# THE FILIPPOV MOMENTS SOLUTION ON THE INTERSECTION OF TWO AND THREE MANIFOLDS

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Fabio V. Difonzo

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# THE FILIPPOV MOMENTS SOLUTION ON THE INTERSECTION OF TWO AND THREE MANIFOLDS

Approved by:

Professor Luca Dieci, Advisor
School of Mathematics
*Georgia Institute of Technology*

Professor Rafael de la Llave
School of Mathematics
*Georgia Institute of Technology*

Professor Wassim M. Haddad
School of Aerospace Engineering
*Georgia Institute of Technology*

Professor Sung Ha Kang
School of Mathematics
*Georgia Institute of Technology*

Professor Hao Min Zhou
School of Mathematics
*Georgia Institute of Technology*

Date Approved: 2 April 2015

*Ibam forte via Sacra...*

*nescio quid meditans nugarum, totus in illis...*

*(Quintus Horatius Flaccus, Satira, I, 9)*

*Alla mia famiglia e a Sandèrme,*

*la cui lontananza*

*ha tutto il necessario tenuto vicino.*

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*Un paese ci vuole, non fosse che per il gusto di andarsene via. Un paese vuol dire non essere soli, sapere che nella gente, nelle piante, nella terra c’è qualcosa di tuo, che anche quando non ci sei resta ad aspettarti.* (La luna e i falò, *C. Pavese*).

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# SUMMARY

In this thesis, we study the Filippov moments solution for differential equations with discontinuous right-hand side. In particular, our aim is to define a suitable Filippov sliding vector field on a co-dimension 2 manifold $\Sigma$, intersection of two co-dimension 1 manifolds with linearly independent normals, and then study the dynamics provided by this selection. More specifically, we devote Chapter 1 to motivate our interest in this subject, and presenting several problems from control theory, nonsmooth dynamics, vehicle motion, and neural networks. We then introduce the co-dimension 1 case and basic notation, from which we set up, in the most general context, our specific problem. In Chapter 2, we propose and compare several approaches in selecting a Filippov sliding vector field for the particular case of $\Sigma$ nodally attractive. In Chapter 3, we focus on *moments solution*, that is the main and novel mathematical object presented and studied in this thesis. There, we extend the validity of the moments solution to $\Sigma$ attractive under general sliding conditions, proving results about the smoothness of the Filippov sliding vector field on $\Sigma$, tangential exit at first-order exit points, and uniqueness at potential exit points among all other admissible solutions. In Chapter 4, we propose a completely new and different perspective from which one can look at the problems; namely, we study minimum variation solutions for Filippov sliding vector fields in $\mathbb{R}^3$, taking advantage of the relatively easy form of the Euler-Lagrange equation provided by the analysis, and of the orbital equivalence that we have in the eventuality $\Sigma$ does not have any equilibrium points on it. We further remove this assumption and extend our results. In Chapter 5, several examples and numerical implementations are given, with which we corroborate our theoretical results and show that selecting a Filippov sliding vector

field on $\Sigma$ without the required properties of smoothness and exit at first-order exit points ends up dynamics that make no sense, developing undesirable singularities. Finally, Chapter 6 presents an extension of the moments method to co-dimension 3 and higher: this is the first result which provides a unique admissible solution for this problem.

# Chapter I

# INTRODUCTION AND BACKGROUND

## 1.1 Motivation and Scope

Discontinuous dynamical systems arise naturally in a disparate set of engineering, physics and biological applications. For example, in control theory, open-loop bang-bang controllers that switch discontinuously between extreme values of the bounded inputs in order to generate minimum-time trajectories from one state to another, or closed-loop bang-bang controllers that regulate physical states, are governed (see [43, 44]) by discontinuous differential equations. Also, the theory of sliding mode control has developed a systematic approach to the design of discontinuous feedback controllers for stabilization [3, 52, 53, 54]. As Cortes highlights [11], a result due to Brockett [8, 50] implies that many control systems, including driftless systems, cannot be stabilized by means of continuous state-dependent feedbacks. As a result, one is forced to consider either time-dependent or discontinuous feedback (see also [7, 41] and Example 1.1.3 below). In non-smooth mechanics, evolution of rigid bodies undergoing friction, slip, stick or impacts, such as suspension bridges or robotic manipulation of objects with mechanical contacts or, also, motion of vehicles, is described by non-smooth dynamical systems [28, 29, 39, 40]. In biology, piecewise-linear models have been successfully applied to networks of interactions, such as genetic regulatory networks, which are not originally discontinuous, providing an insightful direction for holding together the description and the dynamical analysis of regulatory systems [10, 12, 30, 31, 48].

**Example 1.1.1** ([44]). *Consider the system*

$$\ddot{x} = u, \quad u \in [-1, 1],$$

*which can represent a car with position $x \in \mathbb{R}^2$ with bounded acceleration $u$ acting as the control (negative acceleration corresponds to braking). If we want study the problem of* parking *the car at the origin, i.e., bringing it to rest at $x = 0$, in minimal time, then we will necessarily obtain that the optimal control $u^*$ takes only the values $\pm 1$, and switches between them at most once. The initial sign and the switching time of course depend on the initial condition. This kind of functions is called* bang-bang control.

**Example 1.1.2** ([28]). *Consider a non-smooth dynamical system, the solution of which slides on the intersection of two surfaces. In [28] the author studies a mechanical system composed by two blocks on a moving belt, as depicted in Figure 1. The velocity of the belt is constant and is called the driving velocity $v$. Each block is connected to a fixed support and to the other block by elastic springs. The surface between the blocks and the belt is rough so that the belt exerts a dry friction force on each block that sticks on the belt to the point where the elastic forces due to the springs exceed the maximum static force. At this point the blocks start slipping and the slipping motion will continue to the point where the velocity of the block will equal that of the belt and the elastic forces will be equilibrated by the static friction force. The continuous repetition of this type of motions generates a stick-slip oscillation. This mechanical system may be described in its simplest form by the following set of differential equations:*

$$\begin{cases} m_1 x_1'' = -k_1 x_1 - k_{12}(x_1 - x_2) + f_{k1}(x_1' - v), \\ m_2 x_2'' = -k_2 x_2 - k_{12}(x_2 - x_1) + f_{k2}(x_2' - v) \end{cases} \tag{1.1.1}$$

*where $x_i(t)$ is the displacement, $m_i$ is the mass, $f_{ki}(x_i' - v)$ the kinetic friction force of the i-th block, $k_1$, $k_2$, $k_{12}$ suitable constants. The kinetic force has the form $f_{k2}(x' -$*

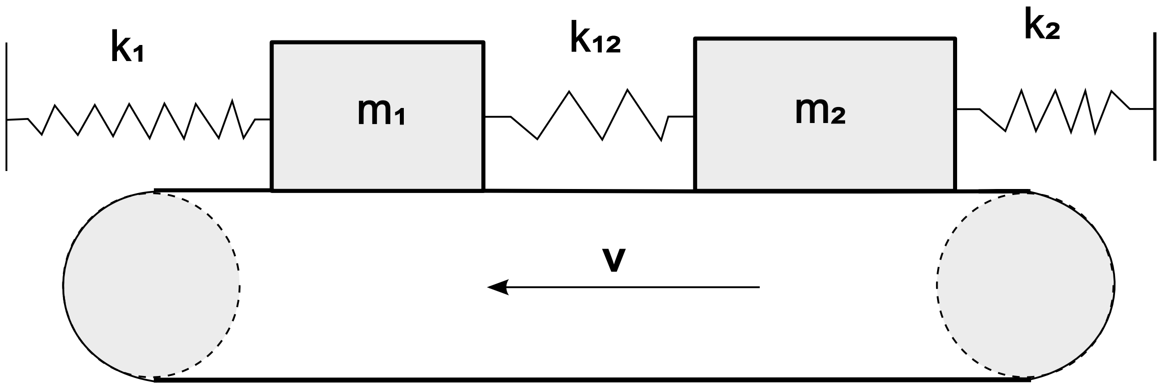


Figure 1: Stick-slip 2 block mechanical system described by (1.1.1).

$v) = \beta f_{k1}(x' - v)$ *with:*

$$f_{k1}(x'-v) := \begin{cases} \frac{1-\delta}{1-\gamma(x'-v)} + \delta + \eta(x'-v)^2, & x' < v, \\ -\frac{1-\delta}{1-\gamma(x'-v)} - \delta - \eta(x'-v)^2, & x' > v, \end{cases} \tag{1.1.2}$$

*where* $\beta$, $\gamma$, $\delta$, $\eta$ *are suitable constants. We will analyze this specific example in Chapter 5.*

**Example 1.1.3.** *In [8], Brockett stated the following problem:*

**Problem 1.1.4.** *Given three matrices* $A$, $B$, $C$, *what conditions ensure the existence of a matrix* $K(t)$ *such that the system*

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + BK(t)y(t), & x \in \mathbb{R}^n, \\ y(t) = Cx(t), \end{cases} \tag{1.1.3}$$

*is asymptotically stable.*

*Stabilizing mechanical systems often necessitates to select specific matrices* $K(t)$. *These matrices could be periodic on* $[0, T]$ *and such that*

$$\int_0^T K(t)\,\mathrm{d}t = 0.$$

*For example, let us consider a linear approximation near an equilibrium point for the pendulum with vertically oscillating suspension point:*

$$\ddot{\theta}(t) + \alpha\dot{\theta}(t) + (K(t) - \omega_0^2)\theta(t) = 0, \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0, \quad t \geq 0. \tag{1.1.4}$$

*where $\alpha$ and $\omega_0$ are positive numbers. Here, the most common choice for $K(t)$ is $\beta \sin \omega t$, or*

$$K(t) = \begin{cases} \beta, & t \in \left[0, \frac{T}{2}\right), \\ -\beta, & t \in \left[\frac{T}{2}, T\right). \end{cases} \tag{1.1.5}$$

*For such functions $K(t)$ as in (1.1.5), the effect of stabilization of the upper equilibrium point is well known for large $\omega$ and, consequently, small $T$. In [41], Leonov gives necessary and sufficient conditions for the stabilization of a system of the type (1.1.3) by periodic piecewise constant functions $K(t)$ that solve the Brockett's problem 1.1.4; moreover, it is shown that low-frequency stabilization ($T \gg 1$) is possible for (1.1.4) with $K(t)$ of the form (1.1.5).*
*Also, in [7] necessary and sufficient conditions for a wider class of stabilizing matrices $K(t)$ relative to (1.1.3) are given.*

In this thesis, we will chiefly focus on discontinuous dynamical systems with two intersecting discontinuity surfaces, being their intersection $\Sigma$ *attractive* for the nearby dynamics: our main purpose is to introduce and analyze a novel way to define a *sliding vector field* on $\Sigma$, that we will call *Filippov moments sliding vector field*, so to keep the same smoothness on $\Sigma$ of the problem initial data, and pursue smooth tangential exits at first order exit points (see Definition 1.4.10).
The plan of this thesis is as follows. In the remainder of this chapter, we present the general problem for the case of one surface of discontinuity (co-dimension 1) and for the case of two intersecting sufaces (co-dimension 2). We introduce Filippov convexification method and the concept of Filippov solutions. In Chapter 2, we will compare several approaches to select Filippov sliding vector fields for the case when $\Sigma$ (co-dimension 2) is nodally attractive, and introduce the moments solution [15]. In Chapter 3, we will completely justify the moments solution under general attractivity conditions [18]. In Chapter 4, we will propose minimum variation solutions to our

problem, restricting ourselves to $\mathbb{R}^3$ [16, 20]. In Chapter 5, we will provide implementation and numerical results. Finally, in Chapter 6, we will propose an extension of the moments method to the nodally attractive case in co-dimension 3 [17].

## 1.2 The problem and Filippov solutions

We are interested in piecewise smooth differential systems of the following type:

$$\dot{x}(t) = f(x(t)) \ , \ \ f(x(t)) = f_i(x(t)) \ , \ \ x \in R_i \ , \ i = 1, \dots, N \ , \ \ t \in [0, T] \ . \tag{1.2.1}$$

Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, so that (locally) $\mathbb{R}^n = \overline{\bigcup R_i}$, and on each region $R_i$ the function $f$ is given by a smooth vector field $f_i$. Further, the regions $R_i$'s are separated by manifolds defined as 0-sets of smooth (at least $\mathscr{C}^2$) scalar functions $h_i$: $\Sigma_i := \{x \in \mathbb{R}^n \ : \ h_i(x) = 0\}$, $i = 1, \dots, p$ (and, for us, $2^p = N$). From (1.2.1), in general the vector field is not properly defined on the boundaries of the $R_i$'s, where a classical solution ceases to exist. A successful definition of generalized solutions for problems as in (1.2.1) is due to Filippov, [26]. These are absolute continuous functions $x(t)$, for $t \in [0, T]$, such that $\dot{x}(t) \in \mathcal{F}(x(t))$ for almost all $t \in [0, T]$, and where $\mathcal{F}(x)$ is the convex hull of the values of $f(x)$ obtained approaching $x$ through a region $R_i$. Formally:

$$\mathcal{F}(x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{\text{co}} \{ f\,(B(x, \delta)) \setminus S\} \ , \tag{1.2.2}$$

$\mu$ being Lebesgue measure on $\mathbb{R}^n$. Under mild conditions (boundedness and upper semicontinuity of $\mathcal{F}$), existence of Filippov solutions is guaranteed, but uniqueness is much more elusive, as it depends on the interaction of neighboring vector fields on the boundaries of the regions $R_i$'s.

## 1.3 Co-dimension 1: attractivity, existence and uniqueness

The basic theory of Filippov (see [26]) covers fully the case of two regions separated by a manifold $\Sigma$ defined as the 0-set of a smooth scalar valued function $h$. One has the following system:

$$
\begin{aligned}
&\dot{x} = f_1(x) \ , \ \ x \in R_1 \ , \quad \text{and} \quad \dot{x} = f_2(x) \ , \ \ x \in R_2 \ , \\
&\Sigma := \{x \in \mathbb{R}^n \ : \ h(x) = 0\} \ , \quad h : \ \mathbb{R}^n \to \mathbb{R} \, ,
\end{aligned}
\tag{1.3.1}
$$

where $h$ is a $\mathcal{C}^k$ function, with $k \geq 2$, $\nabla h$ is bounded away from 0 for all $x \in \Sigma$, hence near $\Sigma$, and (without loss of generality) we label $R_1$ such that $h(x) < 0$ for $x \in R_1$, and $R_2$ such that $h(x) > 0$ for $x \in R_2$.

**Remark 1.3.1.** *We stress that the direction of time, the time arrow, is crucial. In this thesis, we will tacitly assume of proceeding forward in time. For this reason, as we clarify below, and unlike -say- the case of a boundary value problem, we believe it is important to take into account the attractivity properties of the discontinuity surface $\Sigma$, and to have these reflected into the behavior of trajectories on/near $\Sigma$.*

The interesting case is when trajectories reach $\Sigma$ from $R_1$ (or $R_2$), and one has to decide what happens next. To answer this question, it is useful to look at the components of the two vector fields $f_{1,2}$ orthogonal to $\Sigma$:

$$
w_1 := \nabla h(x)^\top f_1(x) \ , \ \ w_2 := \nabla h(x)^\top f_2(x) \ , \ \ x \in \Sigma \, . \tag{1.3.2}
$$

Here, $\Sigma$ is called *attractive in finite time* if for some positive constant $c$, we have

$$
\nabla h(x)^\top f_1(x) \geq c > 0 \ \text{ and } \ \nabla h(x)^\top f_2(x) \leq -c < 0 \ , \tag{1.3.3}
$$

for $x \in \Sigma$ and in a neighborhood of $\Sigma$. In this case, trajectories starting near $\Sigma$ must reach it, transversally, and remain there, giving rise to so-called *sliding motion.* A vector field associated to sliding motion is called *sliding vector field.* Filippov proposal (see (1.2.2)) is to take as sliding vector field on $\Sigma$ a convex combination of $f_1$ and $f_2$, $f_{\mathrm{F}} := (1-\alpha) f_1 + \alpha f_2$, with $\alpha$ chosen so that $f_{\mathrm{F}} \in T_\Sigma$ ($f_{\mathrm{F}}$ is tangent to $\Sigma$ at each $x \in \Sigma$):

$$
\dot{x} = (1-\alpha) f_1 + \alpha f_2 \ , \quad \alpha = \frac{\nabla h(x)^\top f_1(x)}{\nabla h(x)^\top \big(f_1(x) - f_2(x)\big)} \, . \tag{1.3.4}
$$

At the same time, Filippov theory also provides *first order exit conditions*: whenever $\alpha = 0$, respectively $\alpha = 1$, one should expect to leave $\Sigma$ to enter in $R_1$ with vector field $f_1$, respectively enter $R_2$ with vector field $f_2$. [In other words, if the sliding vector field has aligned with either –but not both– $f_1$ or $f_2$, then generically (for smooth $f_1, f_2$) we should leave $\Sigma$ as above].

We note that, during sliding motion, the right-hand side of (1.3.4) is a smooth vector field. This allows to study the dynamics during sliding motion using classical tools from the theory of dynamical systems with smooth vector fields; in particular, stability and bifurcation studies for equilibria on $\Sigma$, and for periodic orbits that may lie at least partly on $\Sigma$, have been extensively studied (e.g., see [13]). □

## 1.4 Co-dimension 2: general attractivity by subsliding

Our specific interest in this thesis is the case of (1.2.1) with $N = 4$. Now we will assume that the $R_i$'s are (locally) separated by two intersecting smooth manifolds of co-dimension 1. That is, we have

$$\Sigma_1 = \{x : \ h_1(x) = 0\} \ , \ \Sigma_2 = \{x : \ h_2(x) = 0\} \ , \ h_i : \mathbb{R}^n \to \mathbb{R} \ , \ i = 1, 2 \ , \ \Sigma = \Sigma_1 \cap \Sigma_2 \ , \tag{1.4.1}$$

and we will also use the following notation

$$\Sigma_1^{\pm} = \{x : \ h_1(x) = 0 \ , \ \ h_2(x) \gtrless 0\} \ , \ \ \Sigma_2^{\pm} = \{x : \ h_2(x) = 0 \ , \ \ h_1(x) \gtrless 0\} \ . \tag{1.4.2}$$

We will always assume that $h_1$, $h_2$ are $\mathscr{C}^k$ functions, with $k \geq 2$, that $\nabla h_1(x) \neq 0, x \in \Sigma_1$, $\nabla h_2(x) \neq 0, x \in \Sigma_2$, and further that $\nabla h_1(x)$ and $\nabla h_2(x)$ are linearly independent for $x$ on (and in a neighborhood of) $\Sigma$.

So, we have four different regions $R_1$, $R_2$, $R_3$ and $R_4$ with the four different smooth vector fields $f_i$, $i = 1, \dots, 4$, in these regions:

$$\dot{x} = f_i(x) \ , \ \ x \in R_i \ , \ \ i = 1, \dots, 4 \ . \tag{1.4.3}$$

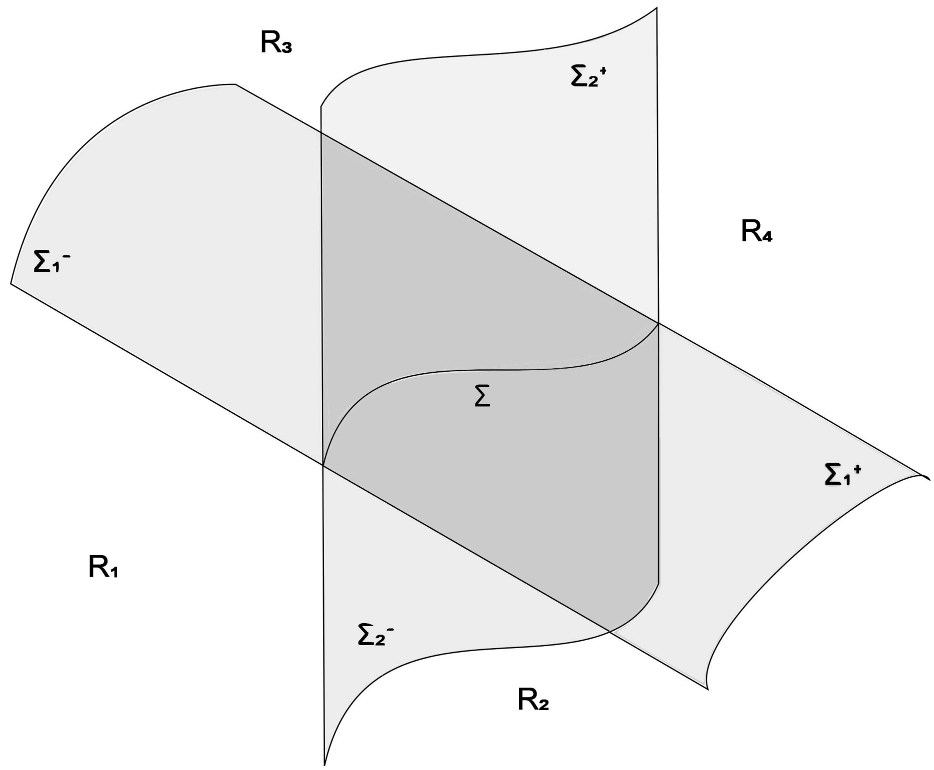


Figure 2: Regions $R_i$'s, subsurfaces $\Sigma^{\pm}_{1,2}$ and the co-dimension 2 manifold $\Sigma$.

Without loss of generality, we will label these regions as follows:

$$\begin{aligned} &R_1: \ f_1 \quad \text{when} \quad h_1 < 0 \ , \ h_2 < 0 \ , \qquad && R_2: \ f_2 \quad \text{when} \quad h_1 < 0 \ , \ h_2 > 0 \ , \\ &R_3: \ f_3 \quad \text{when} \quad h_1 > 0 \ , \ h_2 < 0 \ , \qquad && R_4: \ f_4 \quad \text{when} \quad h_1 > 0 \ , \ h_2 > 0 \ . \end{aligned} \tag{1.4.4}$$

We are specifically interested in the case when trajectories starting near $\Sigma$ will reach it, transversally (and in finite time), a case refereed to as having $\Sigma$ attractive for nearby dynamics. To characterize this situation, it is again convenient to consider the components of the vector fields orthogonal to $\Sigma$. That is, we let (cfr. with (1.3.2))

$$\begin{aligned} &w_1^1 = \nabla h_1^\top f_1 \ , \ w_2^1 = \nabla h_1^\top f_2 \ , \ w_3^1 = \nabla h_1^\top f_3 \ , \ w_4^1 = \nabla h_1^\top f_4 \ , \\ &w_1^2 = \nabla h_2^\top f_1 \ , \ w_2^2 = \nabla h_2^\top f_2 \ , \ w_3^2 = \nabla h_2^\top f_3 \ , \ w_4^2 = \nabla h_2^\top f_4 \ , \end{aligned} \tag{1.4.5}$$

and we will use the notation $w_i = (w_i^1, w_i^2) \in \mathbb{R}^2$, $i = 1, 2, 3, 4$, for those four points in $\mathbb{R}^2$.

Table 1: Nodal Attractivity.

| Component | $i=1$ | $i=2$ | $i=3$ | $i=4$ |
|---|---|---|---|---|
| $w_i^1$, $i=1:4$ | $>0$ | $>0$ | $<0$ | $<0$ |
| $w_i^2$, $i=1:4$ | $>0$ | $<0$ | $>0$ | $<0$ |

**Example 1.4.1.** *The simplest case of attractive $\Sigma$ is when it is* nodally attractive. *This means that on each of $\Sigma_{1,2}^{\pm}$ there is sliding motion toward the intersection $\Sigma$. These sliding motions on $\Sigma_{1,2}^{\pm}$ occur with Filippov sliding vector fields given as in* (1.3.4)*, henceforth labeled $f_{F_{1,2}^{\pm}}$. Namely,*

$$
\begin{aligned}
f_{F_1^+} &= (1-\alpha^+)f_2 + \alpha^+ f_4 \ , \ \alpha^+ = \left[\frac{\nabla h_1^\top f_2}{\nabla h_1^\top (f_2 - f_4)}\right]_{x\in\Sigma_1^+} = \frac{w_2^1}{w_2^1 - w_4^1} \ , \\
f_{F_1^-} &= (1-\alpha^-)f_1 + \alpha^- f_3 \ , \ \alpha^- = \left[\frac{\nabla h_1^\top f_1}{\nabla h_1^\top (f_1 - f_3)}\right]_{x\in\Sigma_1^-} = \frac{w_1^1}{w_1^1 - w_3^1} \ , \\
f_{F_2^+} &= (1-\beta^+)f_3 + \beta^+ f_4 \ , \ \beta^+ = \left[\frac{\nabla h_2^\top f_3}{\nabla h_2^\top (f_3 - f_4)}\right]_{x\in\Sigma_2^+} = \frac{w_3^2}{w_3^2 - w_4^2} \ , \\
f_{F_2^-} &= (1-\beta^-)f_1 + \beta^- f_2 \ , \ \beta^- = \left[\frac{\nabla h_2^\top f_1}{\nabla h_2^\top (f_1 - f_2)}\right]_{x\in\Sigma_2^-} = \frac{w_1^2}{w_1^2 - w_2^2} \ .
\end{aligned}
\tag{1.4.6}
$$

*Finally, at first order, we note that nodal attractivity is guaranteed by the signs of Table 1 for the entries of $w_i^j$, $i=1,\dots,4$, $j=1,2$.* □

The next characterization of attractivity for $\Sigma$ was called *attractivity through sliding* in [19].

**Definition 1.4.2** (Partial Nodal Attractivity; [19])**.** *We say that $\Sigma$ is* partially nodally attractive, *or* attractive through sliding, *if the following conditions hold:*

**(a)** $\begin{bmatrix} w_j^1(x) \\ w_j^2(x) \end{bmatrix}$ *does not have the same sign of* $\begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}$ *for $x \in R_j$, $j=1,2,3,4$ ;*

**(b)** *at least one of the following conditions is satisfied on $\Sigma$, and in a neighborhood of $\Sigma$:*

$(1^+)$ $\det \begin{bmatrix} w_2^1 & w_4^1 \\ 1 & 1 \end{bmatrix} > 0$ *together with* $(1_a^+)$*:* $(1-\alpha^+)w_2^2 + \alpha^+ w_4^2 < 0$*;*

$(1^-)$ $\det \begin{bmatrix} w_3^1 & w_1^1 \\ 1 & 1 \end{bmatrix} < 0$ *together with* $(1_a^-)$*:* $(1-\alpha^-)w_1^2 + \alpha^- w_3^2 > 0$*;*

$(2^+)$ $\det \begin{bmatrix} w_4^2 & w_3^2 \\ 1 & 1 \end{bmatrix} < 0$ *together with* $(2_a^+)$*:* $(1-\beta^+)w_3^1 + \beta^+ w_4^1 < 0$*;*

$(2^-)$ $\det \begin{bmatrix} w_1^2 & w_2^2 \\ 1 & 1 \end{bmatrix} > 0$ *together with* $(2_a^-)$*:* $(1-\beta^-)w_1^1 + \beta^- w_2^1 > 0$*;*

**(c)** *if any of* $(1^\pm)$ *or* $(2^\pm)$ *is satisfied, then* $(1_a^\pm)$ *or* $(2_a^\pm)$ *must be satisfied as well.*

*Above, we note that the quantities* $\alpha^\pm$*,* $\beta^\pm$ *(as given in* (1.4.6)*), are well defined whenever the relevant conditions* $(1^\pm)$*,* $(2^\pm)$ *hold.* □

The next result gives a handy rewriting of $(1_a^\pm)$, $(2_a^\pm)$ in Definition 1.4.2.

**Lemma 1.4.3.** *Let any of* $(1^\pm)$ *and/or* $(2^\pm)$ *in Definition 1.4.2 hold. Then, the corresponding conditions* $(1_a^\pm)$*,* $(2_a^\pm)$ *are equivalent, respectively, to the following:*

$$(\tilde{1_a^+}) : \det \begin{bmatrix} w_2 & w_4 \end{bmatrix} < 0; \qquad (\tilde{1_a^-}) : \det \begin{bmatrix} w_3 & w_1 \end{bmatrix} < 0;$$

$$(\tilde{2_a^+}) : \det \begin{bmatrix} w_4 & w_3 \end{bmatrix} < 0; \qquad (\tilde{2_a^-}) : \det \begin{bmatrix} w_1 & w_2 \end{bmatrix} < 0.$$

*Proof.* Let us prove equivalence between $(1_a^+)$ and $(\tilde{1_a^+})$. The others are analogous.

Since $(1^+)$, $(1_a^+)$, (1.4.6), hold, we get that

$$\frac{-w_4^1 w_2^2 + w_2^1 w_4^2}{\det \begin{bmatrix} w_2^1 & w_4^1 \\ 1 & 1 \end{bmatrix}} < 0 \ , \quad \text{from which} \det \begin{bmatrix} w_2 & w_4 \end{bmatrix} < 0 \ .$$

Conversely, if $\det \begin{bmatrix} w_2 & w_4 \end{bmatrix} < 0$, since $(1^+)$ holds, we get $(1_a^+)$ at once. □

**Remark 1.4.4.** *Partial nodal attractivity (which of course includes nodal attractivity as a special case) implies that one has sliding motion on (at least) one of* $\Sigma^{\pm}_{1,2}$*, directed towards* $\Sigma$*, and no sliding motion on any of* $\Sigma^{\pm}_{1,2}$*, away from* $\Sigma$*. A typical solution trajectory starting near* $\Sigma$ *will approach (in finite time) the intersection* $\Sigma$*, by first sliding on one of* $\Sigma_1$ *or* $\Sigma_2$*, directed towards* $\Sigma$ *(of course, a trajectory may also reach* $\Sigma$ *directly from within one of the regions* $R_i$*'s, but this is a less likely event).*

**Remark 1.4.5.** *We also note that partial nodal attractivity is not an exclusive characterization of attractivity of* $\Sigma$*. Namely,* $\Sigma$ *may also be* spirally attractive*. In this case, there is no attractivity toward* $\Sigma$ *through sliding on any of* $\Sigma^{\pm}_1$*,* $\Sigma^{\pm}_2$*, and trajectories reach* $\Sigma$ *by spiraling around it. See [14] for the characterization of spirally attractive* $\Sigma$*.*

### 1.4.1 Co-dimension 2: general ambiguity

At this point, we may envision having the following scenario for a solution trajectory of a system (1.4.3), with attractive $\Sigma = \Sigma_1 \cap \Sigma_2$.

- It starts in a region $R_i$ for some $i = 1, 2, 3, 4$, until
- it reaches transversally one of $\Sigma^{\pm}_{1,2}$;
- then, it begins sliding on $\Sigma^{\pm}_{1,2}$, until
- it reaches transversally the intersection $\Sigma$. What happens then?

Now, when $\Sigma$ is attractive, a trajectory starting on $\Sigma$ cannot leave $\Sigma$. But, how should a solution trajectory evolve on $\Sigma$? In the class of Filippov solutions, we will need to have that $\dot{x} \in \mathcal{F}(x)$ as in (1.2.2), and further that $\dot{x}$ lies on the tangent plane to $\Sigma$, for any $x \in \Sigma$. That is, Filippov convexification will give

$$\begin{aligned} \dot{x} \;\in & \left\{ \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \ , \ \ \lambda_i \geq 0 \ , \ i = 1, \dots, 4 \ , \ \sum_{i=1}^{4} \lambda_1 = 1 \right\} \ , \\ \nabla h_1^\top \dot{x} \ &= \ \nabla h_2^\top \dot{x} \ = \ 0 \, . \end{aligned} \tag{1.4.7}$$

But, from (1.4.7), it is apparent that there is no uniqueness of a sliding vector field on $\Sigma$, so that sliding motion on $\Sigma$ is not uniquely defined.

In this thesis, we propose a way to select a smooth sliding vector field on $\Sigma$, from the class of Filippov convex combinations (1.4.7), whenever $\Sigma$ is attractive through sliding. In other words, we will select a *smooth Filippov sliding vector field* $f_{\mathrm{F}}$: for $x \in \Sigma$, this is of the form

$$\begin{aligned} f_{\mathrm{F}} &= \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \ , \quad \lambda_i \geq 0 \ , \ i = 1, \dots, 4 \ , \ \sum_{i=1}^{4} \lambda_1 = 1 \ , \\ \nabla h_1^{\top} f_{\mathrm{F}} &= \nabla h_2^{\top} f_{\mathrm{F}} = 0 \ , \end{aligned} \tag{1.4.8}$$

where the coefficients $\lambda_i$'s depend smoothly on $x \in \Sigma$. Therefore, with previous notation, we will have to solve the problem (for $x \in \Sigma$):

$$\begin{bmatrix} W \\ \mathbb{1}^{\top} \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \text{ where } \lambda := \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} , W := \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \end{bmatrix} , \mathbb{1} := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} . \tag{1.4.9}$$

Obviously, (1.4.9) is an underdetermined linear system, reflecting the fact that the mere requirement of $f_{\mathrm{F}}$ being on $T_{\Sigma}$ is not generally sufficient to uniquely[1] characterize a convex combination of the four vector fields $f_1, \dots, f_4$. We propose the following definition of *admissible* solution of (1.4.9).

**Definition 1.4.6.** *Under the conditions of partial nodal attractivity of Definition 1.4.2, we say that a solution $\lambda$ of* (1.4.9) *is* admissible*, if $\lambda \geq 0$ and $\lambda$ depends smoothly on $x \in \Sigma$.* □

**Remark 1.4.7.** *The problem of understanding sliding motion on $\Sigma$ has been of considerable interest in the last 15 years. To date, the choice that has received most*

[1]There are special cases when the aforementioned ambiguity is not present, as when two of the original vector fields are identical (e.g., see [47]), but in general we must expect to have an underdetermined system.

*attention is one based on* bilinear interpolation. *This consists in selecting the Filippov vector field below:*

$$
\begin{aligned}
&(a) \qquad f_B := (1-\alpha)\left((1-\beta)f_1 + \beta f_2\right) + \alpha\left((1-\beta)f_3 + \beta f_4\right) \ , \\
&(b) \qquad (\alpha,\beta) \in (0,1)^2 \ : \ W\lambda_B = 0 \quad \textit{with } \lambda_B := \begin{bmatrix} (1-\alpha)(1-\beta) \\ (1-\alpha)\beta \\ \alpha(1-\beta) \\ \alpha\beta \end{bmatrix} \ .
\end{aligned}
\tag{1.4.10}
$$

*This bilinear interpolation method was originally introduced in [2] for nodally attractive $\Sigma$, it was further mentioned in [13], it was later studied in [23, 19], and it is effectively the sliding technique underpinning the singular perturbation approach of [45] and of [37]. As proven in [19], when the conditions of Definition 1.4.2 hold, this bilinear method gives an admissible solution $\lambda_B$ and a smoothly varying Filippov vector field on $\Sigma$. To be precise, and for later reference, we note that one needs to solve the nonlinear system* (1.4.10)-(b)*, that is $W\lambda_B = 0$, for $(\alpha,\beta)$. In general, this system may have more than one admissible solution; the quoted result in [19] guarantees that there is only one admissible solution (i.e., values of $\alpha$ and $\beta$ in* $[0,1]$*), whenever $\Sigma$ is attractive as in Definition 1.4.2.*

Unfortunately, there are potential difficulties caused by the choice (1.4.10) of vector field. These become apparent when $\Sigma$ loses attractivity at generic *first order exit points* (see below), where one of the sub-sliding vector fields (on $\Sigma_1$ or $\Sigma_2$) has itself become tangent to $\Sigma$. As we will see in Lemma 1.4.9, at generic exit points $\Sigma$ ceases to be attractive, and one might expect a trajectory to exit $\Sigma$ on the lower co-dimension manifold. However, as proven in [19], at generic exit points there could be two solutions of (1.4.10)-(b), giving distinct $(\alpha,\beta)$ in $[0,1]^2$, and different vector fields. Again referring to [19], one such solution always necessarily gives the sliding vector field on the lower co-dimension manifold, but the other solution corresponds to the sliding vector field that the trajectory was obeying. As a consequence, even assuming that one is able to obtain all roots of (1.4.10)-(b) rather than just following one by continuation, in general there is a catch: either one discontinuously changes the value of $(\alpha, \beta)$ in order to exit from $\Sigma$ (and loses smoothness), or the loss of attractivity of $\Sigma$ will go unnoticed to the bilinear vector field one is using (which remains well defined) and one ends up sliding on $\Sigma$, even though $\Sigma$ is no longer attractive (see Section 5.1 for illustration of this fact). To us, this seems undesirable, since -if perturbations off $\Sigma$ obey the dynamics of the original piecewise smooth system (1.4.3)- in general we expect that the perturbed solution trajectories will not return to $\Sigma$, when $\Sigma$ is not attractive.

**Definition 1.4.8** (First order exit points; [19])**.** *Let* $\dot{x}$ *be as in* (1.4.7)*, and let* $f_{\Sigma^{\pm}_{1,2}}$ *be as in* (1.4.6) *(whenever there is a well defined sliding motion on* $\Sigma^{\pm}_{1,2}$*). We say that* $x \in \Sigma$ *is a* generic first order exit point *if one (and just one) of the* $f_{F^{\pm}_{1,2}}$ *is itself in the class* (1.4.7)*, that is it is tangent to* $\Sigma$*. The corresponding* $f_{F^{\pm}_{1,2}}$ *is called an* exit vector field*.* $\square$

As Lemma 1.4.9 below clarifies (see also [19]), first order exit points are points where $\Sigma$ ceases to be partially nodally attractive.

**Lemma 1.4.9.** *If a point* $x_e \in \Sigma$ *is a first order exit point relative to* $\Sigma_1^+$*, then*

$$\det \begin{bmatrix} w_2 & w_4 \end{bmatrix} = 0. \tag{1.4.11}$$

*Analogously, if the first order exit points correspond to a sliding regime on* $\Sigma_1^-$ *we have* $\det \begin{bmatrix} w_3 & w_1 \end{bmatrix} = 0$*, relatively to* $\Sigma_2^+$ *we have* $\det \begin{bmatrix} w_4 & w_3 \end{bmatrix} = 0$*, and relatively to* $\Sigma_2^-$ *have* $\det \begin{bmatrix} w_1 & w_2 \end{bmatrix} = 0$.

*Proof.* If $x_{\mathrm{e}} \in \Sigma$ is a potential exit point for subsliding on $\Sigma_1^+$, then (at $x_{\mathrm{e}}$) $f_{F_1^+}$ is not just in the plane tangent to $\Sigma_1$ but also to $\Sigma$. That is, at $x_{\mathrm{e}}$ we must have

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,$$

which of course implies

$$\begin{bmatrix} w_2 & w_4 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ hence } \det \begin{bmatrix} w_2 & w_4 \end{bmatrix} = 0 ,$$

since $f_{\mathrm{F}1}^{+} = \lambda_2 f_2 + \lambda_4 f_4$, $\lambda_2 + \lambda_4 = 1$, is the Filippov sliding vector field on $\Sigma_1^+$. Similarly for the other cases. $\square$

As a consequence of Lemma 1.4.9, at a generic first order exit point for one of the $\Sigma_{1,2}^{\pm}$, we would like a solution trajectory to leave $\Sigma$ and to begin sliding (away from $\Sigma$) on the relevant sub-manifold $\Sigma_{1,2}^{\pm}$ with corresponding exit vector field. For this reason, we will further restrict our search for admissible $\lambda$, solutions of (1.4.9), in such a way that they will render the exit vector field at generic first order exit points.

**Definition 1.4.10** (Smooth Exits)**.** *Let $\lambda$ in* (1.4.8)-(1.4.9) *be admissible and such that, at a generic first order exit point, $\lambda$ renders also the exit vector field*[2]*. Then, $f_F$ will be called a* smoothly exiting *vector field.* $\square$

### 1.4.2 General form of coefficients

The following result is helpful in order to write the general form of an admissible solution $\lambda$ in (1.4.8), and will be proven, in a more generally setting, in Lemma 3.2.6.

**Lemma 1.4.11** ([18])**.** *When $\Sigma$ is attractive, or we are at a generic first order exit point, the matrix* $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ *in* (1.4.9) *has full rank* 3*. Furthermore, there is a nontrivial*

[2] *this means that two of the four entries of $\lambda$ are* 0

*vector* $v$*, as smooth as* $W$*, spanning* $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$*, and* $v$ *can be chosen as the eigenvector relative to the* 0*-eigenvalue of* $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}^\top \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. $\square$

In light of Lemma 1.4.11, clearly any admissible solution of $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ can be written as

$$\lambda \;=\; \mu + cv\,, \tag{1.4.12}$$

where $\mu$ is any (smooth) particular solution of $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \mu = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $v$ (smoothly) spans $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. We note that, since $\mathbb{1}^\top v = 0$, then $v$ cannot have all components of the same sign. In particular, in order for $\lambda$ to be admissible, we must have that the function $c$ satisfies

$$\alpha \le c \le \beta\ ,\ \alpha := \max\left\{-\frac{\mu_i}{v_i} : v_i > 0\right\}\ ,\ \beta := \min\left\{-\frac{\mu_i}{v_i} : v_i < 0\right\}\ , \tag{1.4.13}$$

for each $x$ in (the sliding portion of) $\Sigma$. Note that $\alpha \le 0$ and $\beta \ge 0$. Of course, $\alpha$ and $\beta$ are functions of $x$ (since so are $\mu$ and $v$), and in general are only continuous functions (even if $\mu$ and $v$ are smooth). Finally, we note that, by the nature of the solution set in (1.4.12), although the admissible region for $c$ in (1.4.13) depends on the specific choices of $\mu$ and $v$, the admissible set of coefficients $\lambda$ does not. Further, the topological properties (say, connectedness) of the admissibility region in (1.4.13) are preserved by choosing different $\mu$ and $v$.

To sum up, in our present context, all possible admissible smooth sliding vector fields of Filippov type (i.e., with smooth and positive coefficients) arise from (1.4.12), for given smooth $\mu$ and $v$ as above, and selecting a smooth function $c$ satisfying (1.4.13).

In what follows, and particularly in Chapter 2 and Chapter 4, we will review or introduce various techniques: in order to compare them, we will use the following example, which is sufficiently simple to allow hand calculations, yet rich enough to illustrate all desired features.

**Example 1.4.12** (A model example)**.** *We take the following vector fields $f_i$, $i = 1,2,3,4$, taking values in $\mathbb{R}^3$:*

$$f_1(x) := \begin{bmatrix} 2x_1+1 \\ -x_1+x_2x_3+1 \\ x_1+x_2+1 \end{bmatrix}, \; x \in R_1 \;, \quad f_2(x) := \begin{bmatrix} 2x_1-1 \\ -x_1+x_3-1 \\ x_1+x_2x_3+2 \end{bmatrix}, \; x \in R_2 \;,$$
$$f_3(x) := \begin{bmatrix} 2x_1-3 \\ -x_1+x_2+2 \\ x_1+x_2x_3-1 \end{bmatrix}, \; x \in R_3 \;, \quad f_4(x) := \begin{bmatrix} 2x_1+2 \\ -x_1+x_3-2 \\ x_1+x_3-2 \end{bmatrix}, \; x \in R_4 \;,$$

*where the regions $R_i$'s are as in* (1.4.4) *and*

$$h_1(x) := x_3, \;\; h_2(x) := x_2 \;.$$

*Therefore, $\Sigma = \{x \in \mathbb{R}^3 : \;\; x_2 = x_3 = 0\}$, we have the two unit normals $n_1(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $x \in \Sigma_1$, $n_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $x \in \Sigma_2$, and we can write the matrix $W$ for $x \in \Sigma$ as:*

$$\begin{bmatrix} W(x) \\ \mathbb{1}^\top \end{bmatrix} = \begin{bmatrix} x_1+1 & x_1+2 & x_1-1 & x_1-2 \\ -x_1+1 & -x_1-1 & -x_1+2 & -x_1-2 \\ 1 & 1 & 1 & 1 \end{bmatrix} . \tag{1.4.14}$$

*Observe that the sign pattern of Table 1 for nodal attractivity holds for $x_1 \in (-1,1)$. At the same time, we also note that the more comprehensive attractivity conditions*

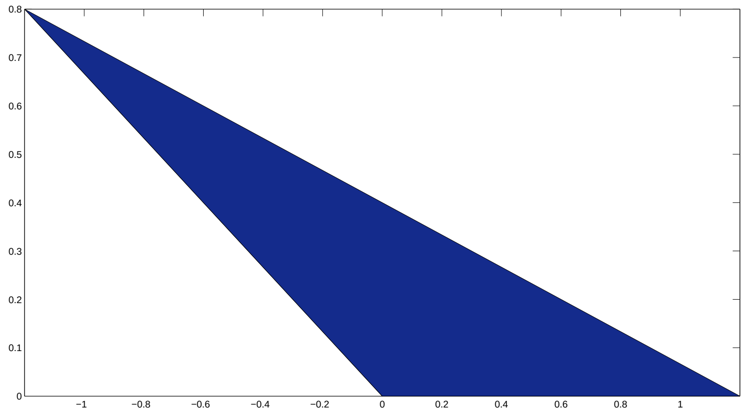


Figure 3: Admissible region $(x_1, c)$ in (1.4.15).

*of Definition 1.4.2 hold also outside of this interval, namely for* $|x_1| \leq 1.2$*, and that when* $x_1 = \pm 1.2$ *the exit conditions of Definition 1.4.8 hold,* $\Sigma$ *is no longer attractive, and one should exit* $\Sigma$ *by sliding on* $\Sigma_1$*, respectively* $\Sigma_2$*. On account of this, we would surely value any technique able to provide smoothly varying solutions* $\lambda$ *for all* $|x_1| \leq 1.2$*, relatively to the present example, and further one which when* $x_1 = \pm 1.2$ *renders two coefficients in* $\lambda$ *equal to* $0$*. As we will see below, there are not many such choices. Finally, one can easily obtain the general form of the admissible solutions* (1.4.12)*, for example written as*

$$\lambda = \begin{bmatrix} \frac{2}{3} - \frac{5}{9}x_1 \\ 0 \\ \frac{2}{3}x_1 \\ \frac{1}{3} - \frac{1}{9}x_1 \end{bmatrix} + c \begin{bmatrix} -\frac{5}{3} \\ 1 \\ 1 \\ -\frac{1}{3} \end{bmatrix} , \tag{1.4.15}$$

*which is admissible for* $(x_1, c)$ *in the shaded region in Figure 3*

# Chapter II

# A COMPARISON OF FILIPPOV SLIDING VECTOR FIELDS IN CO-DIMENSION 2

In this chapter, we consider several possibilities on how to define a Filippov sliding vector field on a co-dimension 2 singularity surface $\Sigma$, intersection on two co-dimension 1 surfaces. As underlying assumption, we consider the case of nodally attractive $\Sigma$.
We broadly classify the various possibilities in two groups: algebraic/analytic and geometric. In the first group, we consider three possible ways to define a Filippov vector field: a mean-field formulation, two approaches based on minimizing the 2-norm, and two different averaging techniques.
The geometric approaches we consider are a generally viable mean to select a Filippov sliding vector field. In particular, the techniques which can be cast in the framework of "barycentric coordinates" methods deliver a uniquely defined and smoothly varying vector field on a nodally attractive $\Sigma$. Specifically, we reinterpret the bilinear method as introduced in (1.4.10) (that has been extensively analyzed in [19, 23] under general attractivity assumptions on $\Sigma$), introduce the moments method and review other techniques already present in literature.

