Final Report for Period: 07/2007 - 06/2008 **Submitted on:** 07/15/2008

Principal Investigator: Randall, Dana J. **Award ID:** 0515105

Organization: GA Tech Res Corp - GIT

Submitted By:

Randall, Dana - Principal Investigator

Title:

Analysis of Markov Chains and Algorithms for Ad-Hoc Networks

Project Participants

Senior Personnel

Name: Randall, Dana

Worked for more than 160 Hours: Yes

Contribution to Project:

Post-doc

Graduate Student

Name: Bhatnagar, Nayantara

Worked for more than 160 Hours: Yes

Contribution to Project:

Undergraduate Student

Technician, Programmer

Other Participant

Research Experience for Undergraduates

Organizational Partners

Other Collaborators or Contacts

Collaborators during this period include: Nayantara Bhatnagar, Mary Cryan, Martin Dyer, Raissa D'Sousa, David Galvin, Sam Greenberg, Jeff Kahn, Russell Martin, Cris Moore, Amanda Pascoe, Gregory Sorkin, Vijay Vazirani, Eric Vigoda, Peter Winkler.

Activities and Findings

Research and Education Activities:

Algorithms for randomly sampling elements of a large set are used extensively in practive, although often without any performance guarantees. My research over the grant period has concentrated on identifying the underlying properties that cause a Markov chain to converge slowly, and developing new mathematical tools in order to make these observations rigorous.

We have made progress in this field by showing, for instance, that independent sets undergo a phase change as the fugacity of the system drops (SODA 2006). Moreover, we demonstrated that local chains for sampling 3-colorings become slow in sufficiently high dimensions; the dimension plays the role of the fugacity for this physical system (SODA 2007).

The proposed work on ad hoc systems was to consider various topology control algorithms that would allow a sensor network to achieve connectivity while operating well below maximum energy capacities. We demonstrated a distributed algorithm that would achieve connectivity, if it is achievable, by slowly ramping up power until certain angular constraints are satisfied at each sensor (IPSN 2006). We have extended this work recently by generalizing the results to various classes of wireless footprints that had not been preivously considered.

Over the last few years I have been very active in education by co-organizing a joint Georgia Tech - DIMACS special focus on 'Discrete Random Systems.' There were five workshops the first year, organized by Laci Lovasz and Benny Sudakov in October 2006, Milena Mihail, Ashish Goel, Fan Chung and Chris Wiggins in January 2007, by Gregory Sorkin and Eric Vigoda in March 2007, by Jim Fill (and a committee) in June 2007, and by Graham Brightwell, Tom Trotter and me in June 2007. In addition, we have held one working groups, organized by Eric Vigoda and me in March 2007, and a second is upcoming Fall 2008. These activities have been split between Rutgers and Georgia Tech. Last year there was one workshop held at Rutgers, and we are currently planning the activities for the final year at Georgia Tech, including a workshop on Randomized Algorithms in the Real World and a subsequent working group.

Findings:

1. Slow mixing of Glauber dynamics via topological obstructions:
Over the last two years, I've introduced a method for showing slow mixing based on the presence or absence of certain topological obstructions in the independent sets. The key new idea was redefining how to partition the state space into pieces to demonstrate that there is a bottleneck obstructing flow from one side to the other. Instead of basing this on relative numbers of odd or even vertices in each independent set, as had become the norm for such proofs, the new method instead bases the partition on the presence of paths of vertices from one side of the region to the other. Then, using elementary arguments, we showed that Glauber dynamics will be slow for sampling independent sets in 2 dimensions when the fugacity lambda > 8.066, improving on the best previous bound by a factor of 10. We also showed that it is slow on the torus when lambda > 6.1824. (In SODA 2006.)

In recent work with my student Sam Greenberg, we extended the topological obstruction technique to considering an 'ocean' of sites comprising a fault line. With this new argument were able to show slow mixing of local Markov chains for independent sets on the triangular lattice, as well as showing that the so-called '8-vertex model' of statistical physics becomes slow beyond a certain fugacity (RANDOM 2007, to appear in Algorithmica). We have also applied these ideas to show the behavior of a model of DNA-based self-assembly is fast at high temperature and slow at low temperature (FNANO 2007, to appear in Theoretical Computer Science).

