# GRACEFUL CONNECTIONS IN DYNAMICAL SYSTEMS – AN APPROACH TO GAIT TRANSITIONS IN ROBOTICS

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# GRACEFUL CONNECTIONS IN DYNAMICAL SYSTEMS – AN APPROACH TO GAIT TRANSITIONS IN ROBOTICS

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*To my parents, Bilquis and Ghulam Mustafa Memon.*

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# SUMMARY

Gaits have become an integral part of the design method of robots heading to complex terrains. But research into optimal ways to transition between different gaits is still lacking, and is the primary motivation behind this research. An essential characteristic of gaits is periodicity, and considering that a novel notion of graceful transition is proposed: a graceful transition is one that has maximally persisting periodicity. This particular notion of persistence in the characteristic behavior can be generalized. Therefore, a comprehensive framework for the general problem of connecting any two trajectories of a dynamical system, with an underlying characteristic behavior, over a finite time interval and in a manner that the behavior persists maximally during the transition, is developed and presented. This transition is called the Gluskabi Raccordation, and the characteristic behavior is defined by a kernel representation. Along with establishing this framework, the kernel representations for some interesting characteristic behaviors are also identified. The problem of finding the Gluskabi Raccordations is then solved for different combinations of characteristic behaviors and dynamical systems, and compact widely applicable results are obtained. Lastly, the problem of finding graceful gait transitions is treated within this newly established broader framework, and these graceful gait transitions are obtained for the case of a two-piece worm model.

# CHAPTER 1

# INTRODUCTION

Animals have the exceptional ability to move efficiently in complex environments and so it is only natural that the same behavior is desirable in robotics. It is for this reason that biomimetic robots have gathered popularity over their wheeled counterparts [2]. However, animal locomotion is a complex process involving the animal's neural, sensory and motor systems, their muscle-body dynamics, and finally the environment [3]. Considering the complex nature of the animal locomotion process, the replication of this entire system and gracefulness of animal motion has not yet been achieved. But in part, the problem of traversing complex terrains is approached by granting distinct patterns of locomotion to the robot. Hence, the problem of transitioning gracefully between these patterns is of significant interest in the grand scheme of mimicking animal locomotion. The problem of designing gait transitions has received some attention in the research community, but the focus has been on simplifying the control function of the transition. The motivating factor behind this thesis is addressing the problem of transitions between different gaits or locomotion patterns of a biomimetic robot such that the resulting transitions are graceful. Towards that end, a novel approach for graceful gait transitions is proposed in this dissertation.

Investigation into the gait transition problem led to the discovery that the problem is actually part of a larger class of problems, specifically that of connecting any two trajectories of a dynamical system, with an underlying common characteristic behavior, in finite time and in such a way that the characteristic behavior persists maximally during the transition. This can be seen as connecting two different modes of operation of the system. The trajectories being connected could, for instance, be stationary solutions or limit cycles of the dynamical system, and the underlying characteristic behavior is that the time derivative of the trajectories is zero for the former case and periodicity for the latter. The problem is related to the idea of quasistatic transitions in thermodynamics between two equilibrium points. It is desirable there to remain stationary but a transition cannot occur if you remain stationary so one settles for a quasistatic transition, which is accomplished by letting the transition happen infinitely slowly. For the problem of gait transitions, the characteristic behavior is periodicity and it is desirable that the transition be as periodic looking as possible. Prior research on this problem has been conducted by Yeung and Verriest [4, 5, 6, 7, 8], and extensions to their work are carried out in this research. The term "*Gluskabi raccordation*" was coined for these transitions in [4] and it will be adopted in this work as well. Raccordation is a french term, meaning 'to connect', and has its roots in civil engineering where two planes have to be connected by some smooth and easily constructed surface. An example of that would be when a vertical canal wall is connected to a sloping river bank by a piece of hyperbolic paraboloid. Gluskabi is a being from Native Indian mythology who made himself from dust and has the ability to transform animals. It fits in this context because one trajectory is being transformed to another.

The question of existence of a connection between the two trajectories of a dynamical system relates to the controllability of the dynamical system in the context of behavioral theory, a new paradigm in systems theory introduced by Willems [1]. If the system is controllable then a transition exists but it may not be unique. The Gluskabi problem is that of choosing the transition which *maximally preserves the desired behavior*, and this is equivalent to choosing a cost function, penalizing any deviation from this behavior, and solving a problem in optimal control. But the goals of this thesis go beyond that; specifically we are interested in constructing a comprehensive framework for this problem, solidifying our vague notions about these transitions, and further identifying the different interesting behaviors and solving the transition problem for these interesting behaviors with various classes of dynamical systems. In other words, what is desirable here is a complete theory or a set of cases whereby one identifies a transition problem with one of these cases and subsequently the solution can simply be looked up or obtained by solving some simple equations in a systematic manner.

A special focus of this thesis is on the interesting behavior of periodicity. Periodic phenomena are highly prevalent in natural as well as artificial systems. For instance many biological processes ranging from the beating of the heart to locomotion, occur with periodic patterns [9, 10]. Moreover, a single system may exhibit different types of periodic behavior. This leads to the natural question of transitions between different periodic behaviors. Consider the example of animal locomotion. Most animals employ a variety of gaits such as one for walking and a different one for running [11]. To switch from one gait to another, one necessarily has to employ an aperiodic transition but animals do this naturally in a graceful manner. It is our hypothesis that this translates to the transient motion remaining as close as possible to a periodic behavior. The gait transition problem for biomimetic robots has already been mentioned. Chemical reactors may also operate in periodic cycles since it results in better yields. In chemical process control, it is desired to transfer from one operating point (periodic cycle) to another smoothly so as to avoid drastic changes [12]. It can be argued here that the transition has to be maximally persistent in periodicity. These and many other examples justify the special attention paid to periodic behavior in this research.

The Gluskabi problem can be related to a number of problems in other areas. For instance, a problem in the dynamical systems' literature, akin to the Gluskabi problem, is the problem of finding heteroclinic orbits, i.e., paths in phase space that join two different equilibrium points or two different periodic orbits. This problem has received widespread attention in recent years in the mathematics community [13, 14] due to its application in astrodynamics, such as finding low energy trajectories for moving a satellite from one geosynchronous orbit to another or the problem of capturing near-Earth asteroids [15]. However, almost all of the prior work is dedicated to finding these connections for autonomous and non-parametrized dynamical systems or systems without input, and no optimality criterion is considered. Contrastingly, the authors of [16] have obtained a Newton method to compute orbits of a specifically perturbed system that connect points near two equilibria of the unperturbed system, and also under the assumption of constant control. Moreover, a key difference from this research is that a heteroclinic orbit connects the two limit sets asymptotically. Another related body of work is due to Sultan on the deployment of tensigrity structures, where the structure changes from one equilibrium configuration to another [17, 18, 19]. The author defines an equilibrium manifold containing both the equilibrium configurations, and the chosen transition minimizes the deployment time and lies within a predefined threshold of the equilibrium manifold at all times. A related problem can also be found in the area of image processing. Image morphing is the process in which one digital image is fluidly transformed into another, by applying transformations to the images to retain geometric alignment between their features. A key sub-process in this is warp generation, which computes mapping functions between the two images under the constraints of feature correspondence and is similar to the Gluskabi problem considered in this research. A number of image morphing schemes have been developed such as mesh warping, field morphing, radial basis functions, energy minimization etc and have been discussed in [20]. A related problem that occurs in manufacturing is of determining the transformation needed to bend a flat sheet of metal into the required shape using minimal energy, as outlined in [21]. In medical imaging, image matching is used for detection of abnormalities by comparing two images of the organ taken at different times. An approach to this problem is finding the transformation that minimizes a distortion functional [22]. The mathematical problem underlying all of the previous examples is outlined in [23], and is that of finding a diffeomorphism between two Riemannian manifolds $(M, g_M)$ and $(N, g_N)$, that minimizes a deformation energy functional,

$$\Phi(h) = \int_M ||h^* g_N - g_M||^2 \omega_M.$$

This can be thought of as a Gluskabi problem in higher dimensions.

The dual of the raccordation problem in estimation, as considered in [24], is also of interest. Incidentally while exploring this problem, another interesting and unsolved problem came to light. This is the problem of estimation of the state of a discrete-time stochastic multi-mode switched linear system, with a finite number of modes. The parameters in the different modes or the system matrices are assumed to be known, but the mode switching is random. The objective here is to obtain a good causal estimate of the state process from the past of the observed sequence and perhaps some prior information, such as the mode switching probabilities. Although this problem can be formulated in the exact least squares sense, with the conditional expectation as solution, finding this optimal solution is prohibitively complex due to the Bayesian explosion as indicated in [25, 26, 27, 28, 29] and more recently [30, 31]. Therefore, a simple suboptimal scheme for the estimation of state sequences of multi-mode systems with random switching has been obtained, in this research, by deriving the linear least squares filter for this problem [32, 33]. As a special case, the filter for a linear system, with observations degraded by intermittent sensor failure is also derived and its performance is analyzed. This model is an alternative to the packet loss model, where the information regarding the presence or absence of a packet is part of the observed data.

The organization of this thesis is as follows: First of all it is divided into two parts. The first part deals with the Gluskabi raccordation problem and the second part with the estimation problem. The first part begins with a review of the behavioral approach to system theory by Willems in Chapter 2, followed by rigorous formulation of the Gluskabi problem in Chapter 3. Some interesting behaviors are introduced and studied in Chapter 4. Next, a two step approach is taken to the Gluskabi problem: at first the simplified problem of finding transitions between functions, not necessarily trajectories of some dynamical system, with a characteristic behavior is dealt with in Chapter 5, and this is named the "*signal raccordation problem*"; the next step is when the same problem is solved for the trajectories of some dynamical system in Chapter 6, and this is called the "*dynamical raccordation problem*".

# PART-I

# CHAPTER 2

# BEHAVIORAL APPROACH TO SYSTEM THEORY

This chapter reviews some of the relevant concepts from the behavioral approach to system theory. These ideas will be used in the later chapters. An excellent and brief exposition of the subject can be found in [1], while a detailed version for the continuous time case is [34] and the discrete time case can be found in [35]. The behavioral approach provides a language that respects the physics of the system. The basic idea of behavioral approach to modeling is to consider a set that forms the totality of feasible events and then the mathematical model of the phenomenon restricts the outcomes to a subset called the behavior.

In the study of dynamical systems this broad idea is solidified with the following definitions.

- The set of independent variables is called the **time axis**, $\mathbb{T}$. For continuous time systems, the time axis is taken to be $\mathbb{T} = \mathbb{R}$, and for discrete systems, $\mathbb{T} = \mathbb{Z}$.

- The set of dependent variables, $\mathbb{W}$, in which outcomes of the signals being modeled take their values is called the **signal space**. Typically, $\mathbb{W} = \mathbb{R}^n$, $n \geq 1$.

- The **universum** is the collection of all maps from the time axis to the signal space, denoted by $\mathbb{W}^{\mathbb{T}}$. It defines the totality of feasible events.

- A **dynamical system**, $\Sigma$, is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$. The *behavior*, $\mathcal{B}$, is a suitable subset of $\mathbb{W}^{\mathbb{T}}$, for instance the piecewise smooth functions, compatible with the laws governing $\Sigma$.

- The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ is said to be **linear** if $\mathbb{W}$ is a vector space , and the behavior $\mathcal{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$. Linearity means that if $w_1, w_2 \in \mathcal{B}$ then $\alpha w_1 + \beta w_2 \in \mathcal{B}$ for all scalars $\alpha, \beta$.

- The dynamical system, $\Sigma$, is said to be **shift invariant** if $\mathbb{T}$ is closed under addition and $w \in \mathcal{B}$ implies that $\mathbf{S}_\tau w \in \mathcal{B}$ for all $\tau \in \mathbb{T}$. The evaluation functional $\sigma_t$ is defined as $\sigma_t(w) = w(t)$ a.e. (exception where $w$ is not defined). The shift operator, $\mathbf{S}_\tau$, is then defined by $\sigma_t(\mathbf{S}_\tau w) = \sigma_{t+\tau} w$.

- The behavior restricted to a small open interval $(-\epsilon, \epsilon)$ is defined by $\mathcal{B}_\epsilon = \{\tilde{w} : (-\epsilon, \epsilon) \to \mathbb{W} \,|\, \exists w \in \mathcal{B}$ such that $\sigma_t \tilde{w} = \sigma_t w$ for all $-\epsilon < t < \epsilon\}$. The continuous time system $\Sigma$ is called **locally specified** if for all $\epsilon > 0$,

$$(w \in \mathcal{B}) \Leftrightarrow (\mathbf{S}_\tau w|_{(-\epsilon,\epsilon)} \in \mathcal{B}_\epsilon \text{ for all } \tau \in \mathbb{R}).$$

In most applications, the behavior is usually described by a system of equations, which in continuous time cases is typically a system of differential equations and in discrete time case a system of difference equations. Let $\mathbf{D} = \frac{d}{dt}$ be the differentiation operator. Then, a behavior described by a system of differential equations could have three representations:

- **Kernel:** $\mathcal{B} = \{w : \mathbb{R} \to \mathbb{W} \,|\, f(w, \mathbf{D}\, w, \cdots, \mathbf{D}^n\, w) = 0$ for all $t \in \mathbb{R}\}$.

- **Manifest-Latent Variables:**
$\mathcal{B} = \{w : \mathbb{R} \to \mathbb{W} \,|\, \exists\, l : \mathbb{R} \to \mathbb{L}$ such that $f(w, \mathbf{D}\, w, \cdots, \mathbf{D}^n\, w, l, \mathbf{D}\, l, \cdots, \mathbf{D}^n\, l) = 0$ for all $t \in \mathbb{R}\}$.

- **Image:** $\mathcal{B} = \{w : \mathbb{R} \to \mathbb{W} \,|\, w = f(l, \cdots, \mathbf{D}^n\, l) = 0$ for any $l : \mathbb{R} \to \mathbb{L}\}$.

In the following section, linear time-invariant systems and specifically some of their properties are discussed.

## 2.1 Linear Time Invariant (LTI) systems

LTI systems in continuous time are described by systems of constant coefficient differential equations. Thus, the smooth solutions of an LTI behavior can be defined in terms of matrices $R_0, R_1, \cdots, R_n \in \mathbb{R}^{\cdot \times \cdot}$ as all $w$ that are solution to

$$R_0 w + R_1 \mathbf{D}\, w + \cdots + R_n \mathbf{D}\, w = 0.$$

This can be compactly written as,

$$R(\mathbf{D})w = 0,$$

where $R(\xi) \in \mathbb{R}^{p \times n}[\xi]$ is a matrix of polynomials with real coefficients and $\mathbf{D}$ is the differentiation operator, represents a system of $p$ linear time invariant (LTI) ordinary differential equations (ODE) in $n$ scalar variables. A system described by behavioral differential equations is locally specified. In order to verify if a trajectory $w$ belongs to the behavior, it suffices to look at the trajectory in an infinitesimal neighborhood about each point.

### 2.1.1 Equivalence of two systems

There is a surjective map from the space of LTI systems to the space of polynomial matrices. Two LTI systems given by polynomial matrices $R_1$ and $R_2$ are equivalent or are the same behavior if the matrices are related by a left unimodular transformation $U$, i.e. $R_1 = U R_2$. In general, $R_1 = X R_2 \Leftrightarrow \mathcal{B}_2 \subset \mathcal{B}_1$. A unimodular polynomial matrix is one whose inverse is also a polynomial matrix or equivalently whose determinant is a unit of the polynomial ring. Remember that in general the inverse of polynomial matrix is a rational matrix.

### 2.1.2 Minimality

An LTI behavior can have multiple representations. A representation is **minimal** if it has the least number of rows among all representations. Given a behavior $\mathcal{B}$ represented by a polynomial matrix $R$, there exists a representation $\tilde{R}$ of $\mathcal{B}$ with full row rank, i.e.

$$R = U \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix},$$

where $U$ is a unimodular matrix. Consequently, a representation is minimal if and only if it has full row rank.

The behavior defined by the full row rank, $p$, polynomial matrix $R$, i.e. $R(\mathbf{D})w = 0$, admits an Input/Output representation $P(\mathbf{D})y = Q(\mathbf{D})u$ such that $\det P \neq 0$ and $P^{-1}Q$ is a proper rational matrix, where $P \in \mathbb{R}^{p\times p}[\xi]$, $Q \in \mathbb{R}^{p\times m}[\xi]$, $y = (w_1, \cdots, w_p)$, and $u = (w_{p+1}, \cdots, w_{p+m})$.

### 2.1.3 Controllability

A behavior is called **autonomous** if for all $w_1, w_2 \in \mathcal{B}$ $w_1(t) = w_2(t)$ for $t \leq 0$ implies $w_1(t) = w_2(t)$ for almost all $t$. For an autonomous system, the future is entirely determined by its past. Behaviors defined by square full row rank polynomial matrices are autonomous.

The notion of *controllability* is an important concept in the behavioral theory. Let $\mathcal{B}$ be the behavior of a linear time invariant system. This system is called **controllable** if for any two trajectories $w_1$ and $w_2$ in $\mathcal{B}$, there exists a $\tau \geq 0$ and a trajectory $w \in \mathcal{B}$ such that

$$\sigma_t(w) = \begin{cases} \sigma_t(w_1) & t \leq 0 \\ \sigma_t(\mathbf{S}_{-\tau} w_2) & t \geq \tau \end{cases}$$

i.e., one can switch from one trajectory to the other, with perhaps a delay, $\tau$. This is illustrated in Figure 1. Note that an autonomous system cannot get off a trajectory once it is on

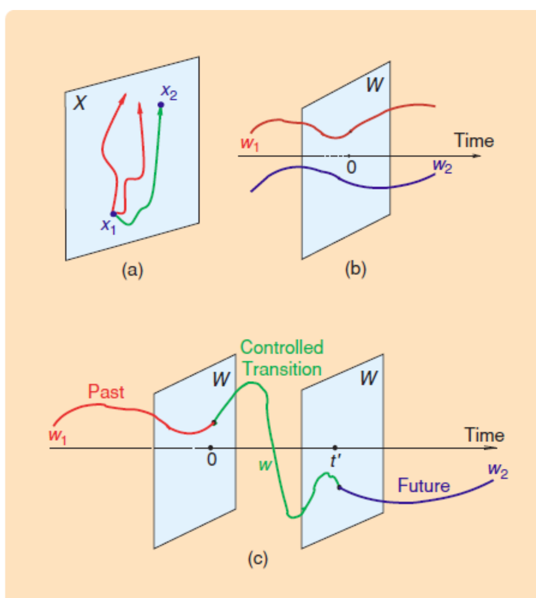



**Figure 1. State controllability shown in (a). Behavioral Controllability shown in (b) and (c). Source: [1]**

it. Hence, a non-trivial autonomous system is not controllable (The behavior $\mathcal{B} = \{0\}$ is autonomous and controllable).

The system defined by $R(\mathbf{D})\,w = 0$ is controllable if and only if the rank of the polynomial matrix $R(s)$ is the same for all $s \in \mathbb{C}$. If the system is instead given by an Input/Output representation $P(\mathbf{D})\,y = Q(\mathbf{D})\,u$, then the rank condition for controllability translates to coprimeness of the polynomial matrices $P$ and $Q$. Moreover, for the behavior of smooth solutions to an LTI system, the $\tau$ in the definition of controllability is independent of the two trajectories being connected and can be taken to be arbitrarily small.

An LTI system admits an image representation if and only if it is controllable. If a state representation of an LTI system is given then the notions of state controllability and behavioral controllability are equivalent, i.e. either one implies the other.

#### *2.1.3.1 Decomposition Theorem*

**Theorem 2.1.1.** *The LTI behavior, $\mathcal{B}$, defined by a full row rank polynomial matrix, $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$, as $R(\mathbf{D})\,w = 0$ can be decomposed into the direct sum of an autonomous behavior and a controllable one, i.e.*

$$\mathcal{B} = \mathcal{B}_{aut} \oplus \mathcal{B}_{cont}.$$

**PROOF.** Given the matrix $R$, one can find unimodular matrices $U$ and $V$ that transform $R$ into the Smith form, i.e.

$$\tilde{R}(\xi) = U(\xi)R(\xi)V(\xi) = \begin{bmatrix} D(\xi) & 0 \end{bmatrix},$$

with $\det(D) \neq 0$. The matrix $\tilde{R}$ defines a new behavior: $\tilde{\mathcal{B}} = V^{-1}(\mathbf{D})\mathcal{B}$. The trajectories $\tilde{w} \in \mathcal{B}$ can be partitioned according to $\tilde{R}$ into $\tilde{w}_1$ and $\tilde{w}_2$. If $\det(D)$ is constant then $\mathcal{B}_{cont} = \mathcal{B}$ and $\mathcal{B}_{aut} = \{0\}$. Otherwise, $\tilde{\mathcal{B}}$ can be easily decomposed as follows:

$$\tilde{\mathcal{B}}_{aut} = \{\tilde{w} \in C^\infty(\mathbb{R}, \mathbb{R}^q) \,|\, D(\mathbf{D})\,\tilde{w}_1 = 0, \tilde{w}_2 = 0\}$$
$$\tilde{\mathcal{B}}_{cont} = \{\tilde{w} \in C^\infty(\mathbb{R}, \mathbb{R}^q) \,|\, \tilde{w}_1 = 0\}$$

It is clear that $\tilde{\mathcal{B}}_{aut}$ and $\tilde{\mathcal{B}}_{cont}$ are autonomous and controllable respectively and that $\tilde{\mathcal{B}} =$

$\tilde{\mathcal{B}}_{aut} \oplus \tilde{\mathcal{B}}_{cont}$. The corresponding polynomial matrices describing these behaviors are:

$$\tilde{R}_{aut} = \begin{bmatrix} D(\xi) & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad \tilde{R}_{cont} = \begin{bmatrix} I & 0 \end{bmatrix}.$$

By transforming back, we obtain a decomposition of the behavior $\mathcal{B}$:

$$\mathcal{B}_{aut} = V(\mathbf{D})\,\tilde{\mathcal{B}}_{aut} \quad \text{and} \quad \mathcal{B}_{cont} = V(\mathbf{D})\,\tilde{\mathcal{B}}_{cont},$$

$$R_{aut} = \tilde{R}_{aut}(\xi)V^{-1}(\xi) \quad \text{and} \quad R_{cont} = \tilde{R}_{cont}(\xi)V^{-1}(\xi).$$

Since, the rank of a matrix is invariant under multiplication by a unimodular matrix, $R_{aut}$ and $R_{cont}$ are autonomous and controllable respectively. □

The decomposition of a behavior into an autonomous and a controllable part is not unique. This is evident from the above procedure. If there exists a unimodular matrix $W$ such that,

$$\begin{bmatrix} D(\xi) & 0 \end{bmatrix} W(\xi) = \begin{bmatrix} D'(\xi) & 0 \end{bmatrix}$$

then we will obtain a different decomposition, by employing the unimodular matrix $VW$ instead of $V$ now. Such polynomial matrices $W$ exist and are in fact of the form,

$$W = \left[\begin{array}{c|c} U_1 & 0 \\ \hline \times & U_2 \end{array}\right],$$

where $U_1$ and $U_2$ are unimodular matrices. However, it can be shown that the controllable part is unique. Once a decomposition of the behavior has been chosen, i.e. a polynomial matrix $V$ has been found, then the decomposition of every trajectory in the behavior into a controllable and an autonomous part is unique.

# CHAPTER 3

# THE GLUSKABI PROBLEM

In this chapter, the requisite nomenclature that has been developed for the raccordation problem in [36], using the behavioral approach, is presented, culminating in a rigorous formulation of the Gluskabi problem. To begin with, the key problem of this research, namely the Gluskabi problem, can be stated as follows:

> Given two trajectories $w_1$ and $w_2$ of the same behavior, the objective is to construct a transition, $w$, over some finite time interval $[a, b]$ such that $w = w_1$ for $t \leq a$, $w = w_2$ for $t \geq b$, and the characteristic behavior persists maximally during the transition.

After the Gluskabi nomenclature has been defined, this problem statement will be made clearer and restated in terms of the new nomenclature.

## 3.1 The Gluskabi Framework

We begin by defining a behavior which restricts the universum to just the ones which are interesting.

**Definition 3.1.1.** *The* **Base Behavior** *($\mathcal{B}_0$) is a subset of the universum, $\mathcal{B}_0 \subset \mathbb{W}^{\mathbb{T}}$, that defines the set of all allowable functions of interest. For any particular problem, the functions being connected lie in this set and the search for a connection[1] between the two is also conducted in this set, or in other words this is also the space of admissible controls.*

For example, if we want to work with real smooth functions entirely then $\mathcal{B}_0 = C^\infty(\mathbb{R}, \mathbb{R}^n)$. Or, if we are interested in the smooth trajectories of an LTI differential system then $\mathcal{B}_0 = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^n) \text{ s.t. } R(\mathbf{D})w = 0\}$, where $R(\xi)$ is a matrix of polynomials with real coefficients and $\mathbf{D}$ is the differentiation operator.

[1] This usage of the term connection is different from a connection defined in differential geometry.

In the following discussion only real-valued functions are considered and so the signal space is real. For the most part, the time axis is also real except on occasions it is the set of integers. From this point onwards $\mathbb{W} = \mathbb{R}^n$ for some $n \geq 1$.

It is also remarked here that changing the base behavior may possible change the solution of the Gluskabi problem. This is evident from the example that a differential system has different solutions depending on the solution space [34]. For instance, the ODE $\mathbf{D}\, w_2 = w_2 + w_1$ has the usual smooth solution but it also has the weak solution,

$$(w_1(t), w_2(t)) = \begin{cases} (0,0) & t < 0 \\ (1, e^t - 1) & t \geq 0 \end{cases}.$$

Even more, the solution may change by the choice of time axis [37]. For instance, the system of differential equations

$$w_1 - \mathbf{D}\, w_2 = 0, \quad w_2 = 0$$

has just the zero solution in either the space of smooth solutions or distributions over the entire real line $\mathbb{R}$; But if a solution is sought over the half-line $[0, \infty)$ then $(w_1, w_2) = (b\delta, 0)$ with $w_2(0^-) = b$ for any $b \in \mathbb{R}$ is also a solution. Hence, the base behavior is of special importance and needs to be chosen wisely.

**Definition 3.1.2.** *A **Type** ($\mathcal{T}$) is a strict subset of the base behavior ($\mathcal{T} \subset \mathcal{B}_0$) such that all of its elements share a property, described by an associated operator,* $\mathbf{Op} : \mathcal{A} \to \mathcal{V}$*, in the following way:*

$$\mathcal{T} = \{w \in \mathcal{A} \;\; s.t. \;\; \mathbf{Op}\, w = 0\},$$

*where $\mathcal{A} \subset \mathcal{B}_0$ is the maximal linear space on which the operator is well defined, $\mathcal{A} \subset$ Domain*$(\mathbf{Op})$*, and $\mathcal{V}$ is a linear space as well.*

The Type behavior defines the set of trajectories possessing a desired quality, which we want to connect. The set $\mathcal{A}$ is required to be a linear space because of the variational problem to be defined later. This way, when a subset of $\mathcal{A}$ is chosen by restricting the domain of functions to a compact set, it is a normed space since it is clearly a vector space also and it always admits the maximum norm ($L_\infty$) at the very least. The requirement of a normed space gives sense to the idea of local extrema. This condition can be relaxed by just requiring that the subset formed by restricting the domain be a normed space. The set $\mathcal{V}$ on the other hand need not be a vector space. It is only required that it contain the zero element and that it be a metric space so that a cost function can be defined as the distance from zero. These conditions may need to be revised if non-real valued or generalized functions are considered.

**Definition 3.1.3.** *A **Trait** ($\mathcal{T}_\theta$) is a subtype of the type i.e., it is a subset of the type such that it has its own characteristic behavior, given by some operator $\mathbf{Op}_\theta$.*

$$\mathcal{T}_\theta = \{w \in \mathcal{T} \textit{ such that } \mathbf{Op}_\theta w = 0\}$$

For instance, a trait could be specified by some (or all) boundary conditions, or some intermediate values and their derivatives.

Given the obvious similarities, we call this the *Kernel* representation of the type and trait irrespective of whether the operator **Op** is linear or nonlinear. A type may admit representations other than the kernel representation, one such being the image representation. This work only considers the kernel representation of types but for the sake of completeness, the *Image* representation is also defined here.

**Definition 3.1.4.** *The image representation is described by an associated mapping, $\Phi$, from a parameter space to the space of operators, yielding a family of operators. The type is then the union of kernels of all the operators in the family. The parameter space is typically some subset of $\mathbb{R}^n$.*

$$\mathcal{T} = \cup_{\theta\in\Theta}\{w \in \mathcal{A} \;\; s.t. \;\; \Phi(\theta)\, w = 0\}$$

*The type given by the image of a particular parameter is a trait.*

**Example 3.1.1** (Constants)**.** *Let $\mathcal{B}_0 = C^0(\mathbb{R}, \mathbb{R})$. Then, the operator* **Op** := **D***, the differentiation operator, defines the type of constants in $\mathcal{B}_0$. The domain of the operator is typically* *a proper subset of the base behavior and in this case* $\mathcal{A} = C^1(\mathbb{R},\mathbb{R})$*, i.e. the space of continuously differentiable functions. An example of a particular trait in this type could be the constant c i.e.,* $\mathcal{T}_c = \{w \in \mathcal{T} \text{ s.t. } w = c\}$.

**Example 3.1.2** (Polynomials)**.** *Let* $\mathcal{B}_0 = C^0(\mathbb{R},\mathbb{R})$*. Then, the operator* $\mathbf{Op} := \mathbf{D}^3$ *defines the second order polynomials type in* $\mathcal{B}_0$*. The domain of the operator,* $\mathbf{Op}$*, in this case is* $\mathcal{A} = C^3(\mathbb{R},\mathbb{R})$*, i.e. the space of thrice differentiable functions with continuous derivatives. An example of a trait in this type is the subtype of first order polynomials or constants. Another example of trait in this type is polynomials that vanish at* $t = 0$*.*

**Example 3.1.3** (Periodic signals with period $\tau$)**.** *The operator* $\mathbf{Op} := (\mathbf{I} - \mathbf{S}_\tau)$ *where* $\mathbf{I}$ *is the identity operator and* $\mathbf{S}$ *is the shift operator, defines the periodic type in* $\mathcal{B}_0$*. A smooth periodic function can be seen as a sum of harmonic signals of integer multiples of the base frequency. Thus, the periodic type in* $\mathcal{B}_0 = C^\omega(\mathbb{R},\mathbb{R})$ *may also be characterized by the infinite product operator* $\left[\mathbf{D}\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2\omega^2}\mathbf{D}^2\right)\right]$*, which can also be written as* $\sinh\left(\frac{\pi}{\omega}\mathbf{D}\right)$ *([38]), where* $\omega = 2\pi/\tau$*. This representation defines a number of traits in terms of the number of finite product terms and these traits serve as various levels of approximation to the periodic functions.*

The above definitions form the basic nomenclature of our problem but we will need one more definition to rigorously define a connection later on. Given any type we can extend it to create a collection of related types in the following manner.

**Definition 3.1.5.** *The* ***Equation Error System*** *(*$\mathcal{T}_{ee}$*) of a type* $\mathcal{T}$*, defined by the kernel of the operator* $\mathbf{Op}$*, is a union of behaviors* $\mathcal{T}_e := \{(w, e) \in \mathcal{A} \times \{e\} \text{ s.t. } \mathbf{Op}\, w = e\}$*.*

$$\mathcal{T}_{ee} := \cup_{e \in \mathbf{Op}(\mathcal{A})} \mathcal{T}_e = \{(w, e) \in \mathcal{A} \times \mathcal{V} \text{ s.t. } \mathbf{Op}\, w = e\},$$

*where* $\mathcal{V}$ *is the vector space where the image of* $\mathbf{Op}$ *lies i.e.,* $\mathbf{Op}(\mathcal{A}) \subset \mathcal{V}$*.*

Notice that the original type $\mathcal{T}$ is the projection onto $\mathcal{A}$ of the behavior $\mathcal{T}_0$ in this collection, where $\mathcal{T}_0 = \{(w, 0) \in \mathcal{A} \times \mathcal{V} \text{ s.t. } \mathbf{Op}\, w = 0\}$. It is also worth noticing that the Equation Error System lies in an extended base behavior $\Sigma = (\mathbb{T}, \mathbb{W}\times\mathbb{E}, \mathcal{B}_0)$, where $\mathcal{V} \subset \mathbb{E}^{\mathbb{T}}$.

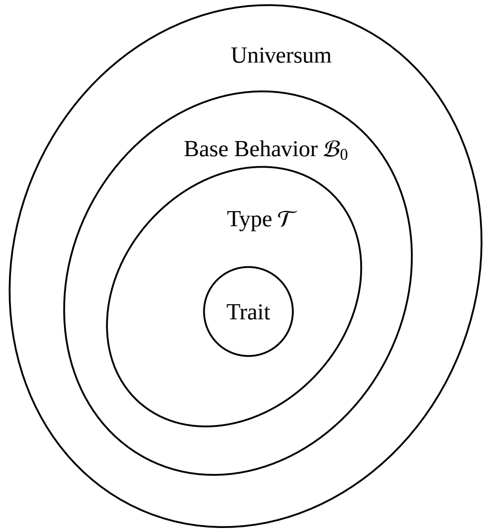



**Figure 2. Relation of terms to the Universum**

**Example 3.1.4.** *Consider the type in $C^{\infty}(\mathbb{R},\mathbb{R})$ defined by the operator $\mathbf{Op} := (\mathbf{D}-\lambda\mathbf{I})$, i.e., the type of multiples of the exponential $e^{\lambda t}$. Then the equation error system corresponding to this type is the set of solutions w to the non-homogeneous ODE ($\mathbf{D}w - \lambda w = e$), for some forcing function $e \in C^{\infty}(\mathbb{R},\mathbb{R})$.*

Now equipped with this suitable terminology, the Gluskabi problem can be formulated. Given a type $\mathcal{T}$, the objective is to find a mapping that assigns to any two elements $w_1$ and $w_2$ in the said type, a unique element, $w$, in the base behavior which connects $w_1$ and $w_2$ in finite time, i.e., over the given interval $[a, b]$, and in such a manner that the defining quality of the type persists maximally. We will call this mapping the "*Gluskabi map*". Using the established idea that a type is given by the kernel of some operator **Op**, the Gluskabi map and the notion of persistence of a trajectory are defined in the following manner.

**Definition 3.1.6.** *Given a type $\mathcal{T}$ with the associated operator **Op**, an element $\bar{w} \in \mathcal{A} \subset \mathcal{B}_0$ is said to be* ***persistent*** *with respect to the norm $||.||$, defined on the space $\mathcal{V}$ restricted to some interval $\mathcal{I}$, if $||\mathbf{Op}\,\bar{w}|| = 0$. This means that $\bar{w}$ restricted to the interval $\mathcal{I}$ is contained in the restricted type behavior, i.e. $\bar{w}|_{\mathcal{I}} \in \mathcal{T}|_{\mathcal{I}}$.*

**Definition 3.1.7.** *Given a type $\mathcal{T}$ with the associated operator **Op**, an element $\bar{w} \in \mathcal{A} \subset \mathcal{B}_0$*

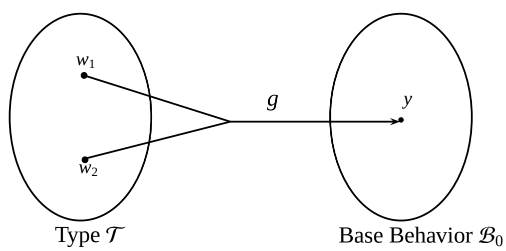
*is said to be **maximally persistent** over an interval $\mathcal{I} \subset \mathbb{T}$ with respect to the norm $\|.\|$, defined on the space $\mathcal{V}$ restricted to the interval $\mathcal{I}$, if $\bar{w}$ minimizes the norm of* **Op** *$w$ in that interval, i.e.,*

$$\bar{w} = arg\min_{w \in \mathcal{A}} \|\mathbf{Op}\, w|_{\mathcal{I}}\|.$$

**Definition 3.1.8.** *Given a type $\mathcal{T}$ with the associated operator* **Op** *and a compact interval $[a, b] \subset \mathbb{T}$, the **Gluskabi map** $g : \mathcal{T} \times \mathcal{T} \to \mathcal{B}_0$ with respect to the norm $\|.\|$, is defined as follows*

$$g(w_1, w_2)(t) = \begin{cases} w_1(t) & t \leq a \\ \bar{w}(t) & a < t < b \\ w_2(t) & t \geq b \end{cases}$$

*where $\bar{w} \in \mathcal{A}$ is such that $g(w_1, w_2)$ is maximally persistent over the support of* **Op** *$g(w_1, w_2)$ with respect to the given norm, and $\bar{w}$ agrees with $w_1$ and $w_2$ at the boundaries, i.e., at $t = a$ and $t = b$.*




Clearly, this requires that $\mathcal{V}$ restricted to the support of **Op** $g(w_1, w_2)$ be a normed space. Since, we are dealing with real valued functions in the present publication, i.e. $\mathcal{V} \subset \mathbb{R}^n$, if the space $V$ is also a vector space then its restriction to the compact interval can be easily made into a normed space. It definitely admits one norm, namely the maximum norm.

