Active

Project #: E-24-623   Cost share #: E-24-320   Rev #: 1
Center # : 10/24-6-R7052-0A0   Center shr #: 10/22-1-F7052-0A0   OCA file #:
                                                            Work type : RES
Contract#: DDM-9012020   Mod #: AMENDMENT 01   Document   : GRANT
Prime #:   Contract entity: GTRC

Subprojects ? : N   CFDA: 47.041
Main project #:   PE #: N/A

Project unit: ISYE   Unit code: 02.010.124
Project director(s):
    GOLDSMAN D M   ISYE   (404)894-2365

Sponsor/division names: NATL SCIENCE FOUNDATION   / GENERAL
Sponsor/division codes: 107   / 000

Award period: 900901   to   930228 (performance)   930531 (reports)

Sponsor amount   New this change   Total to date
    Contract value   0.00   130,000.00
    Funded   65,000.00   130,000.00
Cost sharing amount   6,542.00

Does subcontracting plan apply ?: N

Title: CHOOSING THE BEST SIMULATED SYSTEM WITH APPLICATIONS TO MANUFACTURING

PROJECT ADMINISTRATION DATA

OCA contact: Mildred S. Heyser   894-4820

 Sponsor technical contact   Sponsor issuing office

 THOM J HODGSON   MARIA VALERIO
 (202)357-5167   (202)357-9626

 NATIONAL SCIENCE FOUNDATION   NATIONAL SCIENCE FOUNDATION
 1800 G STREET, N.W.   1800 G STREET, N.W.
 WASHINGTON, D.C. 20550   WASHINGTON, D>C> 20550

Security class (U,C,S,TS) : U   ONR resident rep. is ACO (Y/N): N
Defense priority rating   :   N/A   NSF supplemental sheet
Equipment title vests with: Sponsor   GIT X

Administrative comments -
    AMEND 1 ADDS $65K AND EXTENDS PERFORMANCE PERIOD ENDING DATE THROUGH
    FEBRUARY 28, 1993.

NOTICE OF PROJECT CLOSEOUT

Closeout Notice Date 08/05/93

Project No. E-24-623_____   Center No. 10/24-6-R7052-0A0_

Project Director GOLDSMAN D M_____   School/Lab ISYE_____

Sponsor NATL SCIENCE FOUNDATION/GENERAL_____

Contract/Grant No. DDM-9012020_____   Contract Entity GTRC

Prime Contract No. _____

Title CHOOSING THE BEST SIMULATED SYSTEM WITH APPLICATIONS TO MANUFACTURING____

Effective Completion Date 930228 (Performance) 930531 (Reports)

Closeout Actions Required:	Y/N	Date Submitted
Final Invoice or Copy of Final Invoice	Y	_____
Final Report of Inventions and/or Subcontracts	N	_____
Government Property Inventory & Related Certificate	N	_____
Classified Material Certificate	N	_____
Release and Assignment	N	_____
Other _____	N	_____

CommentsEFFECTIVE DATE 9-1-90. CONTRACT VALUE $130,000._____

Subproject Under Main Project No. _____

Continues Project No. _____

Distribution Required:

Project Director	Y
Administrative Network Representative	Y
GTRI Accounting/Grants and Contracts	Y
Procurement/Supply Services	Y
Research Property Managment	Y
Research Security Services	N
Reports Coordinator (OCA)	Y
GTRC	Y
Project File	Y
Other CARL BAXTER-FMD_____	Y
FRED CAIN-OOD_____	Y

Georgia Tech

June 2, 1991

Mrs. Carol Guido, Administrative Officer
DDM, Room 1128
National Science Foundation
1800 G Street., N.W.
Washington, DC 20550

Dear Mrs. Guido,

This is a letter formally requesting my continuing grant increment on NSF Project DDM-90-12020, "Choosing the Best Simulated System, with Applications to Manufacturing." I have held this grant since last September, and I think that I have been very productive so far.

I have also enclosed my yearly report; it contains a short abstract of current and projected activities as you requested. I was wondering if you would like any copies of papers for which my NSF grant was acknowledged?

Thanks very much for your time.

Sincerely,

David Goldsman
(404)894-2365
dgoldsma@gtri01.bitnet

Annual Report on NSF Project DDM–90–12020: Choosing the Best Simulated System, with Applications to Manufacturing

David Goldsman
School of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0205

June 2, 1991

Abstract

The ongoing research concerns the problem of choosing the best of a number of simulated systems. Specific areas of research include the development and evaluation of: new variance estimators for use in ranking and selection procedures in the simulation environment, particularly in manufacturing systems; normal means selection procedures (for the cases of common known variance, common unknown variance, arbitrary unknown variance, and correlated normal populations); two-factor normal means procedures; and one- and two-factor nonparametric selection procedures (which are based on procedures for selecting the most probable multi-nomial cell).

1 Introduction

This report contains a survey of the work supported under NSF grant DDM–90–12020 during the last nine months. The work studies procedures for choosing the best of a number of simulated systems, particularly with respect to manufacturing simulations. The main contributions are the development and evaluation of:

- New variance estimators for use in ranking and selection procedures in the simulation environment (§2). Since simulations rarely produce independent and identically distributed normal observations, we cannot use the usual sample variance to estimate variances in simulations. Instead, we must develop new estimators. Our new estimators have a number of desirable properties.

1

- Normal means selection procedures (§3). We have studied a number of new procedures for selecting the normal population having the largest (or smallest) mean. Our new procedures are quite a bit more efficient than competing procedures, both in terms of the numbers of stages and observations that the procedures must take in order to guarantee the same probability requirement.

- Two-factor normal means procedures (§4). It is well known that factorial experimentation when employed in ranking and selection problems can result in considerable savings in total sample size relative to independent single-factor experimentation when both guarantee comparable probability requirements. Indeed, we have developed new two-factor normal means procedures which are much more efficient than previous methods.

- One- and two-factor multinomial selection procedures (§5). Multinomial selection procedures are very useful in the simulation environment because they can be used as nonparametric procedures. We have a fast technique for evaluating the performance of many multinomial selection procedures. Further, we have developed a number of two-factor generalizations of existing one-factor multinomial selection procedures.

2 New Variance Estimators

We have developed and analyzed a number of new variance estimators for use in the simulation environment.

2.1 Spaced Batch Means

The most commonly-used variance estimator in simulations is the familiar batch means estimator. This estimator divides a long simulation run into contiguous, equal-length batches, and then (incorrectly) assumes that the resulting batch means are independent and identically distributed normal random variables. The method of *spaced* batch means simply inserts spacers between adjacent batches in the hopes of reducing any correlation which might be present between the batch means. In terms of such criteria as variance estimator bias and confidence interval coverage, we have shown in [9] that the use of spacers never really hurts the experimenter; sometimes, especially in the presence of negative serial correlation, spaced batch means achieves dramatic improvements over regular batch means.

2.2   Orthonormal Standardized Time Series Estimators

In [8], we generalize Schruben's standardized time series area estimator in a way which yields an estimator with higher degrees of freedom. This results in a new estimator with very small variance as well as other desirable properties. Since the estimator has tractable distributional properties, it can be used to form confidence intervals for the mean of a stationary simulation process.

2.3   Cramér-von Mises Estimators

Other standardized time series estimators resemble generalizations of the Cramér-von Mises statistic. In [13], we find that these estimators also have a number of desirable properties, including comparatively low variances. Unfortunately, the estimators' distributional forms are nontrivial, and so we have not yet used them to form confidence intervals for the mean of a stationary simulation process.

2.4   Overlapping Estimators

Building on the work of Meketon, Schmeiser, and others, we have developed and analyzed a number of so-called *overlapping* estimators; these estimators are closely related to spectral estimators. In [10] and [12], we study overlapping batch means estimators, overlapping standardized time series area estimators, and overlapping Cramér-von Mises estimators. In all cases, the overlapping versions have the same bias but smaller variance than their non-overlapping counterparts.

3   Normal Means Procedures

We have developed and analyzed a number of normal means procedures, i.e., procedures for finding that one of a number of competing normal populations which has the largest (or smallest) mean.

