume that there is a finite m-tuple $\bar{c}$ in X and an n-isometric linear map $f : \langle \bar{c} \rangle \to Z$ such that $\psi \geq f_{|\bar{c}} + \varepsilon$. (Recall our convention of identifying partial isomorphisms between $\mathcal{L}$-structures with the corresponding approximate isomorphisms.) Denote by $\bar{b}\bar{c}$ the concatenation of the tuples $\bar{b}$ and $\bar{c}$. By assumption if $\delta > 0$, then we can extend f to a linear map $f : X \to Z$ satisfying $||f||_n ||f^{-1}||_n < 1 + \delta$. In view of Proposition 3.2.4 by choosing δ small enough one can ensure that

$$d_{\mathcal{M}_n}\left(\bar{b}\bar{c}, f\left(\bar{b}\bar{c}\right)\right) < \varepsilon.$$

Therefore by the definition of the Fraïssé metric $d_{\mathcal{M}_n}$ (Definition 3.1.4) there is

$$\varphi \in \mathrm{Stx}_{\mathcal{M}_n}\left(\left\langle \bar{b}\bar{c} \right\rangle, \left\langle f\left(\bar{b}\bar{c}\right) \right\rangle\right) \subset \mathrm{Stx}_{\mathcal{M}_n}(X, Z)$$

such that $\varphi(b_i, f(b_i)) < \varepsilon$ for $i \leq l$ and $\varphi(c_j, f(c_j)) < \varepsilon$ for $j \leq m$. Observe that such a φ satisfies

$$\psi \geq f_{|\bar{c}} + \varepsilon > \varphi_{|\bar{c} \times f(\bar{c})} \geq \varphi.$$

This concludes the proof. □

In view of Proposition 3.2.6 the following theorem is an immediate consequence of 3.1.9 and the fact that $\mathcal{M}_n$ is a complete Fraïssé class.

Theorem 3.2.7. *There is a separable M_n-space $\mathbb{G}_n$ with the following property: If $X \subset Y$ are finite-dimensional M_n-spaces, $\psi : X \to \mathbb{G}_n$ is a linear n-isometry, and $\varepsilon > 0$, then there is a linear function $\phi : Y \to \mathbb{G}_n$ extending ϕ such that $||\phi||_n \, ||\phi^{-1}||_n < 1 + \varepsilon$. Any two separable M_n-spaces with such a property are n-isometrically isomorphic. Moreover $\mathbb{G}_n$ contains any separable M_n-space as a subspace, and has the following homogeneity property: If $X \subset \mathbb{G}_n$ is finite-dimensional, $\phi : X \to \mathbb{G}_n$ is a linear n-isometry, and $\varepsilon > 0$, then there is a surjective n-isometry $\psi : \mathbb{G}_n \to \mathbb{G}_n$ such that $\left\|\psi_{|X} - \phi\right\|_n < \varepsilon$.*

Clearly for $n = 1$ one obtains a Banach space which is isometrically isomorphic to the Gurarij space. The same proof as [82, Theorem 1.1] where one replaces [82, Lemma 2.1] with Lemma 3.2.1 yields the following homogeneity result for $\mathbb{G}_n$.

Theorem 3.2.8. *Suppose that $X \subset \mathbb{G}_n$ is a finite-dimensional subspace and $\phi : X \to \mathbb{G}_n$ is an invertible linear map such that $\|\phi\| < 1 + \varepsilon$ and $\left\|\phi^{-1}\right\| < 1 + \varepsilon$. Then there exists a surjective linear n-isometry $\psi : \mathbb{G}_n \to \mathbb{G}_n$ such that $\left\|\psi_{|X} - \phi\right\|_n < \varepsilon$.*

3.3 The noncommutative Gurarij space

3.3.1 MIN and MAX spaces

Clearly any operator space can be canonically regarded as an M_n-space. Conversely if X is an M_n-space, then there are two canonical ways to regard X as an operator space. It is natural to call an operator space structure $\widehat{X}$ on X *compatible* if the map $X \mapsto \widehat{X}$ is an n-isometry. The *minimal and maximal* compatible operator space structures $\mathrm{MIN}_n(X)$ and $\mathrm{MAX}_n(X)$ on an M_n-space are defined by the formulas

$$\|x\|_{M_k(\mathrm{MIN}_n(X))} = \sup_{\phi} \|(id_{M_k} \otimes \phi)(x)\|_{M_k \otimes M_n}$$

where ϕ varies among all n-contractions from X to M_n, and

$$\|x\|_{M_k(\mathrm{MAX}_n(X))} = \sup_u \|(id_{M_k} \otimes u)(x)\|_{M_k(B(H))}$$

where u varies among all n-contractions from X to $B(H)$. These are introduced in [86, Section I.3] as a generalization of the minimal and maximal quantization of a Banach space as in [26, Section 3.3]; see also [105, Section 2]. If X is an operator space then we define $\mathrm{MIN}_n(X)$ and $\mathrm{MAX}_n(X)$ to be the structures defined above starting from X regarded just as M_n-space. This is consistent with the terminology used in [105, 104].

The names MIN and MAX are suggestive of the following property; see [86, Proposition I.3.1]. If $\widehat{X}$ is a compatible operator space structure on X then the identity maps

$$\mathrm{MAX}_n(X) \to \widehat{X} \to \mathrm{MIN}_n(X)$$

are completely bounded. The operator space structures MIN and MAX are characterized by the following universal property; see [86, Proposition I.3.6 and Proposition I.3.7]. If Z is an operator space and $u : Z \to X$ is a linear map, then $u : Z \to X$ is n-bounded if and only if $u : Z \to \mathrm{MIN}_n(X)$ is completely bounded, and in such case

$$\|u : Z \to \mathrm{MIN}_n(X)\|_{cb} = \|u : Z \to X\|_n .$$

Similarly if Z is an operator space and $u : X \to Z$ is a linear map, then $u : X \to Z$ is n-bounded if and only if $u : \mathrm{MAX}_n(X) \to Z$ is completely bounded, and in such case

$$\|u : \mathrm{MAX}(X) \to Z\|_{cb} = \|u : X \to Z\|_n .$$

Remark 3.3.1. In the following we will always consider an M_n-space X as an operator system endowed with its minimal compatible operator system structure.

It is worth noting at this point that all the proofs of Section 3.2 go through without change when M_n-spaces are regarded as operator spaces with their minimal compatible operator space structure. This easily follows from the properties of the minimal quantization recalled above.

3.3.2 Exact and 1-exact operator spaces

Suppose that E and F are two finite-dimensional operator spaces. Define $d_{cb}(E, F)$ to be the infimum of $||\phi||_{cb} \, ||\phi^{-1}||_{cb}$ when ϕ ranges over all linear isomorphisms from E to F. The *exactness constant* $ex\,(E)$ of a finite-dimensional operator space is the infimum of $d_{cb}\,(E, F)$ where F ranges among all subspaces of M_n for $n \in \mathbb{N}$. Equivalently one can define $ex(E)$ to be the limit for $n \to +\infty$ of the decreasing sequence

$$\|id_E : \mathrm{MIN}_n(E) \to E\|_{cb},$$

where id_E denotes the identity map of E. If X is a not necessarily finite-dimensional operator space, then its exactness constant $ex\,(X)$ is the supremum of $ex\,(E)$ where E ranges over all finite-dimensional subspaces of E.

An operator space is *exact* if it has finite exactness constant, and 1-*exact* if it has exactness constant 1. For C*-algebras exactness is equivalent to 1-exactness, which is in turn equivalent to several other properties; see [11, Section IV.3.4]. Exactness is a fundamental notion in the theory of C*-algebras and operator spaces. It is a purely noncommutative phenomenon: there is no Banach space analog of nonexactness. In fact every Banach space—and in fact every M_n-space—is 1-exact. More information and several equivalent characterizations of exactness can be found in [114] and [115, Chapter 17].

In the following we will denote by $\mathcal{E}_1$ the class of finite-dimensional 1-exact operator spaces. Moreover we will denote by $\mathcal{M}_\infty^0 \subset \mathcal{E}_1$ the class of operator spaces that admit a completely isometric embedding into M_n for some $n \in \mathbb{N}$. Our goal is to show that $\mathcal{E}_1$ is a Fraïssé class.

3.3.3 Amalgamation of 1-exact operator spaces

It is clear that $\mathcal{E}_1$ has (HP) from Definition 3.1.2. It remains to verify that $\mathcal{E}_1$ satisfies (AP). This will give (JEP) as consequence, since the trivial operator space $\{0\}$ embeds in every element of $\mathcal{E}_1$.

We recall that if (Z_n) is a direct sequence of operator spaces with completely isometric linear maps $\phi_n : Z_n \to Z_{n+1}$ one can define the *direct limit* $\lim_{(\phi_n)} Z_n$ with canonical completely isometric linear maps $\sigma_k : Z_k \to \lim_{(\phi_n)} Z_n$ in the following way. Let $W = \ell^\infty\,(\mathbb{N}, (Z_n))$ be the space of sequences $(z_n) \in \prod_n Z_n$ with $\sup_n \|z_n\| < +\infty$. Define an operator seminorm structure on $\widehat{W}$ in the sense of [12, 1.2.16] by setting

$$\rho_k\,((z_n)_{n\in\mathbb{N}}) = \limsup_{n\to+\infty} \|z_n\|_{M_k(Z_n)}$$

for $k \in \mathbb{N}$ and $z_n \in M_k\,(Z_n)$. Finally define W to be the operator space associated with such an operator seminorm structure on $\widehat{W}$. For n, m let $\phi_{n,n} = id_{Z_n}$, $\phi_{n,m} = \phi_{m-1} \circ \cdots \circ \phi_n$

if $n < m$, and $\phi_{n,m} = 0$ otherwise. Define the maps $\sigma_k : Z_k \to W$ by

$$\begin{array}{rcl} Z_k & \to & W \\ x & \mapsto & (\phi_{k,n}(x))_{n\in\mathbb{N}} \,. \end{array}$$

Finally set $\lim_{(\phi_n)} Z_n$ to be the closure inside W of the union of the images of Z_k under σ_k for $k \in \mathbb{N}$. It is clear that if for every $k \in \mathbb{N}$ the space Z_k is 1-exact, then $\lim_{(\phi_n)} Z_n$ is 1-exact.

The proof of the following proposition is inspired by [24, Theorem 4.7] and [104, Theorem 1.1].

Proposition 3.3.2. *Suppose that $X_0 \subset X$ and Y are finite-dimensional 1-exact operator spaces, $\delta > 0$, and $f : X_0 \to Y$ is such that $\|f\|_{cb} < 1 + \delta$ and $\left\|f^{-1}\right\|_{cb} < 1 + \delta$. Then there exists a 1-exact separable operator space Z and linear complete isometries $j : Y \to Z$ and $i : X \to Z$ such that $\left\|j \circ f - i_{|X_0}\right\|_{cb} < \delta$.*

Proof. Fix $\delta' < \delta$ such that $\|f\|_{cb} < 1 + \delta'$ and $\left\|f^{-1}\right\|_{cb} < 1 + \delta'$ and $\varepsilon > 0$ such that $\delta' + 4\varepsilon < \delta$. We will construct by recursion on k sequences $(n_k)_{k\in\mathbb{N}}$, $(Z_k)_{k\in\mathbb{N}}$, $i_k : X \to Z_k$, $j_k : Y \to Z_k$, $\phi_k : Z_k \to Z_{k+1}$ such that

1. $(n_k)_{k\in\mathbb{N}}$ is nondecreasing,
2. Z_k is an M_{n_k}-space,
3. i_k and j_k are injective completely contractive linear maps,
4. ϕ_k is a completely isometric linear map,
5. $\left\|i_k^{-1}\right\|_{cb} \leq 1 + \varepsilon 2^{-k}$, $\left\|j_k^{-1}\right\|_{cb} \leq 1 + \varepsilon 2^{-k}$,
6. $\|\phi_k \circ i_k - i_{k+1}\|_{cb} \leq 1 + \varepsilon 2^{-k}$, $\|\phi_k \circ j_k - j_{k+1}\|_{cb} \leq 1 + \varepsilon 2^{-k}$, and
7. $\left\|j_k \circ f - (i_k)_{|X_0}\right\|_{cb} \leq \delta' + (2\varepsilon) \sum_{i<k} 2^{-i}$.

We can apply Lemma 3.2.1 and Lemma 3.1.11 to define n_1, Z_1, i_1, and j_1. Suppose that n_k, Z_k, i_k, j_k, and ϕ_{k-1} have been defined for $k \leq m$. By Lemma 3.1.11 we can pick $n_{m+1} \geq n_m$ and injective completely contractive maps $\theta_X : X \to M_{n_{m+1}}$ and $\theta_Y : Y \to M_{n_{m+1}}$ such that $\left\|\theta_X^{-1}\right\|_{cb} \leq 1 + \varepsilon 2^{-2(m+1)}$ and $\left\|\theta_Y^{-1}\right\|_{cb} \leq 1 + \varepsilon 2^{-(m+1)}$. By injectivity of $M_{n_{m+1}}$ there are complete contractions $\alpha_X, \alpha_Y : Z_m \to M_{n_{m+1}}$ such that

$$\alpha_X \circ i_m = \frac{1}{1 + \varepsilon 2^{-m}} \theta_X \quad \text{and} \quad \alpha_Y \circ j_m = \frac{1}{1 + \varepsilon 2^{-m}} \theta_Y.$$

Define W to be $\mathrm{MIN}_{n_{m+1}}(Z_m \oplus^\infty M_{n_{m+1}})$. Define linear maps

$$\begin{array}{rccl} \widehat{\theta}_X : & X & \to & W \\ & x & \mapsto & \left(i_m\left(x\right), \theta_X\left(x\right)\right), \end{array}$$

$$\begin{array}{rccl} \widehat{\theta}_Y : & Y & \to & W \\ & y & \mapsto & \left(j_m\left(y\right), \theta_Y\left(y\right)\right), \end{array}$$

and

$$\begin{array}{rccl} \widehat{\alpha}_X : & Z_m & \to & W \\ & z & \mapsto & \left(z, \alpha_X(z)\right), \end{array}$$

$$\begin{array}{rccl} \widehat{\alpha}_Y : & Z_m & \to & W \\ & z & \mapsto & \left(z, \alpha_Y(z)\right). \end{array}$$

Observe that $\widehat{\alpha}_X, \widehat{\alpha}_Y$ are completely isometric, while $\widehat{\theta}_X$ and $\widehat{\theta}_Y$ are completely contractive with

$$\left\| \widehat{\theta}_X^{-1} \right\|_{cb} \leq \left\| \theta_X^{-1} \right\|_{cb} \leq 1 + \varepsilon 2^{-(m+1)}$$

and

$$\left\| \widehat{\theta}_Y^{-1} \right\|_{cb} \leq \left\| \theta_Y^{-1} \right\|_{cb} \leq 1 + \varepsilon 2^{-(m+1)}.$$

Note also that

$$\left\| \widehat{\theta}_X - \widehat{\alpha}_X \circ i_m \right\|_{cb} \leq \varepsilon 2^{-m} \quad \text{and} \quad \left\| \widehat{\theta}_Y - \widehat{\alpha}_Y \circ i_m \right\|_{cb} \leq \varepsilon 2^{-m}.$$

Define now

$$N = \left\{ \left(-\left(z_0 + z_1\right), \widehat{\alpha}_X(z_0), \widehat{\alpha}_Y(z_1) \right) \in Z_m \oplus W \oplus W : z_0, z_1 \in Z_m \right\}.$$