It is possible that the function $g(w_1, w_2)$ may not be contained in the domain of the operator **Op**. This is the reason that the base behavior is typically chosen to be a larger set than the set $\mathcal{A}$ which contains the type behavior. However, the function $g(w_1, w_2)$ is piecewise contained in $\mathcal{A}$ and there exists a partition of $\mathbb{T}$ with finitely many parts (intervals) such that each part is contained in $\mathcal{A}$ restricted to the respective interval. Typically, the end points, $a$ and $b$, of the given connection interval partition the time axis $\mathbb{T}$, and the operator **Op** then acts on the function $g(w_1, w_2)$ in the following piecewise fashion, i.e.

$$\mathbf{Op}\, g(w_1, w_2)(t) = \begin{cases} \mathbf{Op}\, w_1(t) & t \leq a \\ \mathbf{Op}\, \bar{w}(t) & a < t \leq b \\ \mathbf{Op}\, w_2(t) & t > b \end{cases}.$$

The support of $\mathbf{Op}\, g(w_1, w_2)$ only depends on the operator. If the operator is memoryless (the type is locally specified) then the support is compact, since $w_1$ and $w_2$ are persistent in the type, i.e. $\mathbf{Op}\, w_1 = \mathbf{Op}\, w_2 = 0$, and so $\mathbf{Op}\, g(w_1, w_2) = 0$ for $t < a$ and $t > b$. For instance, if **Op** is some differential operator then the support is $[a, b]$. If **Op** is a finite memory operator then the support is still compact, but now $\mathbf{Op}\, g(w_1, w_2) = 0$ for $t < a - \epsilon_a$ and $t > b + \epsilon_b$. Since the interval is compact, the norm is well-defined. Furthermore, of all the maximally persistent trajectories, $\bar{w}$ is chosen by matching the boundary conditions, formed by some attributes of $\bar{w}$, to $w_1$ and $w_2$. Depending on the operator, the attributes may include the value of the function at the boundary points as well as any number of its derivatives.

The connection in the interval $[a, b]$ will be called the "*Gluskabi raccordation*". As evident from the definition of the Gluskabi map, the element $w$ corresponding to $w_1, w_2 \in \mathcal{T}$ may not lie in the type $\mathcal{T}$ and is constructed piecewise from elements in $\mathcal{A}$. A new behavior can now be constructed by collecting all the elements $w = g(w_1, w_2)$ corresponding to any two elements $w_1$ and $w_2$ in the type $\mathcal{T}$, i.e., this behavior is the image of the Gluskabi map. This behavior will be called the "*Gluskabi Extension*" and can also be defined using the extended types $\mathcal{T}_{ee}$ in the following way.

**Definition 3.1.9.** *Given a type $\mathcal{T}$ with the associated operator* **Op** *and a compact interval $[a, b] \subset \mathbb{T}$, the* ***Gluskabi Extension*** *($\mathcal{G}_\mathcal{T}$) with respect to the norm $\|.\|$, is defined as*

$$\mathcal{G}_\mathcal{T} := \{w \in \mathcal{B}_0 \textit{ s.t. } \exists w_1, w_2 \in \mathcal{T} \textit{ with } \Pi_- w = \Pi_- w_1,\ \Pi_+ w = \Pi_+ w_2,\ \textit{and } \exists (u, e) \in \mathcal{T}_{ee}$$
$$\textit{s.t. } \Pi_{[a,b]} w = \Pi_{[a,b]} u \textit{ with } \|e\| \textit{ minimal}\}$$

*where* $\Pi$ *is the projection operator i.e.* $\Pi_{-}w$ *is the restriction of* $w$ *to the interval* $(-\infty, a]$*,* $\Pi_{+}w$ *is the restriction to the interval* $[b, \infty)$*, and* $\Pi_{[a,b]}w$ *is the restriction to the interval* $[a, b]$.

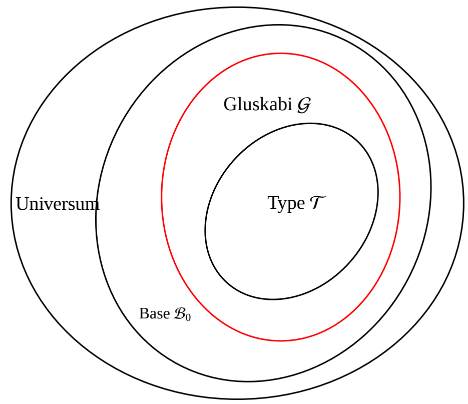



**Figure 3. Relation of Gluskabi Extension to other behaviors**

Notice that the type $\mathcal{T}$ is in the Gluskabi Extension $\mathcal{G}_{\mathcal{T}}$. Since the space $\mathcal{V}$ restricted to an interval generally admits multiple norms, the Gluskabi map and extension will in general depend on the chosen norm and the raccordation interval. Thus, a suitable norm in conjunction with the operator **Op** completely characterizes the desired persistence. For instance, if **Op** is a differential operator of some order acting on functions then any Sobolev norm of compatible degree can be used to get the required level of smoothness. Say the time interval is $[a, b]$ and the $\mathbf{Op} : C^r(\mathbb{R}, \mathbb{R}) \to C^s(\mathbb{R}, \mathbb{R})$, then the Sobolev norm $||.||_W$ on $e \in \mathcal{V} = C^s([a, b], \mathbb{R})$ is given by,

$$||e||_W = \sum_{i=0}^{n} \rho_i ||\mathbf{D}^i e||_{L^2} \quad \text{where } \rho_i > 0,\ n \leq s,\ \text{and } ||x||^2_{L^2} = \int_a^b x^2(t)dt.$$

# CHAPTER 4

# SOME INTERESTING TYPES

In this chapter, two interesting types are introduced in the context of the Gluskabi framework. The goal here is to obtain a kernel representation for these types, i.e. obtain an operator such that the kernel of that operator is the type in question. One of the types studied here is the periodic type – one of the special focuses of this thesis. Two equivalent representations are presented for the $\tau$-periodic type. Finally, the general periodic type that includes periodic signals of all periods, is discussed.

## 4.1 Linear Time Invariant Differential (LTID) Type

In this section, we focus our attention on an interesting type namely the linear time invariant differential (LTID) behavior, $\mathcal{L}_n^k$, of some order $n$ , i.e., the set of all solutions to any system of $k$ constant coefficient homogeneous differential equations of $n$th order. The goal here is to find a kernel representation for this type $\mathcal{L}_n^k$, i.e., find the operator characterizing this type in accordance with Definition-3.1.2. This type was first introduced in [24], where the operator was derived for the scalar $n$-th order differential equation case i.e., when $k = 1$, and the operator for the vector case was later derived in [36]. Using Willems's approach, this behavior is represented as,

$$\mathcal{L}_n^k = \left\{ w \in C^n(\mathbb{R}, \mathbb{R}^k) \mid \exists R \in \mathbb{R}[\xi]^{k\times k} \text{ for which } \; R(\mathbf{D})w = 0 \right\}$$

where $\mathbf{D}$ is the differentiation operator and $R$ is a polynomial matrix

$$R(\xi) := R_0\xi^n + R_1\xi^{n-1} + \cdots + R_n\xi.$$

The kernel representation will be derived for the scalar case first and it will be shown that the operator derived forms both necessary and sufficient condition. This is followed by the derivation of the kernel representation for the vector case.

### 4.1.1 Scalar LTID Types ($k = 1$)

The idea is illustrated first with the help of the simplest type in this category and then generalized. Consider the type comprised of solutions, $w$, to the first-order ODE, $\dot{w}(t) - a\,w(t) = 0$, with the associated operator $\mathbf{Op} = \mathbf{D} - a\mathbf{I}$. This is the type of all scalar multiples of the exponential $e^{at}$. The LTID type $\mathcal{L}_1^1$ is the type comprised of solutions to any first-order ODE, i.e. the solutions to the previous differential equation for all values of $a$. This section deals with the question of finding the kernel representation for such types. To obtain the said representation, notice that if $w \in \mathcal{L}_1^1$, then

$$\frac{\dot{w}}{w} = a \qquad \text{for some } a \in \mathbb{R}. \tag{1}$$

Differentiating the previous equation we get,

$$\frac{\ddot{w}w - \dot{w}^2}{w^2} = 0. \tag{2}$$

This results in the kernel of the nonlinear operator $\mathbf{Op}\,w = \ddot{w}w - \dot{w}^2 = \begin{vmatrix} w & \dot{w} \\ \dot{w} & \ddot{w} \end{vmatrix}$, which is the Wronskian of the functions $w$ and $\dot{w}$, characterizing the type $\mathcal{L}_1^1$. This is generalized in the following theorem.

**Theorem 4.1.1.** *The LTID type, $\mathcal{L}_n^1$, is characterized by the kernel of a Wronskian operator,* $\mathbf{W}$*, specifically the operator associated with the type is defined as*

$$\mathbf{Op}\,w = \mathbf{W}\left(w, \dot{w}, \cdots, w^{(n)}\right) = \begin{vmatrix} w & \dot{w} & \cdots & w^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(n)} & w^{(n+1)} & \cdots & w^{(2n)} \end{vmatrix}.$$

**PROOF.** Say it's given that $\mathbf{Op}\,w = 0$, i.e. the Wronskian is zero. It is known that if the Wronskian of a finite family of analytic functions is zero, then the functions are linearly dependent [39]. Therefore, assuming that $w$ is analytic, if $\mathbf{Op}\,w = 0$ then

$$r_0 w^{(n)} + r_1 w^{(n-1)} + \cdots + r_{n-1}\dot{w} + r_n w = 0,$$

for some real coeffcients $r_0, r_1, \cdots, r_n$. This means that if $\mathbf{Op}\, w = 0$ then $w$ is an analytic solution to an $n$th order constant coefficient homogeneous differential equation. It is also known that any constant coefficient homogeneous differential equation of any order can be converted into a first order differential equation in vector form, the smooth solution to which is the matrix exponential times a constant vector. The matrix exponential is an analytic function as shown in Appendix A.1 and so all smooth solutions are actually analytic. This means that if $\mathbf{Op}\, w = 0$ then $w$ is a smooth solution to some $n$th order differential equation and is subsequently contained in the LTID type $\mathcal{L}_n^1$.

In the other direction, if $w \in \mathcal{L}_n^1$, then there exist $r_i \in \mathbb{R}$ for $i \in \{0, 1, \cdots, n\}$ such that

$$r_0 w^{(n)} + r_1 w^{(n-1)} + \cdots + r_{n-1}\dot{w} + r_n w = 0. \tag{3}$$

This implies that the functions $\{w, \dot{w}, \cdots, w^{(n)}\}$ are linearly dependent and hence their Wronskian is zero, i.e. $\mathbf{W}\left(w, \dot{w}, \cdots, w^{(n)}\right) = 0$. □

**AN ALTERNATE PROOF.**

A different proof is presented here that follows the idea outlined earlier for the $\mathcal{L}_1^1$ case and offers some insights into the problem. It was shown earler that if $w \in \mathcal{L}_1^1$, then $\mathbf{W}(w, \dot{w}) = 0$. Now let's assume that this holds for the case $n-1$, i.e. if $w \in \mathcal{L}_{n-1}^1$, then $\mathbf{W}\left(w, \dot{w}, \cdots, w^{(n-1)}\right) = 0$, and proceed by induction. If $w \in \mathcal{L}_n^1$, then there exist $r_i \in \mathbb{R}$ for $i \in \{0, 1, \cdots, n\}$ such that (3) holds. Let's assume that $r_0 \neq 0$. Then (3) can be written as,

$$w^{(n)} + r_1 w^{(n-1)} + \cdots + r_{n-1}\dot{w} + r_n w = 0 \tag{4}$$

$$w^{(n)} + \begin{bmatrix} r_n & \cdots & r_1 \end{bmatrix} \begin{bmatrix} w \\ \vdots \\ w^{(n-1)} \end{bmatrix} = 0. \tag{5}$$

The idea is to separate the coefficients $r_i$ from $w$ and its derivatives. This can be accomplished by differentiating (4), and then (5) is transformed into,

$$-\begin{bmatrix} r_n & \cdots & r_1 \end{bmatrix} \begin{bmatrix} w & \dot{w} & \cdots & w^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(n-1)} & w^{(n)} & \cdots & w^{(2n-2)} \end{bmatrix} = \begin{bmatrix} w^{(n)} & \cdots & w^{(2n-1)} \end{bmatrix}. \tag{6}$$

Let's call the square matrix in (6) by $W$ and name the matrix on the right hand side as $Z$. Assume that $W$ is invertible, i.e. the determinant of $W$ is not zero. Then,

$$-\begin{bmatrix} r_n & \cdots & r_1 \end{bmatrix} = Z\,W^{-1}. \tag{7}$$

Differentiate (7) to obtain the representation for an arbitrary coefficient vector.

$$0 = \dot{Z}\,W^{-1} - Z\,W^{-1}\,\dot{W}\,W^{-1} \tag{8}$$

$$0 = \dot{Z} - Z\,W^{-1}\,\dot{W}. \tag{9}$$

It turns out that all entries on the right hand side of (9) are zero except one. This can be seen by noticing that the first $n-1$ columns of $\dot{W}$ are simply the last $n-1$ columns of $W$, i.e. these first columns of $\dot{W}$ are obtained by shifting $W$ to the left. And so the first $n-1$ columns of $W^{-1}\,\dot{W}$ are the last $n-1$ columns of the identity matrix, i.e.

$$W^{-1}\,\dot{W} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \times \\ 1 & 0 & \cdots & 0 & \times \\ 0 & 1 & \cdots & 0 & \times \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \times \end{bmatrix} \tag{10}$$

And so the right hand side of (9) is

$$\dot{Z} - Z\,W^{-1}\,\dot{W} = \begin{bmatrix} w^{(n+1)} & \cdots & w^{(2n)} \end{bmatrix} - \begin{bmatrix} w^{(n)} & \cdots & w^{(2n-1)} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & \times \\ 1 & 0 & \cdots & 0 & \times \\ 0 & 1 & \cdots & 0 & \times \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \times \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & \times \end{bmatrix}.$$

Therefore, (9) simplifies to just the entry in the last column which is

$$w^{(2n)} - \begin{bmatrix} w^{(n)} & \cdots & w^{(2n-1)} \end{bmatrix} \begin{bmatrix} w & \dot{w} & \cdots & w^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(n-1)} & w^{(n)} & \cdots & w^{(2n-2)} \end{bmatrix}^{-1} \begin{bmatrix} w^{(n)} \\ \vdots \\ w^{(2n-1)} \end{bmatrix} = 0. \tag{11}$$

If we enlarge the matrix $W$ to a new matrix $W'$ by adding a column and a row of higher order derivatives then the expression in (11) is in fact the Schur complement of the bottom right element $W$ in $W'$. Expressly, if $w \in \mathcal{L}_n^1$ then Schur($W$) = 0 in $W'$, where

$$W' = \begin{bmatrix} w & \dot{w} & \cdots & w^{(n-1)} & w^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \\ w^{(n-1)} & w^{(n)} & \cdots & w^{(2n-2)} & w^{(2n-1)} \\ w^{(n)} & w^{(n+1)} & \cdots & w^{(2n-1)} & w^{(2n)} \end{bmatrix}$$

In arriving at this result, two assumptions were made: $r_0 \neq 0$ and $det(W) \neq 0$. If either of these conditions were not true then the minimal order of the differential equation for which $w$ is a solution is less than $n$. If $r_0 = 0$, then $w$ satisfies $r_1 w^{(n-1)} + \cdots + r_{n-1}\dot{w} + r_n w = 0$ for some real coefficients $r_1, \cdots, r_n$, and therefore $w \in \mathcal{L}_{n-1}^1$ and consequently $det(W) = 0$ and vice versa. Thus, if $w \in \mathcal{L}_n^1$ then either Schur($W$) = 0 in $W'$ or $det(W) = 0$. These two conditions are combined in the following manner: An LDU decomposition of the matrix $W'$ where the Schur complement of $W$ appears as shown in Appendix A.2 can be obtained.

$$W' = \begin{bmatrix} I & O \\ ZW^{-1} & I \end{bmatrix} \begin{bmatrix} W & O \\ O & \text{Schur}(W) \end{bmatrix} \begin{bmatrix} I & W^{-1}Z^T \\ O & I \end{bmatrix}$$

Using the previous expression, the determinant of $W'$ is given by,

$$\det(W') = \det(W)\ \det(\text{Schur}(W))\,.$$

Therefore, If Schur$(W) = 0$ then $\det W' = 0$. Now, if $w \in \mathcal{L}^1_{n-1}$ then as mentioned before $w$ satisfies $r_1 w^{(n-1)} + \cdots + r_{n-1}\dot{w} + r_n w = 0$ for some real coefficients $r_1, \cdots, r_n$, and so $r_1 w^{(2n-1)} + \cdots + r_n w^{(n)} = 0$ also. Consequently, $\det(W') = 0$. This leads to the aforestated condition that if $w \in \mathcal{L}^1_n$ then the determinant of $W'$ is zero, since either $Schur(W) = 0$ in $W'$ or $det(W) = 0$ and both of these lead to the fact that the determinant of $W'$ is zero.

### 4.1.2 Vector LTID Types ($k > 1$)

In this section, the previous result is extended to the vector case, i.e. the goal here is to derive a kernel representation for the general LTID type, $\mathcal{L}^k_n$. It is assumed that the system of differential equations is not underdetermined or overdetermined, particularly if the function $w$ is $k$-dimensional then the number of equations in the system is $k$. The operator characterizing the type is a Schur complement as stated in the following theorem.

**Theorem 4.1.2.** *The LTID type, $\mathcal{L}^k_n$, is characterized by the kernel of a Schur complement, specifically the operator associated with the type is defined as*

$$\mathbf{Op}\, w = Schur\left(\widehat{W}\right)\ in\ W,$$

*where*

$$W = \left[\begin{array}{ccc|c} w & \cdots & w^{(nk-1)} & w^{(nk)} \\ \vdots & & \vdots & \vdots \\ w^{(n-1)} & \cdots & w^{(n-1+nk-1)} & w^{(n-1+nk)} \\ \hline w^{n} & \cdots & w^{(n-1+nk)} & w^{(n+nk)} \end{array}\right] \tag{12}$$

*and*

$$\widehat{W} = \begin{bmatrix} w & \cdots & w^{(nk-1)} \\ \vdots & & \vdots \\ w^{(n-1)} & \cdots & w^{(n-1+nk-1)} \end{bmatrix}.$$

**PROOF.** If $w \in \mathcal{L}_n^k$ then there exist $R_i \in \mathbb{R}^{k\times k}$ such that the following holds true:

$$R_0 w^{(n)} + R_1 w^{(n-1)} + \cdots + R_n w = 0, \tag{13}$$

where $w = \begin{bmatrix} w_1 & \cdots & w_k \end{bmatrix}^T$. Assume that $\det(R_0) \neq 0$, then (13) can be multiplied on the left by $R_0^{-1}$ to obtain the following equation, and the solution set remains unchanged because this multiplication is a left unimodular transformation [34].

$$w^{(n)} + R_1 w^{(n-1)} + \cdots + R_n w = 0. \tag{14}$$

If $\det(R_0) = 0$, then the system (13) contains atleast one equation of order $(n-1)$ and cases of this type are excluded from $\mathcal{L}_n^k$. Differentiating (14) a number of times yields,

$$\left(\mathbf{D}^n + R_1\mathbf{D}^{n-1} \cdots + R_n\right)\begin{bmatrix} w & \dot{w} & \cdots & w^{(nk)} \end{bmatrix} = 0. \tag{15}$$

$$\begin{bmatrix} R_n & \cdots & I \end{bmatrix}\begin{bmatrix} w & \dot{w} & \cdots & w^{(nk)} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(n)} & w^{(n+1)} & \cdots & w^{(n+nk)} \end{bmatrix} = 0 \tag{16}$$

The square matrix in (16) is named $W$, as indicated in the theorem statement, and is partitioned in the manner shown in (12). Let's name the upper left and the upper right block of this partitioned matrix as $\widehat{W}$ and $\widetilde{W}$ respectively. i.e.,

$$\widehat{W} = \begin{bmatrix} w & \cdots & w^{(nk-1)} \\ \vdots & & \vdots \\ w^{(n-1)} & \cdots & w^{(n-1+nk-1)} \end{bmatrix} \tag{17}$$

$$\widetilde{W} = \begin{bmatrix} w^{(nk)} \\ \vdots \\ w^{(n-1+nk)} \end{bmatrix}. \tag{18}$$

Assuming that $\det(\widehat{W}) \neq 0$, an LDU decomposition of the matrix $W$ is obtained using the formula shown in Appendix A.2, and (16) is transformed to,

$$\begin{bmatrix} R_n & \cdots & R_1 & I \end{bmatrix}\left[\begin{array}{c|c} I & O \\ \hline \begin{bmatrix} w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix}\widehat{W}^{-1} & I \end{array}\right]\left[\begin{array}{c|c} \widehat{W} & O \\ \hline O & \text{Schur}\left(\widehat{W}\right) \end{array}\right] = 0. \tag{19}$$

$$\Rightarrow \begin{cases} \begin{bmatrix} R_n & \cdots & I \end{bmatrix} \widehat{W} + R_0 \begin{bmatrix} w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix} = 0 \\ \text{Schur}(\widehat{W}) = 0 \end{cases} \tag{20}$$

The first equation in (20) is just a subset of the original set of equations (16), specifically the ones formed by using the columns to the left of the partition in (12). Thus, if $w \in \mathcal{L}_n^k$ then a necessary condition for $w$ is that Schur($\widehat{W}$) in $W$ is zero, i.e.,

$$\begin{bmatrix} w^{(n+nk)} \end{bmatrix} - \begin{bmatrix} w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix} \widehat{W}^{-1} \widetilde{W} = 0 \tag{21}$$

where $\widehat{W}$ and $\widetilde{W}$ are as defined in (17) and (18). Notice that

$$\det(\widehat{W}^T) = \mathbf{W}\left(w_1, \cdots, w_k, \dot{w}_1, \cdots, \dot{w}_k, w_1^{(n-1)}, \cdots, w_k^{(n-1)}\right),$$

and so if the det($\widehat{W}$) = 0 then these functions are linearly dependent. Therefore, the condition that det($\widehat{W}$) = 0 corresponds to the existence of atleast one $(n-1)$th order differential equation among the system of $k$ differential equations (13). Cases of this type are excluded from $\mathcal{L}_n^k$.

On the other hand, if Schur($\widehat{W}$) = 0 then

$$w^{(n+nk)} - \begin{bmatrix} w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix} \widehat{W}^{-1} \begin{bmatrix} w^{(nk)} \\ \vdots \\ w^{(n-1+nk)} \end{bmatrix} = 0. \tag{22}$$

The previous equation can be extended to a larger $(k \times nk)$ matrix equation, where (22) occupies entries in the last column and the other entries are just zeroes.

$$\begin{bmatrix} w^{(n+1)} & \cdots & w^{(n+nk)} \end{bmatrix} - \begin{bmatrix} w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix} \widehat{W}^{-1} \begin{bmatrix} \dot{w} & \cdots & w^{(nk)} \\ \vdots & \cdots & \vdots \\ w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix} = 0. \tag{23}$$

Let $Z = \begin{bmatrix} w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix}$. Then (23) is

$$\begin{aligned} \dot{Z} - Z\,\widehat{W}^{-1}\,\dot{\widehat{W}} &= 0 \\ \Rightarrow \dot{Z}\,\widehat{W}^{-1} - Z\,\widehat{W}^{-1}\,\dot{\widehat{W}}\,\widehat{W}^{-1} &= 0 \\ \Rightarrow \frac{d}{dt}\left(Z\,\widehat{W}^{-1}\right) &= 0 \end{aligned} \tag{24}$$

The previous equation (24) implies that $Z\,\widehat{W}^{-1}$ is constant, i.e.

$$\begin{bmatrix} w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix} \begin{bmatrix} w & \cdots & w^{(nk-1)} \\ \vdots & & \vdots \\ w^{(n-1)} & \cdots & w^{(n-1+nk-1)} \end{bmatrix}^{-1} = R = \left[\begin{array}{c|c|c} R_n & \cdots & R_1 \end{array}\right] \tag{25}$$

for some $R \in \mathbb{R}^{k\times nk}$. If $R$ is partitioned as shown, then multiplying both sides by $\widehat{W}$ on the right and considering the first column,

$$w^{(n)} = R_1 w^{(n-1)} + \cdots + R_{n-1}\dot{w} + R_n w.$$

Hence, if $\text{Schur}(\widehat{W}) = 0$ then $w \in \mathcal{L}_n^k$. □

### 4.1.3 Constrained LTID Types

Within the $\mathcal{L}_n^k$ type, different traits can be defined by specifying more constraints on the coefficients of the differential equations. There may be situations when one is interested in a specific trait rather than the entire type. The traits can be defined by individually specifying the constraint each coefficient must satisfy. Remember from the previous sections that the coefficients can be expressed in terms of the function $w$ and its derivatives as in (25). The idea is depicted here by considering the specific trait of harmonics in the second order LTID type $\mathcal{L}_2^1$.

The trait of harmonics is the behavior of all functions $w$ such that it satisfies $\ddot{w} + \omega^2\, w = 0$, where $\omega \in \mathbb{R}$. Directly employing the idea of separating the coefficients,

$$\frac{\ddot{w}}{w} = -\omega^2. \tag{26}$$

Differentiating (26) we get,

$$\frac{w\, w^{(3)} - \dot{w}\,\ddot{w}}{w^2} = 0. \tag{27}$$

So, the trait of harmonics could be defined by (27), but then the solution set also includes the exponentials. In order to completely characterize the trait of harmonics, an additional constraint needs to be specified, namely the coefficient is positive or $\frac{\ddot{w}}{w} < 0$. Therefore, the trait of harmonics is characterized by,

$$\begin{cases} w\,w^{(3)} - \dot{w}\,\ddot{w} = 0 \\ \dfrac{\ddot{w}}{w} < 0 \end{cases}. \tag{28}$$

Alternatively, consider the LTID type $\mathcal{L}_2^1$, characterized by

$$\ddot{w} = r_1\dot{w} + r_2 w. \tag{29}$$

The coefficients in (29) can now be expressed as,

$$\begin{aligned} \begin{bmatrix} r_2 & r_1 \end{bmatrix} &= \begin{bmatrix} \ddot{w} & w^{(3)} \end{bmatrix} \begin{bmatrix} w & \dot{w} \\ \dot{w} & \ddot{w} \end{bmatrix}^{-1} \\ &= \frac{1}{w\,\ddot{w} - \dot{w}^2} \begin{bmatrix} \ddot{w}^2 - \dot{w}\,w^{(3)} & w\,w^{(3)} - \dot{w}\,\ddot{w} \end{bmatrix}. \end{aligned} \tag{30}$$

For the trait of harmonics, it is required that $r_1$ be zero, $r_2$ be a constant, and that $r_2$ be negative. These conditions correspond to,

$$\begin{cases} w\,w^{(3)} - \dot{w}\,\ddot{w} = 0 \\ \dfrac{d}{dt}\left(\dfrac{\ddot{w}^2 - \dot{w}\,w^{(3)}}{w\,\ddot{w} - \dot{w}^2}\right) = 0 \\ \dfrac{\ddot{w}^2 - \dot{w}\,w^{(3)}}{w\,\ddot{w} - \dot{w}^2} < 0 \end{cases}. \tag{31}$$

Again remember from the previous section that derivatives of the coefficients are given by the expression,

$$\begin{aligned} \begin{bmatrix} \dot{r}_2 & \dot{r}_1 \end{bmatrix} &= \begin{bmatrix} 0 & Schur\left(\begin{bmatrix} w & \dot{w} \\ \dot{w} & \ddot{w} \end{bmatrix}\right) \end{bmatrix} \begin{bmatrix} w & \dot{w} \\ \dot{w} & \ddot{w} \end{bmatrix}^{-1} \\ &= \frac{1}{w\,\ddot{w} - \dot{w}^2} \begin{bmatrix} -\dot{w} & w \end{bmatrix} Schur\left(\begin{bmatrix} w & \dot{w} \\ \dot{w} & \ddot{w} \end{bmatrix}\right). \end{aligned} \tag{32}$$

Thus, the second condition in (31) can be eliminated because if $r_1 = 0$ then $\dot{r}_1 = 0$, which from (32) implies that $Schur(\widehat{W}) = 0$ since $w \neq 0$ and consequently $\dot{r}_2 = 0$. The third condition in (31) can be simplified to the form in (28 by expressing the condition as,

$$\frac{\frac{d}{dt}\left(\frac{\dot{w}}{\ddot{w}}\right)}{\frac{d}{dt}\left(\frac{\dot{w}}{w}\right)} \frac{w^2}{\ddot{w}^2} < 0.$$

$$\Rightarrow \frac{\frac{d}{dt}\left(\frac{\dot{w}}{\ddot{w}}\right)}{\frac{d}{dt}\left(\frac{\dot{w}}{w}\right)} < 0$$

This implies that,

$$\frac{d}{dt}\left(\frac{\dot{w}}{\ddot{w}}\right) = -C\frac{d}{dt}\left(\frac{\dot{w}}{w}\right),$$

for some positive $C$. And so,

$$\frac{\dot{w}}{\ddot{w}} = -C\left(\frac{\dot{w}}{w}\right) + D.$$

Since neither $w$ nor $\dot{w}$ is zero otherwise the third condition in (31) is violated, the expression can be multiplied by $\frac{w}{\dot{w}}$ to obtain

$$\frac{w}{\ddot{w}} = -C + D\left(\frac{w}{\dot{w}}\right).$$

The first condition in (31) states that $\frac{\ddot{w}}{w}$ is a constant and so $D$ must be zero because if $\frac{w}{\dot{w}}$ is a constant then $w$ is an element of a first order type. Thus, the third condtition in (31) reduces to $\frac{\ddot{w}}{w} < 0$.

## 4.2 The $\tau$-Periodic Type

In this section, the periodic type is introduced in context of the Gluskabi framework. The periodic types are defined by the kernel of operators involving the shift operator, $\mathbf{S}$. For instance the $\tau$-periodic type, which is the behavior of periodic functions of period $\tau$, is defined by the kernel of the operator $\mathbf{Op} := (\mathbf{I} - \mathbf{S}_{-\tau})$ where $\mathbf{I}$ is the identity operator and $\mathbf{S}_{-\tau} f(t) = f(t-\tau)$. The operator for the periodic type could also be defined using an advance - $\mathbf{S}_{\tau} f(t) = f(t+\tau)$. Throughout this thesis, we will define the operator for the $\tau$-periodic type using the lag.

A periodic function whose Fourier series exists can also be seen as a sum of harmonic signals of integer multiples of the base frequency. Inspired by this observation, the $\tau$-periodic type in the base behavior $\mathcal{B}_0 = C^{\omega}(\mathbb{R}, \mathbb{R})$ may also be characterized by the following infinite product operator:

**Theorem 4.2.1.** *The $\tau$-periodic type associated with the operator* **Op** *can also be characterized by the infinite product operator* $\frac{\pi}{\omega}\mathbf{D}\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2\omega^2}\mathbf{D}^2\right)$*, which can also be written as* $\sinh\left(\frac{\pi}{\omega}\mathbf{D}\right)$ *[38], where* $\omega = 2\pi/\tau$.

Before delving into the proof of this theorem, some results are presented about the hyperbolic sine function. It is known that the function $sinh(x)$ admits both an infinite product expression and a Maclaurin series representation, i.e.

$$\sinh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = x\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2\pi^2}\right), \tag{33}$$

for all $x \in \mathbb{R}$. The infinite product in (33) can be expanded into an infinite polynomial, where the coefficient of each of the powers of $x$ is an infinite series, i.e.

$$x\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2\pi^2}\right) = x + \frac{1}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)x^3 + \frac{1}{\pi^4}\left[\frac{1}{2^2}+\frac{1}{3^2}\left(1+\frac{1}{2^2}\right)+\cdots\right]x^5+\cdots \tag{34}$$

Let's name the coefficient of $x^i$ in (34) as $a_{\frac{i-1}{2}}$. Then each $a_k$ is a series, i.e. $a_k = \sum_{j=0}^{\infty} a_{kj}$. For instance,

$$a_1 = \frac{1}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}.$$

By comparing the coefficients of the powers of $x$ in the two series in (33), it is concluded that each of the $a_k$'s is a convergent series whose value is the coefficient of the $x^{2k+1}$ term in the Maclaurin series, i.e.

$$a_k = \frac{1}{(2k+1)!}.$$

To show the equivalence of two operators, a known result about power series is needed and so will be presented here.

**Lemma 4.2.2.** *Let* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *be a power series. Then there exists and* $R \in \mathbb{R}$ *such that the power series is absolutely convergent for x satisfying* $0 \le x < R$.

**PROOF.** This result can be found in any analysis textbook but the proof will be reproduced here for completeness. Suppose the power series converges for some $x_0 \neq 0 \in \mathbb{R}$. Then the terms of the series converge to zero and are bounded, i.e. there exists $M > 0$ such that

$$|a_n(x_0 - c)^n| \leq M \qquad \text{for } n = 0, 1, 2, \cdots$$

If $|x - c| < |x_0 - c|$, then

$$|a_n(x - c)^n| = |a_n(x_0 - c)^n| \left| \frac{x - c}{x_0 - c} \right|^n \leq M \left| \frac{x - c}{x_0 - c} \right|^n .$$

This means that absolute value of each term of the power series is bounded above by corresponding term of the convergent geometric series. Thus, if the power series converges for some $x_0 \in \mathbb{R}$ then it converges absolutely for every $x \in \mathbb{R}$ with $|x - c| < |x_0 - c|$. Finally, let

$$R = \sup \left\{ |x - c| \text{ s.t. } \sum a_n(x - c)^n \text{ converges} \right\}. \quad \square$$

**PROOF.** **(Theorem 4.2.1)** Let's name the two expressions for the operator $\sinh\left(\frac{\pi}{\omega}\mathbf{D}\right)$ as

$$\mathbf{Op}_1 = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\tau}{2}\right)^{2k+1} \mathbf{D}^{2k+1} = \frac{\tau}{2}\mathbf{D} + \frac{1}{3!}\left(\frac{\tau}{2}\right)^3 \mathbf{D}^3 + \frac{1}{5!}\left(\frac{\tau}{2}\right)^5 \mathbf{D}^5 + \cdots$$

$$\mathbf{Op}_2 = \frac{\tau}{2}\mathbf{D} \prod_{k=1}^{\infty} \left[\mathbf{I} + \frac{1}{k^2\pi^2}\left(\frac{\tau}{2}\right)^2 \mathbf{D}^2\right] = \frac{\tau}{2}\mathbf{D}\left[1 + \frac{1}{\pi^2}\left(\frac{\tau}{2}\right)^2 \mathbf{D}^2\right]\left[1 + \frac{1}{4\pi^2}\left(\frac{\tau}{2}\right)^2 \mathbf{D}^2\right]\cdots$$

The equivalence between the two operators, $\mathbf{Op}_1$ and $\mathbf{Op}_2$ will be shown first, and then it will be shown that $\mathbf{Op}_1$ also characterized the $\tau$-periodic type. Two operators are said to be equivalent if their domains are equal and the images are equal for every element in the domain, i.e. $\sigma_t \, \mathbf{Op}_1 \, f = \sigma_t \, \mathbf{Op}_2 \, f$ for every $f \in C^\omega(\mathbb{R}, \mathbb{R})$ and for every $t \in \mathbb{R}$ where $\sigma_t$ is the evaluation functional.

