A class of asymmetric gapped Hamiltonians on quantum spin chains and its characterization

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Classification of gapped Hamiltonians

We would like to bulk-classify gapped Hamiltonians by the following criterion.

Two translation invariant gapped Hamiltonians are equivalent if there exists a path of translation invariant gapped Hamiltonians connecting them.

It would be nice if we can classify all the gapped Hamiltonians in the world to know that which ones are essentially the same, and which ones are essentially different.

This talk is about a class of Hamiltonians found in a trial about this problem.

Local Hamiltonians

A quantum spin chain is given by $\mathcal{A}_{\mathbb{Z}} = \bigotimes_{\mathbb{Z}} \mathrm{M}_n$.

We consider translation invariant finite range interactions.

Let h be an interaction with the interaction length m.

We define local Hamiltonians out of it, with open boundary conditions:

$$H_\Lambda(h) := \sum_{I \subset \Lambda} h_I$$

h_I indicates a copy of h acting on the interval I with length m. The dynamics α_h in the thermodynamic limit is given by

$$\alpha_h^t(A) := \lim_{\Lambda \nearrow \mathbb{Z}} e^{itH_\Lambda(h)} A e^{-itH_\Lambda(h)}, \quad A \in \mathcal{A}_{\mathbb{Z}}, \quad t \in \mathbb{R}.$$

We denote the net of local Hamiltonians by $H(h) = \{H_\Lambda(h)\}_\Lambda$ and call it a Hamiltonian given by h.

Ground states

Definition (Ground states in the bulk)

Let δ_h be the generator of α_h. A state ω on $\mathcal{A}_{\mathbb{Z}}$ is called an α_h-ground state if the inequality

$$-i\omega\left(A^*\delta_h\left(A\right)\right) \geq 0$$

holds for any element A in the domain $\mathcal{D}(\delta_h)$ of δ_h.

Remark

Any thermodynamic limit of ground states on finite intervals with arbitrary boundary conditions satisfies this condition.

To decide all the ground states is a nontrivial problem.

- One-dimensional XY -model : Araki-Matsui '85
- One-dimensional XXZ-models : Matsui '96, Koma-Nachtergaele '98
- Kitaev's quantum double models : Cha-Naaijkens-Nachtergaele '16 preprint

Gapped in the bulk

Let $(\mathcal{H}, \pi, \Omega)$ be the GNS triple of an α_h-ground state ω. There exists a unique positive operator $H_{\omega,h}$ on $\mathcal{H}$ such that

$$e^{itH_{\omega,h}}\pi(A)\, e^{-itH_{\omega,h}} = \pi\left(\alpha_h^t(A)\right), \quad e^{itH_{\omega,h}}\Omega = \Omega,$$

for all $A \in \mathcal{A}_{\mathbb{Z}}$ and $t \in \mathbb{R}$.
Let us call $H_{\omega,h}$, the bulk Hamiltonian associated to ω.

Definition

We say the Hamiltonian $H(h)$ given by h is gapped in the bulk if

1. *For any α_h-ground state φ , 0 is the non-degenerate eigenvalue of the bulk Hamiltonian $H_{\varphi,h}$.*
2. *There is a constant $\gamma > 0$, such that*

$$\sigma(H_{\varphi,h}) \setminus \{0\} \subset [\gamma, \infty),$$

for any α_h-ground state φ.

Bulk classification

Definition (Bulk Classification)

We say that the Hamiltonians $H(h_0)$, $H(h_1)$ gapped in the bulk, given by h_0, h_1 are bulk-equivalent ($H(h_0) \simeq_B H(h_1)$), if the followings hold.

1. *There exist an $m \in \mathbb{N}$ and a continuous path of interactions $h(s)$, $s \in [0, 1]$ with interaction length less than m, such that $h(0) = h_0$, and $h(1) = h_1$.*
2. *Hamiltonians $H(h(s))$ given by $h(s)$ are gapped in the bulk, and the gap is bounded from below uniformly in $s \in [0, 1]$.*

Existence of gap

Existence of the gap implies :

- Stability under shallow perturbations:

Theorem (Michalakis-Zwolak '13)

Assume some additional conditions on h. Then for any perturbation V, there exists an $\varepsilon_0 > 0$ such that $H(h + sV)$ is gapped in the bulk, for all $|s| < \varepsilon_0$.

- Exponential decay of correlation functions

Theorem (Hastings-Koma '06, Nachtergaele-Sims '09)

Suppose that ω is a unique α_h-ground state. If $H_{\omega,h}$ has a spectral gap, then the correlation functions decay exponentially fast.