## *2.1 Analytic-Algebraic methods*

In this section, we introduce some techniques to select $\lambda$ in (1.4.12) for the case of $\Sigma$ nodally attractive. As far as we know, the construction behind the method(s) of Section 2.1.1 is new. The idea of Section 2.1.2.1 is patterned on general minimum variation principles, and the second method in that section is already in [1]. Finally, the techniques examined in Section 2.1.3 are patterned after a successful interpretation of the Filippov sliding vector field in co-dimension 1.

### 2.1.1 Mean field methods

Given the form of (1.4.12), and the restriction on $c$ given by (1.4.13), we define a *uniform* mean field method by selecting $c$ to be the midpoint of $[a, b]$ (recall that $a$ and $b$ depend on $\mu$, $v$, and $x \in \Sigma$):

$$\lambda_{\mathrm{MF}} := \mu + \frac{a+b}{2} v \ . \tag{2.1.1}$$

Note that, in (2.1.1), we are taking the expected value of the random variable $\Xi$ according to the uniform distribution over $[a, b]$. This suggests a useful generalization, based on the following definition.

**Definition 2.1.1** (Mean Field Methods)**.** *Let $\mu$ be a particular solution of* (1.4.9)*, and $v$ be also given. Assume that the random variable $\Xi$ obeys a probability distribution over $[a, b]$, with pdf (probability density function) $g(\xi)$. Then, we define the family of mean field methods according to*

$$c := \int_a^b \xi g(\xi)\, d\xi \quad \textit{and} \quad \lambda_g := \mu + \left( \int_a^b \xi g(\xi)\, d\xi \right) v \ . \tag{2.1.2}$$

We have the following result, telling us that the (pointwise) value of $\lambda_g$ is independent of $\mu$.

**Lemma 2.1.2.** *For given $v$, the value of $\lambda_g$ in* (2.1.2) *is independent of the particular solution $\mu$. Moreover, choosing $c$ and $\lambda_g$ as in* (2.1.2) *always gives an admissible solution.*

*Proof.* Suppose that we have chosen $c$ as in (2.1.2) for a given $\mu$, and let $\tilde{\mu}$ be another solution of (1.4.9), giving admissibility interval $\tilde{c} \in [\tilde{a}, \tilde{b}]$.

Then, there exists a value $\tau \in [a, b]$ such that $\tilde{\mu} = \mu + \tau v$. But

$$\tilde{\mu} + \tilde{c} v \geq 0 \Leftrightarrow \mu + (\tilde{c} + \tau) v \geq 0 \Leftrightarrow \tilde{c} + \tau \in [a, b] \Leftrightarrow \tilde{c} \in [a - \tau, b - \tau] \ .$$

In particular, $[\tilde{a}, \tilde{b}]$ and $[a, b]$ have the same length. From this, it follows that if $\xi$ has pdf $g(\xi)$ over $[a, b]$, then $\tilde{\xi}$ will have pdf $\tilde{g}(\tilde{\xi}) := g(\tilde{\xi} + \tau)$, $\tilde{\xi} \in [\tilde{a}, \tilde{b}] = [a - \tau, b - \tau]$. Therefore, by (2.1.2),

$$\begin{aligned}
\tilde{\lambda}_g &= \tilde{\mu} + \left( \int_{a-\tau}^{b-\tau} \tilde{\xi} \tilde{g}(\tilde{\xi}) \, d\tilde{\xi} \right) v \\
&= \tilde{\mu} + \left( \int_{a-\tau}^{b-\tau} \tilde{\xi} g(\tilde{\xi} + \tau) \, d\tilde{\xi} \right) v \\
&= \tilde{\mu} + \left( \int_{a}^{b} (\xi - \tau) g(\xi) \, d\xi \right) v \\
&= \tilde{\mu} + \left( \int_{a}^{b} \xi g(\xi) \, d\xi - \tau \int_{a}^{b} g(\xi) \, d\xi \right) v \\
&= \tilde{\mu} - \tau v + \left( \int_{a}^{b} \xi g(\xi) \, d\xi \right) v \\
&= \mu + \left( \int_{a}^{b} \xi g(\xi) \, d\xi \right) v \\
&= \lambda_g \ .
\end{aligned}$$

Finally, that choosing $c$ and $\lambda_g$ as in (2.1.2) produces an admissible solution is clear. $\square$

The following example shows that, in general, $\lambda_{\mathrm{MF}}$ (i.e., where the probability distribution function is the uniform distribution), although obviously admissible, and trivially continuous in case $\mu$ is, is not as smooth as $W$.

**Example 2.1.3.** *Let us refer to Example 1.4.12. By the configuration of this problem, it is easy to obtain*

$$\frac{a(x_1) + b(x_1)}{2} = \begin{cases} \frac{1}{6} x_1 - \frac{1}{15}, & \text{if } x_1 \in \left[-\frac{6}{5}, 0\right], \\ \frac{1}{18} x_1 - \frac{1}{15}, & \text{if } x_1 \in \left[0, \frac{6}{5}\right], \end{cases}$$

*which gives $\lambda_{MF}$ not differentiable at $x_1 = 0$, whereas $W$ is analytic for all $x_1 \in \left[-\frac{6}{5}, \frac{6}{5}\right]$.* $\square$

So, it is natural to ask: "How can we choose a distribution function $g$ in order to make $\lambda_g$ in (2.1.2) as smooth as $W$? "

We propose to consider the following family of distribution functions:

$$g_\alpha(\xi) := \frac{\alpha(\xi - a)^{\alpha-1}}{(b-a)^\alpha}, \quad \xi \in [a,b],\ \alpha \in (0, +\infty)\ . \tag{2.1.3}$$

This family of pdf's belongs to the Beta distribution family with parameters $(\alpha, 1)$, and we restrict to this family of pdf's because of their natural formulation on compact intervals.

For (2.1.3), we have

$$g_\alpha \geq 0, \quad \int_a^b g_\alpha(\xi)d\xi = 1, \quad \int_a^b \xi g_\alpha(\xi)d\xi = \frac{1}{\alpha+1}a + \frac{\alpha}{1+\alpha}b,$$

from which $c$ in (2.1.2) is given by

$$c = (1-\gamma)a + \gamma b\ ,\ \gamma = \frac{\alpha}{\alpha+1}\ , \tag{2.1.4}$$

that is, for every $\alpha \in (0, +\infty)$, the expectation of the random variable $\xi$ with measure $g_\alpha(\xi)$ is the convex combination of $a$, $b$ with weights $\frac{1}{\alpha}$, $\frac{\alpha}{1+\alpha}$.

Although not necessarily any choice of $\alpha$ in (2.1.3) gives an admissible solution as smooth as $W$ (e.g., taking $\alpha = 1$ gives $\lambda_{\text{MF}}$), we will see in Section 2.2 that in fact it is possible to choose $\alpha$ to obtain a smoothly varying, admissible, $\lambda_g$.

### 2.1.2 Minimum norm

Here we look at two very natural approaches: to choose the Filippov sliding vector field $f_F$ in such a way to minimize $\|\lambda\|$, or to minimize $\|f_F\|$ directly. Below, the norm is the 2-norm.

#### *2.1.2.1 Minimizing $\lambda$*

Here we seek the minimum norm solution of (1.4.9).

Without directly imposing the positivity constraints, it is simple to obtain the minimum 2-norm solution; e.g., by using the SVD (singular value decomposition) of

$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$:

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} = USV^\top,$$

where $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{4\times 4}$ are orthogonal and $S = [\Sigma,\ 0]$ with $\Sigma = \mathrm{diag}(\sigma_i, i = 1, 2, 3)$ (note, $\sigma_i \neq 0$):

$$\lambda_{\min} = Vy\ ,\ \ y = \begin{bmatrix} u_1/\sigma_1 \\ u_2/\sigma_2 \\ u_3/\sigma_3 \\ 0 \end{bmatrix}\ ,\ \ u := U^\top \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ ,$$

which can also be rewritten from the form (1.4.12) as

$$\lambda_{\min} := (I - vv^\top)\mu\ . \tag{2.1.5}$$

It can be shown that $\lambda_{\min}$ is as smooth as $W$.[1] However, this solution may be not admissible (i.e., it is not generally true that $\lambda_{\min} \geq 0$).

Using again the structure (1.4.12), the min 2-norm admissible solution $\hat{\lambda}_{\min}$ is simply given by $\lambda_{\min}$ above if $\lambda_{\min}$ is admissible, and by whichever of $\mu + av$ or $\mu + bv$ gives minimum 2-norm otherwise. Unfortunately, in this case $\hat{\lambda}_{\min}$ may fail to vary smoothly.

**Example 2.1.4.** *Take Example 1.4.12, at $x_1 = -0.9$.*

*Then, $\lambda_{min} = \begin{bmatrix} \frac{1}{4} \\ \frac{11}{20} \\ -\frac{1}{20} \\ \frac{1}{4} \end{bmatrix}$, which is clearly not admissible. In this case, the admissible*

[1]Use the argument in the proof of Lemma 1.4.11

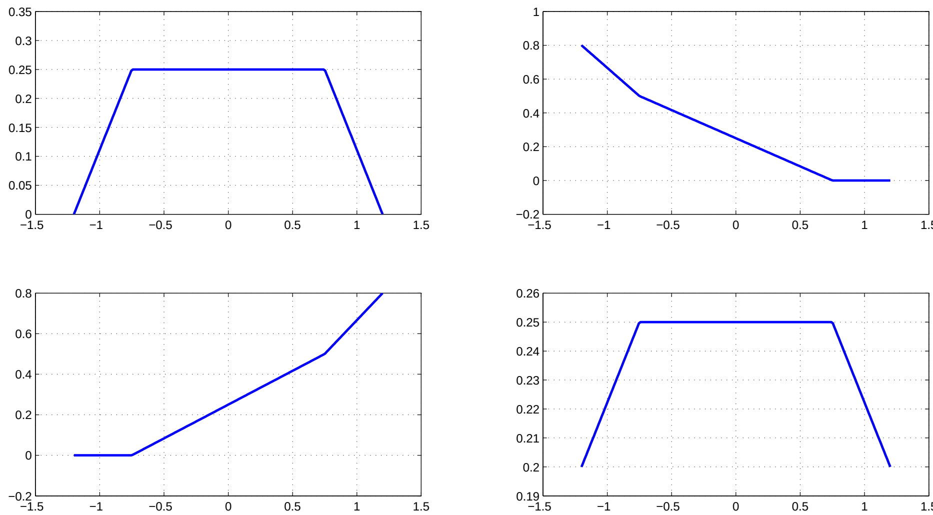


Figure 4: Components of $\hat{\lambda}_{\min}$ for Example 1.4.12.

*solution of minimum 2-norm is* $\hat{\lambda}_{min} = \begin{bmatrix} \frac{1}{6} \\ \frac{3}{5} \\ 0 \\ \frac{7}{30} \end{bmatrix}$*, with* $f_{min} = \begin{bmatrix} -1.7667 \\ 0 \\ 0 \end{bmatrix}$*. (Coincidentally, these correspond to* $\lambda_{ave}$ *and* $f_{ave}$ *as in Example 2.1.11). However, as can be seen in Figure 4,* $\hat{\lambda}_{min}$ *is not as smooth as* $W$*.*

#### *2.1.2.2 Minimizing f*

[Minimum Variation] This approach was already suggested in [1]. The goal is to find $f$ as in (1.4.8) of minimal norm. That is, one solves

$$\min \|f\|^2, \text{ subject to } \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Writing $\lambda = \mu + cv$ as in (1.4.12), then we have to determine the minimum of

$$\|F_\mu\|^2 + 2cF_\mu^\top F_v + c^2\|F_v\|^2 \ , \ \text{where} \ \ F_\mu := \sum_{i=1}^{4} \mu_i f_i \ , \ F_v := \sum_{i=1}^{4} v_i f_i \ .$$

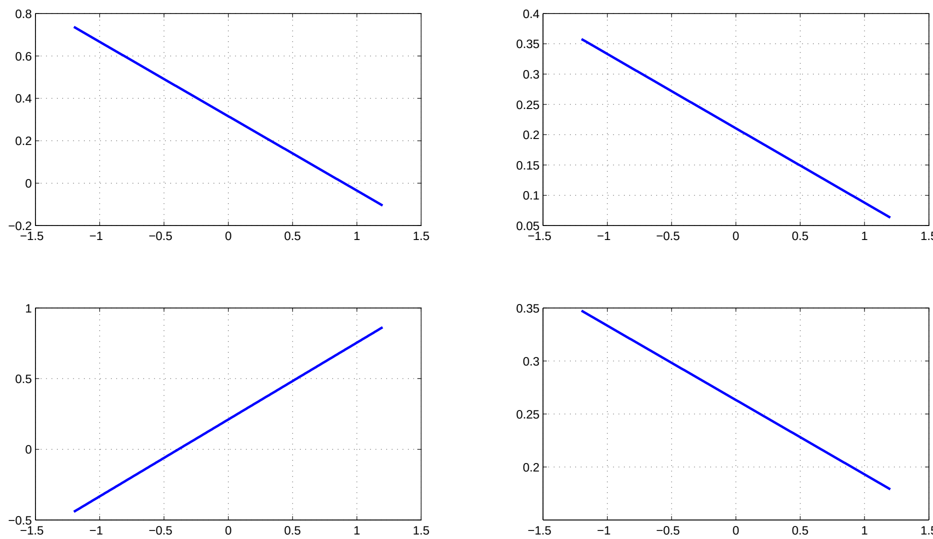


Figure 5: Components of $\lambda$ for (2.1.7).

The minimum is attained for $c = -\frac{F_\mu^\top F_v}{\|F_v\|^2}$, and so the vector field afforded by this approach is

$$f_{\mathrm{MV}} := F_\mu^\top \left( I - \frac{F_v}{\|F_v\|} \frac{F_v^\top}{\|F_v\|} \right) F_\mu \, , \tag{2.1.6}$$

which can be fit into the class of vector fields (1.4.8) by taking

$$\lambda_{\mathrm{MV}} = \mu - \frac{F_\mu^\top F_v}{\|F_v\|^2} v \, . \tag{2.1.7}$$

Unfortunately, this approach is also affected by similar limitations as those encountered for $\lambda_{\min}$. To be precise, now it may happen that $f_{\mathrm{MV}}$ is not a Filippov vector field (in the sense that $\lambda_{\mathrm{MV}}$ in (2.1.7) is not admissible), and by restricting the minimization search so that $\lambda_{\mathrm{MV}}$ is admissible may render a non-smooth $f_{\mathrm{MV}}$.

**Example 2.1.5.** *Consider again Example 1.4.12. Here, the resulting $f_{MV} = 0$. Looking at the $\lambda_{MV}$ components in Figure 5, we notice that they are smooth, but not always positive for $x_1 \in (-1, 1)$. By imposing positivity constraints, that is solving*

$$\min \|f\|^2, \; \textit{subject to} \; \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; \lambda \geq 0 \, , \tag{2.1.8}$$

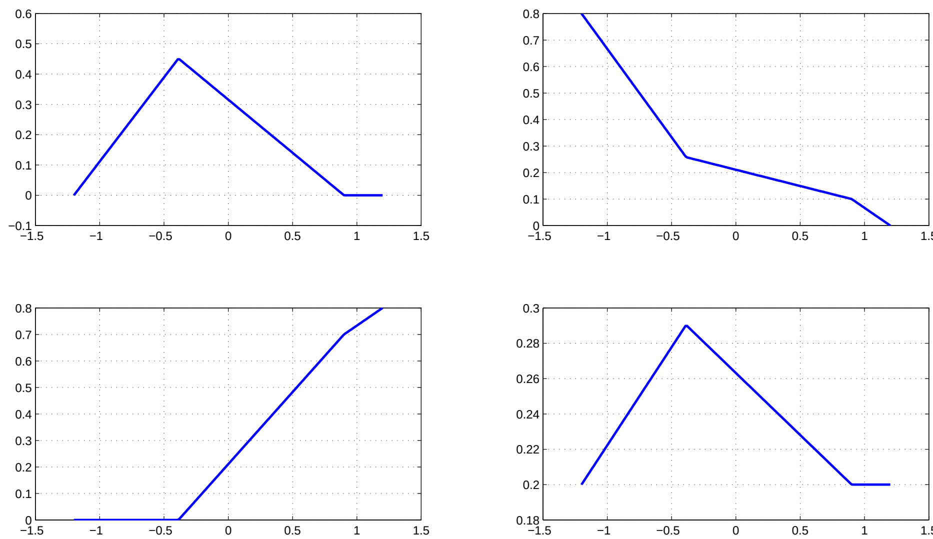

Figure 6: Components of $\lambda$ for (2.1.8).

*we highlight in the Figures 6 and 7 how this generally produces a lack of smoothness in $\lambda$ and a resulting lack of smoothness in $f$.*

For completeness, we remark that –in general– it is not true that $f_{\mathrm{MV}} = 0$ even without imposing the admissibility constraints.

**Remark 2.1.6.** *A natural, related problem about the minimization techniques presented above is the selection of a suitable norm so to obtain a smoothly varying admissible solution for the definition of a Filippov sliding vector field on $\Sigma$. A refined approach would be using a norm more demanding on the regularity of the minimum variation solution. This approach seems indeed to be promising, and requires a deeper and more specific analysis: we dedicate Chapter 4 to this task. There, we will analyze minimum variation techniques with respect to the $H^1$-norm.*

### 2.1.3 Averaging

Here we attempt to indirectly define a Filippov sliding vector field by averaging the dynamics near $\Sigma$ in a similar way to what has proven to be successful in the case of

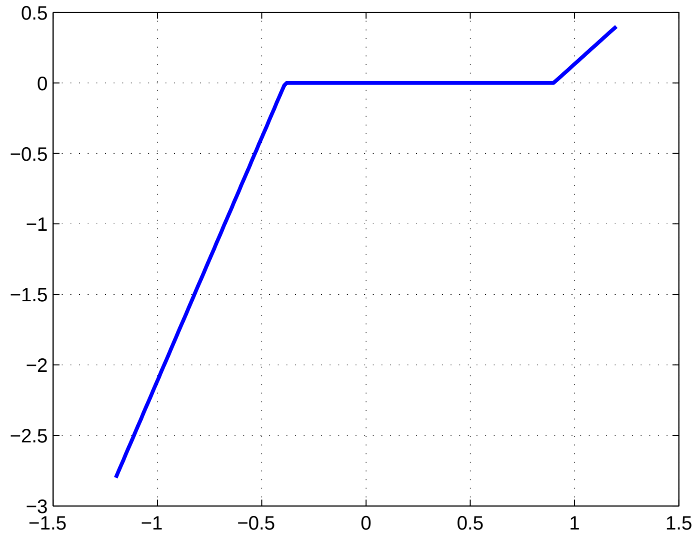


Figure 7: First component of $f$ relatively to (2.1.8).

sliding motion on a co-dimension 1 surface.

We recall that when $\Sigma$ has co-dimension 1, a simple averaging process of the Euler discretization method converges to the Filippov sliding vector field in (1.3.4). In that case, the idea seems to have been originally introduced by Utkin in [54] (see also [24, 49] for added generality). The idea is simple, but we need to re-interpret it appropriately in order to appreciate how we may extend it.

Let $x_0 \in \Sigma$, let $n(x_0)$ be the (unit) normal to $\Sigma$ at $x_0$ and represent points in a $\delta$-neighborhood of $x_0$, of base point $x_0$ (i.e., whose orthogonal projection is $x_0$), as $\{x \in \mathbb{R}^n \ : \ x = x_0 + n(x_0)c(x)\}$, where the scalar valued function $c(x)$ represent the distance along the normal direction, hence $c(x) = h(x)$. This way we can define a strip $\mathcal{C}$ of width $2\delta$ around $\Sigma$.

Now, suppose we have fields $f_1$ and $f_2$, defined on and around $\Sigma$. Take a point $x^{(0)} \in R_1$, of base point $x_0 \in \Sigma$, such that $h(x^{(0)}) = -\delta$, and consider the value given by a Euler step, $x^{(1)} = x^{(0)} + \tau_0 f_1(x^{(0)})$, with $\tau_0$ chosen so that $x^{(1)}$ is in $R_2$ and $h(x^{(1)}) = \delta$ (this is always possible, given that $h_x^T f_1 > 0$). From $x^{(1)}$, we take another Euler step, $x^{(2)} = x^{(1)} + \tau_1 f_2(x^{(1)})$, with $\tau_1$ so that $x^{(2)} \in R_1$ and $h(x^{(2)}) = -\delta$. Now

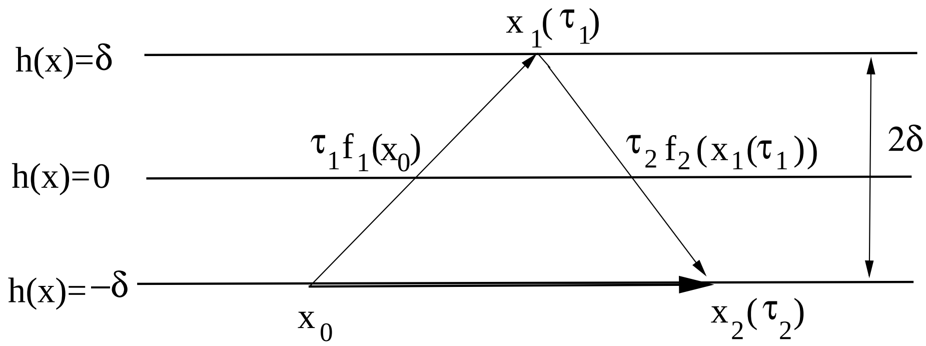


Figure 8: Euler steps as explained in Remarks 2.1.7.

consider $(x^{(2)} - x^{(0)})/(\tau_0 + \tau_1) = \frac{\tau_0}{\tau_0+\tau_1} f_1(x^{(0)}) + \frac{\tau_1}{\tau_0+\tau_1} f_2(x^{(1)})$. A standard calculation (e.g., see [24]) gives that

$$\lim_{\delta \to 0} (x^{(2)} - x^{(0)})/(\tau_0 + \tau_1) = \alpha f_1(x_0) + (1 - \alpha) f_2(x_0) \ ,$$
$$\alpha = h^T(x_0) f_1(x_0)/(h^T(x_0)(f_1(x_0) - f_2(x_0)) \ ,$$

that is (1.3.4).

**Remarks 2.1.7.**

*(i) We note that this averaging process is logically one-dimensional, since the iterates are effectively controlled by the scalar values $h(x)$, rather than by $x$.*

*(ii) We also note that the limiting value is the same for any point at distance $\delta$ from $\Sigma$, relatively to the same base point $x_0 \in \Sigma$. In other words, we could have started just as well from the point $x_0 + n(x_0)\delta$.*

*(iii) Finally, we stress that the process is (and must be) stopped after two Euler steps.*

We can visualize this process as if it is taking place on an interval of length $2\delta$ for the $h$-axis around the origin ($h = 0$), and we bounce from one end of the interval to the other. See Figure 8.

In co-dimension 2, we attempt to generalize the above approach by working with the Euclidean distance. So, we consider a "cylinder-like" region $\mathcal{C}$ surrounding $\Sigma$

(which serves as the "axis" of the cylinder) and "radius" $\delta$, as defined by the requirement that

$$x \in \mathcal{C} \iff \|h(x)\|^2 = (h_1(x))^2 + (h_2(x))^2 = \delta^2 .$$

It will be useful to better explain the structure of $\mathcal{C}$ by considering points within distance $\delta$ from a base point $x_0 \in \Sigma$. In other words, if $N(x_0) = [n_1, n_2]_{x_0}$ represent the matrix of the unit normals at $x_0 \in \Sigma$, we will have $x = x_0 + N(x_0)c(x)$, and $\|x - x_0\|^2 \leq \delta^2$. Hence, all points in $\mathcal{C}$ (hence, at distance $\delta$ from $\Sigma$), of same base point (orthogonal projection) $x_0 \in \Sigma$, will belong to a section $R_\delta(x_0)$ of $\mathcal{C}$, for which we will have

$$c = \delta\big(N^T(x_0)N(x_0)\big)^{-1} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} , \quad \theta \in S^1 . \tag{2.1.9}$$

Through (2.1.9), we can thus bijectively map all points in $\mathcal{C}$ of same base point $x_0$ to points on the unit circle, i.e., to angles $\theta$. [Note that, in general, the neighborhood is ellipsoidal].

**Example 2.1.8.** *Consider Example 1.4.12. Here, $\Sigma$ is a plane, and the two normals are $n_1 = e_3$ and $n_2 = e_2$. From (2.1.9) we get $c = \delta \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, that is a circular neighborhood. All points in $\mathcal{C}$ are distinguished by the value of the first component $x_1$, and by the angle $\theta$, and the vector fields, evaluated on $\mathcal{C}$, assume the form*

$$f_1(x) = \begin{bmatrix} 2x_1 + 1 \\ -x_1 + 1 + \delta^2\cos\theta\sin\theta \\ x_1 + 1 + \delta\cos\theta \end{bmatrix} , \quad f_2(x) = \begin{bmatrix} 2x_1 - 1 \\ -x_1 - 1 + \delta\sin\theta \\ x_1 + 2 + \delta^2\cos\theta\sin\theta \end{bmatrix} ,$$

$$f_3(x) = \begin{bmatrix} 2x_1 - 3 \\ -x_1 + 2 + \delta\cos\theta \\ x_1 - 1 + \delta^2\cos\theta\sin\theta \end{bmatrix} , \quad f_4(x) = \begin{bmatrix} 2x_1 + 2 \\ -x_1 - 2 + \delta\sin\theta \\ x_1 - 2 + \delta\sin\theta \end{bmatrix} .$$

With the above in mind, we will now distinguish between two different averaging processes: (i) averaging the dynamics induced by the original vector fields $f_{1,2,3,4}$, or

(ii) averaging the dynamics induced by the sub-sliding vector fields of (1.4.6), $f^{\pm}_{F_{1,2}}$.

*2.1.3.1 Averaging Original Dynamics*

Here we look at the dynamics of the Euler map under the original vector fields, by requiring successive iterates to remain in $\mathcal{C}$.

We generate points on $\mathcal{C}$ by the following iterative process.

**Algorithm 1**.

(i) Given a point $x^{(0)} \in \mathcal{C}$, let $x^{(0)} \in R_{i_0}$ (one of the regions $R_1, R_2, R_3, R_4$) and let $f_{i_0}$ be the corresponding vector field. Then, take a Euler step with stepsize $\tau_0$ so that the value

$$x^{(1)} = x^{(0)} + \tau_0 f_{i_0}(x^{(0)}) \tag{2.1.10}$$

is also in $\mathcal{C}$ (see Lemma 2.1.9 below). [In the (measure 0) eventuality that $x^{(0)}$ or one of the iterates below is on $\Sigma_1$ or $\Sigma_2$, we modify this construction by taking the Filippov sliding vector field $f^{\pm}_{F_{1,2}}$ on these co-dimension 1 surfaces.]

(ii) Repeat this process. That is, for $k = 0, 1, 2, \dots,$ let

$$x^{(k+1)} = x^{(k)} + \tau_k f_{i_k}(x^{(k)}) \ , \ \ \tau_k : \ \ \|h(x^{(k+1)})\|^2 = \delta \, . \tag{2.1.11}$$

**Lemma 2.1.9.** *Let the assumptions on* $\Sigma$ *of Table 1 hold. Then, for given* $\delta > 0$*, the iteration* (2.1.11) *is well defined, and hence, there exists a unique* $\tau_k > 0$ *in* (2.1.11).

*Proof.* We consider the first step, assuming that $x^{(0)}$ is not on either of $\Sigma_1$, $\Sigma_2$. The other steps, as well as the case of $x^{(0)} \in \Sigma_{1,2}$, are handled similarly. We have $\|h(x^{(0)})\|^2 = \delta^2$, and seek $\tau_0$ such that $\|h(x^{(1)})\|^2 = \delta^2$. From Taylor expansion with remainder in Lagrange form, we have

$$h(x^{(1)}) = h(x^{(0)}) + \tau_0 \nabla h^T(\eta_0) f_{i_0}(\eta_0) \ , \ \ (\eta_0)_j \in \left[(x^{(0)})_j, (x^{(1)})_j\right] \ , \ \ j = 1, \dots, n \, .$$

Now, requiring $h(x^{(1)})^T h(x^{(1)}) = \delta^2$, gives $\tau_0 = 0$, which is unacceptable, or

$$\tau_0 = -2\delta \frac{h^T(x^{(0)}) \left[\nabla h^T(\eta_0) f_{i_0}(\eta_0)\right]}{\|\nabla h^T(\eta_0) f_{i_0}(\eta_0)\|^2} \ ,$$

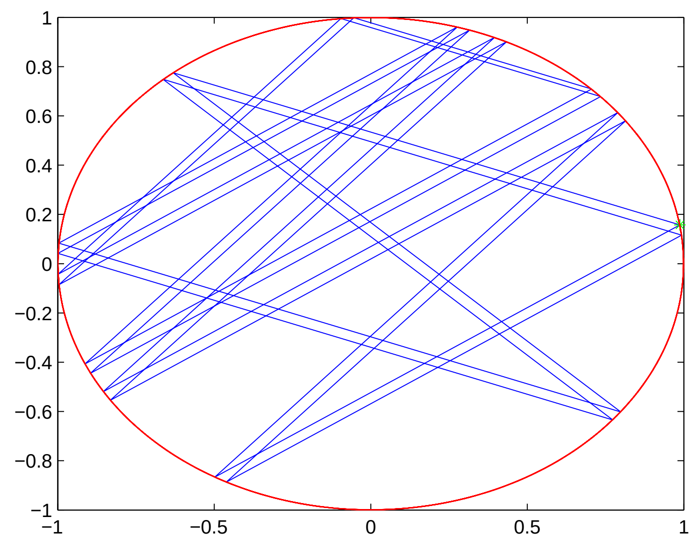


Figure 9: Iterative process as in Lemma 2.1.9.

which is strictly positive on account of Table 1 and of the labeling of the regions $R_1, \dots, R_4$. □

It is insightful to visualize this iterative process as if we bounce from point to point on a circle of radius $\delta$ around the origin by taking Euler steps of appropriate stepsizes; see Figure 9. In order to obtain an average vector field from the above iteration, we now collect together in four different groups all stepsizes generated in (2.1.11) above, according to which one is the vector field for which they are being Euler steps. That is, from (2.1.11) we will call $\tau_k = \tau_k^{(1)}$, if $f_{i_k} = f_1$, and similarly for $\tau_k^{(2)}, \tau_k^{(3)}, \tau_k^{(4)}$, with the obvious modification required if we are using one of the $f^{\pm}_{F_{1,2}}$. It must be appreciated that the values of the $\tau_k$'s depend on $\delta$.

Suppose[2] that the trajectory generated by $x^{(0)}$ is periodic in the angle $\theta$; that is, suppose that we generate iterates whose associated angles satisfy $\theta(x^{(0)}), \dots, \theta(x^{(N_0-1)})$, $\theta(x^{(N_0)}) = \theta(x^{(0)})$, and note that $N_0$ itself generally may depend on $\delta$. Under this

[2]We conjecture that, for fixed $\delta > 0$, and constant vector fields, this supposition is correct, but lack a complete proof of this fact; based on what follows, we lack motivation to embark in such possible proof.

situation, it is reasonable to consider the following quantity:

$$\lambda^i_{\text{ave}}(x^{(0)},\delta) := \frac{\sum_{k=0}^{N_0-1}\tau_k^{(i)}}{\sum_{k=0}^{N_0-1}\tau_k}\ , \quad i=1,2,3,4\ . \tag{2.1.12}$$

Note that this would give an admissible solution. But, as we said, we need the orbits to be periodic. Moreover, we must demand that (2.1.12) has a limit as $\delta \to 0$, a property which is not clear at all if it is true. In fact, both periodicity and existence of the limit are quite hard to prove in general and/or to verify in a practical problem. Furthermore, as we see in Example 2.1.10 below, even if the orbit is periodic and the limit exists, in general the value of points in $\mathcal{C}$ with same projection $x_0 \in \Sigma$ differ. As a consequence, this averaging technique turns out to be unsatisfactory as a way to define a Filippov sliding vector field. We say this because an obvious requirement of this way of proceeding must be that the limiting values of $\lambda_{\text{ave}}(x^{(0)},\delta)$ be the same for all $x^{(0)} \in R_\delta(x_0)$.

**Example 2.1.10.** *Consider Example 1.4.12, with* $x_1 = 0.5$ *there; so, we let* $x_0 = (0.5, 0, 0) \in \Sigma$. *We take two different points in* $R_\delta(x_0)$, *namely (see Example 2.1.8) corresponding to: (a)* $\theta = \texttt{eps}$, *and (b)* $\theta = 0.7815$ *(here,* `eps` *is the machine precision, and* $\texttt{eps} \approx 2.2204e-016$*). In these cases, the generated orbits are periodic and for* $\lambda_{ave}$ *given in* (2.1.12) *the limiting values as* $\delta \to 0$ *exist and give:*

$$(a)\ \begin{bmatrix} 0 \\ 0.2333 \\ 0.5667 \\ 0.2000 \end{bmatrix}, \qquad (b)\ \begin{bmatrix} 0.3889 \\ 0 \\ 0.3333 \\ 0.2778 \end{bmatrix},$$

*with average periods of* 95.2704 *and* 96.2323 *respectively.*

To move out of the impasse above, we also considered a second averaging process, over the angle $\theta$, for all points with same base point on $\Sigma$. That is, calling $x(\theta)$ the points in $\mathcal{C}$ with same base point $x_0$, and subject to the same limitations previously

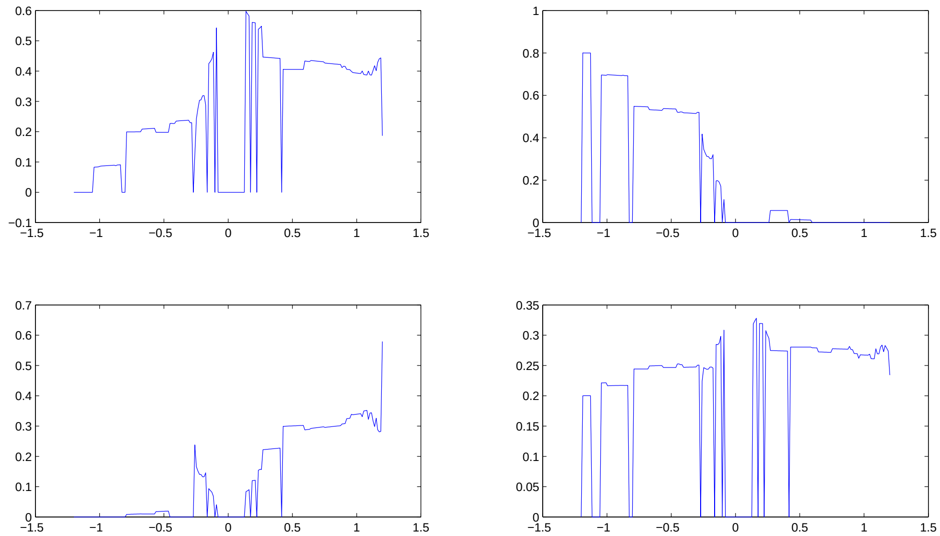


Figure 10: Components of $\lambda_{\text{ave}}$ for Example 1.4.12.

mentioned on the proper definition of $\lambda_{\text{ave}}(x(\theta))$, we considered the following quantity,

$$\lambda_{\text{ave}}(x_0) := \frac{1}{2\pi}\int_0^{2\pi} \lambda_{\text{ave}}(x(\theta))\, d\theta \ , \tag{2.1.13}$$

which –as long as it is well defined– is surely giving an admissible solution, identical for all points in $\mathcal{C}$ with same base point $x_0$. Alas, even when well defined, the above turns out to be unsatisfactory.

**Example 2.1.11.** *Let us refer again to Example 1.4.12, with $x_1 = -0.9$.*

*In this case we obtain* $f_{ave} = \begin{bmatrix} -1.7667 \\ 0 \\ 0 \end{bmatrix}$, *and* $\lambda_{ave} = \begin{bmatrix} 0.1667 \\ 0.6000 \\ 0 \\ 0.2333 \end{bmatrix}$, *which is surely admissible. But, as Figure 10 exemplifies, this $\lambda_{ave}$ solution is clearly not differentiable in $x_1$, despite $W$ being analytic in it. As a consequence, this possible way to interpret how to select a Filippov sliding vector field does not appear to be a viable choice.*

#### *2.1.3.2 Averaging Sub-Sliding Dynamics*

In the nodally attractive case considered in this work, we can take also an alternative point in view in order to build an average sliding vector field. As before, we consider the 2-norm to define the cylinder $\mathcal{C}$ around $\Sigma$, of radius $\delta$.

The point of the construction below is to realize that –because of nodal attractivity– a trajectory of the dynamical system (1.2.1) starting at a point in $\mathcal{C}$ will typically **hit** one of the sub-sliding surfaces $\Sigma_{1,2}^{\pm}$ before reaching $\Sigma$ itself. This allows us to effectively reduce the dimensionality of the averaging process, by looking at the points in $\mathcal{C}$ which end up first on one of $\Sigma_{1,2}^{\pm}$. At that point, the averaging process will be the same as we had in co-dimension 1.

Recalling (1.4.6), we will look for a sliding vector field on $\Sigma$ of the following form

$$f := c_1^+ f_{F_1^+} + c_1^- f_{F_1^-} + c_2^+ f_{F_2^+} + c_2^- f_{F_2^-} \; . \tag{2.1.14}$$

To understand how to select the coefficients $c_{1,2}^{\pm}$, we reason as follows.

Let $x_0 \in \Sigma$ be given, and consider the $\delta$-section $R_\delta(x_0)$ in $\mathcal{C}$, defined as before; see (2.1.9). For fixed value of $\delta$, consider the Euler segments starting at a point $x^{(0)} \in R_\delta(x_0)$, defined so to remain in $\mathcal{C}$, but monitoring the first time that any such segment crosses one of the $\Sigma_{1,2}^{\pm}$. In other words, we define (see (2.1.10)) $x^{(1)}(\tau) = x^{(0)} + \tau f_{i_0}(x^{(0)})$, $\tau \le \tau_0$; if this segment reaches $\mathcal{C}$ without first having crossed one of the $\Sigma_{1,2}^{\pm}$, then we take $\tau = \tau_0$ as in (2.1.10), and continue by taking Euler segments (see (2.1.11)) to generate $x^{(k+1)}(\tau) = x^{(k)} + \tau f_{i_k}(x^{(k)})$, $\tau \le \tau_k$, until the first time one of these segments crosses one of the $\Sigma_{1,2}^{\pm}$. [The probability 0 eventuality that one of these segments first reaches $\Sigma$ directly is presently ignored, and see Remark 2.1.12-(i) below.] It is quite easy to see that, because of nodal attractivity, for any starting point in $R_\delta(x_0)$ there is a first Euler segment crossing one of $\Sigma_{1,2}^{\pm}$. We stress that this process generally depends on $\delta$.

By doing what described above, and recalling the form of $R_\delta(x_0)$, we effectively obtain a partition of $S^1$, that is of $[0, 2\pi]$, into arcs: an angle from each of these arcs is associated to whichever sub-surface $\Sigma^{\pm}_{1,2}$ is reached first by the Euler segments starting from that angle in $R_\delta(x_0)$. So, for given $\delta$, we will have four arc-lengths, which we call $\theta^{\pm}_{1,2}$; e.g., $\theta^+_1$ is the length of the arc of $S^1$ whose associated points have a Euler segment first reaching $\Sigma^+_1$, etc.. Again, let us stress that these $\theta^{\pm}_{1,2}$ generally depend on $\delta$.

Now, as soon as one of the sub-surfaces $\Sigma^{\pm}_{1,2}$ is reached by a Euler segment, we reduce the dimensionality of the process and go back to the case of co-dimension 1. For example, suppose that for a certain angle $\theta$, the Euler iterates starting with $x^{(0)} \in R_\delta(x_0)$ reach $\Sigma^+_1$ first; then, we restrict consideration to the co-dimension 1 surface $\Sigma_1$, with Filippov vector fields given by $f^+_{F_1}$ and $f^-_{F_1}$ in (1.4.6); but, in co-dimension 1 the averaging process is well understood, and in this case it will give a Filippov sliding vector field at $x_0 \in \Sigma$. With this, we will now have (all quantities below generally depend on $\delta$)

$$\begin{aligned} &f_{F_1} := (1-a_1) f_{F_1^+} + a_1 f_{F_1^-} \ , \quad f_{F_2} := (1-a_2) f_{F_2^+} + a_2 f_{F_2^-} \ , \\ &a_1 := \frac{n_2^\top f_{F_1^+}}{n_2^\top (f_{F_1^+} - f_{F_1^-})} \ , \quad a_2 := \frac{n_1^\top f_{F_2^+}}{n_1^\top (f_{F_2^+} - f_{F_2^-})} \ . \end{aligned} \tag{2.1.15}$$

Next, we compute the following ratios, defining the percentage of points in $R_\delta(x_0)$ contributing to $f_{F_1}$, respectively to $f_{F_2}$, see Figure 11. We make the dependence on $\delta$ explicit:

$$\begin{aligned} L_1(\delta) &:= \frac{\theta^+_1(\delta) + \theta^-_1(\delta)}{2\pi} \ , \\ L_2(\delta) &:= \frac{\theta^+_2(\delta) + \theta^-_2(\delta)}{2\pi} \ . \end{aligned} \tag{2.1.16}$$

Finally, we let $\delta \to 0$, and propose taking

$$L_1 = \lim_{\delta \to 0} L_1(\delta) \ , \quad L_2 = \lim_{\delta \to 0} L_2(\delta) \ , \tag{2.1.17}$$

and from this the overall sliding vector field at $x_0 \in \Sigma$ as

$$f_{\text{mean}} = L_1 f_{F_1} + L_2 f_{F_2} \ .$$

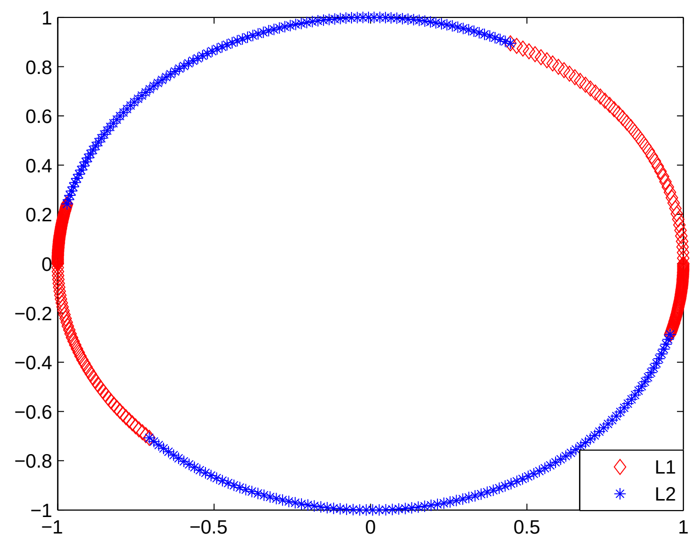


Figure 11: Geometric visualization for $L_1$, $L_2$.

With this rewriting, the coefficients $c^{\pm}_{1,2}$ in (2.1.14) are:

$$c_1^+ := L_1(1-a_1)\ , \quad c_1^- := L_1 a_1\ , \quad c_2^+ := L_2(1-a_2)\ , \quad c_2^- := L_2 a_2\ . \tag{2.1.18}$$

Therefore, by making definition (2.1.14) explicit in terms of the $f_i$'s, this "average" solution of (1.4.9) is

$$\lambda_{\text{mean}} := \begin{bmatrix} (1-\alpha^-)c_1^- + (1-\beta^-)c_2^- \\ (1-\alpha^+)c_1^+ + \beta^- c_2^- \\ \alpha^- c_1^- + (1-\beta^+)c_2^+ \\ \alpha^+ c_1^+ + \beta^+ c_2^+ \end{bmatrix}. \tag{2.1.19}$$

**Remarks 2.1.12.**

*(i)* *The case in which a Euler segment crosses $\Sigma$ directly, ahead of crossing either (but not both) $\Sigma_1$ or $\Sigma_2$, is not a concern in defining the values in* (2.1.16)*, and then* (2.1.17)*, because, for each given $\delta$, there are just four angles giving this eventuality. Hence, they do not contribute to the arc lengths we used.*

*(ii)* *The limit in* (2.1.17) *as $\delta \to 0$ exists as consequence of the fact that (for any $i = 1, 2, 3, 4$) $\|f_i(x) - f_i(x_0)\|$ is arbitrarily small for $x \in R_\delta(x_0)$.*

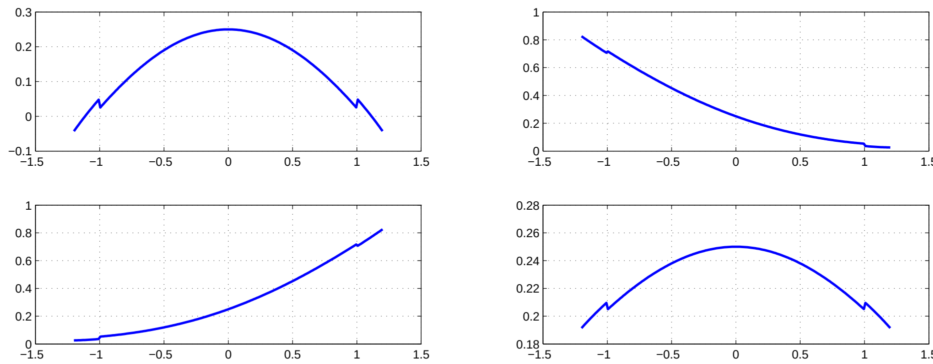


Figure 12: Components of $\lambda_{\text{mean}}$ for Example 1.4.12.

(iii) *In principle, it is possible to attempt averaging for neighborhoods of $\Sigma$ defined by norms other than the 2-norm we used. We made some (limited) experiments also with the $\infty$-norm and the 1-norm, and our results were qualitatively similar to those we reported for the 2-norm.*

**Example 2.1.13.** *Let us consider again Example 1.4.12, with $x_1 = -0.9$.*

*In this case we obtain*

$$\begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \end{bmatrix} = \begin{bmatrix} 0.1992 \\ 0.0383 \\ 0.2179 \\ 0.5446 \end{bmatrix}, \quad \lambda_{mean} = \begin{bmatrix} 0.0636 \\ 0.6618 \\ 0.0618 \\ 0.2127 \end{bmatrix}$$

*and*

$$f_{mean} = \begin{bmatrix} -2.1582 \\ 0 \\ 0 \end{bmatrix},$$

*whereas a plot of all components of $\lambda_{mean}$ in function of $x_1$ is given on the right. As Figure 12 makes clear, the components vary smoothly as long as the nodal attractivity assumptions hold; i.e., $x_1 \in (-1, 1)$. But, they do not extend nicely outside of this interval, a fact which appears to limit this averaging process and the construction of $\lambda_{mean}$ to purely nodally attractive configurations.*

## *2.2 Geometric methods*

Here we look at techniques which can be naturally framed within the context of rebuilding polygons in the plane, and finding a representation (i.e., coordinates) for points internal to the polygon in terms of convex combination of the vertices. As it turns out, these are the most interesting techniques. Chapter 3 will be devoted to completely justify and analyze this geometric construction.