The first step is showing that the operator $\mathbf{Op}_1$ is well-defined in $\mathcal{B}_0 = C^\omega(\mathbb{R}, \mathbb{R})$, i.e. $\mathbf{Op}_1 \, f$ exists for all analytic functions $f$. It is further shown that $\mathbf{Op}_1$ is equivalent to the shift operator $\frac{1}{2}\left(\mathbf{S}_{\tau/2} - \mathbf{S}_{-\tau/2}\right)$. It is given that the Taylor series of the function $f$ exists at all points. So, the Taylor series of $f$ about the point $t$ and evaluated at the points $\left(t + \frac{\tau}{2}\right)$ and $\left(t - \frac{\tau}{2}\right)$ are given by,

$$f\left(t + \frac{\tau}{2}\right) = 1 + \frac{\tau}{2}\sigma_t \dot{f} + \frac{1}{2!}\left(\frac{\tau}{2}\right)^2 \sigma_t f^{(2)} + \frac{1}{3!}\left(\frac{\tau}{2}\right)^3 \sigma_t f^{(3)} + \cdots \tag{35}$$

$$f\left(t - \frac{\tau}{2}\right) = 1 - \frac{\tau}{2}\sigma_t \dot{f} + \frac{1}{2!}\left(\frac{\tau}{2}\right)^2 \sigma_t f^{(2)} - \frac{1}{3!}\left(\frac{\tau}{2}\right)^3 \sigma_t f^{(3)} + \cdots \tag{36}$$

By the above lemma, both of the series above are absolutely convergent. Consequently, the negative of either of these series is also absolutely convergent and so is their sum. Therefore,

$$\begin{aligned}\frac{1}{2}f\left(t+\frac{\tau}{2}\right)-\frac{1}{2}f\left(t-\frac{\tau}{2}\right) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\tau}{2}\right)^{k}\sigma_t f^{(k)}-\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{\tau}{2}\right)^{k}\sigma_t f^{(k)}\\ &= \left(\frac{1}{2}-\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{2}\right)\frac{\tau}{2}\dot{f}+\left(\frac{1}{2}-\frac{1}{2}\right)\frac{1}{2!}\left(\frac{\tau}{2}\right)^{2}f^{(2)}\\ &\quad+\left(\frac{1}{2}+\frac{1}{2}\right)\frac{1}{3!}\left(\frac{\tau}{2}\right)^{3}f^{(3)}+\cdots\\ &= \frac{\tau}{2}\dot{f}+\frac{1}{3!}\left(\frac{\tau}{2}\right)^{3}f^{(3)}+\frac{1}{5!}\left(\frac{\tau}{2}\right)^{5}f^{(5)}+\cdots\\ &= \sigma_t\,\mathbf{Op}_1\,f.\end{aligned}$$

Hence, $\sigma_t\,\mathbf{Op}_1\,f$ is also an absolutely convergent series for all $t$ and

$$\mathbf{Op}_1\,f=\frac{1}{2}\left(e^{\frac{\tau}{2}\mathbf{D}}-e^{-\frac{\tau}{2}\mathbf{D}}\right)f=\frac{1}{2}\left(\mathbf{S}_{\frac{\tau}{2}}-\mathbf{S}_{-\frac{\tau}{2}}\right)f.$$

The next step is showing that $\sigma_t\,\mathbf{Op}_2\,f$ is an absolutely convergent series for all $t$. The partial sum $S_n$ of the corresponding absolute series of $\sigma_t\,\mathbf{Op}_2\,f$ is given by,

$$S_n=\sum_{k=0}^{n}\sum_{j=0}^{n-k+1}a_{kj}\left(\frac{\tau}{2}\right)^{2k+1}\left|\sigma_t\,f^{(2k+1)}\right|,$$

where $a_{kj}$ are the terms described above for the sinh series. Then,

$$S_n\le\sum_{k=0}^{n}\frac{1}{(2k+1)!}\left(\frac{\tau}{2}\right)^{2k+1}\left|\sigma_t\,f^{(2k+1)}\right|.$$

This shows that the partial sums are actually bounded above by the partial sums of the absolute series corresponding $\sigma_t\,\mathbf{Op}_1\,f$, which is absolutely convergent and so the partial sums are bounded above. Since the partial sums are nonnegative this implies that $S_n$ is a convergent sequence and consequently $\sigma_t\,\mathbf{Op}_2\,f$ is absolutely convergent. Hence, the terms of the series $\sigma_t \,\mathbf{Op}_2\, f$ can be rearranged as follows and the equivalence can be shown.

$$\begin{aligned}\sigma_t \,\mathbf{Op}_2\, f &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{kj}\left(\frac{\tau}{2}\right)^{2k+1} \sigma_t\, f^{(2k+1)} \\ &= \sum_{k=0}^{\infty} a_k \left(\frac{\tau}{2}\right)^{2k+1} \sigma_t\, f^{(2k+1)} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\left(\frac{\tau}{2}\right)^{2k+1} \sigma_t\, f^{(2k+1)} \\ &= \sigma_t \,\mathbf{Op}_1\, f\end{aligned}$$

Finally, it is shown that the kernel of the two operators $\left(\mathbf{S}_{\frac{\tau}{2}} - \mathbf{S}_{-\frac{\tau}{2}}\right)$ and $(\mathbf{I} - \mathbf{S}_{-\tau})$ are equal. In fact,

$$\begin{aligned}&\left(\mathbf{S}_{\frac{\tau}{2}} - \mathbf{S}_{-\frac{\tau}{2}}\right) f = 0 \\ \Leftrightarrow\ & S_{-\frac{\tau}{2}}\left(\mathbf{S}_{\frac{\tau}{2}} - \mathbf{S}_{-\frac{\tau}{2}}\right) f = 0 \\ \Leftrightarrow\ & (\mathbf{I} - \mathbf{S}_{-\tau})\, f = 0.\end{aligned}$$

Thus, the periodic type with period $\tau$ in the base behavior of analytic functions can be characterized by either the kernel of $(\mathbf{I} - \mathbf{S}_{-\tau})$ or $\sinh\left(\frac{\pi}{\omega}\mathbf{D}\right)$. □

## 4.3 The General Periodic Type

Finding a kernel representation for the general periodic type, i.e., type characterizing periodic signals of any period, is a difficult problem. For a finite set of periods, $\{\tau_1, \tau_2, \cdots, \tau_n\}$, an operator **Op** characterizing signals of periods belonging to this set is,

$$\mathbf{Op}\, w = [w(t) - w(t-\tau_1)]\,[w(t) - w(t-\tau_2)] \cdots [w(t) - w(t-\tau_n)]\,.$$

Extending this idea to signals of any period requires the concept of products over an uncountable set. The idea of a product integral can be utilized here which can be thought of as a continuous version of discrete product analogous of going to the Riemann integral from the sum. So a possible operator could be,

$$\mathbf{Op}\, w = \lim_{\Delta\tau\to 0} \prod\, [w(t) - w(t-\tau)]^{2\Delta\tau} = \exp\left(\int_0^{\infty} \ln\,[w(t) - w(t-\tau)]^2 d\tau\right).$$

For continuous functions, $w(t) - w(t - \tau)$ is a continuous function of $\tau$ for every $t$. For a periodic function $w$ this difference goes to zero at every integer multiple of the period, and so the logarithm of this difference goes to $-\infty$ at every integer multiple of the period. Consequently, the integral is $-\infty$ and $\mathbf{Op}\, w = 0$. Thus, $\mathbf{Op}\, w = 0$ is definitely a necessary condition for $w$ to be periodic. For unbounded functions, this operator is not well-defined.

# CHAPTER 5

# SIGNAL RACCORDATION PROBLEM

The objective of this chapter is to present results about solution to the Gluskabi problem, specifically the signal raccordation problem when the types are described by the kernel of some operator. Using the established terminology from Section 3.1, the Gluskabi problem is restated here: Given a type $\mathcal{T}$, with the associated operator **Op**, and a finite time interval $[a, b] \subset \mathbb{T}$, solving the Gluskabi problem is to assign an element from the base behavior to every pair of elements from the type such that the raccordation is maximally persistent in the type. So, the objective is to find the Gluskabi extension or alternatively the Gluskabi map. For the signal raccordation problem, the base behavior is some appropriately chosen function space. For the rest of the chapter, it will be assumed that the raccordation is sought over the interval $[a, b]$ unless stated otherwise.

The Gluskabi problem is an optimal control problem and it is solved for the combination of a number of types and base behaviors in this section. This chapter is organized as follows: At first a new compact method is presented for finding the Gluskabi extension for a broad collection of types. This collection includes stationary, periodic, and the newly introduced LTID types. Next, the focus is shifted to the periodic type and the Gluskabi map is determined for this specific type for two different choices of base behaviors.

As mentioned previously, this research is an extension of the previous body of work by Yeung and Verriest. The focus of their work is on the motivation of the Gluskabi problem, and so each individual problem is dealt with separately as opposed to the generalized approach in this research. Two ways are used to characterize the types – the kernel representation, i.e. as the kernel of some operator just as done in this thesis, and the image representation when the type is parametrized resulting in a one to one correspondence between the type and some Euclidean space, and then each trajectory is a point in the parameter space. Using the latter characterization, the raccordation is the inverse image of the geodesic between the two parameter points under the parametrizing function, with respect to the Riemannian metric chosen to make the parameter space isometric to the type. The kernel approach has been formalized and systematically developed in this thesis.

The Gluskabi problem for the periodic type was also treated in these prior works: [6] and [7]. The image approach for the periodic type is to solve this problem using Fourier series expansions. However, to use this method practically the Fourier series needs to be truncated to a finite number of terms and so a compromise has to be made between accuracy and computational complexity, when choosing the number of terms. The kernel approach presented in these earlier works uses impulsive approximations to arrive at a result. In what follows, the Gluskabi framework is employed to arrive at an alternative but simpler solution for finding a raccordation between two periodic trajectories. The method proposed in this research is simpler, does not require any approximations, and does not impose any restrictions on the length of the raccordation interval, unlike the prior works where the length of the interval had to be a multiple of the period and much greater than the period.

## 5.1 Gluskabi Extension for General Types

In this section, the Gluskabi extension is derived for a number of types that satisfy only two assumptions. The two assumptions are stated as follows:

(S1) The range of the operator associated with the type or $\mathcal{V}$, restricted to the support of the Gluskabi map is an inner product space, with inner product $\langle\cdot,\cdot\rangle$.

(S2) The operator associated with the type admits an adjoint.

These assumptions are not extremely restrictive and are satisfied by a number of interesting operators such as the differential operators and shift operators. Further comments will be made about the assumption (S2) in Section 5.1.1. For the rest of this thesis, it will be assumed that the two assumptions, (S1) and (S2), always hold. The Gluskabi extension for these types is found in the following theorem.

**Theorem 5.1.1.** *Given a type* $(\mathcal{T}, \mathbf{Op})$*, the Gluskabi extension with respect to the norm* $\|.\|_{\mathbf{Q}}$*, where* $\mathbf{Q}$ *is a self-adjoint operator, is given by*

$$\mathcal{G}_{\mathcal{T}} = \{w \in \mathcal{B}_0 \ \textit{ such that } \exists w_1, w_2 \in \mathcal{T} \textit{ with } \Pi_- w = w_1,\ \Pi_+ w = w_2, \textit{ and } \mathbf{Op}_w^* \mathbf{Q} \mathbf{Op}\, w = 0\},$$

*where the norm is computed as* $\|.\|_{\mathbf{Q}}^2 = \langle \mathbf{Q}(.), (.)\rangle$*, and* $\mathbf{Op}_w$ *is the linearized form (Gâteaux derivative) of the operator* $\mathbf{Op}$ *about* $w$*.*

**PROOF.** This can be easily proved using variational calculus [40]. Given the type operator $\mathbf{Op} : \mathcal{A} \to \mathcal{V}$ and the norm $\|.\|_{\mathbf{Q}}^2$, which is defined on the support of the operator $\mathbf{Op}$, the cost functional to be minimized can be written as

$$J(w) = \|\mathbf{Op} w\|_{\mathbf{Q}}^2 = \langle \mathbf{Q}\mathbf{Op}\, w, \mathbf{Op}\, w\rangle. \tag{37}$$

Now using the assumption that $\mathbf{Op}$ is Gâteaux differentiable, it is shown that the first variation of $J$ exists and its expression in terms of $\mathbf{Op}$ is computed as follows:

$$\begin{aligned}\Delta J = J(w + th) - J(w) = &\langle \mathbf{Q}\mathbf{Op}(w + th), \mathbf{Op}(w + th)\rangle \\ &- \langle \mathbf{Q}\mathbf{Op}\, w, \mathbf{Op}\, w\rangle\end{aligned}$$

It is given that $\mathbf{Op}(w + th) = \mathbf{Op}\, w + \mathbf{Op}_w\, th + O\left(t^2\right)$, being the definition of Gâteaux differentiability (See Appendix B.1). So,

$$\begin{aligned}\Delta J = &\langle \mathbf{Q}\mathbf{Op}\, w, \mathbf{Op}_w\, th\rangle + \langle \mathbf{Q}\mathbf{Op}_w\, th, \mathbf{Op}\, w\rangle + \langle \mathbf{Q}\mathbf{Op}_w\, th, \mathbf{Op}_w\, th\rangle + \left\langle \mathbf{Q}\mathbf{Op}\, w, O\left(t^2\right)\right\rangle \\ &+ \left\langle \mathbf{Q}\, O\left(t^2\right), \mathbf{Op}\, w\right\rangle + \left\langle \mathbf{Q}\mathbf{Op}_w\, th, O\left(t^2\right)\right\rangle + \left\langle \mathbf{Q}O\left(t^2\right), \mathbf{Op}_w\, th\right\rangle + \left\langle O\left(t^2\right), O\left(t^2\right)\right\rangle.\end{aligned} \tag{38}$$

Using the given assumptions that $\mathbf{Q}$ is self adjoint, $\mathbf{Op}_w$ is linear, and the inner product is symmetric, (38) can be written as

$$\begin{aligned}\Delta J = &t\langle \mathbf{Q}\mathbf{Op}\, w, \mathbf{Op}_w\, th\rangle + t^2\langle \mathbf{Q}\mathbf{Op}_w\, h, \mathbf{Op}_w\, h\rangle + 2\left\langle \mathbf{Q}\mathbf{Op}\, w, O\left(t^2\right)\right\rangle + 2\left\langle \mathbf{Q}\mathbf{Op}_w\, th, O\left(t^2\right)\right\rangle \\ &+ \left\langle O\left(t^2\right), O\left(t^2\right)\right\rangle.\end{aligned} \tag{39}$$

Then, the first variation is given by

$$
\begin{aligned}
\delta J(w; h) &= \lim_{t \to 0} \frac{\Delta J}{t} \\
&= 2\langle \mathbf{QOp}\, w, \mathbf{Op}_w\, h \rangle \\
&= 2\langle \mathbf{Op}_w^* \mathbf{QOp}\, w, h \rangle
\end{aligned} \tag{40}
$$

since each of the other terms in (39) goes to zero as $t \to 0$, employing the lemma in Appendix B.2. If $w$ is the minimizer of the functional (37) then the first variation $\delta J(w; h)$ is zero at $w$. Thus a necessary condition for all raccordations in the Gluskabi extension $\mathcal{G}_{\mathcal{T}}$ is that

$$
\mathbf{Op}_w^* \mathbf{QOp}\, w = 0 \quad \forall w \in \mathcal{G}_{\mathcal{T}}. \quad \square \tag{41}
$$

If there exists an operator $\mathbf{Op}^*$ such that

$$
\mathbf{Op}^*(w + \delta w) - \mathbf{Op}^*\, w = \mathbf{Op}_w^*\, \delta w \quad \forall w \in \mathcal{A} \tag{42}
$$

then the above condition for the Gluskabi Extension (41) can be written as the following nested form.

$$
\mathbf{Op}^*(w + \mathbf{QOp}\, w) = \mathbf{Op}^*\, w \quad \forall w \in \mathcal{G}_{\mathcal{T}} \tag{43}
$$

Furthermore, an example of the norms that can be employed is the Sobolev norm. The operator $\mathbf{Q}$ corresponding to the $k$th order Sobolev norm can be found as follows:

$$
\begin{aligned}
\|w\|_W^2 &= \langle c_0\, w, c_0\, w \rangle + \langle c_1 \mathbf{D} w, c_1\, \mathbf{D} w \rangle + \cdots + \langle c_k \mathbf{D}^k w, c_k\, \mathbf{D}^k w \rangle \\
&= \langle c_0^2 w, w \rangle + \langle -c_1^2 \mathbf{D}^2 w, c_1\, w \rangle + \cdots + \langle c_k^2 (-\mathbf{D}^2)^k w, c_k\, w \rangle \\
&= \langle [c_0^2 - c_1^2 \mathbf{D}^2 + \cdots + c_k^2 (-\mathbf{D}^2)^k] w, w \rangle \\
&= \langle W\left(-\mathbf{D}^2\right) w, w \rangle = \langle \mathbf{Q}\, w, w \rangle,
\end{aligned}
$$

where $W$ is a polynomial with positive coefficients, $W(\xi) = c_0^2 + c_1^2 \xi + c_2^2 \xi^2 + \cdots + c_k^2 \xi^k$, and $\mathbf{D}$ is the differentiation operator. It is implicitly assumed here that the base behavior is controllable, as defined in Chapter 2. Controllability is one of the sufficient conditions for the Gluskabi raccordation to exist for all possible pairs in the type. The controllability condition can be relaxed and this will be explored further in the next chapter about dynamical raccordations. The typical choices of base behaviors such as the piecewise continuous functions or continuous functions are controllable behaviors. It is further remarked that the Gluskabi raccordation typically lies in an even smaller subset of the domain of the operator, $\mathcal{A}$.

To round up this section, an interesting corollary to the previous theorem is now presented.

**Corollary 5.1.2.** *Let* $\mathbf{Op} : \mathcal{A} \to \mathcal{V}$ *be the linear operator associated with the type* $\mathcal{T}$ *and its adjoint be* $\mathbf{Op}^* : \mathcal{V} \to \mathcal{A}$*, i.e.*

$$\langle \mathbf{Op}\, w, y \rangle = \langle w, \mathbf{Op}^*\, y \rangle \quad \textit{for all } w \in \mathcal{A} \textit{ and for all } y \in \mathcal{V}.$$

*Then, the Gluskabi raccordation for any pair of elements in the type,* $\mathcal{T}$*, if exists is also an element of the same type* $\mathcal{T}$*.*

**PROOF.** It is claimed that the kernel of $\mathbf{Op}^*$ is equal to the orthogonal complement of the image of $\mathbf{Op}$ in $\mathcal{V}$. For if $y \in \ker \mathbf{Op}^*$, then

$$0 = \langle w, \mathbf{Op}^*\, y \rangle = \langle \mathbf{Op}\, w, y \rangle \quad \forall w \in \mathcal{A},$$

or $y \in (\operatorname{im} \mathbf{Op})^{\perp}$. On the other hand, if $y \in (\operatorname{im} \mathbf{Op})^{\perp}$ then $\mathbf{Op}^*\, y$ is orthogonal to every element $w \in \mathcal{A}$ which means that $\mathbf{Op}^*\, y = 0$.

Now if $w$ is in the Gluskabi extension then it satisfies

$$\mathbf{Op}^* \mathbf{Op}\, w = 0.$$

This implies that $\mathbf{Op}\, w$ is in the kernel of $\mathbf{Op}^*$ or the orthogonal complement of the image of $\mathbf{Op}$. But $\mathbf{Op}\, w$ is also in the image of $\mathbf{Op}$. This implies that

$$\mathbf{Op}\, w \in (\operatorname{im} \mathbf{Op}) \cap (\operatorname{im} \mathbf{Op})^{\perp}.$$

$$\Rightarrow \mathbf{Op}\, w = 0$$

Thus, $w$ is in the kernel of **Op** or in other words $w$ is in the type $T$. □

So for instance if the base behavior is a finite dimensional space and **Op** is a finite operator then the Gluskabi raccordation if it exists at all lies in the type. This is not the case for differential operators like the ones described earlier. And so, for the differential operators the Gluskabi raccordation lies outside the type in general.

### 5.1.1 Adjoint of the operator $\mathbf{Op}_w$

The adjoint of a linear operator, **Op**, in an inner product space is defined as the operator $\mathbf{Op}^*$ that satisfies

$$\langle \mathbf{Op}\, w_1, w_2 \rangle = \langle w_1, \mathbf{Op}^*\, w_2 \rangle.$$

Since the space is a general inner product space it is not necessary that a non-trivial adjoint of the operator exists. Non-triviality means that the domain of the operator $\mathbf{Op}^*$ does not just contain the zero element, since the above equation is clearly satisfied if $w_2 = \mathbf{Op}^*\, w_2 = 0$. The domain of the adjoint is all elements $w_2$ such that the equation above holds for all $w_1$. In the present case, it is required for Theorem 5.1.1 that the operator $\mathbf{Op}_w$ restricted to the domain $\mathcal{H}$, i.e. the set of all elements in $\mathcal{A}$ such that the boundary conditions that are to be satisfied by the Gluskabi raccordation are all zero, admits a non-trivial adjoint. Additionally, it is required that $\mathbf{QOp}\, w$ be contained in the domain of the adjoint $\mathbf{Op}_w^*$ for all $w \in \mathcal{A}$. Because the adjoint is only required for a restricted domain, the number of operators that admit an adjoint increases, and we get non-trivial solutions unlike what happens in Corollary 5.1.2.

For instance, say $\mathbf{Op} : \mathcal{A} \to \mathcal{V}$ is a differential operator $f(\mathbf{D})$ of highest order $k$ and the $L^2$ inner product is considered. The boundary conditions in this case will be the value of the function and $(k-1)$ derivatives at each of the boundary points, $a$ and $b$. Consequently, the space $\mathcal{H}$ is the space of all functions in $\mathcal{A}$ such that their value and the first $(k-1)$ derivatives are zero at $a$ and $b$. The linearized operator $\mathbf{Op}_w$ in this case will be of the type

$\sum_{i=0}^{k} \mathbf{Q}_i \mathbf{D}^i$, where $\mathbf{Q}_i$ is a multiplication operator whose action is multiplication by some function of $w$. Then, using integration by parts

$$\begin{aligned}
\langle w_1, \mathbf{Op}_w\, h\rangle &= \int_a^b w_1(t) \sum_{i=0}^{k} q_i(w(t)) h^{(i)}(t)\, dt \\
&= \int_a^b h(t) \sum_{i=0}^{k} (-1)^i \left[w_1(t) q_i(w(t))\right]^{(i)}\, dt \\
&\quad + \sum_{i=1}^{k} \sum_{j=1}^{i} (-1)^{i-1} \left[w_1(t) q_i(w(t))\right]^{(j-1)} h^{(i-j)}(t) \Big|_a^b \\
&= \int_a^b h(t) \sum_{i=0}^{k} (-1)^i \left[w_1(t) q_i(w(t))\right]^{(i)}\, dt \\
&= \langle \sum_{i=0}^{k} (-1)^i \mathbf{D}^i \mathbf{Q}_i\, w_1, h\rangle
\end{aligned}$$

The boundary terms are zero because of the fact that $h \in \mathcal{H}$ and so $h^{(i-j)}(b) = h^{(i-j)}(a) = 0$, for $1 \leq i, j \leq k$. Therefore, contrary to the usual scenario for differential operators, they admit adjoints over here because of the restricted domain.

#### *5.1.1.1 Adjoint of the scalar LTID Type Operator*

This section is concluded with finding the adjoint of the operator for the scalar LTID type, $\mathcal{L}_n^1$, introduced in Section 4.1.1. Remember that the operator for this type is $\mathbf{Op}\, w = \det(W)$, where

$$W(w) = \begin{bmatrix} w & \dot{w} & \cdots & w^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(n)} & w^{(n+1)} & \cdots & w^{(2n)} \end{bmatrix}.$$

The linearized version of this operator can be found as,

$$\begin{aligned}
\mathbf{Op}_w\, h &= \frac{d}{dt} \mathbf{Op}(w + th) \Big|_{t=0} \\
&= |W| \operatorname{tr}\left( W^{-1} \left. \frac{dW(w+th)}{dt} \right|_{t=0} \right) \\
&= |W| \operatorname{tr}\left[ \frac{C_w^T}{|W|} W(h) \right] \\
&= \operatorname{tr}\left[ C_w^T W(h) \right],
\end{aligned}$$

where $C_w^T$ is the adjugate matrix of $W$, i.e. the transpose of the matrix of cofactors.

Now the adjoint of this operator, which is linear in $h$, is obtained by realizing that this operator is of the same form, $\sum_{i=0}^{k} \mathbf{Q}_i \mathbf{D}^i$, as discussed above. Thus, the adjoint of the operator $\mathbf{Op}_w$ restricted to the domain $\mathcal{H}$ is

$$\mathbf{Op}_w^* = \operatorname{tr}\left(\begin{bmatrix} \mathbf{I} & -\mathbf{D} & \cdots & (-1)^n \mathbf{D}^n \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^n \mathbf{D}^n & (-1)^{n+1} \mathbf{D}^{n+1} & \cdots & (-1)^{2n} \mathbf{D}^{2n} \end{bmatrix} \mathbf{C}_w^T\right),$$

where $\mathbf{C}_w^T$ is now a multiplication operator, i.e. each element of $\mathbf{C}_w^T$ is a multiplication operator that acts by multiplying the argument by the corresponding element of $C_w^T$.

### 5.1.2 Vector-valued Operator Op

The operator **Op** associated with the type can be vector-valued or matrix-valued and the element $w$ in the type can be vectors in general as well, i.e.

$$\mathbf{Op}\, w = \begin{bmatrix} \mathbf{Op}_{11}\, w & \cdots & \mathbf{Op}_{1n}\, w \\ \vdots & & \vdots \\ \mathbf{Op}_{m1}\, w & \cdots & \mathbf{Op}_{mn}\, w \end{bmatrix},$$

where $\mathbf{Op}_{ij}$ are all scalar valued operators. It is clear that Theorem 5.1.1 can still be applied to this case if there exists a valid inner product on $\mathcal{V}$, the operator **Op** is Gâteaux differentiable, and then the subsequent linearized operator admits an adjoint.

It is evident from the definition of Gâteaux differentiability in Appendix B.1 that the operator **Op** is Gâteaux differentiable if and only if all of its component scalar operators are Gâteaux differentiable.

$$\begin{aligned} \delta\mathbf{Op}(w;h) &= \lim_{t\to 0} \frac{\mathbf{Op}(w+th) - \mathbf{Op}(w)}{t} \\ &= \begin{bmatrix} \delta\mathbf{Op}_{11}(w;h) & \cdots & \delta\mathbf{Op}_{1n}(w;h) \\ \vdots & & \vdots \\ \delta\mathbf{Op}_{m1}(w;h) & \cdots & \delta\mathbf{Op}_{mn}(w;h) \end{bmatrix} \end{aligned}$$

So, as seen from the previous expression the variation is linear in the second argument if and only if each of the variations $\delta\mathbf{Op}_{ij}$ is linear in the second argument.

A possible inner product that can be employed for this vector case is presented here. It is based on any scalar inner product, i.e. inner product for the case when $\mathbf{Op}\, w$ is scalar. Say $\langle\cdot,\cdot\rangle_s$ is such an inner product and $\mathbf{Op}\, w$ is $(m \times n)$ dimensional, then the inner product for this multidimensional case is

$$\langle x, y\rangle = \sum_{i=1,j=1}^{m,n} \langle x_{ij}, y_{ij}\rangle_s.$$

It can be shown that all the axioms of an inner product are satisfied. Since we are dealing with real spaces, the inner product is because the scalar one is. Similarly, it is linear:

$$\langle \alpha x + \beta z, y\rangle = \sum_{i=1,j=1}^{m,n} \langle \alpha x_{ij} + \beta z_{ij}, y_{ij}\rangle_s = \sum_{i=1,j=1}^{m,n} \alpha\langle x_{ij}, y_{ij}\rangle_s + \beta\langle z_{ij}, y_{ij}\rangle_s = \alpha\langle x, y\rangle + \beta\langle z, y\rangle.$$

Finally, it is positive definite because $\langle x_{ij}, x_{ij}\rangle_s \geq 0$ for all $i$ and $j$. Also, if $\langle x, x\rangle = 0$ then $x_{ij} = 0$ for all $i$ and $j$ since the inner product here is the sum of nonnegative numbers, and consequently $x = 0$.

If each of the component operators' Gâteaux derivative $\delta\mathbf{Op}_{ij}$ admits an adjoint, then the operator $\mathbf{Op}$ also admits an adjoint. This adjoint can be constructed in the following manner:

$$\begin{aligned}
\langle x, \delta\mathbf{Op}(w; h)\rangle &= \sum_{i=1,j=1}^{m,n} \langle x_{ij}, \delta\mathbf{Op}(w; h)_{ij}\rangle_s \\
&= \sum_{i=1,j=1}^{m,n} \langle \delta\mathbf{Op}^*_{ij}(w; x_{ij}), h\rangle_s \\
&= \left\langle \sum_{i=1,j=1}^{m,n} \delta\mathbf{Op}^*_{ij}(w; x_{ij}), h \right\rangle_s .
\end{aligned}$$

Thus, if $w$ is $p$-dimensional then the adjoint is $p$-dimensional vector valued operator and is defined as

$$\delta\mathbf{Op}^*(w; x) = \sum_{i=1,j=1}^{m,n} \delta\mathbf{Op}^*_{ij}(w; x_{ij}).$$

If $\mathbf{Op}\, w$ is vector valued, i.e. it's column dimension $n$ is one, then a $Q$ weighted norm can also be defined. In this case, the operator $\mathbf{Q}$ is a diagonal operator:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & & & \\ & \mathbf{Q}_2 & & \\ & & \ddots & \\ & & & \mathbf{Q}_m \end{bmatrix},$$

where the diagonal element $\mathbf{Q}_i$ is the weighting operator for the $i$th component of $\mathbf{Op}\, w$ respectively. It is clear that $\mathbf{Q}$ is self-adjoint since each of the components $\mathbf{Q}_i$ is self-adjoint.

$$\begin{aligned} \langle \mathbf{Q}\, x, y \rangle &= \sum_{i=1}^{m} \langle \mathbf{Q}_i\, x_i, y_i \rangle_s \\ &= \sum_{i=1}^{m} \langle x_i, \mathbf{Q}_i\, y_i \rangle_s \\ &= \langle x, \mathbf{Q}\, y \rangle \end{aligned}$$

#### *5.1.2.1 Adjoint of the Vector LTID Operator*

In this section, the operator associated with the vector LTID type $\mathcal{L}_n^k$, from Section 4.1.2, is first linearized and then its adjoint is found. Remember that the operator for this type is the Schur complement, i.e.

$$\begin{aligned} \mathbf{Op}\, w &= \mathrm{Schur}(\widehat{W}) \text{ in } W, \\ &= w^{(n+nk)} - \overline{W}\widehat{W}^{-1}\widetilde{W} \end{aligned}$$

where

$$W = \left[\begin{array}{ccc|c} w & \cdots & w^{(nk-1)} & w^{(nk)} \\ \vdots & & \vdots & \vdots \\ w^{(n-1)} & \cdots & w^{(n-1+nk-1)} & w^{(n-1+nk)} \\ \hline w^{(n)} & \cdots & w^{(n-1+nk)} & w^{(n+nk)} \end{array}\right], \quad \widehat{W}(w) = \begin{bmatrix} w & \cdots & w^{(nk-1)} \\ \vdots & & \vdots \\ w^{(n-1)} & \cdots & w^{(n-1+nk-1)} \end{bmatrix},$$

$$\overline{W}(w) = \begin{bmatrix} w^{(n)} & \cdots & w^{(n-1+nk)} \end{bmatrix}, \quad \text{and} \quad \widetilde{W}(w) = \begin{bmatrix} w^{(nk)} \\ \vdots \\ w^{(n-1+nk)} \end{bmatrix}.$$

The linearized version of this operator can be found as,

$$
\begin{aligned}
\mathbf{Op}_w\, h &= \frac{d}{dt}\mathbf{Op}(w+th)\Big|_{t=0} \\
&= h^{(n+nk)} - \overline{W}(h)\widehat{W}^{-1}(w)\widetilde{W}(w) - \overline{W}(w)\widehat{W}^{-1}(w)\widetilde{W}(h) \\
&\quad + \overline{W}(w)\widehat{W}^{-1}(w)\widehat{W}(h)\widehat{W}^{-1}(w)\widetilde{W}(w).
\end{aligned}
$$

The adjoint of this operator is computed by taking the adjoint of each of the terms. The adjoint operator turns out to be,

$$
\begin{aligned}
\mathbf{Op}_w^* = (-1)^{n+nk}\,\mathbf{D}^{n+nk} &- \left[(-1)^n\,\mathbf{D}^n \quad \cdots \quad (-1)^{n-1+nk}\,\mathbf{D}^{n-1+nk}\right]\widehat{\mathbf{W}}^{-1}\widetilde{\mathbf{W}}\,\mathbf{I}_k \\
&- \left[(-1)^{nk}\,\mathbf{D}^{nk}\,\mathbf{I}_k \quad \cdots \quad (-1)^{n+nk-1}\,\mathbf{D}^{n+nk-1}\,\mathbf{I}_k\right]\widehat{\mathbf{W}}^{-T}\,\overline{\mathbf{W}}^T \\
&+ \left[I_k \otimes \left(\mathbf{I} \quad \cdots \quad (-D)^{nk-1}\right) \quad \cdots \quad I_k \otimes \left((-D)^{n-1} \quad \cdots \quad (-D)^{n+nk-2}\right)\right] \\
&\quad \left[\left(\overline{\mathbf{W}}\,\widehat{\mathbf{W}}^{-1}\right)^T \otimes \left(\widehat{\mathbf{W}}^{-1}\,\widetilde{\mathbf{W}}\right)\right].
\end{aligned}
$$

### 5.1.3 Examples

In this section, the results presented in the previous sections are illustrated with the help of some examples. In all of these signal raccordation problems, Theorem 5.1.1 will be applied.

Suppose we are interested in the raccordation between constants. This is a very simple example but it illustrates all the relevant concepts exquisitely. The base behavior is chosen to be the space of continuous functions, i.e. $\mathcal{B}_0 = C^0(\mathbb{R},\mathbb{R})$. The type $\mathcal{T}$ is that of constants and the associated operator is $\mathbf{Op} := \mathbf{D} : C^1(\mathbb{R},\mathbb{R}) \to C^0(\mathbb{R},\mathbb{R})$. Notice that the base behavior is chosen to be larger than the domain of the type. The raccordation interval is chosen to be $[a,b]$ and the constants to be connected are 1 and 2. The support of the Gluskabi map in this case is the same interval $[a,b]$ and so an inner product is needed on the space of continuous functions restricted to this interval. Let's choose the $L^2$ inner product. The adjoint of the differentiation operator $\mathbf{D}$ is $-\mathbf{D}$. Therefore, according to Theorem 5.1.1, the raccordation $w$ satisfies the equation $\mathbf{D}^2\, w = 0$, and so is a ramp whose value is 1 at 0 and 2 at 1. This is shown in Figure 4. Notice that the connected function does not belong to

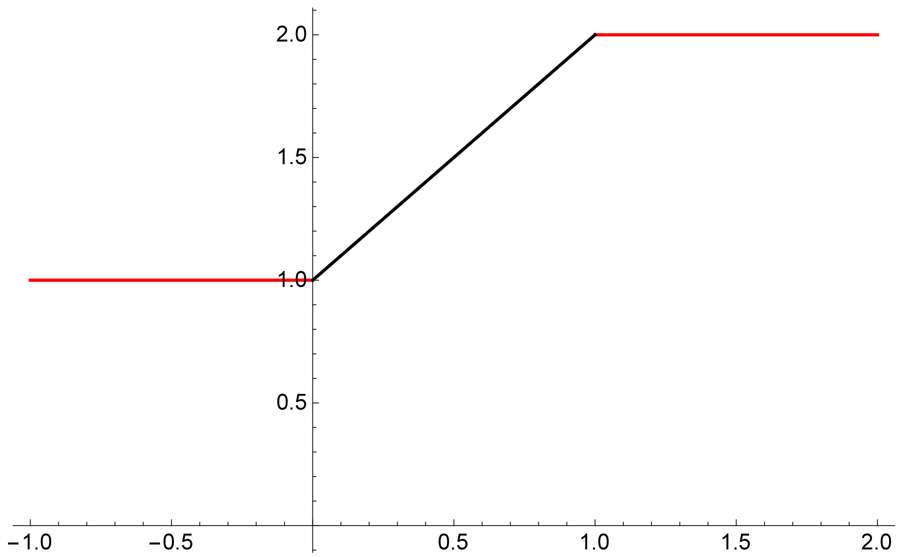



**Figure 4. Raccordation between the two constants 1 and 2.**

the type. It is in fact not even differentiable; this is why the base behavior is usually chosen to be a larger set. Furthermore, notice that the raccordation actually lies in a subset of the domain of the type, i.e. the set of twice differentiable functions.

*5.1.3.1 Scalar LTID Type*

In the following example, the goal is to connect two exponentials. Let's choose our base behavior to be $\mathcal{B}_0 = C^0(\mathbb{R}, \mathbb{R})$ and the type to be scalar first order LTID type $\mathcal{L}_1^1$ i.e. the set of all exponentials $ce^{\lambda t}$ for all values of $c \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Looking back at Section 4.1, the operator for this type is found to be $\mathbf{Op}\, w = \ddot{w}w - \dot{w}^2$. Then, the linearized form of this operator $\mathbf{Op}$ is $\mathbf{Op}_w = w\mathbf{D}^2 - 2\dot{w}\mathbf{D} + \ddot{w}I$. Say the raccordations are sought over the interval $[0, 1]$ and the norm to be minimized is the usual $L^2$ norm. Then according to Theorem 5.1.1, the raccordation $w$ over the interval $[0, 1]$ must be the solution to the differential equation $\mathbf{Op}_w^* \mathbf{Op}\, w = 0$ or

$$\left(\mathbf{D}^2\, w + 2\mathbf{D}\, \dot{w} + \ddot{w}\right)\left(\ddot{w}w - \dot{w}^2\right) = 0$$
$$w^{(4)}w^2 + 4w^{(3)}\dot{w}w + 3\ddot{w}^2\dot{w} - 8\ddot{w}\dot{w}^2 = 0.$$

This gives a generalized solution and then the specific raccordation connecting say $w_1$ and $w_2$ is obtained by using the boundary conditions i.e. $w^{(i)}(0) = w_1^{(i)}(0)$ and $w^{(i)}(1) = w_2^{(i)}(1)$

for $i = 0$ and $i = 1$. The raccordation for the case when $w_1 = 5e^{-2t}$ and $w_2 = 0.1e^{4t}$ is shown in Figure 5.