Let Z_{m+1} be

$$\mathrm{MIN}_{n_{m+1}}((Z_m \oplus^1 W \oplus^1 W)/N).$$

Consider the first coordinate inclusion $\phi_m : Z_m \to Z_{m+1}$ of Z_m into Z_{m+1}. Similarly define $\psi_X, \psi_Y : W \to Z_{m+1}$ to be the second and third coordinate inclusions. Arguing as in the proof of Lemma 3.2.1 one can verify directly that ϕ_m, ψ_X, ψ_Y are complete isometries. Alternatively one can use [104, Lemma 2.4] together with the properties of MIN. Observe that $\widehat{\alpha}_X \circ \phi_m = \psi_X$ and $\widehat{\alpha}_Y \circ \phi_m = \psi_Y$. Define now linear complete contractions

$$i_{m+1} := \psi_X \circ \widehat{\theta}_X : X \to Z_{m+1} \quad \text{and} \quad j_{m+1} := \psi_Y \circ \widehat{\theta}_Y : Y \to Z_{m+1}.$$

Observe that

$$\left\| i_{m+1}^{-1} \right\|_{cb} \leq \left\| \widehat{\theta}_X^{-1} \right\|_{cb} < 1 + \varepsilon 2^{-(m+1)}$$

and

$$\left\| j_{m+1}^{-1} \right\|_{cb} \leq \left\| \widehat{\theta}_Y^{-1} \right\|_{cb} < 1 + \varepsilon 2^{-(m+1)}.$$

Moreover

$$\begin{aligned} \left\| \phi_m \circ i_m - i_{m+1} \right\|_{cb} &= \left\| \phi_m \circ i_m - \psi_X \circ \widehat{\theta}_X \right\|_{cb} \\ &\leq \left\| \phi_m \circ i_m - \psi_X \circ \widehat{\alpha}_X \circ i_m \right\|_{cb} + \varepsilon 2^{-m} \\ &= \varepsilon 2^{-m}. \end{aligned}$$

Similarly

$$\left\| \psi_m \circ j_m - j_{m+1} \right\|_{cb} \leq \varepsilon 2^{-m}.$$

Finally we have

$$\begin{aligned} \left\| i_{m+1} - j_{m+1} \circ f \right\|_{cb} &= \left\| \psi_X \circ \widehat{\theta}_X - \psi_Y \circ \widehat{\theta}_Y \circ f \right\|_{cb} \\ &\leq \left\| \psi_X \circ \widehat{\alpha}_X \circ i_m - \psi_Y \circ \widehat{\alpha}_Y \circ j_m \circ f \right\|_{cb} + (2\varepsilon)\, 2^{-m} \\ &\leq \left\| \phi_m \circ i_m - \phi_m \circ j_m \circ f \right\| + (2\varepsilon)\, 2^{-m} \\ &\leq \left\| i_m - j_m \circ f \right\| + (2\varepsilon)\, 2^{-m} \\ &\leq \delta' + (2\varepsilon) \sum_{i \leq m} 2^{-i}. \end{aligned}$$

This concludes the recursive construction. Let now Z be $\lim_{(\phi_k)} Z_k$ with canonical linear complete isometries $\sigma_k : Z_k \to Z$. Consider also the embeddings $i : X \to Z$ and $j : Y \to Z$ defined by

$$i := \lim_{k \to +\infty} \sigma_k \circ i_k \quad \text{and} \quad j := \lim_{k \to +\infty} \sigma_k \circ j_k.$$

It is easily seen as in the proof of [24, Theorem 4.7] that Z is a 1-exact separable operator space, and i, j are well defined completely isometric linear maps such that $\left\| j \circ f - i_{|X_0} \right\|_{cb} \leq \delta + 2\varepsilon$. $\square$

In particular Proposition 3.3.2 for $\delta = 0$ shows that the class $\mathcal{E}_1$ has (NAP). It is not difficult to modify the proof above to show that the conclusions of Proposition 3.3.2 hold even when $X_0 \subset X$ and Y are not necessarily finite-dimensional separable 1-exact operator spaces. Moreover one can obtain Z to be an $\mathcal{OL}_{\infty,1+}$ space in the sense of [25, 65].

3.3.4 The Fraïssé metric space

Fix $k \in \mathbb{N}$ and denote by $\mathcal{E}_1(k)$ the space of pairs $(\bar{a}, X)$ such that X is a k-dimensional 1-exact operator space and $\bar{a}$ is a basis of X. Two such pairs $(\bar{a}, X)$ and $\left(\bar{b}, Y\right)$ are identified if there is a complete isometry ϕ from X to Y such that $\phi(\bar{a}) = \bar{b}$. To simplify the notation the pair $(\bar{a}, X)$ will be simply denoted $\bar{a}$, where we set $X = \langle \bar{a} \rangle$. Denote by $d_{\mathcal{E}_1}$ the Fraïssé metric on $\mathcal{E}_1(k)$ as in Definition 3.1.4. We further denote by $\mathcal{M}_\infty^0(k)$ the subspace of $\mathcal{E}_1(k)$ consisting of pairs $(\bar{a}, X)$ such that X admits a completely isometric embedding into M_n for some $n \in \mathbb{N}$. Let $\ell^1(k)$ be the k-fold 1-sum of $\mathbb{C}$ by itself in the category of operator spaces. A similar proof as the one of Proposition 3.2.2 gives the following:

Proposition 3.3.3. *Suppose that $\bar{a}, \bar{b} \in \mathcal{E}_1(k)$ and $M > 0$. If $d_{\mathcal{E}_1}(\bar{a}, \bar{b}) \leq M$ then for every $n \in \mathbb{N}$ and every completely contractive $u : X \to M_n$ there is a completely contractive $v : Y \to M_n$ such that the linear function $w : \ell^1(k) \to B(H)$ defined by $w(e_i) = u(a_i) - v(b_i)$ has completely bounded norm at most M, and vice versa.*

Corollary 3.3.4. *Suppose that $\bar{a}, \bar{b} \in \mathcal{E}_1(k)$ and $M > 0$. If $d_{\mathcal{E}_1}(\bar{a}, \bar{b}) \leq M$ then for every $n \in \mathbb{N}$ and $\alpha_i \in M_n$*

$$\left| \left\| \sum_i \alpha_i \otimes a_i \right\| - \left\| \sum_i \alpha_i \otimes b_i \right\| \right| \leq M \left\| \sum_i \alpha_i \otimes e_i \right\|$$

where $\bar{e}$ denotes the canonical basis of $\ell^1(k)$.

As in Subsection 3.2.2 we define an element $\bar{a}$ of $\mathcal{E}_1$ to be N-*Auerbach* if $\|a_i\| \leq N$ and $\left\|a_i'\right\| \leq N$ for every $i \leq k$, where $\bar{a}'$ denotes the dual basis of $\bar{a}$. We denote by $\mathcal{E}_1(k, N)$ the set of N-Auerbach $\bar{a} \in \mathcal{E}_1(k)$. Observe that in view of Corollary 3.2.3 the set $\mathcal{E}_1(k, N)$ is closed in $\mathcal{E}_1(k)$. Moreover it can be easily verified that if $\bar{a} \in \mathcal{E}_1(k, N)$ and $\alpha_i \in M_n$ then

$$\left\| \sum_i \alpha_i \otimes a_i \right\| \leq N \left\| \sum_i \alpha_i \otimes e_i \right\|$$

and

$$\left\| \sum_i \alpha_i \otimes e_i \right\| \leq kN \left\| \sum_i \alpha_i \otimes e_i \right\|.$$

If $\bar{a}, \bar{b} \in \mathcal{E}_1(k)$, denote as in Subsection 3.2.2 by $\iota_{\bar{a},\bar{b}}$ the linear isomorphism from $\langle \bar{a} \rangle$ to $\langle \bar{b} \rangle$ such that $\iota_{\bar{a},\bar{b}}(a_i) = b_i$ for $i \leq k$. Define the completely bounded distance $d_{cb}(\bar{a}, \bar{b})$ to be $||\iota_{\bar{a},\bar{b}}||_{cb}\ ||\iota_{\bar{a},\bar{b}}^{-1}||_{cb}$.

Proposition 3.3.5. *Suppose that $\bar{a}, \bar{b} \in \mathcal{E}_1(k, N)$. Then*

$$d_{cb}(\bar{a}, \bar{b}) \leq \left(1 + kN d_{\mathcal{E}_1}(\bar{a}, \bar{b})\right)^2$$

and

$$d_{\mathcal{E}_1}(\bar{a}, \bar{b}) \leq d_{cb}(\bar{a}, \bar{b}) - 1.$$

Proof. The first inequality can be inferred from Proposition 3.3.3; see also the proof of the first inequality in Proposition 3.2.4. The second inequality is an immediate consequence of Proposition 3.3.2. □

Using Proposition 3.3.5 one can show that $(\mathcal{E}_1(k), d_{\mathcal{E}_1})$ is a separable metric space. The proof is similar to [114, Proposition 12]. Recall that $\mathcal{M}_\infty^0 \subset \mathcal{E}_1$ denotes the class of operator spaces that admit a completely isometric embedding into M_n for some $n \in \mathbb{N}$.

Proposition 3.3.6. *For every $k \in \mathbb{N}$, $(\mathcal{E}_1(k), d_{\mathcal{E}_1})$ is a complete metric space, and $\mathcal{M}_\infty^0(k)$ is a dense subset of $\mathcal{E}_1(k)$.*

Proof. Suppose that $\left(\bar{a}^{(m)}\right)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}_1(k)$. By Proposition 3.3.8 there is $N \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$, $\bar{a}^{(m)} \in \mathcal{E}_1(k, N)$. Moreover

$$\limsup_{k,m\to+\infty} d_{cb}(\bar{a}^{(m)}, \bar{a}^{(k)}) = 1$$

Fix a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$. Define X to be the ultraproduct $\prod_{\mathcal{U}} \langle \bar{a}^{(m)} \rangle$ as in [26, Section 10.3]. Let a_i be the element of X having $(a_i^{(m)})_{m\in\mathbb{N}}$ as representative sequence. Observe that for every $n, m \in \mathbb{N}$ and $\alpha_i \in M_n$,

$$\begin{aligned} & \left|\left\|\sum_i \alpha_i \otimes a_i^{(m)}\right\| - \left\|\sum_i \alpha_i \otimes a_i\right\|\right| \\ = & \lim_{k\to\mathcal{U}} \left|\left\|\sum_i \alpha_i \otimes a_i^{(m)}\right\| - \left\|\sum_i \alpha_i \otimes a_i^{(k)}\right\|\right| \\ \leq & \sup_{k\geq m} d_{cb}(\bar{a}^{(m)}, \bar{a}^{(k)}). \end{aligned}$$

Therefore $d_{cb}(\bar{a}^{(m)}, \bar{a}) \to 1$. Hence again by Proposition 3.3.8 we have that $d_{\mathcal{E}_1}(\bar{a}^{(m)}, \bar{a}) \to 0$. This shows that $\bar{a}$ is the limit of the Cauchy sequence $\left(\bar{a}^{(m)}\right)_{m\in\mathbb{N}}$. Observe now that by Proposition 3.3.8 and by the definition of 1-exact operator space, $\mathcal{M}_{\infty}^{0}(k)$ is dense in $(\mathcal{E}_1(k), d_{\mathcal{E}_1})$. It follows from Proposition 3.2.5 that $\mathcal{M}_{\infty}^{0}(k)$ is a separable subspace of $\mathcal{E}_1(k)$. Alternatively one can observe that the algebra $\mathcal{K}$ of compact operators contains all elements of $\mathcal{M}_{\infty}^{0}(k)$, and then apply Corollary [115, Corollary 2.13.3] and Proposition 3.3.5. In any case one can conclude that $\mathcal{E}_1(k)$ is separable as well. □

3.3.5 The noncommutative Gurarij space as a Fraïssé limit

We have shown in Subsection 3.3.3 that $\mathcal{E}_1$ is a Fraïssé class. Therefore we can consider its corresponding Fraïssé limit according to Theorem 3.1.9. First we observe that using Proposition 3.3.3 one can reformulate the property of being limit.

The following notion has been introduced by Oikhberg in [104].

Definition 3.3.7. An operator space Z is *noncommutative Gurarij* if for any completely isometric embedding $\phi : X \to Z$ of a 1-exact operator space $X \subset Y$, and for any $\varepsilon > 0$, there is an injective linear map $\psi : Y \to Z$ extending ϕ such that $||\psi||_{cb}\, ||\psi^{-1}||_{cb} < 1 + \varepsilon$.

The following proposition characterizes noncommutative Gurarij spaces as limits of the Fraïssé class $\mathcal{E}_1$.

Proposition 3.3.8. *Suppose that Z is a separable 1-exact. The following conditions are equivalent:*

1. *Z is a Fraïssé limit of the class $\mathcal{E}_1$;*
2. *For every $n \in \mathbb{N}$, subspace X of M_n, complete isometry $\phi : X \to Z$, and $\varepsilon > 0$ there is a complete isometry $\psi : M_n \to Z$ such that $\left\|\psi_{|X} - \phi\right\| < \varepsilon$;*
3. *Z is noncommutative Gurarij;*
4. *Z satisfies the universal property of Definition 3.3.7 when $Y = M_n$ for some $n \in \mathbb{N}$;*
5. *For every $n, k, l \in \mathbb{N}$, for every subspace X of M_n, finite l-tuple $\bar{b}$ in X, $\psi \in \mathrm{Stx}_{\mathcal{E}_1}(X, Z)$, and $\varepsilon > 0$, there is $\varphi \in \mathrm{Stx}_{\mathcal{E}_1}^{<\psi}(X, Z)$ such that for every $i \leq l$ there is $y \in Z$ such that $\varphi(b_i, y) < \varepsilon$.*

Proof. The implications (1)⇒(2) and (3)⇒(4) are obvious. The implications (2)⇒(4) and (1)⇒(3) can be proved using Proposition 3.3.5 similarly as the implication (1)⇒(2) in Proposition 3.2.6. The implication (4)⇒(5) can be proved as (2)⇒(1) of Proposition 3.2.6, or [8, Theorem 3.3]. Finally (5)⇒(1) is a consequence of [8, Lemma 2.16] and the fact that $\mathcal{M}_{\infty}^{0}(k)$ is dense in $(\mathcal{E}_1(k), d_{\mathcal{E}_1})$; see Proposition 3.3.6. □

With such a characterization of a limit of the Fraïssé class $\mathcal{E}_1$ at hand, we can finally state the main result of this chapter, which is an immediate consequence of Theorem 3.1.9 and Proposition 3.3.8.

Theorem 3.3.9. *There is a separable* 1*-exact operator space* $\mathbb{NG}$ *which is noncommutative Gurarij. Such a space is unique up to completely isometric isomorphism. Every separable* 1*-exact operator space can be completely isometrically embedded into* $\mathbb{NG}$*.*

Theorem 1.1 of [104] shows that any two noncommutative Gurarij spaces are *approximately* completely isometric. The uniqueness assertion in Theorem 3.3.9 improves such a result, showing that any two noncommutative Gurarij spaces are (exactly) completely isometric. Assuming uniqueness, one can also deduce universality from [104, Theorem 1.1] together with the fact that every separable 1-exact operator space embeds into a separable $\mathcal{OL}_{\infty,1+}$ space; see [24, Theorem 4.7].