The idea is to think of the values $w_j^i$, $i = 1, 2$, $j = 1, 2, 3, 4$, in (1.4.5) as giving the four different points $w_j = (w_j^1, w_j^2)$, $j = 1, 2, 3, 4$, then consider the polygon made up by joining the vertices in the following order

$$\Pi := w_1 w_2 w_4 w_3 \ .$$

Given our assumptions on the $w_j^i$'s, it is easy to realize that the origin is inside the polygon. Thus, our task is to find the *coordinates* of the origin with respect to the given vertices.

Although not derived from this interpretation, the technique in [23, 19] belongs to this class of methods. The appropriate framework within which to interpret these techniques, and to derive another very promising one, turns out to be that of *barycentric coordinates*, widely used in computer graphics.

**Definition 2.2.1** (Barycentric Coordinates)**.** *Let $\Omega$ be a closed convex polygon in the plane, with vertices $w_1, \ldots, w_n$, $n \geq 3$, and let $z \in \Omega$. The functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, are called* barycentric coordinates *for $z$, if they satisfy the three properties of positivity, convexity, and interpolation:*

$$(a) \quad \lambda_i(z) \geq 0, \ \ i = 1, \ldots, n, \quad (b) \quad \sum_{i=1}^{n} \lambda_i(z) = 1, \quad (c) \quad \sum_{i=1}^{n} \lambda_i(z) w_i = z. \quad (2.2.1)$$

In the special case of $n = 3$, barycentric coordinates are unique and are called triangular coordinates. For $n \geq 4$, there is no unique choice of barycentric coordinates. In the context of interest to us, we have $n = 4$, $z = 0$, and we seek $\lambda_i(0)$ to be smoothly varying functions of the vertices $w_1, \ldots, w_4$.

Even though barycentric coordinates are not unique for $n \geq 4$, they share some general properties that follow from the three defining axioms (2.2.1). In particular, they satisfy the Lagrange property $\lambda_i(w_j) = \delta_{ij}$, and they are linear along each edge of $\Omega$. To see this, observe that the axioms (2.2.1) imply linear precision, i.e. for any linear function $f$ one has $\sum_{i=1}^{n} \lambda_i(z) f(w_i) = f(z)$.

Below, we will look at three instances of quadrilateral barycentric coordinates of the origin relatively to the polygon of vertices $w_1$, $w_2$, $w_4$, $w_3$ (in this order). Note that, under nodal attractivity assumption, the origin is inside the polygon.

### 2.2.1 Bilinear interpolation

An important choice of barycentric coordinates is based upon bilinear interpolation. In this case, one seeks $\lambda$ in (2.2.1) of the form:

$$\lambda = \begin{bmatrix} (1-\alpha)(1-\beta) \\ (1-\alpha)\beta \\ \alpha(1-\beta) \\ \alpha\beta \end{bmatrix}, \qquad \alpha, \beta \in [0,1]. \tag{2.2.2}$$

We will call $\lambda_{\mathrm{B}}$ the choice above. In our context, this choice was first proposed in [2], and then throughly investigated and justified in [19], where it was proven to give a smoothly varying solution $\lambda$ so that the Filippov sliding vector field in (1.4.8) is well defined. [The results in [19] validate this choice under more general attractivity assumptions than just nodal attractivity.]

Quite clearly, the structure (2.2.2) derives from the convexity requirement on the solution components,

$$\begin{aligned}(\lambda_1 + \lambda_2) + (\lambda_3 + \lambda_4) &= (1-\alpha) + \alpha \\ &= (1-\alpha)(1-\beta) + (1-\alpha)\beta + \alpha(1-\gamma) + \alpha\gamma,\end{aligned}$$

where $\alpha$, $\beta$, $\gamma \in [0,1]$, and then $\lambda_{\mathrm{B}}$ is obtained by selecting $\gamma = \beta$. This choice can be understood as a (nonlinear) regularization of the system (1.4.9), as below.

**Definition 2.2.2.** *A vector* $\lambda \in \mathbb{R}^4$ *is said to satisfy the B-condition if* $\lambda_1\lambda_4 = \lambda_2\lambda_3$.

*Equivalently, letting* $R := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, *one has* $\lambda^\top R\lambda = 0$.

**Lemma 2.2.3.** *A solution* $\lambda$ *of* (1.4.9) *is* $\lambda_B$ *if and only it satisfies the B condition.*

*Proof.* It is straightforward from the construction that $\lambda_{\mathrm{B}}$ satisfies the B condition. Now, suppose $\lambda$ verifies the B condition. Then, let us define

$$\alpha := \frac{\lambda_3}{\lambda_1 + \lambda_3}, \;\; \beta := \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

A trivial computation gives

$$(1-\alpha)(1-\beta) = \frac{\lambda_1}{\lambda_1 + \lambda_3}\frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{(\lambda_1)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)} = \lambda_1,$$

and similarly for the other components. □

This $\lambda_{\mathrm{B}}$ can be also obtained by appropriate choices of $c$ in (1.4.12), and as a mean field solution associated to a special value of $\alpha$ in the pdf (2.1.3).

**Theorem 2.2.4.** *Consider the form* (1.4.12), $\lambda = \mu + cv$, *where* $\mu$ *is any particular solution of* (1.4.9), $v$ *spans* $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$, *and* $c \in [a, b]$ *(admissibility interval). Then, the bilinear interpolant solution* $\lambda_B$ *is obtained with* $c = \frac{-\mu^\top Rv \pm \sqrt{(\mu^\top Rv)^2 - (\mu^\top R\mu)(v^\top Rv)}}{v^\top Rv}$, *and it is the mean-field solution associated to the pdf* (2.1.3) *with* $\alpha = \gamma/(1-\gamma)$, $\gamma := -\frac{1}{b-a}\left(a + \frac{\mu^\top R\mu}{v^\top Rv}\right)$.

*Proof.* One needs to solve for $c$ from the relation $\lambda_{\mathrm{B}}^\top R\lambda_{\mathrm{B}} = 0$. This gives the quadratic equation for $c$:

$$c^2 v^\top Rv + 2c\mu^\top Rv + \mu^\top R\mu = 0 \;,$$

and the appropriate root is the one identified above. □

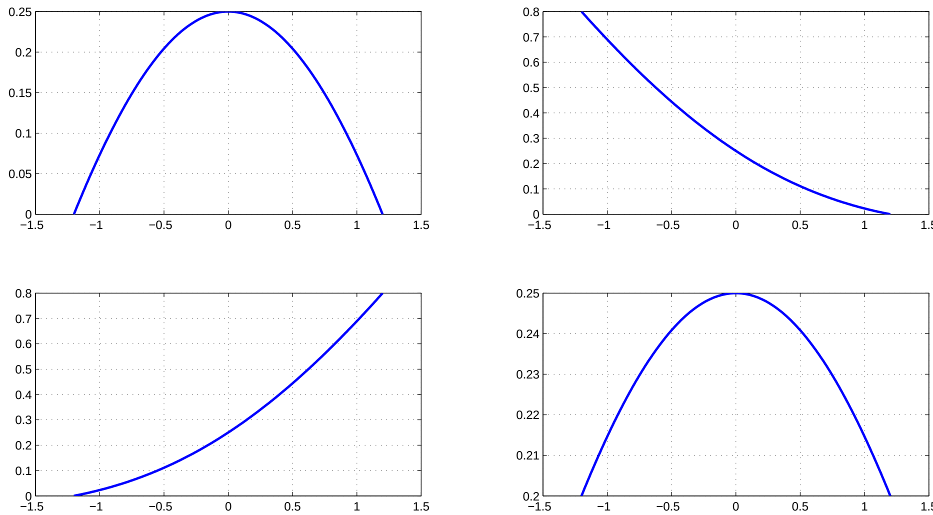


Figure 13: Components of $\lambda_{\mathrm{B}}$ for Example 1.4.12.

**Example 2.2.5.** *In Example 1.4.12 with $x_1 = -0.9$ we have*

$$\lambda_B = \begin{bmatrix} 0.1056 \\ 0.6367 \\ 0.0367 \\ 0.2211 \end{bmatrix} \quad \textit{and} \quad f_{\mathrm{B}} = \begin{bmatrix} -1.9989 \\ 0 \\ 0 \end{bmatrix},$$

*whereas a plot of $\lambda_B$ as function of $x_1$ is shown in Figure 13. Note that two of the components of $\lambda_B$ vanish at $\pm 1.2$ (see Example 1.4.12).*

### 2.2.2 Moments solution: mean value coordinates

Another instance of barycentric coordinates is obtained upon selecting the $\lambda_i$'s in such a way that the total moment of $w_1$, $w_3$ equals the total moment of $w_2$, $w_4$, all taken with respect to the origin. More precisely, we regularize (1.4.9) by adding to it the following condition:

$$d_1\lambda_1 - d_2\lambda_2 - d_3\lambda_3 + d_4\lambda_4 = 0 \ , \ \text{where } d_i := \sqrt{(w_i^1)^2 + (w_i^2)^2} \ , \ \ i = 1, \ldots, 4.$$

So, we are looking for a solution of the system

$$\begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ d_1 & -d_2 & -d_3 & d_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} W \\ d^\top \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.2.3}$$

Below, we will show that there is always a unique solution of (2.2.3), as smooth as $W$. We will call this solution the *moments solution* and label it as $\lambda_{\mathrm{M}}$.

First, we have the following Lemma.

**Lemma 2.2.6.** *Under the nodal attractivity assumptions of Table 1, the matrix* $\begin{bmatrix} W \\ d^\top \end{bmatrix}$ *has full rank* 3*, and thus its kernel is 1-dimensional.*

*Proof.* The sign pattern of the above matrix is

$$\begin{bmatrix} + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}.$$

Then, we claim that any linear combination with coefficients $a_1$, $a_2$ of the first and second rows cannot match the third row. Obviously, the claim is correct if either of $a_1$ or $a_2$ is 0. Now, if $a_1$, $a_2 > 0$, then $d_4$ cannot be obtained; if $a_1 > 0$, $a_2 < 0$, then $d_2$ cannot be obtained; if $a_1 < 0$, $a_2 > 0$, then it is $d_3$ that cannot be obtained, and if $a_1$, $a_2 < 0$, then $d_1$ cannot be obtained. $\square$

To prove that (2.2.3) gives an admissible solution, it is convenient to establish the equivalence of (2.2.3) to the so-called mean value coordinates introduced by Floater; see [27].

**Definition 2.2.7** (Mean Value Coordinates)**.** *Let* $\Omega$ *be a planar polygon of vertices*

$w_1, \ldots, w_n$. *For* $x \in \Omega$, *let*

$$\lambda_i(x) := \frac{\nu_i(x)}{\sum_{j=1}^{n} \nu_j(x)}, \quad \nu_i(x) := \frac{\tan\left(\frac{\alpha_{i-1}(x)}{2}\right) + \tan\left(\frac{\alpha_i(x)}{2}\right)}{\|w_i - x\|}, \tag{2.2.4}$$

*and* $\alpha_i(x)$ *is the angle at* $x$ *in the triangle* $[x, w_i, w_{i+1}]$. *Then, the* $\lambda_i(x)$ *are called mean value coordinates of* $x$.

We refer to the cited work of Floater [27] for a proof that mean value coordinates are well defined for points inside the polygon. Here, we show that they are equivalent to the moments solution in our context, where we have the polygon of vertices $w_1$, $w_2$, $w_4$ and $w_3$, and seek mean value coordinates of the origin.

**Lemma 2.2.8.** *The mean value coordinates satisfy* (2.2.3).

*Proof.* We already know that the mean value coordinates verify (1.4.9), so we are left to prove that they fulfill the third equation of (2.2.3). But this follows immediately from (2.2.4), by noting that

$$\begin{aligned} d_1\lambda_1 - d_2\lambda_2 - d_3\lambda_3 + d_4\lambda_4 &= \tan\left(\frac{\alpha_2}{2}\right) + \tan\left(\frac{\alpha_1}{2}\right) \\ &\quad - \left(\tan\left(\frac{\alpha_4}{2}\right) + \tan\left(\frac{\alpha_2}{2}\right)\right) \\ &\quad - \left(\tan\left(\frac{\alpha_1}{2}\right) + \tan\left(\frac{\alpha_3}{2}\right)\right) \\ &\quad + \left(\tan\left(\frac{\alpha_3}{2}\right) + \tan\left(\frac{\alpha_4}{2}\right)\right) \\ &= 0. \end{aligned}$$

□

Finally, we have

**Theorem 2.2.9.** *The mean value coordinates* (2.2.4) *are the unique solution of* (2.2.3). *In particular,* (2.2.3) *is a nonsingular system.*

*Proof.* From Lemma 2.2.8, we know that the mean value coordinates vector $\lambda_{\mathrm{M}}$ is a solution of (2.2.3), with positive components, and –in particular– it is a nontrivial solution of $\begin{bmatrix} W \\ d^\top \end{bmatrix} \lambda = 0$. Hence, see Lemma 2.2.6, $\lambda_{\mathrm{M}}$ spans the kernel of $\begin{bmatrix} W \\ d^\top \end{bmatrix}$. Since any solution $\mu$ of (2.2.3) must satisfy $\mu \in \ker\left(\begin{bmatrix} W \\ d^\top \end{bmatrix}\right)$, and $\mathbb{1}^\top \mu = 1$, then (2.2.3) has the unique solution $\lambda_{\mathrm{M}}$. $\square$

**Remarks 2.2.10.**

*(i) An important consequence of the above is that $\lambda_M$ is as smooth as $W$. In fact, $\lambda_M$ is solution of* (2.2.3)*, which –on account of Theorem 2.2.9– is an invertible linear system, and so its solution is as smooth as the coefficients, that is as $W$. See also Example 2.2.12.*

*(ii) In light of the above equivalence, we favor implementing the moments method as we proposed in this work, that is solving* (2.2.3)*, rather than by forming* (2.2.4)*. Indeed, in the present context, solving* (2.2.3) *is much simpler.*

The following result summarizes the relation between the moments solution, the general form of admissible solution in (1.4.12), and the mean field solution associated to a special value of $\alpha$ in the pdf (2.1.3).

**Theorem 2.2.11.** *Consider the form* (1.4.12)*, $\lambda = \mu + cv$, where $\mu$ is any particular solution of* (1.4.9)*, $v$ spans* $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$*, and $c \in [a, b]$ (admissibility interval). Then, the moments solution $\lambda_M$ is obtained with $c = -\frac{d^\top \mu}{d^\top v}$, where $d := \begin{bmatrix} d_1 \\ -d_2 \\ -d_3 \\ d_4 \end{bmatrix}$, in* (1.4.12)*,*

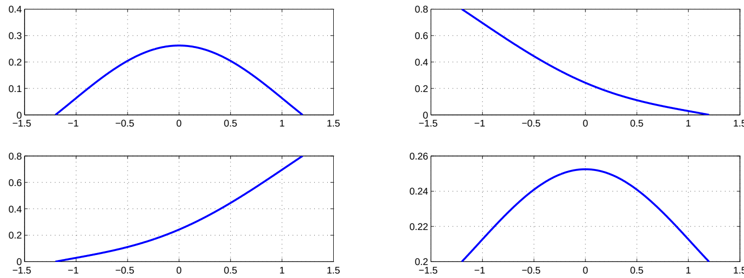

Figure 14: Components of $\lambda_{\mathrm{M}}$ for Example 1.4.12.

*and is the mean-field solution associated to the pdf* (2.1.3) *with* $\alpha = \gamma/(1-\gamma)$, $\gamma := -\frac{1}{b-a}\left(a + \frac{d^\top \mu}{d^\top v}\right)$.

*Proof.* Since $\lambda_{\mathrm{M}}$ is a solution of (1.4.9), then $d^\top \lambda_{\mathrm{M}} = 0$. Therefore, the value of $c$ in (1.4.12) is $-\frac{d^\top \mu}{d^\top v}$, as stated.

From (2.1.4) and the above, we must then have $(1-\gamma)a + \gamma b = -\frac{d^\top \mu}{d^\top v}$, from which the result follows. $\square$

**Example 2.2.12.** *Let us consider Example 1.4.12, with* $x_1 = -0.9$.

*In this case we get*

$$\lambda_M = \begin{bmatrix} 0.0949 \\ 0.6431 \\ 0.0431 \\ 0.2190 \end{bmatrix}, \; f_M = \begin{bmatrix} -2.0395 \\ 0 \\ 0 \end{bmatrix},$$

*whereas a plot of* $\lambda_M$ *in function of* $x_1$ *is shown in Figure 14. Note that two of the components of* $\lambda_M$ *vanish at* $\pm 1.2$ *(see Example 1.4.12).*

### 2.2.3 Wachspress solution

Another choice of planar barycentric coordinates is due to Wachspress (see [27, 55]). Rephrased in our context, this gives an admissible value of $\lambda$ in (1.4.9), which we will

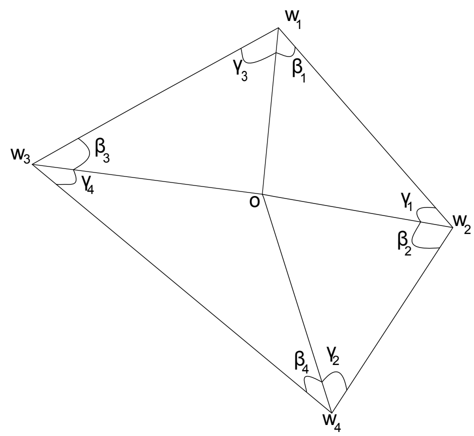


Figure 15: Figure for the definition of Wachspress solution

call $\lambda_{\mathrm{W}}$, defined by the requirement (see Figure 15):

$$\lambda_i = \frac{\mu_i}{\sum_{i=1}^{4} \mu_i} \ , \ \mu_1 := \frac{\cot \gamma_3 + \cot \beta_1}{d_1^2} \ , \ \text{etc. .} \tag{2.2.5}$$

We refer to the original derivation of Wachspress [55] for a justification of this choice.

**Example 2.2.13.** *Let us consider Example 1.4.12, with $x_1 = -0.9$.*

*In this case we get*

$$\lambda_W = \begin{bmatrix} 0.0832 \\ 0.6483 \\ 0.0506 \\ 0.2180 \end{bmatrix}, \ f_{\mathrm{W}} = \begin{bmatrix} -2.0833 \\ 0 \\ 0 \end{bmatrix},$$

*whereas a plot of $\lambda_W$ in function of $x_1$ is shown in Figure 16. We note that Wachspress coordinates extend smoothly beyond the nodal attractivity interval $(-1, 1)$, but the plot of the third component betrays that Wachspress coordinates are not well defined when the origin belongs to a side of the polygon, a fact already remarked by Floater in [27]. This fact makes $\lambda_{\mathrm{W}}$ less appealing than $\lambda_B$ and $\lambda_M$ beyond the case of nodally attractive $\Sigma$.*

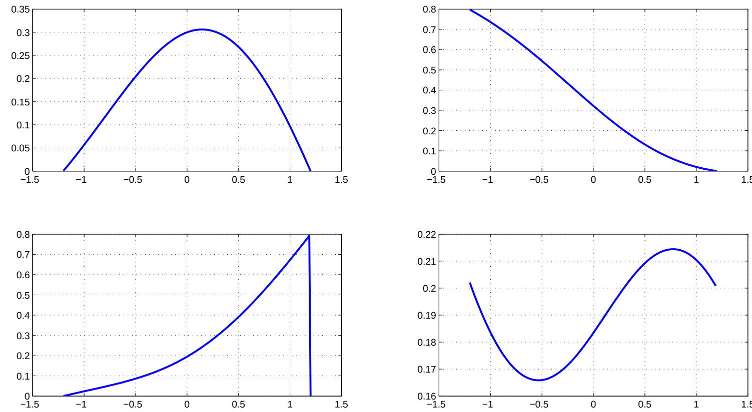

Figure 16: Components of $\lambda_{\mathrm{W}}$ for Example 1.4.12.

### 2.2.4 Another geometric solution

A final choice of geometric coordinates is the one based on the construction adopted in [22]. This choice does not generally give a Filippov solution (that is, it does not select a value of $\lambda$ in (1.4.9)), but still selects a value of $\lambda$ giving a smoothly varying vector field on $\Sigma$. The difference with respect to the standard Filippov choice is that one first projects the vector fields onto the tangent plane at $x_0 \in \Sigma$, then seeks a convex combination of the same. In our notation, calling $\lambda_{\mathrm{P}}$ the resulting values of these convex coefficients, one proceeds as follows.

One seeks a sliding vector field (not necessarily of Filippov type) of the form

$$f_{\mathrm{P}} := \sum_{i=1}^{4} \lambda_i v_i \ , \quad v_i = f_i - N(N^\top N)^{-1} w_i \ , \quad N = \begin{bmatrix} \nabla h_1 & \nabla h_2 \end{bmatrix} \ .$$

In its simplest form, in [22], selection of $\lambda$ was done as follows:

$$\lambda_i = \frac{\mu_i}{\sum_j \mu_j} \ , \quad \text{where} \quad \mu_i = \frac{\prod_{j\neq i} a_j^\top w_j}{\prod_{j\neq i} a_j^\top w_j - a_i^\top w_i} \ , \quad i = 1, \dots, 4$$

$$a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \ a_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \ , \ a_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ , \ a_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ .$$

**Example 2.2.14.** *Let us consider again Example 1.4.12, with* $x_1 = -0.9$.

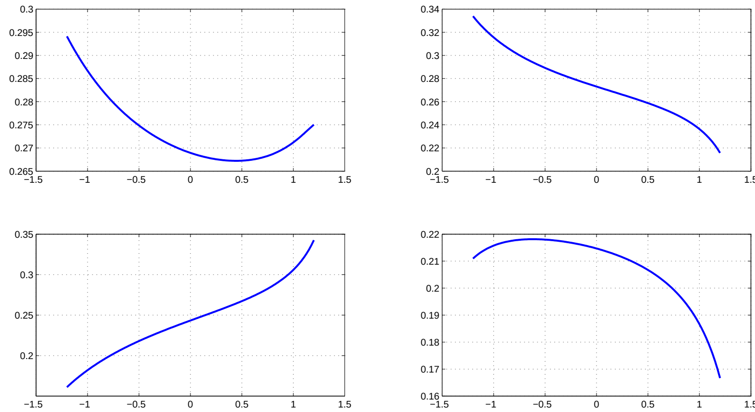

Figure 17: Components of $\lambda_{\mathrm{P}}$ for Example 1.4.12.

*In this case we get* $\lambda_{\mathrm{P}} = \begin{bmatrix} 0.2789 \\ 0.2940 \\ 0.2021 \\ 0.2250 \end{bmatrix}$, *and* $f_{\mathrm{P}} = \begin{bmatrix} -1.9713 \\ 0 \\ 0 \end{bmatrix}$, *whereas a plot of* $\lambda_{\mathrm{P}}$ *in function of* $x_1$ *is shown in Figure 17: we note that these coordinates extend smoothly beyond the nodal attractivity interval* $(-1, 1)$. *However, note that none of the components of* $\lambda_{\mathrm{P}}$ *is* $0$ *at* $\pm 1.2$ *(see Example 1.4.12). So, although this choice does not generally give a Filippov sliding vector field, it may be of some (limited) interest in the nodally attractive case.*

## 2.3 Nodal attractivity and stochastic basis

In this final section, we adopt the rewriting of a Filippov vector field in terms of the sub-sliding vector fields (cfr. (2.1.14)). Indeed, we can rewrite $\lambda$ as:

$$\lambda = Sq \ , \quad \text{where} \quad q := \begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \end{bmatrix} , \quad \text{and} \quad S := \begin{bmatrix} 0 & 1-\alpha^- & 0 & 1-\beta^- \\ 1-\alpha^+ & 0 & 0 & \beta^- \\ 0 & \alpha^- & 1-\beta^+ & 0 \\ \alpha^+ & 0 & \beta^+ & 0 \end{bmatrix} .$$

Observe that $S$ is column stochastic, hence we may call any $\lambda$ derived from this form a *stochastic subsliding solution*.

This implies that we can obtain a solution of (1.4.9) by solving the following problem:

$$B \begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{s.t.} \quad \lambda = S \begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \end{bmatrix} > 0, \tag{2.3.1}$$

where $B := \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} S$. Moreover, letting for $i, j = 1, 2$, $i < j$, $D_{ij} := \det \begin{bmatrix} w_i & w_j \end{bmatrix}$, then $B$ can be written as

$$B := \begin{bmatrix} 0 & 0 & -b_{13} & b_{14} \\ -b_{21} & b_{22} & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

where

$$b_{13} := \frac{D_{34}}{w_3^2 - w_4^2}, \;\; b_{14} := -\frac{D_{12}}{w_1^2 - w_2^2}, \;\; b_{21} := -\frac{D_{24}}{w_2^1 - w_4^1}, \;\; b_{22} := \frac{D_{13}}{w_1^1 - w_3^1}.$$

Under nodal attractivity assumption, Table 1 assures that these $b_{ij}$'s are positive, so that the sign pattern of $B$ results $\begin{bmatrix} 0 & 0 & - & + \\ - & + & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, and obviously $\operatorname{rank}(B) = 3$. So, from (2.3.1) we have

$$c_1^+ = x b_{22} \ , \;\; c_1^- = x b_{21} \ , \;\; c_2^+ = y b_{14} \ , \;\; c_2^- = y b_{13} \ ,$$

for some $x$ and $y$ such that

$$(b_{13} + b_{14}) y + (b_{21} + b_{22}) x = 1,$$

and thus, for some $\gamma \in \mathbb{R}$, $y = \dfrac{1-\gamma}{b_{13} + b_{14}}$, $x = \dfrac{\gamma}{b_{21} + b_{22}}$.

In particular, we can write every solution of (2.3.1) as

$$q = \begin{bmatrix} 0 \\ 0 \\ \frac{b_{14}}{b_{13}+b_{14}} \\ \frac{b_{13}}{b_{13}+b_{14}} \end{bmatrix} + \gamma \begin{bmatrix} \frac{b_{22}}{b_{21}+b_{22}} \\ \frac{b_{21}}{b_{21}+b_{22}} \\ -\frac{b_{14}}{b_{13}+b_{14}} \\ -\frac{b_{13}}{b_{13}+b_{14}} \end{bmatrix} = (1-\gamma) \begin{bmatrix} 0 \\ 0 \\ \frac{b_{14}}{b_{13}+b_{14}} \\ \frac{b_{13}}{b_{13}+b_{14}} \end{bmatrix} + \gamma \begin{bmatrix} \frac{b_{22}}{b_{21}+b_{22}} \\ \frac{b_{21}}{b_{21}+b_{22}} \\ 0 \\ 0 \end{bmatrix}. \tag{2.3.2}$$

Setting

$$s_1 := \begin{bmatrix} 0 \\ 0 \\ \frac{b_{14}}{b_{13}+b_{14}} \\ \frac{b_{13}}{b_{13}+b_{14}} \end{bmatrix}, \quad s_2 := \begin{bmatrix} \frac{b_{22}}{b_{21}+b_{22}} \\ \frac{b_{21}}{b_{21}+b_{22}} \\ 0 \\ 0 \end{bmatrix},$$

then (2.3.2) rewrites as

$$q = (1-\gamma)s_1 + \gamma s_2. \tag{2.3.3}$$

Now, let us determine the largest admissibility interval for $\gamma$. From (2.3.3), we have

$$Sq = Ss_1 + \gamma S(s_2 - s_1). \tag{2.3.4}$$

But, both $Ss_1$ and $Ss_2$ are admissible solutions of (1.4.9), and so $S(s_2 - s_1)$ belongs to $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. Therefore, we can use (1.4.12) with

$$\mu := Ss_1,$$

$$v := S(s_2 - s_1).$$

From this, we can find the admissibility interval for $c$: $\lambda = \mu + cv$, call it $(a_S, b_S)$, see (1.4.13). Hence, from (2.3.4) we get that $\gamma \in (a_S, b_S)$ if and only if $q$ as in (2.3.3) provides a strictly positive solution $Sq$ of (1.4.9).

**Example 2.3.1.** *Consider again Example 1.4.12, with* $x_1 = -0.9$. *We have* $(a_S, b_S) = (-0.3039..., 1.1144...)$ *and the values of* $\gamma$ *giving all the solutions we have derived so* *far are:*

$$\gamma_B = 0.5944,$$

$$\gamma_{MF} = 0.4052,$$

$$\gamma_{min} = 1.8235,$$

$$\gamma_m = 0.5034,$$

$$\gamma_{mean} = 0.2375,$$

$$\gamma_{\mathrm{W}} = 0.4052,$$

$$\gamma_{\mathrm{P}} = 1.4690.$$

*Note that* $\gamma_{mean} = L_1$ *in* (2.1.16). *Also, note that* $\gamma_{min}$ *and* $\gamma_{\mathrm{P}}$ *produce values outside of the admissibility interval, betraying that the corresponding approaches either produce Filippov solutions which are not admissible (namely,* $\lambda_{min}$*), or do not produce Filippov solutions (namely,* $\lambda_{\mathrm{P}}$*).*

# Chapter III

# THE MOMENTS SLIDING VECTOR FIELD ON THE INTERSECTION OF TWO MANIFOLDS

## *3.1 Introduction*

In Section 2.2.2, we have introduced the *moments method* in (2.2.3), and showed it is well defined under nodal attractivity of $\Sigma$. Let us stress that our proposal was based on a rather general principle: *To regularize the system* (1.4.9) *by adding to it one extra condition, linear in $\lambda$, so to obtain an invertible system giving a solution $\lambda$ enjoying specific properties*. For our scopes, these properties amount to having that $\lambda$ be positive and smooth.

For later convenience, let us repeat that we consider the following system (cfr. with (1.4.9) and (2.2.3)) to be satisfied for $x \in \Sigma$:

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \text{ where } M := \begin{bmatrix} W \\ \mathbb{1}^\top \\ d^\top \end{bmatrix} , \tag{3.1.1}$$

with $W$ defined in (1.4.9) and

$$d := \begin{bmatrix} d_1 \\ -d_2 \\ -d_3 \\ d_4 \end{bmatrix} , \text{ where } d_i := \|w_i\|_2 , \quad i = 1, \dots, 4 . \tag{3.1.2}$$

**Definition 3.1.1** (Moments method)**.** *We call* moments method *the method resulting from solving* (3.1.1) *for $\lambda$, and using this in the selection of sliding vector field in*

(1.4.8). *We call* moments solution *the solution* $\lambda$ *of* (3.1.1)*, call* moments vector field *the resulting vector field* (1.4.8)*, and call* moments trajectory *the solution of the differential equation on* $\Sigma$ *obtained when using the moments vector field.* □

Below, we validate the moments method, by showing that, for $x \in \Sigma$ and $\Sigma$ attractive as in Definition 1.4.2, the matrix $M$ in (3.1.1) is non-singular, that the unique solution of (3.1.1) is admissible, and that the resulting smoothly varying Filippov sliding vector field $f_{\mathrm{F}}$ is further smoothly exiting at generic first order exit points. Let us emphasize that our construction will give **a** Filippov solution (1.4.8) of the general piecewise smooth system (1.4.3). Let us also emphasize that the overall solution trajectory, in general, will only be piecewise smooth: our concern is that it be smooth on the intersection $\Sigma$, but of course –in general– it will be only continuous at entry points in a sliding region.

**Remark 3.1.2.** *Of course, the formulation of the moments method we validate in this paper is valid precisely for the case of* $\Sigma$ *of co-dimension 2 examined herein. The extension of the moments method to the case of* $\Sigma$ *of co-dimension* 3 *(intersection of three co-dimension 1 surfaces) requires an appropriately modified formulation; details are in [17].*

A plan of Chapter 3 is as follows. In Section 3.2, we associate a quadrilateral to the attractivity configuration of $\Sigma$, extending and rigorously explaining what we have already done in Section 2.2. In Section 3.3, this geometrical configuration is exploited to prove invertibility of the matrix $M$ in (3.1.1), and admissibility of the unique solution $\lambda$. In Section 3.4, we rigorously prove that the moments vector field is smoothly exiting at generic first order exit points, and we briefly discuss other possibilities enjoying this property.

## 3.2 Geometrical pattern for the dynamical problem

In this section, we give a useful geometrical reinterpretation of the algebraic problem (1.4.9), when $\Sigma$ is attractive. Later, this configuration will be exploited to establish solvability of the system (3.1.1).

We begin by observing that the general Filippov convexification construction based on (1.4.8)-(1.4.9) is effectively saying that the origin must be in the convex hull of the four points $w_i$, $i = 1, \dots, 4$. However, the convex hull of the four points $w_i$'s is a very large set, and may fail to give a good geometrical correspondence with the dynamics of the problem.

**Example 3.2.1.** *Consider the following model problem of the type* (1.4.3)*:*

$$\dot{x} = f_i(x)\ ,\ \ i = 1, 2, 3, 4,\ \textit{where}$$

$$f_1 = \begin{bmatrix} x_3 - 1 \\ x_3 \\ x_1 - 1 \end{bmatrix}\ ,\ \ f_2 = \begin{bmatrix} 2 \\ -1 \\ x_2 - 1 \end{bmatrix}\ ,\ \ f_3 = \begin{bmatrix} -1 \\ 2 \\ x_1 x_2 - 1 \end{bmatrix}\ ,\ \ f_4 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\ ,$$

*where (see* (1.4.1)*)* $\Sigma_1 = \{x_1 = 0\}$*,* $\Sigma_2 = \{x_2 = 0\}$*, and so* $\Sigma = \{x_1 = x_2 = 0\}$*, and therefore (see* (1.4.5)*)*

$$w_1 = \begin{bmatrix} x_3 - 1 \\ x_3 \end{bmatrix}\ ,\ \ w_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\ ,\ \ w_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}\ ,\ \ w_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}\ .$$

*In this case, on $\Sigma$, there is a unique Filippov sliding vector field:* $\dot{x}_3 = -1$.

*Consider the initial condition* $(0, 0, 2)$ *and the time interval* $0 \le t \le 2$*. In Figure 18, we show the four snapshots of the vertices $w_i$'s, at times $t = 0$, $t = 1$, $t = 5/3$, and $t = 9/5$. For $t = 0$, we are in a configuration of nodal attractivity, which persists for as long as $t < 1$. However, as soon as $t \ge 1$, the vertex $w_1$ plays no role in the convex hull of the four points (dotted segment). Also, observe that as soon as $t > 5/3$, $\Sigma$ is no longer attracting nearby trajectories (hence, a perturbation off $\Sigma$ will move away from $\Sigma$), though the convex hull has not changed.* □

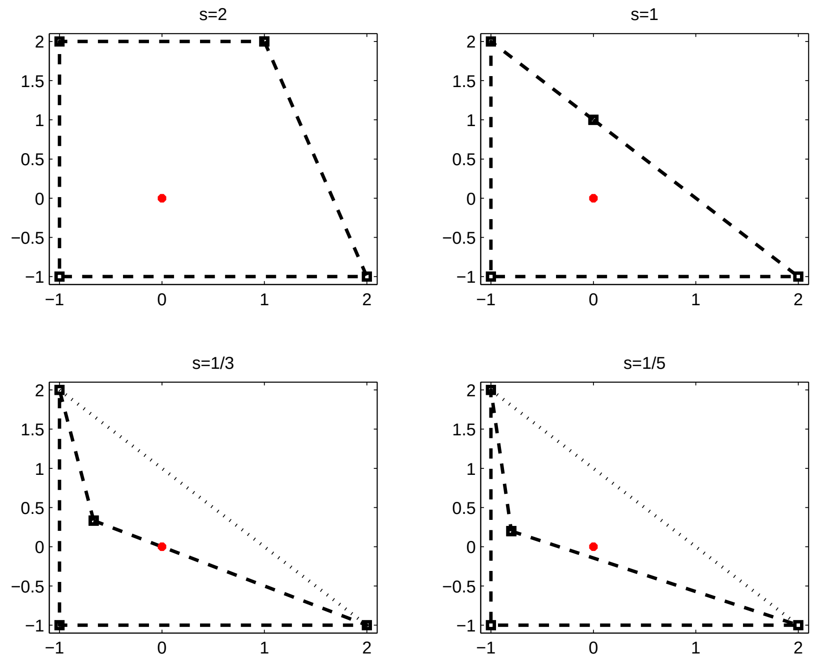


Figure 18: Dynamics of Example 3.2.1: Convex hull versus quadrilateral $Q$.

Motivated by the above, our goal is to consider a geometric configuration that better reflects the dynamics of the problem (and attractivity of $\Sigma$). To this end, we propose to consider the quadrilateral $Q$, determined by $w_1$, $w_2$, $w_4$, $w_3$, in this order. Accordingly, we are proposing to reinterpret an admissible Filippov solution as one that obtains weights $\lambda$ to be put on the vertices of $Q$ in such a way that the origin be the *barycenter* of $Q$ relative to $\lambda$ [1]. For later reference, we summarize our proposal of quadrilateral $Q$.

**Definition 3.2.2.** *Given the four points $w_1$, $w_2$, $w_3$, $w_4$, as in* (1.4.5)*, we define the quadrilateral $Q$ associated to $W$ to be the quadrilateral obtained by joining the four points in the order $w_1$ to $w_2$, to $w_4$, to $w_3$, and back to $w_1$.* □

The following result is a simple consequence of the characterizations of attractivity of $\Sigma$ and the definition of quadrilateral $Q$. [For part (i), in the case of $\Sigma$ attractive

[1] In this context, we can reinterpret (3.1.1) as a physical equilibrium requirement about the moments provided by the weights $\lambda$ with respect to origin, hence the proposed name of *moments method* we adopted for our technique.

through sliding, the result follows at once from Definition 1.4.2. In the case of spiral attractivity, it follows immediately from [14, Table 3 or 4]). For part (ii), see Lemma 1.4.3.] Also, note that, in case (i), sliding motion on $\Sigma$ should be taking place.

**Lemma 3.2.3.** *Let $W$ and $Q$ be defined as above, for $x \in \Sigma$.*

*(i) If $\Sigma$ is attractive (through sliding, or by spiraling), then the origin is in the interior of $Q$. In particular, if the origin is external to $Q$, then $\Sigma$ cannot be attractive.*

*(ii) If $x$ is a generic first order exit point, then the origin belongs to one side (and one only) of $Q$.* □

We emphasize that that the quadrilateral $Q$ tells us that "if $0 \notin \bar{Q}$ then $\Sigma$ is not attractive, and a trajectory with initial conditions off $\Sigma$ will not be attracted to $\Sigma$": this is our key reason to consider $Q$.

Below, we give some results on the interplay between the quadrilateral $Q$ and the algebraic problem (3.1.1). These results will be used in Section 3.3 to establish solvability of (3.1.1).

**Definition 3.2.4.** *The quadrilateral $Q$ is called* non-degenerate*, if and only if these two conditions hold:*

*(a) the vertices are not all aligned (equivalently, at most three vertices are aligned), and*

*(b) if one vertex of $Q$ is at the origin, then there cannot be two other vertices aligned with it; in particular, no two vertices can be at the origin.* □

**Remark 3.2.5.** *In agreement with Lemma 3.2.3, it is an important observation that, in each of the sliding configurations allowed by Definition 1.4.2,[2] the points $w_i$, $i =$*

[2] *there are 13, not equivalent ones, [19]*

$1, \dots,,$ *will always give that $Q$ is non-degenerate. In fact, the origin is always in the interior of $Q$. Furthermore, at generic first order exit points, the origin is along one edge (and one only) of $Q$, and in particular the origin cannot be a vertex of $Q$.*

Next, we give a key algebraic result that will be used in Section 3.3.

**Lemma 3.2.6.** *If $Q$ is non-degenerate, then the matrix $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ in* (1.4.9) *has full rank* 3*. Furthermore, there is a nontrivial vector $v$, as smooth as $W$, spanning* $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$.

*Proof.* Since we are assuming the quadrilateral relative to $W$ to be non-degenerate, then there exist three vectors in $\{w_i : i = 1, 2, 3, 4\}$ such that the corresponding triangle has nonzero area: this implies that the columns corresponding to those three vectors in $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ are linearly independent. The statement about the span of $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ is because the symmetric function $\begin{bmatrix} W^\top & \mathbb{1} \end{bmatrix} \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ has exactly one zero eigenvalue which is simple (of algebraic multiplicity 1). Therefore, the eigenvector associated to this 0 eigenvalue can be chosen smooth (e.g., see [36]), and it provides a basis for $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. □

We next give a more precise algebraic characterization of the vector $v \in \ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ relatively to non-degenerate quadrilaterals. This result will be used in Section 3.4.

**Notation 3.2.7.** *We will write $\mathcal{A}_{ijk}$ for the signed area of the triangle of vertices $w_i$, $w_j$, $w_k$, in this order, $i, j, k = 1, 2, 3, 4$, and where the indices are distinct. For example, $\mathcal{A}_{123} = \frac{1}{2} \det \begin{bmatrix} w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{bmatrix}$, and the sign of the determinant indicates whether the triangle is traced clockwise or counterclockwise.* □

**Lemma 3.2.8.** *Let $Q$ be non-degenerate, and let $W$ be the usual matrix: $W = [w_1\ w_2\ w_3\ w_4]$. Then, if $v \in \ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$, $v$ can have at most one zero component.*

*Proof.* By Lemma 3.2.6, there is at least one triangle determined by vertices of $Q$ with nonzero area: without loss of generality, we assume it to be $\mathcal{A}_{123}$. Therefore, by Cramer's rule (and elementary rules of the determinant), we can write this vector $v$ as

$$v = \begin{bmatrix} \mathcal{A}_{243} \\ \mathcal{A}_{134} \\ \mathcal{A}_{142} \\ \mathcal{A}_{123} \end{bmatrix} . \tag{3.2.1}$$

If, by contradiction, more than one of these components were zero, then the four vertices would be aligned: but this contradicts that $Q$ be non-degenerate. $\square$

Additionally, (3.2.1) also shows smoothness of $v$, because the (signed) area of a triangle is a smooth function of the triangle vertices (that is, the determinant is a smooth function of the matrix entries).

**Remark 3.2.9.** *In light of Lemmata 3.2.6 and 3.2.8, clearly any solution of $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ can be written as*

$$\lambda = \lambda_p + cv \,,$$

*where $\lambda_p$ is any particular solution, and $v \in \ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$, and thus we note that $v$ cannot have all components of the same sign. Therefore, in particular, if $\lambda_p$ is admissible (hence $\lambda_p \geq 0$), in order for $\lambda$ to be admissible we must have $a \leq c \leq b$, where $a \leq 0$*

*and* $b \geq 0$ *are defined as (see 1.4.13)*

$$a := \max\left\{-\frac{\lambda_{p,i}}{v_i} : v_i > 0\right\} \ , \quad b := \min\left\{-\frac{\lambda_{p,i}}{v_i} : v_i < 0\right\} \ .$$

## 3.3 Moments Solution under general attractivity conditions

Assume that the quadrilateral $Q$ is non-degenerate and the origin is internal to it or on at most one of its edges. In particular, this is the situation when $\Sigma$ is attractive through sliding. Then, we will show that $M$ in (3.1.1) is nonsingular, and the moments solution $\lambda$ is admissible. In Section 3.4, we will further show that the moments vector field is smoothly exiting at generic first order exit points.

Consider system (3.1.1), repeated here for convenience:

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \ , \text{ where } M := \begin{bmatrix} W \\ \mathbb{1}^\top \\ d^\top \end{bmatrix} \ , \tag{3.3.1}$$

and recall that, see Lemma 3.2.6, $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ has dimension 1 and it is smoothly spanned by a vector $v$, which we will take as in (3.2.1).

The following general result will be used below.

**Lemma 3.3.1.** *Let* $A \in \mathbb{R}^{(n-1)\times n}$ *be of rank* $(n-1)$, *and let its null space be spanned by the vector* $v$. *Let* $d \in \mathbb{R}^n$ *be given and consider the matrix* $B = \begin{bmatrix} A \\ d^\top \end{bmatrix}$. *Then,* $B$ *is nonsingular if and only if* $d^\top v \neq 0$.

*Proof.* Suppose $B$ is nonsingular, and by contradiction that $d^\top v = 0$. Then $Bv = 0$, and hence $B$ would be singular. If $d^\top v \neq 0$, since $\ker(A)$ is spanned just by $v$, then there cannot be any vector $y \in \mathbb{R}^n$ such that $By = 0$. □

Using Lemma 3.3.1 and Laplace expansion of the determinant with respect to the fourth row of $M$, from (6.1.10) we get (for $v$ in (3.2.1)):

$$\det M = d^\top v \ . \tag{3.3.2}$$

Now, let $M_{\text{adj}}$ be the adjugate[3] of $M$. Since $MM_{\text{adj}} = M_{\text{adj}}M = \det(M)I$, if $M$ is invertible, to obtain the unique solution of (6.1.10) we must look at the third row, $M_{\text{adj}}(3,:)$, of $M_{\text{adj}}$.

Direct computation gives

$$M_{\text{adj}}(3,:) = \left[\det\begin{bmatrix} w_2 & w_3 & w_4 \\ -d_2 & -d_3 & d_4\end{bmatrix},\ -\det\begin{bmatrix} w_1 & w_3 & w_4 \\ d_1 & -d_3 & d_4\end{bmatrix},\ \det\begin{bmatrix} w_1 & w_2 & w_4 \\ d_1 & -d_2 & d_4\end{bmatrix},\ -\det\begin{bmatrix} w_1 & w_2 & w_3 \\ d_1 & -d_2 & -d_3\end{bmatrix}\right],$$

and further

$$\sum_{j=1}^{4} M_{\text{adj}}(3,j) = d^\top v = \det M \ ,$$

and therefore the unique solution of (6.1.10), if indeed it exists unique, must be given by

$$\lambda_{\text{M}} := \frac{1}{d^\top v} M_{\text{adj}}(3,:)^\top \ . \tag{3.3.3}$$

What we will prove below is that each entry in $M_{\text{adj}}(3,:)$ has the same sign (some entries may be 0, but not all of them can be), from which it will follow that $\det M \neq 0$, and further that the entries of $\lambda_{\text{M}}$ are all nonnegative (and sum to 1), which is what we had set out to prove.

We use a geometrical technique. To begin with, assume that for all $i = 1,2,3,4$, $w_i \neq 0$, and express each $w_i$ in polar coordinates:

$$w_i = d_i \hat{w}_i, \quad \hat{w}_i := \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix},\ i = 1,2,3,4 \ . \tag{3.3.4}$$

Note that just as the original vertices $w_i$'s gave us the quadrilateral $Q$, now we have obtained the quadrilateral $\hat{Q}$ defined by the vertices $\hat{w}_1, \hat{w}_2, \hat{w}_4, \hat{w}_3$ (in this order)

[3] the transpose of the matrix of cofactors of $M$

on the unit circle; in so doing, we have respected the signs of the original vertices coordinates. In particular, if $Q$ was non-degenerate, so is the associated quadrilateral $\hat{Q}$ on the unit circle, and if the origin was internal to $Q$, it is still internal to the new quadrilateral $\hat{Q}$.