#### *5.1.3.2 Vector LTID Type*

In the following example, two trajectories from the vector LTID type $\mathcal{L}_1^2$ are connected. The operator **Op** for this type is the Schur operator (See Section 4.1.2). The raccordation is then the solution of $\mathbf{Op}_w^* \, \mathbf{Op} \, w = 0$ where the adjoint is obtained using the formula from Section 5.1.2.1. Each trajectory vector $w = (w_1, w_2) \in \mathcal{L}_1^2$ is the solution of $\dot{w} = A\,w$, for some $A \in \mathbb{R}^{2\times2}$. The example illustrated in Figure 6 connects the trajectories for $A_l = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ to $A_r = \begin{pmatrix} -4.67 & 1.76 \\ -14.25 & 7.09 \end{pmatrix}$.

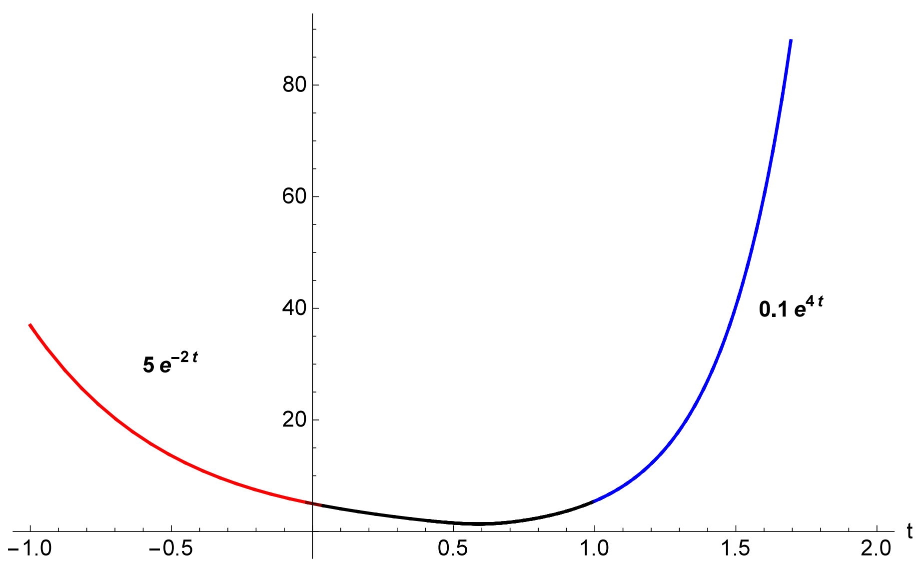



**Figure 5. A raccordation (black line) connecting $w_1 = 5e^{-2t}$ (red line) in the $\mathcal{L}_1^1$ type to $w_2 = 0.1e^{4t}$ (blue line) in the same type.**

## 5.2 Periodic Type

The problem considered in this section is that of finding a maximally persistent raccordation between two periodic trajectories $w^a$ and $w^b$ over a finite time interval $[a, b]$. This problem can also be stated as finding a trajectory $w(t)$ such that $w = w^a$ for $t \leq a$ and $w = w^b$ for $t \geq b$ and the trajectory in the interval $[a, b]$ is maximally persistent in the periodic type with respect to a specified norm. The results in this section were presented in [41].

The problem of finding raccordations for periodic trajectories is a specific case of the much broader result stated in the previous section. This section is organized as follows: The case of $\tau$-periodic type is first considered. This case is further divided into two sub-cases – when the interval is an integer multiple of the period $\tau$ and when it's not. Then, the case of a sub-type of $\tau$-periodic type when the functions are continuous is considered with the two sub-cases again. Finally, the case of general periodic type, i.e. the behavior that includes functions of all periods, is considered.

### 5.2.1 $\tau$-Periodic Type

In the context of the Gluskabi framework, the $\tau$-periodic type, which is the behavior of periodic functions of period $\tau$, is defined by the kernel of the operator $\mathbf{Op} := (\mathbf{I} - \mathbf{S}_{-\tau})$ where $\mathbf{I}$ is the identity operator and $\mathbf{S}$ is the shift operator, as mentioned in Section 4.2. The problem of finding the Gluskabi raccordation between two trajectories of the $\tau$-periodic type is split up into two cases. At first the raccordation for the $\tau$-periodic type over an interval that is an integer multiple of the period is found. This result is then generalized to intervals of arbitrary length.

For deriving the Gluskabi raccordations, the following two additional assumptions have

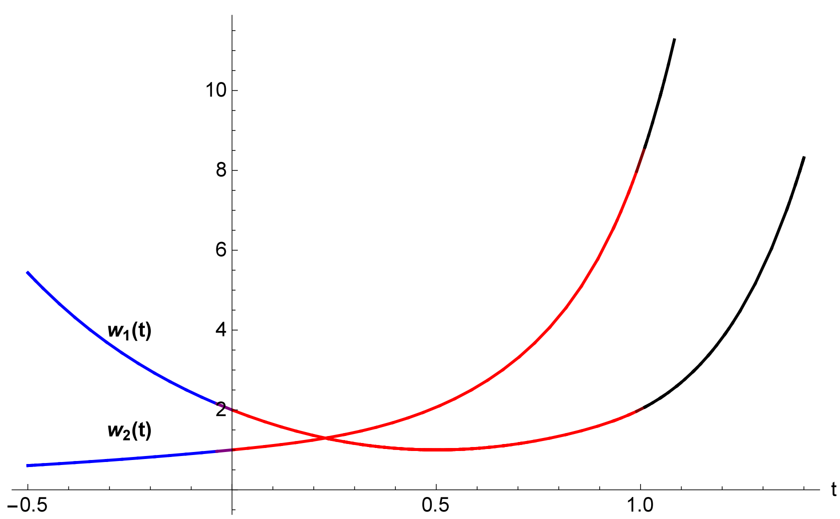



**Figure 6. Raccordation (red lines) for $\mathcal{L}_1^2$ type connecting trajectory of $A_l$ (blue lines) to $A_r$ (black lines).**

been made in this section.

(S3) The base behavior is the space of piecewise continuous functions, i.e., $\mathcal{B}_0 = PC(\mathbb{R}, \mathbb{R})$.

(S1a) The specific inner product assumed in (S1) is chosen to be the $L^2$ inner product.

Consequently, the $L^2$ norm will be minimized. Since the raccordation interval is $[a, b]$, the support of $\mathbf{Op}\, g(w^a, w^b)$ is the interval $[a, b + \tau]$, for some $w^a$ and $w^b$ in the $\tau$-periodic type, and so the minimization will be carried out over this interval. The Gluskabi map or the extension can now be found by applying Theorem 5.1.1, but in order to gain some insight into the problem, optimal control theory will be directly applied to the problem [42].

**Theorem 5.2.1.** *Given two trajectories $w^a$ and $w^b$ from the $\tau$-periodic type, the Gluskabi raccordation, $w$, between the two is given by*

$$w(t) = w^a(t) + \left(1 + \left\lfloor \frac{t-a}{\tau} \right\rfloor\right) \frac{w^b(t) - w^a(t)}{n+1},$$

*where $\lfloor \cdot \rfloor$ is the floor operation and provided $b = a + n\tau$ for some $n \in \mathbb{Z}_+$.*

**PROOF.** Let $u = (\mathbf{I} - \mathbf{S}_{-\tau})w$, or

$$u(t) = w(t) - w(t - \tau) \tag{44}$$

for any $w$ in the base behavior. Notice that if $w$ is periodic then $u = 0$. Then, the Gluskabi raccordation is the argument of the following optimization problem:

$$\min_{w(t)} J = \min_{w(t)} \frac{1}{2} \int_a^{b+\tau} u^2(t)\, dt. \tag{45}$$

Now adjoining the definition of $u$ with the Lagrange multiplier $\lambda$ to the cost function (45) gives:

$$J(u) = \int_a^{b+\tau} \frac{u^2(t)}{2} + \lambda(t)\, [w(t) - w(t-\tau) - u(t)]\, dt.$$

Perturbing $u$ by $\delta u$ changes the cost function to

$$J(u + \delta u) = \int_a^{b+\tau} \frac{1}{2}\, [u(t) + \delta u(t)]^2 + \lambda(t)\, [w(t) + \delta w(t) - w(t-\tau) - \delta w(t-\tau) - u(t) - \delta u(t)]\, dt \tag{46}$$

and so

$$\begin{aligned}\delta J &= J(u+\delta u) - J(u)\\ &\approx \int_a^{b+\tau} u(t)\delta u(t) + \lambda(t)\left[\delta w(t) - \delta w(t-\tau) - \delta u(t)\right] dt\\ &= \int_a^{b+\tau} [u(t)-\lambda(t)]\delta u(t)\, dt + \int_a^{b+\tau} \lambda(t)\delta w(t)\, dt - \int_a^{b+\tau} \lambda(t)\delta w(t-\tau)\, dt\\ &= \int_a^{b+\tau} [u(t)-\lambda(t)]\delta u(t)\, dt + \int_a^{b+\tau} \lambda(t)\delta w(t)\, dt - \int_{a-\tau}^{b} \lambda(t+\tau)\delta w(t)\, dt. \end{aligned} \tag{47}$$

A necessary condition for a $w$ to minimize (45) is that $\delta J$ be zero for any arbitrary $\delta u$. The perturbation $\delta w$ in (47) is zero in the intervals $[b, b+\tau]$ and $[a-\tau, a]$, since by the definition of the Gluskabi map $w = w^b$ and $w = w^a$ in these respective intervals. So, the integrals involving $\delta w$ are zero in these intervals. The multiplier $\lambda(t)$ is chosen in the following way in the interval $(a, b)$ to avoid computing $\delta w$ in this interval:

$$\lambda(t) = \lambda(t+\tau) \qquad \forall t \in (a, b). \tag{48}$$

Then, the necessary condition for optimality is

$$\lambda(t) = u(t) \qquad \forall t \in [a, b+\tau]. \tag{49}$$

Notice that $\lambda(t)$ is free in the interval $[b, b+\tau]$ and as a consequence of (48), choosing $\lambda$ in any $\tau$ length sub-interval in $[a, b+\tau]$ completely determines it for all time. The initial definition (44) of $u$ in the interval $[a, b+\tau]$ can be written as follows, using (49) and (48): For any $\theta \in [a, a+\tau]$ and $i \in \{0, 1, \cdots, n\}$,

$$\begin{aligned} w(\theta+i\tau) &= u(\theta+i\tau) + w(\theta+(i-1)\tau)\\ &= u(\theta+i\tau) + u(\theta+(i-1)\tau) + w(\theta+(i-2)\tau) \end{aligned}$$

$$= (i + 1)\ u(\theta) + w(\theta - \tau)$$
$$= (i + 1)\ u(\theta) + w^a(\theta - \tau). \tag{50}$$

When $i = n$, we have that $w(\theta + n\tau) = w^b(\theta + n\tau)$ and so (50) becomes

$$w^b(\theta + n\tau) = (n + 1)\ u(\theta) + w^a(\theta - \tau).$$

Since $w^a$ and $w^b$ are both periodic with period $\tau$, the above equation yields the following simplified expression for $u(\theta)$ in the interval $[a, a + \tau]$:

$$u(\theta) = \frac{w^b(\theta) - w^a(\theta)}{n + 1}. \tag{51}$$

This consequently defines $u(t)$ in the entire interval $[a, b + \tau]$ by (48) and (49). Thus, the Gluskabi raccordation for this periodic type is obtained from (50) and is as follows:

$$w(t) = w^a(t) + \left(1 + \left\lfloor \frac{t - a}{\tau} \right\rfloor\right) u(t). \quad \square$$

Alternatively, if one wishes to apply Theorem 5.1.1 then the adjoint of the shift operator $\mathbf{S}_{-\tau}$ is required. The inner product is the $L^2$ inner product, i.e. $\langle \cdot, \cdot \rangle : PC([a, b + \tau], \mathbb{R}) \times PC([a, b + \tau], \mathbb{R}) \to \mathbb{R}$. The restricted domain $\mathcal{H}$ for which the adjoint must hold is the set of all functions $h$ such that $h(t) = 0$ when $t \in [b, \infty)$ or $(-\infty, a]$. The shift operator $S_{-\tau}$ acts on the functions in $\mathcal{H}$ by shifting $h$ to the right and appending zero function in interval $[a, a + \tau]$. Then, the adjoint of $\mathbf{S}_{-\tau}$ is,

$$\begin{aligned}
\langle x, \mathbf{S}_{-\tau} h \rangle &= \int_a^{b+\tau} x(t)\mathbf{S}_{-\tau}h(t)\, dt \\
&= \int_{a+\tau}^{b+\tau} x(t)h(t - \tau)\, dt \\
&= \int_a^{b} x(t + \tau)h(t)\, dt \\
&= \int_a^{b+\tau} x(t + \tau)h(t)\, dt \\
&= \langle \mathbf{S}_\tau\, x, h \rangle.
\end{aligned}$$

Therefore, applying Theorem 5.1.1 the Gluskabi raccordation must satisfy,

$$\mathbf{Op}^*\mathbf{Op}\, w = (\mathbf{I} - \mathbf{S}_{\tau})\,(\mathbf{I} - \mathbf{S}_{-\tau})\, w = 0$$

$$\Rightarrow (2\mathbf{I} - \mathbf{S}_{-\tau} - \mathbf{S}_{\tau})\, w = 0 \quad \text{in the interval } (a, b)$$

which yields the same result as Theorem 5.2.1.

This result indicates that the Gluskabi raccordation basically takes the difference between the values of the two trajectories within an aligned period, and covers this difference in $n+1$ periods. This result is generalized in the next theorem for the case where the raccordation interval is of arbitrary length, i.e., its length is not restricted to be an integer multiple of the period.

**Theorem 5.2.2.** *Given two trajectories $w^a$ and $w^b$ from the $\tau$-periodic type, the Gluskabi raccordation, $w$, between the two is given by,*

$$w(t) = w^a(t) + \left(1 + \left\lfloor \frac{t-a}{\tau} \right\rfloor\right) u(t)$$

*where*

$$u(t) = \begin{cases} \frac{1}{n+2}\left[w^b(t) - w^a(t)\right] & a \leq t' \leq (b - n\tau) \\ \frac{1}{n+1}\left[w^b(t) - w^a(t)\right] & (b - n\tau) \leq t' \leq (a + \tau) \\ & t' = (t - a) \quad \text{mod } \tau \end{cases}$$

**PROOF.** The proof of this theorem is exactly along the lines of Theorem 5.2.1 since we employ the same cost function and we obtain the same Euler-Lagrange equation and optimality equation, i.e.,

$$\lambda(t) = \lambda(t + \tau) \qquad \forall t \in (a, b) \tag{52}$$

$$\lambda(t) = u(t) \qquad \forall t \in [a, b + \tau]. \tag{53}$$

Again, it holds that $\lambda(t)$ is free in the interval $[b, b + \tau]$, and as a consequence of (52) choosing $\lambda$ in any $\tau$ length sub-interval in $[a, b + \tau]$ completely determines it for all time.

The raccordation interval $[a, b]$ can be split up into intervals of length $\tau$ with possibly one remaining interval of length less than $\tau$. Let $n = \left\lfloor \frac{b-a}{\tau} \right\rfloor$. Then, $b - a = n\tau + (b - a - n\tau)$. From the definition of $u$ we have that,

$$w(t) = u(t) + w(t - \tau).$$

Or for any $\theta \in [a, a + \tau]$ and $i \in \{0, 1, \cdots, n\}$,

$$\begin{aligned} w(\theta + i\tau) &= u(\theta + i\tau) + w(\theta + (i - 1)\tau) \\ &= (i + 1)\, u(\theta) + w^a(\theta - \tau) \end{aligned} \tag{54}$$

by making use of (52) and (53). There are two separate cases to be dealt with here. When $i = n$ and $\theta \in [b - n\tau, a + \tau]$, we have that $w = w^b$, and so (54) becomes

$$w^b(\theta + n\tau) = (n + 1)u(\theta) + w^a(\theta - \tau). \tag{55}$$

On the other hand when $i = n + 1$ and $\theta \in [a, b - n\tau]$, (52) still holds since the argument $(\theta + i\tau)$ is in the interval $[b, b + \tau]$ and so (54) becomes

$$w^b(\theta + (n + 1)\tau) = (n + 2)u(\theta) + w^a(\theta - \tau). \tag{56}$$

Using the fact that both $w^a$ and $w^b$ are periodic with period $\tau$, the above equations (55) and (56) yield the following simplified expression for $u(\theta)$ in the interval $[a, a + \tau]$:

$$u(\theta) = \begin{cases} \frac{1}{n+2}\left[w^b(\theta) - w^a(\theta)\right] & a \leq \theta \leq (b - n\tau) \\ \frac{1}{n+1}\left[w^b(\theta) - w^a(\theta)\right] & (b - n\tau) < \theta \leq (a + \tau) \end{cases}. \tag{57}$$

This consequently defines $u(t)$ in the entire interval $[a, b + \tau]$ by (52) and (53). Thus the Gluskabi raccordation over the interval $[a, b]$ is obtained by substituting the above expression in (54) as follows:

$$w(t) = w^a(t) + \left(1 + \left\lfloor \frac{t - a}{\tau} \right\rfloor\right) u(t). \quad \square$$

Contrary to Theorem 5.2.1, the raccordation interval now is comprised of $n$ whole periods and one partial period. This result indicates that every period in the raccordation interval is split up into two parts. The racordation in the first part is the previous period plus $(n + 2)$th of the difference between the values of the two functions in the first part of any period. And, in the second part it is the previous period plus $(n+1)$th of the difference. This will be elucidated in the coming example.

Before proceeding to the next result, some remarks will be made about the previous two theorems. The same results are obtained for the Gluskabi raccordation if the alternate definition is used for the periodic operator, employing an advance instead of a lag. This indicates that the Gluskabi raccordation obtained is truly associated with the periodic type. The two theorems can also be viewed in the Fourier domain. The Gluskabi raccordation there is similar to the time domain and the difference between the Fourier coefficients is equally covered in $n + 1$ or $n + 2$ steps.

The previous results are illustrated with the help of an example. The considered problem is that of finding raccordation between $\cos 2\pi t$ and the triangle wave with period one. Notice that both these trajectories are from the periodic type with period one, and this type can be characterized by the operator $(\mathbf{I} - \mathbf{S}_{-\tau})$, where $\tau = 1$. The raccordation is sought over the interval $[0, 3.5]$ and so the length of the interval is not a multiple of the period. The result from Theorem 5.2.2 is applied and the resultant raccordation is shown in Figure 7. Notice the discontinuities at multiples of the period as well as at distance $3.5 \mod 1 = \frac{1}{2}$ or at the midpoint within each period. The discontinuities at the multiples of the period is due to the discrepancy in the values of the functions at the end points of a single period, i.e., the cosine function at 0 is one but the triangle wave being considered is zero at 0. The discontinuity at the middle point in every period can be attributed to the discrepancy in the values of the two functions at that particular point in every period. Therefore considering one aligned period, if the two continuous function are equal at the end points and at the distance equal to the raccordation interval length modulo the period, then the raccordation will be continuous.

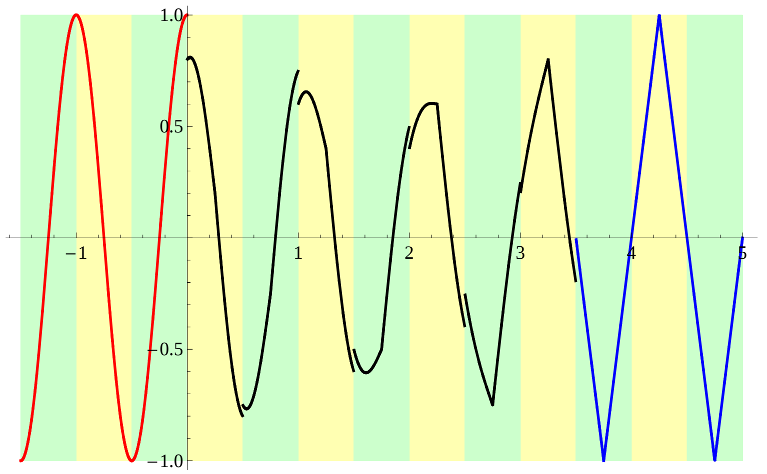



**Figure 7. The function is** $\cos 2\pi t$ **for** $t \leq 0$ **and a triangle wave for** $t \geq 3.5$**. Raccordation is over the interval** $[0, 3.5]$**.**

### 5.2.2 Continuous $\tau$-Periodic Type

The chosen base behavior, $\mathscr{B}_0$, in the previous section was the space of piecewise continuous functions. As the results indicate the Gluskabi raccordation is piecewise continuous even if the trajectories being connected are continuous, just as in Figure 7. If the base behavior is the space of continuous functions, then the optimization does not yield a solution since no continuous function satisfies the necessary conditions found in the previous section, in general. The only case when a continuous minimum exists is when the signals being connected match up at the boundary of the interval, i.e. $w(a) = w(b)$, and match up at the point $(b - a - n\tau)$ within the period. Expressly, the infimum of this optimization problem lies on the boundary of the space of continuous functions, in general.

A way to impose continuity of the Gluskabi raccordation is to employ a different norm, specifically by appending a cost on the derivative of the trajectory to the usual cost function, i.e., instead of minimizing $||\mathbf{Op}w||_2$ one can minimize $||\mathbf{Op}w||^2 = ||\mathbf{Op}w||_2^2 + \rho^2||\mathbf{D}\,\mathbf{Op}w||_2^2$, with $\rho^2$ being the weighting factor. This corresponds to choosing the $\mathbf{Q}$ operator in Theorem 5.1.1 as $\mathbf{Q} = \left(\mathbf{I} - \rho^2\mathbf{D}^2\right)$. In order to do this, the assumption (S3), from the previous section, is modified to the following:

(S3') The base behavior $\mathcal{B}_0$ is the space of continuous functions that are piecewise differentiable, or $\mathcal{B}_0 = C^0 \cap PWC^1$.

The assumption (S1a) still holds. The Gluskabi raccordation is now found for the $\tau$-periodic type in this new base behavior in the following theorem.

**Theorem 5.2.3.** *Given two trajectories $w^a$ and $w^b$ from the continuous $\tau$-periodic type, the Gluskabi raccordation, $w$, between the two, with respect to the norm $\|\cdot\|^2 = \|\cdot\|_2^2 + \rho^2 \|\mathbf{D}(\cdot)\|_2^2$, is given by solving the following set of equations:*

$$
\begin{aligned}
f_0 &= \rho\,(w_1 + w_a - 2w_0) \\
f_k &= \rho\,(w_{k+1} + w_{k-1} - 2w_k) \qquad 1 \le k \le n-2 \\
f_{n-1} &= \rho\,(w_b + w_{n-2} - 2w_{n-1}) \\
f_i &= c_0\, e^{\frac{1}{\rho}(t+i\tau)} - d_0\, e^{-\frac{1}{\rho}(t+i\tau)} \quad 0 \le i \le n-1
\end{aligned} \tag{58}
$$

*with the associated boundary conditions $w(t) = w^a(t)$ when $t \in (-\infty, a]$ and $w(t) = w^b(t)$ when $t \in [b, \infty)$, and then $w(t) = w_k(t - a - k\tau)$ with $k = \left\lfloor \frac{t-a}{\tau} \right\rfloor$ where $b = a + n\tau$ for some $n \in \mathbb{Z}_+$.*

**PROOF.** The Gluskabi raccordation in this case is the argument of the following optimization problem:

$$\min_{w(t)} \int_a^{b+\tau} [w(t) - w(t-\tau)]^2 + \rho^2 [\dot{w}(t) - \dot{w}(t-\tau)]^2\, dt.$$

To simplify the problem, let's define a set of shifted functions $w_k(\theta) = w(a + k\tau + \theta)$ for $k \in \{0, \cdots, n-1\}$ and $\theta \in [0, \tau]$. Also, let $u_k = \dot{w}_k$, $w_a(\theta) = w^a(a - \tau + \theta)$ and $w_b(\theta) = w^b(b + \theta)$. These shifted functions cover the entire interval $[a - \tau, b + \tau]$, and the cost function $J$ can now be written as,

$$
\begin{aligned}
J = \frac{1}{2} \int_0^{\tau} & [w_0(\theta) - w_a(\theta)]^2 + \rho^2\, [u_0(\theta) - u_a(\theta)]^2 + \sum_{k=1}^{n-1} [w_k(\theta) - w_{k-1}(\theta)]^2 \\
& + \rho^2\, [u_k(\theta) - u_{k-1}(\theta)]^2 + [w_b(\theta) - w_{n-1}(\theta)]^2 + \rho^2\, [u_b(\theta) - u_{n-1}(\theta)]^2\, d\theta,
\end{aligned} \tag{59}
$$

and the boundary conditions take the form of $w_0(0) = w_a(\tau)$ and $w_{n-1}(\tau) = w_b(0)$. Since the raccordation, $w$, has to be continuous, additional constraints are imposed on the boundaries of the interior shifted functions, specifically $w_k(0) = w_{k-1}(\tau)$ for $k \in \{1, \cdots, n-1\}$. Now adjoining the constraint equations $u_k - \dot{w}_k = 0$ along with the Lagrange multipliers, $\lambda_k$, for $k \in \{0, \cdots, n-1\}$ to the cost function (59), and employing the usual techniques of optimal control the following set of Euler-Lagrange equations are obtained:

$$\begin{aligned} \dot{\lambda}_0 &= w_1 + w_a - 2w_0 \\ \dot{\lambda}_k &= w_{k+1} + w_{k-1} - 2w_k \qquad 1 \leq k \leq n-2 \\ \dot{\lambda}_{n-1} &= w_b + w_{n-2} - 2w_{n-1}. \end{aligned} \tag{60}$$

Boundary conditions on the Lagrange multipliers are also obtained stemming from the fact that to preserve continuity, the perturbations at the end points of the shifted functions are the same, i.e., $\delta w_k(0) = \delta w_{k-1}(\tau)$. These boundary conditions turn out to be $\lambda_k(0) = \lambda_{k-1}(\tau)$ for $k \in \{1, \cdots, n-1\}$, which has similar form to the boundary conditions on shifted functions. The set of optimality conditions is as follows:

$$\begin{aligned} \frac{\lambda_0}{\rho^2} &= u_1 + \dot{w}^a - 2u_0 \\ \frac{\lambda_k}{\rho^2} &= u_{k+1} + u_{k-1} - 2u_k \qquad 1 \leq k \leq n-2 \\ \frac{\lambda_{n-1}}{\rho^2} &= \dot{w}^b + u_{n-2} - 2u_{n-1}. \end{aligned} \tag{61}$$

Differentiating (60) once and comparing it to (61) yields the following set of second order differential equations for the Lagrange multipliers:

$$\ddot{\lambda}_k - \frac{1}{\rho^2}\lambda_k = 0 \qquad 0 \leq k \leq n-1. \tag{62}$$

The solutions $\lambda_k$ are a linear combination of exponential modes $e^{\pm t/\rho}$ and can be written as,

$$\lambda_k = c_k\, e^{t/\rho} + d_k\, e^{-t/\rho} \tag{63}$$

for some constants, $c_k$ and $d_k$. This results in $2n$ unknowns in the form of these constants that can be solved for using the $n-1$ boundary conditions for Lagrange multipliers, the $n-1$

boundary conditions for functions $w_k$, and the two end point conditions $w_0(0) = w_a(\tau)$ and $w_{n-1}(\tau) = w_b(0)$. From the conditions $\lambda_k(0) = \lambda_{k-1}(\tau)$ for $k \in \{1, \cdots, n-1\}$, the following condition is obtained,

$$c_k + d_k = c_{k-1}\, e^{\tau/\rho} + d_{k-1}\, e^{-\tau/\rho}$$

$$\Rightarrow c_k = c_{k-1}\, e^{\tau/\rho} + d_{k-1}\, e^{-\tau/\rho} - d_k. \tag{64}$$

Additionally, the conditions $x_k(0) = x_{k-1}(\tau)$ for $k \in \{1, \cdots, n-1\}$ on the shifted functions, translate to boundary conditions on $\dot{\lambda}_k$ in (60) i.e., $\dot{\lambda}_k(0) = \dot{\lambda}_{k-1}(\tau)$ for $k \in \{1, \cdots, n-1\}$ or that

$$\frac{1}{\rho}\,(c_k - d_k) = \frac{1}{\rho}\left(c_{k-1}\, e^{\tau/\rho} - d_{k-1}\, e^{-\tau/\rho}\right).$$

Substituting (64) in this yields,

$$d_k = d_{k-1}\, e^{-\tau/\rho} \tag{65}$$

and then (64) becomes

$$c_k = c_{k-1}\, e^{\tau/\rho}. \tag{66}$$

Therefore, the expressions for all the Lagrange multipliers (63) can be written in terms of just two constants $c_0$ and $d_0$ as follows:

$$\lambda_k = c_0\, e^{\frac{1}{\rho}(t+k\tau)} + d_0\, e^{-\frac{1}{\rho}(t+k\tau)}. \tag{67}$$

Substituting this in (60) gives us the set of equations (58) to be solved for finding the Gluskabi raccordation.

Given that the functions $w^a$ and $w^b$ are continuous, one can conclude from the resultant set of linear equations that the functions, $w_k$, are continuous. Subsequently, the interior point constraints, $w_k(0) = w_{k-1}(\tau)$, and the boundary constraints, $w_0(0) = w^a(a)$ and $w_{n-1}(\tau) = w^b(b)$, guarantee that the raccordation, $w$, is continuous. □

Alternatively, if one wishes to apply Theorem 5.1.1 then the adjoint of the shift operator is the same as before. Additionally, $\mathbf{Q} = (\mathbf{I} - \mathbf{D}^2)$. The inner product is the $L^2$ inner product but now on the new base behavior restricred to the interval $[a, b+\tau]$. The restricted domain $\mathcal{H}$ is the same. Therefore, applying Theorem 5.1.1 the Gluskabi raccordation must satisfy,

$$\mathbf{Op}^*\mathbf{QOp}\, w = (\mathbf{I} - \mathbf{S}_\tau)\left(\mathbf{I} - \mathbf{D}^2\right)(\mathbf{I} - \mathbf{S}_{-\tau})\, w = 0$$

$$\Rightarrow (2\mathbf{I} - \mathbf{S}_{-\tau} - \mathbf{S}_\tau)\left(\mathbf{D}^2 - \mathbf{I}\right) w = 0 \qquad \text{in the interval } (a, b).$$

Applying the operator $(\mathbf{D}^2 - \mathbf{I})$ to (58) shows that $(\ddot{w} - w)$ is in the kernel of $(2\mathbf{I} - \mathbf{S}_{-\tau} - \mathbf{S}_\tau)$, and hence the two results agree with each other.

**Theorem 5.2.4.** *Given two trajectories $w^a$ and $w^b$ from the continuous $\tau$-periodic type, the Gluskabi raccordation, $w$, between the two, with respect to the norm $\|\cdot\|^2 = \|\cdot\|_2^2 + \rho^2\|\mathbf{D}(\cdot)\|_2^2$, is given by solving the following set of equations:*

$$\begin{aligned}
f_0 &= \rho\,(w_1 + w_a - 2w_0) \\
f_k &= \rho\,(w_{k+1} + w_{k-1} - 2w_k) \qquad 1 \le k \le n-2 \\
f_{n-1} &= \begin{cases} w_n + w_{n-2} - 2w_{n-1} & \text{if } \theta \in [0, t'] \\ w_b + w_{n-2} - 2w_{n-1} & \text{if } \theta \in [t', \tau] \end{cases} \\
f_n &= w_b + w_{n-1} - 2w_n \qquad \theta \in [0, t'] \\
f_i &= c_0\, e^{\frac{1}{\rho}(t+i\tau)} - d_0\, e^{-\frac{1}{\rho}(t+i\tau)} \qquad 0 \le i \le n
\end{aligned} \tag{68}$$

*with the associated boundary conditions $w(t) = w^a(t)$ when $t \in (-\infty, a]$ and $w(t) = w^b(t)$ when $t \in [b, \infty)$, and then $w(t) = w_k(t - a - k\tau)$ with $k = \left\lfloor \frac{t-a}{\tau} \right\rfloor$, and $t' = (b - a - n\tau)$ with $n = \left\lfloor \frac{b-a}{\tau} \right\rfloor$.*

**PROOF.** The proof is along the same lines as Theorem 5.2.3. The same cost function is employed. The raccordation interval $[a, b]$ can be split up into intervals of length $\tau$ with possibly one remaining interval of length less than $\tau$. Let $n = \left\lfloor \frac{b-a}{\tau} \right\rfloor$. Then, $b - a = n\tau + t'$. To simplify the problem, let's define shifted functions again but now,

$$\begin{aligned}
&\text{for } k \in \{0, \cdots, n-1\} \qquad & w_k(\theta) &= w(a + k\tau + \theta) \qquad \text{and } \theta \in [0, \tau] \\
& & w_n(\theta) &= w(a + n\tau + \theta) \qquad \text{and } \theta \in [0, t'].
\end{aligned}$$

The $u_k = \dot{w}_k$ are similarly defined, $w_a(\theta) = w^a(a-\tau+\theta)$ and $w_b(\theta) = w^b(b+t'+\theta)$. Notice the shift in the definition of $w_b$. These shifted functions cover the entire interval $[a-\tau, b+\tau]$, and the cost function $J$ can now be written as,

$$\begin{aligned}
J = \frac{1}{2}\int_0^{\tau} & [w_0(\theta) - w_a(\theta)]^2 + \rho^2\,[u_0(\theta) - u_a(\theta)]^2 \; d\theta \\
& + \frac{1}{2}\int_0^{\tau}\sum_{k=1}^{n-1}[w_k(\theta) - w_{k-1}(\theta)]^2 + \rho^2\,[u_k(\theta) - u_{k-1}(\theta)]^2\, d\theta \\
& + \frac{1}{2}\int_0^{t'}[w_n(\theta) - w_{n-1}(\theta)]^2 + \rho^2\,[u_n(\theta) - u_{n-1}(\theta)]^2\, d\theta \\
& + \frac{1}{2}\int_{t'}^{\tau}[w_b(\theta) - w_{n-1}(\theta)]^2 + \rho^2\,[u_b(\theta) - u_{n-1}(\theta)]^2\, d\theta \\
& + \frac{1}{2}\int_0^{t'}[w_b(\theta) - w_n(\theta)]^2 + \rho^2\,[u_b(\theta) - u_n(\theta)]^2\, d\theta, \qquad (69)
\end{aligned}$$

and the boundary conditions take the form of $w_0(0) = w_a(\tau)$ and $w_n(t') = w_b(t')$. The interior constraints originating from the continuity requirement have an addition, specifically $w_k(0) = w_{k-1}(\tau)$ for $k \in \{1, \cdots, n\}$ and $w_{n-1}(t'^-) = w_{n-1}(t'^+)$. Adjoining constraints and the subsequent optimal control machinery yields the following set of Euler-Lagrange equations:

$$\begin{aligned}
\dot{\lambda}_0 &= w_1 + w_a - 2w_0 \\
\dot{\lambda}_k &= w_{k+1} + w_{k-1} - 2w_k \qquad 1 \le k \le n-2 \\
\dot{\lambda}_{n-1} &= \begin{cases} w_n + w_{n-2} - 2w_{n-1} & \text{if } \theta \in [0, t'] \\ w_b + w_{n-2} - 2w_{n-1} & \text{if } \theta \in [t', \tau] \end{cases} \qquad (70) \\
\dot{\lambda}_n &= w_b + w_{n-1} - 2w_n \qquad \theta \in [0, t']
\end{aligned}$$

The boundary conditions on the Lagrange multipliers turn out to be $\lambda_k(0) = \lambda_{k-1}(\tau)$ for $k \in \{1, \cdots, n\}$ and $\lambda_{n-1}(t'^-) = \lambda_{n-1}(t'^+)$. The set of optimality conditions is as follows:

$$\begin{aligned}
\frac{\lambda_0}{\rho^2} &= u_1 + \dot{w}^a - 2u_0 \\
\frac{\lambda_k}{\rho^2} &= u_{k+1} + u_{k-1} - 2u_k \qquad 1 \le k \le n-2
\end{aligned}$$

$$\frac{\lambda_{n-1}}{\rho^2} = \begin{cases} u_n + u_{n-2} - 2u_{n-1} & \text{if } \theta \in [0, t'] \\ \dot{w}_b + u_{n-2} - 2u_{n-1} & \text{if } \theta \in [t', \tau] \end{cases} \tag{71}$$

$$\frac{\lambda_n}{\rho^2} = \dot{w}_b + u_{n-1} - 2u_n \qquad \theta \in [0, t']$$

The multipliers $\lambda_k$ are:

$$\lambda_k = c_k\, e^{t/\rho} + d_k\, e^{-t/\rho} \qquad 0 \le k \le n \tag{72}$$

for some constants, $c_k$ and $d_k$. This results in $(2n + 4)$ unknowns in the form of these constants that can be solved for using the $n + 1$ boundary conditions for Lagrange multipliers, the $n + 1$ boundary conditions for functions $w_k$, and the two end point conditions. From the Lagrange multiplier conditions, the following condition is obtained,

$$c_k + d_k = c_{k-1}\, e^{\tau/\rho} + d_{k-1}\, e^{-\tau/\rho}$$

And so,

$$\begin{aligned} c_k &= c_{k-1}\, e^{\tau/\rho} + d_{k-1}\, e^{-\tau/\rho} - d_k \qquad\qquad 1 \le k \le n-1 \\ c'_{n-1}\, e^{t'/\rho} &= c_{n-1}\, e^{t'/\rho} + d_{n-1}\, e^{-t'/\rho} - d'_{n-1}\, e^{-t'/\rho} \\ c_n &= c'_{n-1}\, e^{\tau/\rho} + d'_{n-1}\, e^{-\tau/\rho} - d_n. \end{aligned} \tag{73}$$

Similarly, the interior constraints on the shifted functions, translate to boundary conditions on $\dot{\lambda}_k$ in (70) i.e.,

$$\begin{aligned} c_k - d_k &= c_{k-1}\, e^{\tau/\rho} - d_{k-1}\, e^{-\tau/\rho} \qquad 1 \le k \le n-1 \\ c'_{n-1}\, e^{t'/\rho} - d'_{n-1}\, e^{-t'/\rho} &= c_{n-1}\, e^{t'/\rho} - d_{n-1}\, e^{-t'/\rho} \\ c_n - d_n &= c'_{n-1}\, e^{\tau/\rho} - d'_{n-1}\, e^{-\tau/\rho} \end{aligned} \tag{74}$$

Substituting (73) in this yields,

$$d_{n-1} = d'_{n-1} \quad \text{and} \quad d_k = d_{k-1}\, e^{-\tau/\rho} \tag{75}$$

and then (73) becomes

$$c_{n-1} = c'_{n-1} \quad \text{and} \quad c_k = c_{k-1}\, e^{\tau/\rho}. \tag{76}$$

Therefore, the expressions for all the Lagrange multipliers (72) are:

$$\lambda_k = c_0\, e^{\frac{1}{\rho}(t+k\tau)} + d_0\, e^{-\frac{1}{\rho}(t+k\tau)}. \tag{77}$$

Substituting this in (70) gives us the set of equations (68) to be solved for finding the Gluskabi raccordation. □

These results for the continuous $\tau$-periodic type are now illustrated with the help of the same problem of finding raccordation between $\cos 2\pi t$ and the triangle wave. But now a continuous raccordation is sought over the interval $[0, 6]$. The result from Theorem 5.2.3 will be used in this case and the resultant raccordation is depicted in Figure 8, where the regularization factor $\rho = 1$. This regularization factor means that both the discrepancy in the trajectory and the discrepancy in the derivative are equally weighted. Increasing the value of $\rho$ would weigh the derivative more and so smoothen the raccordation. Notice the decrease in the magnitude of the raccordation followed by an increase, as the raccordation progresses. This pinching effect can be attributed to the difference in phase of the two trajectories being connected. It was also observed previously in the problem of connecting harmonics in [8]. The effect is most pronounced when the two trajectories are 180 degrees out of phase. Equivalently, there is no pinching at all when the two trajectories are phase aligned. This is illustrated in Figure 9 and Figure 10. It was shown in [8] using the image method that if the phase difference is greater than a certain threshold then pinching occurs since the raccordation passes through zero in the parameter space. These Figures 9 and 10 also depict that the raccordation is also dependent on the locations within the period chosen as the points of connection.