Recall that an operator space X is an $\mathcal{OL}_{\infty,1+}$ space as defined in [25] if for every finite-dimensional subspace E of X and every $\varepsilon > 0$ there is a finite-dimensional C*-algebra A and a subspace F of A such that $d_{cb}(E, F) < 1 + \varepsilon$. This notion provides the noncommutative analog of $\mathcal{L}_{\infty,1+}$ spaces as in [87]. Clearly $\mathcal{OL}_{\infty,1+}$ spaces are closed under direct limits. Therefore from Remark 3.1.10 and Proposition 3.3.8 one can deduce the following fact, already observed by Oikhberg in [104].

Proposition 3.3.10. *The noncommutative Gurarij space is an* $\mathcal{OL}_{\infty,1+}$ *space.*

The following homogeneity property of $\mathbb{NG}$ follows from the homogeneity statement in Theorem 3.1.9 and Proposition 3.3.8: If $X \subset \mathbb{NG}$ is finite-dimensional, $\phi : X \to \mathbb{NG}$ is a complete isometry, and $\varepsilon > 0$, then there is a completely isometric surjection $\psi : \mathbb{NG} \to \mathbb{NG}$ such that $\left\|\psi_{|X} - \phi\right\| < \varepsilon$. We now observe that one can get ψ to be close to ϕ is cb-norm. The following lemma can be easily obtained from Proposition 3.3.2, as Lemma 2.2 is derived from Lemma 2.1 in [83].

Lemma 3.3.11. *If* $X \subset \mathbb{NG}$ *is finite-dimensional,* Y *is finite-dimensional and* 1*-exact, and* $f : X \to Y$ *is an invertible linear map such that* $\|f\|_{cb} < 1+\delta$ *and* $\left\|f^{-1}\right\|_{cb} < 1+\delta$ *then for every* $\varepsilon > 0$ *there exists* $g : Y \to \mathbb{NG}$ *such that* $\|g\|_{cb} \left\|g^{-1}\right\|_{cb} < 1+\varepsilon$ *and* $\|g \circ f - id_X\|_{cb} < \delta$*.*

One can then run the same argument as the proof of Theorem 1.1 in [83], where [83, Lemma 2.2] is replaced by Lemma 3.3.11, to show that $\mathbb{NG}$ has the following homogeneity property.

Theorem 3.3.12. *If $X \subset \mathbb{NG}$ is a finite-dimensional subspace and $\phi : X \to \mathbb{NG}$ is an invertible linear map such that $\|\phi\|_{cb} < 1 + \delta$ and $\left\|\phi^{-1}\right\|_{cb} < 1 + \delta$ then there exists a surjective complete isometry $\psi : \mathbb{NG} \to \mathbb{NG}$ such that $\left\|\psi_{|X} - \phi\right\|_{cb} < \delta$.*

Chapter 4

The Gurarij operator system

A unital operator space is a closed subspace X of $B(H)$ containing the identity operator. Unital operator spaces can also be defined abstractly as operator spaces with a distinguished unitary element; see [13, Theorem 2.1]. Particularly important among unital operator spaces are operator systems. These are the unital operator spaces $X \subset B(H)$ that are closed by taking adjoints. Again these can be abstractly characterized as those unital operator spaces that are spanned by their hermitian elements; see [13, Proposition 3.2]. An operator system X inherits from $B(H)$ a notion of positivity for self-adjoint elements of $M_n(X)$. Operator systems can be equivalently characterized in terms of the $*$-vector space structure together with the unit and the matricial positive cones; see [19, Section 2] and [107, Chapter 13]. A linear map between operator systems is *positive* if it maps positive elements to positive elements, and *completely positive* if all its amplifications are positive. A unital linear map between operator systems is completely positive if and only if it is completely contractive [12, 1.3.3], and in such a case it is automatically self-adjoint. A surjective unital complete isometry between operator systems is called a *complete order isomorphism*.

To every unital operator space $X \subset B(H)$ one can canonically assign the operator system $X + X^{\star} = \operatorname{span}\{x, x^{*} : x \in X\}$. Such an operator system does not depend (up to complete order isomorphism) from the unital completely isometric realization of X as a subspace of $B(H)$; see [12, 1.3.7]. Moreover any unital completely contractive (respectively completely isometric) linear map $\phi : X \to Y$ between unital operator systems has a unique extension to a map $\widetilde{\phi} : X + X^{\star} \to Y + Y^{\star}$ with the same properties. Therefore in some sense there is no real loss of generality in only considering operator systems rather than arbitrary unital operator spaces.

In this chapter we consider the natural operator system analog the notion of operator space of universal disposition. This is obtained by replacing 1-exact operator spaces with 1-exact operator systems, and considering *unital* linear maps instead of arbitrary linear maps. (An operator system is 1-exact if it is 1-exact as an operator space; see [70, Section 5] for equivalent characterizations.) Therefore we say that a separable 1-exact operator system $\mathbb{GS}$ to is of almost universal disposition if whenever $E \subset F$ are finite-dimensional 1-exact operator systems, $\phi : E \to \mathbb{GS}$ is a unital complete isometry, and $\varepsilon > 0$, there is an extension $\widehat{\phi} : F \to \mathbb{GS}$ of ϕ such that $\left\|\widehat{\phi}\right\|_{cb}\left\|\widehat{\phi}^{-1}\right\|_{cb} < 1+\varepsilon$.

In this chapter we prove that the class $\mathcal{E}_1^{sy}$ of finite-dimensional 1-exact operator systems is a Fraïssé class. Moreover an operator system is of almost universal disposition if and only if it is a limit of $\mathcal{E}_1^{sy}$. As a consequence we conclude that there exist a unique (up to complete order isomorphism) operator system $\mathbb{GS}$ of almost universal disposition, which we call the Gurarij operator system. Furthermore any separable 1-exact operator system admits a unital completely isometric embedding into $\mathbb{GS}$. The homogeneity property of $\mathbb{GS}$ asserts that for any unital complete isometry $\phi : E \to F$ between finite-dimensional subspaces of $\mathbb{GS}$ and any $\varepsilon > 0$ there is a complete order automorphism α of $\mathbb{GS}$ such that $\left\|\alpha_{|E} - \phi\right\|_{cb} < 1+\varepsilon$. The Gurarij operator system is nuclear in the sense of [57, Theorem 3.1], and in fact it is an inductive limit of full matrix algebras with unital completely isometric connecting maps. In particular $\mathbb{GS}$ is a C*-system in the sense of [79], i.e. the second dual $\mathbb{GS}^{**}$ of $\mathbb{GS}$ is a C*-algebra and the canonical embedding of $\mathbb{GS}$ into $\mathbb{GS}^{**}$ is unital and completely isometric. Finally show that $\mathbb{GS}$ is a universal C*-system as defined in [79, Section 3]. This means that the canonical $*$-homomorphism from the universal C*-algebra $C_u^*(\mathbb{GS})$ to the C*-envelope $C_e^*(\mathbb{GS})$ is a $*$-isomorphism.

The rest of the chapter is organized as follows. In Section 4.1 we recall some facts about Fraïssé limits for metric structures and about operator systems. Section 4.2 contains the proof that finite-dimensional 1-exact operator systems form a Fraïssé class. Finally Section 4.3 contains the proof of the main result, charactering the Gurarij operator systems $\mathbb{GS}$ as the Fraïssé limit of finite-dimensional 1-exact operator systems.

4.1 Preliminary notions

4.1.1 Operator systems

An *operator system* is a unital operator space X such that there exists a complete isometry $\phi : X \to B(H)$ mapping the distinguished unitary to the identity operator and mapping X onto a self-adjoint subspace of $B(H)$. Operator systems are precisely the unital operator space that are spanned by their hermitian elements; see [13, Definition 3.1 and Proposition

3.2]. This provides an intrinsic characterization of operator systems among unital operator spaces.

A unital complete isometry $\phi : X \to B(H)$ induces an involution $x \mapsto x^*$ on X, as well as a notion of positivity in $M_n(X)$ for every $n \in \mathbb{N}$. We will regard operator systems as structures in the language $\mathcal{L}_{OSy}$ of unital operator spaces with a unitary function symbol for the involution. The intrinsic characterization [13, Proposition 3.2] shows that an $\mathcal{L}_{OSy}$-structure is an operator system if and only if all its finitely generated substructures are operator systems.

An earlier and more commonly used abstract description of operator spaces is due to Choi and Effros [19]. Such a description only involves the unit, the involution, and the positive cones in all the matrix amplifications. The operator space structure is completely recovered by the relation

$$\|x\|_{M_n(X)} \leq 1 \quad \text{iff} \quad \begin{bmatrix} I_n & x \\ x^* & I_n \end{bmatrix} \in M_{2n}(X) \text{ is positive.}$$

Suppose that X and Y are operator systems. A map $\phi : X \to Y$ is *unital* provided that it maps the unit to the unit, and *self-adjoint* if $\phi(x^*) = \phi(x)^*$ for every $x \in X$. The positive cones in X and Y also define a notion of positivity for maps. Thus $\phi : X \to Y$ is *positive* if it maps positive elements to positive elements, and *completely positive* if its amplifications $\phi^{(n)}$ are positive for every $n \in \mathbb{N}$. Every positive map is automatically self-adjoint. It is well known that a unital linear map between operator systems is completely positive if and only if it is completely contractive; see [12, 1.3.3]. In the following we will abbreviate "unital completely positive" with *ucp*, as it is customary. We will often tacitly use Arveson's extension theorem, asserting that $B(H)$ is an injective element in the category of operator systems and with ucp maps as morphisms; see [107, Theorem 7.5].

Definition 4.1.1. An operator space is *matricial* if it admits a completely isometric embedding into M_n for some $n \in \mathbb{N}$. An operator system is matricial if it admits a complete order embedding into M_n for some $n \in \mathbb{N}$.

The theory of injective envelopes of operator spaces has been developed independently by Hamana [55, 54] and Ruan [124]. It follows from the main theorem of [130] that an operator space is matricial if and only if it its injective envelope is finite-dimensional. Moreover by [12, Corollary 4.2.8] an operator system is matricial if and only if it is matricial as an operator space.

4.1.2 Exactness

If $X \subset B(H)$ and $Y \subset B(H)$ are operator spaces, then their minimal tensor product $X \otimes_{\min} Y$ is the completion of the algebraic tensor product with respect to the norm induced by the inclusion $X \otimes Y \subset B(H) \otimes B(H) \subset B(H \overline{\otimes} H)$ where $H \overline{\otimes} H$ denotes the Hilbertian tensor product. The minimal tensor product is canonically endowed with an operator space structure induced by the inclusion $X \otimes_{\min} Y \subset B(H \overline{\otimes} H)$.

An operator space X 1-*exact* whenever for every C*-algebra B and ideal I of B the sequence

$$0 \to I \otimes_{\min} X \to B \otimes_{\min} X \to B/I \otimes_{\min} X \to 0$$

is exact, and moreover the induced isomorphisms

$$\frac{B \otimes_{\min} X}{I \otimes_{\min} X} \to B/I \otimes_{\min} X$$

is a complete isometry.

If E and F are operator spaces, the *completely bounded distance* $d_{cb}(E, F)$ of E and F is the infimum of $\|\phi\|_{cb} \left\|\phi^{-1}\right\|_{cb}$ when ϕ ranges among all isomorphisms from E to F. Theorem 1 of [114] provides the following equivalent reformulation of 1-exactness in terms of the completely bounded distance. An operator space X is 1-exact if and only if for every finite dimensional subspace E of X and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and a subspace F of M_n such that $d_{cb}(E, F) \leq 1 + \varepsilon$.

Suppose now that X, Y are operator systems, B is a *unital* C*-algebra, and I is an ideal of B. The minimal tensor product $X \otimes_{\min} Y$ is an operator system obtained as before as a completion of the algebraic tensor product via the inclusion $X \otimes Y \subset B(H \overline{\otimes} H)$. It is shown in [70] that the quotient

$$\frac{B \otimes_{\min} X}{I \otimes_{\min} X}$$

is canonically endowed with an operator system structure. An operator system X is 1-*exact* [70, Definition 5.4] if for every *unital* C*-algebra B and every ideal I of B the sequence

$$0 \to I \otimes_{\min} X \to B \otimes_{\min} X \to B/I \otimes_{\min} X \to 0$$

is exact, and moreover the induced (unital) isomorphism

$$\frac{B \otimes_{\min} X}{I \otimes_{\min} X} \to B/I \otimes_{\min} X$$

is a complete isometry. It is then shown in [70, Proposition 5.5] that an operator system X is 1-exact if and only if it is 1-exact as an operator space. Moreover the same proof as [114, Theorem 1] shows that if X is a 1-exact operator system then for every finite subset E of X and for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$, a subspace F of M_n, and a ucp invertible map $\phi : E \to F$ such that $\left\|\phi^{-1}\right\|_{cb} < 1 + \varepsilon$. Summarizing we can list the following equivalent characterizations of 1-exactness for operator spaces.

Proposition 4.1.2 (Kavruk-Paulsen-Todorov-Tomforde, Pisier)**.** *Suppose that E is a finite-dimensional operator system. The following statements are equivalent:*

1. *E is* 1*-exact;*
2. *for every $\varepsilon > 0$ there is a matricial operator space F such that $d_{cb}(E, F) < 1 + \varepsilon$;*
3. *for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and an injective ucp map $\phi : E \to M_n$ such that $\left\|\phi^{-1}\right\|_{cb} < 1 + \varepsilon$;*

An operator system X is 1*-exact if and only if all its finite-dimensional subspaces are* 1*-exact.*

The implications $3 \Rightarrow 2$ in Proposition 4.1.2 is obvious. The implication $2 \Rightarrow 1$ follows from [114, Theorem 1] and [70, Proposition 5.5]. Finally the implication $1 \Rightarrow 3$ follows from the proof of [114, Theorem 1].

4.1.3 Pushouts of operator systems

A *$*$-vector space* is a complex vector space V endowed with a conjugate linear involutive map $v \mapsto v^*$ from V to V. A unital $*$-vector space is a $*$-vector space V endowed with a distinguished element 1_V such that $(1_V)^* = 1_V$. Clearly any operator system is in particular a unital $*$-vector space. A map $\phi : V \to B(H)$ is unital and self-adjoint if $\phi(1_V)$ is the identity operator and $\phi(v^*) = \phi(v)^*$ for every $v \in V$.

Suppose that $\mathcal{P}$ is a collection of unital self-adjoint maps $\phi : V \to B(H_\phi)$ such that $\sup_\phi \|\phi(x)\| < +\infty$ for every $x \in V$. Let $J_\mathcal{P}$ be the subspace

$$\bigcap_{\phi \in \mathcal{P}} \operatorname{Ker}(\phi)$$

of V. Define on $V / J_\mathcal{P}$ the norm

$$\|x + J_\mathcal{P}\| = \sup_{\phi \in \mathcal{P}} \|\phi(x)\|_{B(H_\phi)}.$$

Then the completion of $V_{\mathcal{P}}$ of $V/J_{\mathcal{P}}$ is an operator system with unit $1_V + J_{\mathcal{P}}$ and matrix norms

$$\|[x_{ij} + J_{\mathcal{P}}]\| = \sup_{\phi \in \mathcal{P}} \|\phi(x_{ij})\|_{B(H_\phi)}.$$

Equivalently $V_{\mathcal{P}}$ can be defined as the closure inside $\oplus_\phi B(H_\phi) \subset B(\oplus_\phi H_\phi)$ of the image of V under the map

$$v \mapsto (\phi(v))_{\phi \in \mathcal{P}}.$$

By definition any element ϕ of $\mathcal{P}$ induces a ucp map from $V_{\mathcal{P}}$ to $B(H_\phi)$.