In this new notation, we have (note the changes of sign on the second equality)

$$M_{\text{adj}}(3,:)^\top = \begin{bmatrix} d_2d_3d_4 \det \begin{bmatrix} \cos\theta_2 & \cos\theta_3 & \cos\theta_4 \\ \sin\theta_2 & \sin\theta_3 & \sin\theta_4 \\ -1 & -1 & 1 \end{bmatrix} \\ -d_1d_3d_4 \det \begin{bmatrix} \cos\theta_1 & \cos\theta_3 & \cos\theta_4 \\ \sin\theta_1 & \sin\theta_3 & \sin\theta_4 \\ 1 & -1 & 1 \end{bmatrix} \\ d_1d_2d_4 \det \begin{bmatrix} \cos\theta_1 & \cos\theta_2 & \cos\theta_4 \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_4 \\ 1 & -1 & 1 \end{bmatrix} \\ -d_1d_2d_3 \det \begin{bmatrix} \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_3 \\ 1 & -1 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} d_2d_3d_4 \det \begin{bmatrix} -\hat{w}_2 & -\hat{w}_3 & \hat{w}_4 \\ 1 & 1 & 1 \end{bmatrix} \\ d_1d_3d_4 \det \begin{bmatrix} \hat{w}_1 & -\hat{w}_3 & \hat{w}_4 \\ 1 & 1 & 1 \end{bmatrix} \\ -d_1d_2d_4 \det \begin{bmatrix} \hat{w}_1 & -\hat{w}_2 & \hat{w}_4 \\ 1 & 1 & 1 \end{bmatrix} \\ -d_1d_2d_3 \det \begin{bmatrix} \hat{w}_1 & -\hat{w}_2 & -\hat{w}_3 \\ 1 & 1 & 1 \end{bmatrix} \end{bmatrix},$$

from which (again, note the changes of sign) we get

$$M_{\text{adj}}(3,:)^\top = \begin{bmatrix} -d_2d_3d_4 \det \begin{bmatrix} -\hat{w}_2 & \hat{w}_4 & -\hat{w}_3 \\ 1 & 1 & 1 \end{bmatrix} \\ -d_1d_3d_4 \det \begin{bmatrix} \hat{w}_1 & \hat{w}_4 & -\hat{w}_3 \\ 1 & 1 & 1 \end{bmatrix} \\ -d_1d_2d_4 \det \begin{bmatrix} \hat{w}_1 & -\hat{w}_2 & \hat{w}_4 \\ 1 & 1 & 1 \end{bmatrix} \\ -d_1d_2d_3 \det \begin{bmatrix} \hat{w}_1 & -\hat{w}_2 & -\hat{w}_3 \\ 1 & 1 & 1 \end{bmatrix} \end{bmatrix}. \tag{3.3.5}$$

Now, each determinant in the components of the vector in (3.3.5) above represents the (signed) area of one of the four triangles in which the quadrilateral on the unit circle $\hat{w}_1\hat{w}_2\hat{w}_4\hat{w}_3$ is divided by its diagonals. We want to show that they all have the same signs.

The following result from convex geometry will be helpful to us.

**Proposition 3.3.2.** *[46, Theorem 4.4.1 and Exercise 4.4.1] A non-degenerate quadrilateral $Q$ is convex if and only if its diagonals intersect in its closure.* $\square$

Next, we prove that, for any given quadrilateral on the unit circle, containing the origin and non-degenerate, its transformed quadrilateral obtained by reflecting one of its diagonals with respect to the origin is always convex. See Figure 19 for an illustration of this fact.

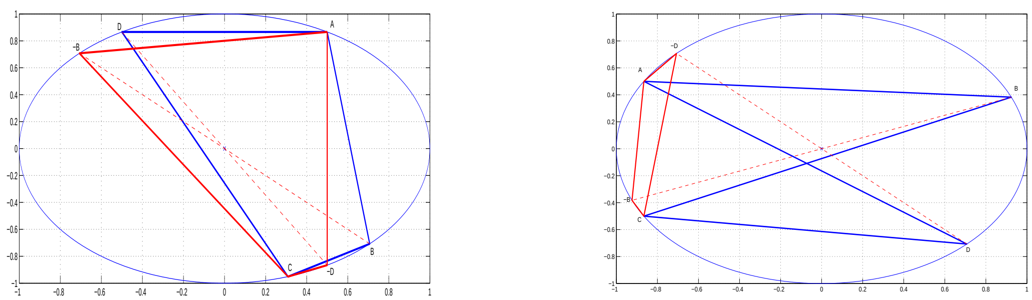


Figure 19: Illustration of Proposition 3.3.3. Transformation of the quadrilateral: left, convex case, right, nonconvex case.

**Proposition 3.3.3.** *Given a non-degenerate quadrilateral $\hat{Q} = ABCD$ with vertices on the unit circle, and containing the origin, the transformed quadrilateral $\tilde{Q} := A(-B)C(-D)$ is convex.*

*Proof.* Note that if $Q$ reduces to a triangle, the result is trivially true. So, let us assume that all vertices of $Q$ are distinct.

If $\hat{Q}$ is convex, the reflected diagonal $(-B)(-D)$ still intersects the other diagonal $AC$ in the closure of $\tilde{Q}$.

If $\hat{Q}$ is not convex, then it is necessarily self-intersecting (on the unit circle we can connect four points in two different ways only: to create a convex quadrilateral following any clockwise direction, or a self-intersecting one). Up to relabeling, we can assume that the origin is inside the triangle $ABD$. Call $\tilde{B} := -B$, $\tilde{D} := -D$, and consider the quadrilateral of vertices $A, \tilde{D}, C, \tilde{B}$, in this order.

Now, since the two angles $A\hat{B}C$ and $A\hat{\tilde{D}}C$ subtend the same arc $AC$, being the origin inside the triangle $ABD$, then

$$A\hat{B}C = A\hat{\tilde{D}}C = \alpha + \beta \ ,$$

where $\alpha$ is the angle at $B$ in the right triangle $AB\tilde{B}$, and $\beta$ is the angle in $B$ in the right triangle $CB\tilde{B}$. Therefore:

$$A\hat{\tilde{B}}B = \frac{\pi}{2} - \alpha \ , \quad C\hat{\tilde{B}}B = \frac{\pi}{2} - \beta \ ,$$

and so

$$A\hat{\tilde{B}}C = A\hat{\tilde{B}}B + B\hat{\tilde{B}}C = \pi - (\alpha + \beta) \ ,$$

whereas $A\hat{\tilde{D}}C = \alpha + \beta$. Therefore $\tilde{B}$ and $\tilde{D}$ are on opposite sides with respect to $AC$ because, otherwise, it would be $A\hat{\tilde{B}}C = A\hat{\tilde{D}}C$: so $AC$ intersects $\tilde{B}\tilde{D}$ in the closure of $\tilde{Q}$. By Proposition 3.3.2, $\tilde{Q}$ is convex. $\square$

With the help of Proposition 3.3.3, we can now give our main result.

**Theorem 3.3.4.** *Let* $\Sigma$ *be defined in* (1.4.1), $w_i$, $i = 1, \ldots, 4$, *be given in* (1.4.5), *and let* $Q$ *be the quadrilateral of Definition 3.2.2. Assume that* $Q$ *is non-degenerate, that* $w_i \neq 0$, *for all* $i = 1, \ldots, 4$, *and that* $0 \in \overline{Q}$, *as* $x \in \Sigma$. *Then, the matrix* $M$ *of the moments method in* (3.1.1) *is nonsingular and the moments solution* $\lambda_M$ *of* (3.3.3) *is admissible as* $x$ *varies in* $\Sigma$.

*Proof.* Since $Q$ is non-degenerate, the origin is not a vertex, and $0 \in \overline{Q}$, then the quadrilateral $\hat{Q}$ on the unit circle obtained by using the polar representation of (3.3.4) is non-degenerate and the origin is either internal to $\hat{Q}$ or on just one edge. Recall that $\hat{Q}$ is the quadrilateral $\hat{w}_1, \hat{w}_2, \hat{w}_4, \hat{w}_3$.

From Proposition 3.3.3, the quadrilateral obtained from $\hat{Q}$ reflecting with respect to the origin the diagonal joining $\hat{w}_2$ and $\hat{w}_3$ is convex. That is, the quadrilateral of vertices $\hat{w}_1, -\hat{w}_2, \hat{w}_4, -\hat{w}_3$, is convex. This means that the signed areas of the triangles $(\hat{w}_1, -\hat{w}_2, -\hat{w}_3)$, $(\hat{w}_1, -\hat{w}_2, \hat{w}_4)$, $(\hat{w}_1, \hat{w}_4, -\hat{w}_3)$, and $(-\hat{w}_2, \hat{w}_4, -\hat{w}_3)$, all have the same sign. [Since $\hat{Q}$ is non-degenerate, some but not all of these areas may be 0].

By looking at the determinants appearing in (3.3.5), we recognize them exactly as the areas of the aforementioned triangles, and therefore all the components of $M_{\text{adj}}(3,:)$ have the same sign, and then, by (3.3.3), $\lambda_{\text{M}}$ is the unique solution of (6.1.10), further admissible since $\sum_{j=1}^{4} M_{\text{adj}}(3,j) = d^\top v$.

The fact that $\lambda_{\text{M}}$ varies smoothly with $x \in \Sigma$ is a consequence of the smoothness of the determinant with respect to the matrix entries. □

**Corollary 3.3.5.** *If the quadrilateral $Q$ is non-degenerate, and the origin is internal to $Q$, then $\lambda_M > 0$; i.e., all components of $\lambda_M$ are positive.*

*Proof.* Let the origin be in the interior of $Q$. By Theorem 3.3.4, $\lambda_{\text{M}}$ is therefore admissible. Let us assume, by contradiction, that for some $i = 1, 2, 3, 4$, $(\lambda_{\text{M}})_i = 0$: without loss of generality, let $(\lambda_{\text{M}})_1 = 0$. Looking at (3.3.5), this happens if and only if the area of the triangle on the unit circle with vertices $-\hat{w}_2$, $\hat{w}_4$, $-\hat{w}_3$ is zero; but this is equivalent to say that either $-\hat{w}_2 = \hat{w}_4$ or $\hat{w}_4 = -\hat{w}_3$, which in turn is true if and only if the origin belongs to either $w_2 w_4$ or $w_4 w_3$, that is to the boundary of $Q$, which contradicts the assumption.
A similar argument holds for the other cases. □

**Remark 3.3.6.** *Suppose that the origin is on the segment $w_1 w_2$ and it is not a vertex (similarly, for any other side of the quadrilateral). Then, the unique solution of*

(6.1.10), *under the assumptions of Theorem 3.3.4, is*

$$\lambda := \begin{bmatrix} \frac{d_2}{d_1+d_2} \\ \frac{d_1}{d_1+d_2} \\ 0 \\ 0 \end{bmatrix}.$$

**Remark 3.3.7.** *As we said, our motivation was in validating the moments method under the conditions of partial nodal attractivity. Theorem 3.3.4 does achieve this. But in fact, it does more, only needing nondegeneracy of $Q$ and that the origin be either inside $Q$ or on at most one edge. In particular, Theorem 3.3.4 validates the moments method also in the case of $\Sigma$ being spirally attractive, see [14]. This is simply because, when $\Sigma$ is spirally attractive, the origin is inside $Q$, see Lemma 3.2.3.*

As a consequence of Theorem 3.3.4, we have the following result, which will be useful in Section 3.4.

**Theorem 3.3.8.** *Let $x \in \Sigma$, let $w_i$, $i = 1, \ldots, 4$, be given in* (1.4.5) *(these vertices of course depend on $x$), let $Q$ be the quadrilateral of Definition 3.2.2, and let $M$ be given in* (3.1.1). *Assume that $Q$ is non-degenerate and that $w_i \neq 0$, for all $i = 1, \ldots, 4$.*

*Then, for each $\epsilon > 0$ sufficiently small, if dist$(0, Q) := \min_{y \in \overline{Q}} \|y\| < \epsilon$, the matrix $M$ in* (3.1.1) *is invertible. Moreover, if $0 \notin \overline{Q}$, then the unique solution of* (3.1.1) *is not admissible.*

*Proof.* Since the determinant function is continuous as a function of the entries of $W$, and $\det(M) \neq 0$ as $0 \in \overline{Q}$, then $\det(M) \neq 0$ if 0 is sufficiently close to $\overline{Q}$.
If $0 \notin \overline{Q}$, then since $M$ is nonsingular the unique solution $\lambda_{\mathrm{M}}$ of (3.1.1) is still given by (3.3.3). But, looking at the signed areas in (3.3.5), we see that two of them are negative, making $\lambda_{\mathrm{M}}$ not admissible. $\square$

### 3.3.1 One vertex of $Q$ at the origin

Our results, particularly the construction of the quadrilateral $\hat{Q}$ and therefore Theorem 3.3.4, have relied on the assumption that $w_i \neq 0$, for every $i = 1, 2, 3, 4$. As we will clarify below, this is a very mild and natural assumption, both in terms of the problem dynamics and of the geometrical interpretation of the same. At the same time, let us consider here the case when this assumption is violated, and what it implies.

First of all, *if two or more* of the $w_i$'s were zero, then the quadrilateral $Q$ would be degenerate, and as a consequence (see (1.4.9) and (6.1.10)) $W$ would be of rank 2, and $M$ would be singular; so, the moments regularization would not be of any use. Moreover, the problem dynamics would be inherently ambiguous since two of the $w_i$'s being 0 (say $w_1 = w_2 = 0$), implies that there are two admissible exit vector fields in two different regions $R_i$'s (say, in $R_1$ and $R_2$). Finally, note that this case of two $w_i$'s equal to 0 is a co-dimension 4 phenomenon.

Suppose now that *there is just one* index $i = 1, 2, 3, 4$, for which $w_i = 0$. In this case, something more can be said. Without loss of generality, suppose that we are at a point $x \in \Sigma$ where $w_1 = 0$, and $w_i \neq 0$, $i = 2, 3, 4$.

(a) In terms of the problem's dynamics, $w_1 = 0$ means that the vector field $f_1$ is itself tangent to $\Sigma$, and therefore $f_1$ is an exit vector field. Clearly, this is not a first order exit condition (which is a co-dimension 1 phenomenon), and it is a co-dimension 2 phenomenon. Moreover, it is not clear that we can *predict* the dynamics after this situation occurs. See Example 3.3.11 below.

(b) In terms of the quadrilateral $Q$, if $Q$ is non-degenerate, then there is still a unique solution to (3.1.1), as we show below.

**Lemma 3.3.9.** *If $w_1 = 0$, and $Q$ is non-degenerate, then the matrix*

$$N = \begin{bmatrix} w_2 & w_3 & w_4 \\ -d_2 & -d_3 & d_4 \end{bmatrix}$$

*is invertible.*

*Proof.* Suppose not. Then, without loss of generality we have $\begin{bmatrix} w_2 \\ -d_2 \end{bmatrix} = \alpha \begin{bmatrix} w_3 \\ -d_3 \end{bmatrix} + \beta \begin{bmatrix} w_4 \\ d_4 \end{bmatrix}$ for some $\alpha, \beta$, not both 0. Then, we have $w_2 = \alpha w_3 + \beta w_4$ and $-d_2 = \alpha d_3 + \beta d_4$. From the first relation, we get

$$d_2^2 = \alpha^2 d_3^2 + \beta^2 d_4^2 + 2\alpha\beta w_3^\top w_4$$

and from the second one we get

$$d_2^2 = \alpha^2 d_3^2 + \beta^2 d_4^2 - 2\alpha\beta d_3 d_4 \,.$$

Comparing these two expressions for $d_2^2$, we get the following.

(i) If both $\alpha$ and $\beta$ are nonzero, then we must have $w_3^\top w_4 = -d_3 d_4$. From the Cauchy-Schwartz inequality, this implies that $w_3$ and $w_4$ are aligned with 0 and so $Q$ would be degenerate, which is a contradiction.

(ii) Now suppose just one of $\alpha$ or $\beta$ is 0. If $\alpha = 0$, then $w_2$ and $w_4$ would need to be aligned with the origin. If $\beta = 0$, then $w_2$ and $w_3$ would need to be aligned with the origin. Either way, $Q$ would be degenerate and we reach a contradiction.

$\square$

As a consequence of Lemma 3.3.9, we have the following result.

**Theorem 3.3.10.** *Let $x \in \Sigma$ and $w_i$, $i = 1, 2, 3, 4$, and $Q$ be defined as usual. Suppose that, at such $x$, $w_i = 0$ for an index $i$, and $w_j \neq 0$, $j \neq i$, and let $Q$ be non-degenerate.*

*Then, the moments matrix $M$ is invertible, and* (3.1.1) *has a unique admissible solution $\lambda_M$:*

$$(\lambda_M)_j = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases} \quad , \quad j = 1, 2, 3, 4.$$

*Moreover, as long as $Q$ remains non-degenerate, the solution $\lambda_M$ is continuous, but not differentiable, in $x \in \Sigma$.*

*Proof.* Without loss of generality, let $w_1 = 0$, so that (3.1.1) rewrites as:

$$\begin{bmatrix} 0 & w_2 & w_3 & w_4 \\ 1 & 1 & 1 & 1 \\ 0 & -d_2 & -d_3 & d_4 \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} .$$

Clearly, $\lambda_{\mathrm{M}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ solves this system. The solution is further unique since $N$ (defined as in Lemma 3.3.9) is invertible.

Continuity of $\lambda_{\mathrm{M}}$ is a consequence of continuity and invertibility of $M$ with respect to $x$. Lack of differentiability is due to lack of smoothness at the origin for the square root function (viz., for $\|\cdot\|$). $\square$

The above lack of smoothness is responsible for the difficulties one may have in locating an exit point where $w_1 = 0$, and hence to properly predict the dynamics past such an exit point.

**Example 3.3.11.** *Consider the dynamics on $\Sigma$ embodied by the points*

$$w_1 = \begin{bmatrix} -t \\ t \end{bmatrix} , \quad w_2 = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix} , \quad w_3 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} , \quad w_4 = \begin{bmatrix} -2 \\ -1 \end{bmatrix} , \quad \text{and } -1 \leq t \leq 0.5 . \tag{3.3.6}$$

*As long as $t < 0$, $\Sigma$ is attractive and we have well defined sliding motion on $\Sigma$. At $t = 0$, $w_1 = 0$ and for $t > 0$ the origin exits the quadrilateral $Q$: attractivity is violated, and a co-dimension 2 exit phenomenon from $\Sigma$ into $R_1$ should be taking place. However, suppose we continue following the trajectory on $\Sigma$ (this can be done because of Theorem 3.3.8). The components of the moments solution $\lambda_M$ behave as in Figure 20, and we observe that two of them (here, $\lambda_3$ and $\lambda_4$) change of sign through this non-generic exit point. A naive application of Theorem 3.4.7 below may lead us to believe that exiting and sliding on $\Sigma_2^-$ with $f_{\Sigma_2^-}$ should be taking place past the exit point, rather than exiting onto $R_1$.*

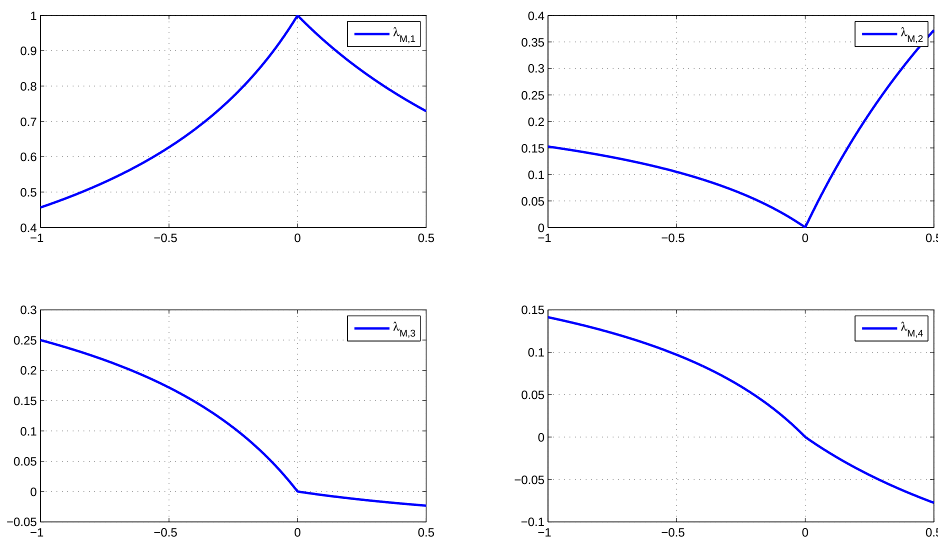


Figure 20: Solution components of $\lambda_{\mathrm{M}}$ for the dynamics given by (3.3.6).

$\square$

## *3.4 Smooth exits for the moments method and extensions*

In this section, we first show that –at generic first order exit points on $\Sigma$– the moments solution renders (automatically) the coefficients for the exit vector field. Then, we briefly discuss other possibilities to regularize the underdetermined system (1.4.9), by appending to it a linear constraint, similarly to what we did in (3.1.1), and ascertain when/how this will render an admissible solution $\lambda$.

### 3.4.1 Smooth exits

As shown in Figure 25 relative to the Example 5.1.1, when the moments trajectory reached a generic first order exit point, two components of the moments solution (i.e., of the vector $\lambda_{\mathrm{M}}$ ) became zero, and the other two gave the coefficients of cthe exit vector field. In fact, more was observed to be true. Since the matrix $M$ remained invertible (see Theorem 3.3.8), the solution of (6.1.10) could be continued past the exit point, and a trajectory sliding on $\Sigma$ according to $f_{\mathrm{M}}$ continued to exist; however, the moments solution was no longer admissible, since the two components that had become 0 at the exit point eventually became negative. This is a general behavior, that here we are going to justify rigorously. It is also a very important and useful fact, because it allows us to **detect** that an exit point is reached, and thus to eventually leave $\Sigma$ smoothly at the exit point.

First, we have the following simple result.

**Lemma 3.4.1.** *Let $T = ABC$ be a planar triangle of vertices $A$, $B$, and $C$, joined in this order. Then,*

$$sgn\,\mathcal{A}(AB\hat{C}) = -sgn\,\mathcal{A}(ABC)\ ,$$

*where $\hat{C}$ is the reflection of $C$ with respect to the origin, and $\mathcal{A}$ indicates the signed area.*

*Proof.* The result follows from the fact that if $ABC$ proceeds clockwise, then $AB\hat{C}$ has counterclockwise ordering, and vice versa. □

Next, we need the following concept.

**Definition 3.4.2** (Origin exiting along an edge)**.** *Let $x(t)$, $0 \leq t \leq T$, be the smooth trajectory on $\Sigma$ associated to the moments vector field, where the time interval is a* *time interval for which the trajectory is well defined (i.e., the associated matrix $M$ in* (3.1.1) *is invertible). Assume that there is a neighborhood of the trajectory, $U(x)$, such that $\Sigma \cap U(x)$ is attractive for values of $t$ in some interval $0 \le t \le t_0$, $0 < t_0 \le T$. Let $Q(x(\cdot))$ be the quadrilateral associated to this trajectory, and let $Q$ be non-degenerate, and such that that none of the vertices of $Q$ be at the origin.*

*Then, we say that* the origin is exiting $Q$ along *the edge $w_1w_2$ if and only if, by definition, the following occur:*

*(i) there exists a time $t_e > 0$ such that $x(t_e) \in \Sigma$, $\mathcal{A}_{120}(x(t_e)) = 0$, and for all $t$ : $0 \le t < t_e$, $\mathcal{A}_{120}(x(t)) \neq 0$, $\mathcal{A}_{240}(x(t)) \neq 0$, $\mathcal{A}_{430}(x(t)) \neq 0$, and $\mathcal{A}_{310}(x(t)) \neq 0$. Here, $\mathcal{A}_{120}$ is the signed area of the triangle with vertices $w_1$, $w_2$ and the origin, and similarly for $\mathcal{A}_{240}$ and so forth;*

*(ii) there exists an open interval $\mathcal{I}_e$ centered at $t_e$ and contained in $[0, T]$, such that for all $t_1, t_2 \in \mathcal{I}_e$, with $t_1 < t_e < t_2$, then the following inequality holds:*

$$\mathcal{A}_{120}(x(t_1)) < 0 < \mathcal{A}_{120}(x(t_2)) \ ; \tag{3.4.1}$$

*(iii) for all $t \in \mathcal{I}_e$, $\mathcal{A}_{240}(x(t)) \neq 0$, $\mathcal{A}_{430}(x(t)) \neq 0$, and $\mathcal{A}_{310}(x(t)) \neq 0$.*

*Analogous definitions hold for the origin exiting along the other edges of the quadrilateral $Q$, that is along $w_2w_4$, $w_4w_3$, $w_3w_1$. The value of $t_e$ above is called* (first) exit time *for the moments trajectory.* □

**Remark 3.4.3.** *The above definition characterizes the situation when –following the moments solution trajectory on $\Sigma$– the origin ends up outside the quadrilateral $Q$ after having encountered a first order exit point. In this case, since at $t_e$ we have $w_i(t_e) \neq 0$, $i = 1, 2, 3, 4$, then it is meaningful to determine along which edge of $Q$ the origin exited. See Lemmata 1.4.3 and 1.4.9 for motivation on the inequality* (3.4.1).

In the Lemma below, we will use normalized barycentric coordinates of the origin with respect to a triangle. Let us recall these.

**Notation 3.4.4.** *For a given planar triangle $T_{ABC}$ of distinct vertices $A \equiv (x_A, y_A)$, $B \equiv (x_B, y_B)$, $C \equiv (x_C, y_C)$, the normalized barycentric coordinates of the origin are given by the triplet $(\tau_A, \tau_B, \tau_C)$ satisfying the system*

$$\begin{cases} \tau_A \begin{bmatrix} x_A \\ y_A \end{bmatrix} + \tau_B \begin{bmatrix} x_B \\ y_B \end{bmatrix} + \tau_c \begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \tau_A + \tau_B + \tau_C = 1 \ . \end{cases} \tag{3.4.2}$$

*In particular, all coordinates are in $[0,1]$ whenever $0 \in \overline{T_{ABC}}$, and if any of them is negative then $0$ is external to the triangle. Finally, if we need to specify the coordinates of a vertex with respect to the specific triangle $T_{ABC}$, we will write $(\tau_A^{ABC}, \tau_B^{ABC}, \tau_C^{ABC})$.*

□

**Lemma 3.4.5.** *With the notation of Definition 3.4.2, suppose that the origin exited $Q$ along $w_1 w_2$. Let $t \in \mathcal{I}_e$, $t > t_e$, so that $0 \notin \overline{Q(x(t))}$. For any such $t$, let $w_i$, $i = 1, 2, 3, 4$, be the vertices of $Q$, and let $T_{ijk}$ be the triangles of vertices $w_i$, $w_j$, $w_k$ (in this order), for different indices $i, j, k \in 1, 2, 3, 4$.*

*Then:*

$$0 \in \overline{T}_{123^\dagger} \ , \quad \text{or} \quad 0 \in \overline{T}_{124^\dagger} \ ,$$

*where $w_3^\dagger$ and $w_4^\dagger$ are, respectively, the reflections of $w_4$ and $w_3$ with respect to the origin.*

*Proof.* For simplicity, below we will omit writing the dependence on the point $x(t)$, and simply write $Q$ for $Q(x(t))$, and so forth.

Since $Q$ is not degenerate, and $0 \notin \overline{Q}$, then $0 \notin \overline{T}_{123}$ or $0 \notin \overline{T}_{124}$ (both could be true, of course). Suppose that $0 \notin \overline{T}_{123}$.

Consider the triangle $T_{124^\dagger}$ of vertices $w_1, w_2, w_4^\dagger$, and look at the normalized barycentric coordinates of the origin with respect to $T_{124^\dagger}$. Note that $T_{124^\dagger}$ cannot be degenerate. (In fact, assume it was: then $w_4^\dagger \in w_1 w_2$, and hence the entire segment with extrema $w_4^\dagger$ and its transformed with respect to the origin, that is $w_3$, would be contained in $T_{123}$. In particular, this would imply that $0 \in T_{123}$, which is a contradiction.)

Therefore, from (3.4.2), using Cramer's rule and Lemma 3.4.1, we get

$$\tau_{4^\dagger}^{124^\dagger} = \frac{\det \begin{bmatrix} w_1 & w_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}{\mathcal{A}(w_1 w_2 w_4^\dagger)} = \frac{\det \begin{bmatrix} w_1 & w_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}{|\mathcal{A}(w_1 w_2 w_4^\dagger)| \operatorname{sgn} \mathcal{A}(w_1 w_2 w_4^\dagger)} =$$

$$= -\frac{|\mathcal{A}(w_1 w_2 w_3)|}{|\mathcal{A}(w_1 w_2 w_4^\dagger)|} \frac{\det \begin{bmatrix} w_1 & w_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}{|\mathcal{A}(w_1 w_2 w_3)| \operatorname{sgn} \mathcal{A}(w_1 w_2 w_3)} = -\frac{|\mathcal{A}(w_1 w_2 w_3)|}{|\mathcal{A}(w_1 w_2 w_4^\dagger)|} \tau_3^{123} > 0 \ ,$$

since $\tau_3^{123} < 0$, being $0 \notin \overline{T}_{123}$. Similarly for the other possibilities. $\square$

**Corollary 3.4.6.** *With same notation as in Lemma 3.4.5, let $0 \notin \overline{Q}$ and assume the origin exited along $w_1 w_2$. Then, the origin is in the interior of $\hat{Q}$, where $\hat{Q}$ has vertices $w_1$, $w_2$, $w_4^\dagger$, $w_3^\dagger$, and $w_3^\dagger$, $w_4^\dagger$ are, respectively, the reflections of $w_4$, $w_3$ with respect to the origin.*

*Proof.* This is a direct consequence of Lemma 3.4.5, and the fact that the origin cannot be on the edge $w_1 w_2$. $\square$

We are now ready for the anticipated result, stating that two components of $\lambda_{\mathrm{M}}$ change sign as the moments' trajectory continues on $\Sigma$ past an exit point (cfr. Theorem 3.3.8).

**Theorem 3.4.7.** *With the notation of Definition 3.4.2, suppose that the origin exited $Q$ along $w_1 w_2$, relatively to a moments solution trajectory $x(\cdot)$.*

*Let $t \in \mathcal{I}_e$, $t > t_e$, and sufficiently close to $t_e$, so that $0 \notin \overline{Q(x(t))}$. Then, the 3rd and 4th components of $\lambda_M$ are negative at such $t$: $\lambda_{M,3} < 0$ and $\lambda_{M,4} < 0$.*

*Proof.* For ease of notation, we omit writing the explicit dependence of $t$, but all quantities below must be understood to be relative to the value $x(t)$ of the trajectory.

We prove the result by contradiction. In particular, we assume that $\lambda_{\mathrm{M},3} < 0$ and $\lambda_{\mathrm{M},4} \geq 0$; the other two cases are dealt with analogously (i.e., $\lambda_{\mathrm{M},3} \geq 0$ and $\lambda_{\mathrm{M},4} < 0$, or $\lambda_{\mathrm{M},3} \geq 0$ and $\lambda_{\mathrm{M},4} \geq 0$).

As usual, below $M$ is the matrix of the moments' method: $M = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ 1 & 1 & 1 & 1 \\ d_1 & -d_2 & -d_3 & d_4 \end{bmatrix}$, which under the stated assumptions is invertible. Therefore, there is a unique solution $\lambda_{\mathrm{M}} = \begin{bmatrix} \lambda_{\mathrm{M},1} \\ \lambda_{\mathrm{M},2} \\ \lambda_{\mathrm{M},3} \\ \lambda_{\mathrm{M},4} \end{bmatrix}$ to (3.1.1), for which, in particular, $\lambda_{\mathrm{M},3} < 0$ and $\lambda_{\mathrm{M},4} \geq 0$. Next, consider the matrix

$$\hat{M} := \begin{bmatrix} w_1 & w_2 & -w_4 & -w_3 \\ 1 & 1 & 1 & 1 \\ d_1 & -d_2 & -d_4 & d_3 \end{bmatrix},$$

and let $\hat{Q}$ be the quadrilateral associated to $w_1, w_2, -w_3, -w_4$ (taken in this order). By Corollary 3.4.6, the origin is in the interior of $\hat{Q}$, and so (by Theorem 3.3.4) there exists a unique admissible moments solution $\hat{\lambda}$ such that $\hat{M}\hat{\lambda} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, which by Corollary 3.3.5 has all components strictly positive.

Now, set $\tilde{\lambda} := \begin{bmatrix} \lambda_{\mathrm{M},1} \\ \lambda_{\mathrm{M},2} \\ -\lambda_{\mathrm{M},4} \\ -\lambda_{\mathrm{M},3} \end{bmatrix}$, and note that

$$\begin{bmatrix} w_1 & w_2 & -w_4 & -w_3 \\ d_1 & -d_2 & -d_4 & d_3 \end{bmatrix} \tilde{\lambda} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the origin is exiting along $w_1 w_2$, and at $t_e$ we have $\lambda_{\mathrm{M},3} = \lambda_{\mathrm{M},4} = 0$, by continuity of $\lambda_{\mathrm{M}}$, possibly restricting the interval $\mathcal{I}_e$, we can assume that

$$\lambda_{\mathrm{M},3} + \lambda_{\mathrm{M},4} < \frac{1}{2} \,,$$

so that

$$\lambda_{\mathrm{M},1} + \lambda_{\mathrm{M},2} - \lambda_{\mathrm{M},3} - \lambda_{\mathrm{M},4} > 0.$$

Thus,

$$\tilde{\tilde{\lambda}} := \frac{1}{\lambda_{\mathrm{M},1} + \lambda_{\mathrm{M},2} - \lambda_{\mathrm{M},3} - \lambda_{\mathrm{M},4}} \tilde{\lambda}$$

is solution of

$$\hat{M}\tilde{\tilde{\lambda}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} .$$

But $\hat{M}$ is non-singular, and so we get $\tilde{\tilde{\lambda}} = \hat{\lambda}$, which contradicts the fact that $\hat{\lambda}$ is positive, whereas $\hat{\lambda}_3 = \tilde{\tilde{\lambda}}_3 = -\lambda_{\mathrm{M},4} \leq 0$. $\square$

### 3.4.2 Extensions

Here we consider other possible regularizations, besides that giving the moments method, of the system (1.4.9), still obtained enlarging the system (1.4.9) by appending to it a linear constraint (as we did in (3.1.1)). Namely, for $x \in \Sigma$, we consider the enlarged system

$$\begin{bmatrix} W \\ \mathbb{1}^\top \\ a^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \tag{3.4.3}$$

where $a$ is a smoothly varying function of $x \in \Sigma$, taking values in $\mathbb{R}^4$.

First, we have the following result, which restricts the search for possible functions $a$, in order to obtain an admissible solution $\lambda$ of (3.4.3).

**Theorem 3.4.8.** *Let the quadrilateral $Q$ be defined as usual, let it be non-degenerate, and assume that $0 \in Q$. Define*

$$\mathcal{A} := \{a \ : \ 0_3 \in \mathcal{T}_{W_a}\} \ ,$$

*where $\mathcal{T}_{W_a}$ is the tetrahedron with vertices the columns of $W_a$, and*

$$W_a := \begin{bmatrix} W \\ a^\top \end{bmatrix} \ .$$

*Let $\lambda$ be any solution of the underdetermined system* (1.4.9)*.*

*Then, $\lambda$ is admissible if and only if there exists $a \in \mathcal{A}$ such that $a^\top \lambda = 0$.*

*Proof.* Let $\lambda$ be any given solution of the underdetermined system (1.4.9).

If there is $a \in \mathcal{A}$ such that $a^\top \lambda = 0$, then $\lambda$ is a solution of

$$M_a \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \ , \quad \text{where} \quad M_a = \begin{bmatrix} W \\ \mathbb{1}^\top \\ a^\top \end{bmatrix} \ . \tag{3.4.4}$$

Looking at the third row of the adjugate of $M_a$, similarly to what we did in Section 3.3, we observe that its entries are the volumes of the tetrahedra that any three vertices of $\mathcal{T}_{W_a}$ form with the origin of $\mathbb{R}^3$. Since $0 \in \mathcal{T}_{W_a}$, these entries are all positive, hence $M_a$ is invertible, and there is a unique solution, call it $\lambda_a$, of (3.4.4), which is further admissible (nonnegative entries, and smoothly varying).

Next, suppose that $\lambda$ is an admissible solution of (1.4.9). Therefore, since $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ has rank 3, there exists a smoothly varying function $a$ such that $a^\top \lambda = 0$. Further, since $\lambda$ is admissible, from $\lambda \geq 0$, one has that $0 \in \mathcal{T}_{W_a}$, hence $a \in \mathcal{A}$. □

Below, call $\lambda_a$ the solution of (3.4.3). In Theorem 3.4.9, we consider $\lambda_a$ at generic first order exit points, and show that $\lambda_a$ has to be the moments solution $\lambda_{\mathrm{M}}$, if this $\lambda_a$ renders the exit vector field.

**Theorem 3.4.9.** *Let the quadrilateral $Q$ be defined as usual, and let it be non-degenerate. Let $v$ span* $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$*, and let $a$ in* (3.4.3) *be such that $a^\top v \neq 0$. Then, considering the unique solution $\lambda_a$ of* (3.4.3)*, there holds one of the following alternatives:*

1. *either $\lambda_a$ is not admissible; or*

2. *if $\lambda_a$ is admissible, and if $x_e$ is a generic first order exit point, then at $x_e$ either $\lambda_a = \lambda_M$, or $\lambda_a$ does not give the exit vector field, hence the trajectory associated to $\lambda_a$ cannot exit $\Sigma$ smoothly at $x_e$.*

*Proof.* For any given $x \in \Sigma$, since $a^\top v \neq 0$, by Lemma 3.3.1, (3.4.3) has a unique solution. Therefore, there exists a unique $c_a \in \mathbb{R}$ (of course, $c_a$ depends on $x$) such that

$$\lambda_a = \lambda_M + c_a v \ ,$$

where $\lambda_{\mathrm{M}}$ is the moments' solution associated to (3.1.1). Denote with $[a_M, b_M]$ the admissibility interval determined by $\lambda_M$ (note that $a_M < 0$ and $b_M > 0$):

$$a_M := \max\left\{-\frac{\lambda_{M,i}}{v_i} : v_i > 0\right\} \ , \quad b_M := \min\left\{-\frac{\lambda_{M,i}}{v_i} : v_i < 0\right\} \ .$$

Since $a^\top \lambda_a = 0$ and $d^\top \lambda_M = 0$, then $c_a$ is uniquely determined as

$$c_a = \frac{d^\top \lambda_a}{d^\top v} = -\frac{a^\top \lambda_M}{a^\top v}.$$

Therefore, if $c_a \notin [a_M, b_M]$, then $\lambda_a$ is not admissible.

If $c_a \in [a_M, b_M]$, and $\lambda_a$ is admissible, let $x_e$ be a generic first order exit point, and without loss of generality [4] let $f_{\mathrm{F}_2^-}$ be the associated exit vector field, that is

[4] of course, any other choice of exit vector field is handled similarly

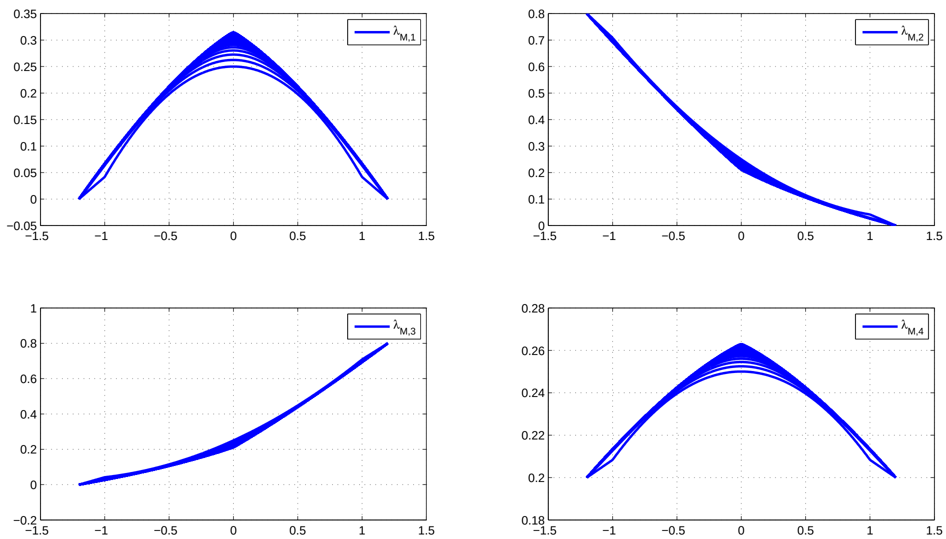


Figure 21: Solution components of $\lambda_{\mathrm{M}}$ for Example 3.4.11, using $\|\cdot\|_i$, $i = 2, \ldots, 100$.

$0 \in w_1 w_2$. Suppose by contradiction that $\lambda_a \neq \lambda_{\mathrm{M}}$ (at $x_e$), but that $\lambda_a$ leads to the exit vector field $f_{\mathrm{F}_2^-}$ at $x_e$. Then, $\lambda_{a,3} = \lambda_{a,4} = 0$, and, as we know, we also have $\lambda_{M,3} = \lambda_{M,4} = 0$. By Lemma 3.2.8, either $v_3 \neq 0$ or $v_4 \neq 0$, and therefore $c_a = 0$, giving $\lambda_a = \lambda_M$, which is a contradiction. $\square$

**Remark 3.4.10.** *Of course, Theorem 3.4.9 does not say that there are no other solutions as in* (3.4.3) *–beside the moments solution– which enjoy the property of rendering the exit vector field at a first order generic exit point. Indeed, we regularized* (1.4.9) *with a vector $d$ as in* (3.1.2)*, using the Euclidean distance from the origin of the vertices of $Q$ (i.e., the 2-norm), but we could have used different norms. We illustrate this in Example 3.4.11 below.*

**Example 3.4.11.** *With usual notation, consider $f_i$, $i = 1, 2, 3, 4$, below:*

$$f_1(x) := \begin{bmatrix} 2x_1 + 1 \\ -x_1 + x_2x_3 + 1 \\ x_1 + x_2 + 1 \end{bmatrix}, \ x \in R_1 \ , \quad f_2(x) := \begin{bmatrix} 2x_1 - 1 \\ -x_1 + x_3 - 1 \\ x_1 + x_2x_3 + 2 \end{bmatrix}, \ x \in R_2 \ ,$$

$$f_3(x) := \begin{bmatrix} 2x_1 - 3 \\ -x_1 + x_2 + 2 \\ x_1 + x_2x_3 - 1 \end{bmatrix}, \ x \in R_3 \ , \quad f_4(x) := \begin{bmatrix} 2x_1 + 2 \\ -x_1 + x_3 - 2 \\ x_1 + x_3 - 2 \end{bmatrix}, \ x \in R_4 \ ,$$

*where*

$$h_1(x) := x_3, \ h_2(x) := x_2 \ .$$

*Here $\Sigma$ is the $x_1$-axis, and the matrix $W$ for $x \in \Sigma$ is:*

$$W(x) = \begin{bmatrix} x_1 + 1 & x_1 + 2 & x_1 - 1 & x_1 - 2 \\ -x_1 + 1 & -x_1 - 1 & -x_1 + 2 & -x_1 - 2 \end{bmatrix} \ .$$

*There is attractive sliding motion (in the direction of increasing $x_1$) for $|x_1| \leq 1.2$. The value $x_1 = 1.2$ is a first order exit point, and one should exit $\Sigma$ at $x_1 = 1.2$, with exit vector field $f_{F_2}^{+}$.*

*As illustration, consider the following family of regularizations of* (1.4.9)*:*

$$\begin{bmatrix} W \\ \mathbb{1}^\top \\ a^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{where} \quad a = \begin{bmatrix} \|w_1\|_p \\ -\|w_2\|_p \\ -\|w_3\|_p \\ \|w_4\|_p \end{bmatrix} , \ p \geq 2 \ .$$

*In Figure 21, we show the plots of the solutions $\lambda$ of this system, relative to different choices of the p-norm, for $p = 2, \ldots, 100$. Clearly, the qualitative behavior of different solutions $\lambda$'s relative to different norms is quite similar.*

*In conclusion, although there are alternatives to using the 2-norm when forming the vector $d$ in* (3.1.2)*, for the class of regularized system of the type* (3.4.3) *it seems* *natural to simply use* $d$ *as we did in* (3.1.2)*, using* $\|\cdot\|_2$*, and compute* $\lambda_M$*. This choice allowed us to retain the geometrical flavor of "moments" for the entries of* $\lambda_M$*.* $\square$

# Chapter IV

# MINIMUM VARIATION SOLUTIONS FOR SLIDING VECTOR FIELDS ON THE INTERSECTION OF TWO SURFACES IN $\mathbb{R}^3$

In Chapter 2, we have started the study of minimum variation techniques to select a smooth varying admissible solution of (1.4.9) (see also [1, 15]). As highlighted in Remark 2.1.6, though, this approach is challenging and requires a careful analysis. Chapter 4 is devoted to this task, suggesting novel techniques susceptible for generalization to approach the problem in a more general setting, being it largely unexplored.

Here, we restrict ourselves to model scenarios in $\mathbb{R}^3$ in order to understand how to properly define a smooth *minimum variation sliding vector field* in the case of sliding on a co-dimension 2 discontinuity manifold $\Sigma$, intersection of two co-dimension 1 discontinuity surfaces. Whereas our model problems are sufficiently simple to allow explicit computations, the process we propose is rather general. All the results and theorems presented here will be clarified, through several examples, in Chapter 5.

Our idea is to select a smooth Filippov sliding vector field as solution of a minimum variation problem. As far as we know, in this context, this idea is new. At the same time, minimum variation techniques have proven quite powerful in Mathematics and Engineering studies, notably in Optimal Control applications (see [33, 50]), and in studying stick-slip motion phenomena for solid/solid interactions (see [5, 6]).

We will be interested in the situation in which $\Sigma$ is an arc which attracts the dynamics of the given piecewise smooth system, with endpoints corresponding to isolated values where $\Sigma$ ceases to be attractive (generic first order exit points). This way, we will be able to set up the boundary value problem corresponding to the minimality conditions of a minimum variation solution (Euler-Lagrange equation).

## *4.1 An example: Minimum variation solutions*

Here we consider the model problem in $\mathbb{R}^3$ introduced in Example 1.4.12, and give details of the construction of a *minimum variation* Filippov solution for it. Later, we will consider a different model, and give a new interpretation of other admissible Filippov solutions also as minimum variation solutions, but with respect to a different minimization task and ultimately with respect to a different parametrization of time. For convenience, let us recall here the setup of Example 1.4.12.