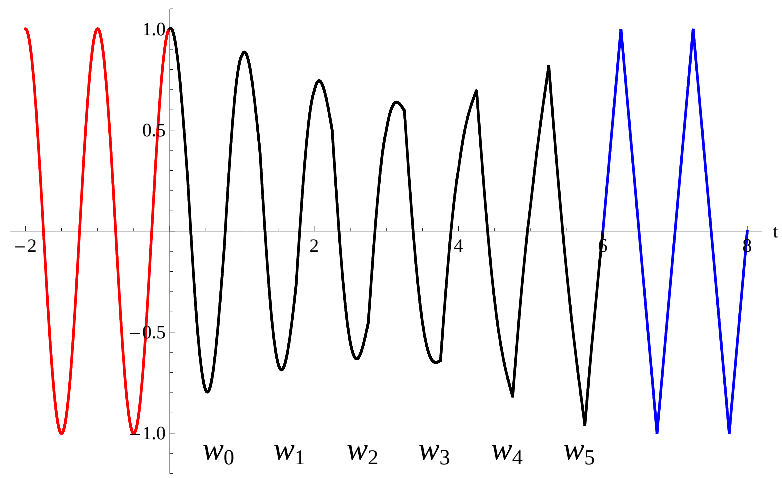



**Figure 8. The function is** $\cos 2\pi t$ **for** $t \leq 0$ **and the triangle wave for** $t \geq 6$**. Raccordation is over the interval** $[0, 6]$**.**

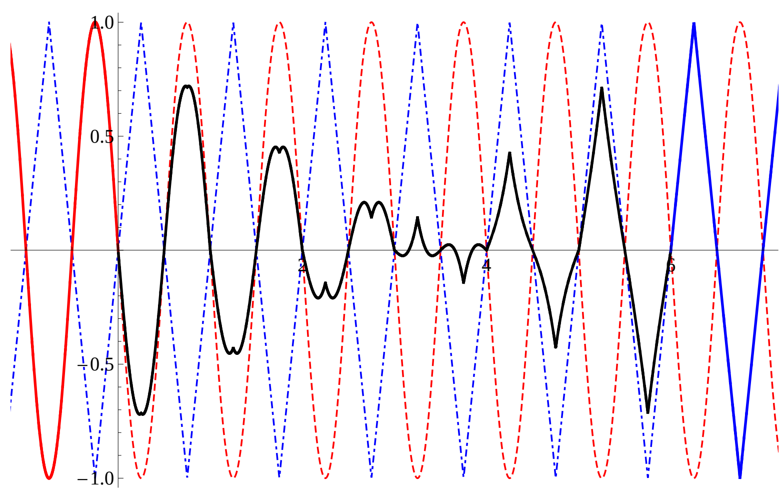



**Figure 9. The function is** $\cos\left(2\pi t + \frac{\pi}{2}\right)$ **for** $t \leq 0$ **and the triangle wave for** $t \geq 6$**. The two functions are** $180°$ **out of phase. Dashed lines show the functions being connected.**

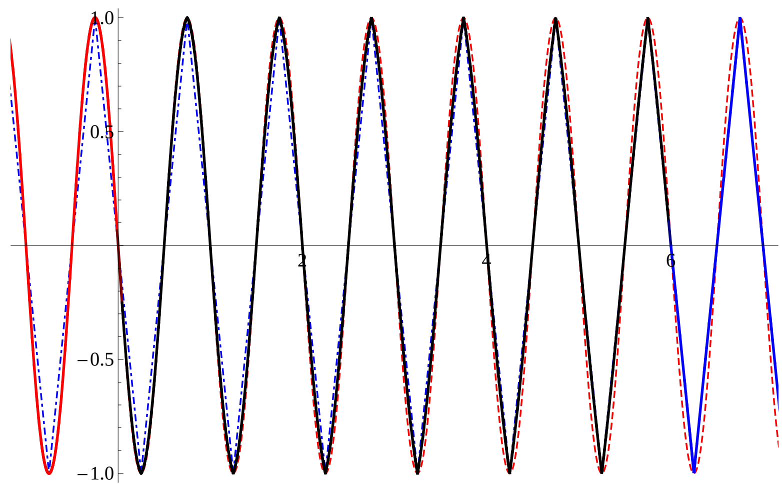



**Figure 10. The function is $\cos\left(2\pi t - \frac{\pi}{2}\right)$ for $t \leq 0$ and the triangle wave for $t \geq 6$. The two functions are phase aligned. Dashed lines show the functions being connected.**

### *5.2.2.1 More analysis of Theorem 5.2.3*

Let $y_k = w_k - w_{k-1}$. Then from (58),

$$\begin{aligned} w_0 &= w_a + y_0 \\ w_1 &= w_0 + y_1 = w_0 + y_0 + f_0 = w_a + 2y_0 + f_0 \\ w_2 &= w_1 + y_0 + f_0 + f_1 = w_a + 3y_0 + 2f_0 + f_1 \\ &\vdots \quad \vdots \\ w_k &= w_{k-1} + y_0 + f_0 + \cdots + f_{k-1} = w_a + (k+1)y_0 + kf_0 + (k-1)f_1 + \cdots + f_{k-1} \end{aligned} \tag{78}$$

This yields an expression for $y_0$ by realizing that $w_n = w_b$, specifically

$$y_0 = \frac{w_b - w_a - [nf_0 + (n-1)f_1 + \cdots + f_{n-1}]}{n+1}. \tag{79}$$

Let $S = [nf_0 + (n-1)f_1 + \cdots + f_{n-1}]$. Again, slightly modified from (58)

$$f_i = \frac{1}{\rho} c_0 \, e^{\frac{1}{\rho}(t+i\tau)} - \frac{1}{\rho} d_0 \, e^{-\frac{1}{\rho}(t+i\tau)}$$

where $t \in [0, \tau]$. And so, the sum $S$ can be computed as,

$$
\begin{aligned}
S(t) = &\frac{1}{\rho} c_0\, e^{\frac{1}{\rho}t} \left[ n + (n-1)e^{\frac{1}{\rho}\tau} + (n-2)e^{\frac{1}{\rho}2\tau} + \cdots + e^{\frac{1}{\rho}(n-1)\tau} \right] - \\
&\frac{1}{\rho} d_0\, e^{-\frac{1}{\rho}t} \left[ n + (n-1)e^{-\frac{1}{\rho}\tau} + (n-2)e^{-\frac{1}{\rho}2\tau} + \cdots + e^{-\frac{1}{\rho}(n-1)\tau} \right] \\
= &\frac{1}{\rho} c_0\, e^{\frac{1}{\rho}t}\, \frac{n\left(1 - e^{\frac{1}{\rho}\tau}\right) + e^{\frac{1}{\rho}\tau}\left(e^{\frac{1}{\rho}n\tau} - 1\right)}{\left(1 - e^{\frac{1}{\rho}\tau}\right)^2} - \frac{1}{\rho} d_0\, e^{-\frac{1}{\rho}t}\, \frac{n\left(1 - e^{-\frac{1}{\rho}\tau}\right) + e^{-\frac{1}{\rho}\tau}\left(e^{-\frac{1}{\rho}n\tau} - 1\right)}{\left(1 - e^{-\frac{1}{\rho}\tau}\right)^2} \\
= &\frac{1}{\rho} c_0\, e^{\frac{1}{\rho}t}\, A - \frac{1}{\rho} d_0\, e^{-\frac{1}{\rho}t}\, B.
\end{aligned}
\tag{80}
$$

From (78), one notices that the difference terms can be expressed in terms of $y_0$ as,

$$
\begin{aligned}
y_k &= y_0 + f_0 + f_1 + \cdots + f_{k-1} \\
&= y_0 + \frac{1}{\rho} c_0\, e^{\frac{1}{\rho}t} \left( \frac{1 - e^{\frac{1}{\rho}k\tau}}{1 - e^{\frac{1}{\rho}\tau}} \right) - \frac{1}{\rho} d_0\, e^{-\frac{1}{\rho}t} \left( \frac{1 - e^{-\frac{1}{\rho}k\tau}}{1 - e^{-\frac{1}{\rho}\tau}} \right).
\end{aligned}
\tag{81}
$$

The values of the constant $c_0$ and $d_0$ can be found by utilizing the endpoint conditions – $w_{n-1}(\tau) = w_b(0)$ and $w_0(0) = w_a(0)$. From the former condition,

$$
\begin{aligned}
0 &= w_b(\tau) - w_{n-1}(\tau) = y_n(\tau) \\
&= y_0(\tau) + \frac{1}{\rho} c_0\, e^{\frac{1}{\rho}\tau} \left( \frac{1 - e^{\frac{1}{\rho}n\tau}}{1 - e^{\frac{1}{\rho}\tau}} \right) - \frac{1}{\rho} d_0\, e^{-\frac{1}{\rho}\tau} \left( \frac{1 - e^{-\frac{1}{\rho}n\tau}}{1 - e^{-\frac{1}{\rho}\tau}} \right) \\
&= \frac{w_b(\tau) - w_a(\tau) - S(\tau)}{n+1} + \frac{1}{\rho} c_0\, e^{\frac{1}{\rho}\tau}\, C - \frac{1}{\rho} d_0\, e^{-\frac{1}{\rho}\tau}\, D \\
&= \frac{w_b(\tau) - w_a(\tau)}{n+1} + \frac{1}{\rho} c_0\, e^{\frac{1}{\rho}\tau} \left( C - \frac{A}{n+1} \right) - \frac{1}{\rho} d_0\, e^{-\frac{1}{\rho}\tau} \left( D - \frac{B}{n+1} \right),
\end{aligned}
\tag{82}
$$

and from the latter,

$$
\begin{aligned}
0 &= w_0(0) - w_a(0) = y_0(0) \\
&= \frac{w_b(0) - w_a(0) - S(0)}{n+1} \\
&= \frac{w_b(0) - w_a(0)}{n+1} - \frac{1}{\rho} c_0 \left( \frac{A}{n+1} \right) + \frac{1}{\rho} d_0 \left( \frac{B}{n+1} \right).
\end{aligned}
\tag{83}
$$

Multiply (83) by $e^{\frac{1}{\rho}\tau}\left(C - \frac{A}{n+1}\right)$, and (82) by $\left(\frac{A}{n+1}\right)$ and add to get,

$$0 = \frac{1}{\rho} d_0 \left[ (n+1)\left(BC\, e^{\frac{1}{\rho}\tau} - AD\, e^{-\frac{1}{\rho}\tau}\right) - AB\left(e^{\frac{1}{\rho}\tau} - e^{-\frac{1}{\rho}\tau}\right)\right]$$
$$+ \left[(n+1)C\, e^{\frac{1}{\rho}\tau} + A\left(1 - e^{\frac{1}{\rho}\tau}\right)\right] [w_b(0) - w_a(0)]$$
$$\Rightarrow d_0 = \frac{\rho\, [w_b(0) - w_a(0)] \left[(n+1)C\, e^{\frac{1}{\rho}\tau} + A\left(1 - e^{\frac{1}{\rho}\tau}\right)\right]}{AB\left(e^{\frac{1}{\rho}\tau} - e^{-\frac{1}{\rho}\tau}\right) - (n+1)\left(BC\, e^{\frac{1}{\rho}\tau} - AD\, e^{-\frac{1}{\rho}\tau}\right)}. \tag{84}$$

And now substituting in (83),

$$c_0 = \frac{\rho\, [w_b(0) - w_a(0)] \left[(n+1)D\, e^{-\frac{1}{\rho}\tau} + B\left(1 - e^{-\frac{1}{\rho}\tau}\right)\right]}{AB\left(e^{\frac{1}{\rho}\tau} - e^{-\frac{1}{\rho}\tau}\right) - (n+1)\left(BC\, e^{\frac{1}{\rho}\tau} - AD\, e^{-\frac{1}{\rho}\tau}\right)}. \tag{85}$$

As evident from the figures in the previous section the alignment or the phase difference between the two periodic signals affects the Gluskabi cost considered and consequently the raccordation. It is then reasonable to ask that given two signals of the periodic type and a fixed raccordation interval, what is the optimum raccordation with respect to the starting and ending points. As of now, this does not yield a general solution. The cost function using the notation from this section is,

$$J = \int_0^\tau \sum_{k=0}^{n} \left[ y_k^2(t) + \dot{y}_k^2(t) \right] dt.$$

Let the phase shift for $w_a$ and $w_b$ be $\nu_a$ and $\nu_b$ respectively, i.e. the functions $w_a(\theta + \nu_a)$ and $w_b(\theta + \nu_b)$ are expressed as functions of the parameters $\nu_a$ and $\nu_b$, and the cost function is to be minimized with respect to these parameters. Intuitively, there are two factors affecting the cost – the correlation between a period of the two functions $w_a$ and $w_b$, and the difference between the end point values $w_a(0) - w_b(0)$.

### 5.2.3 General Periodic Type

The problem of finding the connection between two periodic trajectories of different periods in general is more complicated. The complication arises from the fact that there is no simple operator such that the kernel of this operator is the behavior of all periodic trajectories of arbitrary period. An alternate approach has been devised for solving the raccordation problem between trajectories of differing periods that yields interesting results.

The approach involves stretching the signals so that they both belong to the type with the same period, say the $\tau$-periodic type for some $\tau$. For example, if $w$ is a signal of period $\tau_1$, then choosing the time scaling factor to be $\tau_1/\tau$ will make the period $\tau$. The stretching function is chosen in the spirit of the Gluskabi problem, specifically it is required that the stretching function's velocity or the time scaling factor be maximally persistent in the constants type, so that during the raccordation interval the time scaling factor is as constant like as possible.

Given two trajectories $w^a$ and $w^b$ from the $\tau_a$-periodic type and the $\tau_b$ periodic type respectively, the Gluskabi raccordation, $w$, between the two over the interval $[a,b]$, with respect to the norm $\|\cdot\|^2 = \|\cdot\|_2^2 + \rho^2\|\mathbf{D}(\cdot)\|_2^2$ and in $\mathcal{B}_0 = C^0 \cap PWC^1$, is found in the following manner.

Let $\eta$ be the new stretched-time variable. Let's choose an arbitrary common period $\tau$. Let $f : \eta \to t$ be the stretching function, going from the new time scale to the old time scale. Let the raccordation interval in the new time scale be $[c,d]$. Then, the following conditions are required of the stretching function $f$:

1. $f(c) = a$ and $f(d) = b$.
2. The rate of change $\dot{f}(\eta)$ is constant for $\eta \leq c$ and for $\eta \geq d$. In fact, for the present problem $\dot{f}(\eta) = \tau_a/\tau$ for $\eta \leq c$ and $\dot{f}(\eta) = \tau_b/\tau$ for $\eta \geq d$.
3. The function $f$ is monotonous.

This problem is formulated in the spirit of the Gluskabi problem by required the persistence of the rate of change in the constants type, characterized by the operator $\mathbf{Op} = \mathbf{D}$. So, employing Theorem (5.1.1) the rate of change in the raccordation interval is given by $\mathbf{Op}^*\mathbf{Op}\,\dot{f} = 0$ or $\mathbf{D}^2\,\dot{f} = 0$, with the aforementioned boundary conditions. Thus,

$$\dot{f}(\eta) = c_1\eta + c_2 \qquad \eta \in (c,d)$$

for some $c_1, c_2 \in \mathbb{R}$, that can be solved for using the boundary conditions.

$$\dot{f}(c) = c_1\, c + c_2 = \tau_a/\tau$$

$$\dot{f}(d) = c_1\, d + c_2 = \tau_b/\tau$$

Consequently, the following values for the constants are obtained.

$$c_1 = \frac{\tau_b - \tau_a}{\tau(d-c)}$$
$$c_2 = \frac{d\tau_a - c\tau_b}{\tau(d-c)}$$

Therefore,

$$\dot{f}(\eta) = \frac{\tau_b - \tau_a}{\tau(d-c)}\eta + \frac{d\tau_a - c\tau_b}{\tau(d-c)}.$$

Integrating the equation above we obtain,

$$f(\eta) = \frac{\tau_b - \tau_a}{2\tau(d-c)}\eta^2 + \frac{d\tau_a - c\tau_b}{\tau(d-c)}\eta + C.$$

Condition 1 needs to be satisfied still but notice that we have an extra degree of freedom. So, let's just choose the starting point in the new time scale $c = 0$. Then,

$$f(c) = f(0) = a \quad \Rightarrow \quad C = a,$$

and

$$f(d) = b \quad \Rightarrow \quad d = \frac{2(b-a)\tau}{\tau_a + \tau_b}.$$

Therefore, the stretching function is

$$f(\eta) = \frac{\tau_b^2 - \tau_a^2}{4(b-a)\tau^2}\eta^2 + \frac{\tau_a}{\tau}\eta + a \qquad \text{for } \eta \in (0, d).$$

Now, let's look at the original periodic functions. Let $\bar{w}^a(\eta) = w^a\left(\frac{\tau_a}{\tau}\eta\right)$ and $\bar{w}^b(\eta) = w^b\left(\frac{\tau_b}{\tau}(\eta - d)\right)$. Then, $\bar{w}^a$ and $\bar{w}^b$ both belong to the $\tau$-periodic type. The raccordation is sought over the interval $[0, d]$ for these two functions and can be found by applying Theorem 5.2.3 or Theorem 5.2.4 from the previous section, with the operator $\mathbf{Op} = (\mathbf{I} - \mathbf{S}_{-\tau})$ and $\mathbf{Q} = (\mathbf{I} - \rho^2\tau\mathbf{D}^2)$. So, the cost function to be minimized is

$$\int_0^{d+\tau} [\bar{w}(\eta) - \bar{w}(\eta - \tau)]^2 + \rho^2\tau\,[\dot{\bar{w}}(\eta) - \dot{\bar{w}}(\eta - \tau)]^2\, d\eta$$

The raccordation $\bar{w}$ will be in the new time $\eta$ and so can be transformed back into a raccordation between the original periodic functions, $w^a$ and $w^b$, of different periods by using the inverse function of the stretching function $f$. The inverse $g : t \to \eta$ can be constructed as follows, by completing the square:

$$\frac{\tau_b^2 - \tau_a^2}{4(b-a)\tau^2}\eta^2 + \frac{\tau_a}{\tau}\eta + a = f(\eta) = t$$

$$\eta^2 + \frac{4(b-a)\tau\tau_a}{\tau_b^2 - \tau_a^2}\eta + \frac{4a(b-a)\tau^2}{\tau_b^2 - \tau_a^2} = \frac{4(b-a)\tau^2 t}{\tau_b^2 - \tau_a^2}$$

$$\left[\eta + \frac{2(b-a)\tau\tau_a}{\tau_b^2 - \tau_a^2}\right]^2 = \frac{4(b-a)\tau^2 t}{\tau_b^2 - \tau_a^2} - \frac{4a(b-a)\tau^2}{\tau_b^2 - \tau_a^2} + \left[\frac{2(b-a)\tau\tau_a}{\tau_b^2 - \tau_a^2}\right]^2$$

$$\Rightarrow \eta = -\frac{2(b-a)\tau\tau_a}{\tau_b^2 - \tau_a^2} + \sqrt{\frac{4(b-a)\tau^2 t}{\tau_b^2 - \tau_a^2} - \frac{4a(b-a)\tau^2}{\tau_b^2 - \tau_a^2} + \left[\frac{2(b-a)\tau\tau_a}{\tau_b^2 - \tau_a^2}\right]^2}$$

The idea is illustrated with the help of the following example. The Gluskabi problem is solved for the periodic functions $\sin 2\pi t$ and $\sin 4\pi t$, with periods 1 and $1/2$ respectively. The raccordation is sought over the interval $[0, 3]$. Using the method discussed above, the raccordation is shown in Figure 11.

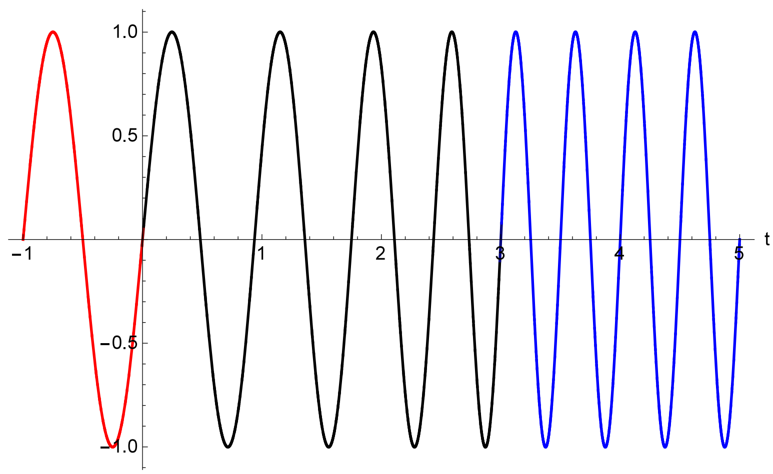



**Figure 11. The function is** $\sin 2\pi t$ **for** $t \le 0$ **and** $\sin 4\pi t$ **for** $t \ge 3$**. Raccordation is over the interval** $[0, 3]$**.**

It can be shown that the raccordation in the $t$-space is independent of the choice of the period $\tau$ in the $\eta$-space. Say $\bar{w}$ is the solution to the raccordation problem in the $\eta$-space when the chosen period is $\tau$. Then $\bar{w}$ minimizes the cost function:

$$J = \int_0^{d+\tau} \left[\bar{w}(\eta) - \bar{w}(\eta - \tau)\right]^2 + \rho^2\tau\left[\dot{\bar{w}}(\eta) - \dot{\bar{w}}(\eta - \tau)\right]^2 \, d\eta.$$

And the solution in the $t$-domain is $\bar{w}(g(t))$. Now, say there is another period $\tau'$ and the raccordation corresponding to it is $w'$, the stretching function is $f'$ and the interval length is $d'$. Say $M = \tau'/\tau$. Then, looking at the equations above the stretching functions $f$ and $f'$ are related as follows:

$$f(\eta) = f'(M\eta) = f'(\eta').$$

Similarly, $d' = Md$. It will be shown that two cost functions in $\eta$ and $\eta'$ space are equivalent.

$$\begin{aligned}
J &= \int_0^{d+\tau} \left[\bar{w}(\eta) - \bar{w}(\eta-\tau)\right]^2 + \rho^2\tau \left[\dot{\bar{w}}(\eta) - \dot{\bar{w}}(\eta-\tau)\right]^2 \, d\eta \\
&= \int_0^{d+\tau} \left[\bar{w}\left(\frac{\eta'}{M}\right) - \bar{w}\left(\frac{\eta'}{M} - \frac{M\tau}{M}\right)\right]^2 + \rho^2\tau \left[\dot{\bar{w}}\left(\frac{\eta'}{M}\right) - \dot{\bar{w}}\left(\frac{\eta'}{M} - \frac{M\tau}{M}\right)\right]^2 \, d\eta \\
&= \int_0^{d+\tau} \left[w'\left(\eta'\right) - w'\left(\eta'\tau'\right)\right]^2 + \rho^2\tau \left[\dot{\bar{w}}\left(\eta'\right)\frac{1}{M} - \dot{\bar{w}}'\left(\eta'-\tau'\right)\frac{1}{M}\right]^2 M^2 \, d\eta \\
&= \int_0^{d'+\tau'} \left[w'\left(\eta'\right) - w'\left(\eta'\tau'\right)\right]^2 + \rho^2\tau \left[\dot{w}'\left(\eta'\right) - \dot{w}'\left(\eta'-\tau'\right)\right]^2 M^2 \, \frac{d\eta}{M} \\
&= \int_0^{d'+\tau'} \left[w'\left(\eta'\right) - w'\left(\eta'\tau'\right)\right]^2 + \rho^2\tau' \left[\dot{w}'\left(\eta'\right) - \dot{w}'\left(\eta'-\tau'\right)\right]^2 \, d\eta
\end{aligned}$$

This shows that if $\bar{w}$ minimizes the Gluskabi cost function in the $\eta$ space then $w'$ minimizes it's respective cost function in the $\eta'$ space. Consequently, in the $t$ space, the raccordation is:

$$\begin{aligned}
w(t) &= w'(g'(t)) \\
&= w'(M\eta) \\
&= \bar{w}(\eta) \\
&= \bar{w}(g(t))
\end{aligned}$$

# CHAPTER 6

# DYNAMICAL RACCORDATION PROBLEM

This chapter is devoted to the study of the dynamical raccordation case, i.e. when the trajectories in the base behavior are constrained by the dynamics of a system. Since one is never allowed to step out of the base behavior, the dynamical system constraints can be called "*hard constraints*", where as the type constraints can then be called "*soft constraints*". In th following sections, the Gluskabi extension is derived for linear time invariant dynamical systems, in both the continuous time and discrete time case. The relation of the controllability of the system to the existence of a raccordation is also investigated.

## 6.1 Continuous-Time LTI Systems

The first result here deals with continuous time LTI systems and types that are described by polynomial differential operators, where as the next result deals with shift types. The time axis is chosen to be the set of real numbers in this section. The base behavior is the set of smooth trajectories of an LTI dynamical system, and so for this section the assumption (S3) can be stated using Willems's notation as,

(S3*) The base behavior is $\mathcal{B}_0 = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ s.t. $R(\mathbf{D})\, w = 0\}$, where $R$ is a polynomial, $R \in \mathbb{R}^{g\times q}[\xi]$, and $g < q$.

Every LTI system has an equivalent minimal representation, $R(\mathbf{D})w = 0$, that can also be expressed in the input/output form by partitioning $w$ as:

$$P(\mathbf{D})y = N(\mathbf{D})u, \tag{86}$$

where $y$ and $u$ are called the output and input respectively, $P \in \mathbb{R}^{g\times g}[\xi]$, $N \in \mathbb{R}^{g\times(q-g)}[\xi]$, $\det P \neq 0$, and $P^{-1}N$ is a proper matrix ([34]). It is this I/O representation that will be used in the following theorems to define an LTI system.

### 6.1.1 Polynomial Differential Types

A polynomial differential type, $\mathcal{T}$, is one that is described by the kernel of an operator **Op** that is a polynomial in the differentiation operator **D**. Since $u$ and $y$ are simply obtained by some partition of $w$, $w \in \mathcal{T}$ implies that both $u$ and $y$ are of the same type. Hence we are looking for connections of input/output pairs of the same type $\mathcal{T}$.

The presentation of the main result is preceded by some necessary remarks. The definition of the LTI system above suggests that $w$ is $q$-dimensional and so in general a vector operator is needed to describe the type, $\mathcal{T}$, such as in Section 5.1.2. However, an extension of scalar operator is used in this section. Given a scalar type $(\mathcal{T}, \mathbf{Op})$ i.e., defined on signal space $\mathbb{W} = \mathbb{R}$, it can be correspondingly defined for vector trajectories i.e., $\mathbb{W} = \mathbb{R}^q$ by extending **Op** as $\mathbf{Op}^e\, w = \left(\mathbf{Op}\, w_1, \cdots, \mathbf{Op}\, w_q\right)^T$ for $w = (w_1, \cdots, w_q)^T \in \mathbb{W}^{\mathbb{R}}$. In the following result **Op** will be understood to be $\mathbf{Op}^e$ wherever appropriate. The inner product is appropriately extended as well, and defined in the same manner as in Section 5.1.2. The inner product is defined on the support of the operator **Op** acting on the Gluskabi map and since **Op** is a differential operator, the support is the same as the raccordation interval. The adjoint of operators is defined as in earlier sections. Furthermore, assumption (S1a) about using the $L^2$ inner product will be carried on in this section. Consequently, any Sobolev norm of order $k$ can be used by choosing $\mathbf{Q} = W(-\mathbf{D}^2)$, where $W$ is a polynomial, $W(\xi) = c_0 + c_1\xi + c_2\xi^2 + \cdots + c_k\xi^k$, for any $k \in \mathbb{Z}_+$ and $c_k \in \mathbb{R}_+$. It can be easily shown that this **Q** is self-adjoint. For the following result, the weighting operator **Q** is also extended to a vector version, in a similar fashion to Section 5.1.2.

**Theorem 6.1.1.** *Given a minimal and controllable linear time invariant dynamical system (86) and a polynomial differential type* $(\mathcal{T}, \mathbf{Op})$*, the trajectories in the Gluskabi extension* *with respect to the Sobolev norm* $\|.\|_{\mathbf{Q}}$*, are given by the following equations:*

$$\begin{aligned} (\mathbf{U}_{12}^{*}\,\mathbf{Op}^{u*}\,\mathbf{Q}^{u}\,\mathbf{Op}^{u}\,\mathbf{U}_{12} + \mathbf{U}_{22}^{*}\,\mathbf{Op}^{y*}\,\mathbf{Q}^{y}\,\mathbf{Op}^{y}\,\mathbf{U}_{22})\;\eta &= 0 \\ -\mathbf{U}_{12}\,\eta &= u \\ \mathbf{U}_{22}\,\eta &= y \end{aligned}$$

*where* $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \mathbb{R}^{q\times q}[\xi]$ *is a unimodular matrix such that* $\begin{bmatrix} N & P \end{bmatrix} U = \begin{bmatrix} I & O \end{bmatrix}$.

**PROOF.** The cost function to be minimized along with the adjoined constraints is given in the inner product form:

$$J(u) = \frac{1}{2}\langle \mathbf{Q}^{u}\mathbf{Op}^{u}\,u, \mathbf{Op}^{u}\,u\rangle + \frac{1}{2}\langle \mathbf{Q}^{y}\mathbf{Op}^{y}\,y, \mathbf{Op}^{y}\,y\rangle + \langle \lambda, P(\mathbf{D})y - N(\mathbf{D})u\rangle \tag{87}$$

where $\mathbf{Op}^{u}$, $\mathbf{Op}^{y}$, $\mathbf{Q}^{u}$ and $\mathbf{Q}^{y}$ are the appropriately extended forms of the operators $\mathbf{Op}$ and $\mathbf{Q}$, depending on the dimensions of $u$ and $y$, respectively. The first variation of the cost function due to a perturbation in $u$ can be computed as follows,

$$\begin{aligned} \delta J(u;\delta u) &= \frac{1}{2}\langle \mathbf{Q}^{u}\mathbf{Op}^{u}\,u, \mathbf{Op}^{u}\,\delta u\rangle + \frac{1}{2}\langle \mathbf{Q}^{u}\mathbf{Op}^{u}\,\delta u, \mathbf{Op}^{u}\,u\rangle + \frac{1}{2}\langle \mathbf{Q}^{y}\mathbf{Op}^{y}\,y, \mathbf{Op}^{y}\,\delta y\rangle \\ &\quad + \frac{1}{2}\langle \mathbf{Q}^{y}\mathbf{Op}^{y}\,\delta y, \mathbf{Op}^{y}\,y\rangle + \langle \lambda, P(\mathbf{D})\,\delta y\rangle - \langle \lambda, N(\mathbf{D})\,\delta u\rangle \\ &= \langle \mathbf{Q}^{u}\mathbf{Op}^{u}\,u, \mathbf{Op}^{u}\,\delta u\rangle + \langle \mathbf{Q}^{y}\mathbf{Op}^{y}\,y, \mathbf{Op}^{y}\,\delta y\rangle + \langle \lambda, P(\mathbf{D})\,\delta y\rangle - \langle \lambda, N(\mathbf{D})\,\delta u\rangle \\ &= \langle \mathbf{Op}^{u*}\mathbf{Q}^{u}\mathbf{Op}^{u}\,u, \delta u\rangle + \langle \mathbf{Op}^{y*}\mathbf{Q}^{y}\mathbf{Op}^{y}\,y, \delta y\rangle + \langle P(\mathbf{D})^{*}\,\lambda, \delta y\rangle - \langle N(\mathbf{D})^{*}\,\lambda, \delta u\rangle \\ &\quad + \text{ boundary terms} \end{aligned} \tag{88}$$

where we have used the fact that $\mathbf{Q}^{u}$ and $\mathbf{Q}^{y}$ are self adjoint, since $\mathbf{Q} = \mathbf{Q}^{*}$, in deriving the last expression. All the adjoints exist because the boundary terms can be safely ignored since the functions $u$ and $y$ over the interval $[a, b]$ are to be matched to their respective given trajectories at the boundaries. Thus, the variations $\delta u$ and $\delta y$ and the required number of their derivatives are zero at the end points. This leads to the Euler-Lagrange equations,

$$\mathbf{Op}^{y*}\mathbf{Q}^{y}\mathbf{Op}^{y}\,y + P(\mathbf{D})^{*}\,\lambda = 0. \tag{89}$$

The necessary condition for optimality is,

$$\mathbf{Op}^{u*}\mathbf{Q}^{u}\mathbf{Op}^{u}\, u - N(\mathbf{D})^{*}\, \lambda = 0. \tag{90}$$

To find the Gluskabi extension it is required to eliminate $\lambda$ from the above two equations and solve the resultant equations along with the dynamical system equation for $u$ and $y$ and the given boundary conditions. In other words one needs to find the behavior given by the representation,

$$\begin{bmatrix} N^{*} & X & O \\ P^{*} & O & Z \\ O & N & P \end{bmatrix} (\mathbf{D}) \begin{bmatrix} \lambda \\ -u \\ y \end{bmatrix} = 0 \tag{91}$$

where $X$, $Z$, $N^{*}$, and $P^{*}$ are polynomial matrices such that $X(\mathbf{D}) = \mathbf{Op}^{u*}\mathbf{Q}^{u}\mathbf{Op}^{u}$, $Z(\mathbf{D}) = \mathbf{Op}^{y*}\mathbf{Q}^{y}\mathbf{Op}^{y}$, $N^{*}(\mathbf{D}) = N(\mathbf{D})^{*}$, and $P^{*}(\mathbf{D}) = P(\mathbf{D})^{*}$. The behavior in (91) will be unchanged under any left unimodular transformation on the polynomial matrix ([34]). Since the system is controllable, the rank of the matrix $\begin{bmatrix} P(s) & -N(s) \end{bmatrix}$ is the same for all $s \in \mathbb{C}$ and because of minimality, the matrix has full row rank for almost all $s$. This implies that this matrix has full rank for all $s$ and the polynomial matrices $P$ and $N$ are left coprime ([34] and [43]). Thus, there always exists a unimodular matrix $U$ such that

$$\begin{bmatrix} N & P \end{bmatrix} U = \begin{bmatrix} I & O \end{bmatrix}. \tag{92}$$

It also holds that

$$U^{*} \begin{bmatrix} N^{*} \\ P^{*} \end{bmatrix} = \begin{bmatrix} I \\ O \end{bmatrix} \tag{93}$$

where $U^{*}(s) = U(-s)^{T}$. The matrix $U^{*}$ is also unimodular since $\det U = \det U^{T}$, and since the determinant is a polynomial in the entries of the matrix, which are themselves polynomials for the matrix $U$ and so the determinant is a polynomial in the indeterminate $s$ and is some constant since $U$ is unimodular and so changing the indeterminate to $-s$ doesn't change the determinant. If the matrix $U$ is partitioned as $\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$, then $U^{*} = \begin{bmatrix} U_{11}^{*} & U_{21}^{*} \\ U_{12}^{*} & U_{22}^{*} \end{bmatrix}$.