For two Hamiltonians $H(h_0)$ and $H(h_1)$, the equivalence $H(h_0) \simeq_B H(h_1)$ means they can be connected keeping these normal properties. What we would like to do is to group the gapped Hamiltonians by this criterion.

Classification of gapped Hamiltonians

We denote by $\mathcal{J}_{FB}$, the set of h satisfying the followings.

1. $H(h)$ is gapped in the bulk.
2. There exists a unique α_h-ground state ω on $\mathcal{A}_{\mathbb{Z}}$.
3. (We may assume $h \geq 0$.) There exists a constant $d \in \mathbb{N}$ such that

$$1 \leq \dim\ker\left(H_{[1,N]}(h)\right) \leq d,$$

for all $N \in \mathbb{N}$.

Theorem (O '16 preprint)

For any $h_0, h_1 \in \mathcal{J}_{FB}$, we have $H(h_0) \simeq_B H(h_1)$.

MPS-Hamiltonian

The first step of the proof is the reduction of the problem to the classification problem of MPS (Matrix product state)-Hamiltonians. The origin of MPS is AKLT model. [Affleck-Kennedy-Lieb-Tasaki '87]. Generalizing it, Fannes-Nachtergaele-Werner '92 introduced a recipe to construct gapped Hamiltonians.

The recipe construct an interaction $h_{\mathbf{v}}$, out of an n-tuple of $k \times k$-matrices $\mathbf{v} = (v_1, \ldots, v_n) \in \mathrm{M}_k^{\times n}$, via some concrete formula. Here, k is some ancillary introduced degree of freedom which is associated to the model. We call a Hamiltonian given by this recipe, an MPS (matrix product state)-Hamiltonian.

The procedure in the recipe allows us to construct an interaction out of any n-tuple $\mathbf{v}$. However, if we would like the resulting Hamiltonian to be gapped, we need to require some additional condition on $\mathbf{v}$.

The sufficient condition introduced in [FNW 92'] is that $\mathbf{v}$ is primitive.

Primitive $\mathbf{v}$

Theorem

For $\mathbf{v} := (v_1, \ldots, v_n) \in \mathrm{M}_k^{\times n}$, *let* $T_{\mathbf{v}} : \mathrm{M}_k \to \mathrm{M}_k$ *be the completely positive map given by* $T_{\mathbf{v}}(A) = \sum_{i=1}^{n} v_i A v_i^*$, $A \in \mathrm{M}_k$. *Let* $r_{\mathbf{v}} > 0$ *be the spectral radius of* $T_{\mathbf{v}}$. *The following properties are equivalent.*

1. *The spectrum of* $r_{\mathbf{v}}^{-1} T_{\mathbf{v}}$ *satisfies* $\sigma\left(r_{\mathbf{v}}^{-1} T_{\mathbf{v}}\right) \cap \mathbb{T} = \{1\}$. 1 *is a non degenerate eigenvalue of* $r_{\mathbf{v}}^{-1} T_{\mathbf{v}}$. *There exist a faithful* $r_{\mathbf{v}}^{-1} T_{\mathbf{v}}$*-invariant state* $\varphi_{\mathbf{v}}$ *and a strictly positive* $r_{\mathbf{v}}^{-1} T_{\mathbf{v}}$*-invariant element* $e_{\mathbf{v}} \in \mathrm{M}_k$.
2. *For* $m \in \mathbb{N}$ *large enough, we have*
   $\mathcal{K}_m(\mathbf{v}) := \mathrm{span}\left\{ v_{\mu_1} v_{\mu_2} \cdots v_{\mu_m} \mid (\mu_1, \mu_2, \ldots, \mu_m) \subset \{1, \ldots, n\}^{\times m} \right\} = \mathrm{M}_k$

When these conditions hold, $\mathbf{v}$ *is pimitive.*

Definition

Out of primitive $\mathbf{v}$, *we can construct a matrix product state* $\omega_{\mathbf{v}}$, *which is given by an explicit formula, using* $\mathbf{v}$, $e_{\mathbf{v}}$, $\varphi_{\mathbf{v}}$.

Primitive $\mathbf{v}$

Theorem (FNW 92')

If $\mathbf{v}$ is primitive, the Hamiltonian given by $h_{\mathbf{v}}$ is gapped. Furthermore, the matrix product state $\omega_{\mathbf{v}}$ is an $\alpha_{h_{\mathbf{v}}}$-ground state.