3.1   Common Known Variance Case

In [4], we conducted a large Monte Carlo study which compared the performances of various procedures for selecting the normal population having the largest mean when the variances are *known* and equal; namely, we compared Bechhofer's original single-stage procedure, a two-stage elimination procedure due to Tamhane and Bechhofer, a truncated version of the Bechhofer, Kiefer, and Sobel sequential procedure, and the Hartmann sequential procedure with elimination. During the last few months, we have studied in [6] the performance of the Kao and Lai sequential procedure with elimination.

3.2 Common Unknown Variance Case

In [1], we conducted a large Monte Carlo study which compared the performances of various procedures for selecting the normal population having the largest mean when the variances are *unknown* and equal; clearly, this is a more realistic and useful problem than that studied in the previous subsection. Specifically, we compared the original Bechhofer, Dunnett, and Sobel two-stage procedure, the Gupta and Kim two-stage elimination procedure, and a sequential procedure with elimination due to Hartmann. Recently, we have also studied in [6] the performance of the Kao and Lai sequential elimination procedure for the common unknown variance case.

3.3 Unknown Variance Case

We are currently investigating the more realistic normal means scenario in which the variances are unknown and not necessarily equal. The idea here is to modify normal means procedures (e.g., that of Dudewicz and Dalal) for use in the presence of serial correlation.

3.4 Multivariate Case

Variance reduction techniques (VRTs) are often used in simulation to help distinguish among alternative populations. One common VRT is common random numbers, which attempts to induce positive correlation among the competing populations. In order to take advantage of such VRTs, we have begun development of multivariate normal selection procedures.

4 Two-Factor Normal Means Procedures

Suppose Π_{ij}, $i = 1, \ldots, a$, $j = 1, \ldots, b$, represent ab normal populations in a two-factor setup without interaction, i.e., the expected value of an observation from Π_{ij} is $\mu + \alpha_i + \beta_j$. The goal is to find from one two-factor experiment that population having the largest α_i and β_j.

We have shown in an earlier paper [3] that factorial experimentation when employed in ranking and selection problems can result in considerable savings in total sample size relative to independent single-factor experimentation when both guarantee comparable probability requirements. In [5], we discuss the performance of a sequential elimination procedure due to Hartmann. We find that Hartmann's procedure performs better than its predecessors in terms of the expected number of observations it takes; unfortunately, it performs more poorly in terms of the expected number of sample stages.

5   Multinomial Procedures

There are circumstances under which it might not be prudent to use an adaptation of a normal means procedure when trying to find the best of a number of simulated systems. For instance, suppose the simulation runs are rather short. Then any resulting batch means might not be independent and identically distributed normal random variables; in such a case, the normal means results outlined above would not be appropriate. Instead, the experimenter might be advised to use a *nonparametric* procedure for selecting the best system. It is well known that procedures for selecting the most probable multinomial cell can be interpreted as nonparametric.

We have developed an extremely efficient technique [11] for calculating performance characteristics of a number of existing multinomial procedures (e.g., [2]); this technique is efficient in its use of both computing time and space. This summer, we shall also work on two-factor multinomial problems in the spirit of [7] and the procedures discussed previously in §4.

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[12] ROBERT G. SARGENT, KEEBOM KANG AND DAVID GOLDSMAN (1988). "An Investigation of Small Sample Size Behavior of Confidence Interval Estimation Procedures." Technical Report, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA. Submitted for publication.

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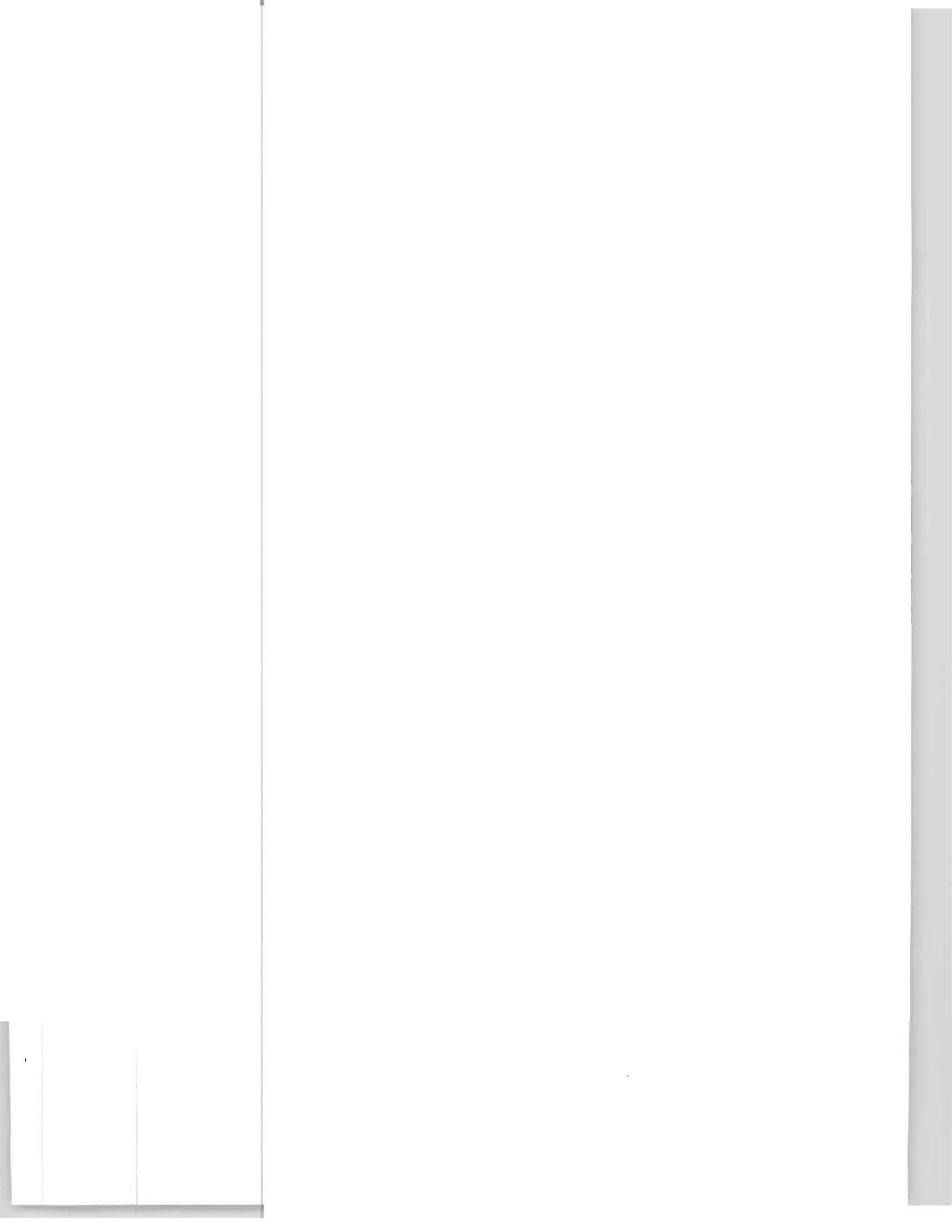
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NATIONAL SCIENCE FOUNDATION
FINAL PROJECT REPORT

PART I - PROJECT IDENTIFICATION INFORMATION

1. Program Official/Org.	DR. HANK GRANT, DR. THOMAS HODGSON
2. Program Name	DDM
3. Award Dates (MM/YY)	**From:** 9/90 **To:** 3/93
4. Institution and Address	School of ISyE Georgia Institute of Technology Atlanta, GA 30332
5. Award Number	DDM-9012020
6. Project Title	

Choosing the Best Simulated System, with Applications to Manufacturing

This Packet Contains
NSF Form 98A
And 1 Return Envelope

NSF Grant Conditions (Article 17, GC-1, and Article 9, FDP-11) require submission of a Final Project Report (NSF Form 98A) to the NSF program officer no later than 90 days after the expiration of the award. Final Project Reports for expired awards must be received before new awards can be made (NSF Grants Policy Manual Section 677).