**Example 4.1.1.** *We have $f_i$, $i = 1, 2, 3, 4$, taking values in $\mathbb{R}^3$:*

$$f_1(x) := \begin{bmatrix} 2x_1 + 1 \\ -x_1 + x_2x_3 + 1 \\ x_1 + x_2 + 1 \end{bmatrix}, \ x \in R_1 \ , \quad f_2(x) := \begin{bmatrix} 2x_1 - 1 \\ -x_1 + x_3 - 1 \\ x_1 + x_2x_3 + 2 \end{bmatrix}, \ x \in R_2 \ ,$$

$$f_3(x) := \begin{bmatrix} 2x_1 - 3 \\ -x_1 + x_2 + 2 \\ x_1 + x_2x_3 - 1 \end{bmatrix}, \ x \in R_3 \ , \quad f_4(x) := \begin{bmatrix} 2x_1 + 2 \\ -x_1 + x_3 - 2 \\ x_1 + x_3 - 2 \end{bmatrix}, \ x \in R_4 \ ,$$

*where $\Sigma_1 = \{x : \ x_3 = 0\}$, $\Sigma_2 = \{x : \ x_2 = 0\}$, so that $\Sigma = \Sigma_1 \cap \Sigma_2$ is the $x_1$-axis. Here, the matrix $W$ of (1.4.5) is*

$$W(x) = \begin{bmatrix} x_1 + 1 & x_1 + 2 & x_1 - 1 & x_1 - 2 \\ -x_1 + 1 & -x_1 - 1 & -x_1 + 2 & -x_1 - 2 \end{bmatrix}, \tag{4.1.1}$$

*and it is simple to verify that $\Sigma$ is attractive in the segment $|x_1| < 1.2$ and the values $x_1 = \pm 1.2$ are generic first order exit points, at which point $\Sigma$ is no longer attractive. Since $W(-1.2) = \begin{bmatrix} -0.2 & 0.8 & -2.2 & -3.2 \\ 2.2 & 0.2 & 3.2 & -0.8 \end{bmatrix}$ then one should exit $\Sigma$ at $x = -1.2$ by* *sliding on $\Sigma_1^+$; similarly, since* $W(1.2) = \begin{bmatrix} 2.2 & 3.2 & 0.2 & -0.8 \\ -0.2 & -2.2 & 0.8 & -3.2 \end{bmatrix}$ *then one should exit $\Sigma$ at $x = 1.2$ by sliding on $\Sigma_2^+$.*

*The general form of the solution $\lambda$ to* $W\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *can be written as*

$$\lambda = \mu + cv \text{ or, explicitly: } \lambda = \begin{bmatrix} \frac{2}{3} - \frac{5}{9}x_1 \\ 0 \\ \frac{2}{3}x_1 \\ \frac{1}{3} - \frac{1}{9}x_1 \end{bmatrix} + c \begin{bmatrix} -\frac{5}{3} \\ 1 \\ 1 \\ -\frac{1}{3} \end{bmatrix}, \qquad (4.1.2)$$

*which is admissible for $(x_1, c)$ in the triangular region in Figure 22.*

Note that, in particular, we must have $c(-1.2) = 0.8$ and $c(1.2) = 0$. For any admissible $\lambda$, we will get a Filippov sliding vector field of the form:

$$f_{\mathrm{F}} = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 , \quad \text{or}$$

$$f_{\mathrm{F}} = \begin{bmatrix} \lambda_1(2x_1+1) + \lambda_2(2x_1-1) + \lambda_3(2x_1-3) + \lambda_4(2x_1+2) \\ 0 \\ 0 \end{bmatrix}. \qquad (4.1.3)$$

Hence, on $\Sigma$, the differential equation to solve is simply:

$$\dot{x}_1 = \frac{4}{3} - \frac{7}{9}x_1 - \frac{19}{3}c , \qquad (4.1.4)$$

and we observe that there is an equilibrium on $\Sigma$ at the value $x_1$ for which $c(x_1) = \frac{4}{19} - \frac{7}{57}x_1$. Given the admissibility region of Figure 22, any smooth selection of $c$ will give an equilibrium, which will be unstable. Different ways to select $c$, in general will give a different location for the equilibrium.

Both the moments and bilinear solutions of (3.1.1), (1.4.10), are well defined for this problem, exit smoothly at $x = \pm 1.2$, and select (similar) $c$-curves; see Figure

22 below. For this problem, there is also another obvious solution, the so-called *triangular solution*, namely the solution obtained choosing for $c$ the straight line segment $c_{\text{tr}}(x_1) = \frac{8}{20} - \frac{x_1}{3}$, $-1.2 \leq x_1 \leq 1.2$, joining the boundary values, that is the longest side of the triangle in Figure 22.

Next, we consider new types of solutions, still on Example 4.1.1, obtained via a variational formulation.

### 4.1.1 Minimum variation solutions for model problem

Recall that we want to have $c$ (hence $\lambda$) smooth functions of $x_1$. Further, recall that we have a family of solutions, depending on how we select an admissible function $c$. The choice of an admissible $c$ impacts the choice of the coefficients $\lambda_i$'s, and clearly the resulting sliding vector field in (4.1.4).

So, a natural idea is to seek an admissible function $c$ that, for $-1.2 \leq x_1 \leq 1.2$, minimizes the $H^1$-norm of either $\lambda$ or of the sliding vector field itself.

**Remark 4.1.2.** *A version of Weierstrass' Theorem (e.g., see [38]) states that, if $A \subset \mathbb{R}^n$ is closed and $f : A \longrightarrow \mathbb{R}$ is continuous and coercive, then $f$ has a minimum in $A$. This justifies all the minimization problems we examine below. In particular, the well posedness of Problems* (4.1.5) *and* (4.1.8) *below, as well as* (4.2.5) *and* (4.2.9) *in Section 4.2. This is because all of these problems amount to minimization of the functional given by $\|\cdot\|_{H^1}$ over the compact set of $\lambda \in \mathbb{R}^4$ with nonnegative components adding to* 1.

#### *4.1.1.1 Minimum variation for $\lambda$*

Accounting for the fact that we want the solution to be defined from $x_1 = -1.2$ to $x_1 = 1.2$, we seek the value of the function $c$ such that the following functional is minimized:

$$\min_{\lambda \in \mathscr{C}^1, \lambda \geq 0} \int_a^b \left[\|\lambda(x_1)\|^2 + \|\lambda'(x_1)\|^2\right] \mathrm{d}x_1 \,, \quad a = -1.2 \,,\, b = 1.2 \,. \tag{4.1.5}$$

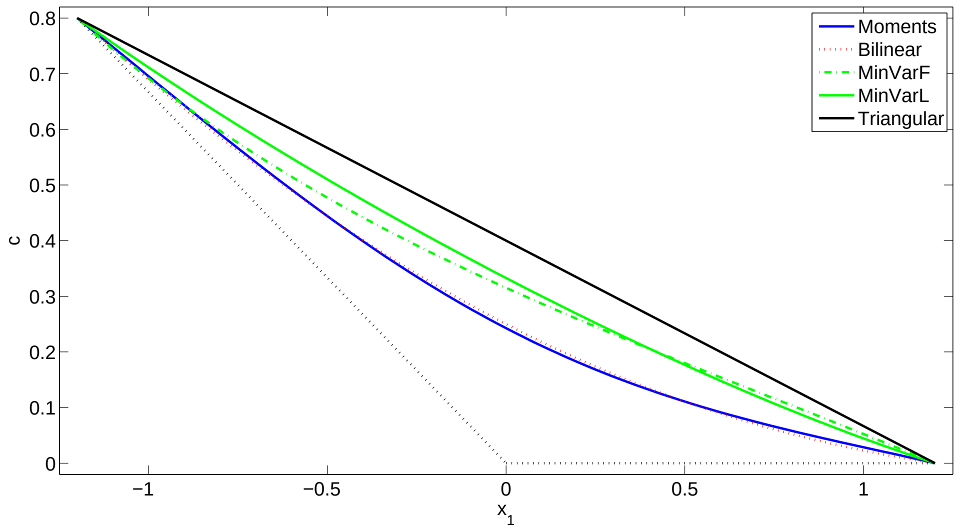


Figure 22: Admissible region $(x_1, c)$ in (4.1.2), and moments, bilinear, triangular, and minimum variations values of $c$.

With the Lagrangian given by the integrand, next we write down the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial c} - \frac{\mathrm{d}}{\mathrm{d}x_1}\frac{\partial \mathcal{L}}{\partial c'} \;=\; 0 \;.$$

With a little algebra, and using the exit conditions, this gives the boundary value problem for $c$:

$$c'' - c \;=\; x_1/3 - 1/4 \;, \quad c(-1.2) = 0.8 \;, \;\; c(1.2) = 0 \;, \tag{4.1.6}$$

which has the solution

$$c \equiv c_{\mathrm{MV},\lambda}(x_1) \;=\; \frac{0.15}{e^{1.2} + e^{-1.2}}\left(e^{x_1} + e^{-x_1}\right) - \frac{x_1}{3} + \frac{1}{4} \;. \tag{4.1.7}$$

With this value of $c_{\mathrm{MV},\lambda}$, we obtain what we call *minimum variation* solution with respect to $\lambda$. See Figure 22 for a plot of $c_{\mathrm{MV},\lambda}$.

#### *4.1.1.2 Minimum variation for $f_F$*

Now we consider the general form of the smooth sliding vector field $f_{\mathrm{F}}$ and seek the function $c$ in order to minimize the $H^1$ norm of $f_{\mathrm{F}}$, still considering the model problem of Example 4.1.1.

In other words, we seek the (smooth) function $c$ such that the following functional is minimized among smooth admissible functions $c$:

$$\min_{c} \int_a^b \left[ \|f_{\mathrm{F}}(x_1)\|^2 + \|{f_{\mathrm{F}}}'(x_1)\|^2 \right] \mathrm{d}x_1\,, \quad a=-1.2\ ,\, b=1.2\,. \tag{4.1.8}$$

Given the simple expression (4.1.4), this reduces to minimizing

$$\int_a^b \mathcal{L}(x_1,c,c')\mathrm{d}x_1\ ,\ \ \mathcal{L} = \left(\frac{4}{3}-\frac{7}{9}x_1-\frac{19}{3}c\right)^2+\left(\frac{7}{9}+\frac{19}{3}c'\right)^2\ .$$

The Euler-Lagrange equation gives the following boundary value problem for $c$:

$$c''-c \ =\ \frac{7}{57}x_1-\frac{12}{57}\ ,\quad c(-1.2)=0.8\ ,\ \ c(1.2)=0\ , \tag{4.1.9}$$

which has the solution

$$\begin{aligned} c\equiv c_{\mathrm{MV},f_{\mathrm{F}}}(x_1) \ &=\ A_1e^{x_1}+B_1e^{-x_1}+\frac{12-7x_1}{57}\ , \\ \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \ &=\ \frac{6}{95(e^{-2.4}-e^{2.4})}\begin{bmatrix} 7e^{-1.2}+e^{1.2} \\ -e^{-1.2}-7e^{1.2} \end{bmatrix}\ . \end{aligned} \tag{4.1.10}$$

With this value of $c_{\mathrm{MV},f_{\mathrm{F}}}$, we obtain what we call *minimum variation* solution with respect to the $H^1$-variation of $f_{\mathrm{F}}$. See Figure 22 for a plot of $c_{\mathrm{MV},f_{\mathrm{F}}}$.

**Remark 4.1.3.** *It is a simple computation to verify that the minimum variation solutions we obtained, both with respect to $\lambda$ and with respect to the vector field $f_F$, in the end give parameters values $\lambda$, in an independent way of how we chose $\mu$ and $\upsilon$ in* (4.1.2)*.*

**Questions 4.1.4.** *The above example suggests several questions, which we will address in the next section.*

*(i) In Example 4.1.1, in spite of the different expressions for the functions $c$ we obtained, in the end all sliding vector fields have a similar behavior: there is an equilibrium on $\Sigma$, and –depending on where one enters $\Sigma$– motion goes to the right/left until an exit point is reached. Different choices of admissible functions $c$ determine the position of the equilibrium. See Figure 23.*

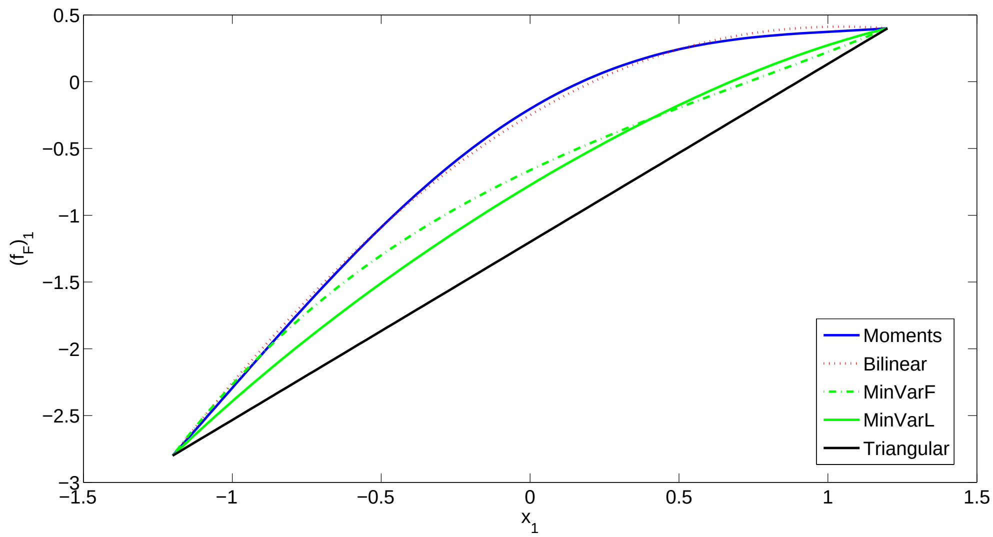


Figure 23: Sliding vector fields for moments, bilinear, triangular, and minimum variations solutions. All have an equilibrium.

*(ii) Below, we will consider a similar model, for which no smooth Filippov vector field has an equilibrium on $\Sigma$. In this case, according to the results in [20], we know that all possible smooth Filippov sliding motions are orbitally equivalent. Are there functionals, related to the change of time variable in the aforementioned orbital equivalency, whose minimizers give –say– the moments, or the bilinear solutions?*

*(iii) Finally, how can one extend our construction to a broader class of problems?*

## *4.2* **Orbital equivalence and weighted minimum variation**

In this section, we consider another pattern of sliding motion, which has the key features outlined below.

**Conditions 4.2.1.**

*(a) The state space is $\mathbb{R}^3$.*

*(b) The sliding surface $\Sigma$ is a smooth arc: $\Sigma = \{x \in \mathbb{R}^3 : \ x = \gamma(s) \ , \ a \le s \le b\}$.*

*(c) For $a < s < b$, $\Sigma$ is attractive, there are no equilibria on $\Sigma$ for any smooth Filippov sliding vector field, and motion on $\Sigma$ proceeds from $x_a := \gamma(a)$ to $x_b := \gamma(b)$.*

*(d) The point $x_b$ is a generic first order exit point, and the point $x_a$ is a generic first order exit point for the time reversed problem.*

When Conditions 4.2.1 hold (in particular $\Sigma$ is attractive), the function $W$ (which depends solely on the parameter $s$), is of full rank. Therefore, the general form of an admissible solution $\lambda$ in (1.4.8), can be written as (see Section 1.4.2)

$$\lambda(s) = \mu(s) + c(s)v(s)\,, \quad a \leq s \leq b\,, \tag{4.2.1}$$

where $\mu$ is any given (smooth) particular solution, $v$ is a given (smooth) vector spanning $\ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$, and the function $c$ is subject to restrictions as in (1.4.13).

**Note**. We will want to select an admissible function $c(s)$, $a \leq s \leq b$, so that the resulting $\lambda(s)$ in (4.2.1) at the endpoints $s = a$ and $s = b$ gives the respective "exiting" vector fields. We know that this is possible, since it is achieved, for example, by the moments method. Indeed, as proved elsewhere (see [18] and [19]), both moments and bilinear solutions give well defined Filippov sliding vector fields, the moments vector field further being guaranteed to give coefficients that render the exit vector field at first order exit points. Below, we show how to formally define a minimum variation solution in this general case.

Now, in light of the results in [20], for a problem with the above characteristics, all smooth sliding vector fields on $\Sigma$ are orbitally equivalent. That is, if we have two different smooth sliding vector fields, say $f_{\mathrm{F}1}$ and $f_{\mathrm{F}2}$, then the solutions associated to these vector fields are tracing the same orbit, but at different speeds. In other words, we must have

$$f_{\mathrm{F}1} \;=\; \omega(x) f_{\mathrm{F}2}\,, \;\; \omega \in \mathscr{C}^1\,, \;\; \omega > 0\,, \tag{4.2.2}$$

and therefore

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_{\mathrm{F}1} \iff \frac{\mathrm{d}x}{\mathrm{d}\tau} = f_{\mathrm{F}2} \text{ and } \omega(x) = \frac{\mathrm{d}t}{\mathrm{d}\tau} \,.$$

This being the case, and the system being autonomous, it means that we can interpret the two distinct vector fields above as follows:

$$\begin{aligned} \text{If } f_{\mathrm{F}1} &= \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \ , \\ \text{then } f_{\mathrm{F}2} &= \lambda_1(\omega f_1) + \lambda_2(\omega f_2) + \lambda_3(\omega f_3) + \lambda_4(\omega f_4) \ , \end{aligned} \tag{4.2.3}$$

which means that "*Any sliding vector field can be interpreted as having modified* **all** *vector fields* $f_i$*,* $i = 1, 2, 3, 4$*, through the reparametrization of time*".

Observe that –under Conditions 4.2.1– we can assume that $\omega$ is parametrized by $s$. Therefore, for all orbitally equivalent smooth vector fields, further smoothly aligning at the exit points with the exit vector fields, we must have $\omega|_{s=a} = \omega|_{s=b} = 1$.

### 4.2.1 Weighted Minimum Variation

Motivated by the above, we are thus lead to consider a generalization of the approach in Section 4.1.1.2, and seek minimization of functionals more general than those in Section 4.1.1.2. Namely, we will seek the function $c$ so that in the end we will minimize either

(i) the $H^1$-variation of $w\lambda$, or

(ii) the $H^1$-variation of the sliding vector field $wf_{\mathrm{F}}$.

Above, the function $w$ –which we will call *weight function*– is required to satisfy these properties:

$$\begin{aligned} &\text{(i)} \quad w \ \text{ is smooth (at least } \mathscr{C}^2) \ \ \forall s \in (a, b) \\ &\text{(ii)} \quad w > 0 \ \forall s \in [a, b] \ , \ \text{ and } \ w|_{s=a} = w|_{s=b} = 1 \ . \end{aligned} \tag{4.2.4}$$

Each of the above $H^1$-minimization tasks has its merits, though minimization of $\|wf_{\mathrm{F}}\|_{H^1}$ is more in tune with the previously mentioned reparametrization of time.

**Remark 4.2.2.** *In all cases, the value of $c$ will be* **required** *to take the values $c(a) = c_a$, and $c(b) = c_b$, specified so that $\lambda(a)$ and $\lambda(b)$ give the exiting vector fields at $\gamma(a)$ and $\gamma(b)$. Therefore, we emphasize that, with the choices we made for the weight function $w$ and the values of $c(a)$ and $c(b)$, the solutions of our minimization problems (when solvable) will give smoothly exiting solutions.*

*4.2.1.1 Minimum variation for $\lambda$*

With the function $w$ as in (4.2.4), we seek $c$ such that

$$\min_c \int_a^b \left[\|w\lambda\|^2 + \|(w\lambda)'\|^2\right] \mathrm{d}s \ , \quad c(a) = c_a \ , \ c(b) = c_b \, . \tag{4.2.5}$$

Consider the Lagrangian associated to (4.2.5), that is

$$\mathcal{L}(s, c, c') = \|w\lambda\|^2 + \|(w\lambda)'\|^2 = w^2\|\lambda\|^2 + (w')^2\|\lambda\|^2 + w^2\|\lambda'\|^2 + 2ww'\lambda^\top\lambda' \ .$$

The Euler-Lagrange equation on this functional (with some algebra), gives the following boundary value problem to be solved for $c$ (note that $\|v\| \neq 0$)

$$\begin{cases} \big[c''w\|v\|^2 + 2c'\big(w'\|v\|^2 + w(v^\top v')\big) - \\ \qquad c\big((w - w'')\|v\|^2 - w(v^\top v'') - 2w'(v^\top v')\big)\big] \ = \\ \qquad (w - w'')(v^\top \mu) - wv^\top\mu'' - 2w'(v^\top\mu') \ , \\ \qquad c(a) = c_a \ , \ \ c(b) = c_b \ . \end{cases} \tag{4.2.6}$$

**Remark 4.2.3.** *In general, it is not clear how to obtain the exact solution of the boundary value problem* (4.2.6)*. However, there is an important special case where* (4.2.6) *can be solved exactly. This is when the null vector $v \in \ker \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ is constant. In fact, in this case* (4.2.6) *becomes*

$$\begin{aligned} &c''w\|v\|^2 + 2c'w'\|v\|^2 - c(w - w'')\|v\|^2 \ = \\ &\quad (w - w'')v^\top\mu - wv^\top\mu'' - 2w'v^\top\mu' \ , \qquad c(a) = c_a \ , \ \ c(b) = c_b \ . \end{aligned} \tag{4.2.7}$$

*The differential equation in* (4.2.7) *rewrites as*

$$y'' \;=\; y + g(s) \ , \quad \textit{where} \quad y = cw\|v\|^2 + wv^\top\mu \ , \ \ \textit{and} \ \ g(s) = 2v^\top\mu'(w' - w'') \ .$$

*For this, letting* $y_1(s) = e^s$ *and* $y_2(s) = e^{-s}$, *the solution can be written as*

$$y(s) = Ay_1(s) + By_2(s) + y_p(s) \ .$$

*The associated Wronskian is* $\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = -2$, *and using the variation of constants formula gives*

$$y_p(s) = \frac{1}{2}\left[e^s \int e^{-s}g(s)ds - e^{-s}\int e^s g(s)ds\right] \ ,$$

*from which one can obtain the solution of* (4.2.7)*:*

$$\begin{cases} c(s) \;=\; \dfrac{Ae^s + Be^{-s} + y_p(s) - w(s)v^\top(s)\mu(s)}{w(s)\|v(s)\|^2} \ , \ \ a \le s \le b \ , \\ A, B : c(a) = c_a \ , \ \ c(b) = c_b \ . \end{cases} \tag{4.2.8}$$

*Observe that since* $w(a) = w(b) = 1$, *the values of* $A$ *and* $B$ *in* (4.2.8) *are independent of the weight function* $w$.

#### *4.2.1.2 Minimum variation for* $f_F$

Now, with the weight function $w$ as above, we seek $c$ such that

$$\min_c \int_a^b \left[\|wf_{\mathrm{F}}\|^2 + \|(wf_{\mathrm{F}})'\|^2\right] \mathrm{d}s \ , \quad c(a) = c_a \ , \, c(b) = c_b \, . \tag{4.2.9}$$

Again, $c(a) = c_a$, and $c(b) = c_b$, must be assigned to make sure that $\lambda(a)$ and $\lambda(b)$ give the exiting vector fields at $\gamma(a)$ and $\gamma(b)$.

For a general sliding vector field $f_{\mathrm{F}}$, given the form of $\lambda$ (4.2.1), we will use the notation

$$f_{\mathrm{F}} = F_\mu + cF_v \ ,$$

where $F_\mu = \mu_1 f_1 + \mu_2 f_2 + \mu_3 f_3 + \mu_4 f_4$, and $F_v = v_1 f_1 + v_2 f_2 + v_3 f_3 + v_4 f_4$.

We will **assume** that $F_v \neq 0$, for all $s \in [a,b]$ (see Remark 4.2.8 below when this is violated).

The Lagrangian associated to (4.2.9) is

$$\mathcal{L}(s,c,c') = \|wf_{\mathrm{F}}\|^2 + \|(wf_{\mathrm{F}})'\|^2 = w^2\|f_{\mathrm{F}}\|^2 + (w')^2\|f_{\mathrm{F}}\|^2 + w^2\|f_{\mathrm{F}}{}'\|^2 + 2ww' f_{\mathrm{F}}{}^{\top} f_{\mathrm{F}}{}' \ .$$

The Euler-Lagrange equation on this functional (with some algebra), gives the following boundary value problem to be solved for $c$:

$$\begin{cases} \big[c''w\|F_v\|^2 + 2c'\big(w'\|F_v\|^2 + w(F_v^{\top}F_v')\big) - \\ \qquad c\big((w-w'')\|F_v\|^2 - w(F_v^{\top}F_v'') - 2w'(F_v^{\top}F_v')\big)\big] \ = \\ \qquad (w-w'')(F_v^{\top}F_\mu) - wF_v^{\top}F_\mu'' - 2w'(F_v^{\top}F_\mu') \ , \\ \qquad c(a) = c_a \ , \ \ c(b) = c_b \ . \end{cases} \tag{4.2.10}$$

**Remark 4.2.4.** *Again, in general, it is not clear how to obtain the exact solution of the boundary value problem* (4.2.10)*. However, there is an important special case when in fact it can be solved exactly, that is when the discontinuity surfaces $\Sigma_1$ and $\Sigma_2$ are given by coordinates' planes*[1]*.*

*So, without loss of generality, in this case we can take $\Sigma_1 = \{x: \ x_2 = 0\}$ and $\Sigma_2 = \{x: \ x_3 = 0\}$. Then, $\Sigma$ is (a segment on) the $x_1$-axis, and one has that both $F_v$ and $F_\mu$ have only the first components different from* 0*, on $\Sigma$:*

$$F_\mu = \begin{bmatrix} f_\mu \\ 0 \\ 0 \end{bmatrix} \ , \quad F_v = \begin{bmatrix} f_v \\ 0 \\ 0 \end{bmatrix} \ ,$$

*and we are requiring that $f_v \neq 0$ for all $x_1 \in [a,b]$.*

*Using this in* (4.2.10)*, and dividing by $f_v$, we get the boundary value problem*

[1] *In fact, through a simple change of variable, the same reasoning holds true whenever $\Sigma_{1,2}$ are planes. More complicated discontinuity surfaces would require a nonlinear change of variable.*

*(differentiation is with respect to* $x_1$*):*

$$\begin{aligned} c''wf_v + 2c'\big(w'f_v + wf_v'\big) - c\big((w-w'')f_v - wf_v'' - 2w'f_v'\big) \ &= \\ (w-w'')f_\mu - wf_\mu'' - 2w'f_\mu' \ , \qquad c(a) = c_a \ , \ c(b) = c_b \ . \end{aligned} \tag{4.2.11}$$

*The point is that now the differential equation in* (4.2.11) *rewrites as*

$$\big[(cwf_v) + (wf_\mu)\big]'' \ = \ (cwf_v) + (wf_\mu) \ ,$$

*from which we get the solution of* (4.2.11)*:*

$$\begin{cases} c(x_1) \ = \ \dfrac{Ae^{x_1} + Be^{-x_1} - w(x_1)f_\mu(x_1)}{w(x_1)f_v(x_1)} \ , \ a \le x_1 \le b \ , \\ A, B : c(a) = c_a \ , \ c(b) = c_b \ . \end{cases} \tag{4.2.12}$$

*Note that since* $w(a) = w(b) = 1$*, the values of* $A$ *and* $B$ *in* (4.2.12) *are independent of the weight function* $w$*. Also, note that, as long as the value of* $c$ *in* (4.2.12) *is admissible, and hence* $\lambda$ *as in* (4.2.1) *gives an admissible Filippov sliding vector field, then we must have*

$$Ae^{x_1} + Be^{-x_1} \neq 0, \ \textit{for all } x_1 \in [a,b] \ , \tag{4.2.13}$$

*as otherwise the resulting vector field would be* 0 *at some point, giving an equilibrium, which is excluded.*

Now, with respect to either of the above minimization tasks (that is, minimizing either the $H^1$ norm of $w\lambda$ or of $wf_{\mathrm{F}}$), the following questions are natural.

**Questions 4.2.5.**

*(i) Can we choose* $w$ *so that the solution of* (4.2.5)-(4.2.9) *gives us the bilinear and moments solutions? More generally, can we interpret a given admissible solution as the minimum variation solution of* (4.2.5)-(4.2.9) *for some* $w$*?*

*(ii) Can we relate to each other the weight* $w$ *and the reparametrization of time performed by* $\omega$*?*

As already remarked, in general, the boundary value problems (4.2.6) and (4.2.10) do not appear to be easy to solve exactly, and probably one would need to solve them numerically. However, in the important special cases of Remarks 4.2.3 and 4.2.4 they can be solved exactly. We will clarify in Example 5.2.1 how we use these exact solutions to derive minimum variation solutions, and answer the above questions on that concrete Example.

### 4.2.2 General result

As Example 5.2.1 will make clear, the process used there is fully general, and it can be leveraged, for example, anytime the situation of Remark 4.2.4 applies.

With the previous notation, we then state and prove following theorem.

**Theorem 4.2.6.** *Let Conditions 4.2.1 hold. Let* $\Sigma_1 = \{x : \ x_2 = 0\}$, $\Sigma_2 = \{x : \ x_3 = 0\}$. *In the notation of Conditions 4.2.1, let* $\Sigma$ *be the segment* $(a, b)$ *on the* $x_1$*-axis. Let the general solution for* $\lambda$ *be as in* (1.4.12), *with the particular solution* $\mu$ *and the vector* $v$ *smoothly varying in* $\Sigma$ *(for example,* $\mu$ *could be the moments solution* $\lambda_M$*), and let the smooth function* $c$ *in* (1.4.12) *be subject to the constraints* $\alpha(x_1) \leq c(x_1) \leq \beta(x_1)$, *for all* $a \leq x_1 \leq b$. *Let* $\hat{f}_F$ *be any smooth Filippov sliding vector field on* $\Sigma$, *obtained from smooth, admissible coefficients (for example, the moments' vector field* $f_M$*), in particular with a smooth admissible function* $\hat{c}$ *in* (1.4.12) *so that* $\hat{\lambda} = \mu + \hat{c}v$ *at the exit points render the coefficients of the smoothly exiting Filippov vector field.*

*Assume that* $f_v \neq 0$ *on* $\Sigma$, *and consider the boundary value problem* (4.2.11) *with solution* (4.2.12), *and with* $A$ *and* $B$ *as there. Assume that* (4.2.13) *holds.*

*(i) If* $\left(Ae^{x_1} + Be^{-x_1}\right)\left(\hat{f}_F(x_1)\right)_1 > 0$, *for all* $x_1 \in \Sigma$, *then the function*

$$\hat{w}(x_1) \ = \ \frac{Ae^{x_1} + Be^{-x_1}}{\left(\hat{f}_F(x_1)\right)_1} \ , \tag{4.2.14}$$

*is the weight function associated to* $\hat{f}_F$. *That is, this weight function* $\hat{w}$ *is such* *that the $H^1$ minimization problem for $wf_F$ gives the function $\hat{c}$ as solution of* (4.2.11).

*(ii) On the other hand, let $w$ be an arbitrary weight function as in* (4.2.4)*, and let $c$ be the smooth function in* (4.2.12)*. This will be admissible if and only if, for all $x_1 \in \Sigma$, we have*

$$\hat{c}(x_1) + \hat{w}(x_1)\frac{f_\mu(x_1)}{f_v(x_1)} - \beta(x_1) \le w(x_1)\frac{f_\mu(x_1)}{f_v(x_1)} \le \hat{c}(x_1) + \hat{w}(x_1)\frac{f_\mu(x_1)}{f_v(x_1)} - \alpha(x_1) \ , \tag{4.2.15}$$

*where $\hat{w}$ and $\hat{c}$ are an admissible weight and its associated solution in* (4.2.11)*. When $c$ is admissible, the resulting vector field is orbitally equivalent to that associated to $\hat{c}$, with orbital equivalence factor $1/w$.*

*(iii) If* (4.2.13) *is violated, that is $\left(Ae^{x_1} + Be^{-x_1}\right) = 0$ at some $x_1 \in \Sigma$, then there is no admissible sliding vector field obtained as solution of the Euler Lagrange equation, by minimization of the $H^1$ norm of $wf_F$, for any weight function $w$.*

*Proof.* Statement (i) holds by construction. Indeed, since $\left(\hat{f}_{\mathrm{F}}(x_1)\right)_1 = f_\mu + \hat{c}f_v$, we seek the function $\hat{w}$ for which (4.2.11) holds. That is, we want $\hat{w}$ such that

$$\hat{c}(x_1) \ = \ \frac{Ae^{x_1} + Be^{-x_1} - \hat{w}(x_1)f_\mu(x_1)}{\hat{w}(x_1)f_v(x_1)} \ ,$$

which gives (4.2.14). Note that, since $\hat{c}$ is admissible and the resulting $\hat{\lambda}$ at the exit points give the coefficients of the smoothly exiting vector fields, then we have $w(a) = w(b) = 1$ because of the way $A$ and $B$ were found.

To verify (4.2.15), we need to check whether or not the function $c$ one finds is admissible. Because of (4.2.12), we always have (for all $x_1 \in \Sigma$):

$$\begin{aligned} \left(c(x_1)f_v(x_1) + f_\mu(x_1)\right) w(x_1) \ &= \ Ae^{x_1} + Be^{-x_1} \ , \text{ and} \\ \left(\hat{c}(x_1)f_v(x_1) + f_\mu(x_1)\right) \hat{w}(x_1) \ &= \ Ae^{x_1} + Be^{-x_1} \ . \end{aligned}$$

from which we get

$$c(x_1) \ = \ (\hat{w}(x_1) - w(x_1))\frac{f_\mu(x_1)}{f_v(x_1)} + \hat{c}(x_1) \ .$$

The constraint $\alpha(x_1) \leq c(x_1) \leq \beta(x_1)$ can thus be rewritten as in (4.2.15). The statement on orbital equivalence is obvious.

Finally, validity of the statement (iii) is simply because in case (4.2.13) is violated the resulting minimum variation vector field would give an equilibrium, which is excluded. $\square$

**Remark 4.2.7.** *We note that the point (iii) of Theorem 4.2.6 does not contradict Remark 4.1.2. In fact, in order to find a minimum solution for* (4.2.9)*, we have solved its associated Euler-Lagrange equation without enforcing the constraint on c (ensuring that the corresponding $\lambda = \mu + cv$ has nonnegative components adding to one). Therefore, it could happen that the unconstrained solution does not lie completely in the admissibility set, as it happens when, as proven above,* (4.2.13) *is violated. In other words, the unique solution of the constrained minimization problem would be a boundary solution with respect to the admissibility set, thus not solving the Euler-Lagrange equation associated to the unconstrained problem.*

**Remark 4.2.8.** *When $F_v = 0$ in* (4.2.10)*, and in particular $f_v = 0$ in* (4.2.11)*, the technique based on minimization of the $H^1$-norm of $wf_F$ gives a singular differential equation. We have not explored in details this situation (which would require analyzing the nature of the singular points), but observe that in the case of $F_v \equiv 0$ for all $a \leq s \leq b$ in* (4.2.10)*, then the minimization task for $wf_F$ is surely ill-posed. Example 5.2.2 in Chapter 5 will clarify this statement.*

### 4.2.3 Revisiting Example 4.1.1: Singular weights

We conclude our discussion on minimization of the $H^1$ variation of admissible solutions, with some considerations on the case of sliding vector fields with equilibria on $\Sigma$. In particular, we reconsider Example 4.1.1. That was a situation where -unlike the scenario of Conditions 4.2.1- **every** smooth sliding vector field of Filippov type had an equilibrium on $\Sigma$. Suppose that this is indeed the case, and thus consider the following scenario, still in $\mathbb{R}^3$, and still considering as discontinuity surfaces $\Sigma_1 = \{x : \; x_2 = 0\}$, and $\Sigma_2 = \{x : \; x_3 = 0\}$ (see Remark 4.2.4).

**Conditions 4.2.9** (Equilibrium on $\Sigma$)**.**

*(i) The sliding surface is the segment* $\Sigma = \{x_1 : \;\; a \le x_1 \le b\}$.

*(ii) For* $a < x_1 < b$, $\Sigma$ *is attractive, any smooth Filippov sliding vector field* $f_F$ *has one -and just one- equilibrium* $\bar{x}$ *on* $\Sigma$ *(the value of* $\bar{x}$ *depends on the choice of vector field), which is unstable and generic* [2]. *Let* $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ 0 \\ 0 \end{bmatrix}$, *so that motion on* $\Sigma$ *proceeds from any left neighborhood of* $\bar{x}_1$ *to* $a$ *(right-to-left) and from any right neighborhood of* $\bar{x}_1$ *to* $b$ *(left-to-right).*

*(iii) The points* $x_1 = a$ *and* $x_1 = b$ *are generic first order exit points.*

Obviously, under Conditions 4.2.9, different sliding vector fields cannot be orbitally equivalent, and the dynamics on $\Sigma$ differ (unless all possible sliding vector field share the same equilibrium). Indeed, in the case of Conditions 4.2.9, and with the above notation, we have this result.

**Theorem 4.2.10.** *Assume that* $f_v \neq 0$ *for* $x_1 \in [a, b]$, *and that, for* $w = 1$, *the solution* $c_{MV,f_F}$ *in* (4.2.12) *of the boundary value problem* (4.2.11) *is well defined and gives an admissible smooth Filippov sliding vector field* $f_{F1}$. *Then, the following holds.*

*(i) The function*

$$Ae^{x_1} + Be^{-x_1}$$

*is* 0 *at the point* $\bar{x}_1$, *equilibrium of* $(f_{F1})_1$ *(cfr. with* (4.2.13)*).*

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[2] *By this, we mean that* $\frac{\mathrm{d}}{\mathrm{d}x_1}(f_F)_1\Big|_{x_1=\bar{x}_1} \neq 0$

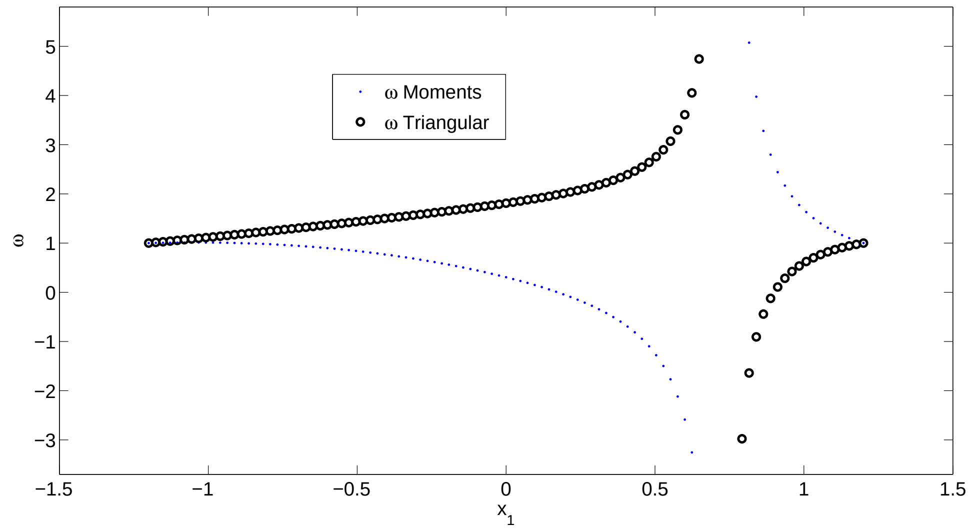


Figure 24: Moments and triangular orbital pseudo-equivalence factors, with respect to $f_{\mathrm{F}1}$ for Example 4.1.1.

*(ii) The only admissible weight functions $w$, satisfying* (4.2.4) *and giving an admissible solution $c$ of* (4.2.12)*, are those for which the resulting vector field has the equilibrium at $\bar{x}$.*

*(iii) To any other sliding vector field $f_F$ formed from an admissible $c$, we can associate a* singular weight *$w$, namely one which goes through* 0 *and changes sign at the value $\bar{x}_1$, and that has a first order pole at the zero of $(f_F)_1$. As a consequence, there is a singular* orbital pseudo-equivalence *factor $\omega$, relating $f_F$ and $f_{F1}$, given by $1/w$; $\omega$ is* 0 *at the equilibrium of $(f_F)_1$ and has a first order pole at $\bar{x}_1$.*

*Proof.* By hypothesis, we have that $c_{MV,f_{\mathrm{F}}} = \frac{Ae^{x_1}+Be^{-x_1}-f_\mu(x_1)}{f_v(x_1)}$, and therefore,

$$f_{\mathrm{F}1} = \begin{bmatrix} f_\mu(x_1) + c_{MV,f_{\mathrm{F}}}(x_1) f_v(x_1) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Ae^{x_1} + Be^{-x_1} \\ 0 \\ 0 \end{bmatrix},$$

from which point (i) follows.

To verify point (ii), suppose there were a weight function $w$ satisfying (4.2.4), giving an admissible solution $c_w$ of (4.2.12), and such that the resulting vector field has an equilibrium at a point different from $\bar{x}$. Then, we must have

$$w(x_1) = \frac{Ae^{x_1} + Be^{-x_1}}{f_\mu(x_1) + c_w(x_1) f_v(x_1)} \ . \tag{4.2.16}$$

But, the denominator of this expression vanishes at the equilibrium of the vector field $f_\mu(x_1) + c_w(x_1) f_v(x_1)$, and since -by hypothesis- this is different from $(\bar{x})_1$, we reach the contradiction that $w$ satisfies (4.2.4), and the claim follows.

Finally, point (iii) follows at once from the expression (4.2.16). □

In Figure 24 we illustrate Theorem 4.2.10, by considering the orbital pseudo-equivalence factors for the moments and the triangular solutions of Example 4.1.1.

# Chapter V

# IMPLEMENTATION AND NUMERICAL RESULTS

This chapter is devoted to examples and numerical implementations that will explain and validate our theoretical results stated and proven in previous chapters; in particular, in the first section of this chapter we will exemplify on results from Chapter 2 and Chapter 3, where we have introduced and completely justified the moments solution: aim of these examples will be to show that selecting different vector fields (namely, for us, the bilinear solution) could make dynamics develop undesirable singularities. In the second section of this chapter, we present motivating examples for the techniques and results on minimum variation solutions from Chapter 4.

## 5.1 *Examples: Comparing bilinear and moments solutions*

Our purpose in this section is to show some numerical experiments with the moments method and compare it (qualitatively) to the bilinear interpolation technique (see Remark 1.4.7) insofar as sliding on $\Sigma$.

The basic numerical integration scheme is a $4^{\text{th}}$ order embedded Runge-Kutta pair based on the $\frac{3}{8}$-th Runge-Kutta method, with Butcher's tableau

$$\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 0 & 0 & 0 & 0 \\
2/3 & -1/3 & 1 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 0 & 0 \\
\hline
b & 1/8 & 3/8 & 3/8 & 1/8 & 0 \\
1 & 1/8 & 3/8 & 3/8 & 3/8 & 0 \\
\hline
\hat{b} & 1/12 & 1/2 & 1/4 & 0 & 1/6
\end{array} .$$

Adaptive step size control is done as suggested in [34]:

$$h_{\text{new}} := h \cdot \min\left\{ facmax, \max\left\{ facmin, fac \cdot \left(\frac{1}{err}\right)^{\frac{1}{q+1}} \right\}\right\},$$

where $h$ is the current step size, $q := \min\{p, \hat{p}\}$, being $p$ the order of the Runge-Kutta scheme and $\hat{p}$ the order of the error estimator, and we have chosen

$$facmax = 5\ ,\ \ facmin = \text{eps}\ \ (\approx 10^{-16})\ ,\ \ fac = 0.8\ ,$$

and

$$Err := \left(\frac{|x_i - \hat{x}_i|}{1 + |x_i| \cdot tol}\right)_{i=1:n}\ , \quad err := \|Err\|_\infty\ ,$$

where $tol$ is a given error tolerance (below, $tol = 10^{-6}$).

The overall method is an *event driven method* (according to the naming in [1]), whereby different regimes (entering and exiting from the discontinuity manifolds) are monitored, and the appropriate vector fields are integrated. Integration in the regions $R_i$'s ($i = 1, 2, 3, 4$) is standard, and follows the above scheme. Integration during sliding motion is done with a projected version of the basic integration scheme to guarantee that the stage values and the computed approximations remain on the discontinuity manifold(s).

More precisely, we can have two different possibilities:

1. after a previously fixed time $T_{\max}$, we remain in $R_1$;

2. we hit either $\Sigma_1$,$\Sigma_2$ or $\Sigma$.

In the first case, the algorithm stops.

In the second case, the algorithm precisely locates the event point at which the dynamics enters on, say, $\Sigma_1$: this detection uses the one-sided numerical method developed in [25]. Once we get $\Sigma_1$ at $x_1$, we need to figure out what to do next: it is necessary to analyze the behaviors of the two vector fields, $f_1$ and $f_2$, acting on $x_1$: if transversal intersection occurs, then we need to integrate one step further the dynamics, and go back to the previous case, namely integration in one of the regions $R_i$'s.

If sliding mode occurs, then, following Filippov theory, we construct $f_F$ as in (1.3.4), and integrate over $\Sigma_1$ using the chosen projected Runge-Kutta scheme presented above. Once the dynamics evolves on $\Sigma_1$, each stage and each step of the method is projected on it to avoid the numerical solution to prematurely leave the surface (see [22]).

Now, if within $T_{\max}$ the numerical solution does not hit $\Sigma$ or any potential exit point, the algorithm stops.

If a potential exit points is reached, then we are back to the previous case, and we need to continue integration in some region $R_i$.

If $\Sigma$ is reached, we choose to slide on it following the moments solution, and the integration proceeds until $T_{\max}$ is reached, or a tangential exit point is detected. The location of potential exit points on $\Sigma$ is based on Theorem 3.4.7 (see also [18]). If $T_{\max}$ is attained, the algorithm stops.

Otherwise, it accurately locates the tangential exit point and follows the correct sliding vector field, that will lead the dynamics on the correct co-dimension 1 surface. At this point, if $T_{\max}$ has not been reached yet, we are back to one of the previous cases, and the algorithm restarts.

For our purposes, and to better visualize the differences between the method proposed here and other different ones, in all problems below integration on $\Sigma = \Sigma_1 \cap \Sigma_2$ will proceed according to two different choices of convex combination coefficients, and the associated vector fields: the coefficients $\lambda_{\mathrm{B}}$ used to form the bilinear vector field in (1.4.10)-(a), and the moments coefficients $\lambda_{\mathrm{M}}$ used to form the moments vector field. Let us stress that $\lambda_{\mathrm{B}}$ is found by solving the nonlinear system (1.4.10)-(b) for $\alpha$ and $\beta$; as the bilinear trajectory evolves on $\Sigma$, the coefficients $\alpha, \beta$, are updated by continuation with respect to the value at the previous integration step.