We use this construction to define *pushouts* in the category of operator systems with ucp maps. Suppose that Z, X, Y are operator systems and $\alpha_X : Z \to X$ and $\alpha_Y : Z \to Y$ are ucp maps. Let V be the quotient of the algebraic direct sum $X \oplus Y$ by the subspace $N = \operatorname{span}\{(-1_X, 1_V)\}$. Consider the collection $\mathcal{P}$ of unital self-adjoint maps $\phi : V \to B(H)$ of the form

$$(x, y) + N \mapsto \phi_X(x) + \phi_Y(y)$$

where ϕ_X and ϕ_Y are ucp maps on X and Y such that $\phi_X \circ \alpha_X = \phi_Y \circ \alpha_Y$. Let W be the operator system associated with the collection $\mathcal{P}$ as in the paragraph above. The canonical morphisms $\psi_X : X \to W$ and $\psi_Y : Y \to W$ are obtained from the first and second coordinate inclusions of X and Y into $(X \oplus Y)/N$. When moreover the maps $\alpha_X : Z \to X$ and $\alpha_Y : Z \to Y$ are unital complete isometries then $\psi_X : X \to W$ and $\psi_Y : Y \to W$ are unital complete isometries. It follows easily from the definition that this construction is indeed a pushout in the category of operator systems with ucp maps as morphisms.

As shown in [52, Appendix B] operator systems can be regarded as structures in a suitable language in the logic for metric structures as defined in [35, Section 2]; see also [9]. In particular if $(X_n)_{n \in \mathbb{N}}$ is a sequence of operator systems and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, then one can define the ultraproduct $\prod_{\mathcal{U}} X_n$ of the X_n's in such a language with respect to the ultrafilter $\mathcal{U}$. The ultraproduct can also be explicitly constructed as follows. Define

$$\ell^\infty(X_n) := \left\{ (x_n) \in \prod_n X_n : \sup_n \|x_n\| < +\infty \right\}.$$

Such a space has a natural unital $*$-vector space structure with unit $(1_{X_n})_{n \in \mathbb{N}}$ and involution $(x_n)^* = (x_n^*)$. Let then $\mathcal{P}_{\mathcal{U}}$ be the collection of ucp maps of $\ell^\infty(X_n)$ of the form

$$\begin{array}{rcl} \ell^\infty(X_n) & \to & \prod_{\mathcal{U}} B(H_{\phi_n}) \\ (x_n) & \mapsto & [\phi_n(x_n)], \end{array}$$

where $\phi_n : X_n \to B(H_{\phi_n})$ is a ucp map and $\prod_{\mathcal{U}} B(H_{\phi_n})$ denote the C*-algebraic ultraproduct; see [11, II.8.1.7]. The ultraproduct $\prod_{\mathcal{U}} X_n$ is the operator system obtained from the collection ucp maps $\mathcal{P}_{\mathcal{U}}$ on the unital $*$-vector space $\ell^\infty(X_n)$ as at the beginning of this subsection.

We recall that the category of operator systems also admits inductive limits, which can be defined as in [77, Section 2]; see also the proof of Proposition 16 in [79].

4.1.4 M_n-systems

Here we recall the definition of M_n-spaces and the functor MIN_n defined and studied in [86]; see also [105, Section 2] and [104, Subsection 2.1]. Let X be an operator space and $n \in \mathbb{N}$. The operator space $\mathrm{MIN}_n(X)$ has the same underlying vector space as X with matrix norms

$$\|x\|_{M_k(\mathrm{MIN}_n(X))} = \sup_{\phi} \left\| \phi^{(k)}(x) \right\|_{M_k(M_n)}$$

where ϕ ranges over all completely contractive maps from X to M_n. This defines an operator space structure on X such that the identity map $id_X : X \to \mathrm{MIN}_n(X)$ is completely contractive. Such a space is characterized by the following property. If Z is an operator space and $\phi : Z \to X$ is a linear map, then $\phi : Z \to \mathrm{MIN}_n(X)$ is completely bounded if and only if $\phi^{(n)} : M_n(Z) \to M_n(X)$ is bounded, and in such a case

$$\|\phi : Z \to \mathrm{MIN}_n(X)\|_{cb} = \left\| \phi^{(n)} : M_n(Z) \to M_n(X) \right\|.$$

This is a consequence of Smith's lemma [129, Theorem 2.10].

An operator space X is called an M_n-space provided that $id_X : \mathrm{MIN}_n(X) \to X$ is a complete isometry. It follows easily from the above mentioned property that M_n-spaces form a full subcategory of the category of operator spaces. Moreover the inclusion functor implements an equivalence of categories with inverse the functor MIN_n. Assume now that E is a finite-dimensional operator space. Then E is 1-exact if and only if for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

$$\|id_X : \mathrm{MIN}_n(E) \to E\|_{cb} \leq 1 + \varepsilon;$$

see [105, Lemma 2.2]. In particular all M_n-spaces are 1-exact.

The natural analogs of the above notions in the category of operator systems have been defined in [136]. Suppose that X is an operator system. Then $\mathrm{OMIN}_n(X)$ is the operator system having the same unital $*$-vector space structure as X and matrix norms

$$\|x\|_{M_k(\mathrm{OMIN}_n(X))} = \sup_{\phi} \left\| \phi^{(k)}(x) \right\|_{M_k(M_n)}$$

where ϕ ranges over all ucp maps from X to M_n. This is a particular case of the construction of operator systems presented in Subsection 4.1.3 where $\mathcal{P}$ is the collection of ucp maps from X to M_n. It is shown in [136] that OMIN_n has analogous properties as MIN_n where one replaces operator spaces with operator systems and (completely) contractive maps with unital (completely) positive maps. In the proofs the use of Smith's lemma for operator spaces [26, Proposition 2.2.2] is replaced by [107, Theorem 6.1]. It follows from the proof of the abstract characterization of operator systems [107, Theorem 13.1] that X and $\mathrm{OMIN}_n(X)$ have the same matricial positive cones up to the n-th amplification. Moreover since a unital 2-positive map is contractive, $\mathrm{OMIN}^u_{2n}(X)$ and X are n-isometric.

It follows from Proposition 4.1.2 and the properties of OMIN_n that a finite-dimensional operator system E is 1-exact if and only if for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

$$\|id_E : \mathrm{OMIN}_n(E) \to E\|_{cb} \leq 1 + \varepsilon.$$

Definition 4.1.3. An M_n-system is an operator system X such that $id_X : \mathrm{OMIN}_n(X) \to X$ is a complete isometry.

Equivalently X is an M_n-system if and only if it admits a complete order embedding into $C(K, M_n)$ for some compact Hausdorff space K. As before M_n-systems form a full subcategory of the category of operator systems with ucp maps as morphisms, and the inclusion functor is an equivalence of categories with inverse OMIN_n. This readily implies in view of Subsection 4.1.3 that the category of M_n-systems has pushouts. Moreover such a pushout can be obtained by applying OMIN_n to the pushout in the category of operator systems. In the following we will tacitly use the fact that every M_n-system can be approximated by a matricial M_n-system. In fact if X is an M_n-system and $\varepsilon > 0$ then there is a ucp map $\phi : X \to M_n \oplus^{\infty} \cdots \oplus^{\infty} M_n$ such that $\left\|\phi^{-1}\right\|_{cb} < 1 + \varepsilon$. It is a consequence of [107, Theorem 6.1] that a unital linear map between operator systems is completely contractive if and only if it is n-contractive if and only if its n-th amplification is positive. Finally we observe that the ultraproduct $\prod_{\mathcal{U}} C(K_i, M_n)$ can be identified with $C(T, M_n)$ where T is the compact Hausdorff space $\sum_{\mathcal{U}} K_i$ defined as in [6]. It follows that the class of M_n-systems is closed under ultraproducts.

4.2 Finite-dimensional 1-exact operator systems

We regard operator systems as structures in the language $\mathcal{L}_{OSy}$ defined in Subsection 4.1.1. We denote by $\mathcal{E}_1^{sy}$ the class of finite-dimensional 1-exact operator systems. It follows easily from the characterization of 1-exactness provided by Proposition 4.1.2 together with the abstract characterization of operator systems among unital operator spaces [13, Proposition 3.2] that the $\mathcal{E}_1^{sy}$-structures in the sense of Subsection 3.1.3 are precisely the 1-exact operator systems. We define also $\mathcal{M}_\infty^0 \subset \mathcal{E}_1^{sy}$ to be the class of matricial operator systems in the sense of Definition 4.1.1. In this section we will show that $\mathcal{E}_1^{sy}$ is a Fraïssé class, and $\mathcal{M}_\infty^0$ is a dense subclass.

4.2.1 Amalgamation of 1-exact operator systems

In this subsection we will show that the class $\mathcal{E}_1^{sy}$ of finite-dimensional 1-exact operator systems satisfies (NAP) from Definition 3.1.2. We will start by considering matricial operator systems. If X, Y are operator systems and $f : X \to Y$ is a linear map, define $f^* : X \to Y$ by setting

$$f^*(x) = f(x^*)^*.$$

Define then $\mathrm{Re}(f) = \frac{1}{2}(f + f^*)$ and $\mathrm{Im}(f) = \frac{1}{2i}(f - f^*)$.

The following lemma is an approximate version of [107, Lemma 2.10].

Lemma 4.2.1. *Suppose that X is an operator system, $\delta \in (0,1]$, and ϕ is a unital linear functional on X such that $\|\phi\| \leq 1 + \delta$ for $\delta \in [0,1]$. Then $|\mathrm{Im}(\phi(x))| \leq 2\sqrt{\delta}\,\|x\|$ whenever $x \in X$ is self-adjoint.*

Proof. Suppose that $x \in X$ is self-adjoint and $\|x\| \leq 1$. Denote by $\sigma(x)$ the spectrum of x. If $M > 0$ then $\sigma(x)$ is contained in the disc of center iM and radius $\left(M^2+1\right)^{\frac{1}{2}}$. Therefore $\sigma(x - iM)$ is contained in the disc of center 0 and radius $\left(M^2+1\right)^{\frac{1}{2}}$. Thus

$$\|\phi(x) - iM\| \leq (1+\delta)\|x - iM\| \leq (1+\delta)\left(M^2+1\right)^{\frac{1}{2}}.$$

Therefore

$$\mathrm{Im}(\phi(x)) \geq M - (1+\delta)\left(M^2+1\right)^{\frac{1}{2}}.$$

Observe that, for $M \geq 0$,

$$\begin{aligned}(1+\delta)\left(M^2+1\right)^{\frac{1}{2}}-x &= (1+\delta)\,M\left(1+\frac{1}{M^2}\right)^{\frac{1}{2}}-M \\ &\leq (1+\delta)\,M\left(1+\frac{1}{2M^2}\right)-M \\ &\leq \frac{1}{M^2}+\delta M.\end{aligned}$$

Setting $M=\frac{1}{\sqrt{\delta}}$, this yields $\mathrm{Im}\left(\phi\left(x\right)\right)\geq -2\sqrt{\delta}$. A similar argument shows that $\mathrm{Im}\left(\phi\left(x\right)\right)\leq 2\sqrt{\delta}$. □

Lemma 4.2.2. *Suppose that X,Y are operator systems, $\delta\in(0,1]$, and $f:X\to Y$ is a unital linear map such that $\|f\|_{cb}\leq 1+\delta$. Then $\|\mathrm{Im}\left(f\right)\|_{cb}\leq 4\sqrt{\delta}$.*

Proof. Suppose that $n\in\mathbb{N}$ and ρ is a state of $M_n\left(Y\right)$. Applying Lemma 4.2.1 to $\phi=\rho\circ f^{(n)}$ one obtains that, whenever $x\in M_n\left(X\right)$ is self-adjoint,

$$\left|\left(\rho\circ\mathrm{Im}\left(f\right)^{(n)}\right)(x)\right|=\left|\mathrm{Im}\left(\left(\rho\circ f^{(n)}\right)(x)\right)\right|\leq 2\sqrt{\delta}.$$

Therefore $\left\|\mathrm{Im}\left(f\right)^{(n)}\right\|\leq 4\sqrt{\delta}$, and $\|\mathrm{Im}\left(f\right)\|_{cb}\leq 4\sqrt{\delta}$. □

We will need the following perturbation lemma, which is a minor variation of [23, Theorem 2.5].

Lemma 4.2.3. *Suppose that E is a finite-dimensional operator system, Y is an operator system, and $\delta\in[0,1]$. Denote by $\dim\left(E\right)$ the dimension of E as a vector space. If $f:E\to Y$ is unital linear map such that $\|f\|_{cb}\leq 1+\delta$, then there is a ucp map $\psi:E\to Y$ such that $\|f-\psi\|_{cb}\leq 20\left(\dim\left(E\right)+1\right)\sqrt{\delta}$.*

Proof. For $\delta=0$ this follows from [107, Lemma 2.10]. Suppose now that δ is nonzero. Consider Y as a subsystem of $B\left(H\right)$, and let n be $\dim\left(E\right)$. By Lemma 4.2.2 we have that $\|\mathrm{Im}\left(f\right)\|_{cb}\leq 4\sqrt{\delta}$. By Wittstock's decomposition theorem [135]—see also [108, Corollary 2.6]—there are completely positive maps $\phi_1,\phi_2:E\to B\left(H\right)$ such that $\mathrm{Re}\left(f\right)=\phi_1-\phi_2$ and $\|\phi_1+\phi_2\|_{cb}\leq\|\mathrm{Re}\left(f\right)\|_{cb}$. Let $a_i=\phi_i\left(1\right)$ for $i=1,2$. By [107, Proposition 3.6] we have that

$$\|a_1\|\leq\|a_1+a_2\|=\|\phi_1+\phi_2\|_{cb}\leq\|\mathrm{Re}\left(f\right)\|_{cb}\leq 1+\delta+4\sqrt{\delta}.$$

If $\xi \in H$ has norm 1, then

$$\langle a_2\xi, \xi\rangle = \langle (a_1 - 1)\,\xi, \xi\rangle \leq \delta + 4\sqrt{\delta} \leq 5\sqrt{\delta}.$$

Therefore $\|\phi_2\|_{cb} = \|a_2\| \leq 5\sqrt{\delta}$. By [23, Lemma 2.4] there is a linear functional θ on E—which we will regard as a function from E to Y—such that $\|\theta\| \leq 5n\sqrt{\delta}$ and $\theta - \phi_2$ is completely positive. Define $\psi' = \mathrm{Re}\,(f) + \theta = \phi_1 - \phi_2 + \theta$ and observe that ψ' is completely positive, being sum of completely positive maps. Moreover $\|\psi' - f\|_{cb} \leq \|\mathrm{Im}\,(f)\|_{cb} + \|\theta\| \leq 4\sqrt{\delta} + 5n\sqrt{\delta}$ and $\|\psi'\|_{cb} \leq 1 + 5\,(n+1)\sqrt{\delta}$. Set $\psi'' = \frac{1}{1+5(n+1)\sqrt{\delta}}\psi'$ and observe that ψ'' is completely positive and completely contractive. Moreover

$$\left\|\psi''(1) - 1\right\| \leq \left\|\psi''(1) - \psi'(1)\right\| + \left\|\psi'(1) - f(1)\right\| \leq 10\,(n+1)\sqrt{\delta}.$$

Let ρ be a state on E and set $\psi(x) = \psi''(x) + \rho(x)\,(1 - \psi''(1))$. Observe that ψ is completely positive being sum of completely positive maps. Moreover ψ is unital and

$$\begin{aligned} \|\psi - f\|_{cb} &\leq \left\|1 - \psi''(1)\right\| + \left\|\psi' - f\right\|_{cb} + \left\|\psi' - \psi''\right\|_{cb} \\ &\leq 20\,(n+1)\sqrt{\delta}. \quad \square \end{aligned}$$