2. 3-colorings in high dimensional lattices:
There has been a lot of interest in understanding when local Markov chains are rapidly mixing on the set of k-colorings of a given graph. Likewise, in statistical physics there is interest in generating k-colorings on bipartite lattice graphs, in particular grid graphs. Here they arise in the context of the k-state Potts model, a fundamental model of antiferromagnetism. While we showed in previous work that local dynamics converge quickly in two dimensions, it was conjectured that they should converge slowly in

sufficiently high dimensions.

David Galvin and I recently proved this, building on related work we are doing with Jeff Kahn and Greg Sorkin. The argument builds on sophisticated work of Galvin and Kahn showing the existence of multiple Gibbs states for independent sets in high dimensions. We used combinatorial arguments to show that the independent set combrised of the of red points would behave similarly to a random independent set on the grid, and then adapted their previous argument. This appeared in SODA 2007.

3. The effect of boundary conditions on mixing rates of lattice models:
In the context of independent sets and colorings, we saw a change in mixing rates depending on a parameter lambda that is inversely related to temperature: at high temperatures systems are well-behaved and sampling is efficient, while at low temperatures they are rigid and slow. In addition, it is believed that even at low temperature, the rate of convergence is strongly dependent on the environment in which the spin system is placed: we expect that there are boundary conditions such that one causes a local chain to mix quickly while others can cause the same chain to mix slowly. This had been verified previously only for spin systems on trees.

With my students Bhatnagar and Greenberg (RANDOM 2006), we defined a local chain for Ising configurations on the Cartesian lattice and show that with one boundary condition it will mix in polynomial time, while for another it will mix exponentially slowly. We show an analogous result for matchings on the square-octagon lattice, where, strikingly, the two boundaries differ only by the inclusion or exclusion of four vertices. These are the first rigorous proofs of this phenomenon on lattice graphs.

4. Many combinatorial and physical models on lattices have a height function representation, including several 2-dimensional tilings and 3-colorings on Z^d. These height functions map a d-dimensional configuration to a surface in (d+1) dimensions that has well-defined local properties. For example, lozenge tilings on the triangular lattice correspond to a monotonic surface on the 3-dimensional Cartesian lattice that looks like the upper envelope of a set of cubes, where each cube is supported by three other cubes. Previously local Markov chains have been studied to uniformly generate these objects, and it is known that they converge quickly in 2 and 3 dimensions.

Recently there has been a lot of interest in biased version of this chain where we are more likely to add unit cubes than remove them. It was previously known that the 2-dimensional chain is rapidly mixing for any constant bias, and that the 3-dimensional chain is rapidly mixing when the bias is O(n^3). With my students Sam Greenberg and Amanda Pascoe we introduced a much simpler method for analyzing these biased chains, and show that for all biases in 2-dimensions (including when the bias is less than a constant), as well as any bias exceeding d^2 in d dimensions, the local chain is rapidly mixing. This paper was just submitted.

5. We have also made progress on various applications of Markov chains in computing. In work with Bhatnagar, Vazirani and Vigoda, we showed how to sample 'bichromatic perfect matchings' where the input graph has edges that are one of two colors and we specify how many edges of the perfect matching should be each color. We apply this to sampling perfect matchings on lattice graph regions, where vertical edges are one color and horizontal the other, a problem of interest in physics.

In work with Cryan and Dyer, we considered gereralizations of the problem of sampling contingency tables to the 'cell-bounded' case. In the standard problem, row and column sums of a non-negative, integer valued matrix are given as input, and the goal is to count (or sample) from the set of consistent tables. Polynomial-time algorithms are only known when there are a constant number of rows, or when the row and column sums are all large. Cell-bounded tables generalize the standard model, and are interesting partly because this formulation is self-reducible (implying that counting and sampling are reducible to one another) whereas the standard formulation is not. We place the cell-bounded version on the same footing as the specialized standard model, and show that we can efficiently sample in exactly the same settings.

Training and Development:

The results achieved during this project was the major part of the Ph.D. dissertations of Nayantara Bhatnagar (graduated Summer 2007) and Sam Greenberg (graduated Summer 2008). In addition some results will be part of the dissertation of Amanda Pascoe, who has just completed her second year.