A new unimodular matrix can now be constructed using $U^*$, specifically $\begin{bmatrix} U^* & O \\ O & I \end{bmatrix}$, and applying it as a left unimodular transformation to (91) yields

$$\begin{bmatrix} U^* & O \\ O & I \end{bmatrix} \begin{bmatrix} N^* & X & O \\ P^* & O & Z \\ O & N & P \end{bmatrix} (\mathbf{D}) \begin{bmatrix} \lambda \\ -u \\ y \end{bmatrix} = \begin{bmatrix} I & U_{11}^* X & U_{21}^* Z \\ O & U_{12}^* X & U_{22}^* Z \\ O & N & P \end{bmatrix} (\mathbf{D}) \begin{bmatrix} \lambda \\ -u \\ y \end{bmatrix} = 0 \tag{94}$$

The behavior corresponding to the equation above is equivalent to

$$\begin{bmatrix} I & U_{11}^* X & U_{21}^* Z \\ O & U_{12}^* X & U_{22}^* Z \\ O & N & P \end{bmatrix} \begin{bmatrix} I & O \\ O & U \end{bmatrix} \begin{bmatrix} I & O \\ O & U \end{bmatrix}^{-1} (\mathbf{D}) \begin{bmatrix} \lambda \\ -u \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} I & \mathbf{U}_{11}^* \mathbf{X} \mathbf{U}_{11} + \mathbf{U}_{21}^* \mathbf{Z} \mathbf{U}_{21} & \mathbf{U}_{11}^* \mathbf{X} \mathbf{U}_{12} + \mathbf{U}_{21}^* \mathbf{Z} \mathbf{U}_{22} \\ O & \mathbf{U}_{12}^* \mathbf{X} \mathbf{U}_{11} + \mathbf{U}_{22}^* \mathbf{Z} \mathbf{U}_{21} & \mathbf{U}_{12}^* \mathbf{X} \mathbf{U}_{12} + \mathbf{U}_{22}^* \mathbf{Z} \mathbf{U}_{22} \\ O & I & O \end{bmatrix} \begin{bmatrix} \lambda \\ \nu \\ \eta \end{bmatrix} = 0 \tag{95}$$

where the bold font corresponds to the differential operator of the respective polynomial e.g. $\mathbf{X} = X(\mathbf{D})$ and so on, $\begin{bmatrix} I & O \\ O & U \end{bmatrix}^{-1} \begin{bmatrix} \lambda \\ -u \\ y \end{bmatrix} = \begin{bmatrix} \lambda \\ \nu \\ \eta \end{bmatrix}$, $\nu$ is a $g \times 1$ vector, and $\eta$ is a $(q - g) \times 1$ vector. The third row of (95) simplifies to $\nu = 0$ and the second row simplifies to the equation,

$$(\mathbf{U}_{12}^* \mathbf{X} \mathbf{U}_{12} + \mathbf{U}_{22}^* \mathbf{Z} \mathbf{U}_{22})\, \eta = 0$$

$$(\mathbf{U}_{12}^*\, \mathbf{Op}^{u*}\, \mathbf{Q}^u\, \mathbf{Op}^u\, \mathbf{U}_{12} + \mathbf{U}_{22}^*\, \mathbf{Op}^{y*}\, \mathbf{Q}^y\, \mathbf{Op}^y\, \mathbf{U}_{22})\, \eta = 0 \tag{96}$$

and the subsequent substitution yields

$$u = -\mathbf{U}_{12}\, \eta \tag{97}$$

$$y = \mathbf{U}_{22}\, \eta \tag{98}$$

□

It is of course assumed that the set $\{(u, y)$ s.t. $P(\mathbf{D})y = N(\mathbf{D})u$ and $\mathbf{Op}\, u = 0$ and $\mathbf{Op}\, y = 0\}$ is nonempty i.e., there exist input/output pairs of the dynamical system of the required type $\mathcal{T}$. Otherwise the question of finding the Gluskabi extension is moot. Also, the controllability assumption (as defined in Chapter 2) is one sufficient condition for the solution to exist. It guarantees that there exist trajectories of the dynamical system connecting the left trajectory to the right one in some finite time. Furthermore, for smooth solutions to an LTID system the time can be taken to be arbitrarily small ([34]) and so the length of the interval $[a, b]$ does not matter. This result can be further generalized to the case when only the input or the output is of the type and needs to be connected, or to the case when the persistence of output is more important then the input. Either of these cases can be viewed as an extension of the previous result by changing the operator $\mathbf{Q}$ of the inner product. For instance, the first case can be accomplished by choosing $\mathbf{Q}^u = 0$ or $\mathbf{Q}^y = 0$.

*6.1.1.1 Examples*

The previous result, Theorem 6.1.1, for finding the raccordation between two trajectories of an LTI system is illustrated with the help of two examples.

**Example-1.**

We have a scalar first order LTI system given by the input-output differential equation $(\mathbf{D} + 1)y = u$. We are interested in transitioning from one constant steady state to another. So our type is constants and $\mathbf{Op} = \mathbf{D}$. Notice that elements of this type satisfy the hard constraint i.e., if $y = c$ where $c$ is some constant then $u = c$. The transfer function for this system is $H(s) = \frac{1}{s+1}$ and so at steady state $y_{ss} = u_{ss}$, by the final value theorem. The chosen norm is again the $L^2$ norm and the raccordation time interval is $[0, 1]$. The numerator and denominator polynomials are $N(s) = 1$ and $P(s) = s + 1$ respectively. And so $U = \begin{bmatrix} 1 & -(s+1) \\ 0 & 1 \end{bmatrix}$ is the unimodular matrix required by Theorem 6.1.1 and to find the

Gluskabi extension the following system of equations need to be solved.

$$[(\mathbf{D}+1)^*\mathbf{Op}^*\mathbf{Op}(\mathbf{D}+1)+\mathbf{Op}^*\mathbf{Op}]\,\eta = 0 \tag{99}$$

$$(\mathbf{D}+1)\,\eta = u \tag{100}$$

$$\eta = y \tag{101}$$

The equation (99) is simplified to get

$$\left(\mathbf{D}^4 - 2\mathbf{D}^2\right)\eta = 0 \tag{102}$$

Solving these differential equations yields,

$$y(t) = Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t} + C + Dt \tag{103}$$

$$u(t) = (1+\sqrt{2})Ae^{\sqrt{2}t} + (1-\sqrt{2})Be^{-\sqrt{2}t} + C + D(1+t) \tag{104}$$

Again, the specific raccordation is obtained by using the boundary conditions i.e. $u(0)$, $y(0)$, $u(1)$, and $y(1)$. The raccordation for the case when $u = y = 0$ for $t \leq 0$ and $u = y = 1$ for $t \geq 1$ is illustrated in Figure 12.

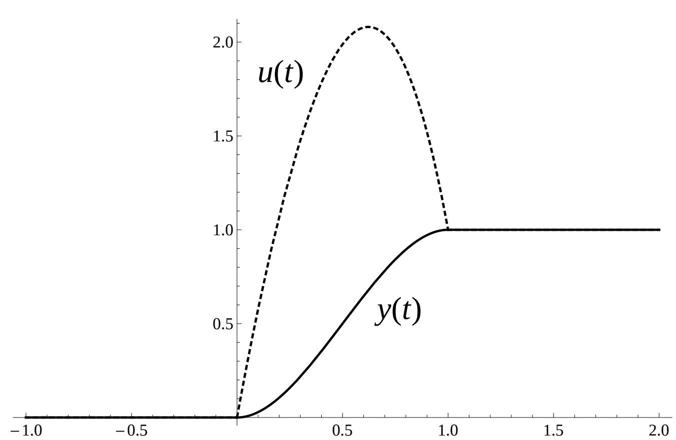



**Figure 12. The raccordation from constant 0 to the constant 1. The input is dashed line and output is the solid one.**

**Example-2.**

We end this section by looking at the cyber-physical problem of charging a capacitor. We consider the simplest series RC circuit shown in Figure 13. The objective here is to put a charge $Q$ on the capacitor in time interval $[0, T]$. So the type to be considered for this case is the "type of constants" and again the $L^2$ norm is minimized. The dynamical system equation associated with the circuit is $\dot{q} + \frac{1}{RC}q = \frac{1}{R}u$, where $q$ is the charge on the capacitor and $u$ is the source voltage as well as the input over here. The type constraint is only imposed on the output i.e. $q$ and so in terms of Theorem 6.1.1, $\mathbf{Q}^u = 0$. The resulting trajectory of charge and the input voltage is illustrated in Figure 14. An interesting parallel has been found that the resulting minimizing trajectory obtained from applying Theorem 6.1.1 is the same trajectory obtained when minimizing the heat generated in the resistor as shown in [44]. This points to a possible correlation between our theory and minimization of entropy for thermodynamic systems.

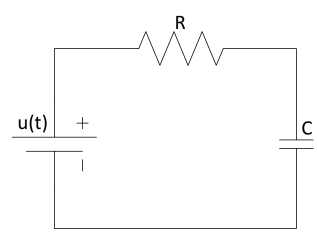



**Figure 13. Charging of a capacitor in an RC circuit**

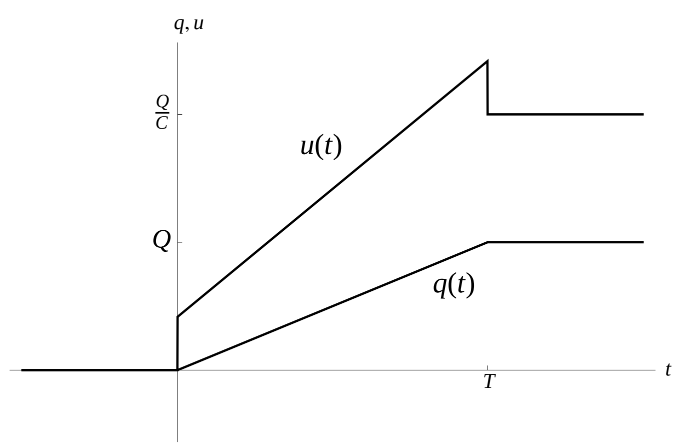



**Figure 14. Charge and input voltage trajectories of the RC circuit**

### 6.1.2 Linear Shift Types with Commensurate Delays

In this section, the Gluskabi extension is found for types described by linear shift operators with commensurate shifts. The extension is found for the trajectories of a continuous time LTI system, and so the base behavior is the same set of smooth solutions to an LTI system, (S3*). For these types, the same result as Theorem 6.1.1 holds and is presented here. The support of the operator is now longer than the raccordation interval, in general. For instance, if the raccordation interval is $[a, b]$ then for the $\tau$-periodic type, included in this class, the support of the Gluskabi map is $[a, b + \tau]$. The inner product is then obviously defined on this longer interval which is the support of the Gluskabi map.

**Theorem 6.1.2.** *Given a minimal and controllable linear time invariant dynamical system (86) and a linear shift type* $(\mathcal{T}, \mathbf{Op})$*, the trajectories in the Gluskabi extension are given by the following equations:*

$$\begin{aligned} \left(\mathbf{U}_{12}^{*}\, \mathbf{Op}^{u*}\, \mathbf{Op}^{u}\, \mathbf{U}_{12} + \mathbf{U}_{22}^{*}\, \mathbf{Op}^{y*}\, \mathbf{Op}^{y}\, \mathbf{U}_{22}\right) \eta &= 0 \\ -\mathbf{U}_{12}\, \eta &= u \\ \mathbf{U}_{22}\, \eta &= y \end{aligned}$$

*where* $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \mathbb{R}^{q \times q}[\xi]$ *is a unimodular matrix such that* $\begin{bmatrix} N & P \end{bmatrix} U = \begin{bmatrix} I & O \end{bmatrix}$.

**PROOF.** Notice that the Gluskabi extension is given by exactly the same equations as Theorem 6.1.1. The proof is along the same lines and so it will not be completely replicated here. The same cost function is minimized in this case as well and so the same set of Euler-Lagrange equations and optimal conditions are obtained, i.e.,

$$\begin{bmatrix} N^{*} & X & O \\ P^{*} & O & Z \\ O & N & P \end{bmatrix} (\mathbf{D}, \mathbf{S}) \begin{bmatrix} \lambda \\ -u \\ y \end{bmatrix} = 0, \tag{105}$$

where $N$, $P$, $N^{*}$, and $P^{*}$ are polynomial matrices in the indeterminate $s$, and $X$ and $Z$ are polynomial matrices in the indeterminates $z$ and $z^{-1}$, so that a polynomial in $s$ corresponds to a differential operator and a polynomial in $z$ and $z^{-1}$ to a polynomial in shift operators – $X(\mathbf{S}) = \mathbf{Op}^{u*}\mathbf{Op}^{u}$, $Z(\mathbf{S}) = \mathbf{Op}^{y*}\mathbf{Op}^{y}$. Thus, (105) is a delay-differential system of equations.

Notice that the entries of the matrix in (105) belong to the ring $\mathbb{R}[s, z, z^{-1}]$. However, it has been shown in [45, 46, 47] that if a certain extension, $\mathcal{K}$, of this ring is considered then the properties of delay-differential systems with commensurate delays can be studied algebraically in the same manner as is done for system of ODEs. The extended ring is specifically

$$\mathcal{K} = \left\{ \frac{p}{\phi} \text{ s.t. } p \in \mathbb{R}[s, z, z^{-1}], \phi \in \mathbb{R}[s] \setminus \{0\}, \frac{p^*}{\phi} \in H(\mathbb{C}) \right\},$$

where $p^* = p(s, e^{-s})$ and $H(\mathbb{C})$ is the ring of entire functions on the whole complex plane.

With this ring $\mathcal{H}$, it turns out that two delay-differential behaviors are again equivalent if they are left unimodularly related. The determinant of the unimodular matrix is now a unit of the ring $\mathcal{H}$. The polynomial matrix $U$, in the indeterminate $s$, that transforms the matrix $\begin{bmatrix} N & P \end{bmatrix}$ to $\begin{bmatrix} I & O \end{bmatrix}$ is still unimodular with respect to the new ring. Thus, the rest of the arguments in the proof of Theorem 6.1.1 still hold and we arrive at the same result for the Gluskabi extension. □

## 6.2 Discrete LTI Systems

The dynamical raccordation problem can be considered in discrete time as well. The universum in this case is $(\mathbb{R}^q)^{\mathbb{Z}}$, i.e., the space of all infinite sequences that take values in $\mathbb{R}^q$. The analogue for LTI systems in the discrete time case are Autoregressive systems, i.e., systems expressed by a set of linear difference equations ([35] Theorem 3.1), and the trajectories of such a system will form the base behavior in this section. The assumption (S3) takes the following form:

(S3”) The base behavior is $\mathcal{B}_0 = \{w \in (\mathbb{R}^q)^{\mathbb{Z}} \text{ s.t. } R(\sigma, \sigma^{-1})\, w = 0\}$, where $R \in \mathbb{R}^{g \times q}[z, z^{-1}]$ and $\sigma$ is the left shift operator and $\sigma^{-1}$ the right shift operator.

These systems are linear, time invariant, and complete. Just as in the continuous case, there is a surjection from the space of polynomial matrices, with entries in the ring $\mathbb{R}[z, z^{-1}]$, to the set of LTI complete discrete time systems, and this induces an equivalence relation on the polynomial matrices. Every such system also admits a minimal representation, having full row rank, and equivalent minimal representations are related by left unimodular transformations ([35] Prop. 3.3). It is worth mentioning here that a unimodular matrix is one whose determinant is a unit of the ring, which is the ring of polynomials in two indeterminates $z$ and $z^{-1}$ here, so the determinant is $\alpha z^k$ where $\alpha \neq 0$ and $k \in \mathbb{Z}$. This translates to one more elementary operation than the continuous case and that is multiplication by $z^k$. A system is said to be controllable if it allows an AR-representation with $R$ left prime ([35] Theorem 5.2). In the following result, I/O dynamical systems will be considered given by the AR-representation

$$P(\sigma, \sigma^{-1})y = Q(\sigma, \sigma^{-1})u, \tag{106}$$

where $P \in \mathbb{R}^{p\times p}[z, z^{-1}]$, $Q \in \mathbb{R}^{p\times(q-p)}[z, z^{-1}]$, and $\det P \neq 0$. The I/O system is controllable if and only if $P$ and $Q$ are left coprime ([35]).

### 6.2.1 AR Types

An AR type is one that is described by the kernel of an operator **Op** that is a polynomial in the difference operator. Furthermore, the **Op** has been defined for the scalar sequence case, but it can be extended to the multidimensional case in a similar manner to the continuous time. The inner product that will be used here is the canonical one over the finite interval $\mathbb{T}$, i.e.,

$$\langle x, y\rangle_{\mathbb{T}} = \sum_{i\in\mathbb{T}} x_i\, y_i.$$

The maximal persistence of the raccordation is sought over the support of the Gluskabi map, which will be typically longer than the raccordation interval since the operator **Op** contains shifts. Say the raccordation interval is $[a, b]$, then the support is $\mathbb{T} = [c, d]$ where $c = (a -$ max. power of $\sigma$ in **Op**) and $d = (b +$ max. power of $\sigma^{-1}$ in **Op**). The adjoint of the operator **Op** now must exist for a restricted set, specifically the set of functions $h : [c, d] \to \mathbb{R}^q$ such that $h_k = 0$ for $k \leq a$ and for $k \geq b$. Since, the scalar **Op** is a polynomial in $\sigma$, it suffices to just show the adjoint of the operator $\sigma^k$.

$$\begin{aligned}\langle \sigma^k h, y\rangle_{\mathbb{T}} &= \sum_{i=c}^{d} h[i+k]\, y[i] \\ &= \sum_{i=c+k}^{d+k} h[i]\, y[i-k] \\ &= \sum_{i=c}^{d} h[i]\, y[i-k] \\ &= \langle h, \sigma^{-k} y\rangle_{\mathbb{T}}\end{aligned}$$

Notice that, $|k|$ is less than or equal to the maximum power of $\sigma$ and $\sigma^{-1}$, and so by the definition of $c$ and $d$, $(c+k) \leq a$ and $(d+k) \geq b$. Consequently, since $h[i] = 0$ for $i$ outside $(a, b)$, the limits of the sum can be changed to $[c, d]$ in the above derivation. Therefore, $\mathbf{Op} = R(\sigma, \sigma^{-1})$ admits the adjoint – $\mathbf{Op}^* = R(\sigma^{-1}, \sigma)$. Now, the adjoint can be extended to multidimensional case in the same manner as the continuous version.

**Theorem 6.2.1.** *Given a minimal and controllable LTI discrete system (106) and an AR-type* $(\mathcal{T}, \mathbf{Op})$*, the trajectories in the Gluskabi extension are given by the following equations:*

$$\begin{aligned}\left(\mathbf{U}_{12}^* \, \mathbf{Op}_u^* \, \mathbf{Op}_u \, \mathbf{U}_{12} + \mathbf{U}_{22}^* \, \mathbf{Op}_y^* \, \mathbf{Op}_y \, \mathbf{U}_{22}\right) \eta &= O \\ \mathbf{U}_{12}\, \eta &= u \\ -\mathbf{U}_{22}\, \eta &= y\end{aligned}$$

*where* $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ *is a unimodular matrix such that* $\begin{bmatrix} Q & P \end{bmatrix} U = \begin{bmatrix} I & O \end{bmatrix}$.

**PROOF.** The proof is exactly the same as the continuous time case since everything required for that proof still holds in this case. □

## 6.3 Controllability

For both the signal and dynamical raccordation problems, the controllability of the base behavior guarantees that for any two elements of the type there exists at least one trajectory in the base behavior connecting them. So, controllability of the base behavior is a sufficient condition for the non-emptiness of the search space of the Gluskabi optimization problem. In this section, the controllability condition will be relaxed, and instead a necessary condition will be obtained for continuous time LTI systems.

A minimal LTI behavior can be decomposed into the direct sum of a controllable behavior, $\mathcal{B}_{cont}$, and an autonomous behavior, $\mathcal{B}_{aut}$, (See Section 2.1.3.1). This decomposition is not unique, but it can be shown that the controllable part, however, is unique. Using this decomposition, the condition of controllability of the base behavior can be relaxed to the following condition.

**Theorem 6.3.1.** *Given a minimal LTI system (86), and a linear differential type* $(\mathcal{T}, \mathbf{Op})$*, a necessary and sufficient condition for the existence of at least one trajectory connecting any pair of elements in the type over any given finite time interval is that the type behavior be included in the controllable part of the base behavior,* $\mathcal{B}_{cont}$.

**PROOF.** Let's first show that the aforementioned condition is necessary. Say there exists at least one trajectory between any pair of elements in the type over any given finite interval. Since, the type is linear it always contains the zero trajectory. So we have that for any given trajectory in the type, there exists at least one trajectory, in the base behavior, connecting it to the zero trajectory over the given finite time interval, and vice versa. But, the zero trajectory is included also in the controllable part of the base behavior. This implies that every trajectory in the type behavior can be connected over some finite interval to any trajectory in the controllable part of the base behavior, and vice versa. Consequently, by the definition of controllability, the trajectories in the type behavior are included in the controllable part of the base behavior or $\mathcal{T} \subset \mathcal{B}_{cont}$. Thus, this condition is necessary.

On the other hand, if the type behavior is contained in the controllable part of the base behavior then clearly there exists at least one trajectory connecting any pair of elements in the type, over some finite time interval. The interval can be of arbitrary length because it is an LTI behavior. Thus, the aforementioned condition is also sufficient. □

Given the polynomial matrix $R$, describing the LTI base behavior, a corresponding polynomial matrix $R_{cont}$ can be found such that the controllable part of the base behavior is described by $R_{cont}(\mathbf{D})\, w = 0$ (See Section 2.1.3.1). In terms of this matrix, the above condition that the type be contained in the controllable part of the base behavior translates to the equivalence between the two behaviors, namely the type $\mathcal{T}$ and the intersection between the controllable part and the type or $\mathcal{B}_{cont} \cap \mathcal{T}$. Thus, the necessary and sufficient condition is that the polynomial matrices describing these two behaviors are left-unimodularly related. In other words there exists a unimodular polynomial matrix $U$ such that

$$\begin{bmatrix} R \\ Op \end{bmatrix} = U \begin{bmatrix} R_{cont} \\ Op \end{bmatrix},$$

where $Op$ is the polynomial matrix corresponding to the operator $\mathbf{Op}$.

If $\mathbf{Op}$ is a nonlinear differential operator then the situation is more interesting. In fact, the linear operator is a special case of this general result.

**Theorem 6.3.2.** *Given a minimal LTI system (86), and a differential type* $(\mathcal{T}, \mathbf{Op})$*, a necessary and sufficient condition for the existence of at least one trajectory connecting any pair of elements in the type over any given finite time interval is that the autonomous part of all the trajectories in the type, obtained by decomposing the base behavior, be the same trajectory.*

**PROOF.** The base behavior, $\mathcal{B}$, can be decomposed into the direct sum of a controllable part, $\mathcal{B}_{cont}$, and an autonomous part, $\mathcal{B}_{aut}$ (See Section 2.1.3.1), i.e.,

$$\mathcal{B} = \mathcal{B}_{aut} \oplus \mathcal{B}_{cont}.$$

Using the decomposition of the base behavior, every trajectory, $w$, in the base behavior can be decomposed into an autonomous part and a controllable part, i.e., $w = w^a + w^c$. Now, say there exists at least one trajectory, $w$, between any pair of elements, $w_1$ and $w_2$, in the type over a given finite time interval, $[a, b]$. Let the decomposition of these trajectories be:

$$w_1 = w_1^a + w_1^c \qquad w_2 = w_2^a + w_2^c \qquad w = w^a + w^c.$$

Since, $w(t) = w_1(t)$ for $t \leq a$, it implies that $w^a(t) = w_1^a(t)$ for $t < a$. But then the two trajectories, $w^a$ and $w_1^a$, both in the autonomous behavior $\mathcal{B}_{aut}$ agree with each other for $t < a$, and so they agree for all time, i.e. $w^a = w_1^a$. Since $w$ connects $w_1$ and $w_2$, and so $w^a(t) = w_2^a(t)$ for $t > b$, it yields that $w_1^a = w_2^a$. Thus, a necessary condition for the existence of at least one trajectory connecting any pair of elements in the type is that the autonomous part of all the trajectories in the type be the same.

On the other hand, if the two trajectories, $w_1$ and $w_2$, in the type have the same autonomous part, i.e. $w_1^a = w_2^a$, then since $w_1^c, w_2^c \in \mathcal{B}_{cont}$ there exists a trajectory $w^c \in \mathcal{B}_{cont}$ such that it connects $w_1^c$ and $w_2^c$ over the given interval. Then, $w = w_1^a + w^c$ connects the trajectories $w_1$ and $w_2$. Thus, the condition of the autonomous part being the same is also a sufficient condition.

To prove that this result is independent of the decomposition, it needs to be shown that if the autonomous part of all the trajectories in the type is the same in one decomposition then it is the same, albeit a different trajectory, in every other decomposition of the base behavior. It was shown in Section 2.1.3.1 that if one decomposition is given by the kernels of $R_{cont}(\mathbf{D})$ and $R_{aut}(\mathbf{D})$, where

$$R_{cont}(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} V^{-1} \quad \text{and} \quad R_{aut}(\xi) = \begin{bmatrix} D(\xi) & 0 \\ 0 & I \end{bmatrix} V^{-1}, \tag{107}$$

then another decomposition is given by the kernels of $R_{cont}(\mathbf{D})$ and $\tilde{R}_{aut}(\mathbf{D})$, where

$$\tilde{R}_{aut}(\xi) = \begin{bmatrix} D'(\xi) & 0 \\ 0 & I \end{bmatrix} W^{-1} V^{-1} \quad \text{and} \quad \begin{bmatrix} D(\xi) & 0 \end{bmatrix} W = \begin{bmatrix} D'(\xi) & 0 \end{bmatrix}. \tag{108}$$

Let two trajectories of the same type, say $w_1$ and $w_2$, have the same autonomous part $w^a$ using the decomposition (107), i.e.,

$$w_1 = w^a + w_1^c \quad \text{and} \quad w_2 = w^a + w_2^c.$$

If the matrices $V$ and $V^{-1}$ are partitioned with appropriate dimensions as follows:

$$V = \left[\begin{array}{c|c} V_1 & V_2 \\ \hline V_3 & V_4 \end{array}\right] \quad \text{and} \quad V^{-1} = \left[\begin{array}{c} V_u \\ \hline V_l \end{array}\right],$$

then the autonomous part can be computed as,

$$w^a = V \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \left[\begin{array}{c} V_u \\ \hline V_l \end{array}\right] (\mathbf{D})\, w_1 = V \left[\begin{array}{c} V_u \\ \hline 0 \end{array}\right] (\mathbf{D})\, w_1 = V \left[\begin{array}{c} V_u \\ \hline 0 \end{array}\right] (\mathbf{D})\, w_2.$$

This means that for any $w_1$ and $w_2$ in the type,

$$\left[\begin{array}{c} V_u(\mathbf{D}) \\ \hline 0 \end{array}\right] (w_1 - w_2) \in \ker(V).$$

Since $V$ is unimodular, the kernel of $V$ is trivial, i.e. $\ker V = \{0\}$. Therefore, if the autonomous part of the two trajectories is the same then $V_u(\mathbf{D})\, w_1 = V_u(\mathbf{D})\, w_2$. Now using the other decomposition (108), say the autonomous parts of $w_1$ and $w_2$ are $w_1^a$ and $w_2^a$ respectively. Then,

$$\begin{aligned} w_1^a &= VW \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^{-1} & 0 \\ \times & U_2^{-1} \end{bmatrix} \left[\begin{array}{c} V_u \\ \hline V_l \end{array}\right] (\mathbf{D})\, w_1 \\ &= VW \left[\begin{array}{c} U_1^{-1} V_u \\ \hline 0 \end{array}\right] (\mathbf{D})\, w_1 = VW \left[\begin{array}{c} U_1^{-1} V_u \\ \hline 0 \end{array}\right] (\mathbf{D})\, w_2 \\ &= w_2^a, \end{aligned}$$

where $W = \begin{bmatrix} U_1 & 0 \\ \times & U_2 \end{bmatrix}$. This concludes the proof. □

Lastly, in this section, Theorem 6.1.1 is modified in the light of Theorem 6.3.1 to accommodate the case of finding the Gluskabi extension when the system is not controllable.

**Theorem 6.3.3.** *Given a minimal LTI system (86) and a polynomial-differential type* $(\mathcal{T}, \mathbf{Op})$*, the trajectories in the Gluskabi extension with respect to the Sobolev norm* $\|.\|_{\mathbf{Q}}$ *are given* *by the following equations:*

$$(\mathbf{V}_{12}^* \,\mathbf{Op}^{u*}\, \mathbf{Q}^u \,\mathbf{Op}^u\, \mathbf{V}_{12} + \mathbf{V}_{22}^* \,\mathbf{Op}^{y*}\, \mathbf{Q}^y \,\mathbf{Op}^y\, \mathbf{V}_{22})\ \eta = 0$$
$$\mathbf{V}_{12}\, \eta = u$$
$$\mathbf{V}_{22}\, \eta = y$$

*where* $V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \in \mathbb{R}^{q\times q}[\xi]$ *is a unimodular matrix such that the controllable part of* $\begin{bmatrix} -N & P \end{bmatrix}$ *is described by the polynomial matrix* $R_{cont} = \begin{bmatrix} I & O \end{bmatrix} V^{-1}$.

**PROOF.** It is assumed that the assumption from Theorem 6.3.1 is satisfied, i.e. the type behavior is contained in the controllable part of the base behavior. Then, the hard constraint can be chosen to be the description for the controllable part, $R_{cont}$, instead of the description for the entire LTI system. This is valid because of the fact that any trajectory connecting any two trajectories in the type lies in the controllable part, since they all have the same autonomous parts (See Theorem 6.3.2).

If the minimal LTI system is described by the polynomial matrix $\begin{bmatrix} -N & P \end{bmatrix}$ then from Section 2.1.3.1, its controllable part is described by the polynomial matrix $R_{cont} = \begin{bmatrix} I & O \end{bmatrix} V^{-1}$, where $V$ is unimodular. Using $R_{cont} = \begin{bmatrix} \widehat{N} & \widehat{P} \end{bmatrix}$ as the system description, the proof is the same as Theorem 6.1.1. The same cost function is minimized and it leads to the following set of equations:

$$\begin{bmatrix} \widehat{N}^* & X & O \\ \widehat{P}^* & O & Z \\ O & \widehat{N} & \widehat{P} \end{bmatrix} (\mathbf{D}) \begin{bmatrix} \lambda \\ u \\ y \end{bmatrix} = 0,$$

where $X$, $Z$, $N^*$, and $P^*$ are polynomial matrices such that $X(\mathbf{D}) = \mathbf{Op}^{u*}\mathbf{Q}^u\mathbf{Op}^u$, $Z(\mathbf{D}) = \mathbf{Op}^{y*}\mathbf{Q}^y\mathbf{Op}^y$, $\widehat{N}^*(\mathbf{D}) = \widehat{N}(\mathbf{D})^*$, and $\widehat{P}^*(\mathbf{D}) = \widehat{P}(\mathbf{D})^*$. Using the fact that $\begin{bmatrix} \widehat{N} & \widehat{P} \end{bmatrix} V = \begin{bmatrix} I & 0 \end{bmatrix} V^{-1} V = \begin{bmatrix} I & 0 \end{bmatrix}$, and going along the same track as in the proof of Theorem 6.1.1, the equations above can be simplified to

$$\begin{aligned} \left(\mathbf{V}_{12}^{*}\,\mathbf{Op}^{u*}\,\mathbf{Q}^{u}\,\mathbf{Op}^{u}\,\mathbf{V}_{12} + \mathbf{V}_{22}^{*}\,\mathbf{Op}^{y*}\,\mathbf{Q}^{y}\,\mathbf{Op}^{y}\,\mathbf{V}_{22}\right)\eta &= 0 \\ \mathbf{V}_{12}\,\eta &= u \\ \mathbf{V}_{22}\,\eta &= y. \quad \square \end{aligned}$$

# CHAPTER 7
# GAIT TRANSITIONS

The groundwork laid down in the previous chapter culminates in a contribution to a problem of interest in robotics: Finding a graceful transition between different gaits or locomotion patterns of a biomimetic robot. Animals too have different gaits that they employ in varying scenarios, and they gracefully switch from one gait to another. For instance, it has been observed that a horse typically displays four different gaits: walk, trot, pace, and gallop [48]. Insects on the other hand use wave, tetrapod, and tripod gaits [49]. As mentioned earlier, in order to traverse complex terrains biomimetic robots in practice also require distinct gaits. If the locomotion apparatus of the animal or robot is seen as a dynamical system then a gait is described by the collection of periodic trajectories of some of the states such as the positions and velocities, and is a periodic solution of the dynamical system. Thus, a defining characteristic of a gait is that it is periodic, and a graceful transition in the context of this dissertation is one that maximally preserves this periodicity during the transition. This idea is demonstrated with the help of a two-piece worm, modeled simply as two masses connected by an actuator and with asymmetric dry friction, and of systems formed by interconnection of these two-piece worms in the following sections.

There is a huge body of existing work on the design of different gaits but no significant attention has been paid to the problem of gait transitions. An increasingly popular approach for generating control signals for locomotion gaits is to use central pattern generator (CPG) models, but it has been pointed out in [50] that despite plethora of computational studies into gait transitions in animals, few have been applied to robotics. Some instances of work done in gait transitions for robots using this control approach are included here. In [51], the authors have designed a salamander robot that switches from swimming to walking. They employ a network of coupled oscillators to generate limb motions with just one input, the drive signal. As the drive frequency is increased, the system switches from the swimming gait oscillators to walking gait oscillators, with a transition period where a weighted combination of both is taken. A similar weighted combination approach was taken in [52]. In this case, the authors designed the gaits by extracting kinetic motion primitives from the joint trajectories' data of an actual horse using principal component analysis (PCA), and then the control signals for the limb controllers of a quadruped robot are assumed to be linear combinations of these primitives. To transition from one gait to another, a homotopy of the two gait control trajectories is used and the transition occurs in a fixed time about a pivot point. Using this method, successful transitions were not observed for all gait pairs. Another approach is to manually program each of the transitions, as was done in [53] for a hexapod robot keeping just the static stability in mind. The authors extended this work and designed an interesting scheme for automatically generating gait transition controls in [54], again considering stability. Yet another approach to gaits in a hexapod robots is seen in [55], where a switch from one gait to another is accomplished by simply changing the phase difference between the control signals of the different limbs. However, no attention has been paid as to how this change should occur. In conclusion, the aforementioned examples are indicative of how most of the existent research works concentrate on the design of gaits, based solely on kinematics, and reveal a general lack of analysis into the best way to switch from one gait to another.

## 7.1 Two-Piece Worm Model

A simplified one dimensional model of the worm and its rectilinear motion are considered here. The model comprises of two blocks, $M_1$ and $M_2$, of the same mass $m$ connected by an actuator as shown in Figure 15. The actuator exerts a force, $u$, symmetrically on both masses but in opposite directions. The direction of motion of the worm or the forward direction is towards the right, as indicated in the figure. For the sake of simplicity it is assumed that both masses are equal. The friction is asymmetric, i.e., the friction experienced by a mass when moving forward is not the same as when moving backwards.