Lemma 4.2.4. *Suppose that $E \subset F_0$ and F_1 are matricial operator systems, and $\delta \in [0,1]$. If $f : E \to F_1$ is an invertible unital map such that $\|f\|_{cb} \leq 1 + \delta$ and $\left\|f^{-1}\right\|_{cb} \leq 1 + \delta$, then there exist $d \in \mathbb{N}$ and unital completely isometric embeddings $i : F_0 \to M_d$ and $j : F_1 \to M_d$ such that $\left\|j \circ f - i_{|E}\right\|_{cb} \leq 100 \dim(E)\,\delta^{\frac{1}{2}}$.*

Proof. Without loss of generality we can assume that $F_0 = M_n$ and $F_1 = M_k$ for some $n, k \in \mathbb{N}$. By Lemma 4.2.3 and injectivity of M_k there is a ucp map $\phi : M_n \to M_k$ such that $\left\|\phi_{|E} - f\right\|_{cb} \leq 50 \dim(E)\,\delta^{\frac{1}{2}}$. Similarly there is a ucp map $\psi : M_k \to M_n$ such that $\left\|\psi_{|f[E]} - f^{-1}\right\|_{cb} \leq 50 \dim(E)\,\delta^{\frac{1}{2}}$ and hence $\|\psi \circ f - id_E\| \leq 100 \dim(E)\,\delta^{\frac{1}{2}}$. Set $d = n + k$ and define maps $i : M_n \to M_d$ and $j : M_k \to M_d$ by

$$i(x) = \begin{bmatrix} x & 0 \\ 0 & \phi(x) \end{bmatrix} \quad \text{and} \quad j(y) = \begin{bmatrix} \psi(y) & 0 \\ 0 & y \end{bmatrix}$$

for $x \in M_n$ and $y \in M_k$. Observe that i and j are unital complete isometries such that

$$\left\|j \circ f - i_{|E}\right\|_{cb} = \max\left\{\|\psi \circ f - id_E\|_{cb}, \left\|f - \phi_{|E}\right\|_{cb}\right\} \leq 100 \dim(E)\,\delta^{\frac{1}{2}}.$$

$\square$

We are now ready to bootstrap Lemma 4.2.4 to arbitrary finite-dimensional 1-exact operator systems. The proof is analogous to the proof of [90, Proposition 4.2].

Proposition 4.2.5. *Suppose that $E \subset X$ and Y are finite-dimensional 1-exact operator systems, and $\delta \in [0,1]$. If $f : E \to Y$ is a unital map such that $\|f\|_{cb} < 1+\delta$ and $\left\|f^{-1}\right\|_{cb} < 1+\delta$, then there exist a separable 1-exact operator system Z and unital completely isometric embeddings $i : X \to Z$ and $j : Y \to Z$ such that $\|j \circ f - i\|_{cb} < 100 \dim(E)\, \delta^{\frac{1}{2}}$.*

Proof. Fix $\varepsilon, \delta' \in (0,1)$ such that $\|f\|_{cb} < 1+\delta'$, $\left\|f^{-1}\right\|_{cb} < 1+\delta'$, and $\delta' + 2\varepsilon < \delta$. Set $\eta = \left(\frac{\varepsilon}{100\dim(E)}\right)^2$. We will define by recursion on k sequences $(n_k)_{k\in\mathbb{N}}$, $(Z_k)_{k\in\mathbb{N}}$, $(i_k)_{k\in\mathbb{N}}$, $(j_k)_{k\in\mathbb{N}}$, and $(\phi_k)_{k\in\mathbb{N}}$ such that, for every $k \in \mathbb{N}$,

1. Z_k is an M_{n_k}-system,
2. $i_k : X \to Z_k$ is a ucp map such that $\left\|i_k^{-1}\right\|_{cb} \le 1 + \eta 2^{-2k}$,
3. $j_k : Y \to Z_k$ is a ucp map such that $\left\|j_k^{-1}\right\|_{cb} \le 1 + \eta 2^{-2k}$ and
$$\left\|j_k \circ f - (i_k)_{|E}\right\|_{cb} \le 100 \dim(E)\, \delta^{\frac{1}{2}} + 2\varepsilon \left(\sum_{i<k} 2^{-i}\right),$$
4. $\phi_k : Z_k \to Z_{k+1}$ is a ucp map such that $\left\|\phi_k^{-1}\right\| \le 1 + \eta 2^{-4k}$,
$$\|\phi_k \circ i_k - i_{k+1}\|_{cb} \le \varepsilon 2^{-k} \quad \text{and} \quad \|\phi_k \circ j_k - j_{k+1}\|_{cb} \le \varepsilon 2^{-k}.$$

Granted the construction we can let Z be the inductive limit $\lim_{(\phi_k)} Z_k$ and define the maps
$$i = \lim_k i_k : X \to Z \quad \text{and} \quad j = \lim_k j_k : Y \to Z.$$
It is immediate to verify that conditions (1)–(4) above guarantee that these maps have the desired properties. We will now present the recursive construction. By Lemma 4.2.4 and Proposition 4.1.2 one can find $n_1 \in \mathbb{N}$ and ucp maps $i_1 : X \to M_{n_1}$ and $j_1 : Y \to M_{n_1}$ that satisfy conditions (1)–(3) above for $k = 1$ and $Z_1 = M_{n_1}$. Suppose that n_k, Z_k, i_k, j_k, and ϕ_{k-1} have been defined for $k \le m$. By Proposition 4.1.2 there exist $d \in \mathbb{N}$ and ucp maps $\theta_X : X \to M_d$ and $\theta_Y : Y \to M_d$ such that
$$\max\left\{\left\|\theta_X^{-1}\right\|_{cb}, \left\|\theta_Y^{-1}\right\|_{cb}\right\} \le 1 + \eta 2^{-2(m+1)}.$$

Moreover by Lemma 4.2.3 there are ucp maps $\alpha_X : i_m[X] \subset Z_m \to M_d$ and $\alpha_Y : i_m[Y] \subset Z_m \to M_d$ such that

$$\max\left\{\left\|\alpha_X - \theta_X \circ i_m^{-1}\right\|_{cb}, \left\|\alpha_Y - \theta_Y \circ j_m^{-1}\right\|_{cb}\right\} \leq \varepsilon 2^{-m}.$$

Injectivity of M_d ensures that we can extend α_X and α_Y to ucp maps defined on all Z_m. Set now $n_{m+1} = n_m + d$. Define maps

$$\begin{array}{rccl} \widehat{\theta}_X : & X & \to & M_{n_{m+1}} \\ & x & \mapsto & \begin{bmatrix} i_m(x) & 0 \\ 0 & \theta_X(x) \end{bmatrix}, \end{array}$$

$$\begin{array}{rccl} \widehat{\theta}_Y : & Y & \to & M_{n_{m+1}} \\ & y & \mapsto & \begin{bmatrix} j_m(y) & 0 \\ 0 & \theta_Y(y) \end{bmatrix}, \end{array}$$

and

$$\begin{array}{rccl} \widehat{\alpha}_X : & Z_m & \to & M_{n_{m+1}} \\ & z & \mapsto & \begin{bmatrix} z & 0 \\ 0 & \alpha_X(z) \end{bmatrix}, \end{array}$$

$$\begin{array}{rccl} \widehat{\alpha}_Y : & Z_m & \to & M_{n_{m+1}} \\ & z & \mapsto & \begin{bmatrix} z & 0 \\ 0 & \alpha_Y(z) \end{bmatrix}. \end{array}$$

Observe that $\widehat{\alpha}_X$ and $\widehat{\alpha}_Y$ are unital complete isometries, while $\widehat{\theta}_X$ and $\widehat{\theta}_Y$ are ucp maps such that

$$\max\left\{\left\|\widehat{\theta}_X^{-1}\right\|_{cb}, \left\|\widehat{\theta}_Y^{-1}\right\|_{cb}\right\} \leq \max\left\{\left\|\theta_X^{-1}\right\|_{cb}, \left\|\theta_Y^{-1}\right\|_{cb}\right\} \leq 1 + \eta 2^{-4(m+1)}.$$

Moreover

$$\begin{array}{rl} & \max\left\{\left\|\widehat{\theta}_X - \widehat{\alpha}_X \circ i_m\right\|_{cb}, \left\|\widehat{\theta}_Y - \widehat{\alpha}_Y \circ j_m\right\|_{cb}\right\} \\ \leq & \max\left\{\left\|\alpha_X - \theta_X \circ i_m^{-1}\right\|_{cb}, \left\|\alpha_Y - \theta_Y \circ j_m^{-1}\right\|_{cb}\right\} \leq \varepsilon 2^{-m}. \end{array}$$

Now let Z_{m+1} be the pushout of $\widehat{\alpha}_X : Z_m \to M_{n_{m+1}}$ and $\widehat{\alpha}_Y : Z_m \to M_{n_{m+1}}$ in the category of $M_{n_{m+1}}$-systems with canonical unital complete isometries $\psi_X : M_{n_{m+1}} \to Z_{m+1}$ and $\psi_Y :$

$M_{n_{m+1}} \to Z_{m+1}$; see Subsection 4.1.3 and Subsection 4.1.4. Set $\phi_m := \psi_X \circ \widehat{\alpha}_X = \psi_Y \circ \widehat{\alpha}_Y$, $i_{m+1} := \psi_X \circ \widehat{\theta}_X$, and $j_{m+1} := \psi_Y \circ \widehat{\theta}_Y$. Observe that

$$\max\left\{\left\|i_{m+1}^{-1}\right\|_{cb}, \left\|j_{m+1}^{-1}\right\|_{cb}\right\} \leq \max\left\{\left\|\widehat{\theta}_X^{-1}\right\|_{cb}, \left\|\widehat{\theta}_Y^{-1}\right\|_{cb}\right\} \leq 1 + \eta 2^{-4(m+1)}.$$

Moreover

$$\begin{aligned}
\left\|\phi \circ i_m - i_{m+1}\right\|_{cb} &= \left\|\phi \circ i_m - \psi_X \circ \widehat{\theta}_X\right\|_{cb} \\
&\leq \left\|\phi \circ i_m - \psi_X \circ \widehat{\alpha}_X \circ i_m\right\|_{cb} + \varepsilon 2^{-m} \\
&\leq \varepsilon 2^{-m}
\end{aligned}$$

and similarly $\left\|\phi \circ j_m - j_{m+1}\right\|_{cb} \leq \varepsilon 2^{-m}$. Furthermore

$$\begin{aligned}
\left\|\left(i_{m+1}\right)_{|E} - j_{m+1} \circ f\right\|_{cb} &= \left\|\left(\psi_X \circ \widehat{\theta}_X\right)_{|E} - \psi_Y \circ \widehat{\theta}_Y \circ f\right\|_{cb} \\
&\leq \left\|\psi_X \circ \widehat{\alpha}_X \circ i_m - \psi_Y \circ \widehat{\alpha}_Y \circ j_m \circ f\right\| + \varepsilon 2^{-m+1} \\
&\leq \left\|\phi_m \circ i_m - \phi_m \circ j_m \circ f\right\| + \varepsilon 2^{-m+1} \\
&\leq 100 \dim(E)\, \delta^{\frac{1}{2}} + 2\varepsilon \sum_{k \leq m} 2^{-k}.
\end{aligned}$$

This concludes the recursive construction. $\square$

It is not difficult to modify the proof of Proposition 4.2.5 to cover the case when X and Y are not necessarily finite-dimensional.

4.2.2 Embeddings of 1-exact operator systems

The goal of this section is to show that every 1-exact separable operator system admits a unital completely isometric embedding into an inductive limit of full matrix algebras with unital completely isometric connective maps. The construction of the inductive limit of an inductive sequence of operator systems can be found in [79, Section 2]; see also the proof of Proposition 16 in [77]. The proof of the following proposition is similar to the one of [24, Theorem 4.7].

Proposition 4.2.6. *Suppose that X is a separable 1-exact operator system. Then there are a sequence natural numbers n_k and unital complete isometries $\phi_k : M_{n_k} \to M_{n_{k+1}}$ such that X embeds unitally completely isometrically into the inductive limit $\lim_{(\phi_k)} M_{n_k}$.*

Proof. Let $(E_k)_{k\in\mathbb{N}}$ be an increasing sequence of finite-dimensional subsystems of X with dense union. Set $\varepsilon_k = (50 \dim (E_k))^{-2}$, where $\dim (E_k)$ denotes the dimension of E_k as a vector space. We will define the sequences (n_k) and (ϕ_k) by recursion on $k \in \mathbb{N}$ together with ucp maps $i_k : E_k \to M_{n_k}$ such that $\left\|i_k^{-1}\right\|_{cb} \leq 1 + \varepsilon_k 2^{-2k}$ and $\left\|(i_{k+1})_{|E_k} - \phi_k \circ i_k\right\| \leq 2^{-k}$. Granted this construction we let i be the unital complete isometry

$$\lim_{k\to+\infty} i_k : X \to \lim_{(\phi_k)} M_{n_k}.$$

The map $i_1 : E_1 \to M_{n_1}$ can be defined applying the characterization of 1-exactness provided by Proposition 4.1.2. Suppose that n_k, i_k, and ϕ_{k-1} satisfying the conditions above have been defined for $k \leq m$. Again by Proposition 4.1.2 one can find $d \in \mathbb{N}$ together with a ucp map $\theta : E_{m+1} \to M_d$ such that $\left\|\theta^{-1}\right\|_{cb} \leq 1 + \varepsilon_{m+1} 2^{-2(m+1)}$. By Lemma 4.2.3 there is a ucp map $\alpha : i_m [E_m] \subset M_{n_m} \to M_d$ such that $\left\|\alpha - \theta \circ i_m^{-1}\right\|_{cb} \leq 2^{-m}$. By injectivity of M_n we can extend α to a ucp map from M_{n_m} to M_d. Define now $n_{m+1} = n_m + d$, $\phi_m : M_{n_m} \to M_{n_{m+1}}$ and $i_{m+1} : E_{m+1} \to M_{n_{m+1}}$ by setting

$$\phi_m (z) = \begin{bmatrix} z & 0 \\ 0 & \alpha (z) \end{bmatrix} \quad \text{and} \quad i_{m+1} (x) = \begin{bmatrix} i_m (x) & 0 \\ 0 & \theta (x) \end{bmatrix}$$

for $z \in M_{n_m}$ and $x \in E_{m+1}$. Observe that ϕ_m is a unital complete isometry, and i_{m+1} is a ucp map such that

$$\left\|i_{m+1}^{-1}\right\|_{cb} \leq \left\|\theta^{-1}\right\|_{cb} \leq 1 + \varepsilon_{m+1} 2^{-2(m+1)}.$$

Moreover

$$\begin{aligned} \left\|\phi_m \circ i_m - i_{m+1}\right\|_{cb} &= \left\|\theta - \alpha \circ i_m\right\|_{cb} \\ &\leq \left\|\alpha - \theta \circ i_m^{-1}\right\|_{cb} \\ &\leq 2^{-m}. \end{aligned}$$

This concludes the recursive construction $\square$

4.2.3 The Fraïssé metric space

Recall that we denote by $\mathcal{E}_1^{sy}$ the class of finite-dimensional 1-exact operator systems. It is clear that $\mathcal{E}_1^{sy}$ satisfies (HP) from Definition 3.1.2. Proposition 4.2.5 shows that $\mathcal{E}_1^{sy}$ has (NAP). Finally (JEP) is a consequence of (NAP) and the observation that the operator system $\mathbb{C}$ is an initial object in the category of operator systems. In order to conclude that

$\mathcal{E}_1^{sy}$ is a Fraïssé class it is enough to show that for every $k \in \mathbb{N}$ the space $\mathcal{E}_1^{sy}(k)$ endowed with the Fraïssé metric $d_{\mathcal{E}_1^{sy}}$ from Definition 3.1.4 is complete and separable. Without loss of generality we can assume that $\mathcal{E}_1^{sy}(k)$ only contains the pairs $(\overline{a}, E)$ where E is a finite-dimensional 1-exact operator system and $\overline{a}$ is a linear basis of E. Any two such pairs $(\overline{a}, E)$ and $(\overline{b}, F)$ are identified if there is a complete order isomorphism from E to F mapping $\overline{a}$ to $\overline{b}$. For brevity we denote an element $(\overline{a}, E)$ of $\mathcal{E}_1^{sy}(k)$ simply by $\overline{a}$ and set $E = \langle \overline{a} \rangle$.