**Example 5.1.1.** This is a problem in $\mathbb{R}^3$. We have (1.4.3) with $x(0) = \begin{bmatrix} -0.1 \\ -0.1 \\ -0.1 \end{bmatrix}$, $\Sigma_1 := \{x \in \mathbb{R}^3 : h_1(x) := x_1 = 0\}$, $\Sigma_2 := \{x \in \mathbb{R}^3 : h_2(x) := x_2 = 0\}$, and $\Sigma := \Sigma_1 \cap \Sigma_2$ is just the $x_3$-axis. The vector fields are given by

$$f_1(x) := \begin{bmatrix} \frac{\sqrt{2}}{8}\sin\left(\frac{\pi}{4} - x_3^2\right) \\ \frac{\sqrt{2}}{8}\cos\left(\frac{\pi}{4} - x_3^2\right) \\ x_1^2 + x_2^2 + 1 \end{bmatrix}, \quad f_2(x) := \begin{bmatrix} 2\sqrt{2}\sin\left(\frac{3}{4}\pi - x_3^2\right) \\ \sqrt{2}\cos\left(\frac{3}{4}\pi - x_3^2\right) \\ x_1^2 + x_2^2 + 1 \end{bmatrix},$$

$$f_3(x) := \begin{bmatrix} \sqrt{2}\sin\left(\frac{\pi}{4} - 2x_3^2\right) \\ \sqrt{2}\cos\left(\frac{\pi}{4} - 2x_3^2\right) \\ x_1^2 + x_2^2 + 1 \end{bmatrix}, \quad f_4(x) := \begin{bmatrix} -2 \\ -1 \\ x_1^2 + x_2^2 + 1 \end{bmatrix}.$$

Since $x(0) \in R_1$, we integrate $\dot{x} = f_1(x)$, until we hit $\Sigma_2^-$ transversally at $\xi_1 \approx \begin{bmatrix} -0.0208 \\ 0 \\ 0.6320 \end{bmatrix}$. Notice that $\Sigma_2^-$ is attractive, since (see (1.3.3))

$$f_1(\xi_1) \approx \begin{bmatrix} 0.0665 \\ 0.1638 \\ 1.0004 \end{bmatrix}, \quad f_2(\xi_1) \approx \begin{bmatrix} 2.6204 \\ -0.5324 \\ 1.0004 \end{bmatrix}.$$

Thus, from $\xi_1$, the trajectory starts sliding on $\Sigma_2^-$ directed towards $\Sigma$ with vector field

$$f_{\Sigma_2^-}(x) := (1 - \alpha_{\Sigma_2^-}(x))f_1(x) + \alpha_{\Sigma_2^-}(x)f_2(x)\ , \quad \alpha_{\Sigma_2^-}(x) := \frac{w_1^2(x)}{w_1^2(x) - w_2^2(x)}\ .$$

At $\xi_2 \approx \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.6619 \end{bmatrix}$, the trajectory reaches $\Sigma$ transversally. At this point, $\Sigma$ is nodally attractive, since

$$W(\xi_2) \approx \begin{bmatrix} 0.0602 & 2.6596 & -0.1285 & -2 \\ 0.1662 & -0.4812 & 1.4084 & -1 \end{bmatrix}.$$

Observe that there is a unique Filippov sliding vector field (1.4.8) on $\Sigma$, namely $\dot{x}_3 = 1$; however, $\lambda_{\mathrm{B}}$ and $\lambda_{\mathrm{M}}$ are different.

With both the bilinear and moments methods the solution trajectory eventually reaches the first order exit point

$$\xi_3 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{\pi}{2}} \end{bmatrix}, \quad \text{where } W(\xi_3) = \begin{bmatrix} -\frac{1}{8} & 2 & -1 & -2 \\ \frac{1}{8} & 1 & -1 & -1 \end{bmatrix}.$$

For values of $x_3$ greater than $\sqrt{\frac{\pi}{2}}$, $\Sigma$ looses attractivity, and thus, past this value forward integration (i.e., sliding) on $\Sigma$ does not make much sense anymore. For this reason, at $\xi_3$ we should leave $\Sigma$ sliding on $\Sigma_1^+$. Depending on whether we have $\lambda_{\mathrm{B}}$ or $\lambda_{\mathrm{M}}$, however, we witness very different behaviors as we reach $\xi_3$.

As Figure 25 shows, at $\xi_3$ the bilinear solution $\lambda_{\mathrm{B}}$ has all positive components. Instead, the moments solution $\lambda_{\mathrm{M}}$ at $\xi_3$ has its first and third components equal to zero: these are exactly the components of $\lambda$ that do not play a role when sliding on $\Sigma_1^+$ starts; indeed, at $\xi_3$, $\lambda_{\mathrm{M}}$ provides the exit vector field on the sub-manifold $\Sigma_1^+$, that is $f_{\mathrm{F}_1^+}$ (see (1.4.6)). Moreover, we note that if we force integration on $\Sigma$ past $\xi_3$ for the moments trajectory (note that the moments' matrix remains invertible, at least near $\xi_3$, because of Theorem 3.3.8), then the first and third components become negative past the exit point, hence the moments solution is not admissible. [This fact provides a powerful characterization of first order exit points, and a very useful criterion to detect them numerically.]

As far as the bilinear solution, at $\xi_3$ (1.4.10)-(b) must have multiple roots (see [19]): the solution $(\alpha^*, \beta^*)$ we had been following (which gives $\lambda_{\mathrm{B}}$ in Figure 25), and a new one, necessarily being $(\alpha^+, 1)$ which has "entered" the admissible region. As shown in [19], the nonlinear system (1.4.10)-(b) reduces to a quadratic equation in $\beta$, with the two roots $\beta^*$ and 1. We stress that, by solving the nonlinear system (1.4.10)-(b) by continuation, the "new entering" root goes unnoticed. To sum up, assuming

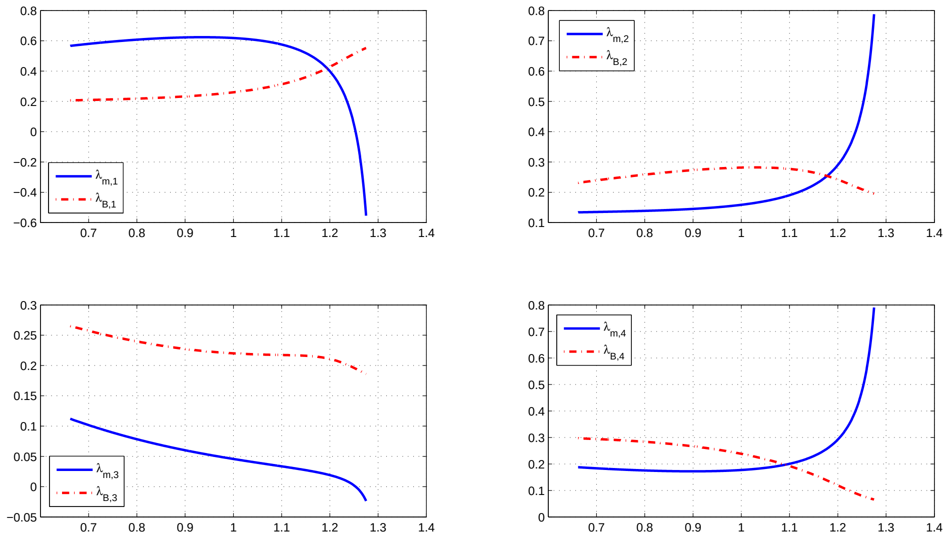


Figure 25: Moments and bilinear solutions for $x_3 \in [0.6619..., 1.275]$.

that, somehow, **all** roots of the nonlinear system (1.4.10)-(b) are monitored, one could force the trajectory to exit at $\xi_3$, but following the solution $(\alpha^*, \beta^*)$ we have been continuing gives no indication that a first order exit point has been reached; all components of $\lambda_\mathrm{B}$ remain positive past $\xi_3$, even though $\Sigma$ is no longer attracting. Moreover, as Figure 26 shows, if we do not exit $\Sigma$ at $\xi_3$ and continue integrating on $\Sigma$ with $f_\mathrm{B}$ (using the continuation of $(\alpha^*, \beta^*)$), then the bilinear solution develops a singularity. Namely, at $x_s \approx \begin{bmatrix} 0 \\ 0 \\ 1.4163 \end{bmatrix}$, $\lambda_\mathrm{B}$ becomes complex valued, and motion on $\Sigma$ with $f_\mathrm{B}$ ceases to make sense. [This last fact is easy to explain, since the roots of the above parabola in $\beta$ collide and become complex valued.]

$\square$

**Remark 5.1.2.** *In Example 5.1.1, we have a system in* $\mathbb{R}^3$*,* $\Sigma$ *is a straight line, and all sliding trajectories satisfy* $\dot{x}_3 = 1$*. In particular, using either* $\lambda_B$ *or* $\lambda_M$*, a sliding trajectory must reach the point* $\xi_3$ *above. Although, in principle, both bilinear and*

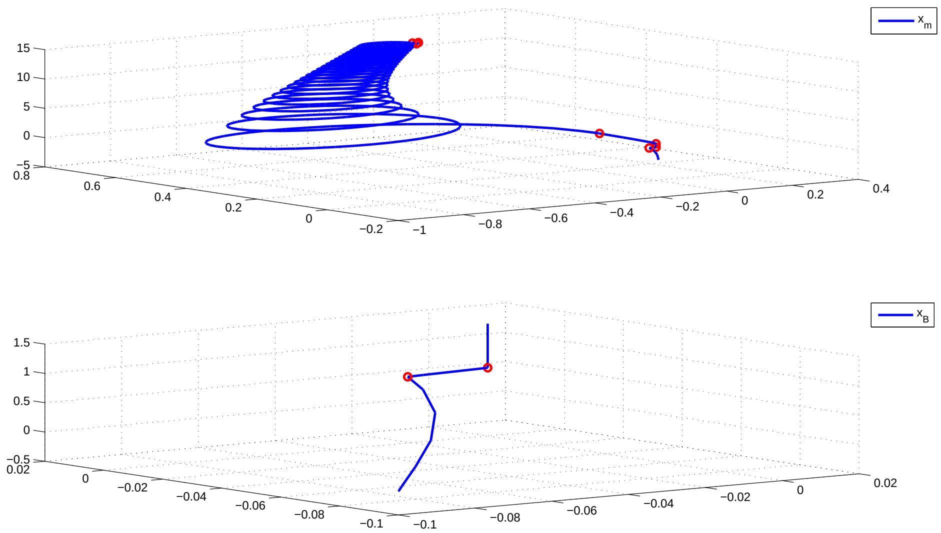


Figure 26: Moments and bilinear trajectories for Example 5.1.1.

*moments trajectories could exit at $\xi_3$, there is a major difference in what happens to $\lambda_B$ or $\lambda_M$ if we let the trajectory continue on $\Sigma$ past $\xi_3$. At first, $\lambda_B$ has all components positive and seemingly well behaved, and it does not betray that the origin has gone outside of the quadrilateral $Q$. On the other hand, $\lambda_M$ has two components going to 0 at $\xi_3$, and then becoming negative. This is an important fact, which betrays that the origin has exited the quadrilateral $Q$, and that allows automatic detection of exit points, as we have elaborated in Section 3.4.*

In the next example, we show that, in general (that is, when the phase space is not $\mathbb{R}^3$, nor $\mathbb{R}^2$), even when they seemingly are both well defined and exit smoothly, the moments and bilinear methods lead to different dynamics, and –again– the bilinear solution may again eventually develops a singularity, similarly to Example 5.1.1.

**Example 5.1.3.** We have (1.4.3) with $x(0) = \begin{bmatrix} -0.1 \\ -0.1 \\ -0.1 \\ 0.1 \end{bmatrix}$, $\Sigma_1 := \{x \in \mathbb{R}^4 \ : \ h_1(x) :=$

$x_1 = 0\}$, $\Sigma_2 := \{x \in \mathbb{R}^4 \,:\, h_2(x) := x_2 = 0\}$, and $\Sigma := \Sigma_1 \cap \Sigma_2$ is the $(x_3, x_4)$ plane. The vector fields are given by

$$f_1(x) := \begin{bmatrix} \frac{\sqrt{2}}{8}\sin\left(\frac{\pi}{4} - x_4 x_3^2\right) \\ \frac{\sqrt{2}}{8}\cos\left(\frac{\pi}{4} - x_3^2\right) \\ x_1^2 + x_2^2 + 1 \\ x_1 + x_2 + x_3 + x_4^2 \end{bmatrix} , \quad f_2(x) := \begin{bmatrix} 2\sqrt{2}\sin\left(\frac{3}{4}\pi - x_3^2\right) \\ \sqrt{2}\cos\left(\frac{3}{4}\pi - x_4 x_3^2\right) \\ x_4 x_1^2 + x_2^2 + 1 \\ e^{x_1} + x_2 + x_3^2 + x_4 \end{bmatrix} ,$$

$$f_3(x) := \begin{bmatrix} \sqrt{2}\sin\left(\frac{\pi}{4} - 2x_3^2\right) \\ \sqrt{2}\cos\left(\frac{\pi}{4} - 2x_3^2\right) \\ x_1^2 + x_2^2 + 1 \\ x_1 + e^{x_2} + x_3 x_4 \end{bmatrix} , \quad f_4(x) := \begin{bmatrix} -2 \\ -1 \\ x_1 + x_2 \\ x_3 x_4^2 + 1 \end{bmatrix} .$$

We integrate $\dot{x} = f_1(x)$ until the trajectory reaches $\Sigma_2^-$ transversally at $\xi_1 \approx \begin{bmatrix} -0.0108 \\ 0 \\ 0.6319 \\ 0.2256 \end{bmatrix}$, where $\Sigma_2^-$ is attractive, since (see (1.3.3))

$$f_1(\xi_1) \approx \begin{bmatrix} 0.1132 \\ 0.1638 \\ 1.0001 \\ 0.6721 \end{bmatrix} , \quad f_2(\xi_1) \approx \begin{bmatrix} 2.6203 \\ -0.9060 \\ 1.0000 \\ 1.6142 \end{bmatrix} .$$

There is sliding motion on $\Sigma_2^-$ directed towards $\Sigma$, with vector field

$$f_{\Sigma_2^-}(x) := (1 - \alpha_{\Sigma_2^-}(x)) f_1(x) + \alpha_{\Sigma_2^-}(x) f_2(x) \ , \quad \alpha_{\Sigma_2^-}(x) := \frac{w_1^2(x)}{w_1^2(x) - w_2^2(x)} .$$

At $\xi_2 \approx \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.6533 \\ 0.2436 \end{bmatrix}$, the trajectory reaches $\Sigma$ transversally (see Figure 27). Since at

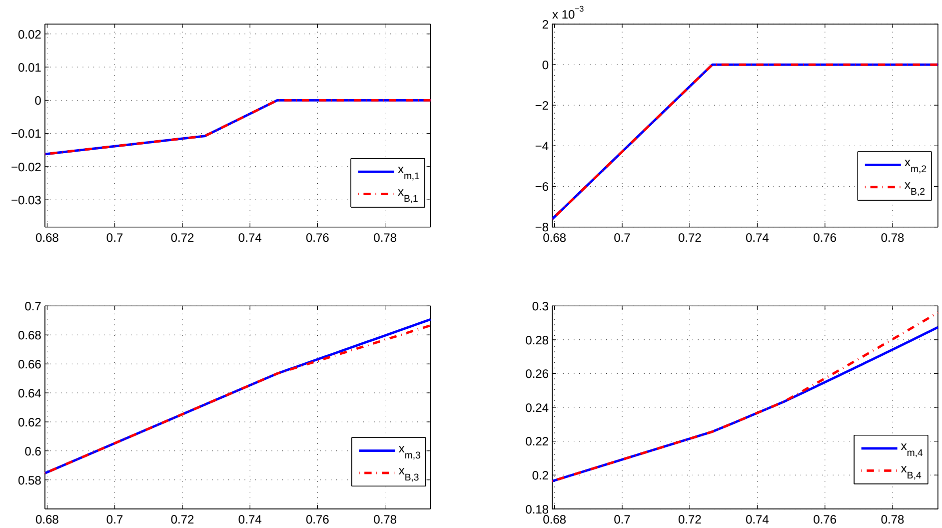


Figure 27: First enter on $\Sigma$ of moments and bilinear trajectories.

$\xi_2$ we have

$$W(\xi_2) = \begin{bmatrix} 0.1114 & 2.6486 & -0.0966 & -2 \\ 0.1655 & -0.8908 & 1.4110 & -1 \end{bmatrix},$$

then $\Sigma$ is (at least, near $\xi_2$) nodally attractive. We slide on $\Sigma$ using either $f_{\mathrm{B}}$ or $f_{\mathrm{M}}$. The respective solution trajectories now follow different paths on $\Sigma$, but eventually both reach the *curve of first order exit points* given by

$$x_4 = -1 + \frac{\pi}{x_3^2} \ .$$

**Remark 5.1.4.** *For this problem, exit curves on $\Sigma$ are directly computable, and are given by:*

$$x_4 = 1 - \frac{\pi + 4k\pi}{2x_3^2} \ , \ k \in \mathbb{Z} \ , \quad \textit{and } x_4 = -1 + \frac{\pi + 2k\pi}{x_3^2} \ , \ k \in \mathbb{Z} \ .$$

As Figure 28 shows, the moments and the bilinear trajectories exit (both of them smoothly) at different positions on the same exit curve. Namely, the moments and

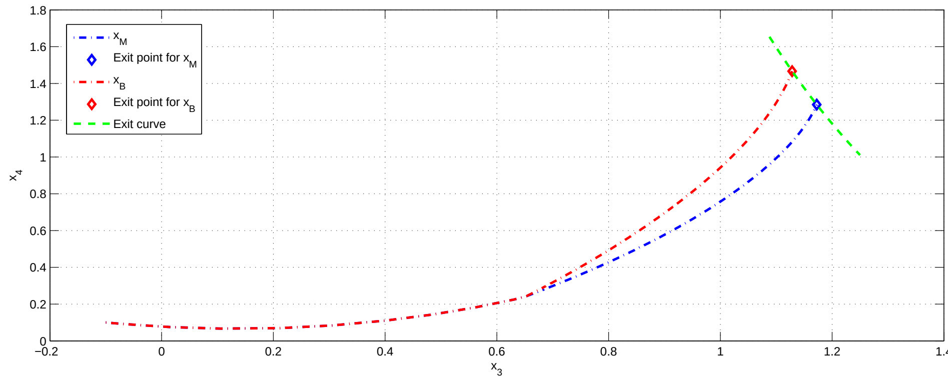


Figure 28: Projection of moments and bilinear trajectories in the $(x_3, x_4)$ plane during sliding motion.

bilinear trajectories exit at

$$\xi_3^{(M)} \approx \begin{bmatrix} 0.0000 \\ 0.0000 \\ 1.1725 \\ 1.2851 \end{bmatrix}, \quad \text{respectively} \quad \xi_3^{(B)} \approx \begin{bmatrix} 0.0000 \\ 0.0000 \\ 1.1285 \\ 1.4670 \end{bmatrix},$$

with coefficients

$$\lambda_{\mathrm{M}}(\xi_3^{(M)}) \approx \begin{bmatrix} 0 \\ 0.4596 \\ 0 \\ 0.5404 \end{bmatrix}, \quad \text{respectively} \quad \lambda_{\mathrm{B}}(\xi_3^{(B)}) \approx \begin{bmatrix} 0 \\ 0.4446 \\ 0 \\ 0.5554 \end{bmatrix}.$$

After they exit, as shown in Figure 29, trajectories evolve in $\Sigma_1^+$ until both of them again reach $\Sigma$ transversally, but at different points: namely, the moments trajectory enters $\Sigma$ at $\xi_4^{(M)} \approx \begin{bmatrix} 0 \\ 0 \\ 1.2923 \\ 2.4231 \end{bmatrix}$, whereas the bilinear trajectory enters $\Sigma$ at $\xi_4^{(B)} \approx$

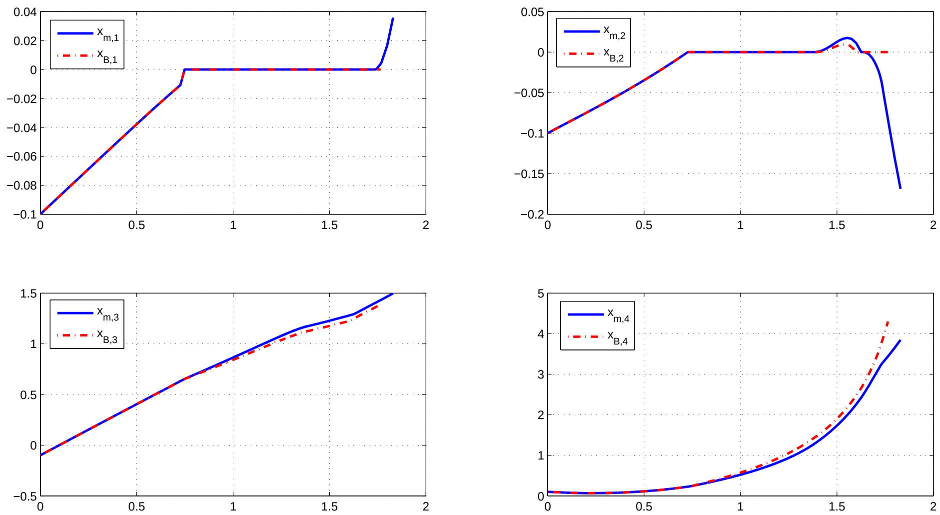


Figure 29: Solution components of moments and bilinear trajectories.

$\begin{bmatrix} 0 \\ 0 \\ 1.2236 \\ 2.4714 \end{bmatrix}$. After a short sliding regime on $\Sigma$, the moments trajectory exits $\Sigma$ smoothly at $\xi_5^{(M)} \approx \begin{bmatrix} 0 \\ 0 \\ 1.3050 \\ 2.5150 \end{bmatrix}$, where $\lambda_{\mathrm{M}}\left(\xi_5^{(M)}\right) \approx \begin{bmatrix} 0.9195 \\ 0 \\ 0.0805 \\ 0 \end{bmatrix}$. On the other hand, during this second sliding motion on $\Sigma$, the bilinear trajectory passes through a first order exit point, and eventually the coefficients become complex valued[1] at $\xi_5^{(B)} \approx \begin{bmatrix} 0 \\ 0 \\ 1.3874 \\ 4.3001 \end{bmatrix}$. See Figure 30 for a magnification of this phenomenon.

[1]The explanation of why the bilinear trajectory we are following does not notice the generic first order exit point, and why the bilinear coefficients eventually become complex valued, is much like the explanation we provided in Example 5.1.1

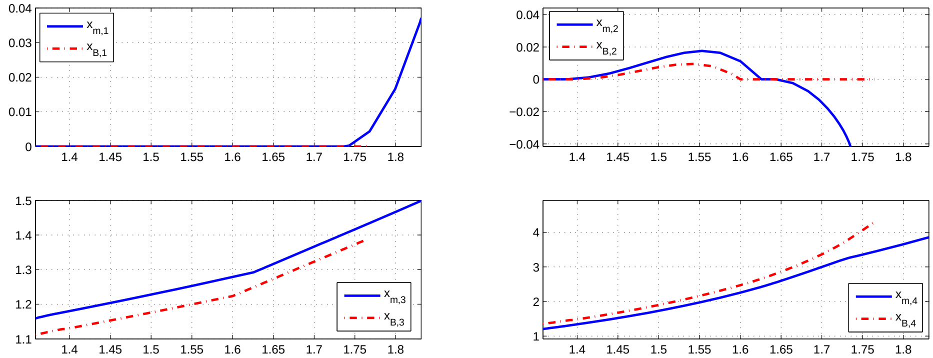


Figure 30: Second sliding on $\Sigma$: moments trajectory evolves properly, whereas bilinear trajectory develops a singularity after passing through a first order exit point.

After $\xi_5^{(M)}$, the moments trajectory begins sliding on $\Sigma_1^-$, from where it exits at $\begin{bmatrix} 0 \\ -0.0427 \\ 1.4022 \\ 3.2765 \end{bmatrix}$, entering $R_3$; once there, the moments trajectory eventually reaches $\Sigma_2^+$ transversally at $\xi_6^{(M)} \approx \begin{bmatrix} 0.3529 \\ 0 \\ 1.9684 \\ 8.6016 \end{bmatrix}$. Then, after sliding on $\Sigma_2^+$, it exits from there at $\xi_7^{(M)} \approx \begin{bmatrix} 0.1363 \\ 0 \\ 2.0794 \\ 164.9307 \end{bmatrix}$ moving into $R_3$. At $\xi_8^{(M)} \approx \begin{bmatrix} 0 \\ -0.0737 \\ 2.1942 \\ 210.5975 \end{bmatrix}$ the moments trajectory reaches transversally $\Sigma_1^-$, slides on it and leaves it at $\xi_9^{(M)} \approx \begin{bmatrix} 0 \\ -0.0738 \\ 2.1953 \\ 266.7981 \end{bmatrix}$ and enters in $R_3$ again. $\square$

The right selection of a sliding vector field on $\Sigma$ and the automatic detection of first order exit points is fundamental in cases where the piecewise smooth dynamical system is expected to provide a periodic orbit. Next example shows that a periodic orbit could be completely destroyed if one does not select a suitable sliding vector field when the dynamics is forced to slide at co-dimension 2.

**Example 5.1.5.** *This example is a slight, but crucial, modification of Example* 16 *in [20]. We have* (1.4.3) *with* $x(0) = \begin{bmatrix} 0.995 \\ 0.2 \\ 0.4 \end{bmatrix}$, $\Sigma_1 := \{x \in \mathbb{R}^3 \,:\, h_1(x) := x_2 - v_1 = 0\}$, $\Sigma_2 := \{x \in \mathbb{R}^3 \,:\, h_2(x) := x_3 - v_2 = 0\}$, *where* $v_1 := 0.2$, $v_2 := 0.4$, *and* $\Sigma := \Sigma_1 \cap \Sigma_2$ *is just the* $x_1$*-axis. The vector fields are given by*

$$f_1(x) := \begin{bmatrix} \frac{x_2+x_3}{2} \\ -x_1 + \frac{1}{1-(x_2-v_1)} \\ -x_1 + \frac{1}{1+\eta-(x_3-v_2)} \end{bmatrix}, \quad f_2(x) := \begin{bmatrix} \frac{x_2+x_3}{2} \\ -x_1 + \frac{1}{1-(x_2-v_1)} \\ -x_1 - \frac{1}{1+(x_3-v_2)} \end{bmatrix},$$

$$f_3(x) := \begin{bmatrix} \frac{x_2+x_3}{2} \\ -x_1 - \frac{1}{1+(x_2-v_1)} + 3 \\ -x_1 + \frac{1}{1-(x_3-v_2)} + \frac{44}{37} \end{bmatrix}, \quad f_4(x) := \begin{bmatrix} \frac{x_2+x_3}{2} + x_1(x_2+0.8)(x_3+0.6) \\ -x_1 - \frac{1}{1+(x_2-v_1)} \\ -x_1 - \frac{1}{1+(x_3-v_2)} \end{bmatrix},$$

*where is chosen to be* $\eta = -0.1$. *In this case, as shown in [20],* $\Sigma$ *is attractive for* $-1 < x_1 < 1$.

*We have compared different solution trajectories according to different selections of the sliding vector field on* $\Sigma$. *Starting at* $x(0)$, *the solution trajectory slides on* $\Sigma$ *according to the moments method, and reaches* $x_1 \approx \begin{bmatrix} 1.004 \\ 0.2 \\ 0.4 \end{bmatrix}$, *then exits* $\Sigma$ *on* $\Sigma_2^-$*; after sliding there, at* $x_2 \approx \begin{bmatrix} 1.112 \\ 0.1756 \\ 0.4 \end{bmatrix}$ *it exits* $\Sigma_2^-$ *and starts evolving in* $R_1$, *eventually*

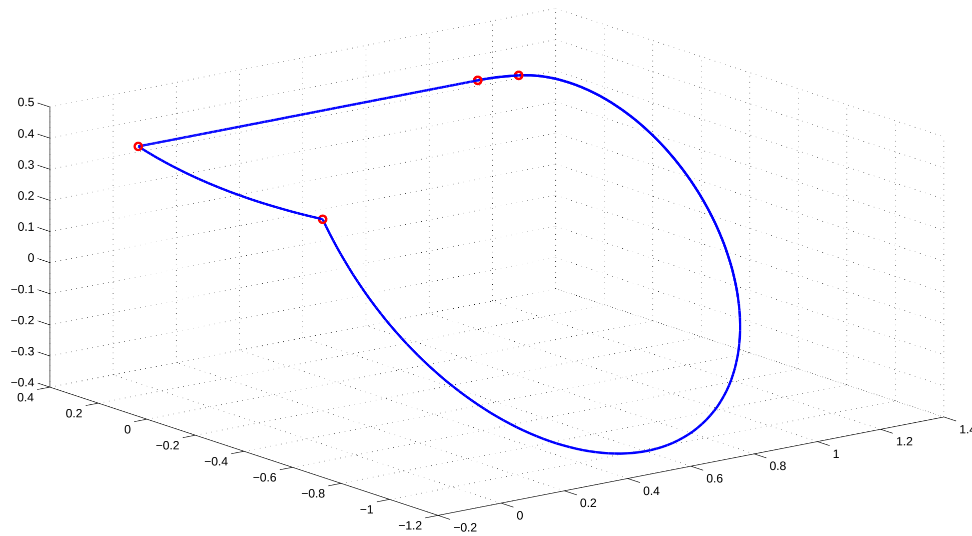


Figure 31: Periodic orbit when the Moments method is selected on $\Sigma$.

*reaching* $\Sigma_2^-$ *at* $x_3 \approx \begin{bmatrix} -0.1456 \\ -0.6536 \\ 0.4 \end{bmatrix}$*; after a co-dimension* 1 *sliding there, the solution trajectory hits* $x_4 \approx \begin{bmatrix} -0.07399 \\ 0.2 \\ 0.4 \end{bmatrix}$*. From now on, sliding according to the moments vector field on* $\Sigma$*, the sliding leads the trajectory reaching* $x(0)$ *and a periodic orbit arises; see Figure 31.*

*If we choose to slide on* $\Sigma$ *using the bilinear sliding vector field, we can see in Figure 32 that, without enforcing the exit at the first order exit point* $x_1$*, the solutions trajectory passes it and the bilinear solution* $\lambda_B$ *eventually develops a singularity at* $x_5 \approx \begin{bmatrix} 1.503 \\ 0.2 \\ 0.4 \end{bmatrix}$.

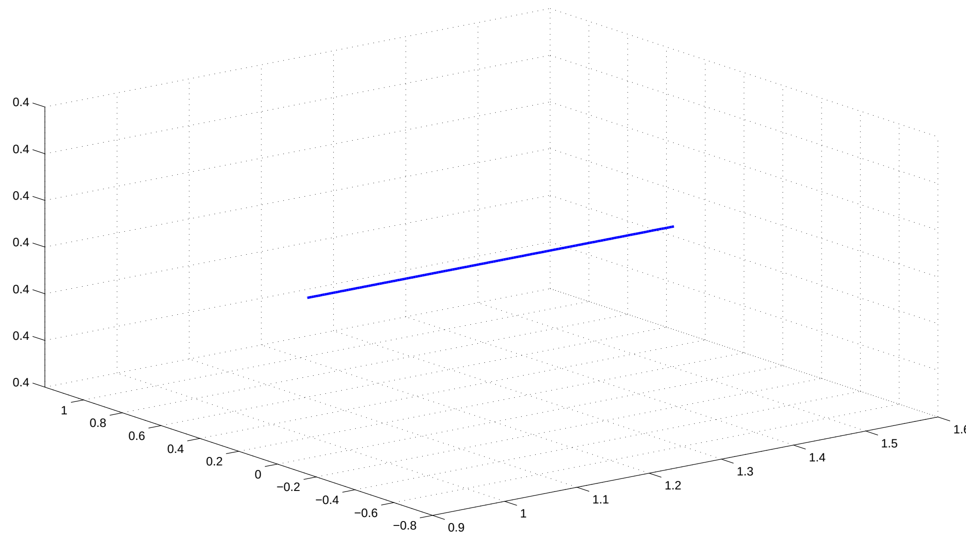


Figure 32: The bilinear method on $\Sigma$ does not detect the exit point automatically and the corresponding solution develops a singularity at $x_5 = (1.503, 0.2, 0.4)$.

## 5.2 Motivating examples for minimum variation techniques

In this section, we present and detailedly analyze the theoretical problems that have led us in studying the minimum variation solutions to determine Filippov sliding vector fields showed in Chapter 4. Example 5.2.1 is a slight modification of Example 4.1.1, where we avoid equilibria on $\Sigma$ in order to relate the weighted $H^1$-norm to the reparametrization of time, that is a natural phenomenon for co-dimension 2 manifolds in $\mathbb{R}^3$, as explained in [20].

Example 5.2.2 shows what happens to the minimum variation solutions when entering or exiting the admissibility region does not happen through points, but through admissibility intervals for $c$ at $a$ and $b$.

**Example 5.2.1** (Another model problem)**.** This is very similar to Example 4.1.1, except for the first component of the vector fields, chosen so that there are no equilibria on the sliding segment. We have $f_i$, $i = 1, 2, 3, 4$, taking values in $\mathbb{R}^3$:

$$f_1(x) := \begin{bmatrix} e^{-x_1} + 1 \\ -x_1 + x_2x_3 + 1 \\ x_1 + x_2 + 1 \end{bmatrix}, \ x \in R_1 \ , \quad f_2(x) := \begin{bmatrix} e^{-x_1} - 1 \\ -x_1 + x_3 - 1 \\ x_1 + x_2x_3 + 2 \end{bmatrix}, \ x \in R_2 \ ,$$
$$f_3(x) := \begin{bmatrix} -e^{-x_1} + 1 \\ -x_1 + x_2 + 2 \\ x_1 + x_2x_3 - 1 \end{bmatrix}, \ x \in R_3 \ , \quad f_4(x) := \begin{bmatrix} -e^{-x_1} + 2 \\ -x_1 + x_3 - 2 \\ x_1 + x_3 - 2 \end{bmatrix}, \ x \in R_4 \ ,$$

where $\Sigma_1 = \{x : \ x_3 = 0\}$, $\Sigma_2 = \{x : \ x_2 = 0\}$, and so $\Sigma = \Sigma_1 \cap \Sigma_2$ is the $x_1$-axis. The admissible region for $c$ is the same as in Example 4.1.1, that is the triangle of Figure 22, hence we have $a = -1.2$, $b = 1.2$, and $c_a = 0.8$, $c_b = 0$, and $\lambda = \mu + cv$ as in (4.1.2). There is sliding motion on $\Sigma$ from $a$ to $b$.

(a) The minimum variation solution with weight $w \equiv 1$, with respect to $\lambda$, that is the solution in (4.2.8), is

$$c_{MV,\lambda} \ = \ \frac{1}{\|v\|^2}\left(Ae^{x_1} + Be^{-x_1} - v^\top \mu\right) \ ,$$

with $v^\top \mu = \frac{44}{27}x_1 - \frac{11}{9}$, $\|v\|^2 = \frac{44}{9}$, and the constants $A, B$, so that $c_{MV,\lambda}(-1.2) = 0.8$ and $c_{MV,\lambda}(1.2) = 0$.

(b) The minimum variation solution with weight $w \equiv 1$, with respect to $f_{\mathrm{F}}$, that is the the solution in (4.2.12) is

$$c_{MV,f_{\mathrm{F}}} \ = \ \frac{1}{f_v}\left(Ae^{x_1} + Be^{-x_1} - f_\mu(x_1)\right) \ ,$$

with $f_v = -\frac{4}{3}e^{-x_1} - \frac{7}{3}$ and $f_\mu = -\frac{10}{9}x_1e^{-x_1} + \frac{1}{3}e^{-x_1} - \frac{1}{9}x_1 + \frac{4}{3}$, and the constants $A, B$, so that $c_{MV,f_{\mathrm{F}}}(-1.2) = 0.8$ and $c_{MV,f_{\mathrm{F}}}(1.2) = 0$.

In Figure 33 we show the five functions $c$ we discussed for this problem: moments, bilinear, triangular, and the two minimum variation solutions (with weight $w = 1$).

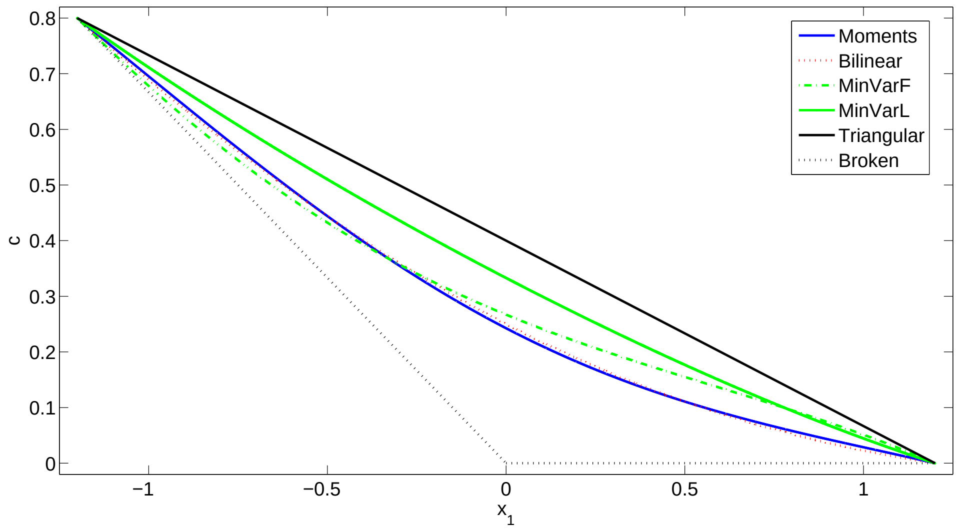


Figure 33: Admissible region $(x_1, c)$ and moments, bilinear, triangular, broken-line, and minimum variation solutions of $c$.

We also show the “broken-line” solution, corresponding to the selection of $c$ given by the path along the two other sides of the triangular region. In this case, all these solutions are admissible (all smooth, except the broken line solution), and give different Filippov sliding vector fields, all smoothly exiting. The corresponding vector fields are shown in Figure 34.

We are finally ready to answer in the positive, on this example, Questions 4.2.5. The reason why we can answer positively those questions is that there are no equilibria, and thus:

$$\left(Ae^{x_1} + Be^{-x_1}\right)(f_{\mathrm{F}})_1 > 0 \ ,$$

where $(f_{\mathrm{F}})_1$ is the first component of any of the above vector fields (the second and third components being 0 in the present case).

(i) In light of the above, we can choose the weight $w$ so that the solution of (4.2.9) gives us any of the above solutions. In fact, for any admissible $c$ giving us a sliding vector field $f_{\mathrm{F}}$, we **define** the weight $w$, which gives $c$ as the minimum

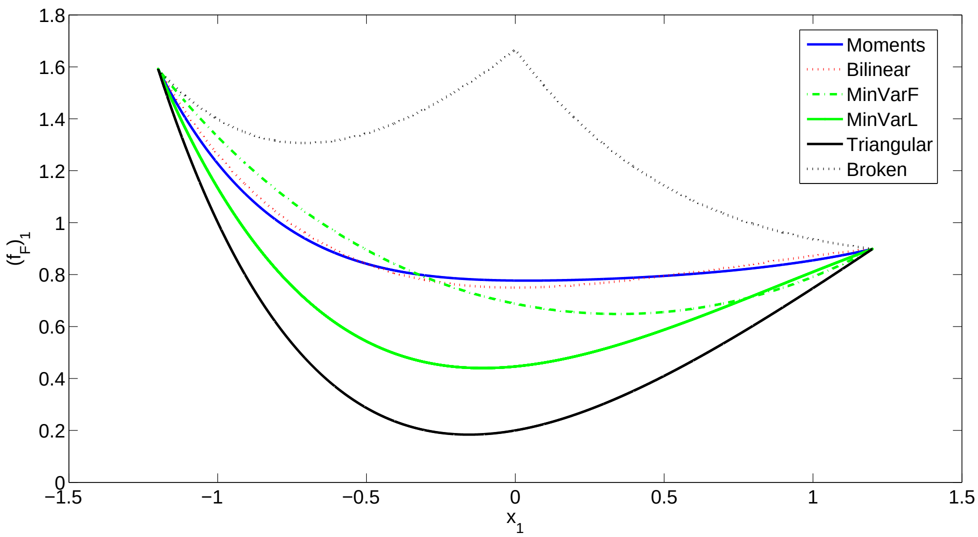


Figure 34: Sliding vector fields for moments, bilinear, triangular, broken-line, and minimum variation solutions.

variation of (4.2.9), from

$$w(x) \;=\; \frac{Ae^{x_1}+Be^{-x_1}}{(f_{\rm F})_1}\;. \tag{5.2.1}$$

By construction, using this weight $w$ in the minimization of (4.2.9) will give us the function $c$ which gave $f_{\rm F}$. In particular, also the bilinear, triangular, and moments solutions are in fact weighted minimum variation solutions. The "broken line" solution, not being smooth, cannot be obtained as solution of (4.2.9) with smooth $w$; nonetheless, we still formally define its associated weight as above (it is attainable as the limit of smooth solutions).

(ii) As we know, the previously displayed vector fields (see Figure 34) are all orbitally equivalent. In particular, it must be true that any of the vector field is a multiple of the vector field obtained as minimum variation with respect to $f_{\rm F}$ with weight 1. Because of (5.2.1), thus we must have

$$\omega(x) \;=\; \frac{1}{w(x)}\;, \tag{5.2.2}$$

where $w(x)$ is the weight associated to the specific choice of $f_{\mathrm{F}}$ under consideration; see (5.2.1). (In other words, in (4.2.2) we are using $f_{\mathrm{F}1} = f_{\mathrm{MV}}$ -minimum variation with respect to $f_{\mathrm{F}}$ with weight $w = 1$- and $f_{\mathrm{F}2}$ any of the previously obtained sliding vector fields). In Figure 35 we show the values of $\omega$ for the vector fields above. We observe that the moments and bilinear solutions give quite similar functions $\omega$. Also, observe that the broken-line solution gives (as expected) a non-smooth factor $\omega$. Looking at Figure 35, we conclude that all possible values of $\omega$ must be within the upper and lower curves, that is in between the functions $\omega$ of the triangular and broken-line solutions.

To conclude our discussion on this example, we observe that the broken-line solution takes the least amount of "time" to travel from $a$ to $b$:

$$t_{\mathrm{broken}} \approx 1.93 < t_m \approx 2.76 < t_b \approx 2.76 < t_{MV,f_{\mathrm{F}}} \approx 2.96 < t_{MV,\lambda} \approx 3.85 < t_{tr} \approx 6.55$$

This was predictable, since –being all vector fields orbitally equivalent– we have that with respect to the time $t$ given by selecting $c_{MV,f_{\mathrm{F}}}$, all other times come from $\mathrm{d}\tau = \frac{1}{\omega}\mathrm{d}t$, and therefore "the larger $\omega$, the shorter the time" (see Figure 35). The fact that the broken-line solution gives the shortest time is also consistent with the general flavor of results in optimal control theory, whereby it is known that, for linear problems with constraints, the optimal control (here, the value of $c$ giving the minimal time solution) lies on the boundary of the admissible region (see [44]). In a specular way, the admissible solution taking the longest time is the triangular solution.

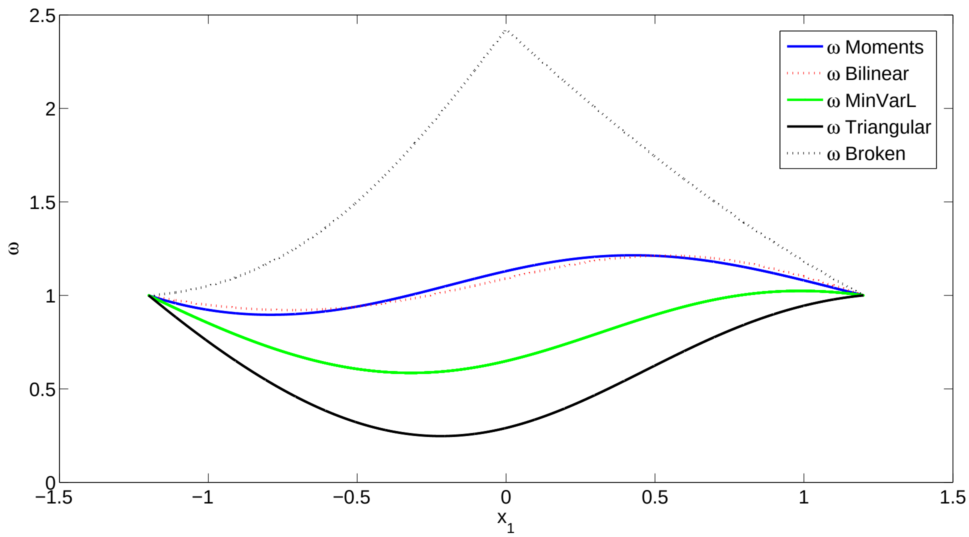


Figure 35: Orbital equivalence factors $\omega$ for moments, bilinear, triangular, broken-line, and minimum variation with respect to $\lambda$.

**Example 5.2.2** ([18]). Consider the following problem in $\mathbb{R}^3$:

$$f_1(x) := \begin{bmatrix} \frac{\sqrt{2}}{8}\sin\left(\frac{\pi}{4}-x_3^2\right) \\ \frac{\sqrt{2}}{8}\cos\left(\frac{\pi}{4}-x_3^2\right) \\ x_1^2+x_2^2+1 \end{bmatrix}, \quad f_2(x) := \begin{bmatrix} 2\sqrt{2}\sin\left(\frac{3}{4}\pi-x_3^2\right) \\ \sqrt{2}\cos\left(\frac{3}{4}\pi-x_3^2\right) \\ x_1^2+x_2^2+1 \end{bmatrix},$$

$$f_3(x) := \begin{bmatrix} \sqrt{2}\sin\left(\frac{\pi}{4}-2x_3^2\right) \\ \sqrt{2}\cos\left(\frac{\pi}{4}-2x_3^2\right) \\ x_1^2+x_2^2+1 \end{bmatrix}, \quad f_4(x) := \begin{bmatrix} -2 \\ -1 \\ x_1^2+x_2^2+1 \end{bmatrix},$$

$\Sigma_1 := \{x \in \mathbb{R}^3 : x_1 = 0\}$, $\Sigma_2 := \{x \in \mathbb{R}^3 : x_2 = 0\}$ and $\Sigma := \Sigma_1 \cap \Sigma_2$ is just the $x_3$-axis, which is in particular attractive in the segment $\gamma := \{-\sqrt{\pi/2} < x_3 < \sqrt{\pi/2}\}$ (the endpoints being generic first order exit points).

In this problem, we stress that $f_v(x_3) = 0$ for all $x_3 \in \gamma$:

$$\begin{bmatrix} x_1^2+x_2^2+1 & x_1^2+x_2^2+1 & x_1^2+x_2^2+1 & x_1^2+x_2^2+1 \end{bmatrix}\Bigg|_{x\in\Sigma} v = \mathbb{1}^\top v = 0\ ,$$

and further –no matter what choice of coefficients we make– all sliding vector fields will

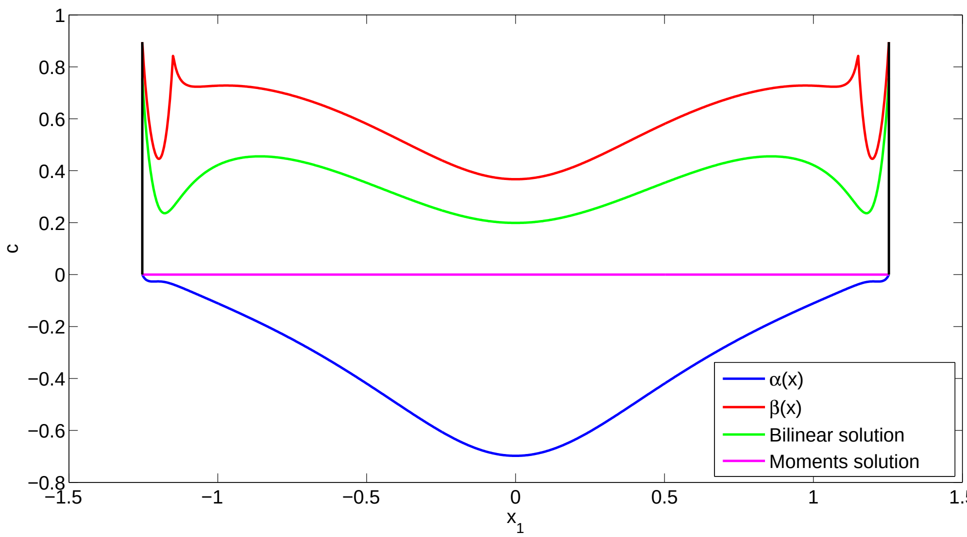


Figure 36: Curves of $\alpha(x_3)$, $\beta(x_3)$ defining the admissible region (1.4.13) for Example 5.2.2, and moments and bilinear solutions.

always be: $f_{\mathrm{F}}(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (that is, $\dot{x}_3 = 1$). As a consequence, the minimum variation requirement in (4.2.9) is ill-posed, as any $\lambda$ solution of (1.4.9) would provide the same sliding vector field. The minimum variation solution requirement in (4.2.5) is feasible, though, and indeed not all different choices of $\lambda$ will provide sets of coefficients that render the exiting vector fields.