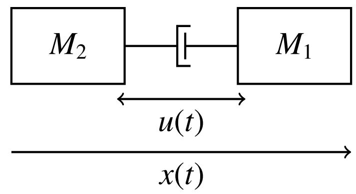



**Figure 15. Two piece worm model**

The behavior of this model has been extensively studied by many researchers, such as [56], [57], and [58]. Additionally, gaits have been designed for this model as well considering different optimality criteria, for instance in [59] and [60]. The dynamics of this model are given by the following equations:

$$\begin{aligned} m\,\ddot{x}_1(t) &= u(t) + f_1(t) \\ m\,\ddot{x}_2(t) &= -u(t) + f_2(t), \end{aligned} \tag{109}$$

where $f_1$ and $f_2$ are the frictional forces experienced by the respective masses. The friction model is as follows:

$$f_i(t) = \begin{cases} -f_F & \text{if } \dot{x}_i(t) > 0 \\ f_B & \text{if } \dot{x}_i(t) < 0 \\ f_O & \text{if } \dot{x}_i(t) = 0 \end{cases} \quad \text{for } i \in \{0, 1\}, \tag{110}$$

where $f_F, f_B > 0$ are constant and are the forward and backward frictional forces respectively. It is also assumed that $f_B > f_F$. This asymmetry in the frictional forces leads to the forward propagation of the worm. The dynamical system is four dimensional, but for the purposes of gait analysis we can exclude the trajectory of the center of mass. Therefore, we consider a three dimensional system with the state vector $y = \{x_1 - x_2, \dot{x}_1 - \dot{x}_2, \dot{x}_1 + \dot{x}_2\}$. So, the dynamical system is a switched system where each of the modes is affine, i.e.,

$$\dot{y} = Ay + Bu + C, \tag{111}$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 \\ f_1 - f_2 \\ f_1 + f_2 \end{bmatrix}$, and the value of the mass $m$ has been subsumed in $u$ and $f$'s.

In terms of the Gluskabi framework, the base behavior is taken to be the space of piecewise-continuous functions with the control being piecewise continuous and the states are continuous. Furthermore, the type to be considered here is the $\tau$-periodic type, described by the operator $\mathbf{Op} = (I - \mathbf{S}_{-\tau})$. Gaits with period $\tau$ are obviously of this type. Therefore, this problem of finding the Glsukabi raccordation between two gaits of period $\tau$ for the two-piece worm, over the interval $[a, b]$ was addressed in [61].

### 7.1.1 Switched Systems with Affine Modes and Periodic Type

The necessary conditions for a transition are derived first in general for the periodic type and any switched system with affine modes, and a predefined switching sequence. The specific case of worm dynamics is considered later.

**Theorem 7.1.1.** *Given a switched system with affine modes, $\dot{y} = A_i y + B_i u + C_i$, and a fixed predefined switching sequence, with $j$ switches, and switching conditions $\psi_k(y(t_k)) = 0$, the Gluskabi raccordation with respect to the $\tau$-periodic type and the $L^2$ norm, over the interval $[a, b]$, must satisfy the following equations:*
*For $t \in (t_{k-1}, t_k)$ and $k \in \{1, \cdots, j+1\}$,*

$$2y(t) - y(t-\tau) - y(t+\tau) + A_k^T \lambda + \dot{\lambda} = 0 \tag{112}$$

$$2u(t) - u(t-\tau) - u(t+\tau) + B_k^T \lambda = 0, \tag{113}$$

*where $t_k$ are the switching times, $t_0 = a$ and $t_{j+1} = b$. And, for $k \in \{1, \cdots, j\}$, $dy_k = (\delta y + \dot{y}\delta t_k)|_{t_k}$, and $g_k = A_k y + B_k u + C_k$, the following boundary conditions must hold,*

$$\left( \lambda^T\big|_{t_k^+} - \lambda^T\big|_{t_k^-} + \nu_k^T \frac{\partial \psi_k}{\partial y} \right) dy_k = 0 \tag{114}$$

$$\left( \lambda^T g_k\big|_{t_k^-} - \lambda^T g_{k+1}\big|_{t_k^+} \right) \delta t_k = 0. \tag{115}$$

**PROOF.** The proof is based on variational calculus and is along the lines of the one outlined in [62] or [63]. The objective function to be minimized here is,

$$J(u) = \frac{1}{2} \int_a^{b+\tau} \|y(t) - y(t-\tau)\|^2 + \|u(t) - u(t-\tau)\|^2 \, dt. \tag{116}$$

Notice that since the trajectories are $\tau$-periodic for $t < a$ and $t > b + \tau$, the cost is zero in these intervals. Now say the switching sequence has $j$ switches at times $\{t_1, t_2, \cdots, t_j\}$, then adjoining the system constraints and the switching condition constraints, the cost function is

$$J = \sum_{k=0}^{j} \frac{1}{2} \int_{t_k}^{t_{k+1}} \|y(t) - y(t-\tau)\|^2 + \|u(t) - u(t-\tau)\|^2 + 2\lambda^T \left[A_{k+1} y + B_{k+1} u + C_{k+1} - \dot{y}\right] dt$$
$$+ \frac{1}{2} \int_{b}^{b+\tau} \|y(t) - y(t-\tau)\|^2 + \|u(t) - u(t-\tau)\|^2 + 2\lambda^T \left[A_t\, y + B_t\, u + C_t - \dot{y}\right] dt$$
$$+ \sum_{k=1}^{j} \nu_k^T \psi_k(y)\Big|_{t_k}. \quad (117)$$

Perturbing the control and switching times, the variation in the cost can be computed as follows,

$$\delta J = \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \left[y(t) - y(t-\tau)\right]^T \left[\delta y(t) - \delta y(t-\tau)\right] + \left[u(t) - u(t-\tau)\right]^T \left[\delta u(t) - \delta u(t-\tau)\right] dt$$
$$+ \int_{b}^{b+\tau} \left[y(t) - y(t-\tau)\right]^T \left[\delta y(t) - \delta y(t-\tau)\right] + \left[u(t) - u(t-\tau)\right]^T \left[\delta u(t) - \delta u(t-\tau)\right] dt$$
$$+ \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \lambda^T \left[A_{k+1} \delta y + B_{k+1} \delta u - \dot{\delta y}\right] dt + \int_{b}^{b+\tau} \lambda^T \left[A_t\, \delta y + B_t\, \delta u - \dot{\delta y}\right] dt$$
$$- \sum_{k=1}^{j} \int_{t_k}^{t_k + \delta t_k} \lambda^T [A_{k+1}(y + \delta y) + B_{k+1}(u + \delta u) + C_{k+1} - \dot{y} - \dot{\delta y}]\, dt$$
$$+ \sum_{k=1}^{j} \int_{t_k}^{t_k + \delta t_k} \lambda^T \left[A_k(y + \delta y) + B_k(u + \delta u) + C_k - \dot{y} - \dot{\delta y}\right] dt + \sum_{k=1}^{j} \nu_k^T \frac{\partial \psi_k}{\partial y} \left[\delta y + \dot{y} \delta t_k\right]\Big|_{t_k}, \quad (118)$$

ignoring higher order terms. By change of variables,

$$\int_{t_i}^{t_{i+1}} y(t)\, \delta y(t-\tau) dt = \int_{t_i - \tau}^{t_{i+1} - \tau} y(t+\tau)\, \delta y(t) dt, \quad (119)$$

$$\int_{t_i}^{t_{i+1}} y(t-\tau)\, \delta y(t-\tau) dt = \int_{t_i - \tau}^{t_{i+1} - \tau} y(t)\, \delta y(t) dt. \quad (120)$$

Also, let's employ the following approximation:

$$\int_{t_k}^{t_k + \delta t_k} g\, dt \approx g(t_k)\, \delta t_k.$$

Now using the above relations, integrating by parts the $\dot{\delta y}$ terms, and ignoring the higher order terms in $\delta y$ and $\delta t$, we get,

$$\delta J = \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \left[2y(t) - y(t-\tau) - y(t+\tau) + A_{k+1}^T \lambda + \dot{\lambda}\right]^T \delta y(t)\, dt + \int_{a-\tau}^{a} [y(t) - y(t+\tau)]^T \delta y(t)\, dt$$

$$+ \int_{b}^{b+\tau} \left[y(t) - y(t-\tau) + A_t^T \lambda + \dot{\lambda}\right]^T \delta y(t)\, dt + \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \left[2u(t) - u(t-\tau) - u(t+\tau) + B_{k+1}^T \lambda\right]^T \delta u(t)\, dt$$

$$+ \int_{a-\tau}^{a} [u(t) - u(t+\tau)]^T \delta u(t)\, dt + \int_{b}^{b+\tau} \left[u(t) - u(t-\tau) + B_t^T \lambda\right]^T \delta u(t)\, dt - \sum_{k=0}^{j} \lambda^T \delta y\Big|_{t_k}^{t_{k+1}} - \lambda^T \delta y\Big|_{b}^{b+\tau}$$

$$+ \sum_{k=1}^{j} \lambda^T \left[A_k y + B_k u + C_k - \dot{y}\right]\Big|_{t_k^-} \delta t_k - \sum_{k=1}^{j} \lambda^T \left[A_{k+1} y + B_{k+1} u + C_{k+1} - \dot{y}\right]\Big|_{t_k^+} \delta t_k$$

$$+ \sum_{k=1}^{j} \nu_k^T \frac{\partial \psi_k}{\partial y} \left[\delta y + \dot{y} \delta t_k\right]\Big|_{t_k}. \quad (121)$$

Notice that $\delta y(t) = 0$ when $t \in [a-\tau, a]$ or $t \in [b, b+\tau]$, and so their corresponding integrals in (121) are zero. Also, since the states are continuous, $\delta y(t_k^-) = \delta y(t_k^+)$. For the objective function to be minimized, its first variation (121) should be zero at the optimal behavior. Thus, the necessary conditions are,

$$2y(t) - y(t-\tau) - y(t+\tau) + A_1^T \lambda + \dot{\lambda} = 0\ ,\ t \in (a, t_1)$$

$$2u(t) - u(t-\tau) - u(t+\tau) + B_1^T \lambda = 0$$

For $t \in (t_{k-1}, t_k)$ and $k \in \{2, \cdots, j\}$,

$$2y(t) - y(t-\tau) - y(t+\tau) + A_k^T \lambda + \dot{\lambda} = 0$$

$$2u(t) - u(t-\tau) - u(t+\tau) + B_k^T \lambda = 0$$

$$2y(t) - y(t-\tau) - y(t+\tau) + A_{j+1}^T \lambda + \dot{\lambda} = 0\ ,\ t \in (t_j, b)$$

$$2u(t) - u(t-\tau) - u(t+\tau) + B_{j+1}^T \lambda = 0$$

$$\left(\lambda^T\Big|_{t_k^+} - \lambda^T\Big|_{t_k^-} + \nu_k^T \frac{\partial \psi_k}{\partial y}\right) dy_k = 0$$

where $dy_k = (\delta y + \dot{y} \delta t_k)|_{t_k}$

$$\left(\lambda^T g_k\Big|_{t_k^-} - \lambda^T g_{k+1}\Big|_{t_k^+}\right) \delta t_k = 0$$

where $A_k y + B_k u + C_k = g_k$. □ (122)

The equations (122) become a bit simpler when the dynamics of worm are considered, and this will be explored in the next section.

### 7.1.2 Graceful Transitions for Two-piece Worm

The worm model is a hybrid system with three possible affine modes, that only differ in the additive vector, $C$, which is dependent on the direction of the velocities of the two masses at that time instant. The mode where both masses have negative velocities is not considered, under the assumption that the velocity of the center of mass is always positive, i.e., the worm is always moving forward. The switch to the next mode happens when one of the mass's velocity hits zero. A gait is any periodic trajectory in the base behavior. There exists a natural switching sequence that repeats during every gait's execution. A table outlining this natural sequence, specifically the mode the system switches to dependent on the current mode and the sign of control, is provided in [60] and will be employed here. So, given a gait and a starting point for the period, a switching sequence for every period of the gait is completely determined. It is assumed that the switching sequences of the starting and ending gaits are perfectly aligned. Consequently, it is assumed that the transition follows the same switching sequence within each period of the transition interval. Clearly, the problem of finding gait transitions for this worm model fits the hypothesis of the result derived in the previous section.

Differentiating (112), and substituting the dynamics (111), we obtain

$$\ddot{\lambda} + A^T\dot{\lambda} + A\left[2y(t) - y(t-\tau) - y(t+\tau)\right] + B\left[2u(t) - u(t-\tau) - u(t+\tau)\right] + \left[2C_t - C_{t^-} - C_{t^+}\right] = 0. \quad (123)$$

Substituting (112) and (113) in (123) yields a differential equation entirely in $\lambda$:

$$\ddot{\lambda} + \left(A^T - A\right)\dot{\lambda} - \left(AA^T + BB^T\right)\lambda + D_t = 0 \quad \text{where } D_t = \left[2C_t - C_{t^-} - C_{t^+}\right]. \quad (124)$$

Equation (124) for $\lambda$ holds in each interval before a switch happens. The constant $D_t$ depends on the time $t$ and the duration for which each mode is in effect, in relation to the mode in effect $\tau$ units forward and backward in time. This yields a number of possible cases and the equation needs to be solved for all these cases. Furthermore, the two possible switching conditions are $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, which translate to $\psi_k = y_2(t_k) + y_3(t_k)$ and $\psi_k = y_2(t_k) - y_3(t_k)$ respectively in our chosen state variables. The jump in the costate, $\lambda$, at the switching instant $t_k$ is dependent on the switching condition at that instant and can be determined using (114) as follows:

$$\lambda|_{t_k^+} - \lambda|_{t_k^-} = \begin{cases} \begin{bmatrix} 0 & -\nu_k & -\nu_k \end{bmatrix}^T & \text{if } \psi_k = y_2 + y_3 \\ \begin{bmatrix} 0 & -\nu_k & \nu_k \end{bmatrix}^T & \text{if } \psi_k = y_2 - y_3 \end{cases} \tag{125}$$

Assuming that the costate $\lambda$ has been computed, the control $u$ can be computed by (113) using the same ideas as in Section 5.2.1. Let $b - a = n\tau + \bar{b}$ and $u_k(t) = u(a + k\tau + t)$ for $t \in [0, \tau]$. Similarly, $\lambda_k(t) = \lambda(a + k\tau + t)$ for $t \in [0, \tau]$. Also let $u_i(t)$ and $u_f(t)$ be the controls corresponding to the initial and final gaits to be connected, i.e., $u(t) = u_i(t)$ for $t < a$ and $u(t) = u_f(t)$ for $t \geq b$. Then (113) can be written as,

$$2u_k - u_{k-1} - u_{k+1} + B^T\lambda_k = 0 \quad \forall k \in \{0, \cdots, n+1\}.$$

Let $h(t) = u_0 - u_i$. Then,

$$\begin{aligned} u_0 &= u_i + h \\ u_1 &= u_0 + h + B^T\lambda_0 = u_i + 2h + B^T\lambda_0 \\ u_k &= u_i + (k+1)h + kB^T\lambda_0 + (k-1)B^T\lambda_1 + \cdots + B^T\lambda_{k-1} \end{aligned} \tag{126}$$

The function $h$ can be determined by realizing that,

$$u_f = \begin{cases} u_{n+1} & \text{if } t \in [0, \bar{b}] \\ u_n & \text{if } t \in [\bar{b}, \tau] \end{cases} .$$

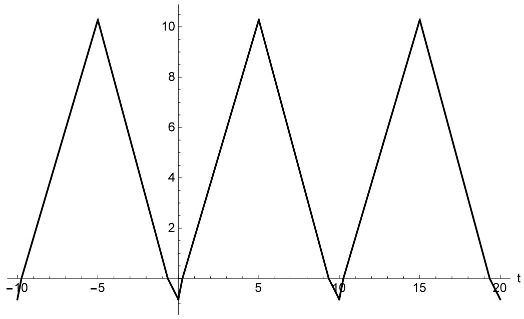
$$h = \begin{cases} \frac{u_f - u_i - (n+1)B^T\lambda_0 - nB^T\lambda_1 \cdots - B^T\lambda_n}{n+2} & \text{if } t \in \left[0, \bar{b}\right] \\ \frac{u_f - u_i - nB^T\lambda_0 - (n-1)B^T\lambda_1 \cdots - B^T\lambda_{n-1}}{n+1} & \text{if } t \in \left[\bar{b}, \tau\right] \end{cases} \tag{127}$$

The states $y$ can be similarly derived. Finally, (115) translates to the following expression and relates the jump $\nu_k$ to the switching time $t_k$.

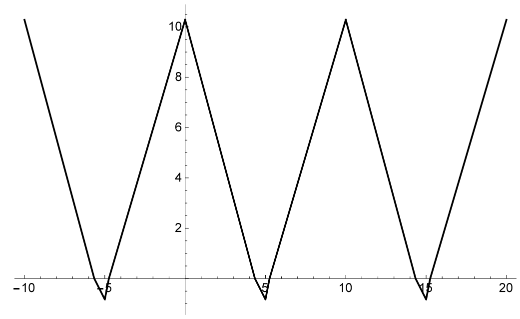
$$\lambda^T(t_k^-)Bu(t_k^-) - \lambda^T(t_k^+)Bu(t_k^+) + \lambda^T(t_k^-)C_k - \lambda^T(t_k^+)C_{k+1} = 0 \tag{128}$$

Solving the equations (124), (125), (126), (127), and (128) yields the expression for the control and state trajectory during the transition in terms of nine constants which can be solved for using (112), $y(a)$ and $y(b)$.

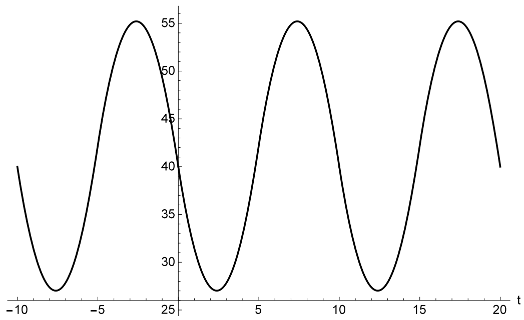
The graceful transition for the worm is illustrated in the following example. The values of the frictional forces are $f_F = 0.1$ and $f_B = 1$. Say the worm is moving with a particular gait 'A', with state trajectories and control depicted in Figures 16a,16b,16c, and 16d. The goal here is to transition gracefully to gait 'B', with corresponding state trajectories and control depicted in Figures 17a,17b,17c, and 17d. The optimal transition between them is depicted in Figures 18a,18b,18c, and 18d, over $[0, 10]$. If the modes are named as: $a : \dot{x}_1 > 0, \dot{x}_2 > 0$; $v : \dot{x}_1 > 0, \dot{x}_2 < 0$; $c : \dot{x}_1 < 0, \dot{x}_2 > 0$, then the switching sequence within each period is $(c, a, b, a, c)$. As illustrated in the figures, the optimal graceful transition strives to preserve the periodic behavior and shape of the gaits being connected.



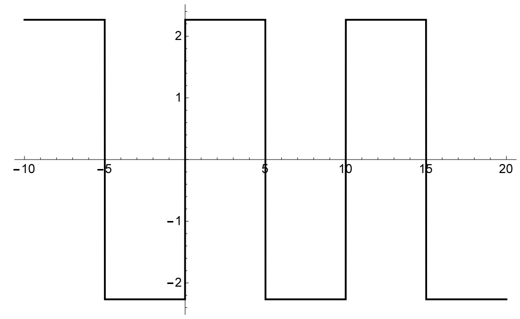

(a) Velocity $\dot{x}_1$ of mass $M_1$ in gait A




(b) Velocity $\dot{x}_2$ of mass $M_2$ in gait A

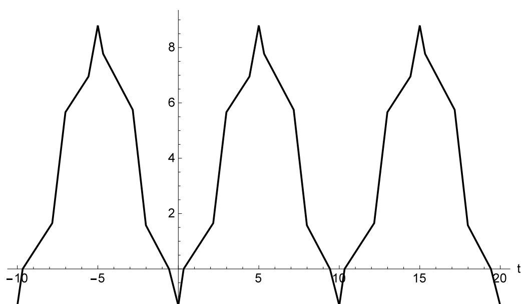



(c) Position difference $x_1 - x_2$ between masses

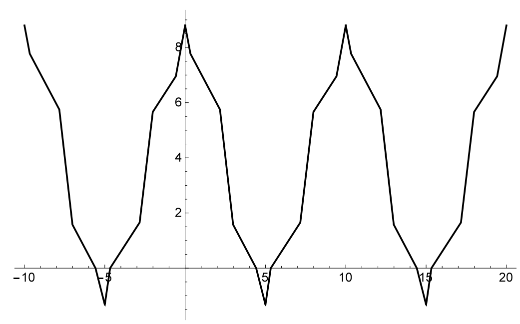



(d) Actuator force $u$ for gait A

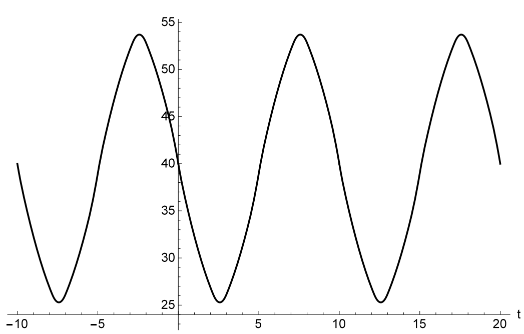
**Figure 16. Velocities, Inter-mass distance, and Actuator force for Gait A**



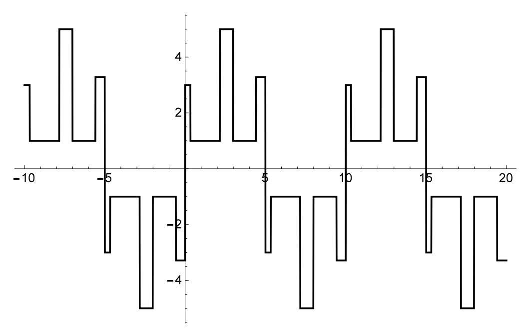

(a) Velocity $\dot{x}_1$ of mass $M_1$ in gait B




(b) Velocity $\dot{x}_2$ of mass $M_2$ in gait B

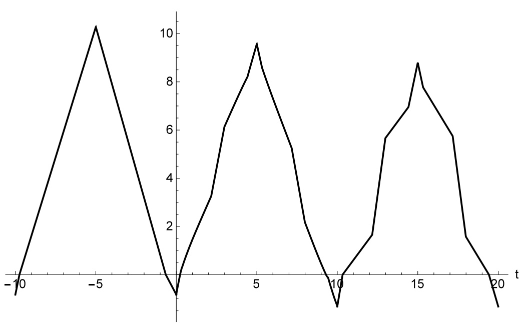



(c) Position difference $x_1 - x_2$ between masses

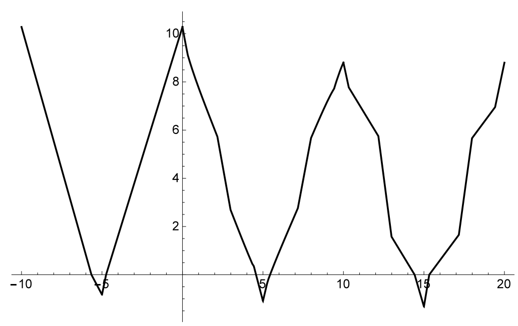



(d) Actuator force $u$ for gait B

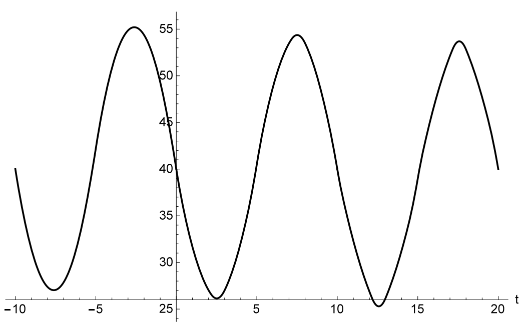
**Figure 17. Velocities, Inter-mass distance, and Actuator force for Gait B**



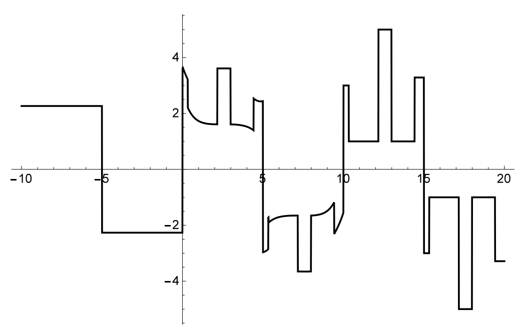

(a) Transition of velocity $\dot{x}_1$




(b) Transition of velocity $\dot{x}_2$




(c) Transition of position difference $x_1 - x_2$




(d) Actuator force $u$ for transition

**Figure 18. Velocities, Inter-mass distance, and Actuator force during transition from Gait A to Gait B**

## 7.2 Interconnection of Two-piece Worms

In this section, an interesting model is considered that is obtained by connecting two of the two-piece worm models from Figure 15 in parallel. This new model is illustrated in Figure 19.

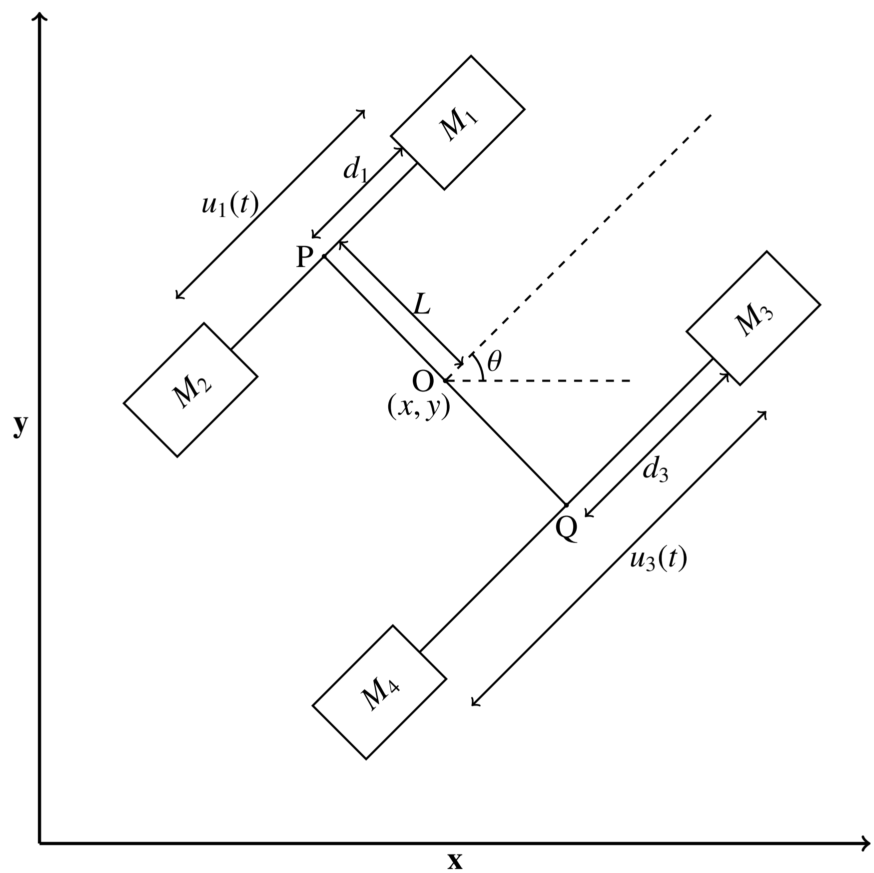



**Figure 19. Parallel Worm Model**

### 7.2.1 The Model

#### *7.2.1.1 Geometry of the Parallel Worm*

The model consists of four blocks $M_1$ through $M_4$ with masses $m_1$ through $m_4$ in the configuration shown in Figure 19. The blocks $M_1$ and $M_2$ form one two-piece worm, considered in the previous section, and the blocks $M_3$ and $M_4$ form the other one. The centers of masses, $P$ and $Q$, of each of these two-piece worms are connected by a link of length

$2L$. The joints at $P$ and $Q$ are rigid, which means that the angles formed by each of the two-piece worms with the connecting link, $\overline{PQ}$, stay the same, i.e., right angles in this case.

#### *7.2.1.2 Actuation*

The actuator force $u_1$ acts symmetrically on the two blocks $M_1$ and $M_2$ and along the line connecting the two blocks, i.e., equal force $u_1$ is experienced by both $M_1$ and $M_2$ along the connecting line but in opposite directions. This is similar to the two-piece worm case from the previous section. The force $u_3$ acts similarly but on blocks $M_3$ and $M_4$.

#### *7.2.1.3 Friction Model*

Unlike the previous section, the worm here traverses on the plane and so a two-dimensional friction model is required. The magnitude friction experienced by each of the blocks, $M_i$, is minimum if that block is moving in the direction of the orientation of the body and maximum if it is moving in the opposite direction of the orientation. The direction of the frictional force is always in the opposite direction of the direction of motion. The magnitude of the frictional force experienced by the block $M_i$ is $f_i$. Let the angle between the direction of motion of block $M_i$ and the orientation vector of the worm body be $\psi$, as depicted in Figure 20. Then, the magnitude of the Coulomb frictional force is:

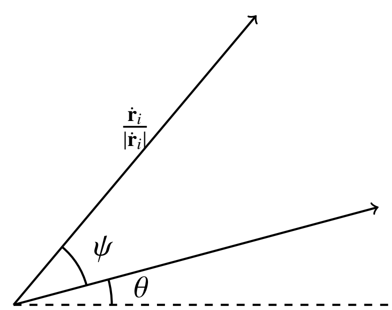



**Figure 20. Friction Model for Block $M_i$**

$$f_i = n_0 - n_1 \cos \psi_i,$$

where $n_0 > n_1 > 0$. The constants $n_0$ and $n_1$ are related to the forward and backward frictions, $f_F$ and $f_B$ respectively, experienced in the direction along the orientation of the body and opposed to the orientation as,

$$f_B = n_0 + n_1 \qquad f_F = n_0 - n_1.$$

### 7.2.2 Equations of Motion

The system can be described by the four pairs of coordinates $(x_i, y_i)$ in the inertial reference frame, each reflecting the position of one of the blocks at a particular time instant. But the geometry of the worm described in the previous section induces the following three constraints:

1. $l_{12} \parallel l_{34}$, where $l_{12}$ is the line passing through $(x_1, y_1)$ and $(x_2, y_2)$.

2. $\overline{PQ} \perp l_{12}$.

3. $\overline{PQ} = 2L$.

These constraints reduce the number of independent coordinates to five, and consequently a choice of generalized coordinates is described here. It is assumed that the masses of all the blocks are the same, say $m$.

The center point, $O$, of the link, $\overline{PQ}$, has coordinates $(x, y)$ in the inertial reference frame. The orientation of the worm is described by the angle $\theta$ that either of the connecting line between two masses makes with the horizontal axis as shown in Figure 19. The distance between the point $P$ and either of the masses $M_1$ or $M_2$ at time $t$ is given by $d_1(t)$. Similarly, the distance between the point $Q$ and either of the masses $M_3$ or $M_4$ at time $t$ is given by $d_3(t)$. Thus, the chosen generalized coordinates are $x$, $y$, $\theta$, $d_1$, and $d_3$. The positions of each of the blocks in the inertial reference frame in terms of the generalized coordinates are:

$$\begin{aligned}
(x_1, y_1) &= (x - L\sin\theta + d_1\cos\theta, y + L\cos\theta + d_1\sin\theta) \\
(x_2, y_2) &= (x - L\sin\theta - d_1\cos\theta, y + L\cos\theta - d_1\sin\theta) \\
(x_3, y_3) &= (x + L\sin\theta + d_3\cos\theta, y - L\cos\theta + d_3\sin\theta) \\
(x_3, y_3) &= (x + L\sin\theta - d_3\cos\theta, y - L\cos\theta - d_3\sin\theta).
\end{aligned} \tag{129}$$

The equations of motion can be derived using the Lagrangian formulation [64]. The Lagrangian, $L$, in the present case is the sum of kinetic energies of each of the blocks, i.e.,

$$L = \frac{1}{2} m \sum_{i=1}^{4} \left( \dot{x}_i^2 + \dot{y}_i^2 \right). \tag{130}$$

Then the equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \tag{131}$$

where $q_j$ is a generalized coordinate and $Q_j$ is the corresponding component of the generalized force obtained using the formula,

$$Q_j = \sum_{i=1}^{4} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}, \tag{132}$$

where $\mathbf{r}_i$ is the position vector of the block $M_i$, $\mathbf{F}_i$ is the sum of all external forces acting on the block $M_i$, and each term is a dot product.

The magnitude of the actuator force acting on block $M_i$ is $u_1$ or $u_3$ and it acts at an angle $\theta$ or $-(\pi + \theta)$ from the horizontal axis. Using (132) then, the component of the generalized force due to the actuator forces is given by,

$$Q_j^u = u_1 \cos\theta \left(\frac{\partial x_1}{\partial q_j} - \frac{\partial x_2}{\partial q_j}\right) + u_1 \sin\theta \left(\frac{\partial y_1}{\partial q_j} - \frac{\partial y_2}{\partial q_j}\right) + u_3 \cos\theta \left(\frac{\partial x_3}{\partial q_j} - \frac{\partial x_4}{\partial q_j}\right) + u_3 \sin\theta \left(\frac{\partial y_3}{\partial q_j} - \frac{\partial y_4}{\partial q_j}\right). \tag{133}$$

The magnitude of the frictional force acting on block $M_i$ is $f_i$ and it acts in the direction opposite to the direction of motion of the block, i.e., along the direction of the vector $-\dot{\mathbf{r}}_i$. Employing (132) again, the component of the generalized force due to the frictional forces is given by,

$$Q_j^f = \sum_{i=1}^{4} -\frac{f_i}{\sqrt{\dot{x}_i^2 + \dot{y}_i^2}} \left( \dot{x}_i \frac{\partial x_i}{\partial q_j} + \dot{y}_i \frac{\partial y_i}{\partial q_j} \right), \tag{134}$$

where $f_i = n_0 - n_1 \cos\psi_i$ and the angle $\psi_i$ is obtained as,

$$\psi_i = \arctan(\dot{x}_i, \dot{y}_i) - \theta.$$

Without loss of generality the mass $m$ can be taken to be one. The Lagrange equations (131) are then explicitly stated as follows.

#### 7.2.2.1 Lagrange equations for unitary masses

When the masses are all equal, the point $O$ corresponds to the center of mass of the body. The equations of motion are as follows where the time dependence is not explicitly stated to avoid clutter.

$$4\ddot{x} = -f_1 \frac{\dot{x}_1}{\sqrt{\dot{x}_1^2 + \dot{y}_1^2}} - f_2 \frac{\dot{x}_2}{\sqrt{\dot{x}_2^2 + \dot{y}_2^2}} - f_3 \frac{\dot{x}_3}{\sqrt{\dot{x}_3^2 + \dot{y}_3^2}} - f_4 \frac{\dot{x}_4}{\sqrt{\dot{x}_4^2 + \dot{y}_4^2}} \tag{135}$$

$$4\ddot{y} = -f_1 \frac{\dot{y}_1}{\sqrt{\dot{x}_1^2 + \dot{y}_1^2}} - f_2 \frac{\dot{y}_2}{\sqrt{\dot{x}_2^2 + \dot{y}_2^2}} - f_3 \frac{\dot{y}_3}{\sqrt{\dot{x}_3^2 + \dot{y}_3^2}} - f_4 \frac{\dot{y}_4}{\sqrt{\dot{x}_4^2 + \dot{y}_4^2}} \tag{136}$$

$$2\ddot{d}_1 = 2d_1\dot{\theta}^2 + 2u_1 - f_1 \frac{\dot{x}_1 \cos\theta + \dot{y}_1 \sin\theta}{\sqrt{\dot{x}_1^2 + \dot{y}_1^2}} + f_2 \frac{\dot{x}_2 \cos\theta + \dot{y}_2 \sin\theta}{\sqrt{\dot{x}_2^2 + \dot{y}_2^2}} \tag{137}$$

$$2\ddot{d}_3 = 2d_3\dot{\theta}^2 + 2u_3 - f_3 \frac{\dot{x}_3 \cos\theta + \dot{y}_3 \sin\theta}{\sqrt{\dot{x}_3^2 + \dot{y}_3^2}} + f_4 \frac{\dot{x}_4 \cos\theta + \dot{y}_4 \sin\theta}{\sqrt{\dot{x}_4^2 + \dot{y}_4^2}} \tag{138}$$

$$\begin{aligned}(4L^2 + 2d_1^2 + 2d_3^2)\,\ddot{\theta} = {} & -4(d_1\dot{d}_1 + d_3\dot{d}_3)\,\dot{\theta} + f_1 \frac{\dot{x}_1\,(L\cos\theta + d_1 \sin\theta) + \dot{y}_1\,(L\sin\theta - d_1\cos\theta)}{\sqrt{\dot{x}_1^2 + \dot{y}_1^2}} \\ & + f_2 \frac{\dot{x}_2\,(L\cos\theta - d_1 \sin\theta) + \dot{y}_2\,(L\sin\theta + d_1\cos\theta)}{\sqrt{\dot{x}_2^2 + \dot{y}_2^2}} \\ & - f_3 \frac{\dot{x}_3\,(L\cos\theta - d_3 \sin\theta) + \dot{y}_3\,(L\sin\theta + d_3\cos\theta)}{\sqrt{\dot{x}_3^2 + \dot{y}_3^2}} \\ & - f_4 \frac{\dot{x}_4\,(L\cos\theta + d_3 \sin\theta) + \dot{y}_4\,(L\sin\theta - d_3\cos\theta)}{\sqrt{\dot{x}_4^2 + \dot{y}_4^2}}\end{aligned} \tag{139}$$

The velocities $\dot{x}_i$ and $\dot{y}_i$ can be expressed in terms of the generalized coordinates using (129):

$$\dot{x}_1 = \dot{x} - L\dot{\theta}\cos\theta + \dot{d}_1\cos\theta - d_1\dot{\theta}\sin\theta \tag{140}$$

$$\dot{y}_1 = \dot{y} - L\dot{\theta}\sin\theta + \dot{d}_1\sin\theta + d_1\dot{\theta}\cos\theta \tag{141}$$

$$\dot{x}_2 = \dot{x} - L\dot{\theta}\cos\theta - \dot{d}_1\cos\theta + d_1\dot{\theta}\sin\theta \tag{142}$$

$$\dot{y}_2 = \dot{y} - L\dot{\theta}\sin\theta - \dot{d}_1\sin\theta - d_1\dot{\theta}\cos\theta \tag{143}$$

$$\dot{x}_3 = \dot{x} + L\dot{\theta}\cos\theta + \dot{d}_3\cos\theta - d_3\dot{\theta}\sin\theta \tag{144}$$

$$\dot{y}_3 = \dot{y} + L\dot{\theta}\sin\theta + \dot{d}_3\sin\theta + d_3\dot{\theta}\cos\theta \tag{145}$$

$$\dot{x}_4 = \dot{x} + L\dot{\theta}\cos\theta - \dot{d}_3\cos\theta + d_3\dot{\theta}\sin\theta \tag{146}$$

$$\dot{y}_4 = \dot{y} + L\dot{\theta}\sin\theta - \dot{d}_3\sin\theta - d_3\dot{\theta}\cos\theta. \tag{147}$$

As seen from the (135) and (136), the translation of center of mass is solely due to the asymmetric nature of friction experienced. The acceleration in the horizontal direction is proportional to the horizontal components of frictional forces and in the vertical direction to the vertical components of frictional forces. The factors contributing to the acceleration of the inter-masses distance, $d_1$ and $d_3$, are the actuation forces, the component of the frictional forces along the $\theta$ direction or the orientation, and the radial component of angular acceleration, as evident from (137) and (138). Finally, (139) basically corresponds to the conservation of angular momentum, i.e.