Recall that a basis $\{x_1, \ldots, x_n\}$ of a Banach space X is a *Auerbach* provided that $\|x_i\| = \|x_i'\| = 1$ for $i \leq n$, where $\{x_1', \ldots, x_n'\}$ denotes the dual basis; see [14, Theorem 13]. In analogy we call an element $\overline{a}$ of $\mathcal{E}_1^{sy}$ with dual basis $\overline{a}'$ N-Auerbach for $N \in \mathbb{N}$ if $\|a_i\| \leq N$ and $\|a_i'\| \leq N$ for $i \leq n$. The following is a standard lemma, known as the small perturbation argument; see [17, Lemma 12.3.15].

Lemma 4.2.7. *Suppose that $\overline{a}, \overline{b} \in \mathcal{E}_1^{sy}(k)$ are N-Auerbach. If $d_{\mathcal{E}_1^{sy}}(\overline{a}, \overline{b}) \leq \frac{1}{4kN}$, then there is a unital isomorphism $f : \langle \overline{a} \rangle \to \langle \overline{b} \rangle$ such that $\|f\|_{cb} \leq 1 + kN d_{\mathcal{E}_1^{sy}}(\overline{a}, \overline{b})$ and $\|f^{-1}\|_{cb} \leq 1 + kN d_{\mathcal{E}_1^{sy}}(\overline{a}, \overline{b})$.*

Proof. Set $\delta = d_{\mathcal{E}_1^{sy}}(\overline{a}, \overline{b})$. We can assume that $\langle \overline{a} \rangle$ and $\langle \overline{b} \rangle$ are unitally contained in a unital C*-algebra A in such a way that $\max_i \|a_i - b_i\| \leq \delta$. Define the map

$$\begin{aligned} \theta : \ & A \ \to \ A \\ & z \ \mapsto \ z + \sum_{i \leq n} b_i'(z)(b_i - a_i). \end{aligned}$$

Observe that $\theta(a_i) = b_i$ for $i \leq k$ and

$$\|id_A - \theta\|_{cb} \leq \sum_{i \leq n} \|b_i'\| \leq kN\delta.$$

Reversing the roles of $\overline{a}$ and $\overline{b}$ allows one to define a map $\widetilde{\theta} : A \to A$ such that $\widetilde{\theta}(b_i) = a_i$ and $\left\|id_A - \widetilde{\theta}\right\|_{cb} \leq kN\delta$. Let now $f : \langle \overline{a} \rangle \to \langle \overline{b} \rangle$ be $\theta_{|\langle \overline{a} \rangle}$, and observe that f satisfies the desired conclusions. □

The following lemma is an immediate consequence of Proposition 4.2.5.

Lemma 4.2.8. *Suppose that $\delta \in [0, \frac{1}{20}]$ and $\overline{a}, \overline{b} \in \mathcal{E}_1^{sy}(k)$. If there is an isomorphism $f : \langle \overline{a} \rangle \to \langle \overline{b} \rangle$ such that $\|f(1) - 1\| \leq \delta$, $\|f\|_{cb} \leq 1 + \delta$ and $\|f^{-1}\|_{cb} \leq 1 + \delta$, then $d_{\mathcal{E}_1^{sy}}(\overline{a}, \overline{b}) \leq 100k\delta^{\frac{1}{2}}$.*

Proof. Let ρ be a state on $\langle \overline{a} \rangle$. Define $g(x) = f(x) + \rho(x)(1 - f(1))$. Observe that $\|g - f\|_{cb} \leq \delta$ and hence $\|g\|_{cb} \leq 1 + 2\delta$. Furthermore g is unital, and an easy calculation shows that g is invertible with $\left\|g^{-1}\right\|_{cb} \leq 1+10\delta$. By Proposition 4.2.5 we can now conclude that $d_{\mathcal{E}_1^{sy}}\left(\overline{a}, \overline{b}\right) \leq 100k\left(10\delta\right)^{\frac{1}{2}} \leq 1000k\delta^{\frac{1}{2}}$. ☐

We are now ready to show that $\left(\mathcal{E}_1^{sy}\left(k\right), d_{\mathcal{E}_1^{sy}}\right)$ is a complete metric space. The proof of this fact involves a standard argument; see also [114, Proposition 12].

Proposition 4.2.9. *Suppose that $k \in \mathbb{N}$. Then $\mathcal{E}_1^{sy}\left(k\right)$ is a complete metric space with respect to the metric $d_{\mathcal{C}}$, and $\mathcal{M}_\infty^0\left(k\right)$ is a dense subspace.*

Proof. Let us first show that $\mathcal{E}_1^{sy}\left(k\right)$ is complete. Suppose that $\left(\overline{a}^{(m)}\right)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}_1^{sy}\left(k\right)$. It is not difficult to verify that there is $N \in \mathbb{N}$ such that $\overline{a}^{(m)}$ is N-Auerbach for every $m \in \mathbb{N}$. Fix a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ and let $E := \prod_{\mathcal{U}}\left\langle\overline{a}^{(m)}\right\rangle$ be the corresponding ultraproduct. Let a_i for $i \leq k$ be the element of X with representative sequence $\left(a_i^{(m)}\right)_{m\in\mathbb{N}}$. Let $\phi_m : \left\langle\overline{a}^{(m)}\right\rangle \rightarrow E$ be the linear map such that

$$\phi_m\left(a_i^{(m)}\right) = a_i.$$

By Lemma 4.2.7 and Łos' theorem for ultraproducts [35, Proposition 4.3] we have that

$$\lim_{m\rightarrow+\infty}\left\|\phi_m\right\|_{cb}\left\|\phi_m^{-1}\right\|_{cb} \rightarrow 1.$$

Therefore X is 1-exact by Proposition 4.1.2. Moreover $d_{\mathcal{E}_1^{sy}}\left(\overline{a}, \overline{a}^{(m)}\right) \rightarrow 0$ by Lemma 4.2.8. The fact that $\mathcal{M}_\infty^0\left(k\right)$ is dense in $\mathcal{E}_1^{sy}\left(k\right)$ is an immediate consequence of Proposition 4.2.6. To prove separability it is enough to show that $\mathcal{M}_\infty^0\left(k\right)$ is separable. For every $n \in \mathbb{N}$ let $D_n \subset M_n$ be a self-adjoint countable dense subset containing the identity. Then the set of $\overline{a} \in \mathcal{M}_\infty^0$ such that $\overline{a} \subset D_n$ for some $n \in \mathbb{N}$ is a countable dense subset of $\mathcal{M}_\infty^0\left(k\right)$. ☐

This concludes the proof that $\mathcal{E}_1^{sy}$ is a Fraïssé class. In view of Theorem 3.1.9 we can consider the corresponding limit. Equivalent characterizations of such a limit will be obtained in Subsection 4.3.1

4.3 The Gurarij operator system

4.3.1 The limit of the class $\mathcal{E}_1^{sy}$

Definition 4.3.1. A separable 1-exact operator system $\mathbb{GS}$ is *Gurarij* if whenever $E \subset F$ are finite-dimensional 1-exact operator systems, $\phi : E \rightarrow \mathbb{GS}$ is a unital complete isometry, and $\varepsilon > 0$, there is a linear map $\widehat{\phi} : F \rightarrow \mathbb{GS}$ extending ϕ such that $\left\|\widehat{\phi}\right\|_{cb}\left\|\widehat{\phi}^{-1}\right\|_{cb} \leq 1 + \varepsilon$.

The following characterization of the limit of $\mathcal{E}_1^{sy}$ follows easily from our results above. The proof is analogous to the ones of [8, Theorem 3.3] and [90, Proposition 4.8].

Proposition 4.3.2. *Suppose that Z is a separable 1-exact operator system. The following statements are equivalent:*

1. *Z is the Fraïssé limit of the class $\mathcal{E}_1^{sy}$ in the sense of Definition 3.1.7;*

2. *Whenever $E \subset F$ are finite-dimensional 1-exact operator systems, $\phi : E \to Z$ is a unital complete isometry, and $\varepsilon > 0$, there is a unital complete isometry $\psi : F \to Z$ such that $\left\|\psi_{|E} - \phi\right\| \leq \varepsilon$;*

3. *Z satisfies the same condition as (2) when $F = M_n$ for some $n \in \mathbb{N}$;*

4. *Z is Gurarij;*

5. *Z satisfies the same conditions as in Definition 4.3.1 when $F = M_n$ for some $n \in \mathbb{N}$;*

6. *For every matricial operator space E, $\psi \in \mathrm{Stx}_{\mathcal{E}_1^{sy}}(E, Z)$, finite subset B of E, and $\varepsilon > 0$, there is $\phi \in \mathrm{Apx}_{\mathcal{E}_1^{sy}}(E, Z)$ such that $\phi \leq \psi$ and for every $b \in B$ there is $z \in Z$ such that $\phi(b, z) \leq \varepsilon$.*

Proof. We present a proof of the nontrivial implications below.

(1)⇒(2) By definition of Fraïssé limit Z is a separable $\mathcal{E}_1^{sy}$-structure, and hence a separable 1-exact operator system by [13, Proposition 3.2]. Observe that the embeddings between structures in the language $\mathcal{L}_{OSy}$ for operator systems are precisely the unital complete isometries. Therefore (2) is a reformulation of being limit as in Definition 3.1.7;

(3)⇒(5) Suppose that $E \subset M_n$ and $\phi : E \to Z$ is a unital complete isometry. Fix a strictly positive real number δ. By hypothesis there is a unital complete isometry $\psi : M_n \to Z$ such that $\left\|\psi_{|E} - \phi\right\| < 1 + \delta$. If δ is small enough, then by Lemma 4.2.7 one can perturb ψ to a unital linear map $\widehat{\phi} : M_n \to Z$ such that $\left\|\widehat{\phi}\right\|_{cb} \left\|\widehat{\phi}^{-1}\right\|_{cb} \leq 1 + \varepsilon$ and $\widehat{\phi}_{|E} = \phi$.

(2)⇒(4) Analogous to the implication (3)⇒(5).

(5)⇒(6) Suppose that E is a matricial operator system, $B \subset E$ is a finite subset, $\psi \in \mathrm{Stx}_{\mathcal{E}_1^{sy}}(E, Z)$, and $\varepsilon > 0$. Lemma 4.2.4 shows that the class $\mathcal{M}_\infty^0$ of matricial operator systems has (HP), (JEP), and (NAP) of Definition 3.1.2. Therefore by [8, Lemma 2.8(iii)], possibly after enlarging E and decreasing ε, we can assume that there is a finite subset A of E and a unital complete isometry

$$f : \mathrm{span}\left(\{1\} \cup A \cup A^*\right) \to Z$$

such that $\psi \geq f_{|A} + \varepsilon$. (Here as in Subsection 3.1.1 we identify a partial morphisms with the approximate morphism given by the distance function from its graph.) Let $\overline{c}$ be linear basis for

$$\mathrm{span}\left(\{1\} \cup A \cup B \cup A^* \cup B^*\right),$$

and k be the length of $\overline{c}$. Fix $\delta \in (0, 1]$. By assumption, since E is matricial we can extend f to a unital linear map $f : E \to Z$ such that $\|f\|_{cb} \leq 1+\delta$ and $\left\|f^{-1}\right\|_{cb} \leq 1+\delta$. This implies that $d_{\mathcal{E}_1^{sy}}\left(\overline{c}, , f\left(\overline{c}\right)\right) \leq 1000k\delta^{\frac{1}{2}}$ by Lemma 4.2.8. Therefore there are a finite-dimensional 1-exact operator system F and unital complete isometries $\phi_0 : \langle\overline{c}\rangle \to F$ and $\phi_1 : \langle f\left(\overline{c}\right)\rangle \to F$ such that $\left\|\phi_0\left(c_i\right) - \left(\phi_1 \circ f\right)\left(c_i\right)\right\| \leq 1000k\delta^{\frac{1}{2}}$ for every i. Let $\phi : \langle\overline{c}\rangle \rightsquigarrow \langle f\left(\overline{c}\right)\rangle$ be the composition of ϕ_1^{-1} and ϕ_0 as approximate morphisms. It is clear that by choosing δ small enough we can ensure that $\phi\left(x, f\left(x\right)\right) \leq \varepsilon$ for every $x \in A \cup B$. Observe finally that

$$\phi \in \mathrm{Apx}\left(\langle\overline{a}\rangle, \langle f\left(\overline{a}\right)\rangle\right) \subset \mathrm{Apx}\left(E, Z\right)$$

and

$$\psi \geq f_{|A} + \varepsilon \geq \phi.$$

This concludes the proof.

(6)⇒(1) Since by Proposition 4.2.9 $\mathcal{M}_\infty^0(k)$ of k-generated matricial operator systems is dense in $\mathcal{E}_1^{sy}(k)$ for every $k \in \mathbb{N}$, the conclusion follows from the characterization of the Fraïssé limit provided by [8, Lemma 2.16].

□

4.3.2 Existence and uniqueness of the Gurarij operator system

Recall that we have shown in Subsection 4.2.1 and Subsection 4.2.3 that $\mathcal{E}_1^{sy}$ is a Fraïssé class. We can therefore consider its limit. Proposition 4.3.2 shows that being limit of $\mathcal{E}_1^{sy}$ is equivalent to being a Gurarij operator system.

The following theorem is now an immediate consequence of Ben Yaacov's main results on Fraïssé limits of metric structures; see Theorem 3.1.9. The following lemma can be obtained from Proposition 4.2.5, similarly as Lemma 2.2 is obtained from Lemma 2.1 in [83].