Below, or on a separate page attached to this form, provide a summary of the completed projects and technical information. Be sure to include your name and award number on each separate page. See below for more instructions.

PART II - SUMMARY OF COMPLETED PROJECT (for public use)

The summary (about 200 words) must be self-contained and intelligible to a scientifically literate reader. Without restating the project title, it should begin with a topic sentence stating the project's major thesis. The summary should include, if pertinent to the project being described, the following items:

The general goal of our research is to develop statistical ranking and selection procedures that can be applied in the manufacturing arena. In particular, we investigate methods for choosing the "best" of a number of simulated processes, where the term "best" refers to some criterion of goodness specified by the experimenter.

Specific areas of interest include the development and evaluation of: new variance estimators for use in ranking and selection procedures in the simulation environment; normal means selection procedures (for the cases of common known variance, common unknown variance, arbitrary unknown variance, and correlated normal populations); two-factor normal means procedures; one- and two-factor nonparametric selection procedures that are based on procedures for selecting the most probable multinomial cell; and other selection procedures involving exponential and Bernoulli populations. We also seek to apply our methods to solve real-world manufacturing problems.

PART III - TECHNICAL INFORMATION (for program management use)

List references to publications resulting from this award and briefly describe primary data, samples, physical collections, inventions, software, etc. created or gathered in the course of the research and, if appropriate, how they are being made available to the research community. Provide the NSF Invention Disclosure number for any invention.

	7/28/93
Principal Investigator/Project Director Signature	Date

I certify to the best of my knowledge (1) the statements herein (excluding scientific hypotheses and scientific opinion) are true and complete, and (2) the text and graphics in this report as well as any accompanying publications or other documents, unless otherwise indicated, are the original work of the signatories or of individuals working under their supervision. I understand that willfully making a false statement or concealing a material fact in this report or any other communication submitted to NSF is a criminal offense (U.S. Code, Title 18, Section 1001).

IMPORTANT MAILING INSTRUCTIONS
Return this entire packet plus all attachments in the envelope attached to the back of this form. Please copy the information from Part I back to the Attention Block on the envelope.

(To be submitted to cognizant Program Officer upon completion of project)

The data requested below are important for the development of a statistical profile on the personnel supported by Federal grants. The information on this part is solicited in response to Public Law 99-383 and 42 USC 1885C. All information provided will be treated as confidential and will be safeguarded in accordance with the provisions of the Privacy Act of 1974. You should submit a single copy of this part with each final project report. However, submission of the requested information is not mandatory and is not a precondition of future award(s). Check the 'Decline to Provide Information' box below if you do not wish to provide this information.

Please enter the numbers of individuals supported under this grant.
Do not enter information for individuals working less than 40 hours in any calendar year.

	Senior Staff		Post-Doctorals		Graduate Students		Under-Graduates		Other Participants[1]	
	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.
A. Total, U.S. Citizens					1	3				
B. Total, Permanent Residents										
U.S. Citizens or Permanent Residents[2]:										
American Indian or Alaskan Native ..										
Asian.						1				
Black, Not of Hispanic Origin.										
Hispanic.										
Pacific Islander										
White, Not of Hispanic Origin					1	2				
C. Total, Other Non-U.S. Citizens					1					
Specify Country 1. Greece					1					
2.										
3.										
D. Total, All participants (A + B + C)					2	3				
Disabled[3]										

☐ Decline to Provide Information: Check box if you do not wish to provide this information (you are still required to return this page along with Parts I-III).

[1] Category includes, for example, college and precollege teachers, conference and workshop participants.

[2] Use the category that best describes the ethnic/racial status to all U.S. Citizens and Non-citizens with Permanent Residency. (If more than one category applies, use the one category that most closely reflects the person's recognition in the community.)

[3] A person having a physical or mental impairment that substantially limits one or more major life activities; who has a record of such impairment; or who is regarded as having such impairment. (Disabled individuals also should be counted under the appropriate ethnic/racial group unless they are classified as Total Non-U.S. Citizens.)

AMERICAN INDIAN OR ALASKAN NATIVE: A person having origins in any of the original peoples of North America and who maintains cultural identification through tribal affiliation or community recognition.

ASIAN: A person having origins in any of the original peoples of East Asia, Southeast Asia or the Indian subcontinent. This area includes, for example, China, India, Indonesia, Japan, Korea and Vietnam.

BLACK, NOT OF HISPANIC ORIGIN: A person having origins in any of the black racial groups of Africa.

HISPANIC: A person of Mexican, Puerto Rican, Cuban, Central or South American or other Spanish culture or origin, regardless of race.

PACIFIC ISLANDER: A person having origins in any of the original peoples of Hawaii; the U.S. Pacific territories of Guam, American Samoa, and the Northern Marianas; the U.S. Trust Territory of Palau; the islands of Micronesia and Melanesia; or the Philippines.

WHITE, NOT OF HISPANIC ORIGIN: A person having origins in any of the original peoples of Europe, North Africa, or the Middle East.

FINAL REPORT — Grant DDM-9012020. Choosing the Best Simulated System, with Applications to Manufacturing

David M. Goldsman   James J. Swain

July 15, 1993

Abstract

This paper reports on the research we have conducted over the last two years. The general goal of our research is to develop statistical ranking and selection procedures that can be applied in the manufacturing arena. In particular, we investigate methods for choosing the "best" of a number of simulated processes, where the term "best" refers to some criterion of goodness specified by the experimenter.

Specific areas of interest include the development and evaluation of: new variance estimators for use in ranking and selection procedures in the simulation environment; normal means selection procedures (for the cases of common known variance, common unknown variance, arbitrary unknown variance, and correlated normal populations); two-factor normal means procedures; one- and two-factor nonparametric selection procedures that are based on procedures for selecting the most probable multinomial cell; and other selection procedures involving exponential and Bernoulli populations. We also seek to apply our methods to solve real-world manufacturing problems.

Authors' addresses: David Goldsman, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, sman@isye.gatech.edu; James J. Swain, Department of Industrial and Systems Engineering, University of Alabama–Huntsville, Huntsville, AL 35899, jswain@ebs330.eb.uah.edu.

1   Introduction

As modern manufacturing systems grow more complex, computer simulation becomes the analysis tool of choice. Simulation is attractive for many reasons: it yields credible

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results when implemented properly, it has general applicability, simulation models are readily constructed, and managers are comfortable with the technique.

An important problem in manufacturing simulation, and the general topic of this research, concerns the determination of the "best" of a number of competing alternative systems or processes. The term "best" is used with respect to that characteristic of a process that the expeimenter deems to be the most important. For instance, we might ask which of a number of alternative manufacturing plant layouts maximizes the expected throughput of a particular product. Or we might wish to find the layout that minimizes product congestion at certain critical workstations.

The above are problems of *ranking and selection*. When applicable, ranking and selection procedures generally are more parsimonious with observations than methods of "classical" statistics, which are designed for fundamentally different problems – e.g., confidence interval estimation and hypothesis testing.

For a number of reasons, it has not heretofore been feasible for the simulation practitioner to exploit the wide variety of ranking and selection techniques. In fact, almost all of the ranking and selection literature has been concerned with the special case in which observations taken from a particular process are independent and identically distributed (i.i.d.) random variables. In the simulation environment, it usually is not possible to obtain such i.i.d. observations since any non-trivial simulated process has *serially correlated* and/or *nonstationary* outputs. Another problem involving the implementation of many ranking and selection techniques concerns the fact that these procedures usually assume that the competing systems have a particular distributional form, for example, normal or Bernoulli. Since simulation output processes almost always have unknown distributional forms, direct implementation of existing ranking and selection methods on such systems is difficult.