The admissibility region for this problem (found from (1.4.13) using the moments solution as particular solution and the smooth eigenvector $v$ of Lemma 3.2.6), is the region comprised between the two curves in Figure 36 (these are $\alpha$ and $\beta$ in (1.4.13)). Looking at Figure 36, it is clear that, when the dynamics enters or exits from sliding motion on $\Sigma$, there are intervals of admissible values for $c$ in (1.4.12). At the same time, for a Filippov vector field to exit smoothly from $\Sigma$, it is necessary that its corresponding $\lambda$ coefficient coincides with $\lambda_{\mathrm{M}}$ at first order exit points (see [18]). Therefore, there is only one way to enter/exit smoothly from $\Sigma$ in this specific problem, and it is given by the end values of $c$ selected by the moments solution in Figure 36. For comparison, we also show the values of $c$ selected by the bilinear solution; since the end values do not coincide with those of the moments solution, we infer that the bilinear solution cannot be a minimum variation solution nor can be smoothly exiting.

## 5.3 A piecewise smooth dynamical system with a periodic orbit

In this section, we analyze in details Example 5.3.1 introduced in Chapter 1. For convenience, we recall it below.

**Example 5.3.1** ([28]). *Let us consider a non-smooth dynamical system, the solution of which slides on the intersection of two surfaces. In [28] the author studies a mechanical system composed by two blocks on a moving belt, as depicted in Figure 1. The velocity of the belt is constant and is called the driving velocity $v$. Each block is connected to a fixed support and to the other block by elastic springs. The surface between the blocks and the belt is rough so that the belt exerts a dry friction force on each block that sticks on the belt to the point where the elastic forces due to the springs exceed the maximum static force. At this point the blocks start slipping and the slipping motion will continue to the point where the velocity of the block will equal that of the belt and the elastic forces will be equilibrated by the static friction force. The continuous repetition of this type of motions generates a stick-slip oscillation. This mechanical system may be described in its simplest form by the following set of differential equations:*

$$\begin{cases} m_1 x_1'' = -k_1 x_1 - k_{12}(x_1 - x_2) + f_{k1}(x_1' - v), \\ m_2 x_2'' = -k_2 x_2 - k_{12}(x_2 - x_1) + f_{k2}(x_2' - v) \end{cases} \tag{5.3.1}$$

*where $x_i(t)$ is the displacement, $m_i$ is the mass, $f_{ki}(x_i' - v)$ the kinetic friction force of the i-th block, $k_1$, $k_2$, $k_{12}$ suitable constants. The kinetic force has the form $f_{k2}(x' -$*

$v) = \beta f_{k1}(x' - v)$ *with:*

$$f_{k1}(x'-v) := \begin{cases} \frac{1-\delta}{1-\gamma(x'-v)} + \delta + \eta(x'-v)^2, & x' < v, \\ -\frac{1-\delta}{1-\gamma(x'-v)} - \delta - \eta(x'-v)^2, & x' > v. \end{cases} \tag{5.3.2}$$

*Now, we fix* $m_1 = m_2 = k_1 = k_2 = -k_{12} = 1$, $\delta = 0$, $\gamma = 3$, $\eta = 0$, $v = 0.295$, $\beta = 1.301$. *Therefore, with these selections, the system in Figure 1 is described by*

$$\dot{x} = \begin{cases} f_1(x), & x_3 < v \text{ and } x_4 < v, \\ f_2(x), & x_3 < v \text{ and } x_4 > v, \\ f_3(x), & x_3 > v \text{ and } x_4 < v, \\ f_4(x), & x_3 > v \text{ and } x_4 > v, \end{cases} \tag{5.3.3}$$

*where* $f_i$, $i = 1, 2, 3, 4$, *are given by:*

$$\begin{aligned} f_1 &= \begin{bmatrix} x_3 \\ x_4 \\ -2x_1 + x_2 + \frac{1}{1-3(x_3-v)} \\ x_1 - 2x_2 + \frac{\beta}{1-3(x_4-v)} \end{bmatrix}, \quad f_2 = \begin{bmatrix} x_3 \\ x_4 \\ -2x_1 + x_2 + \frac{1}{1-3(x_3-v)} \\ x_1 - 2x_2 - \frac{\beta}{1-3(x_4-v)} \end{bmatrix}, \\ f_3 &= \begin{bmatrix} x_3 \\ x_4 \\ -2x_1 + x_2 - \frac{1}{1-3(x_3-v)} \\ x_1 - 2x_2 + \frac{\beta}{1-3(x_4-v)} \end{bmatrix}, \quad f_4 = \begin{bmatrix} x_3 \\ x_4 \\ -2x_1 + x_2 - \frac{1}{1-3(x_3-v)} \\ x_1 - 2x_2 - \frac{\beta}{1-3(x_4-v)} \end{bmatrix}. \end{aligned} \tag{5.3.4}$$

It will be useful for what follows to compute the attractivity region of (5.3.3), as sketched in Figure 37. In order to do so, we will analyze when the four subsliding vector fields $f_{\Sigma^{\pm}_{1,2}}$, determined from (5.3.4), point towards $\Sigma$ altogether. Computations

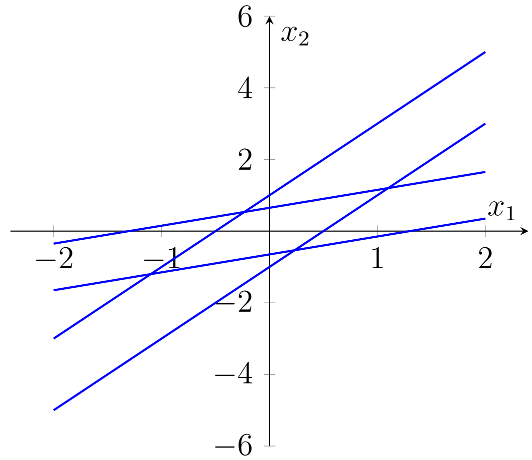


Figure 37: Attractivity region of (5.3.3).

provide the following vector fields:

$$f_{\Sigma_1^-} = \begin{bmatrix} v \\ x_4 \\ 0 \\ x_1 - 2x_2 + \frac{\beta}{1-3(x_4-v)} \end{bmatrix},$$

$$f_{\Sigma_1^+} = \begin{bmatrix} v \\ x_4 \\ 0 \\ x_1 - 2x_2 - \frac{\beta}{1-3(x_4-v)} \end{bmatrix},$$

$$f_{\Sigma_2^-} = \begin{bmatrix} x_3 \\ v \\ -2x_1 + x_2 + \frac{1}{1-3(x_3-v)} \\ 0 \end{bmatrix},$$

$$f_{\Sigma_2^+} = \begin{bmatrix} x_3 \\ v \\ -2x_1 + x_2 - \frac{1}{1-3(x_3-v)} \\ 0 \end{bmatrix}.$$

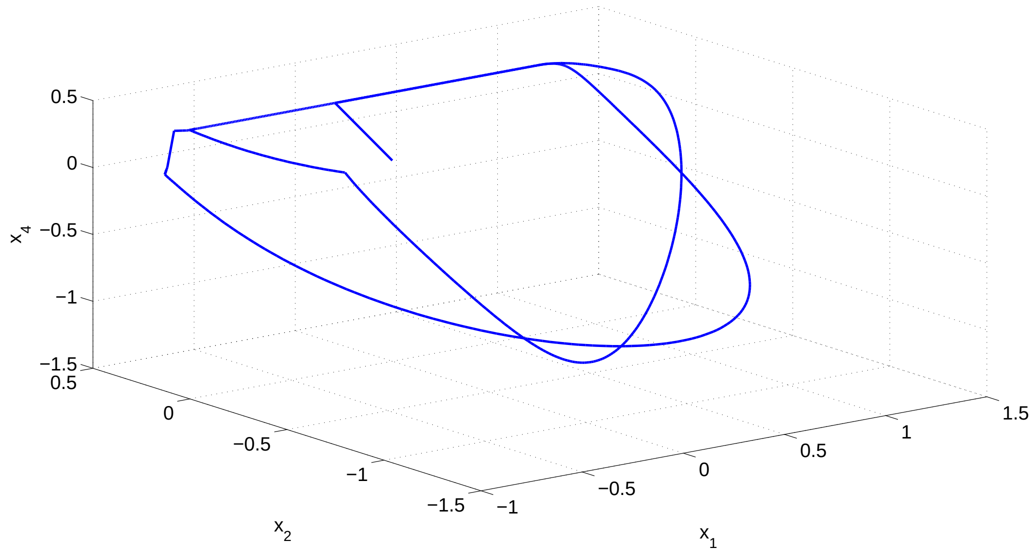


Figure 38: Projected periodic orbit, in the $x_1 - x_2 - x_4$ phase space, for the system (5.3.3) starting at $(0, 0, 0)$.

If we then look at $f_{\Sigma_1^\pm}$ and $f_{\Sigma_2^\pm}$, for the nature of (5.3.4), attractivity of $\Sigma$ requires that

$$\Sigma_1^- : x_1 - 2x_2 + \beta > 0,$$

$$\Sigma_1^+ : x_1 - 2x_2 - \beta < 0,$$

$$\Sigma_2^- : -2x_1 + x_2 + 1 > 0,$$

$$\Sigma_2^+ : -2x_1 + x_2 - 1 < 0.$$

We are going to prove that (5.3.3) provides one periodic orbit, as also shown in Figure 38.

Before embarking in proving the claimed result, we need a technical result.

**Lemma 5.3.2.** *Let $x_E \in \mathbb{R}^n$. Then $x_E \in \Sigma$ is an exit point in $R_1$, i.e. $\lambda_{M,1}(x_E) = 1$ and $\lambda_{M,i}(x_E) = 0$ for all $i \neq 1$, if and only if $w_1(x_E) = 0$.*

*Proof.* Let us assume that $w_1(x_E) = 0$: then $d_1(x_E) = 0$. Since the moments matrix at $x_E$, $M(x_E)$, is nonsingular and $\lambda_{\mathrm{M}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is the unique solution for the moments system, then $x_E$ is an exit point.

Let us assume that $\lambda_{M,1}(x_E) = 1$, $\lambda_{M,i}(x_E) = 0$ for $i \neq 1$. Therefore, because the moments matrix is nonsingular, $\lambda_M$ is the unique solution to the moments system

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Looking at the last row of the moments system, we get $d_1(x_E) = 0$, from which $w_1(x_E) = 0$. □

**Lemma 5.3.3.** *The problem* (5.3.3) *has four distinct first-order exit points on* $\Sigma$*.*

*Proof.* Straightforward computations, in light of Lemma 5.3.2, give that

$$E_1 := \begin{bmatrix} \frac{\beta+2}{3} \\ \frac{2\beta+1}{3} \\ v \\ v \end{bmatrix}, \quad E_2 := \begin{bmatrix} \frac{2-\beta}{3} \\ \frac{1-2\beta}{3} \\ v \\ v \end{bmatrix}, \quad E_3 := \begin{bmatrix} \frac{\beta-2}{3} \\ \frac{2\beta-1}{3} \\ v \\ v \end{bmatrix}, \quad E_4 := \begin{bmatrix} -\frac{\beta+2}{3} \\ -\frac{2\beta+1}{3} \\ v \\ v \end{bmatrix},$$

are the only first-order exit points that provide exits, respectively, in $R_1$, $R_2$, $R_3$ and $R_4$. □

**Lemma 5.3.4.** *If a solution trajectory hits the attractivity region of* (5.3.3)*, then it slides on the line*

$$x_2 - x_1 = \frac{\beta - 1}{3},$$

*that passes through the exit point* $E_1$*.*

*Proof.* Let us assume that the solution trajectory $x(t)$ reaches $\Sigma$ in its attractvity region (see Figure 37), in a point $B$: there the Filippov sliding vector field is

$$f_\Sigma = \begin{bmatrix} v \\ v \\ 0 \\ 0 \end{bmatrix}.$$

Let us stress that $x_1$, $x_2$ are monotonically increasing. Because of attractivity in $B$, it must hold that

$$\begin{cases} n_2^\top f_{\Sigma_1^-} > 0, \\ n_2^\top f_{\Sigma_1^+} < 0, \end{cases}$$

that is

$$\begin{cases} x_1 - 2x_2 + \beta > 0, \\ x_1 - 2x_2 - \beta < 0, \end{cases} \tag{5.3.5}$$

where the evaluation is at $B$. These relations comes from the fact that

$$f_\Sigma = (1 - \alpha_{\Sigma_1}) f_{\Sigma_1^-} + \alpha_{\Sigma_1} f_{\Sigma_1^+}, \tag{5.3.6}$$

where

$$\alpha_{\Sigma_1} := \frac{n_2^\top f_{\Sigma_1^-}}{n_2^\top (f_{\Sigma_1^-} - f_{\Sigma_1^+})}. \tag{5.3.7}$$

Further, it also holds, in $\Sigma$ attractivity region, that

$$\begin{cases} n_1^\top f_{\Sigma_2^-} > 0, \\ n_1^\top f_{\Sigma_2^+} < 0, \end{cases}$$

or, equivalently,

$$\begin{cases} -2x_1 + x_2 + 1 > 0, \\ -2x_1 + x_2 - 1 < 0, \end{cases} \tag{5.3.8}$$

where the evaluation is at $B$. These relations come from the fact that

$$f_\Sigma = (1 - \alpha_{\Sigma_2}) f_{\Sigma_2^-} + \alpha_{\Sigma_2} f_{\Sigma_2^+}, \tag{5.3.9}$$

where

$$\alpha_{\Sigma_2} := \frac{n_1^\top f_{\Sigma_2^-}}{n_1^\top (f_{\Sigma_2^-} - f_{\Sigma_2^+})}. \tag{5.3.10}$$

Since

$$x_1(t) = B_1 + vt, \tag{5.3.11a}$$

$$x_2(t) = B_2 + vt, \tag{5.3.11b}$$

then, at some time $T_B > 0$, first relation in (5.3.5) will be verified as an equality (the second being satisfied for all times):

$$B_1 - 2B_2 - vT_B + \beta = 0. \tag{5.3.12}$$

This says that $\Sigma$ is loosing attractivity: $f_\Sigma$, viewed as in (5.3.6), is aligning with some vector field exiting $\Sigma$ in $R_1$, $R_3$ or on $\Sigma_1^-$ (see (5.3.7)). Since $f_\Sigma$ is uniquely defined on $\Sigma$, at the same time $T_B$, looking at $f_\Sigma$ as in (5.3.9) says that we would get

$$-2B_1 + B_2 - vT_B + 1 = 0 \tag{5.3.13}$$

from combining (5.3.8) and (5.3.11): let us stress that second condition in (5.3.8) is satisfied at all times on $\Sigma$. From (5.3.12) and (5.3.13), we deduce that

$$B_1 - B_2 = \frac{1-\beta}{3}.$$

Since $E_1$ as in Lemma 5.3.3 belongs to the same line, and dynamics on $\Sigma$ is governed by (5.3.11), we conclude that the solution trajectory will slide from $B$ towards $E_1$, where it will exit $\Sigma$, entering the region $R_1$. $\square$

### 5.3.1 Analysis of the dynamics given by (5.3.3)

We are now able to show that the dynamics given by (5.3.3) provides a periodic orbit that passes through $E_1$: we will make also use of plots obtained from a computer-graphics routine. More specifically, we are going to prove the following steps:

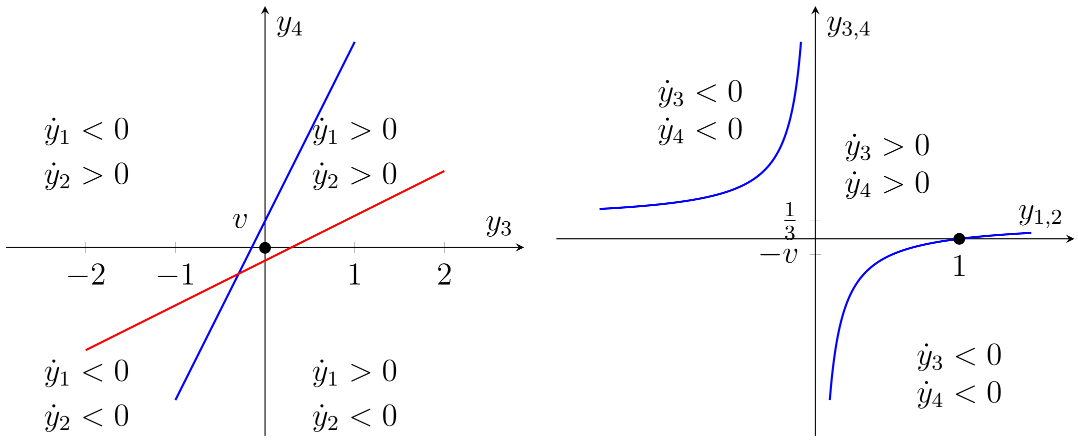


Figure 39: Behaviors of vector fields in the dynamical system (5.3.14).

1. Starting at $E_1$, the dynamics reaches $\Sigma_1^-$ in a region $\mathcal{R}$ that contains the point
$$A \approx \begin{bmatrix} -0.6539 \\ -0.7457 \\ 0.295 \\ -0.4383 \end{bmatrix} \text{ and that is attracted towards } \Sigma;$$

2. starting in $\mathcal{R}$, the dynamics reaches $\Sigma$ in a point $B$ where $\Sigma$ is attractive;

3. the dynamics continues on a line passing through $E_1$, moving towards it.

We refer to Lemma 5.3.3 for the notations below.

1. We first prove that, starting at $E_1$, that represents an exit point in $R_1$ for (5.3.3), we reach $\Sigma_1^-$. Let us then stress that we are looking at the dynamical system
$$\begin{cases} \dot x_1 = x_3, \\ \dot x_2 = x_4, \\ \dot x_3 = -2x_1 + x_2 + \dfrac{1}{1-3(x_3-v)}, \\ \dot x_4 = x_1 - 2x_2 + \dfrac{\beta}{1-3(x_4-v)}, \end{cases}$$

with initial condition $E_1 = \begin{bmatrix} \frac{\beta+2}{3} & \frac{2\beta+1}{3} & v & v \end{bmatrix}^\top$. Applying the change of coordinates

$$
\begin{aligned}
y_1 &:= 2x_1 - x_2, \\
y_2 &:= \frac{1}{\beta}(-x_1 + 2x_2), \\
y_3 &:= x_3 - v, \\
y_4 &:= x_4 - v,
\end{aligned}
$$

provides

$$
\begin{cases}
\dot{y}_1 = 2y_3 - y_4 + v, \\
\beta\dot{y}_2 = 2y_4 - y_3 + v, \\
\dot{y}_3 = -y_1 + \dfrac{1}{1-3y_3}, \\
\dfrac{1}{\beta}\dot{y}_4 = -y_2 + \dfrac{1}{1-3y_4},
\end{cases}
\tag{5.3.14}
$$

with initial condition $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\top$. We can think to split (5.3.14) into two coupled nonlinear oscillators

$$
\begin{cases}
\dot{y}_1 = 2y_3 - y_4 + v, \\
\dot{y}_3 = -y_1 + \dfrac{1}{1-3y_3}, \\
y_1(0) = 1, \\
y_3(0) = 0,
\end{cases}
\qquad
\begin{cases}
\beta\dot{y}_2 = 2y_4 - y_3 + v, \\
\dfrac{1}{\beta}\dot{y}_4 = -y_2 + \dfrac{1}{1-3y_4}, \\
y_2(0) = 1, \\
y_4(0) = 0,
\end{cases}
$$

In a sufficiently small neighborhood $V$ of $(1,0)$, since the vector field acting there is $\begin{bmatrix} v \\ 0 \end{bmatrix}$ for both dynamics, we have that

$$
\dot{y}_3 < \dot{y}_4 < 0 \quad \text{on } V,
\tag{5.3.15}
$$

being $\beta > 1$; therefore, because of the initial conditions, $y_3 < 0$ and $y_4 < 0$ in $V$; since we have already pointed out in (5.3.15) that $y_3$, $y_4$ are strictly decreasing then, in some finite time $t_1$, it will be

$$\dot{y}_1(t_1) = 2y_3(t_1) - y_4(t_1) + v = 0, \text{ and after } t_1 : \dot{y}_1 < 0,$$
$$\beta\dot{y}_2(t_1) = 2y_4(t_1) - y_3(t_1) + v > 0.$$

Therefore, at some time $t_2 > t_1$, $\dot{y}_1(t_2) > 0$, while $\dot{y}_2 = 0$: thus, after $t_2$, $y_2$ starts decreasing, whereas $y_3$ starting increasing hereafter for similar reasons; then, at some other time $t_3 > t_2$, $y_1$ will become decreasing, as well as $y_2$; now, by monotonicity, there will exist $T_1 > t_3$ such that

$$y_3(T_1) = 0, \quad y_4(T_1) < 0,$$

or, in terms of the original coordinates,

$$x_3(T_1) = v, \quad x_4(T_1) < v,$$

that is, the solution trajectory has reached $\Sigma_1^-$. Further, this event happens when $\dot{y}_4 < 0$ (see Figure 39), so that $\varphi^{T_1}(E_1) \in U$, where $U := \widetilde{U} \cap \Sigma_1^-$, being

$$\widetilde{U} := \left\{ x \in \mathbb{R}^4 \,:\, x_1 - 2x_2 + \frac{\beta}{1 - 3(x_4 - v)} > 0 \right\}.$$

It is then evident that $\widetilde{U}$ contains $A$, as defined above, and that it is an attractive subregion for $\Sigma$ in $\Sigma_1^-$.

2. Starting at $A = \begin{bmatrix} -0.6539 \\ -0.7457 \\ 0.295 \\ -0.4383 \end{bmatrix}$, the dynamics is governed by

$$\dot{x} = \begin{bmatrix} v \\ x_4 \\ 0 \\ x_1 - 2x_2 + \frac{\beta}{1-3(x_4-v)} \end{bmatrix}.$$

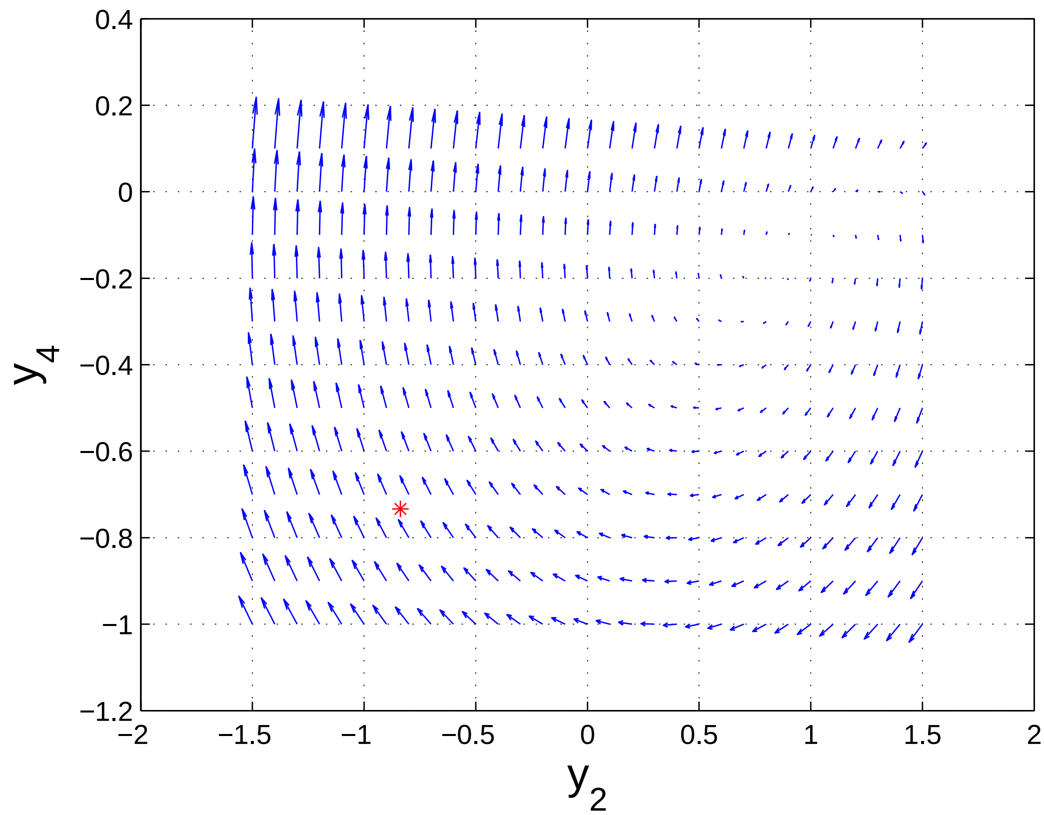


Figure 40: Vector field given by (5.3.16) in $y_2 - y_4$ plane. Point in red is the projection of $A$ in the new coordinates.

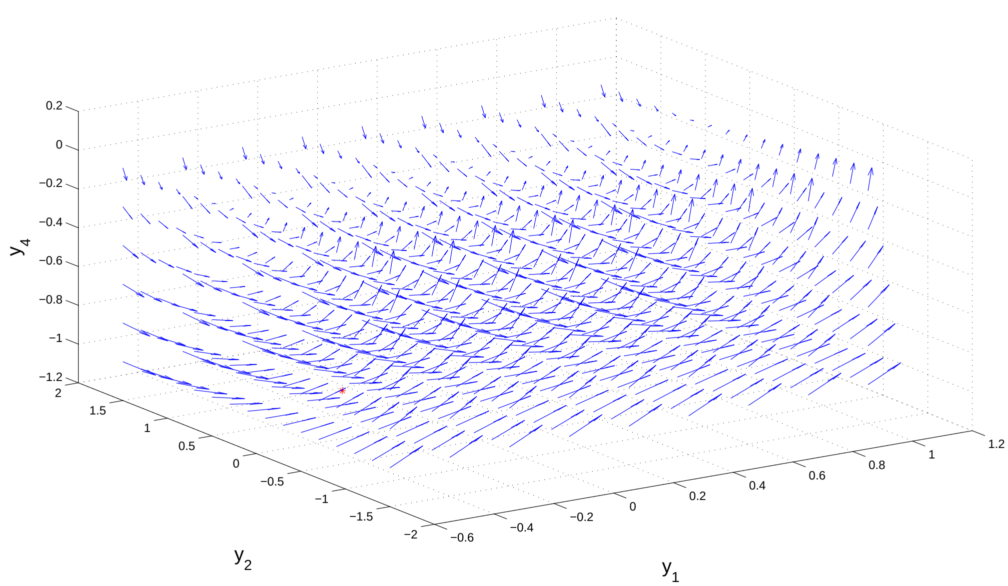


Figure 41: Vector field given by (5.3.16) in the $y_1 - y_2 - y_4$ space. Point in red is the projection of $A$ in the new coordinates.

Let us now define the following:

$$y_1 := 2x_1 - x_2,$$

$$y_2 := -x_1 + 2x_2,$$

$$y_3 := x_3 - v,$$

$$y_4 := x_4 - v.$$

In these new coordinates, dynamics becomes

$$\begin{cases} \dot{y}_1 = v - y_4, \\ \dot{y}_2 = 2y_4 + v, \\ \dot{y}_3 = 0, \\ \dot{y}_4 = -y_2 + \dfrac{\beta}{1 - 3y_4}, \end{cases} \tag{5.3.16}$$

and $A_y = \begin{bmatrix} -0.5621 \\ -0.8375 \\ 0 \\ -0.7333 \end{bmatrix}$.

We want to prove that, the point where the solution trajectory reaches $\Sigma$ belongs to its attractivity region: this is the case if and only if $-1 < 2x_1 - x_2 < 1$ and $-\beta < -x_1 + 2x_2 < \beta$, or, in new coordinates,

$$-1 < y_1 < 1,$$

$$-\beta < y_2 < \beta.$$

As we can see in Figure 40, dynamics of component $y_2$ from a neighborhood of $A$ is led to reach the interval $[-\beta, \beta]$ on $y_2$-axis. From Figure 41, it is also predictable that $y_1$ will fall into $[-1, 1]$. These facts say that, starting at $A \in \Sigma_1^-$, the solution trajectory will reach $\Sigma$ in its attractive region.

3. This step comes directly from Lemma 5.3.4.

# Chapter VI

# MOMENTS SLIDING VECTOR FIELD ON THE INTERSECTION OF THREE MANIFOLDS: NODALLY ATTRACTIVE CASE

In this Chapter we propose and prove an extension the moments' method to the co-dimension 3 case, under nodal attractivity conditions; we attempt, further, a definition of *general attractivity by subsliding* for the co-dimension 3 case. We also prove that our extension of the moments' method, under nodal attractivity conditions, can be further generalized to any co-dimension.

## 6.1 Introduction

Consider the following piecewise smooth system,

$$x'(t) = f_i(x),\ x \in R_i,\ \ i = 1, \dots, 8, \tag{6.1.1}$$

where the regions $R_i$'s are open, disjoint and connected sets of $\mathbb{R}^n$, so that $\mathbb{R}^n = \overline{\bigcup R_i}$, and on each region $R_i$ the function $f_i$ is smooth.

Moreover, the regions $R_i$'s are separated by manifolds defined as 0-sets of smooth (at least $\mathscr{C}^2$) scalar functions $h_i$: $\Sigma_i := \{x \in \mathbb{R}^n \ :\ h_i(x) = 0\}$, $i = 1, 2, 3$, which intersect pairwise and all three of them. For notational convenience, we use

$$\Sigma_{1,2} := \Sigma_1 \cap \Sigma_2\ ,\ \ \Sigma_{1,3} := \Sigma_1 \cap \Sigma_3\ ,\ \ \Sigma_{2,3} := \Sigma_2 \cap \Sigma_3\ ,$$

to describe the three possible co-dimension 2 discontinuity manifolds, and further

$$\Sigma_{1,2}^{\pm} := \{x \in \Sigma_1 \cap \Sigma_2\ :\ h_3(x) \gtrless 0\}$$

and similarly for $\Sigma_{1,3}^{\pm}$ and $\Sigma_{2,3}^{\pm}$. Finally,

$$\Sigma := \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$$

will be the co-dimension 3 manifold of interest to us.

Without loss of generality, we label the regions $R_i$'s as follows (see Figure 42 for an illustration of the situation)

$$R_1 := \{x \in \mathbb{R}^n \,:\, h_1(x) < 0, h_2(x) < 0, h_3(x) < 0\},$$
$$R_2 := \{x \in \mathbb{R}^n \,:\, h_1(x) < 0, h_2(x) < 0, h_3(x) > 0\},$$
$$R_3 := \{x \in \mathbb{R}^n \,:\, h_1(x) < 0, h_2(x) > 0, h_3(x) < 0\},$$
$$R_4 := \{x \in \mathbb{R}^n \,:\, h_1(x) < 0, h_2(x) > 0, h_3(x) > 0\},$$
$$R_5 := \{x \in \mathbb{R}^n \,:\, h_1(x) > 0, h_2(x) < 0, h_3(x) < 0\},$$
$$R_6 := \{x \in \mathbb{R}^n \,:\, h_1(x) > 0, h_2(x) < 0, h_3(x) > 0\},$$
$$R_7 := \{x \in \mathbb{R}^n \,:\, h_1(x) > 0, h_2(x) > 0, h_3(x) < 0\},$$
$$R_8 := \{x \in \mathbb{R}^n \,:\, h_1(x) > 0, h_2(x) > 0, h_3(x) > 0\} \ .$$

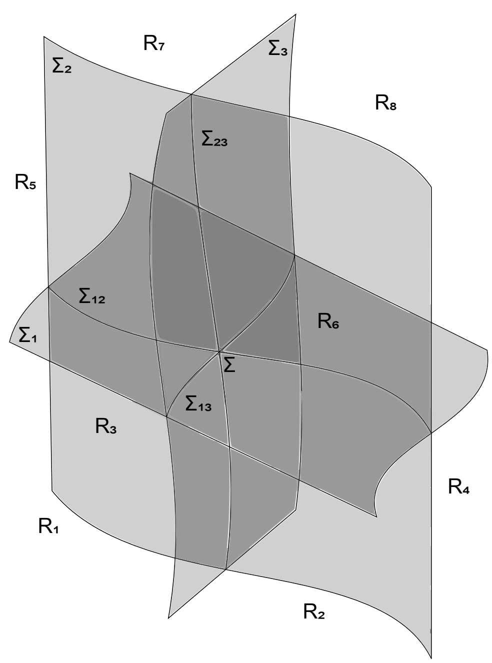


Figure 42: Regions and discontinuity surfaces.

Our goal is to describe a Filippov sliding vector field on $\Sigma$, which extends the moments vector field we proposed in [18] in the co-dimension 2 case.

### 6.1.1 Sliding vector field

We assume that $\{\nabla h_i(x)\}_{i=1,2,3}$ is a linearly independent set at any $x \in \Sigma$ and in a neighborhood of $\Sigma$.

For $x \in \Sigma$, define the projections of the vector fields $f_i$, $i = 1, \dots, 8$, onto the normal directions to the three manifolds:

$$w_i = \begin{bmatrix} w_i^1 \\ w_i^2 \\ w_i^3 \end{bmatrix} := \begin{bmatrix} \nabla h_1^\top f_i \\ \nabla h_2^\top f_i \\ \nabla h_3^\top f_i \end{bmatrix} \quad , \quad i = 1, \dots, 8 \; . \tag{6.1.2}$$

Consider the matrix $W \in \mathbb{R}^{3\times 8}$ (which depends smoothly on $x$):

$$W = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{bmatrix} \; . \tag{6.1.3}$$

Next, we assume that the manifold $\Sigma$ is **nodally attractive**, which we characterize by the following first order condition, that of course depends on the regions' labeling.

**Definition 6.1.1.** *We say that $\Sigma$ is nodally attractive if the matrix $W$ has the following sign pattern:*

$$\begin{bmatrix} + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \end{bmatrix} . \tag{6.1.4}$$

□

On $\Sigma$, we are interested in Filippov solutions of (6.1.1). In particular, we seek a sliding vector field of the form

$$f_{\mathrm{F}} \;=\; \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 + \lambda_5 f_5 + \lambda_6 f_6 + \lambda_7 f_7 + \lambda_8 f_8 \tag{6.1.5}$$

with positive coefficients $\lambda_i$'s adding to 1. Imposing that $f_{\text{F}}$ is tangent to $\Sigma$, gives the following underdetermined linear system

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{6.1.6}$$

where $W \in \mathbb{R}^{3\times 8}$ is defined as in (6.1.3). It is evident that (6.1.6) is an underdetermined system. In Corollary 6.1.7 below we will show that the matrix $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ has a four dimensional kernel; hence, to select a unique Filippov sliding vector field on $\Sigma$, the issue is how "to fix" the four available degrees of freedom. Again, we stress that we are specifically interested in smooth vector fields on $\Sigma$; for this reason, we seek solutions of (6.1.6) with positive components, and with the $\lambda_i$'s smoothly varying with $x \in \Sigma$, which we will call *admissible solutions*.

#### 6.1.1.1 *Trilinear (interpolant) vector field*

A possible choice to determine an admissible solution of (6.1.6), and a vector field as in (6.1.5), is to select $\lambda \in \mathbb{R}^8$ of the form

$$\lambda = \begin{bmatrix} (1-\alpha)(1-\beta)(1-\gamma) \\ (1-\alpha)(1-\beta)\gamma \\ (1-\alpha)\beta(1-\gamma) \\ 1-\alpha)\beta\gamma \\ \alpha(1-\beta)(1-\gamma) \\ \alpha(1-\beta)\gamma \\ \alpha\beta(1-\gamma) \\ \alpha\beta\gamma \end{bmatrix}, \tag{6.1.7}$$

where $\alpha$, $\beta$, $\gamma \in (0,1)$. Since the choice (6.1.7) clearly gives $\sum_i \lambda_i = 1$, one would need that $\alpha, \beta, \gamma \in [0,1]$ to have an admissible solution. Now, the relation (6.1.6) gives a nonlinear system of three equations in the three unknowns $\alpha, \beta, \gamma$. As proven in [23], when $\Sigma$ is nodally attractive, this nonlinear system always has a solution $\alpha,\ \beta,\ \gamma \in (0,1)$. The choice (6.1.7) is the "natural" extension to the co-dimension 3 case of the bilinear interpolant method, and it is important to observe that the choice (6.1.7) is consistent with the bilinear interpolant technique on the lower co-dimension manifolds; indeed, alternately setting one of $\alpha, \beta, \gamma$, to be 0 or 1, gives the 6 possible combinations needed for a sliding vector field on the relevant co-dimension 2 manifolds (namely, on $\Sigma^{\pm}_{1,2}, \Sigma^{\pm}_{1,3}, \Sigma^{\pm}_{2,3}$). For example, when $\gamma = 0$, one obtains the bilinear vector field on $\Sigma^{-}_{12}$, namely

$$(1-\alpha)\left[(1-\beta)f_1 + \beta f_3\right] + \alpha\left[(1-\beta)f_5 + \beta f_7\right] \ . \tag{6.1.8}$$

However, there is a difficulty with the formulation (6.1.7): even when $\Sigma$ is nodally attractive, in general there is more than one admissible solution of the nonlinear system; see Example 6.1.2 below.

**Example 6.1.2.** *Consider the following matrix $W$, which corresponds to a nodally attractive discontinuity surface $\Sigma$ (see Definition 6.1.1 and the sign pattern of* (6.1.4)*)*

$$W := \begin{bmatrix} 1 & 3 & 1 & 11 & -7 & -1 & -3 & -5 \\ 1 & 1 & -11 & -3 & 3 & 11 & -1 & -1 \\ 1 & -9 & 5 & -1 & 1 & -5 & 9 & -1 \end{bmatrix} .$$

*As reported in [21], searching for the trilinear solution* (6.1.7) *relative to the system*

$$W\lambda = 0_3 \ , \tag{6.1.9}$$

*gives two distinct solutions, associated to* $(\alpha, \beta, \gamma) = (1/2, 1/2, 1/2)$ *and to* $(\alpha, \beta, \gamma) \approx$

$(0.3316, 0.2913, 0.3080)$, *namely*

$$\lambda = \begin{bmatrix} 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \\ 1/8 \end{bmatrix} \quad \textit{and} \quad \lambda \approx \begin{bmatrix} 0.3268 \\ 0.1459 \\ 0.1347 \\ 0.06 \\ 0.1626 \\ 0.0724 \\ 0.0668 \\ 0.0298 \end{bmatrix}.$$

*(The Jacobian of the nonlinear system in* $(\alpha, \beta, \gamma)$ *associated to the first root is singular, as that root is double).*

### 6.1.2 Moments method

In case of a discontinuity manifold of co-dimension 2 (intersection of two co-dimension 1 manifolds), in [18] we proposed a methodology to select a uniquely defined sliding vector field of Filippov type, and we called the resulting method the *moments' method*. Here we propose an extension of the *moments' method* as a mean to provide a sliding vector field in case $\Sigma$ is of co-dimension 3.

Let us recall that, if $\Sigma$ from (6.1.1) is a co-dimension 2 manifold, intersection of two co-dimension 1 manifolds $\Sigma_1$, $\Sigma_2$, then computing the moments' solution amounts to solving the linear system

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{6.1.10}$$

where

$$M := \begin{bmatrix} W \\ \mathbb{1}^\top \\ d^\top \end{bmatrix}, \quad W := \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \quad d := \begin{bmatrix} \|w_1\| \\ -\|w_2\| \\ -\|w_3\| \\ \|w_4\| \end{bmatrix}, \tag{6.1.11}$$

with

$$w_i^j := \nabla h_j^\top f_i, \quad i = 1,2,3,4, \quad j = 1,2,$$

being $h_1$ and $h_2$ the event functions of which $\Sigma_1$, and $\Sigma_2$, are the 0-sets.

In [18] it is proven that $M$ is invertible whenever $\Sigma$ is attractive by subsliding, in particular when $\Sigma$ is nodally attractive, and that (6.1.10) provides a unique admissible solution $\lambda_{\mathrm{M}}$. For later reference, we summarize this special case in the following theorem.

**Theorem 6.1.3** ([18]). *Let* $W = \begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \end{bmatrix} \in \mathbb{R}^{2\times 4}$ *have the following sign pattern:*

$$\begin{bmatrix} + & + & - & - \\ + & - & + & - \end{bmatrix}, \tag{6.1.12}$$

*and let* $M$ *be defined as in* (6.1.11)*. Then the linear system* (6.1.10) *is nonsingular and has a unique admissible solution.* □

At this point, the key to understand how to provide the extension of the moments' method is to realize that –alongside the co-dimension 3 manifold $\Sigma$– there are also several lower co-dimension manifolds where solution trajectories can slide, approaching $\Sigma$. Specifically, in a neighborhood of $\Sigma$, there are three co-dimension 1 manifolds (namely, $\Sigma_1$, $\Sigma_2$, $\Sigma_3$), and three co-dimension 2 manifolds, namely $\Sigma_{1,2}$, $\Sigma_{1,3}$, $\Sigma_{2,3}$. Now, under the assumption of nodal attractivity of $\Sigma$, there is a unique Filippov sliding vector field on the co-dimension 1 manifolds, but there is an ambiguity of how to select a Filippov sliding vector field on the co-dimension 2 manifolds.

Therefore, to arrive at an appropriate extension of the moments' method, we will need to insist that on the co-dimension 2 manifolds we are using the moments' vector field as sliding vector field. We will need to further make sure that an appropriate distinction is made between the cases of $\Sigma^+_{1,2}$ and $\Sigma^-_{1,2}$, since different vector fields enter in the convex combination defining the moments sliding vector field in these cases (and similarly for $\Sigma^+_{13}$ and $\Sigma^-_{1,3}$, and $\Sigma^+_{2,3}$ and $\Sigma^-_{2,3}$).

Guided by the above consideration, our idea is to normalize (6.1.6) in the same fashion of co-dimension 2 which leads to consider precisely the matrix of "signed" partial distances (6.1.13). To witness, consider the sub-surface $\Sigma_{2,3}$, that is the subset of $x \in \mathbb{R}^3$ for which $h_2(x) = 0$ and $h_3(x) = 0$. Looking at the sign pattern of $W$ in (6.1.4), we notice that two natural sets of vertices $w_i$'s arise, namely $\{w_1, w_2, w_3, w_4\}$ and $\{w_5, w_6, w_7, w_8\}$, according to the sign of their first component: the first four vertices have $w_i^1 > 0$, $i = 1, 2, 3, 4$; the last four vertices have $w_i^1 < 0$, $i = 5, 6, 7, 8$. Moreover, the sign pattern of $\begin{bmatrix} w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 \end{bmatrix}$ and $\begin{bmatrix} w_5^2 & w_6^2 & w_7^2 & w_8^2 \\ w_5^3 & w_6^3 & w_7^3 & w_8^3 \end{bmatrix}$ is the same as that in (6.1.12), that is the nodal attractivity sign pattern in co-dimension 2. This implies that the two sets $\{w_1, w_2, w_3, w_4\}$ and $\{w_5, w_6, w_7, w_8\}$ are determining subsliding towards $\Sigma$, on $\Sigma^+_{2,3}$ and $\Sigma^-_{2,3}$ respectively. From Theorem 6.1.3, we know that the moments vector fields on $\Sigma^{\pm}_{2,3}$ is well defined. This means that, on $\Sigma^+_{2,3}$, there are unique admissible solutions of

$$\begin{bmatrix} w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 \\ 1 & 1 & 1 & 1 \\ \delta_1^{23} & -\delta_2^{23} & -\delta_3^{23} & \delta_4^{23} \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and of

$$\begin{bmatrix} w_5^2 & w_6^2 & w_7^2 & w_8^2 \\ w_5^3 & w_6^3 & w_7^3 & w_8^3 \\ 1 & 1 & 1 & 1 \\ \delta_5^{23} & -\delta_6^{23} & -\delta_7^{23} & \delta_8^{23} \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} ,$$

where $\delta_i^{23} := \sqrt{(w_i^2)^2 + (w_i^3)^2}$, $i = 1, \ldots, 8$. This implies that –within the moments' method framework– we must regularize those two blocks with the corresponding partial distance vector relative to $\Sigma_{2,3}$: we then choose to append the row

$$\begin{bmatrix} \delta_1^{23} & -\delta_2^{23} & -\delta_3^{23} & \delta_4^{23} & \delta_5^{23} & -\delta_6^{23} & -\delta_7^{23} & \delta_8^{23} \end{bmatrix}$$

to $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ in order to obtain consistency with the moments solution on $\Sigma_{2,3}$. Analogous reasoning relative to $\Sigma_{1,2}^{\pm}$ and $\Sigma_{1,3}^{\pm}$ leads us to regularize $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$ by appending to it the matrix

$$\Delta := \begin{bmatrix} \delta_1^{23} & -\delta_2^{23} & -\delta_3^{23} & \delta_4^{23} & \delta_5^{23} & -\delta_6^{23} & -\delta_7^{23} & \delta_8^{23} \\ \delta_1^{13} & -\delta_2^{13} & \delta_3^{13} & -\delta_4^{13} & -\delta_5^{13} & \delta_6^{13} & -\delta_7^{13} & \delta_8^{13} \\ \delta_1^{12} & \delta_2^{12} & -\delta_3^{12} & -\delta_4^{12} & -\delta_5^{12} & -\delta_6^{12} & \delta_7^{12} & \delta_8^{12} \end{bmatrix} , \tag{6.1.13}$$

where, for each $i = 1, \ldots, 8$,

$$\delta_i^{23} := \sqrt{(w_i^2)^2 + (w_i^3)^2} \ , \ \ \delta_i^{13} := \sqrt{(w_i^1)^2 + (w_i^3)^2} \ , \ \ \delta_i^{12} := \sqrt{(w_i^1)^2 + (w_i^2)^2} \ .$$

Notice that, when $\delta_i^{jk} \neq 0$ for all $i = 1, \ldots, 8$ and $j, k = 1, 2, 3$ (e.g., this is guaranteed when (6.1.4) holds for the signs of the entries of $W$), the sign pattern of $\Delta$ is

$$\begin{bmatrix} + & - & - & + & + & - & - & + \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \end{bmatrix} . \tag{6.1.14}$$

Finally, we assemble the matrix

$$M := \begin{bmatrix} W \\ \Delta \\ d^\top \\ \mathbb{1}^\top \end{bmatrix} , \tag{6.1.15}$$

where $\mathbb{1} \in \mathbb{R}^8$ is the vector of all 1's, reflecting the convexity requirement, and

$$\begin{aligned} d^\top &:= \begin{bmatrix} d_1 & -d_2 & -d_3 & d_4 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix}, \\ d_i &:= \|w_i\|_2,\ i = 1, \dots, 8 \ , \end{aligned} \tag{6.1.16}$$

formally expresses our proposal of *weights* to place on the vertices $w_i$'s, $i = 1, \dots, 8$, to maintain the geometrical flavor of *moments* (so to make the origin the barycenter of the polytope).