Lemma 4.3.3. *Suppose that $E \subset \mathbb{GS}$ is a finite-dimensional subsystem, Y is a finite-dimensional 1-exact operator system, $\delta \in (0,1]$, and $f : E \to Y$ is a unital invertible linear map such that $\|f\|_{cb} < 1+\delta$ and $\left\|f^{-1}\right\|_{cb} < 1+\delta$. Then for every $\eta > 0$ there exists a $g : Y \to \mathbb{NG}$ such that $\|g\|_{cb} < 1+\eta$, $\left\|g^{-1}\right\|_{cb} < 1+\eta$, and $\|g \circ f - id_X\|_{cb} < 100 \dim(E)\, \delta^{\frac{1}{2}}$.*

Theorem 4.3.4. *There is a Gurarij operator system $\mathbb{GS}$ as in Definition 4.3.1. Such an operator system is unique up to complete order isomorphism. Any separable 1-exact operator system embeds unitally and completely isometrically into $\mathbb{GS}$. Furthermore $\mathbb{GS}$ has the following homogeneity property: if E is a finite-dimensional subsystem of $\mathbb{GS}$, $\delta \in (0,1]$, and $\phi : X \to \mathbb{GS}$ is a unital invertible linear map such that $\|\phi\|_{cb} < 1+\delta$ and $\left\|\phi^{-1}\right\|_{cb} < 1+\delta$, then there exists a complete order automorphism α of $\mathbb{GS}$ such that $\left\|\phi - \alpha_{|E}\right\|_{cb} < 100 \dim(X)\, \delta^{\frac{1}{2}}$.*

Proof. The existence, uniqueness, and universality statements follow from Theorem 3.1.9. The proof of the homogeneity property is analogous to the proof of [83, Theorem 1.1] where [83, Lemma 2.2] is replaced by Lemma 4.3.3. □

4.3.3 An explicit construction of $\mathbb{GS}$

Lemma 4.2.4 shows that the class of matricial operator systems is Fraïssé. It follows from this observation and the proof of the existence statement for Fraïssé limits [8, Lemma 2.7] that $\mathbb{GS}$ can be written as a direct limit $\lim_{(\phi_k)} X_k$, where X_k is an operator system completely order isomorphic to M_{n_k} for some $n_k \in \mathbb{N}$ and $\phi_k : X_k \to X_{k+1}$ is a complete order embedding. We will present in this subsection an explicit construction of $\mathbb{GS}$ that makes this fact apparent.

Let us say that a subset D of a metric space X is ε-dense for some $\varepsilon > 0$ if every element of X is at distance at most ε from some element of D. For $m, k \in \mathbb{N}$, let $D_{m,k}$ be a finite 2^{-k}-dense subset of the unit ball of M_m, and $(\overline{a}_{m,k,i})$ be an enumeration of the finite tuples from $D_{m,k}$. Set $E_{m,k,i} = \langle \overline{a}_{m,k,i} \rangle \subset M_m$. We define by recursion on k sequences $(n_k), (X_k), \left(D_k^X\right), (\phi_k), (f_{m,k,i,j})$ such that

1. X_k is an operator system completely order isomorphic to M_{n_k};
2. D_k^X is a 2^{-k}-dense subset of the ball of radius 2 of X_k;

3. $\phi_k : X_k \to X_{k+1}$ is a complete order embedding;

4. $(f_{m,k,i,j})$ enumerates all the unital linear maps f from $E_{m,k,i}$ to X_k such that $f(\overline{a}_{m,k,i}) \subset D_k^X$;

5. for every m, k, i, j such that $m \leq k$ there exists a complete order embedding $g_{m,k,i,j} : M_m \to X_{k+1}$ such that

$$\begin{aligned} & \|g_{m,k,i,j} - \phi_k \circ f_{m,k,i,j}\|_{cb} \\ \leq \quad & 100 \dim (E_{m,k,i}) \left(\max \left\{ \|f_{m,k,i,j}\|_{cb}, \left\| f_{m,k,i,j}^{-1} \right\|_{cb} \right\} - 1 \right)^{\frac{1}{2}}. \end{aligned}$$

Granted the construction define $X = \lim_k X_k$ and identify X_k with its image inside X. Suppose that $m \in \mathbb{N}$, $E \subset M_m$ is a subsystem, $\varepsilon > 0$, and $f : E \to X$ is a complete order embedding. Fix $\eta \in (0,1)$ small enough. By the small perturbation argument [17, Lemma 12.3.15] there exist $k, i, j \in \mathbb{N}$ such that $m \leq k$, $\|f_{m,k,i,j} - f\|_{cb} \leq \eta$, $\|f_{m,k,i,j}\|_{cb} \leq 1 + \eta$, and $\left\| f_{m,k,i,j}^{-1} \right\|_{cb} \leq 1 + \eta$. Therefore

$$\begin{aligned} \|g_{m,k,i,j} - f\|_{cb} &\leq \|g_{m,k,i,j} - f_{m,k,i,j}\|_{cb} + \|f_{m,k,i,j} - f\|_{cb} \\ &\leq 100 m \eta^{\frac{1}{2}} + \eta \leq \varepsilon \end{aligned}$$

for η small enough. This proves that X satisfies Condition (3) of Proposition 4.3.2. Since X is 1-exact, it follows that X is completely order isomorphic to $\mathbb{GS}$. We now show how to recursively define sequences as above. Let $n_1 = 1$, $X_1 = \mathbb{C}$. Suppose that the sequences $(n_k), (X_k), (D_k^X), (\phi_{k-1}), (f_{m,k,i,j})$ have been defined for $k \leq l$. Observe that condition (5) only concerns finitely many functions $f_{m,k,i,j}$. Therefore one can obtain X_{k+1}, $\phi_k : X_k \to X_{k+1}$, and $(g_{m,k,i,j})$ by repeatedly applying Lemma 4.2.4. This concludes the construction.

4.3.4 Unital operator spaces

In this subsection we want to observe that $\mathbb{GS}$ is of universal disposition for 1-exact unital operator spaces. Recall that a unital operator space is an operator space with a distinguished unitary element in the sense of [13]. A unital operator space can be concretely represented as a unital subspace of $B(H)$ with the inherited matrix norms and the identity operator as distinguished unitary. If $X \subset B(H)$ is a unital operator space, define the operator system $X + X^{\star} = \{x_1 + x_2^* : x_1, x_2 \in X\}$. Recall that if $\phi : X \to Y$ is a unital completely contractive (resp. completely isometric) map from X into an operator system Y, then $x_1 + x_2^* \mapsto \phi(x_1) + \phi(x_2)^*$ is a unital completely contractive (resp. completely isometric) map; see [12, Lemma 1.3.6]. This implies that the complete order isomorphism class of $X + X^\star$ does not depend from the concrete representation of X as a unital subspace of $B(H)$.

A unital operator space is 1-exact if it is 1-exact as an operator space. We now want to observe that a unital operator space X is 1-exact if and only if the operator system $X + X^\star$ is 1-exact. The proof of [114] shows that a unital operator space X is 1-exact if and only if for every finite-dimensional unital operator space $E \subset X$ and for every $\delta > 0$ there exist $n \in \mathbb{N}$ and a unital completely contractive map $\phi : E \to M_n$ such that $\left\|\phi^{-1}\right\|_{cb} \leq 1+\delta$. The proof the following lemma is analogous to the proof of Proposition 4.2.6 with the additional ingredient of [12, Lemma 1.3.6].

Lemma 4.3.5. *Suppose that X is a 1-exact unital operator space. Then X admits a unital completely isometric embedding into an inductive limit of full matrix algebras with unital completely isometric connective maps.*

Suppose now that E is a finite-dimensional unital 1-exact operator space. By Lemma 4.3.5 we can assume that $E \subset Z$, where Z is an inductive limit of full matrix algebras with unital completely isometric connective maps. Hence

$$E + E^\star = \{x_1 + x_2^* : x_1, x_2 \in E\} \subset Z.$$

Let $\overline{a}$ be a basis for E and $\varepsilon > 0$. Pick $\delta > 0$ small enough. There exist a subsystem $Z_0 \subset Z$ completely order isomorphic to a full matrix algebra and a tuple $\overline{b}$ in Z_0 such that $\max_i \|a_i - b_i\| \leq \delta$ and, hence, $\max_i \|a_i^* - b_i^*\| \leq \delta$. The proof of Lemma 4.2.7 shows that for δ small enough one has that $d_{cb}(E + E^\star, F) \leq \varepsilon$ for some subsystem F of Z_0. This shows that $E + E^\star$ is a 1-exact operator system. The fact that a unital operator space X is 1-exact if and only if $X + X^\star$ is 1-exact now follows immediately.

Proposition 4.3.6. *If $E \subset F$ are unital 1-exact operator spaces, $\phi : E \to \mathbb{GS}$ is a unital complete isometry, and $\varepsilon > 0$, then there exists a unital complete isometry $\psi : F \to \mathbb{GS}$ such that $\left\|\psi_{|E} - \phi\right\|_{cb} \leq 1 + \varepsilon$.*

Proof. Since F is 1-exact, also $F + F^\star$ is 1-exact. Moreover by [12, Lemma 1.3.6] the map $\widetilde{\phi} : E + E^\star \to \mathbb{GS}$ defined by $x_1 + x_2^* \mapsto \phi(x_1) + \phi(x_2)^*$ is a unital complete isometry. Therefore there is a unital complete isometry $\widetilde{\psi} : F + F^\star \to \mathbb{GS}$ such that $\left\|\widetilde{\psi}_{|E+E^\star} - \widetilde{\phi}\right\|_{cb} \leq 1 + \varepsilon$. Setting $\psi := \widetilde{\psi}_{|E}$ yields a unital complete isometry from F to $\mathbb{GS}$ such that $\|\psi - \phi\|_{cb} \leq 1 + \varepsilon$. ∎

Let us now consider $\mathbb{GS}$ as a unital operator space, i.e. a structure in the language $\mathcal{L}_{uOS}$ of unital operator spaces. Proposition 4.3.6 shows that $\mathbb{GS}$ is a homogeneous $\mathcal{L}_{uOS}$-structure. Therefore the age of $\mathbb{GS}$ in the language $\mathcal{L}_{uOS}$, which is the collection of finite-dimensional 1-exact unital operator spaces, is a Fraïssé class with limit $\mathbb{GS}$.

4.3.5 A unital function system of almost universal disposition

We will now consider the class of M_n-systems defined in Subsection 4.1.4. Let us say that a unital M_n-space is a subspace of an M_n-system containing the unit. Observe that a unital M_n-space is in particular an M_n-space in the sense of [12, 4.1.1]. We can regard unital M_n-spaces as structures in a language $\mathcal{L}_{uM_n}$, which is the language of M_n-spaces described in [90, Subsection 2.5] with the addition of a constant symbol for the unit. Similarly M_n-systems can be regarded as structures in the language $\mathcal{L}_{M_n^{sy}}$ obtained from $\mathcal{L}_{uM_n}$ adding a symbol for the involution.

The proofs of Lemma 4.3.7 and Proposition 4.3.8 is analogous to the proofs of Lemma 4.2.4 and Lemma 4.2.5. We denote the ∞-sum of k copies of M_n by $\ell_k^\infty(M_n)$.

Lemma 4.3.7. *Suppose that $F_0, F_1 \subset \ell_k^\infty(M_n)$, $E \subset F_0$, and $\delta \in [0,1]$. If $f : E \to F_1$ is an invertible unital map such that $\|f\|_{cb} \leq 1+\delta$ and $\left\|f^{-1}\right\|_{cb} \leq 1+\delta$, then there exist $d \in \mathbb{N}$ and unital completely isometric embeddings $i : F_0 \to \ell_k^\infty(M_n)$ and $j : F_1 \to \ell_k^\infty(M_n)$ such that $\left\|j \circ f - i_{|E}\right\|_{cb} \leq 100 \dim(E)\, \delta^{\frac{1}{2}}$.*

Proposition 4.3.8. *Suppose that X and Y are M_q-systems, $E \subset X$ is a finite-dimensional subsystem, $\varepsilon > 0$, and $\delta \in [0,1]$. If $f : E \to Y$ is a unital map such that$\|f\|_{cb} \leq 1+\delta$ and $\left\|f^{-1}\right\|_{cb} \leq 1+\delta$, then there exist a separable M_q-system Z and unital completely isometric embeddings $i : X \to Z$ and $j : Y \to Z$ such that $\|j \circ f - i\|_{cb} \leq 100 \dim(E)\, \delta^{\frac{1}{2}} + \varepsilon$.*

Proposition 4.3.8 shows that M_n-systems form a Fraïssé class. Arguing as before, one can obtain the following characterization for the limit $\mathbb{G}_n^u$ of the Fraïssé class of M_n-systems.

Proposition 4.3.9. *Suppose that Z is a separable M_n-system. The following conditions are equivalent:*

1. *Z is completely order isomorphic to $\mathbb{G}_n^u$;*
2. *whenever $E \subset F$ are finite-dimensional M_n-systems, $\phi : E \to Z$ is a unital complete isometry, and $\varepsilon > 0$, there is a unital complete isometry $\psi : F \to Z$ such that $\left\|\psi_{|E} - \phi\right\|_{cb} \leq \varepsilon$;*

3. *whenever $E \subset F$ are finite-dimensional M_n-systems, $\phi : E \to Z$ is a unital complete isometry, and $\varepsilon > 0$, there is a unital linear map $\psi : F \to Z$ such that $\left\|\widehat{\psi}\right\|_{cb} \left\|\widehat{\psi}^{-1}\right\|_{cb} \leq 1 + \varepsilon$.*

Similar characterizations hold when E and F in (2) and (3) are only assumed to be finite-dimensional unital M_n-spaces. Observe that unital M_1-spaces are precisely *unital function spaces*, i.e. subspaces of $C(K)$ for some compact Hausdorff space K containing the function 1_K constantly equal to 1. Similarly M_1-systems are *function systems*, i.e. subspaces of $C(K)$ for some compact Hausdorff space K containing 1_K and closed under taking adjoints. We can therefore conclude from Proposition 4.3.9 that $\mathbb{G}_1^u$ is a separable unital function system of almost universal disposition for unital function spaces. In other words whenever $E \subset F$ are unital function spaces, $\phi : E \to \mathbb{G}_1^u$ is a unital isometry, and $\varepsilon \geq 0$, there is a unital isometry $\psi : F \to \mathbb{G}_1^u$ such that $\left\|\psi_{|E} - \phi\right\| \leq 1 + \varepsilon$.

4.3.6 The C*-envelope of the Gurarij operator system

We now want to show that the canonical $*$-homomorphism from the universal C*-algebra of $\mathbb{GS}$ to the C*-envelope of $\mathbb{GS}$ is a $*$-isomorphism.

Suppose that X is an operator system. A C*-cover of X is a unital completely isometric embedding ϕ of X into a C*-algebra A such that the image of X under ϕ generates A as a C*-algebra. It was shown by Hamana [55, 54] and, independently, Ruan [124] that there always exists a (projectively) minimal C*-cover $i_X : X \to C_e^*(X)$, called the *C*-envelope* of X (or the regular C*-algebra in [79]). The C*-envelope has the property that whenever $\phi : X \to A$ is a C*-cover, there is a (necessarily unique and surjective) $*$-homomorphism $\pi : A \to C_e^*(X)$ such that $\pi \circ \phi = i_X$.

Similarly it is shown in [79, Section 3] that there always exists a (projectively) maximal C*-cover $u_X : X \to C_u^*(X)$, called the *universal C*-algebra* of X. This has the property that whenever $\phi : X \to A$ is a C*-cover, there is a (necessarily unique and surjective) $*$-homomorphism $\pi : C_u^*(X) \to A$ such that $\pi \circ u_X = \phi$. In particular there is a $*$-homomorphism $\sigma_X : C_u^*(X) \to C_e^*(X)$.

An operator system X for which σ_X is injective (or, equivalently, a $*$-isomorphism) is called *universal* in [79]. In particular this property implies that if $\phi : X \to A$ is any C*-cover of X, then $A \cong C_u^*(X) \cong C_e^*(X)$. It is proved in [79, Theorem 15 and Corollary 18] that if X is a universal operator system of dimension at least 2, then X does not embed unitally completely isometrically into any exact C*-algebra. An example of a nuclear separable universal operator system is constructed in [79, Theorem 17].