2   Research Areas

Our research aims to devise ranking and selection techniques for use with manufacturing simulations. This paper reports on the research we have conducted over the last two years (also see [30] and [31]). We have enjoyed tremendous success in our research efforts on a number of topics; specifically, we discuss the development and evaluation of:

- **Variance estimators for use in ranking and selection procedures in the simulation environment.** Since simulations rarely produce i.i.d. normal observations, we cannot use the usual sample variance to estimate the variance of a sample mean in a simulation. Luckily, it is straightforward to adapt many selection procedures for use with simulations by assuming that the *batch means* arising from a stationary (steady-state) simulation are approximately i.i.d. normal random variables. The key is to accurately and precisely estimate the variances of the batch

2

means. In §3 of this report, we propose and evaluate a number of new variance estimators. The variance estimators are based on *standardized time series* methods; they are shown to be asymptotically unbiased and to have variances that are a great deal smaller than those of competing estimators.

- **Normal means selection procedures.** In §4, we study a number of procedures for selecting the normal population having the largest (or smallest) mean. Our new procedures are quite a bit more efficient than competing procedures, both in terms of the numbers of stages and observations that the procedures must take in order to guarantee the same probability requirement. In the course of writing a practitioner's textbook, [15], on the subject, we have also produced tables that will aid experimenters in their use of our procedures.

- **Two-factor normal means procedures.** It is well known that factorial experimentation when employed in ranking and selection problems can result in considerable savings in total sample size relative to independent single-factor experimentation when both strategies guarantee comparable probability requirements. Indeed, we have investigated new two-factor normal means procedures that are a great deal more efficient than previous methods. This topic is the subject of §5.

- **One- and two-factor multinomial selection procedures.** Multinomial selection procedures are very useful in the simulation environment because they can be used as *nonparametric* procedures. We have recently developed a fast technique for evaluating the performance of many multinomial selection procedures. Further, we have proposed a number of two-factor generalizations of existing one-factor multinomial selection procedures. §6 discusses such multinomial techniques.

- **Procedures for other distributions.** During the course of our work, we devised or evaluated procedures for a variety of other problems, e.g., the selection of the exponential distribution having the largest mean and the selection of the Bernoulli population having the largest success probability. This additional work is detailed in §7.

- **Applications to real-world problems.** The ultimate goal of any applied research project is that it be used to solve real-world problems. To this end, we have begun to apply our research on actual problems in industry. This is discussed in §8 of the report.

3   Some New Variance Estimators

3.1   Motivation and Background

The motivation for this section is the *normal means selection problem*: here we wish to select the "best" one of a number of normal populations—in this case, that population having the largest (or smallest) mean.

Procedures for normal means selection problems generally call for the experimenter to take i.i.d. observations from each of the competing normal processes. Along the way, these procedures usually require the user to estimate the variances of the competing normal processes. Of course, it is almost never possible to directly obtain i.i.d. normal observations from simulation output since any non-trivial simulated process produces *nonnormal* and/or *serially correlated* and/or *nonstationary* output. So it is natural to ask: how can the normal means selection problem have any relevance in the simulation environment?

The answer is simple. Suppose we are interested in selecting that one of k simulated processes having the largest mean value. Consider the outputs from the k competing simulations, $Y_{i1}, Y_{i2}, \ldots, Y_{in}$, $i = 1, 2, \ldots, k$, where n is the (common) simulation run length. By way of example, Y_{ij} could represent the revenue from day j obtained under inventory policy i. With little loss of generality, we shall henceforth assume that $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ is a *stationary* sequence, $i = 1, 2, \ldots, k$; such an assumption is justified if we have already truncated any data suspected of containing initialization bias (cf. [29]). Thus, if $\mu_i \equiv \mathsf{E}[Y_{ij}]$, $i = 1, 2, \ldots, k$, we are interested in selecting that one of the k competing simulations corresponding to the largest of the μ_i's.

For simulation i, we can obtain approximately i.i.d. normal observations by dividing the run $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ into a number of contiguous *batches*. Specifically, suppose that we partition $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ into b nonoverlapping, contiguous batches, each consisting of m Y_{ij}'s. (Assume for convenience that the run length $n = bm$.) The ℓth batch from simulation i consists of the random variables

$$Y_{i,(\ell-1)m+1}, Y_{i,(\ell-1)m+2}, \ldots, Y_{i,\ell m},$$

$i = 1, 2, \ldots, k$, $\ell = 1, 2, \ldots, b$. The ℓth *batch mean* is the sample mean of the m observations from batch ℓ,

$$Z_{i,\ell} \equiv \frac{1}{m} \sum_{j=1}^{m} Y_{i,(\ell-1)m+j},$$

$i = 1, 2, \ldots, k$, $\ell = 1, 2, \ldots, b$.

If m is large enough (and if certain mild moment and mixing conditions hold), a central limit theorem allows us to assume that the batch means from simulation i, $Z_{i,1}, Z_{i,2}, \ldots, Z_{i,b}$, are approximately *i.i.d. normal* random variables, as desired.

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The problem that remains is that the most useful normal means selection procedures call for us to estimate the unknown variances of the observations from the competing normal populations; i.e., we will need to estimate the variances of the batch means from the k simulations—$\mathsf{Var}(Z_{i,\ell})$, $i = 1, 2, \ldots, k$. This is a difficult problem and is the crux of this section.

We now report on progress in developing and evaluating new estimators for the variance of the sample mean of a stationary stochastic process. The new variance estimators are based on *standardized time series* methods. Some necessary background on variance estimators using batch means is given in §3.2. The new standardized time series estimators are introduced and discussed in §3.3. They are shown to be asymptotically unbiased and to have variances that are a great deal smaller than those of competing estimators.

3.2   Batch Means Estimator

Suppose $X_1, X_2, \ldots, X_n$ is a stationary process with mean μ. The estimator of choice for μ is usually the sample mean $\overline{X}_n$, which is unbiased. In order to measure the precision of $\overline{X}_n$, the experimenter often estimates $\sigma_n^2 \equiv n\mathsf{Var}(\overline{X}_n)$ or the so-called *variance parameter*, $\sigma^2 \equiv \lim_{n \to \infty} \sigma_n^2$. The literature contains a great deal of work on the problem of estimating σ^2; see [16] or [38] for surveys.

We first review the popular batch means estimator for σ^2; certainly, it is the simplest to understand. As suggested in §3.1, we divide the n observations into b adjacent, nonoverlapping batches, each of length m (take $n = mb$). For $i = 1, 2, \ldots, b$, define the ith batch mean as

$$\overline{X}_{i,m} \equiv \sum_{j=1}^{m} X_{(i-1)m+j}/m.$$

The batch means estimator for σ^2 (or σ_n^2) is

$$\hat{V}_B \equiv \frac{m}{b-1} \sum_{i=1}^{b} (\overline{X}_{i,m} - \overline{X}_n)^2,$$

where $\overline{X}_n \equiv \sum_{j=1}^{n} X_j/n$ is the sample mean. Notice that $\hat{V}_B$ looks quite similar to the sample variance that one encounters in any elementary statistics text. The batch means estimator has been studied extensively, especially in the case that b is fixed and $m \to \infty$ (see, e.g., [17], [42], and [43]).

Suppose we denote the *covariance function* of the stationary stochastic process by $R_j \equiv \mathsf{Cov}(X_1, X_{1+j})$, $j = 0, \pm 1, \pm 2, \ldots$, and the quantity $\gamma \equiv -2\sum_{j=1}^{\infty} jR_j$. Then under reasonable conditions (see [17], [27], and [44]), it can be shown that

$$\mathsf{E}[\hat{V}_B] = \sigma^2 + \gamma/m + o(1/m)$$

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and

$$\text{Var}(\hat{V}_B) \approx 2\sigma^4/b.$$

So the batch means estimator is a bit biased for σ^2, but this bias decreases in m. Further, the variance of the estimator decreases as the number of batches increases.

We mention that the batch means estimator implicitly (and incorrectly) assumes that the resulting batch means are i.i.d. normal random variables. The method of *spaced* batch means inserts spacers between adjacent batches in the hopes of reducing any correlation that might be present between the batch means. In terms of such criteria as variance estimator bias and confidence interval coverage for μ, we have shown in [22] that the use of spacers never really hurts the experimenter; sometimes, especially in the presence of negative serial correlation, spaced batch means achieves dramatic improvements over regular batch means.

3.3   Standardized Time Series Estimators

We now discuss a number of competitors to the batch means variance estimator. These competitors are all based on the method of *standardized time series*. We first present some background material; then we introduce the new estimators for σ^2.