**Definition 6.1.4.** *The matrix $M$* (6.1.15) *is called the* moments matrix*, and the* moments' method *(on $\Sigma$) consists in solving*

$$M\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.1.17}$$

*for $\lambda$, and then using this $\lambda$ in the construction of the sliding vector field* (6.1.5)*, which will be called* moments vector field*.*

Before stating and proving the fundamental results relative to this construction, we need some preliminary results.

**Lemma 6.1.5.** *Let $A \in \mathbb{R}^{n \times m}$, $n < m$, be full rank, and let $b \in \mathbb{R}^n$. Consider the system*

$$Ax = b \ , \tag{6.1.18}$$

*and let $d \in \mathbb{R}^m$ be a nonzero vector.*

*If there exist* $x$ *and* $y$ *solutions of* (6.1.18)*, such that*

$$d^\top x = \xi \ , \quad \text{and} \quad d^\top y = \eta \ ,$$

*with* $\xi \neq \eta$*, then* $\begin{bmatrix} A \\ d^\top \end{bmatrix}$ *has rank* $n+1$*.*

*Proof.* By hypothesis, $\dim \ker(A) = m - n$. Let then $V \in \mathbb{R}^{m\times(m-n)}$ be such that $\operatorname{range}(V) = \ker(A)$, and by contradiction suppose that $d \in \operatorname{range}(A^\top)$. Then we must have

$$d^\top V c = 0,$$

for all $c \in \mathbb{R}^{m-n}$. Since both $x$ and $y$ are solutions of (6.1.18), then there exists $\overline{c} \in \mathbb{R}^{m-n}$ such that

$$y = x + V\overline{c}.$$

Therefore

$$\eta = d^\top y = d^\top x + d^\top V\overline{c} = \xi \ ,$$

and this contradicts the assumption $\xi \neq \eta$. Hence, $\begin{bmatrix} A \\ d^\top \end{bmatrix}$ has full rank $n+1$. $\square$

Next, we have the following simple result.

**Lemma 6.1.6.** *Let* $W$ *satisfy the sign pattern of* (6.1.4)*. Then*

$$\operatorname{rank} W = 3.$$

*Proof.* By the sign pattern of $W(2:3, 1:2)$, $\operatorname{rank} W \geq 2$. If, by contradiction, $\operatorname{rank} W = 2$, then $w_i \in \operatorname{span}\{w_1, w_2\}$ for all $i = 3, \dots, 8$; nonetheless, no linear combination of $w_1$, $w_2$ can match the signs of all $w_i$, $i = 3, \dots, 8$, at once. $\square$

Finally, we have the anticipated result.

**Corollary 6.1.7.** *Let* $\tilde{W} := \begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$*. Then* $\operatorname{rank} \tilde{W} = 4$*, hence* $\ker(\tilde{W})$ *is 4-dimensional.*

*Proof.* Because of Theorem 6.1.3, the matrix $\begin{bmatrix} w^2 \\ w^3 \\ \mathbb{1}^\top \end{bmatrix}$ contains a non-singular submatrix, hence it must have rank 3. Let us then consider the system

$$\begin{bmatrix} w^2 \\ w^3 \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By Theorem 6.1.3, considering the first four columns and the last four columns of $\begin{bmatrix} w^2 \\ w^3 \\ \mathbb{1}^\top \end{bmatrix}$, there exist the two corresponding moments solutions $\lambda$ and $\mu$ to these system with the following structures:

$$\lambda = \begin{bmatrix} * \\ * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \\ * \\ * \end{bmatrix},$$

and all their entries are nonnegative. Therefore, considering the extended matrix $\tilde{W} = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \\ \mathbb{1}^\top \end{bmatrix}$ and exploiting the sign pattern of $W$ in (6.1.4), we obtain that

$$\tilde{W}\lambda = \begin{bmatrix} > 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \tilde{W}\mu = \begin{bmatrix} < 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, using Lemma 6.1.5, we get that $\tilde{W}$ has rank 4. $\square$

The following results completely justify the moments' method for the co-dimension 3 case under nodal attractivity conditions.

**Theorem 6.1.8.** *Let* $w_i = \begin{bmatrix} w_i^1 \\ w_i^2 \\ w_i^3 \end{bmatrix}$, $i = 1, \ldots, 8$, *be eight vectors in* $\mathbb{R}^3$, *and consider the matrix* $W \in \mathbb{R}^{3\times 8}$ *given by*

$$W := \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{bmatrix} . \tag{6.1.19}$$

*Assume that the entries of* $W$ *are nonzero and have the sign pattern as in* (6.1.4). *Then, the matrix* $M$ *as in* (6.1.15) *is invertible.*

*Proof.* For $i = 1, \ldots, 8$, let $v_i$ be the $i$-th column of $\begin{bmatrix} W \\ \Delta \\ d^\top \end{bmatrix}$. Our proof will consist of showing that the $v_i$'s are affinely independent.

Let also define $\hat{v}_i$ to be the columns given by $\text{sign}(v_i)$, $i = 1, \ldots, 8$; e.g., $\hat{v}_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

Observe that the $\hat{v}_i$'s are affinely independent, since the matrix $\begin{bmatrix} \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_8 \\ 1 & 1 & \cdots & 1 \end{bmatrix}$ is trivially invertible. Indeed, it is immediate to realize that $\sum_{i=1}^8 \hat{v}_i = 0$ and –a fortiori– $\frac{1}{8}\sum_{i=1}^8 \hat{v}_i = 0$. Also, observe that $\hat{v}_i^\top v_i = \|v_i\|_1 > 0$, $i = 1, \ldots, 8$. Define

$$A := \left\{ x \in \mathbb{R}^7 \,:\, \tilde{v}_i^\top x \le 1,\, i = 1, \ldots, 8 \right\} ,$$

where

$$\tilde{v}_i := \frac{\hat{v}_i}{\hat{v}_i^\top v_i}.$$

Observe that $v_i \in A$, $i = 1, \ldots, 8$. To show that the $v_i$'s are affinely independent, our task will be to show that $A$ is a 7-simplex, from which the result will then follow. To reach our scope, we resort to the relation between $A$ and the dual of the set $B$ defined next.

Let

$$B := \text{conv}\{ \tilde{v}_i \,:\, i = 1, \ldots, 8 \} .$$

**Claim 6.1.9.** *The vectors $\tilde{v}_i$, $i = 1, \ldots, 8$, are affinely independent.*

*Proof of Claim 6.1.9.*

(i) Consider the matrices $\hat{V} = \begin{bmatrix} \hat{v}_1 & \cdots & \hat{v}_8 \end{bmatrix} \in \mathbb{R}^{7\times 8}$, which is of rank 7, and $\tilde{V} = \begin{bmatrix} \tilde{v}_1 & \cdots & \tilde{v}_8 \end{bmatrix}$. By the definition of the vectors $\tilde{v}_i$'s, it follows that $\tilde{V} = \hat{V} D^{-1}$, with $D = \operatorname{diag}(\|v_i\|_1,\ i = 1, \ldots 8)$, and hence $\tilde{V}$ is also of rank 7.

(ii) We know that $0 = (1/8)\sum_{i=1}^{8} \hat{v}_i$, and therefore also $0 = \sum_{i=1}^{8} \tau_i \tilde{v}_i$, with $\tau_i = \frac{\|v_i\|_1}{\sum_{j=1}^{8} \|v_j\|_1}$, $i = 1 \ldots, 8$. Let $\tau := \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_8 \end{bmatrix}$, so that $\tilde{V}\tau = 0_7$, and also $\sigma := 2\tau$. Since $1 = \mathbb{1}^\top \tau \neq \mathbb{1}^\top \sigma = 2$, then using Lemma 6.1.5, the matrix $\begin{bmatrix} \tilde{V} \\ \mathbb{1}^\top \end{bmatrix}$ is invertible, and the claim follows. □

Naturally, from Claim 6.1.9 it follows that $B$ is a 7-simplex.

Next, consider $B^\circ$, the polar of $B$ (see [4]):

$$B^\circ := \{x \in \mathbb{R}^7 \,:\, y^\top x \leq 1, \,\forall\, y \in B\}.$$

We claim that $B^\circ = A$. To verify this claim, observe that straightforwardly $B^\circ \subseteq A$. On the other hand, let $x \in A$ and pick an arbitrary $y \in B$: then $y = \sum_{i=1}^{8} \xi_i \tilde{v}_i$, with $\sum_{i=1}^{8} \xi_i = 1$, $\xi_i \geq 0$, $i = 1, \ldots, 8$, and further

$$y^\top x = \sum_{i=1}^{8} \xi_i \tilde{v}_i^\top x \leq \sum_{i=1}^{8} \xi_i = 1$$

saying that $A \subseteq B^\circ$, and the claim is verified.

Next, following Grünbaum, [32, pag.48, Exercise 5.(vii)], we claim that $A$ is bounded if and only if $0 \in \operatorname{int}\operatorname{conv} B$. In fact, let us first prove that, for any set $C \in \mathbb{R}^d$,

$$C \text{ is bounded } \iff 0 \in \operatorname{int} C^\circ. \tag{6.1.20}$$

If $C$ is bounded, then there exists $\delta > 0$ such that $C \subseteq B_\delta(0)$, and by a property of the polar mapping, $B_{\frac{1}{\delta}}(0) \subseteq C^\circ$, from which $0 \in \mathrm{int}C^\circ$. Viceversa, if $0 \in \mathrm{int}C^\circ$, then there exists $\delta > 0$ such that $B_\delta(0) \subseteq C^\circ$, from which $C \subseteq C^{\circ\circ} \subseteq B_{\frac{1}{\delta}}(0)$, meaning that $C$ is bounded.

From this result, using $A^\circ$ instead of $C$, we can conclude that $A^\circ$ is bounded if and only if $0 \in \mathrm{int}A^{\circ\circ}$. Since $A^{\circ\circ} = \mathrm{clconv}(A \cup \{0\})$, then $\mathrm{int}A^{\circ\circ} = \mathrm{intconv}A$.

Therefore, since $0 \in \mathrm{int}B$, then $A$ is a bounded polyhedron, and therefore is a polytope. Next, we will establish that $A$ is a 7-simplex.

To begin with, since $B$ is bounded, then (see (6.1.20)) $0_7 \in \mathrm{int}B^\circ$, and so $0_7 \in \mathrm{int}A$ and hence $A$ is 7-dimensional. A simple computation also shows that $A$ is convex, containing the straight line segment between any of its two points. Therefore, from the Krein-Millman theorem (see [4]), $A$ is the convex hull of $m$ vertices (and $m \geq 8$). Next, since $B$ is a 7-simplex, then it is a polyhedron with 8 facets, and can thus be expressed as

$$B = \{x \in \mathbb{R}^7 \,:\, s_i^\top x \leq 1,\, i = 1, \ldots, 8\}\ .$$

Referring to Barvinok (see [4, p.144, problem 3]), we have that if $B := \{x \in \mathbb{R}^7 : s_i^\top x \leq 1,\, i = 1, \ldots, 8\}$, then $B^\circ = \mathrm{conv}(s_1, \ldots, s_8, 0)$[1]. Using this result, since for us $B^\circ = A$ and $0_7 \in \mathrm{int}A$, then we have that $A = \mathrm{conv}\{s_1, \ldots, s_8\}$. As we already know that $\dim A = 7$, it follows that $s_1, \ldots, s_8$, are affinely independent, and therefore $A$ is a 7-simplex, and $s_i$, $i = 1, \ldots, 8$, are its vertices.

Finally, we are going to show that the vectors $v_i$, $= 1, \ldots, 8$, are affinely independent.

Since each $v_i$ is in $A$, and it is a convex combination of the $s_i$'s (which are affinely independent), then we claim that the $v_i$'s must also be affinely independent. In fact,

[1] In fact, let $B := \{x \in \mathbb{R}^7 \,:\, s_i^\top x \leq 1,\, i = 1, \ldots, 8\}$ and $A := \mathrm{conv}(s_1, \ldots, s_8, 0)$. Then $A^\circ = B$ by a previous result, and so $B^\circ = A^{\circ\circ} = \mathrm{clconv}(A \cup \{0\}) = \mathrm{conv}\{s_1, \ldots, s_8, 0\} = A$.

for all $i = 1, \ldots, 8$, let

$$v_i = \sum_{j=1}^{8} \mu_j^{(i)} s_j, \quad \mu_j^{(i)} \geq 0, \quad \sum_{j=1}^{8} \mu_j^{(i)} = 1 \ .$$

Let, by contradiction, $\{v_i \ : \ i = 1, \ldots, 8\}$ be affinely dependent: then there exist $\xi_1, \ldots, \xi_8$, not all of them zero, such that $\sum_{i=1}^{8} \xi_i = 0$ and $\sum_{i=1}^{8} \xi_i v_i = 0$. Therefore

$$\sum_{i=1}^{8} \xi_i \left( \sum_{j=1}^{8} \mu_j^{(i)} s_j \right) = 0,$$

and then

$$\sum_{j=1}^{8} \left( \sum_{i=1}^{8} \xi_i \mu_j^{(i)} \right) s_j = 0.$$

Moreover

$$\sum_{j=1}^{8} \sum_{i=1}^{8} \xi_i \mu_j^{(i)} = \sum_{i=1}^{8} \xi_i \left( \sum_{j=1}^{8} \mu_j^{(i)} \right) = \sum_{i=1}^{8} \xi_i = 0 \ ,$$

which implies that $\{s_j \ : \ j = 1, \ldots, 8\}$ is affinely dependent, which is not true.

Therefore, since the vectors $v_i$, $i = 1, \ldots, 8$, are affinely independent, the matrix (see (6.1.15))

$$M = \begin{bmatrix} W \\ \Delta \\ d^\top \\ \mathbb{1}^\top \end{bmatrix}$$

is invertible, and the proof of Theorem 6.1.8 is completed. $\square$

From Theorem 6.1.8, (6.1.17) has a unique solution $\lambda_M$. We further notice that this solution can be written using $M_{\text{adj}}$, the adjugate of $M$, as

$$\lambda_M = \frac{1}{\det M} M_{\text{adj}}(8, :)^\top \ ,$$

where

$$M_{\mathrm{adj}}(8,:)^\top = \begin{bmatrix} \det\begin{bmatrix} w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ -d_2 & -d_3 & d_4 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} \\ -\det\begin{bmatrix} w_1 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \Delta_1 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ d_1 & -d_3 & d_4 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} \\ \det\begin{bmatrix} w_1 & w_2 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ d_1 & -d_2 & d_4 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} \\ -\det\begin{bmatrix} w_1 & w_2 & w_3 & w_5 & w_6 & w_7 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ d_1 & -d_2 & -d_3 & -d_5 & d_6 & d_7 & -d_8 \end{bmatrix} \\ \det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_6 & w_7 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_6 & \Delta_7 & \Delta_8 \\ d_1 & -d_2 & -d_3 & d_4 & d_6 & d_7 & -d_8 \end{bmatrix} \\ -\det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_7 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_7 & \Delta_8 \\ d_1 & -d_2 & -d_3 & d_4 & -d_5 & d_7 & -d_8 \end{bmatrix} \\ \det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_8 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_8 \\ d_1 & -d_2 & -d_3 & d_4 & -d_5 & d_6 & -d_8 \end{bmatrix} \\ -\det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 \\ d_1 & -d_2 & -d_3 & d_4 & -d_5 & d_6 & d_7 \end{bmatrix} \end{bmatrix}.$$

In light of this expression, we have the following result.

**Lemma 6.1.10.** *When $W$ has the sign pattern of* (6.1.4) *(i.e., when $\Sigma$ is nodally attractive), it holds that*

$$\lambda_{M,i} \neq 0, \quad \forall i = 1, \dots, 8.$$

*Proof.* Since $M$ is nonsingular by Theorem 6.1.8, then any collection of seven of its columns is linearly independent, and this remains true once their eight-th component equal to one is removed. This implies that none of the entries of $M_{\mathrm{adj}}(8,:)^\top$ can be zero, that is $\lambda_{M,i} \neq 0$ for all $i = 1, \dots, 8$. $\square$

**Theorem 6.1.11.** *With $M$ as in Theorem 6.1.8, consider $\lambda_M$ the unique solution of* (6.1.17). *Then, $\lambda_M$ has all positive components: $\lambda_{M,i} > 0$, $i = 1 \dots, 8$.*

*Proof.* Define the homotopy

$$\tilde{W}(t) := (1-t)\hat{W} + tW \ , \quad 0 \le t \le 1 \ ,$$

where $W$ is the matrix of our problem (see (6.1.19)), and $\hat{W}$ is the matrix with the same sign pattern of $W$, but all entries equal to 1 in absolute value: $\hat{W} = \mathrm{sign}W$. Naturally, for all $t \in [0,1]$, $\tilde{W}(t)$ has the same sign pattern (6.1.4) as the given $W$.

According to the definition of $\tilde{W}(t)$, we define the function

$$\tilde{M}(t) := \begin{bmatrix} \tilde{W} \\ \tilde{\Delta} \\ \tilde{d}^{\top} \\ \mathbb{1}^{\top} \end{bmatrix}$$

where $\tilde{\Delta}$ and $\tilde{d}$ are defined just as $\Delta$ and $d$ were, but relative to $\tilde{W}$. Let $\hat{M} = \tilde{M}(0)$ and note that $\tilde{M}(1) = M$, the given original moments matrix.

Now, $\hat{M}$ is nonsingular because of Theorem 6.1.8, and the unique solution of

$$\hat{M}\hat{\lambda} = \begin{bmatrix} 0_7 \\ 1 \end{bmatrix}$$

is easily found to be

$$\hat{\lambda}_i = \frac{1}{8}, \quad i = 1, \ldots, 8.$$

Moreover, since $\tilde{M}$ is continuous in $t$, and $\tilde{M}$ corresponds to a moments' matrix relative to a nodally attractive configuration, not only $\tilde{M}$ is invertible for all $t \in [0, 1]$, but because of Lemma 6.1.10 no component of the solution $\tilde{\lambda}(t)$ of

$$\tilde{M}(t)\tilde{\lambda}(t) = \begin{bmatrix} 0_7 \\ 1 \end{bmatrix}$$

can be 0 for any $t \in [0, 1]$, and thus have to be positive (since they are so at $t = 0$). But, since $\tilde{M}(1)$ is exactly the moments' matrix $M$ in which we are interested, we thus obtain that

$$\lambda_{M,i} > 0, \quad i = 1, \ldots, 8,$$

which concludes the proof of Theorem 6.1.11. □

**Remark 6.1.12.** *There are several works in linear algebra about sign-invertibility of a matrix, that is relying solely on the signs of the entries of the given matrix; see the works of Thomassen, [51], and the comprehensive treatment in [9]. For example,* *if the matrix $M$ in* (6.1.15) *were an L-matrix, then it would be possible to establish its invertibility and signs of the entries of the inverse by appealing to these results. Unfortunately, however, our matrix $M$ in* (6.1.15) *is not an L-matrix, and none of the existing results on sign-invertibility of matrices can be used to establish that $M$ is invertible (Theorem 6.1.8) nor of course that the solution of the system in Theorem 6.1.11 is positive. For this reason, and motivated by the specific geometric structure of our problems, we have resorted to a proof which uses tools from convex geometry.*

**Remark 6.1.13.** *Our proof of Theorem 6.1.8 (from which Theorem 6.1.11 followed as well) hinged on the key fact that the vectors $\hat{v}_i$, $i = 1, \dots, 8$, were affinely independent, and that the associated vectors $\tilde{v}_i$'s were so as well (see Claim 6.1.9). For us, affine independence of the $\hat{v}_i$'s and $\tilde{v}_i$'s, was a consequence of nodal attractivity of $\Sigma$, and this was the only property we have used that came from the dynamics of the differential system under study. Because of these considerations, the result (i.e., invertibility of the matrix $M = \begin{bmatrix} W \\ \Delta \\ d^\top \\ 1^\top \end{bmatrix}$) would still hold true every time one has a matrix $W$ leading to affinely independent vectors $\hat{v}_i$'s and $\tilde{v}_i$'s. This includes many more cases of attractive $\Sigma$ than just that of nodally attractive $\Sigma$.*

As a consequence of Theorems 6.1.8 and 6.1.11, and under the assumptions therein, we thus have that the moments' method selects a unique solution $\lambda$ with positive entries, and a unique sliding vector field (further, varying smoothly, since so do the entries of the matrix $M$).

**Example 6.1.14.** *With the matrix $W$ as in Example 6.1.2, and forming $M$ as in* (6.1.15)*, the unique moments solution $\lambda_M$, computed according to Theorem 6.1.11 and* *relative to* (6.1.9), *is given by*

$$\lambda_M \approx \begin{bmatrix} 0.4492 \\ 0.0502 \\ 0.0327 \\ 0.1019 \\ 0.0492 \\ 0.0279 \\ 0.0321 \\ 0.2569 \end{bmatrix}.$$

**Remark 6.1.15.** *In the present case of $\Sigma$ of co-dimension 3, to prove our results on the feasibility of the moments method, we are assuming that $\Sigma$ is nodally attractive. Extensive computational evidence indicates that the method proposed herein continues to provide a unique solution with nonnegative entries also under more general attractivity configurations of $\Sigma$. Although we have not attempted a complete proof to include all other possible cases, we note that the proof of Theorem 6.1.8 (and thus also Theorem 6.1.11) holds under more generous assumptions that those of nodal attractivity only; see Remark 6.1.13*

## 6.2 Extensions

### 6.2.1 General attractivity by subsliding

We want now propose a definition for *general attractivity by subsliding* for co-dimension 3, generalizing the same one given for co-dimension 2 presented in [18]. We first label each subportion of $\Sigma_i$:

$$\Sigma_1^{--} := \{x \in \Sigma_1 \,:\, h_2(x) < 0,\, h_3(x) < 0\},$$
$$\Sigma_1^{-+} := \{x \in \Sigma_1 \,:\, h_2(x) < 0,\, h_3(x) > 0\},$$

and similarly for the remaining ten cases.

**Definition 6.2.1.** *We say that* $\Sigma$, *related to* (6.1.1), *is* partially nodally attractive, *or* attractive through sliding, *if the following conditions hold:*

**(a)** $\begin{bmatrix} w_j^1(x) \\ w_j^2(x) \\ w_j^3 \end{bmatrix}$ *does not have the same sign of* $\begin{bmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{bmatrix}$ *for* $x \in R_j$, $j = 1, \ldots, 16$ *;*

**(b)** *at least one co-dimension* 1 *subsliding is taking place on* $\Sigma_i^{**}$, *and in a neighborhood of* $\Sigma_i^{**}$, $i = 1, 2, 3$, *towards at least one adjacent co-dimension* 2 *manifold:*

$(1^{--})$ $\det \begin{bmatrix} w_1^1 & w_5^1 \\ 1 & 1 \end{bmatrix} > 0$ *together with at least one of the following two conditions:*

$(1_2^{--})$ $(1 - \alpha_{\Sigma_1^{--}})w_1^2 + \alpha_{\Sigma_1^{--}} w_5^2 > 0$,

$(1_3^{--})$ $(1 - \alpha_{\Sigma_1^{--}})w_1^3 + \alpha_{\Sigma_1^{--}} w_5^3 > 0$;

$(1^{-+})$ $\det \begin{bmatrix} w_2^1 & w_6^1 \\ 1 & 1 \end{bmatrix} > 0$ *together with at least one of the following two conditions:*

$(1_2^{-+})$ $(1 - \alpha_{\Sigma_1^{-+}})w_2^2 + \alpha_{\Sigma_1^{-+}} w_6^2 > 0$,

$(1_3^{-+})$ $(1 - \alpha_{\Sigma_1^{-+}})w_2^3 + \alpha_{\Sigma_1^{-+}} w_6^3 > 0$;

*similarly for the remaining ten cases;*

**(c)** *at least one co-dimension* 2 *subsliding is taking place on* $\Sigma_{ij}^*$, *and in a neighborhood of* $\Sigma_i^{**}$, $i, j = 1, 2, 3$, $i < j$, *towards* $\Sigma$:

$(23^-)$ *the solution to*

$$M_{\Sigma_{23}^-} \lambda_M = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad M_{\Sigma_{23}^-} := \begin{bmatrix} w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 \\ 1 & 1 & 1 & 1 \\ \delta_1^{23} & -\delta_2^{23} & -\delta_3^{23} & \delta_4^{23} \end{bmatrix},$$

*is admissible, together with:*

$$(23_1^-):\ \sum_{i=1}^{4}\lambda_{M,i}w_i^1 > 0;$$

(23$^+$) *the solution to*

$$M_{\Sigma_{23}^+}\lambda_M = \begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad M_{\Sigma_{23}^+} := \begin{bmatrix} w_5^2 & w_6^2 & w_7^2 & w_8^2\\ w_5^3 & w_6^3 & w_7^3 & w_8^3\\ 1 & 1 & 1 & 1\\ \delta_5^{23} & -\delta_6^{23} & -\delta_7^{23} & \delta_8^{23}\end{bmatrix},$$

*is admissible, together with:*

$$(23_1^+):\ \sum_{i=5}^{8}\lambda_{M,i}w_i^1 < 0;$$

*and similarly for the remaining four cases;*

**(d)** *if any of the conditions in* (*b*) *is satisfied, then the corresponding condition in* (*c*) *must be satisfied as well.*

### 6.2.2 Extension to co-dimension 4 and higher

In this section, we propose the extension of the moments solution to any co-dimension $p \geq 1$, under nodal attractivity conditions. Before doing that, we introduce the differential problem associated to it.

Consider the piecewise smooth system

$$x'(t) = f_i(x),\ x \in R_i,\ \ i = 1,\dots,2^p, \tag{6.2.1}$$

where the regions $R_i$'s are open, disjoint and connected sets of $\mathbb{R}^n$, so that $\mathbb{R}^n = \overline{\bigcup R_i}$, and on each region $R_i$ the function $f_i$ is smooth.

The regions $R_i$'s are separated by manifolds defined as 0-sets of $\mathscr{C}^2$ scalar functions $h_i$: $\Sigma_i := \{x \in \mathbb{R}^n\ :\ h_i(x) = 0\}$, $i = 1,\dots,p$. Assume that the normals $\nabla h_i$'s are linearly independent on (hence in a neighborhood of) $\Sigma$, and let

$$\Sigma := \bigcap_{i=1}^{p}\Sigma_i \tag{6.2.2}$$

be the co-dimension $p$ manifold of interest to us. Letting

$$w_i^j := \nabla h_j(x)^\top f_i(x), \quad i = 1, \dots, 2^p, \; j = 1, \dots, p,$$

we associate the matrix $W = (w_i^j) \in \mathbb{R}^{p \times 2^p}$ to (6.2.1). As before, the linear system to solve in order to determine a sliding vector field on $\Sigma$ is given by

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0_p \\ 1 \end{bmatrix} . \tag{6.2.3}$$

Obviously, this is an undetermined linear system, and in Lemma 6.2.4 and Corollary 6.2.5 we will see that $\left[\begin{smallmatrix} W \\ \mathbb{1}^\top \end{smallmatrix}\right]$ has rank $p+1$, under appropriate attractivity conditions of $\Sigma$. It is this system that we will regularize according to the moments' technique. Once more, we stress that we are interested in admissible solutions of (6.2.3), hence positive and smoothly varying with $x \in \Sigma$.

Let us first recall the sign pattern of $W$ characterizing nodally attractive conditions, as in [23].

**Definition 6.2.2.** *We say that $\Sigma$ in* (6.2.2) *is* nodally attractive*, or equivalently that $W$ satisfies* nodally attractive conditions*, if the sign pattern of $W$ is given by the following recursion relations:*

$$S^{(1)} = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

$$S^{(k)} = \begin{bmatrix} \mathbb{1}_{2^{k-1}}^\top & -\mathbb{1}_{2^{k-1}}^\top \\ S^{(k-1)} & S^{(k-1)} \end{bmatrix}, \quad k = 2, \dots, p.$$

In [23], the authors proved that –when $\Sigma$ is nodally attractive– there always exits a multilinear (interpolant) solution $\lambda$ to the system $W\lambda = 0$. For later reference, we summarize this result without proof.

**Lemma 6.2.3** ([23])**.** *Suppose that $W \in \mathbb{R}^{p \times 2^p}$ satisfies nodally attractive conditions. Then, for any $p \geq 1$, there exist $\alpha_1, \dots, \alpha_p$, all in $(0, 1)$, such that the vector $\lambda \in \mathbb{R}^{2^p}$*

*defined as*

$$\lambda = \begin{bmatrix} (1-\alpha_1)(1-\alpha_2)...(1-\alpha_{p-1})(1-\alpha_p) \\ (1-\alpha_1)(1-\alpha_2)...(1-\alpha_{p-1})\alpha_p \\ (1-\alpha_1)(1-\alpha_2)...\alpha_{p-1}(1-\alpha_p) \\ (1-\alpha_1)(1-\alpha_2)...\alpha_{p-1}\alpha_p \\ \vdots \\ (1-\alpha_1)\alpha_2...(1-\alpha_{p-1})(1-\alpha_p) \\ \vdots \\ (1-\alpha_1)\alpha_2...\alpha_{p-1}\alpha_p \\ \alpha_1(1-\alpha_2)...(1-\alpha_{p-1})(1-\alpha_p) \\ \vdots \\ \alpha_1\alpha_2...\alpha_{p-1}\alpha_p \end{bmatrix}$$

*solves the system* $W\lambda = 0_p$*, and* $\sum_{i=1}^{2^p} \lambda_i = 1$*.* □

With the help of Lemma 6.2.3 we can prove the following.

**Lemma 6.2.4.** *For any* $k \geq 1$*, consider* $W^{(k)} \in \mathbb{R}^{k \times 2^k}$ *satisfying the sign pattern of Definition 6.2.2. Then*

$$\operatorname{rank} W^{(k)} = k \ .$$

*Proof.* The proof is by induction on $k$. The case $k = 1$ is in [26] ($k = 2$ is in [**?**], and $k = 3$ is Corollary 6.1.7).

Let us assume the result true for $k$, and let us consider $W^{(k+1)}$ with sign pattern given as in Definition 6.2.2. Let us pick $w_1, \ldots, w_{2^k}$, the first half of the columns of $W^{(k+1)}$. By Lemma 6.2.3, there exist $\lambda_1, \ldots, \lambda_{2^k} \in (0, 1)$, such that

$$\sum_{i=1}^{2^k} \lambda_i \begin{bmatrix} w_i^2 \\ \vdots \\ w_i^{k+1} \end{bmatrix} = 0_k \ ,$$

and since $w_i^1 > 0$ for $i = 1 \ldots, 2^k$, we also have

$$\sum_{i=1}^{2^k} \lambda_i w_i^1 > 0 \ .$$

Using this linear combination to replace the $(k+1)$-st column of $W^{(k+1)}$ gives the matrix

$$\hat{W}^{(k+1)} := \begin{bmatrix} & & & > 0 \\ & & & 0 \\ w_1 & \cdots & w_k & \\ & & & \vdots \\ & & & 0 \end{bmatrix} .$$

Now, we have that

$$\operatorname{sign}\left(\det \hat{W}^{(k+1)}\right) = \operatorname{sign}\left(\det \begin{bmatrix} w_1^2 & \cdots & w_k^2 \\ \vdots & & \vdots \\ w_1^{k+1} & \cdots & w_k^{k+1} \end{bmatrix}\right) \neq 0 \ ,$$

where the last inference comes from the inductive hypothesis, since $\begin{bmatrix} w_1^2 & \cdots & w_k^2 \\ \vdots & & \vdots \\ w_1^{k+1} & \cdots & w_k^{k+1} \end{bmatrix}$ has the sign pattern of the first $k$ columns of $W^{(k)}$, which is supposed to be full rank. This in turn implies that $\operatorname{rank} W^{(k+1)} = k + 1$. □

Finally, we have

**Corollary 6.2.5.** *For any $k \geq 1$, consider $\tilde{W}^{(k)} := \begin{bmatrix} W^{(k)} \\ \mathbb{1}^\top \end{bmatrix}$, where $W^{(k)} \in \mathbb{R}^{k \times 2^k}$ satisfies the sign pattern of Definition 6.2.2. Then* $\operatorname{rank} \tilde{W}^{(k)} = k + 1$*, hence* $\ker\left(\tilde{W}^{(k)}\right)$ *is $(2^k - k - 1)$-dimensional.*

*Proof.* The case $k = 1$ is elementary. So, proceeding by induction, let $k \geq 2$ be fixed and –using Lemma 6.2.3, and because of the nodally attractive sign pattern– consider multilinear interpolant solutions $\lambda^{(1)}$ and $\lambda^{(2)}$ associated, respectively, to the submatrices $\begin{bmatrix} w_1^2 & \cdots & w_{2^k}^2 \\ \vdots & & \vdots \\ w_1^{k+1} & \cdots & w_{2^k}^{k+1} \\ 1 & \cdots & 1 \end{bmatrix}$, and $\begin{bmatrix} w_{2^k+1}^2 & \cdots & w_{2^{k+1}}^2 \\ \vdots & & \vdots \\ w_{2^k+1}^{k+1} & \cdots & w_{2^{k+1}}^{k+1} \\ 1 & \cdots & 1 \end{bmatrix}$, of $\tilde{W}^{(k+1)}$. Note that

$$\lambda^{(1)} = \begin{bmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } \lambda^{(2)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{bmatrix}. \text{ Then } \tilde{W}^{(k+1)}\lambda^{(1)} = \begin{bmatrix} > 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tilde{W}^{(k+1)}\lambda^{(2)} = \begin{bmatrix} < 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

From inductive hypothesis, since the two submatrices

$$\begin{bmatrix} w_1^2 & \cdots & w_{2^k}^2 \\ \vdots & & \vdots \\ w_1^{k+1} & \cdots & w_{2^k}^{k+1} \\ 1 & \cdots & 1 \end{bmatrix} \text{ and } \begin{bmatrix} w_{2^k+1}^2 & \cdots & w_{2^{k+1}}^2 \\ \vdots & & \vdots \\ w_{2^k+1}^{k+1} & \cdots & w_{2^{k+1}}^{k+1} \\ 1 & \cdots & 1 \end{bmatrix}$$

are full rank $k+1$ having the same sign pattern as $\tilde{W}^{(k)} = \begin{bmatrix} W^{(k)} \\ \mathbb{1}^\top \end{bmatrix}$, using Lemma 6.1.5 gives

$$\operatorname{rank} \begin{bmatrix} W^{(k+1)} \\ \mathbb{1}^\top \end{bmatrix} = k+2 \ .$$

□

**Remark 6.2.6.** *On account of Corollary 6.2.5, for nodally attractive $\Sigma$, it follows that the linear system* (6.2.3)*,*

$$\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix} \lambda = \begin{bmatrix} 0_p \\ 1 \end{bmatrix} \ ,$$

*has rank $p+1$, therefore providing a family of solutions depending on $(2^p - p - 1)$ free parameters. From Lemma 6.2.3, one possibility to fix these is by using the* multilinear interpolant *approach. Needless to say (as already observed in Example 6.1.2 for the case of $p = 3$), there is severe lack of uniqueness of solutions in this case. Below, we will propose the* moments regularization*.*

The moments regularization requires to append a matrix $\Delta$ of signed partial distances and a row $d^\top$ of full distances of $w_1, \dots, w_{2^p}$ to $\begin{bmatrix} W \\ \mathbb{1}^\top \end{bmatrix}$. The matrix $\Delta$ will manage all the subslidings at lower co-dimensions: they happen from co-dimension 2 all the way to co-dimension $p-1$. Therefore, we have

$$\sum_{k=2}^{p-1} \binom{p}{k} = 2^p - p - 2$$

rows of partial distances: thus $\Delta \in \mathbb{R}^{(2^p-p-2)\times 2^p}$. Adding the row $d^\top$, gives $2^p - p - 1$ extra equations, as desired.

In order to decide the sign pattern of $\Delta$, it is necessary to recognize the entire substructures of lower co-dimensions nested within it when a partial distance is selected: then, the sign of each entry is determined by the sign product of the components considered to compute the partial distance. This is better explained by looking at Example 6.2.7 below for the case of co-dimension 4, which clearly indicates how one will proceed in general. About the sign pattern of $d^\top$, our proposal is to consider the following recursion:

$$\begin{aligned} R_1 &:= \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \\ R_{k+1} &:= \begin{bmatrix} R_k & \\ & -R_k \end{bmatrix}, \quad k = 1, \dots, p-1, \end{aligned}$$

and then define

$$d := R_p \begin{bmatrix} \|w_1\| \\ \vdots \\ \|w_{2^p}\| \end{bmatrix}. \tag{6.2.4}$$

Observe that this sign pattern is the same as considering the sign product of all the components in the vectors $\begin{bmatrix} w_i \\ \Delta_i \end{bmatrix}$, $i = 1, \dots, 2^p$.

**Example 6.2.7.** *In co-dimension 4, the sign pattern of $W$ is given by*

$$\mathrm{sign}(W) = \begin{bmatrix} + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \end{bmatrix}.$$

*We split $\Delta$, the matrix of partial distances, as*

$$\Delta = \begin{bmatrix} \mathrm{sign}(\Delta_{\mathrm{III}}) \odot \Delta_{\mathrm{III}} \\ \mathrm{sign}(\Delta_{\mathrm{II}}) \odot \Delta_{\mathrm{II}} \end{bmatrix},$$

*where $\odot$ is the Hadamard (componentwise) product, $\Delta_{\mathrm{III}}$ contains the rows of partial distances over three components of $w_i^j$ at the time, and $\Delta_{\mathrm{II}}$ contains the rows of partial distances over two components of $w_i^j$ at the time. Therefore, choosing components $2, 3, 4$ for the first row, $1, 3, 4$ for the second row, $1, 2, 4$ for the third row, $1, 2, 3$ for the fourth row, we get that the sign pattern of $\Delta_{\mathrm{III}}$ is*

$$\mathrm{sign}(\Delta_{\mathrm{III}}) = \begin{bmatrix} + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\ + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\ + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\ + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \end{bmatrix},$$

*with*

$$\Delta_{\mathrm{III}} = \begin{bmatrix} \delta_{2,3,4}(1) & \cdots & \delta_{2,3,4}(16) \\ \delta_{1,3,4}(1) & \cdots & \delta_{1,3,4}(16) \\ \delta_{1,2,4}(1) & \cdots & \delta_{1,2,4}(16) \\ \delta_{1,2,3}(1) & \cdots & \delta_{1,2,3}(16) \end{bmatrix},$$

*where, for any $h = 1, \ldots, 16$ and suitably chosen $i, j, k = 1, 2, 3, 4$,*

$$\delta_{i,j,k}(h) := \sqrt{(w_i^h)^2 + (w_j^h)^2 + (w_k^h)^2}\ .$$

*Notice that the sign pattern of the first row in $\Delta_{\mathrm{III}}$ is determined this way: since we are considering components $2, 3, 4$, then we look at second, third and fourth row of $W$; those rows present the sign pattern from co-dimension $3$ in columns $1, \ldots, 8$ and $9, \ldots, 16$: we then select the sign pattern of $d$ from the co-dimension $3$ case in the corresponding columns. The same (selecting the corresponding suitable columns) has to be done for the other rows.*

*The same rationale needs to be followed for determining the sign pattern of* $\Delta_{\mathrm{II}}$*, using the sign pattern of* $d$ *from the co-dimension* 2 *case (that is* $\begin{bmatrix} + & - & - & + \end{bmatrix}$*) in the corresponding columns giving the co-dimension* 2 *sign pattern, after we have selected the components to compute the partial distance. Therefore, the sign pattern of* $\Delta_{\mathrm{II}}$ *is*

$$\operatorname{sign}(\Delta_{\mathrm{II}}) = \begin{bmatrix} + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\ + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\ + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\ + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\ + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \end{bmatrix},$$

*with*

$$\Delta_{\mathrm{II}} = \begin{bmatrix} \delta_{1,2}(1) & \cdots & \delta_{1,2}(16) \\ \delta_{1,3}(1) & \cdots & \delta_{1,3}(16) \\ \delta_{1,4}(1) & \cdots & \delta_{1,4}(16) \\ \delta_{2,3}(1) & \cdots & \delta_{2,3}(16) \\ \delta_{2,4}(1) & \cdots & \delta_{2,4}(16) \\ \delta_{3,4}(1) & \cdots & \delta_{3,4}(16) \end{bmatrix},$$

*where, for any* $h = 1, \ldots, 16$ *and suitably chosen* $i, j = 1, 2, 3, 4$,

$$\delta_{i,j}(h) := \sqrt{(w_i^h)^2 + (w_j^h)^2}\ .$$

*Finally, according to* (6.2.4),

$$\operatorname{sign}(d^{\top}) = [+ - - + - + + - - + + - + - - +]\,.$$

*Putting everything together, the sign pattern of the moments matrix* $M_4$ *in co-dimension* 4 *is*

$$\begin{bmatrix} + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\ + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\ + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\ + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\ + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\ + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\ + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\ + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\ + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\ + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

*The proof of invertibility of this matrix, and the fact that the solution of* $M_4\lambda_M = \begin{bmatrix} 0_{15} \\ 1 \end{bmatrix}$ *has all positive components, proceed precisely like the case of co-dimension*

*3 proved in this paper. In particular, the proof of Theorem 6.1.8 when* $p = 4$ *holds unchanged, aside from the obvious changes in the dimensions (we have now 16 vectors* $\hat{v}_i's$*, etc.).*

# Chapter VII

# CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

In this thesis, we have introduced and studied the *moments Filippov sliding vector field* for a co-dimension 2 discontinuity surface under general attractivity condtions by subsliding.

In Chapter 2, we have considered several possibilities on how to define a Filippov sliding vector field on a co-dimension 2 singularity surface $\Sigma$, intersection of two co-dimension 1 surfaces. As underlying assumption, we considered the case of nodally attractive $\Sigma$.

We broadly classified the various possibilities in two groups: algebraic/analytic and geometric. In the first group, we considered three possible ways to define a Filippov vector field: a mean-field formulation, two approaches based on minimizing the 2-norm, and two different averaging techniques. The mean-field approaches depend on the underlying probability density function (pdf), and produce a smoothly varying vector field on $\Sigma$ for an appropriate pdf. The minimization techniques we considered, in general (even if well defined) fail to produce a smoothly varying Filippov sliding vector field. The two averaging techniques we considered behave very differently: (i) averaging the original dynamics appear to have serious difficulties of convergence and smoothness, (ii) averaging the sub-sliding vector fields, instead, delivers a well defined selection; however, this specific interpretation appears to be limited to the case of nodally attractive $\Sigma$.

The geometric approaches we considered are a generally viable mean to select a Filippov sliding vector field. In particular, the techniques which can be cast in the framework of "barycentric coordinates" methods deliver a uniquely defined and smoothly varying vector field on a nodally attractive $\Sigma$. Specifically, we reinterpreted a method based on bilinear interpolation, introduced one which we called moments method, and reviewed Wachspress method. Finally, we also revisited a method introduced in [22].

The most interesting approaches, among all of these, have been the bilinear interpolant and the moments method. The bilinear interpolant method has been extensively analyzed in recent works (e.g., see [19, 23]), under general (not only nodal) attractivity assumptions on $\Sigma$. The moments method, instead, appear to be new in the present context (i.e., to define a Filippov sliding vector field); we further proved that this method is equivalent to the so-called mean value coordinates with which name has been used successfully in the last 10 years in the computer graphics community (see [27, 35]). From the computational point of view, the expense associated with forming the moments and bilinear solution is comparable: the bulk of it is forming the values $w_j^i$'s, which is required for both methods; then, for the moments solution, we need to solve the linear system (3.1.1), whereas for the bilinear solution we need to solve a quadratic equation.

In Chapter 3, we showed that –whenever $\Sigma$ is attractive– the moments regularization gives a well-defined, smoothly varying sets of coefficients, rendering a smooth Filippov sliding vector field on $\Sigma$, which further leads to smooth exits at generic first order exit points. In the process, we introduced (and exploited) a quadrilateral $Q$ which proved to be a useful tool to study sliding vector fields on a co-dimension 2 manifold. We also showed, by numerical experiments presented in Chapter 5, the behavior of the moments method, and the potential dangers associated to selecting a solution $\lambda$ (and an associated sliding vector field) that does not smoothly render the exit vector field at a first order exit point. Finally, we discussed the case of non-generic exit points, and further generalizations of our approach.

To date (and with the exception of trivial modifications), we know of no other constructive technique that provably gives admissible (positive and smooth) coefficients, under general attractivity conditions of $\Sigma$, and that further leads to smooth exits at generic first order exit points. For a relevant result about piecewise linear vector fields, see [42].

In Chapter 4, we have reformulated the problem as one in which we seek a minimum variation solution in the $H^1$-norm for either the coefficients entering in the convex combination, or for the sliding vector field itself. We explicitly solved the resulting Euler-Lagrange equation on some model problems, and compared the resulting minimum variation solution(s) to other sliding vector fields previously considered in the literature (most notably, the bilinear and moments solutions). Moreover, we have also proved, under suitable assumptions, that a properly weighted minimum variation solution coincides with other smoothly varying sliding vector fields (say, the moments method), the weight itself providing a time reparametrization from one vector field to the other. We have exemplified on these concepts in Chapter 5.

Although the methodology proposed in Chapter 4 does not seem to be of trivial, nor universal, applicability (already in $\mathbb{R}^3$), it provides a promising alternative to existing approaches in case the "entry" and "exit" points of sliding motion are known. In fact, it is our opinion that the present minimum-variation ideas can eventually provide insight into appropriate minimality properties of a Filippov sliding vector field.

The extension of our approach to the case of systems in $\mathbb{R}^4$ (and beyond) presents some very interesting and challenging mathematical and modeling issues.

In Chapter 6, we have proposed an extension of the moments method to the case of a nodally attractive co-dimension 3 discontinuity manifold $\Sigma$. This is still a work in progress, and we still lack complete proofs for our conjectures. Moreover, we have described how to compute the moments solution, under nodally attractive conditions, at any co-dimension. We also claim that the moments solution, as proposed here for the co-dimension 3 in nodal attractivity, remains well defined under conditions in Definition 6.2.2, further providing smooth exits at first and second order exit points.

A mostly unexplored problem is to understand and extend minimum variation solutions to problems with a co-dimension 2 discontinuity manifold embedded in $\mathbb{R}^4$ or higher dimensional phase space. Our approach in $\mathbb{R}^3$ suggests a path to follow that seems promising, but little is still known about it.
Further, numerical integration of Filippov systems is getting even more and more attention, and an interesting problem, at the present under investigation, is how to use Newton-type methods to compute periodic solutions for boundary-value problems of Filippov-type. Finally, uniqueness and admissibility of the moments solutions in co-dimension 3 has been proven for the nodally attractive case, but still lacks for the case of general attractivity conditions: we do not know if our proposed definition of these conditions is sufficient to prove the same results as in the nodally attractive case. We think that this could be an appealing problem to pursue, since a full proof for it is not known yet.

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