In the rest of this subsection we will show that the Gurarij operator system is universal in the sense of [79]. The argument is similar to the one in the proof of [79, Theorem 17]. In the following we identify an operator system X with its image inside the C*-envelope $C_e^*(X)$, and we denote by $u_X : X \to C_u^*(X)$ the canonical embedding of X into its universal C*-algebra. It is shown in [79, Proposition 9] that a unital complete isometry $\phi : X \to Y$ between operator system has a unique "lift" to an injective $*$-homomorphism $\overline{\phi} : C_u^*(X) \to C_u^*(Y)$ such that $\overline{\phi} \circ u_X = u_Y \circ \phi$. In particular this allows one to identify the universal C*-algebra of an inductive limit with the inductive limit of the universal C*-algebras of the building blocks.

Theorem 4.3.10. *The canonical $*$-homomorphism $\sigma_{\mathbb{GS}} : C_u^*(\mathbb{GS}) \to C_e^*(\mathbb{GS})$ is a $*$-isomorphism.*

Proof. As observed in Subsection 4.3.3, $\mathbb{GS}$ is the limit of an inductive sequence $(X_k)_{k\in\mathbb{N}}$ with unital completely isometric connective maps $\phi_k : X_k \to X_{k+1}$, where X_k is completely order isomorphic to a full matrix algebra. Denote by $\iota_k : X_k \to \mathbb{GS}$ the canonical inclusion, and observe that the set of elements of the form $\bar{\iota}_k(z)$ for $k \in \mathbb{N}$ and $z \in C_u^*(X_k)$ are dense in $\mathbb{GS}$. Suppose that $\bar{\iota}_k(z)$ is such an element. Fix $\delta > 0$ small enough. By [79, Corollary 7] there exist $d \in \mathbb{N}$ and a unital $*$-homomorphism $\pi : C_u^*(X_k) \to M_d$ such that $\pi \circ u_{X_k}$ is a unital complete isometry and $\|\pi(z)\| \geq (1-\varepsilon)\|z\|$. Set $\theta := \pi \circ u_{X_k} : X_k \to M_d$, and observe that π is the unique *-homomorphism from $C_u^*(X_k)$ to M_d such that $\pi \circ u_{X_k} = \theta$. By the approximate injectivity property of $\mathbb{GS}$—see Proposition 4.3.2—there is a unital complete isometry $\eta : M_d \to \mathbb{GS}$ such that $\|\eta \circ \theta - \iota_k\|_{cb} \leq \delta$. Observe that, for δ small enough, we have

$$\left\| \left(\overline{\eta \circ \theta}\right)(z) - \bar{\iota}_k(z) \right\| \leq \varepsilon.$$

By [18, Theorem 4.1]—see also [79, Lemma 6]—there is a $*$-homomorphism $\mu : C^*(\eta[M_d]) \subset C_e^*(\mathbb{GS}) \to M_d$ such that $\mu \circ \eta = id_{M_d}$. Observe now that

$$\mu \circ \sigma_{\mathbb{GS}} \circ \overline{\eta \circ \theta} : C_u^*(X_k) \to Y$$

is a $*$-homomorphism such that

$$\begin{aligned} \mu \circ \sigma_{\mathbb{GS}} \circ \overline{\eta \circ \theta} \circ u_{X_k} &= \mu \circ \sigma_{\mathbb{GS}} \circ u_{\mathbb{GS}} \circ \eta \circ \theta \\ &= \mu \circ \eta \circ \theta \\ &= \theta. \end{aligned}$$

Therefore $\mu \circ \sigma_{\mathbb{GS}} \circ \overline{\eta \circ \theta} = \pi$. Hence we have that

$$\begin{aligned}
\left\| \sigma_{\mathbb{GS}} \left(\bar{\iota}_k \left(z \right) \right) \right\| &\geq \left\| \left(\sigma_{\mathbb{GS}} \circ \overline{\eta \circ \theta} \right) \left(z \right) \right\| - \varepsilon \\
&\geq \left\| \left(\mu \circ \sigma_{\mathbb{GS}} \circ \overline{\eta \circ \theta} \right) \left(z \right) \right\| - \varepsilon \\
&= \left\| \pi \left(z \right) \right\| - \varepsilon \\
&\geq \left\| \bar{\iota}_k \left(z \right) \right\| - 2\varepsilon.
\end{aligned}$$

This concludes the proof that σ is injective. $\square$

Corollary 4.3.11. *The Gurarij operator system $\mathbb{GS}$ does not admit any complete order embedding into an exact C*-algebra. Moreover if $\mathbb{GS} \subset B(H)$ is a unital completely isometric representation of $\mathbb{GS}$, then the C*-algebra generated by $\mathbb{GS}$ inside $B(H)$ is $*$-isomorphic to $C_e^*(\mathbb{GS})$.*

In particular it follows from Corollary 4.3.11 that $\mathbb{GS}$ is not completely order isomorphic to a unital C*-algebra. (Recall that a C*-algebra is exact if and only if it is 1-exact as an operator system.) In fact by [13, Proposition 4.2] one can also conclude that $\mathbb{GS}$ is not completely isometric to a unital operator algebra (even without assuming that the complete isometry preserves the unit).

4.3.7 The triple envelope of the Gurarij operator space

The (noncommutative) Gurarij operator space $\mathbb{NG}$ has been defined by Oikhberg in [104] and proven to be unique and universal among separable 1-exact operator spaces in [90]. A 1-exact operator space Z is completely isometric to $\mathbb{NG}$ if and only if for every $n \in \mathbb{N}$, $E \subset M_n$, complete isometry $\phi : E \to \mathbb{NG}$, there exists a complete isometry $\psi : F \to \mathbb{NG}$ such that $\left\| \psi_{|E} - \phi \right\|_{cb} \leq \varepsilon$. A similar argument as the one in Subsection 4.3.3 shows that $\mathbb{NG}$ is completely isometric to a direct limit $\lim_{(\phi_k)} X_k$ where X_k is an operator space completely isometric to M_{n_k} for some $n_k \in \mathbb{N}$ and $\phi_k : X_k \to X_{k+1}$ is a complete isometry. Here one needs to replace Lemma 4.2.4 with the following lemma, which can be proved in the same way using injectivity of M_d in the category of operator spaces with completely contractive maps.

Lemma 4.3.12. *Suppose that $k, m \in \mathbb{N}$, F_0, F_1 are matricial operator spaces, $E \subset F_0$, and $\delta > 0$. If $f : E \to F_1$ is an invertible linear map such that $\|f\|_{cb} \leq 1 + \delta$ and $\left\| f^{-1} \right\|_{cb} \leq 1 + \delta$, then there exist $d \in \mathbb{N}$ and unital completely isometric embeddings $i : F_0 \to M_d$ and $j : F_1 \to M_d$ such that $\left\| j \circ f - i_{|E} \right\|_{cb} \leq 2\delta$.*

Here we want to point out that the methods employed in Subsection 4.3.6 can used to show that $\mathbb{NG}$ is not completely isometric to a C*-algebra. This gives an answer to [104, Remark 3.3]. The key idea is to replace the C*-envelope with the triple envelope, and the universal C*-algebras with the universal TRO. Recall that a *ternary ring of operators* (TRO) is a subspace V of $B(H, K)$ for some Hilbert spaces H, K such that $xy^*z \in V$ for any $x, y, z \in V$. The operation $(x, y, z) \mapsto xy^*z$ on V is called *triple product*. A *triple morphism* between TROs is a linear map that preserves the triple product. Observe that (the restriction of) a $*$-homomorphism is in particular a triple morphism. A TRO has a canonical operator space structure, where the matrix norms are uniquely determined by the triple product [69, Proposition 2.1]. A triple morphism between TROs is automatically completely contractive, and it is injective if and only if is completely isometric [12, Lemma 8.3.2].

Suppose that X is an operator space. A *triple cover* of X is a pair (ϕ, V) where V is a TRO and $\phi : X \to V$ is a linear complete isometry. Triple covers naturally form a category, where a morphism from (ϕ, V) to (ϕ', V') is a triple morphism $\psi : V \to V'$ such that $\psi \circ \phi = \phi'$. The (unique up to isomorphism) terminal object in such a category is called the *triple envelope* $\mathcal{T}_e(X)$ of X. The existence of such an object has been proved independently by Ruan [124] and Hamana [56]. The (unique up to isomorphism) initial object in the category of triple covers is the *universal TRO* $\mathcal{T}_u(X)$ of X. We could not find the existence of such an object explicitly stated in the literature. However this can be easily established with a minor modification of the proof of the existence of the universal C*-algebra of an operator system; see [79, Proposition 8]. Moreover the same proof as [79, Proposition 9]—where one replaces Arveson's extension theorem with Wittstock's extension theorem [107, Theorem 8.2]—shows that a complete isometry between TROs "lifts" to an injective triple morphism between the corresponding universal TROs.

The universal property of $\mathcal{T}_u(X)$ yields a (necessarily surjective) canonical triple morphism $\sigma_X : \mathcal{T}_u(X) \to \mathcal{T}_e(X)$. Our goal is to show that such a triple morphism is injective (and hence a triple isomorphism) in the case of the Gurarij operator space $\mathbb{NG}$. First we need an adaptation to TROs of a well known result of Choi and Effros; see [18, Theorem 4.1].

Proposition 4.3.13. *Suppose that V, W are TROs and $\theta : V \to W$ is a linear complete isometry. Then there is a triple morphism $\eta : \mathcal{T}(\theta[V]) \to V$ such that $\eta \circ \theta = id_V$, where $\mathcal{T}(\theta[V])$ is the subTRO of W generated by the image of V under θ.*

Proof. The proof consists in reducing the problem to the unital case via the Paulsen trick;

see [12, 1.3.14]. Represent V as a subspace of $B(H_1, K_1)$ and define the Paulsen system

$$\mathcal{S}(V) = \begin{bmatrix} \mathbb{C}I_{H_1} & V \\ V^* & \mathbb{C}I_{K_1} \end{bmatrix} = \left\{ \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} : x, y \in V, \lambda, \mu \in \mathbb{C} \right\} \subset B(H_1 \oplus K_1).$$

One can similarly define the Paulsen system $\mathcal{S}(W) \subset B(H_2 \oplus K_2)$ of W. The linear complete isometry $\theta : V \to W$ yields a linear map $\widetilde{\theta} : \mathcal{S}(V) \to \mathcal{S}(W)$ defined by

$$\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \mapsto \begin{bmatrix} \lambda & \theta(x) \\ \theta(y)^* & \mu \end{bmatrix}.$$

By [12, Lemma 1.3.15] $\widetilde{\theta}$ is a unital complete isometry. Let $\widetilde{\theta}^{-1} : \widetilde{\theta}[\mathcal{S}(V)] \to \mathcal{S}(V) \subset B(H_1 \oplus K_1)$ be the inverse of $\widetilde{\theta}$. By the Arveson extension theorem [107, Theorem 7.5] there is a unital completely positive map

$$\eta : C^*\left(\widetilde{\theta}[\mathcal{S}(V)]\right) \to B(H_1 \oplus K_1)$$

extending $\widetilde{\theta}$. Here $C^*\left(\widetilde{\theta}[\mathcal{S}(V)]\right)$ is the C*-algebra generated by $\widetilde{\theta}[\mathcal{S}(V)]$ inside $B(H_2 \oplus K_2)$. We claim that η is a $*$-homomorphism. By Choi's multiplicative domain theorem [107, Theorem 3.18] it is enough to show that for every self-adjoint element a of $\mathcal{S}(V)$, one has that $\eta\left(\theta(a)^2\right) = \eta(\theta(a))^2$. This easily follow from the Kadison-Schwartz inequality [12, Proposition 1.3.9] applied to η and θ. Therefore η is a $*$-homomorphism, and its restriction to the subTRO $\mathcal{T}(\theta[V]) \subset B(H_2, K_2)$ of W generated by the image of V under θ is a triple morphism. Since moreover $\eta \circ \theta$ is the identity of V and V is a TRO, it follows that η maps $\mathcal{T}(\theta[V])$ into V. This concludes the proof. □

Theorem 4.3.14. *The canonical triple morphism $\sigma_{\mathbb{NG}} : \mathcal{T}_u(\mathbb{NG}) \to \mathcal{T}_e(\mathbb{NG})$ is a triple isomorphism.*

Proof. The proof is entirely similar to the one of Theorem 4.3.10. Here one needs to use the fact that $\mathbb{NG}$ is the inductive limit of an inductive sequence (X_k) of operator spaces with completely isometric connective maps $\phi_k : X_k \to X_{k+1}$, as observed above. Moreover by [90, Theorem 4.12] $\mathbb{NG}$ satisfies the following approximate injectivity property: whenever $E \subset F$ are 1-exact finite-dimensional operator spaces, $\phi : E \to \mathbb{NG}$ is a linear complete isometry, and $\varepsilon > 0$, then there is a linear complete isometry $\psi : F \to \mathbb{NG}$ such that $\left\|\psi_{|E} - \phi\right\|_{cb} < \varepsilon$. Finally one needs to replace the use of [18, Theorem 4.1] with Proposition 4.3.13. □

We now discuss how Theorem 4.3.14 implies that $\mathbb{NG}$ does not admit any completely isometric embedding into an exact TRO. One can canonically assign to a TRO a C*-algebra called its *linking algebra*. The local properties of a TRO are closely reflected by the local properties of its linking algebra. In particular a TRO is exact if and only if it is 1-exact if and only if its linking algebra is exact [69, Theorem 4.4]. Moreover a (surjective) triple morphism between TROs induces a (surjective) $*$-homomorphism between the corresponding linking algebras. Since the class of exact C*-algebras is closed under quotients [17, Corollary 9.4.3], it follows that the image of an exact TRO under a triple morphism is exact.

Observe now that if X is an operator system, $C_u^*(X)$ is its universal C*-algebra, and $\mathcal{T}_u(X)$ is the triple envelope of X, then the universal property of $\mathcal{T}_u(X)$ implies the existence of a surjective triple morphism from $\mathcal{T}_u(X)$ to $C_u^*(X)$. If $C_u^*(X)$ is not exact, then $\mathcal{T}_u(X)$ is not exact as well. Since the universal C*-algebra of $M_2(\mathbb{C})$ is not exact [79, Section 5], it follows that the universal TRO of $M_2(\mathbb{C})$ is not exact. Recall that, as observed above, completely isometric embeddings between operator spaces "lift" to injective ternary morphisms between the corresponding universal TROs. Since $M_2(\mathbb{C})$ embeds completely isometrically into $\mathbb{NG}$, it follows that $\mathcal{T}_u(M_2(\mathbb{C}))$ embeds as a subTRO of $\mathcal{T}_u(\mathbb{NG}) \cong \mathcal{T}_e(\mathbb{NG})$. Therefore $\mathcal{T}_e(\mathbb{NG})$ is not exact. Now, if $\mathbb{NG}$ embeds completely isometrically into a TRO V, then the universal property of $\mathcal{T}_e(\mathbb{NG})$ implies that $\mathcal{T}_e(\mathbb{NG})$ is the image of under a triple morphism of the subTRO of V generated by (the image of) $\mathbb{NG}$. Hence V is not exact as well.

Corollary 4.3.15. *The Gurarij operator space* $\mathbb{NG}$ *does not admit any unital completely isometric embedding into an exact C*-algebra. Moreover if* $\mathbb{NG} \subset B(H)$ *is a unital completely isometric representation of* $\mathbb{NG}$ *then the TRO generated by* $\mathbb{NG}$ *inside* $B(H)$ *is* $*$*-isomorphic to* $\mathcal{T}_e(\mathbb{NG})$.

Since $\mathbb{NG}$ is 1-exact, it follows in particular that $\mathbb{NG}$ is not completely isometric to a TRO or a C*-algebra.

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