3.3.1   Background

The standardized time series of the stationary stochastic process, $X_1, X_2, \ldots, X_n$, is defined as

$$T_n(t) \equiv \frac{\lfloor nt \rfloor (\overline{X}_n - \overline{X}_{\lfloor nt \rfloor})}{\sigma \sqrt{n}} \text{ for } 0 \leq t \leq 1,$$

where $\overline{X}_j \equiv \sum_{k=1}^{j} X_k / j$, $j = 1, 2, \ldots, n$, and $\lfloor \cdot \rfloor$ is the greatest integer function. Schruben [45] shows that $T_n \Rightarrow \mathcal{B}$, where $\mathcal{B}$ is a standard Brownian bridge process and $\Rightarrow$ denotes weak convergence as $n \to \infty$. We remark that one can break the X_i's into batches, and then form asymptotically independent standardized time series from *each* batch; see §§3.3.3 and 3.3.5.

3.3.2   The Weighted Area Estimator

The first class of estimators we shall consider is related to the area under a Brownian bridge process. Let us define

$$A(n) \equiv \frac{\sum_{j=1}^{n} f(j/n)\sigma T_n(j/n)}{n}$$

and

$$A \equiv \int_0^1 f(t)\sigma\mathcal{B}(t)\,dt,$$

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where (among other technical conditions) $f(t)$ is continuous and normalized so that $\mathsf{Var}(A) = \sigma^2$. One can then show (see, e.g., [27]) that

$$A^2(n) \xrightarrow{\mathcal{D}} A^2 \sim \sigma^2 \chi_1^2,$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution as $n \to \infty$, and χ_d^2 is a χ^2 random variable with d degrees of freedom. We refer to $A^2(n)$ as the *weighted area* estimator for σ^2.

In addition to the covariance function R_j and the quantity γ from §3.2, suppose we define $F \equiv \int_0^1 f(s) \, ds$ and $\overline{F} \equiv \int_0^1 \int_0^t f(s) \, ds \, dt$. Then under mild conditions (see [21], [27], and [44]),

$$\mathsf{E}[A^2(n)] = \sigma^2 + \frac{[(F - \overline{F})^2 + \overline{F}^2]\gamma}{2n} + o\left(\frac{1}{n}\right).$$

Example 1 After a little algebra, we find that the expected value of the "unweighted" area estimator (with $f(t) = \sqrt{12}$ for all t) is $\mathsf{E}[A^2(n)] = \sigma^2 + 3\gamma/n + o(1/n)$. The expected value of the weighted area estimator with weighting function $f(t) = \sqrt{840}(3t^2 - 3t + 1/2)$, $0 \le t \le 1$, is $\mathsf{E}[A^2(n)] = \sigma^2 + o(1/n)$. In this case, we say that $A^2(n)$ is *first-order unbiased* for σ^2.

Further, if $A^4(n)$ is uniformly integrable, then it is straightforward to show that the asymptotic ($n \to \infty$) variance of the weighted area estimator is $\mathsf{Var}(A^2) = 2\sigma^4$.

3.3.3 The Weighted Area Batch Estimator

Our comments so far concerning the area estimators have assumed that we have one long batch of n observations. What if we alternatively break the n observations into b contiguous, nonoverlapping *batches*, each of size m (with $n = bm$)? Let $A_i^2(m)$, $i = 1, 2, \ldots, b$, denote the weighted area estimator formed exclusively from the ith batch of observations, $X_{(i-1)m+1}, X_{(i-1)m+2}, \ldots, X_{im}$. The weighted area batch estimator for σ^2 is $\overline{A}^2(m) \equiv \sum_{i=1}^b A_i^2(m)/b$. It is easily demonstrated that $\overline{A}^2(m) \xrightarrow{\mathcal{D}} \sigma^2 \chi_b^2/b$ as the batch size $m \to \infty$. Further, $\mathsf{E}[\overline{A}^2(m)] = \mathsf{E}[A^2(m)]$ and, if the $A_i^2(m)$'s are approximately independent, $\mathsf{Var}(\overline{A}^2(m)) \approx \mathsf{Var}(A^2(m))/b$.

These results allow us to directly compare the bias and variance of the weighted area batch estimator to the corresponding quantities for the batch means estimator. We see that the bias of the unweighted area batch estimator is about three times that of the batch means estimator; the bias of the weighted area batch estimator with each batch using the "first-order unbiased" weighting function $f(t) = \sqrt{840}(3t^2 - 3t + 1/2)$ (see Example 1) is an order of magnitude better than that of the batch means estimator. The variances of the batch means estimator and the weighted area batch estimator are about the same.

Based on the results of [45], we remark that the weighted area batch estimator and the batch means estimator are asymptotically independent as the common batch size $m \to \infty$. This implies that the two estimators can be combined (by taking the obvious linear combination) so as to yield additional degrees of freedom in the resulting χ^2 limiting random variable.

3.3.4 The Weighted Cramér-von Mises Estimator

We now look at a class of estimators for σ^2 that is related to the area under the square of a Brownian bridge process, and hence is similar to the well-known Cramér-von Mises statistic. Following the development in §3.3.2, let us define the *weighted CvM* estimator for σ^2 by

$$W^2(n) \equiv \frac{\sum_{j=1}^n g(j/n)(\sigma T_n(j/n))^2}{n}.$$

Under mild conditions, it can be demonstrated that

$$W^2(n) \xrightarrow{\mathcal{D}} W^2 \equiv \int_0^1 g(t)(\sigma \mathcal{B}(t))^2 \, dt,$$

where $g(t)$ is continuous on $[0,1]$ and normalized so that $\mathsf{E}[W^2] = \sigma^2 \int_0^1 g(t)t(1-t)\,dt = \sigma^2$. Anderson and Darling [3] derived the distribution of W^2 with $g(t) = [t(1-t)]^{-1}$; the distribution of W^2 with an arbitrary weighting function has not been explicitly determined (see [18]).

Suppose we define $G \equiv \int_0^1 g(s)\,ds$. Then under mild assumptions (see [23] and [24]), we have

$$\mathsf{E}[W^2(n)] = \sigma^2 + \frac{\gamma}{n}(G-1) + o(\frac{1}{n}).$$

Example 2 We find that the expected value of the "unweighted" CvM estimator (with $g(t) = 6$ for all t) is $\mathsf{E}[W^2(n)] = \sigma^2 + 5\gamma/n + o(1/n)$. So this estimator is a bit more biased than the unweighted area estimator (see Example 1). Now suppose $g(t) = 51 - c/2 + ct - 150t^2$, where $t \in [0,1]$ and c is real. Then the expected value of the weighted CvM estimator is $\mathsf{E}[W^2(n)] = \sigma^2 + o(1/n)$. In this case, $W^2(n)$ is *first-order unbiased* for σ^2.

We have seen that it is possible to choose weighting schemes for the weighted area and weighted CvM estimators so that both are first-order unbiased. What sets the CvM estimator apart from the area estimator? The choice of weights clearly affects the asymptotic *variance* of the CvM estimator $W^2(n)$; on the other hand, the choice of weights does *not* affect the asymptotic variance of the area estimator $A^2(n)$, which is always $2\sigma^4$.

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In fact, under reasonable assumptions (including the uniform integrability of $W^4(n)$), one can show that

$$\mathsf{Var}(W^2(n)) \to \mathsf{Var}(W^2) = 4\sigma^4 \int_0^1 g(t)(1-t)^2 \int_0^t g(s)s^2 \, ds \, dt.$$

Example 3 If $g(t) = 6$ (as in Example 2), then $\mathsf{Var}(W^2) = 4\sigma^4/5$. If $g(t) = 51 - c/2 + ct - 150t^2$, where c is real (as in Example 2), then $\mathsf{Var}(W^2) = (c^2 - 300c + 26856)\sigma^4/12600$. This quantity is minimized by $c = 150$, in which case $\mathsf{Var}(W^2) = 1.729\sigma^4$.