Remark

Actually, $\omega_{\mathbf{v}}$ is the unique $\alpha_{h_{\mathbf{v}}}$-ground state. [O '16 preprint]

Recall

Definition (Ground states in the bulk)

Let δ_h be the generator of α_h. A state ω on $\mathcal{A}$ is called an α_h-ground state if the inequality

$$-i\omega\left(A^*\delta_h\left(A\right)\right) \geq 0$$

holds for any element A in the domain $\mathcal{D}(\delta_h)$ of δ_h.

Reduction to MPS-Hamiltonians

In the beginning, our Hamiltonian has nothing to do with the MPS-Hamiltonians. However, the following Theorem connects them.

Theorem (Matsui '98, '13)

Let $h \in \mathcal{J}_{FB}$ and ω its unique α_h-ground state. Then ω is a matrix product state, given by some primitive $\mathbf{v}$.

From this and [FNW'92], the state ω is an α_h-ground state and an $\alpha_{h_{\mathbf{v}}}$-ground state at the same time. Furthermore, the associated bulk Hamiltonians are gapped for both cases. Using these facts, we obtain the following observation.

Lemma

If $h \in \mathcal{J}_{FB}$, there exists a primitive $\mathbf{v}$ such that $H(h) \simeq_B H(h_{\mathbf{v}})$.

It suffices to classify all the MPS-Hamiltonians given by primitive **v**s.

What is our task?

Given primitive $\mathbf{v}_0 \in \mathrm{M}_{k_0}^{\times n}$ and $\mathbf{v}_1 \in \mathrm{M}_{k_1}^{\times n}$, we have to construct a path of Hamiltonians gapped in the bulk connecting $H(h_{\mathbf{v}_0})$ and $H(h_{\mathbf{v}_1})$.
So far, the only way to guarantee the gap (along the path) we know is to take $h_{\mathbf{v}(s)}$ with primitive $\mathbf{v}(s)$.
If $k_0 = k_1 =: k$, then it suffices to show that there exists a path of n-tuples $\mathbf{v}(s) \in \mathrm{M}_k^{\times n}$, such that $\mathbf{v}(s)$ is primitive for all $s \in [0, 1]$.

Theorem (Bachmann-O '15, Szehr-Wolf '15 preprint)

The set of n-tuples

$$\left\{ \mathbf{v} \in \mathrm{M}_k^{\times n} \mid \mathbf{v} : \textit{primitive} \right\}$$

is arcwise connected.

Corollary

Let $\mathbf{v}_0 \in \mathrm{M}_{k_0}^{\times n}$, $\mathbf{v}_1 \in \mathrm{M}_{k_1}^{\times n}$ *be primitive. If* $k_0 = k_1$ *holds, we have*

$$H(h_{\mathbf{v}_0}) \simeq_B H(h_{\mathbf{v}_1}).$$

$k_0 \neq k_1$ case

If $k_0 \neq k_1$, $\mathbf{v}_0$ and $\mathbf{v}_1$ are not living in the same world. Therefore, it is no longer sufficient to think of primitive $\mathbf{v}$.

We introduce a new class of n-tuples to interpolate them.

Recall that the procedure in the recipe of FNW '92 allows us to construct an interaction $h_{\mathbf{v}}$ out of any n-tuple $\mathbf{v}$. However, if we would like the resulting Hamiltonian to be gapped, we need to require some additional conditions on $\mathbf{v}$. Primitivity was one sufficient condition. What we would like to do is to introduce a new condition on n-tuples which guarantees the gap, but also allows the interpolation.

ClassA [O '16]

ClassA is a set of n-tuples of matrices $\mathbb{B} = (B_\mu)_{\mu=1}^n$ which satisfies

$$\mathcal{K}_l(\mathbb{B}) := \mathrm{span}\left\{B_{\mu_1} B_{\mu_2} \cdots B_{\mu_l}\right\} = \mathrm{M}_{n_\mathbb{B}} \otimes \mathcal{D}_\mathbb{B} \Lambda_\mathbb{B}^l \left(\mathbb{1} + Y_\mathbb{B}\right)^l \subset \mathrm{M}_{n_\mathbb{B}} \otimes \mathrm{M}_{k_{L,\mathbb{B}}+k_{R,\mathbb{B}}+1}$$