Throughout all of the Georgia Tech / DIMACS special focus activities I have been organizing, we have placed a strong emphasis on outreach to young researchers and underrepresented groups. Almost all of the workshops have included tutorials to help facilitate the inclusion of new rearchers in the field.

Outreach Activities:

The Georgia Tech / DIMACS activities that are part of the special focus on 'Discrete Random Systems' I am organizing have encouraged participation from underrepresented groups and from young researchers. There have been many tutorials included in these workshops to help graduate students as well as to facilitate interdisciplinary interactions.

I have given a keynote talk at FNANO 2007 (Foundations of NanoScience) giving an overview of Markov chain Monte Carlo methods and their relation to self-assembly based nanosysstems.

I published a survey article on the Markov chain Monte Carlo method in Comptuing in Science and Engineering to help engineers and physicists understand the mathematical requirements of designing a provably efficient algorithm for sampling and related applications.

Journal Publications

Russell Martin and Dana Randall, "Disjoint decomposition of Markov chains and sampling circuits in Cayley graphs.", Combinatorics, Probability and Computing., p. 411, vol. 15, (2006). Published,

Dana Randall, "Rapidly mixing Markov chains with applications in computer science and physics.", Computing in Science and Engineering, p. 30, vol. 8, (2006). Published,

Mary Cryan, Martin Dyer and Dana Randall, "Approximately counting integer flows and cell-bounded contingency tables.", Siam Journal on Computing, p. , vol. , (2008). Accepted,

N. Bhatnagar, D. Randall, V. Vazirani, E. Vigoda, "Random bichromatic matchings", Algorithmica, p. 418, vol. 50, (2008). Published,

S. Greenberg and D. Randall, "Convergence Rates of Markov Chains for Some Self-Assembly and Non-Saturated Ising Models", Theoretical Computer Science, p. , vol. , (2008). Accepted,

S. Greenberg and D. Randall, "Slow Mixing of Markov Chains Using Fault Lines and Fat Contours", Algorithmica, p. , vol. , (2008). Accepted,

Books or Other One-time Publications

Web/Internet Site

Other Specific Products

Contributions

Contributions within Discipline:

The research has produced new methods for showing the inefficiency of certain local Markov
chain algorithms, including 'fault lines' and 'fat faults.' This will help researchers produce similar results in the future, and it also helps guide the design of efficient algorithms by characterizing obstacles in fast mixing for certain classes of algorithms.

We have also made contributions that help expand the set of tools for showing fast mixing of Markov chains. In our work on sampling biased routings, we use a geometrically increasing distance function in order to argue that there is a good coupling. This approach is fine when the space is known to be contracting (i.e., that the expected change in distance between any pair of configurations is negative and bounded away from zero), but the standard coupling theorems are insufficient when the expected change in distance is very small or zero. This is because the maximal distance enters as a factor of the running time, and we require it to be exponentially large. We introduced a modification of the theorem that adds additional conditions under which the chain can still be proven to be rapidly mixing.

Contributions to Other Disciplines:

I presented a keynote address at FNANO 2007 (Foundations of Nanoscience) on the Markov chain Monte Carlo method, its mathematical foundations, and its connection to self-assembly based nanotechnologies. My participation in this conference led to two papers that analyze the behavior of models of self-assembly by viewing the processes as Markov chains.

I wrote a survey article on the design and analysis of Markov chain based algorithms for Computing in Science and Engineering intended for their Engineering and Physics readers.

Contributions to Human Resource Development:

I have worked closely with three Ph.D. students during this period. Nayantara Bhatnagar graudated Summer 2007 and is currently a visiting professor at U.C. Berkeley. Sam Greenberg graduated Summer 2008, and Amanda Pascoe has just completed her second year. They are all very promising students who have already made significant contributions to the discipline.

Contributions to Resources for Research and Education:

The Georgia Tech / DIMACS special focus on 'Discrete Random Systems' has been designed to facilitate interactions between researchers across disciplines, and to encourage participation of young researchers and underrepresented groups.

Contributions Beyond Science and Engineering:

Conference Proceedings

Categories for which nothing is reported:

Organizational Partners

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Contributions: To Any Beyond Science and Engineering

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