Notice that the variances from Example 3 are quite a bit *lower* than $2\sigma^4$, the asymptotic variance of the weighted area estimators.

We would like to choose a weighting function that minimizes the variance of the CvM estimator while satisfying the first-order unbiasedness and normalizing constraints; i.e., we wish find a weighting function $g(t)$ which minimizes $\mathsf{Var}(W^2)$ subject to $G = 1 = \int_0^1 g(t)t(1-t) \, dt$. It is straightforward to show using Lagrangian multipliers that the optimal quadratic *and* cubic polynomial weighting function is $g(t) = -24 + 150t - 150t^2$, the choice studied in Example 3 with $c = 150$. The best quartic turns out to be (after some algebra)

$$g(t) = \frac{-1310}{21} + \frac{19270t}{21} - \frac{25230t^2}{7} + \frac{16120t^3}{3} - \frac{8060t^4}{3},$$

for which $\mathsf{Var}(W^2) = 1.042\sigma^4$.

A number of variants of the CvM estimator are studied in [23], [24], [26], and [48]. These include CvM estimators resulting from batching, *overlapping* estimators (in the spirit of [39]), as well as estimators arising from still other functionals of Brownian bridges; we briefly discuss some of these estimators in the subsequent subsections.

3.3.5 The Weighted CvM Batch Estimator

Similar to our discussion in §3.3.3, we can break the n observations into b contiguous, nonoverlapping batches, each of size m (where we take $n = bm$). Let $W_i^2(m)$, $i = 1, 2, \ldots, b$, denote the CvM estimator formed exclusively from the ith batch of observations, $X_{(i-1)m+1}, X_{(i-1)m+2}, \ldots, X_{im}$. The CvM batch estimator for σ^2 is $\overline{W}^2(m) \equiv \sum_{i=1}^b W_i^2(m)/b$. Obviously, $\mathsf{E}[\overline{W}^2(m)] = \mathsf{E}[W^2(m)]$ and, if the $W_i^2(m)$'s are approximately independent, $\mathsf{Var}(\overline{W}^2(m)) \approx \mathsf{Var}(W^2(m))/b$.

3.3.6 Overlapping Estimators

Building on the work of Meketon and Schmeiser [39] and others, we have also studied a number of so-called *overlapping* estimators; these estimators are closely related to spectral

estimators. In [23], [24], [26], and [42], we consider overlapping batch means estimators, overlapping standardized time series area estimators, and overlapping Cramér-von Mises estimators. For example, let $W^2(i, m)$, $i = 1, \ldots, n - m + 1$, denote the CvM estimator formed exclusively from the observations $X_i, X_{i+1}, \ldots, X_{i+m-1}$. The CvM overlapping estimator for σ^2 is $\widetilde{W}^2(m) \equiv \sum_{i=1}^{n-m+1} W^2(i, m)/(n - m + 1)$. Obviously, $\mathsf{E}[\widetilde{W}^2(m)] = \mathsf{E}[W^2(m)]$; further, in the special case that each $W^2(i, m)$ uses the weighting function $g(t) = 6$, Goldsman and Meketon [26] show that $\mathsf{Var}(\widetilde{W}^2(m)) \approx \frac{11}{21}\mathsf{Var}(\overline{W}^2(m))$. Without going into additional details here, we remark that the overlapping versions have the same bias as and *smaller variance* than their non-overlapping counterparts.

3.3.7 L_p-Norm Estimators

An interesting generalization of the weighted area and CvM estimators is what we refer to as the L_p-norm estimator. Let us first define the Brownian bridge functional

$$L_p \equiv \left(\int_0^1 \mathcal{B}^p(t)\, dt \right)^{2/p},$$

where $p > 0$. Under mild conditions, the continuous mapping theorem implies that

$$L_p^\star(n) \equiv \frac{1}{\mathsf{E}[L_p]} \left(\frac{1}{n} \sum_{j=1}^n \sigma^p T_n^p(j/n) \right)^{2/p} \xrightarrow{\mathcal{D}} \sigma^2 \frac{L_p}{\mathsf{E}[L_p]}.$$

Thus, under uniform integrability,

$$\mathsf{E}[L_p^\star(n)] \rightarrow \sigma^2$$

and

$$\mathsf{Var}(L_p^\star(n)) \rightarrow \sigma^4 \mathsf{Var}(L_p)/(\mathsf{E}[L_p])^2.$$

We see that the estimator $L_p^\star(n)$ is asymptotically unbiased for σ^2 as $n \rightarrow \infty$ for any p of interest. We call $L_p^\star(n)$ the *standardized time series L_p-norm* estimator for σ^2. Notice that the case $p = 1$ is simply the unweighted area estimator, and the case $p = 2$ is the unweighted CvM estimator. Implementation of the above estimators requires that we know the value of $\mathsf{E}[L_p]$ for the given p. This is an easy task for $p = 1$ and 2, but Monte Carlo simulation must be carried out in order to precisely estimate $\mathsf{E}[L_p]$ for other values of p. The relevant details are given in [48].

Given that these estimators are all asymptotically unbiased, one way to compare their performances is to look at their asymptotic variances (divided by σ^4), $v_p \equiv \mathsf{Var}(L_p)/(\mathsf{E}[L_p])^2$, for different values of p. Of course, this quantity is known for $p = 1$ and 2, but must be estimated for other values of p. Table 1 gives actual or estimated values of v_p based on our Monte Carlo work. The entries from Table 1 show that v_p tends to decrease with increasing values of p. In fact, v_{10} is only about 25% of v_1. Since lower variance is desirable, these estimators deserve some closer scrutiny.

Table 1: Actual ($p = 1$ and 2) or Estimated Values of v_p

p	1	2	3	4	5	6	7	8	9	10
v_p	2.0000	0.8000	0.92	0.66	0.71	0.59	0.62	0.56	0.57	0.53

3.3.8 Orthonormally Weighted Area Estimators

In [20] and [21], we generalize Schruben's standardized time series area estimator in a (completely different) way that yields an estimator with higher degrees of freedom. The new estimator has very small variance as well as other desirable properties. Since the estimator has tractable distributional properties, it can be used to form confidence intervals for the mean of a stationary simulation process. In particular, let

$$O_i(n) \equiv \frac{\sum_{j=1}^{n} w_i(j/n)\sigma T_n(j/n)}{n}$$

for continuous weighting functions $w_i(t)$, $i = 1, 2, \ldots, d$. For large n, it can be shown that $(O_1(n), O_2(n), \ldots, O_d(n)) \xrightarrow{\mathcal{D}} (O_1, O_2, \ldots, O_d)$, a multivariate normal random vector with

$$O_i \equiv \int_0^1 w_i(t)\sigma \mathcal{B}(t)\, dt.$$

By the continuous mapping theorem,

$$V_W(n) \equiv \sum_{i=1}^{d} O_i^2(n)/d \xrightarrow{\mathcal{D}} \sum_{i=1}^{d} O_i^2/d.$$

Suppose the functions $w_1, w_2, \ldots, w_d$ are orthonormal with respect to $r(s, t) \equiv (s \wedge t)[1 - (s \vee t)]$ over the unit square, where $\wedge$ denotes minimum and $\vee$ denotes maximum. Then under mild additional conditions, $O_1, O_2, \ldots, O_d$ are i.i.d., normal $(0, \sigma^2)$ random variables. One method of obtaining orthonormal weighting functions is to use the Gram-Schmidt procedure to orthonormalize any set of linearly independent functions $u_1, u_2, \ldots, u_d$.

Example 4 Suppose $u_i(t) = t^{i-1}$, $i = 1, 2, 3$, so that the u_i's are linearly independent. Applying Gram-Schmidt with respect to $r(s, t)$ yields the orthonormal weighting functions

$$\begin{aligned} w_1(t) &= \sqrt{12}, \\ w_2(t) &= \sqrt{720}(t - 1/2), \\ w_3(t) &= \sqrt{25200}(t^2 - t + 1/5). \end{aligned}$$

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Example 5 An infinite sequence of orthonormal weighting functions is $w_i(t) = \sqrt{8}\pi i \cos(2\pi i t)$, $i = 1, 2, \ldots$.