for l large enough, where

- $n_\mathbb{B} \in \mathbb{N}$ and $k_{R,\mathbb{B}}, k_{L,\mathbb{B}} \in \mathbb{N} \cup \{0\}$,
- $\Lambda_\mathbb{B} = \mathrm{diag}(\lambda_{\mathbb{B},i})_{i=-k_{R,\mathbb{B}}}^{k_{L,\mathbb{B}}} \in \mathrm{M}_{k_{L,\mathbb{B}}+k_{R,\mathbb{B}}+1}$, with $\lambda_{\mathbb{B},0} = 1$ and $0 < |\lambda_{\mathbb{B},i}| < 1$, for $i \neq 0$,
- $\mathcal{D}_\mathbb{B}$ is a subalgebra of upper triangular matrices (in $\mathrm{M}_{k_{L,\mathbb{B}}+k_{R,\mathbb{B}}+1}$) with $\mathbb{1} \in \mathcal{D}_\mathbb{B}$, satisfying some additional conditions,
- $Y_\mathbb{B}$ is an upper triangular matrix in $\mathrm{M}_{k_{L,\mathbb{B}}+k_{R,\mathbb{B}}+1}$,
- B_μ is an element of $\mathrm{M}_{n_\mathbb{B}} \otimes \mathcal{D}_\mathbb{B} \Lambda_\mathbb{B} \left(\mathbb{1} + Y_\mathbb{B}\right)$.

Remark

Recall the primitivity: $\mathcal{K}_l(\mathbb{B}) = \mathrm{M}_k$*, for* l *large enough.*
$k_{R,\mathbb{B}} = k_{L,\mathbb{B}} = 0$ *corresponds to the primitivity.*

Properties of Hamiltonians given by $\mathbb{B} \in \mathrm{ClassA}$

Theorem (O '16)

If $\mathbb{B} \in \mathrm{ClassA}$, then we have $h_{\mathbb{B}} \in \mathcal{J}_{FB}$.

We also can consider ground states on left/right infinite chains, out of $h_{\mathbb{B}}$.

Theorem (O '16)

For the Hamiltonian given by some $\mathbb{B} \in \mathrm{ClassA}$, the ground state space on the right/left infinite chain is isomorphic to the state space over $\mathrm{M}_{n_{\mathbb{B}}(k_{R,\mathbb{B}}+1)} / \mathrm{M}_{n_{\mathbb{B}}(k_{L,\mathbb{B}}+1)}$.

Remark

This reflects the upper triangular form of ClassA. If $k_{L,\mathbb{B}} \neq k_{R,\mathbb{B}}$, then the ground state structure is asymmetric.
(Primitive case : $k_{L,\mathbb{B}} = k_{R,\mathbb{B}} = 0$ symmetric)

On the proof of the properties

The proof of the above properties goes almost parallel to the argument of FNW'92. However, in FNW'92, the following nice equivalent properties of primitive $\mathbf{v}$ are used everywhere.

- The spectrum of $r_{\mathbf{v}}^{-1}T_{\mathbf{v}}$ satisfies $\sigma\left(r_{\mathbf{v}}^{-1}T_{\mathbf{v}}\right)\cap\mathbb{T}=\{1\}$. 1 is a non degenerate eigenvalue of $r_{\mathbf{v}}^{-1}T_{\mathbf{v}}$. There exist a faithful $r_{\mathbf{v}}^{-1}T_{\mathbf{v}}$-invariant state $\varphi_{\mathbf{v}}$ and a strictly positive $r_{\mathbf{v}}^{-1}T_{\mathbf{v}}$-invariant element $e_{\mathbf{v}}\in\mathrm{M}_k$.
- For $m\in\mathbb{N}$ large enough, we have $\mathcal{K}_m(\mathbf{v}):=\mathrm{M}_k$

There is a sequence of nontrivial steps to carry out the arguments.

1. For ClassA, initially, we only have $\mathcal{K}_l(\mathbb{B})=\mathrm{M}_{n_{\mathbb{B}}}\otimes\mathcal{D}_{\mathbb{B}}\Lambda_{\mathbb{B}}^{l}(1+Y_{\mathbb{B}})^{l}$, but no information about $T_{\mathbb{B}}$. Therefore, we have to investigate the spectral properties of $T_{\mathbb{B}}$ by our selves, which was not studied before.
2. The proof of the intersection property in FNW'92 uses the injectivity of $\Gamma_{l,\mathbf{v}}$, which we do not have for ClassA. We have to think of an alternative argument for the proof.