If the w_i's are orthonormal, then we easily have that $V_W(n) \xrightarrow{\mathcal{D}} \sigma^2 \chi_d^2/d$. Assuming uniform integrability, we see that $\mathsf{E}[V_W(n)] \to \sigma^2$ (as desired) and $\mathsf{Var}(V_W(n)) = 2\sigma^4/d$, which is quite an improvement over the variance of the weighted area estimator ($d = 1$) from §3.3.2.

Additional details (including a discussion on the effects of batching) and nice properties of the orthonormally weighted area estimator are presented in [21].

4 Normal Means Procedures

We have developed and investigated a number of normal means procedures, i.e., procedures for finding that one of a number of competing normal populations which has the largest (or smallest) mean.

4.1 Motivation and Background

The basic assumptions used throughout this work are as follows:

Statistical Assumptions: Independent random samples of observations $Y_{i1}, Y_{i2}, \ldots$ ($1 \le i \le t$) are taken from $t \ge 2$ normal populations $\Pi_1, \ldots, \Pi_t$. The numbers of observations to be taken from each treatment depend on the goal of the experiment, the probability requirement to be guaranteed, and the particular procedure employed, as explained subsequently. Here Π_i has *unknown* treatment mean μ_i and *known* or *unknown* variance σ_i^2. Varous assumptions concerning the variances can be considered, e.g., $\sigma_1^2 = \cdots = \sigma_t^2$ known; $\sigma_1^2, \ldots, \sigma_t^2$ known but not necessarily equal; $\sigma_1^2 = \cdots = \sigma_t^2$ unknown (the usual Analysis of Variance assumption); $\sigma_1^2, \ldots, \sigma_t^2$ completely unknown but $\max\{\sigma_1^2, \ldots, \sigma_t^2\} \le \sigma_U^2$, where σ_U^2 is known; and $\sigma_1^2, \ldots, \sigma_t^2$ completely unknown and arbitrary.

We denote the vector of treatment means by $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_t)$ and the vector of treatment variances by $\boldsymbol{\sigma}^2 = (\sigma_1^2, \ldots, \sigma_t^2)$. The ordered μ_i-values are denoted by $\mu_{[1]} \le \cdots \le \mu_{[t]}$. Neither the values of the $\mu_{[s]}$ nor the *pairing* of the Π_i with the $\mu_{[s]}$ ($1 \le i, s \le t$) is assumed to be known. The treatment having mean $\mu_{[t]}$ is referred to as the "best" treatment. In cases where $\sigma_1^2 = \cdots = \sigma_t^2$, we denote the common variance by σ^2. Henceforth, let y_{ij} ($1 \le i \le t$, $j \ge 1$) denote the observed values of the Y_{ij}.

For the purposes of this report, the experimental goal and the associated probability (design) requirement are stated in Goal 1 and Equation (1), respectively.

Goal 1 To select the treatment associated with mean $\mu_{[t]}$.

A *correct selection* (CS) is said to be made if Goal 1 is achieved.

Probability Requirement: For constants $(\delta^\star, P^\star)$ with $0 < \delta^\star < \infty$ and $1/t < P^\star < 1$, specified prior to the start of experimentation, we require

$$P\{\text{CS}\} \geq P^\star \quad \text{whenever} \quad \mu_{[t]} - \mu_{[t-1]} \geq \delta^\star. \tag{1}$$

The probability in Equation (1) depends on the differences $\mu_i - \mu_j$ $(i \neq j, 1 \leq i, j \leq t)$, the sample size n, and also on σ^2. The constant $\delta^\star$ can be thought of as the smallest difference worth detecting. If $\mu_{[t-1]}$ and $\mu_{[t]}$ are very "close" in standardized units, i.e., if $(\mu_{[t]} - \mu_{[t-1]})/\sigma$ is small, then the sampling cost required to distinguish between the associated treatments can be prohibitive. Furthermore, if $\mu_{[t-1]}$ is very close to $\mu_{[t]}$, then it may matter little which of the associated treatments is selected. Thus, $\delta^\star$ is the smallest difference which can be detected at a reasonable sampling cost or which is of practical importance.

Clearly, it makes no sense to choose $P^\star \leq 1/t$ since $P^\star = 1/t$ can be achieved without taking *any* observations by rolling a fair t-sided die and selecting the treatment so identified as the best one. Also, we must have $P^\star < 1$ since we cannot guarantee (1) with probability unity.

Parameter configurations $\boldsymbol{\mu}$ satisfying $\mu_{[t]} - \mu_{[t-1]} \geq \delta^\star$ are said to be in the *preference-zone* for a correct selection; configurations satisfying $\mu_{[t]} - \mu_{[t-1]} < \delta^\star$ are said to be in the *indifference-zone*. Any procedure that guarantees (1) is said to be employing the so-called *indifference-zone* approach.

4.2 Common Known Variance Case

A Monte Carlo study conducted in [10] compares the performances of various procedures for selecting the normal population having the largest mean when the variances are *known* and equal; namely, we compare Bechhofer's original single-stage procedure from [5], a two-stage procedure with elimination from [46, 47], a closed version of a sequential procedure from [14], and a closed sequential procedure with elimination from [34]. In [12], we look at the performance of another sequential procedure with elimination from [37]. Subsequent Monte Carlo work, conducted in conjunction with our textbook [15], studies a new sequential procedure with elimination due to Paulson [40]. Among our many findings, we can make the following *general statement* concerning the relative performances of the sequential procedures.

- If the experimenter wishes to minimize the expected number of *total observations*, use Paulson's procedure [40].

- If the experimenter wishes to minimize the expected number of *stages*, use the closed version of the sequential procedure from [14],

4.3 Common Unknown Variance Case

In [7], we conduct a Monte Carlo study that compares the performances of various procedures for selecting the normal population having the largest mean when the variances are *unknown* and equal. This is clearly a more realistic and useful problem than that discussed in §4.2. Specifically, we compare and make recommendations concerning a two-stage procedure from [6], a two-stage procedure with elimination from [33], and a sequential procedure with elimination from [35].

4.4 Unknown Variance Case

We have worked on the still more realistic normal means scenario in which the variances are unknown and not necessarily equal. The idea here is to eventually modify normal means procedures, e.g., that of [41], for use in the presence of serial correlation; see [25] and the application in [28]. In our text [15], we provide easy-to-use tables of constants that are necessary to implement the procedure given in [41].

4.5 Multivariate Case

Variance reduction techniques (VRTs) are often used in simulation to help distinguish among alternative populations. One popular VRT is *common random numbers*, which attempts to induce positive correlation among the competing populations. In order to take advantage of such VRTs, we are also studying multivariate normal selection procedures.

5 Two-Factor Normal Means Procedures

Suppose Π_{ij}, $i = 1, \ldots, a$, $j = 1, \ldots, b$, represent ab normal populations in a two-factor setup without interaction, i.e., the expected value of an observation from Π_{ij} is $\mu + \alpha_i + \beta_j$ for some μ, $\{\alpha_i\}$, and $\{\beta_j\}$. The goal is to find that population having the largest α_i and β_j.

We have shown in [9] that factorial experimentation when employed in ranking and selection problems can result in considerable savings in total sample size relative to independent single-factor experimentation when both strategies guarantee comparable probability requirements. In [11], we discuss the performance of a sequential elimination procedure due to Hartmann [36]. We find that Hartmann's procedure performs better

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than its predecessors in terms of the expected number of observations it takes; unfortunately, it performs more poorly in terms of the expected number of sample stages.

6 Multinomial Procedures

6.1 Motivation and Background

There are circumstances under which it might not be prudent to use an adaptation of a normal means procedure when trying to find the best of a number of simulated systems. For instance, suppose the simulation runs are rather short. Then any resulting batch means might not be approximately i.i.d. normal random variables; in such a case, the normal means results outlined above would not be appropriate. Instead, the experimenter might be well-advised to use a *nonparametric* procedure for selecting the best system. It is well known that procedures for selecting the most probable multinomial cell can be interpreted as being nonparametric. With these comments in mind, we have developed and investigated a number of multinomial selection procedures, i.e., procedures for finding that multinomial cell having the largest probability.

The basic assumptions used throughout this section are as follows:

Statistical Assumptions: Independent vector-observations $\boldsymbol{X}_j = (X_{1j}, \ldots, X_{tj})$ $(j \geq 1)$ are taken from a single multinomial distribution Π having $t \geq 2$ categories with associated unknown probabilities $\boldsymbol{p} = (p_1, \ldots, p_t)$. Let p_i $(0 \leq p_i \leq 1, \sum_{i=1}^{t} p_i = 1)$ be the probability of the event E_i associated with the ith category C_i $(1 \leq i \leq t)$. The events E_i are mutually exclusive and exhaustive; $X_{ij} = 1$ [0] according as E_i does [does not] occur on the jth observation $(1 \leq i \leq t, j \geq 1)$.

We denote the ordered values of $p_1, \ldots, p_t$ by $p_{[1]} \leq \cdots \leq p_{[t]}$. Neither the values of the $p_{[s]}$ nor the pairing of the C_i with the $p_{[s]}$ $(1 \leq i, s \leq t)$ is assumed to be known. The category associated with $p_{[t]}$ is referred to as the "best" category. We denote the observed values of $\boldsymbol{X}_j$ by $\boldsymbol{x}_j = (x_{1j}, \ldots, x_{tj})$ $(j \geq 1)$. The cumulative sum for category C_i after $m \geq 1$ vector-observations have been taken is given by $y_{im} = \sum_{j=1}^{m} x_{ij}$ $(1 \leq i \leq t)$, and the associated ordered values of the y_{im} are given by $y_{[1]m} \leq \cdots \leq y_{[t]m}$.

The purpose of the experiment is stated in Goal 2 and the associated probability (design) requirement is given in (2).

Goal 2 To select the category associated with $p_{[t]}$.

A *correct selection* (CS) is said to be made if Goal 2 is achieved.

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Probability Requirement: For constants $(\theta^\star, P^\star)$ with $1 < \theta^\star < \infty$ and $1/t < P^\star < 1$, specified prior to the start of experimentation, we require

$$P\{\text{CS}\} \geq P^\star \quad \text{whenever} \quad p_{[t]}/p_{[t-1]} \geq \theta^\star. \tag{2}$$

The probability in (2) depends on the entire vector $\boldsymbol{p} = (p_1, \ldots, p_t)$ and on the number n of independent vector-observations taken from Π. The constant $\theta^\star$ can be interpreted as the "smallest $p_{[t]}/p_{[t-1]}$ ratio worth detecting."

6.2 Example of a Multinomial Procedure

By way of example, we now present a truncated sequential procedure from [8] for selecting the most probable multinomial cell.

For given t and $(\theta^\star, P^\star)$, specified prior to the start of sampling, find the truncation number n_0 from a table provided in [8] or [15]. Such truncation numbers are provided for selected $(t; \theta^\star, P^\star)$ and are used to prevent the procedure from requiring a prohibitive number of stages.

Sampling rule: At the jth stage of experimentation ($j \geq 1$), take the random vector-observation $\boldsymbol{X}_j = (X_{1j}, \ldots, X_{tj})$ from Π.

Stopping rule: At stage m ($m \geq 1$), calculate the ordered sample sums $y_{[1]m} \leq \cdots \leq y_{[t]m}$, and then calculate

$$z_m = \sum_{i=1}^{t-1} (1/\theta^\star)^{y_{[t]m} - y_{[i]m}}.$$

Stop sampling when, for the first time, *either*

$$z_m \leq (1 - P^\star)/P^\star \quad \text{or} \quad n = n_0 \quad \text{or} \quad y_{[t]m} - y_{[t-1]m} \geq n_0 - m, \tag{3}$$

whichever occurs first.

Terminal decision rule: Let N denote the value of m at the termination of sampling. Select the category that yielded the largest sample sum, $y_{[t]N}$, as the one associated with $p_{[t]}$. If exactly q of the y_{iN} are tied for largest, i.e., $y_{[t-q]N} < y_{[t-q+1]N} = \cdots = y_{[t]N}$, then select one of the q tied categories as the one associated with $p_{[t]}$ by using a random device that assigns probability $1/q$ to each.

Some artificial examples illustrate the simplicity of the procedure.

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Example 6 For $t = 3$, $P^* = 0.75$ and $\theta^* = 3.0$, entering the table in [8] tells us to truncate sampling at $n_0 = 5$ observations. However, we stop sampling by criterion (3) if

m	x_{1m}	x_{2m}	x_{3m}	y_{1m}	y_{2m}	y_{3m}
1	0	1	0	0	1	0
2	0	1	0	0	2	0

and select category C_2.

Example 7 For $t = 3$, $P^* = 0.75$ and $\theta^* = 3.0$, the truncation number $n_0 = 5$ from the table tells us to stop sampling if

m	x_{1m}	x_{2m}	x_{3m}	y_{1m}	y_{2m}	y_{3m}
1	0	1	0	0	1	0
2	1	0	0	1	1	0
3	0	1	0	1	2	0
4	1	0	0	2	2	0
5	1	0	0	3	2	0

and select category C_1.

We have developed an extremely efficient technique (see [1] and [2]) for calculating performance characteristics of a number of existing multinomial procedures (including that given above); this technique is efficient in its use of both computing time and space. We have also worked on two-factor multinomial problems in the spirit of [13] and the procedures discussed previously in §5.

7 Additional Procedures

During the course of our work, we devised or evaluated procedures for a variety of other problems. For instance, the dissertation of Auclair [4] deals with, among other problems, the selection of the exponential distribution having the largest mean; such a goal is useful in reliability and life studies. In that line of research, we devised a truncated version of the Bechhofer, Kiefer, and Sobel [14] open exponential procedure; the truncated version is somewhat more efficient in terms of sample size than its predecessor. As we undertook the writing of our textbook [15], we also studied the performances of various procedures to select the Bernoulli population having the largest success probability. Such problems have wide applicability in the medical field. In particular, our text compares procedures from [14] and Paulson [40].

8 Real-World Applications

A number of applications for our work are discussed in such sources as [28], [32], and [38]. However, the goal is to use our research in a constructive way in actual manufacturing arenas.

Together with P. A. Farrington and S. L. Messimer, of the University of Alabama in Huntsville, we have been working with a large electronics manufacturer in Huntsville, Alabama. The work consists of building, validating, and using simulation models to assist management at the manufacturing plant in the choice of alternatives for existing and planned high-volume production lines. The work is currently supporting two graduate and one undergraduate research students. Our efforts up to this point have centered on the construction of the simulation models. Work is about to begin on using these models to choose a configuration of a line for a model-year 1996 product. Ranking and selection methodology will be among the techniques used to determine the recommended configuration (see [19]).

The work on this grant will be among the research areas pursued with electronics manufacturers in the Huntsville region. Huntsville is home to approximately five billion dollars in electronics manufacturing annually, and we are discussing ways with local companies to improve facility utilization and planning through design and simulation. A proposed project will be the design of a system for the construction and maintenance of simulation models, and the mechanisms for experimental design when alternatives are to be ranked for decision making. The strategies for ranking and selection investigated here will be a part of the overall strategy being researched. Funding through the local companies, the State of Alabama, or some partnership with these and federal agencies (e.g., ARPA or NSF) is also being pursued.

9 Summary

This paper reported on some of the research we have conducted over the last two years concerning the use of ranking and selection procedures in the discrete-event simulation environment. The motivation for this work concerned problems in the manufacturing arena—problems that are only amenable to solution via simulation. In particular, our research addressed the following rich classes of topics:

- Estimation of the variance of the sample mean.

- Procedures for the normal means selection problem.

- Procedures for two-factor normal means selection problems.

- Procedures for nonparametric (multinomial) selection problems.

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- Procedures for other problems such as exponential or Bernoulli selection.

- Real-world applications of our work.

We have enjoyed a great deal of success with this line of research. We believe that there is a great deal of interesting research left to be done in these areas.

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