Example

Let $n_0 \in \mathbb{N}$, and $k_R, k_L \in \mathbb{N} \cup \{0\}$. We fix $0 < \kappa < 1$, and set $\boldsymbol{\lambda} = (\lambda_i)_{i=-k_R}^{k_L}$ and $\mathbf{r} = (r_\alpha)_{\alpha=1}^{n_0}$ by $r_\alpha = \kappa^{\alpha-1}$, and $\lambda_j = \kappa^{|j| n_0}$.
Let $E_{i,j}$, $i,j = 1, \ldots, -k_R, \ldots, k_L$ be matrix units of $\mathbb{C}^{k_R+k_L+1}$. Set

$$\Lambda_{\mathbf{r}} = \mathrm{diag}(r_\alpha)_{\alpha=1}^{n_0}, \quad \Lambda_{\boldsymbol{\lambda}} = \mathrm{diag}(\lambda_i)_{i=-k_R}^{k_L}, \quad V_R = \sum_{j=-k_R}^{-1} E_{j,j+1}, \quad V_L = \sum_{j=0}^{k_L-1} E_{j,j+1},$$

$$\mathcal{D}_{\mathbb{B}} = \mathrm{span}\left\{ \mathbb{1}, V_R^a,\ V_L^b,\ E_{-a,b} \mid a = 1, \ldots, k_R, \quad b = 1, \ldots, k_L \right\}.$$

Let $e_{\alpha,\beta}$ be the matrix units of M_{n_0}. We define $\mathbb{B} = (B_1, \ldots, B_n)$ by $B_\mu := 0$ for $\mu \geq 3$, and

$$B_1 = \Lambda_{\mathbf{r}} \otimes \Lambda_{\boldsymbol{\lambda}}, \quad B_2 = \sum_{\alpha=2}^{n_0} (e_{1,\alpha} + e_{\alpha,1}) \otimes \Lambda_{\boldsymbol{\lambda}} + \Lambda_{\mathbf{r}} \otimes (V_R + V_L) \Lambda_{\boldsymbol{\lambda}}.$$

Remark

Use Vandermonde determinant to prove $\mathcal{K}_l(\mathbb{B}) = \mathrm{M}_{n_{\mathbb{B}}} \otimes \mathcal{D}_{\mathbb{B}} \Lambda_{\mathbb{B}}^l$. By the use of Vandermonde determinant, we can construct many examples.

Characterization of ClassA

Not only having a lot of examples, actually, ClassA has a qualitative characterization. First we list up some of the properties that is satisfied by $H(h_{\mathbb{B}})$ with $\mathbb{B} \in \mathrm{ClassA}$.

1. $h \in \mathcal{J}_{FB}$.
2. $H(h)$ is gapped with respect to the open boundary conditions.
3. Exponential decay of boundary effect : Let G_N be the spectral projection of $H_{[1,N]}(h)$ onto the lowest eigenvalue 0. There exist $0 < C_1$, $0 < s_1 < 1$, $N_1 \in \mathbb{N}$ and a nice state ω_R on the right infinite chain, such that
$$\left| \frac{\mathrm{Tr}_{[1,N]}(G_N A)}{\mathrm{Tr}_{[1,N]}(G_N)} - \omega_R(A) \right| \leq C_1 s_1^{N-l} \|A\|$$
for all $l \in \mathbb{N}$, $A \in \mathcal{A}_{[1,l]}$, and $N \geq \max\{l, N_1\}$.
4. For any ground state ψ on the right infinite chain, there exists an $l \in \mathbb{N}$ such that
$$\|\psi - \psi \circ \tau_l\| \neq 2$$
Here, τ_l is the space translation.

Characterization of ClassA

The ground state spaces of $H(h)$ satisfying 1-4, can be represented by ClassA.

Theorem (O '16)

Suppose that $H(h)$ satisfies the properties 1-4. Then there exists a $\mathbb{B} \in \mathrm{ClassA}$ satisfying the followings.

1. *The ground states of $H(h)$ and $H(h_{\mathbb{B}})$ on infinite intervals coincide.*
2. *There exist some $0 < s < 1$ and $C > 0$ such that*

$$\|G_{h,N} - G_{h_{\mathbb{B}},N}\| \leq C \cdot s^N, \quad N \in \mathbb{N}.$$

Remark

$G_{h,N}$, $G_{h_{\mathbb{B}},N}$ are the projections onto the ground state spaces of $H_{[1,N]}(h)$ and $H_{[1,N]}(h_{\mathbb{B}})$.

Interpolation via ClassA

Let us come back to the classification problem. From the classification point of view, the advantage of introducing ClassA is that we can accommodate primitive **v**s from different size of matrix algebras.

Using this fact, we can show $H(h_{\mathbf{v}_0}) \simeq_B H(h_{\mathbf{v}_1})$, for any $\mathbf{v}_0, \mathbf{v}_1$.
Hence we prove

Theorem (O '16 preprint)

For any $h_0, h_1 \in \mathcal{J}_{FB}$, we have $H(h_0) \simeq_B H(h_1)$.

Thank you.