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = N,$$

where $N$ are the external torques, $I$ is the moment of inertial, and $\omega = \dot{\theta}$ is the angular velocity. In (139), the bracketed term on the left is the moment of inertia and combined with the first term on the right form the time derivative of the angular momentum. The other terms are torques due to the frictional forces, with the component along the body orientation using the central link as moment arm and the other component using the inter-masses link as the moment arm.

### 7.2.3 Some Interesting Gaits

Gaits are characterized by periodicity in the shape space. In that context, a trajectory of this interconnected worm will be called a gait if the inter-mass distances, $d_1$ and $d_3$, and their derivatives are periodic. Some classes of interesting gaits that we have found are presented in this section. Since the motion is in two dimensions now, rotation can be considered. If no rotation is desired then the effective displacements of two-piee worms on each side need to be matched.

#### *7.2.3.1 Symmetric-Control Gaits*

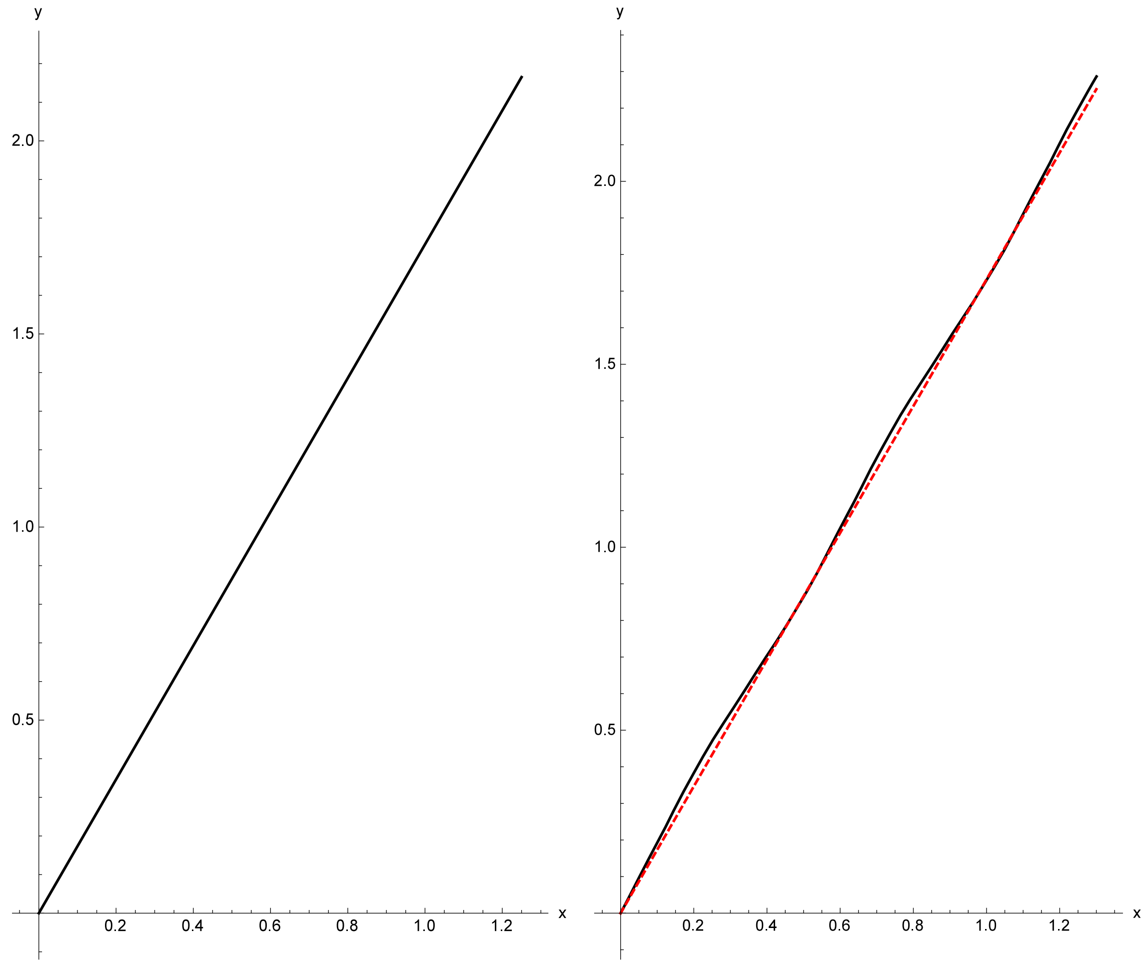
The symmetric gaits are obtained when exactly the same control is applied to both sides, i.e. $u_1 = u_3$. These gaits can be simply viewed as two instances of the two-piece worms. During these gaits, the body translates on the plane in the direction of the initial orientation. The orientation remains the same and no rotation is observed during the entire duration of motion in this gait. This is illustrated by the graphs of the state variables in Figures 21a, 22a, and 23a.

#### *7.2.3.2 Asymmetric-Control Gaits*

These gaits are obtained when there is a phase difference between the controls of the two sides, $u_1$ and $u_3$. The characteristics of these gaits is that they translate the worm body in the direction of the initial orientation of the worm and there is no effective rotation during the motion. However, localized rotations or wobbling is observed. One particular gait in this class is depicted in Figures 21b, 22b, and 23b.

#### *7.2.3.3 Rotation Gaits*

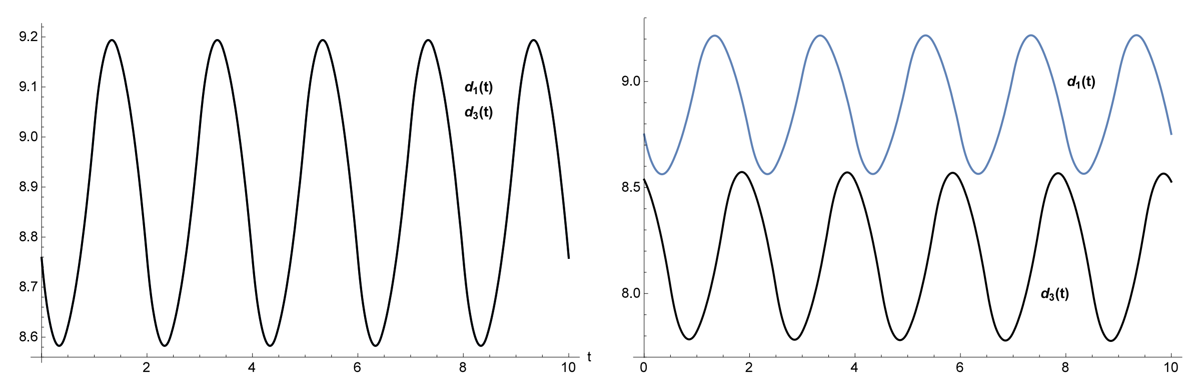
Rotation gaits can be obtained by mismatched controls. These gaits include both translation and rotation. A particular instance of this kind of gait is illustrated in Figures 24, 25a, and 25b.




(a) Symmetric gait.

(b) Asymmetric gait.

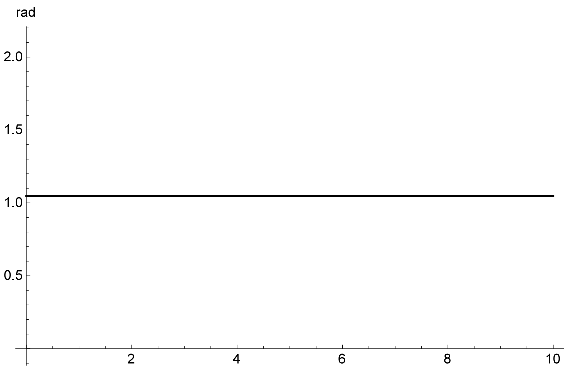
**Figure 21. Path on the plane in symmetric and asymmetric gaits.**



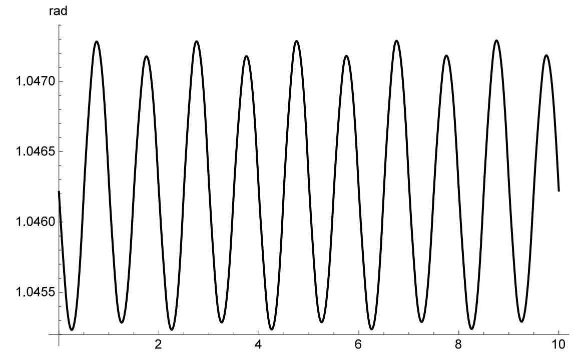

(a) Symmetric gait (Overlapped).

(b) Asymmetric gait ($d_1$: Top and $d_3$: Bottom).

**Figure 22. Inter-mass distances $d_1$ and $d_3$ in symmetric and asymmetric gaits.**




(a) Symmetric gait.




(b) Asymmetric gait.

**Figure 23. Orientation $\theta$ in symmetric and asymmetric gaits.**

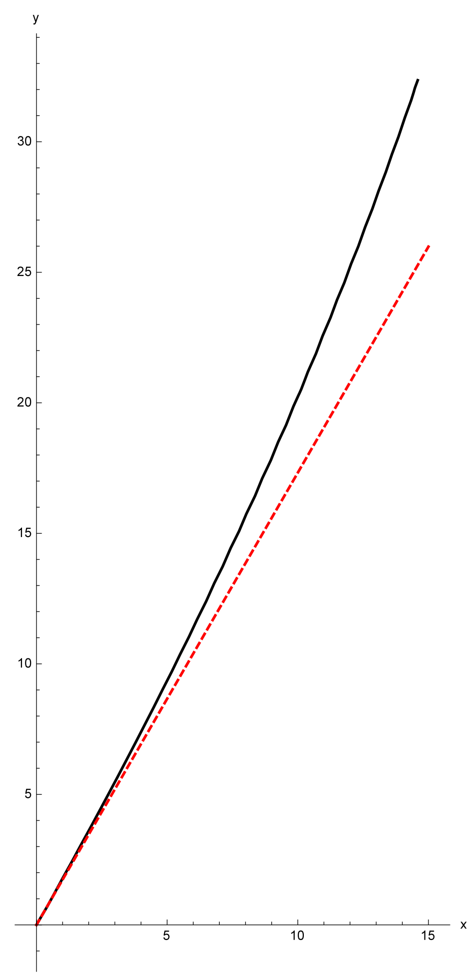
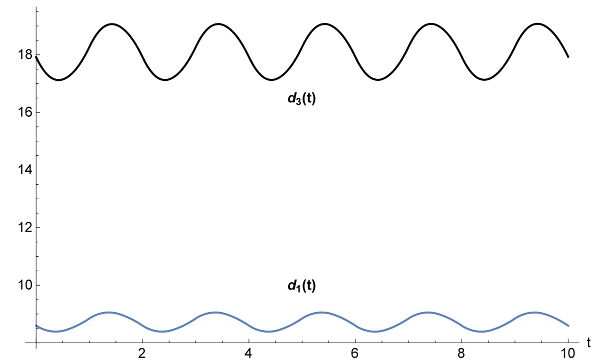
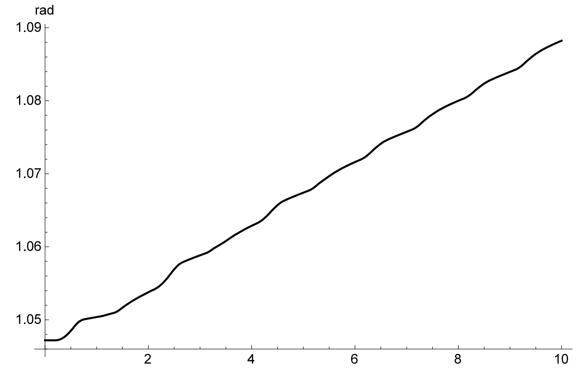



**Figure 24. Path on the plane in a rotation gait. Red dashed line indicates initial orientation.**




(a) Inter-mass distances ($d_3$: Top and $d_1$: Bottom).




(b) Orientation.

**Figure 25. Inter-mass distances, $d_1$ and $d_3$, and Orientation $\theta$ for rotation gait.**

### 7.2.4 Gait Transitions

Similar to the two-piece worm model from Section 7.1, this interconnected system also has multiple modes depending now on the x-direction and the y-direction of velocities of each of the masses. There are four possible combinations for the directions of the velocity pair, $(\dot{x}_i, \dot{y}_i)$, of each of the masses. Consequently, the entire system has 256 possible modes. We can again ignore some of the modes because this interconnected worm is always moving forward relative to its orientation direction. Even so, the problem of finding gait transitions in this case is much more combinatorially complex compared to the two-piece worm model.

The dynamical system is again a hybrid system with autonomous transitions, and the necessary conditions for optimality can be obtained by applying a Hybrid Minimum principle such as the one derived in [65]. Based on the necessary conditions, there exist algorithms to compute the optimal control, including one found by the same authors as above in [66]. However, all of these algorithms are still computationally intensive. An additional level of complexity arises from the fact that the Hamiltonian system is described by delay-differential equations, arising due to the shift appearing in the cost function. At the present time, work on obtaining these gait transitions for this interconnected model is still under progress, and so these results will be presented in a later publication.

# CHAPTER 8

# FUTURE WORK

In the last chapter of this part, future extensions to the material presented in the previous chapters are proposed. The LTID type, $\mathcal{L}_n^k$, was introduced in Chapter 4 but it excluded the cases when one or some of the equations were of order less than $n$. It would be interesting to find a more general characterization that incorporate these cases as well, which would lead to a characterization of differential-algebraic systems too. The constrained LTID type was also introduced in the same chapter. It was shown that the constraints can be simplified for the case of harmonics. This begs the question as to whether this simplification can be carried out in general for the constrained LTID type. Finally, the characterization for the general periodic type needs further analysis.

The signal raccordation problem was solved for types that admit adjoint in Chapter 5, with special attention to the periodic type. A method of stretching the time was used to solve the raccordation problem for the general periodic type. Further study of this method needs to be carried out to discern if it can be used to solver the raccordation problem for other types as well. Throughout, the kernel representation has been used to solve the raccordation problem. It was shown in earlier research that the solutions from the kernel approach and the image approach do not agree in general. Thus, the question of finding the right parametrization for the image approach such that the resulting raccordation agrees with the kernel method is open.

The dynamical raccordation problem was solved in Chapter 6 for LTI systems and polynomial differential types and linear shift types. The raccordation problem can be considered for nonlinear types. Simplified results should be obtained for the raccordation problem for other kinds of systems such as bilinear, delay, and hybrid.

The problem of gait transitions in robotics - the motivation behind this thesis, was approached in Chapter 7. The work in this chapter can be extended in a number of ways. In the problem of gait transitions for the two-piece worm, the optimality of solution with respect to the switching sequence needs to be investigated. The interconnected worm model has not been completely analyzed. Do there exist other type of gaits than the ones mentioned in the chapter? What are the requirements on control and initial conditions for various gaits? The answer to the latter question will possibly also answer the question of what other gaits are in the vicinity of the gait currently being executed. This is an important question for online planning of gait transitions. Moreover, the transitions from rest position to various gaits in the context of our framework is another interesting topic to be considered.

# PART-II

# CHAPTER 9

# SUBOPTIMAL MULTI-MODE STATE ESTIMATION

Multi-mode systems with random switching appear in many areas such as networked control systems, where the jumping characterizes packet loss or delay [67] and in sensor networks, with distributed sensing [68, 69, 70]. Hence communication, and with it communication constraints become an integral part of the estimation problem. Estimation in a network with packet losses is a problem of this scenario: Either normal operating conditions for packet delivery prevail, or there is a failure in packet transmission. This problem has been widely studied by researchers and was also a focus in this research. In [70], a Markovian model was assumed for switching between these two modes. Their emphasis is however not on the optimal filtering but rather an analysis of the stability. Sufficient conditions are given in terms of the failure and recovery rates (for a first order system, the condition also turns out to be necessary). Sinopoli *et al.* [2004] consider and analyze the case of intermittent observations and derive a threshold value for the arrival rate, below which divergence occurs. While Sinopoli et al. [69, 71, 72] assume complete knowledge of the presence or absence of the signal in the observations, we emphasize that this information is a hidden variable in our model and this sets our work apart from the theory of Sinopoli et al. Other problems of this scenario are voice activity detection schemes [73], navigation and tracking [74], vehicle tracking with vision [75], and modeling cortical activity with a view towards improved prosthetics [76]. Typical estimation schemes employed in these cases involve Kalman filter banks such as the Interacting Multiple Models (IMM) algorithm. While the problem at hand could be considered as an adaptive filtering problem, the structure (typically slow adaption) may not be suitable for fast intermittency or switching in the model set. In addition, adaptive filtering typically deals with non discretely varying system parameters.

## 9.1 The General Multi-Mode Filter

### 9.1.1 System Model

Consider a finite set of linear time invariant discrete time systems. Let the set be indexed by $i$. Then each of these systems has dynamics described by:

$$x_{k+1}^{(i)} = F^{(i)} x_k^{(i)} + G^{(i)} w_k^{(i)} \tag{148}$$

$$y_k^{(i)} = H_k^{(i)} x_k^{(i)} + v_k^{(i)}. \tag{149}$$

Let the noise sequences $\{w_k^{(i)}\}$ and $\{v_k^{(i)}\}$, in this model, be stationary white noises with zero mean and covariance matrices $Q^{(i)} \geq 0$ and $R^{(i)} > 0$ respectively and with cross covariance matrix $C^{(i)}$. The system and noise parameters for the $i$-th system will be denoted by $\mathcal{S}_i$. Without loss of generality, we may assume that the dimensions are equal, i.e. $F^{(i)}, G^{(i)}, Q^{(i)} \in \mathbb{R}^{n\times n}, H^{(i)} \in \mathbb{R}^{n\times p}$, and $R^{(i)} \in \mathbb{R}^{p\times p}$. (A theory of multi-mode multi-dimensional systems ($M^3D$) is expounded in [77, 62]). The case considered in this thesis and in [78] is one where the mode switching process is purely random, independent of the state sequence $\{x_k\}$, but can have time varying occupation probabilities. Our compound systems constitute thus a specific class of stochastic hybrid or switched systems. A complete description of the state at time step $k$ of the compound system involves thus not only the *state vector* which is usually denoted by $x_k$, but also the *mode* $\theta_k$ which is active at time $k$.

Let $\pi_k = [\pi_k^1, \pi_k^2, \cdots, \pi_k^q]$ be the probability vector for these different modes, with $\pi^i$ being the probability that mode $i$ is 'active'. The system can be described alternatively by a single $n$-th order system but with time varying parameters, whose values are determined by the 'modal' process $\{\theta_k\}$:

$$x_{k+1} = F(\theta_k)x_k + G(\theta_k)w_k \tag{150}$$

$$y_k = H(\theta_k)x_k + v_k. \tag{151}$$

Also, we have now $Q(\theta_k)$ and $R(\theta_k)$. Without loss of generality we state $Q^{(i)} = I$, since its variation can always be modeled in the $G^{(i)}$ matrices. In this model the modal state $\theta_k$ is a purely random sequence taking integer values. The probability that $\theta_k = i$ is $\pi_k^i$ and if $\theta_k = i$, then the system parameters are $F(i) = F^{(i)}$ and so on. It is also important to note that at each time, only one output vector is available corresponding to the mode at that time ($y_k = y^{(\theta_k)}$).

### 9.1.2 The Filter Equations

We shall be interested in *linear least squares* filters and so the estimates we shall obtain are not the optimal least squares estimates, which are conditional expectations, given the data. However, the computational advantages for the suboptimal linear least squares solutions are great (as well known for the classical Kalman filter for linear stochastic system with non-Gaussian initial conditions). Of the two typical approaches to filtering - the innovations approach and the change of measure method, we will use the innovations approach. The problem of finding the linear least squares filter has also been solved by Costa et al. [30] for the general Markov switching case, but the innovations approach is used here that is different from theirs, and as a result an error has been discovered in their derivation. The approach requires the Hilbert space framework for the stochastic processes involved, and the linear least squares estimate given the data is known to be a linear projection onto the subspace spanned by the data. For this reason, let us first classify the linear information structures.

Let $(\Omega, \mathcal{B}, P)$ be the probability space on which all random variables involved in the state space model are defined: i.e., in particular $x_0 \in \mathcal{B}$, and for all $k$, $\theta_k \in \mathcal{B}$, $w_k \in \mathcal{B}$ and $v_k \in \mathcal{B}$. The following *linear* spaces play an important role in the linear estimation problem:

$$\mathcal{L}_k^0 = \text{span}\{x_0; \theta_0, \cdots, \theta_k; w_0, \cdots, w_k; v_0, \cdots, v_k\} \tag{152}$$

$$\mathcal{L}_k = \text{span}\{\theta_0, \cdots, \theta_k; w_0, \cdots, w_k; v_0, \cdots, v_k\} \tag{153}$$

$$\mathcal{H}_k = \text{span}\{y_0, \cdots, y_k\}, \tag{154}$$

The last space is generated by the observations. Define the innovations, $\epsilon_k = y_k - \mathcal{P}^{\mathcal{H}_{k-1}} y_k$.

Here, $\mathcal{P}^{\mathcal{H}}$ denotes the orthogonal projection onto the closed subspace $\mathcal{H}$. Orthogonality is a well defined concept once an inner product is defined in the spaces: $u \perp v \Leftrightarrow \langle u, v \rangle = 0$ (See [33]). We note that the innovations correspond to the sequential Gram-Schmidt othogonalization of $\{y_0, y_1, y_2, \ldots\}$ (in that order). Introduce also

$$\mathcal{E}_k = \text{span}\{\epsilon_0, \epsilon_1, \ldots, \epsilon_k\}. \tag{155}$$

The spaces $\mathcal{H}_k$ and $\mathcal{E}_k$ are identical, since an invertible transformation between the observation sequence and the innovation sequence up to time $k$ exists. Thus,

$$\mathcal{H}_k = \mathcal{E}_k \subseteq \mathcal{L}_k^0, \quad \mathcal{L}_k \subseteq \mathcal{L}_k^0.$$

Note also the inclusions, for all $k$,

$$\begin{aligned} \mathcal{H}_k &\subseteq \mathcal{H}_{k+1} \\ \mathcal{E}_k &\subseteq \mathcal{E}_{k+1} \\ \mathcal{L}_k &\subseteq \mathcal{L}_{k+1} \\ \mathcal{L}_k^0 &\subseteq \mathcal{L}_{k+1}^0, \end{aligned}$$

expressing that we have *growing* information structures, constituting *linear filtrations*.

Recall first two important facts about projection operators in a Hilbert space.

1. *Smoothing formula*: If $\mathcal{A} \subseteq \mathcal{B}$ are arbitrary subspaces of a Hilbert space $H$, then for any vector $x \in H$:

$$\mathcal{P}^{\mathcal{A}} x = \mathcal{P}^{\mathcal{A}} \mathcal{P}^{\mathcal{B}} x. \tag{156}$$

2. *Pythagorean theorem*: If $\mathcal{A} \perp \mathcal{B}$ are arbitrary orthogonal subspaces of a Hilbert space $H$, then for any vector $x \in H$:

$$\mathcal{P}^{\mathcal{A} \vee \mathcal{B}} x = \mathcal{P}^{\mathcal{A}} x + \mathcal{P}^{\mathcal{B}} x. \tag{157}$$

   Consequently, we also have

$$||\mathcal{P}^{\mathcal{A} \vee \mathcal{B}} x||^2 = ||\mathcal{P}^{\mathcal{A}} x||^2 + ||\mathcal{P}^{\mathcal{B}} x||^2. \tag{158}$$

For simplicity of notation, we introduce the notation

$$\hat{z}_{|k} = \mathcal{P}^{\mathcal{H}_k} z.$$

and note that this is also the projection on the space spanned by the innovations. With this notation, we now proceed with the derivation of the optimal linear filter. See [32] for details.

**i) Innovations:** The *innovations* are defined by

$$\epsilon_k = y_k - \hat{y}_{k|k-1}, \tag{159}$$

where $\hat{y}_{k|k-1}$ is the projection of the measurement $y_k$ onto the subspace $\mathcal{H}_{k-1}$. Its computation is simplified by the smoothing formula (156)

$$\begin{aligned} \mathcal{P}^{\mathcal{H}_{k-1}} y_k &= \mathcal{P}^{\mathcal{H}_{k-1}} \mathcal{P}^{\mathcal{L}^0_{k-1}} \left[ H(\theta_k) x_k + v_k \right] \\ &= \mathcal{P}^{\mathcal{H}_{k-1}} \left[ \sum \pi_i H^{(i)} x_k \right] \end{aligned} \tag{160}$$

If $\chi_i(\theta)$ is the indicator function for mode $i$, then

$$\mathcal{P}^{\mathcal{L}^0_{k-1}} H(\theta_k) = \mathcal{P}^{\mathcal{L}^0_{k-1}} \sum \chi_i(\theta_k) H^{(i)} = \sum \pi_i H^{(i)} \tag{161}$$

Thus the innovations are given by

$$\begin{aligned} \epsilon_k &= y_k - \sum \pi_i H^{(i)} \hat{x}_{k|k-1} \\ &= H(\theta_k)\, \tilde{x}_{k|k-1} + v_k + \left[ H(\theta_k) - \sum \pi_i H^{(i)} \right] \hat{x}_{k|k-1} \end{aligned} \tag{162}$$

where $\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}$.

**ii) Innovations Covariance:** The *innovations covariance* is obtained by 'squaring up' and using (158)

$$\begin{aligned} R^{\epsilon}_k &= \mathbf{E}\, \epsilon_k \epsilon_k' \\ &= \overline{H} P_{k|k-1} \overline{H}' + \sum \pi_i H^{(i)} \Pi_k \left[ H^{(i)} - \overline{H} \right]' + \overline{R} \end{aligned} \tag{163}$$

where $\mathbf{E}$ is the expectation operator and we defined the state covariance $\Pi_k = \mathbf{E}\, x_k x_k'$ and $P_{k|k-1} = \mathbf{E}\, \tilde{x}_{k|k-1}\tilde{x}_{k|k-1}'$ and used the simplified notation for the *averaged* system parameters

$$\overline{H} = \sum \pi_i H^{(i)} \tag{164}$$

$$\overline{R} = \sum \pi_i R^{(i)}. \tag{165}$$

We also used $\Pi_k = \Sigma_{k|k-1} + P_{k|k-1}$, with $\Sigma_{k|k-1}$ being the filtered state covariance.

**iii) Filtered Estimates from Predicted Estimates:** Incorporating the last innovations gives, using (157)

$$\mathcal{P}^{\mathcal{H}_k} x_k = \mathcal{P}^{\mathcal{E}_k} x_k = \mathcal{P}^{\mathcal{E}_{k-1}} x_k + \mathcal{P}^{\text{span}\{\epsilon_k\}} x_k \tag{166}$$

which yields,

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \mathbf{E}\ (x_k \epsilon_k') R_k^{-\epsilon} (y_k - \overline{H}\hat{x}_{k|k-1}) \tag{167}$$

The *Kalman Gain* is

$$\mathbf{E}\, x_k \epsilon_k' = P_{k|k-1}\overline{H}' \tag{168}$$

So,

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}\overline{H}' R_k^{-\epsilon} (y_k - \overline{H}\hat{x}_{k|k-1}) \tag{169}$$

**iv) Predicted Estimate from the Signal Model:**

The time update is obtained from the projection

$$\mathcal{P}^{\mathcal{H}_k} x_{k+1} = \mathcal{P}^{\mathcal{H}_k} [F(\theta_k) x_k + G(\theta_k) w_k]$$

Consider first,

$$\begin{aligned}
\mathcal{P}^{\mathcal{H}_k} F(\theta) x_k &= \mathcal{P}^{\mathcal{E}_{k-1}} F(\theta) x_k + \mathcal{P}^{\text{span}\{\epsilon_k\}} F(\theta) x_k \\
&= \left(\sum \pi_i F^{(i)}\right) \hat{x}_{k|k-1} + \mathbf{E}\ [F(\theta_k) x_k \epsilon_k'] R_k^{-\epsilon} \left(y_k - \overline{H}\hat{x}_{k|k-1}\right) \\
&= \overline{F}\hat{x}_{k|k-1} + \left[\sum \pi_i F^{(i)} P_{k|k-1} H^{(i)'} + \sum \pi_i F^{(i)} \Sigma_{k|k-1} (H^{(i)'} - \overline{H}')\right] R_k^{-\epsilon} \left(y_k - \overline{H}\hat{x}_{k|k-1}\right) \\
&= \overline{F}\hat{x}_{k|k-1} + \left[\overline{F} P_{k|k-1} \overline{H}' - \overline{F}\Pi_k \overline{H}' + \sum \pi_i F^{(i)} \Pi_k H^{(i)'}\right] R_k^{-\epsilon} \epsilon_k
\end{aligned} \tag{170}$$

where we also defined

$$\overline{F} \;=\; \sum \pi_i F^{(i)}, \tag{171}$$

as the average dynamics. Consider next,

$$\begin{aligned}
\mathcal{P}^{\mathcal{H}_k} G(\theta) w_k &= \mathcal{P}^{\mathrm{span}\{\epsilon_k\}} G(\theta) w_k \\
&= \left[\sum \pi_i G^{(i)} C^{(i)}\right] R_k^{-\epsilon} \epsilon_k
\end{aligned} \tag{172}$$

where $C^{(i)} = \mathbf{E}\, w_k^{(i)} v_k^{(i)'}$. This all results in

$$\hat{x}_{k+1|k} = \overline{F}\hat{x}_{k|k-1} + \left[\overline{F} P_{k|k-1}\overline{H}' - \overline{F}\Pi_k \overline{H}' + \sum \pi_i \left(F^{(i)}\Pi_k H^{(i)'} + G^{(i)} C^{(i)}\right)\right] R_k^{-\epsilon} \epsilon_k$$

**v) Error Covariance Matrix:** Finally, we obtain a recursion for the error covariance. The state covariance satisfies:

$$\begin{aligned}
\Pi_{k+1} &= \mathbf{E}\;[F(\theta_k)x_k + G(\theta_k)u_k]\,[F(\theta_k)x_k + G(\theta_k)u_k]' \\
&= \sum \pi_i F^{(i)}\Pi_k F^{(i)'} + \sum \pi_i G^{(i)'} G^{(i)'} \\
&= \sum \pi_i \left[F^{(i)}\Pi_k F^{(i)'} + G^{(i)} G^{(i)'}\right]
\end{aligned} \tag{173}$$

Here we have used that $Q^{(i)} = I$. Replacing $\Pi_{k+1}$ by $\sum \pi_i \Pi_{k+1}^{(i)}$ allows a *decomposition* of this equation to the set

$$\Pi_{k+1}^{(i)} \;=\; F^{(i)}\Pi_k F^{(i)'} + G^{(i)} G^{(i)'} \tag{174}$$

$$\Pi_k \;=\; \sum \pi_i \Pi_k^{(i)} \tag{175}$$

Likewise the covariance of the estimate follows by ‘squaring up’ the predictor equation (173). We find that $\Sigma_{k+1|k}$ equals

$$\Sigma_{k+1|k} = \overline{F}\Sigma_{k|k-1}\overline{F}' +$$

$$\left[\sum \pi_i F^{(i)}\Pi_k H^{(i)'} - \overline{F}\Sigma_{k|k-1}\overline{H}' + \sum \pi_i G^{(i)} C^{(i)}\right] R_k^{\epsilon} \left[\sum \pi_i F^{(i)}\Pi_k H^{(i)'} - \overline{F}\Sigma_{k|k-1}\overline{H}' + \sum \pi_i G^{(i)} C^{(i)}\right]'.$$

The decomposed equations give

$$\Sigma_{k|k-1} = \sum \pi_i \Sigma^{(i)}_{k|k-1}. \tag{176}$$

where

$$\Sigma^{(i)}_{k+1|k} = \overline{F}\Sigma^{(i)}_{k|k-1}\overline{F}' + \left[F^{(i)}\Pi_k H^{(i)'} - \overline{F}\Sigma^{(i)}_{k|k-1}\overline{H}' + G^{(i)}C^{(i)}\right] R^{\epsilon}_k \left[\sum \pi_j \left(F^{(j)}\Pi_k H^{(j)'} + G^{(j)}C^{(j)}\right) - \overline{F}\Sigma_{k|k-1}\overline{H}'\right]'$$

Subtracting both equations yields the recursion for the error covariance:

$$P^{(i)}_{k+1|k} = \Pi^{(i)}_{k+1|k} - \Sigma^{(i)}_{k+1|k}. \tag{177}$$

and, defining $P_{k+1|k} = \sum \pi_i P^{(i)}_{k+1|k}$,

$$P_{k+1|k} = \Pi_{k+1|k} - \Sigma_{k+1|k}. \tag{178}$$

Unfortunately, no *decoupled* recursions (in the form of Riccati equations) exists for the $P^{(i)}_{k+1|k}$.

We summarize:

**Theorem 9.1.1.** *The linear least squares filter for a multi-mode system (MMF) with purely random mode switching for the system given by (150) and (151) is given by (recall: $Q = I$)*

$$\begin{aligned}
\hat{x}_{k+1|k} &= \overline{F}\hat{x}_{k|k-1} + K_k\left(y_k - \overline{H}\hat{x}_{k|k-1}\right) \\
K_k &= \left(\overline{F}P_{k|k-1}\overline{H}' + \overline{GC} + \overline{F\Pi_k H'} - \overline{F}\Pi_k\overline{H}'\right)\mathcal{R}^{-\epsilon}_k \\
\Pi_{k+1} &= \overline{F\Pi_k F'} + \overline{GG'} \\
P_{k+1|k} &= \overline{F}P_{k|k-1}\overline{F}' + \overline{GG'} - K_k\mathcal{R}^{\epsilon}_k K'_k + \overline{F\Pi_k F'} - \overline{F}\Pi_k\overline{F}' \\
\mathcal{R}^{\epsilon}_k &= \overline{H}P_{k|k-1}\overline{H}' + \overline{R} + \overline{H\Pi_k H'} - \overline{H}\Pi_k\overline{H}'
\end{aligned}$$

*The filtered estimates $\hat{x}_{k|k}$ follows from*

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}\overline{H}'\mathcal{R}^{-\epsilon}_k\left(y_k - \overline{H}\hat{x}_{k|k-1}\right).$$

*The 'overline' indicates expectation with respect to $\theta$, i.e., $\overline{X} = \sum_i \pi_i X^{(i)}$.*

It should be emphasized that this multi mode filter is not the Kalman filter for the averaged system.

### 9.1.3 Computational Complexity

In this section, we will discuss the computational complexity of the multi-mode filter (MMF) when compared with algorithms using filter banks such as the IMM algorithm, typically employed to estimate the state of a multi-mode system. We compared the number of scalar multiplication and addition operations required by full implementations of the MMF with a Kalman filter bank. We found out that the number of operations required by MMF is dominated by $n^3$, whereas the number of operations required by a filter bank is dominated by $q(n^2 + p^3)$. Here $n$ is the dimension of the state space, $p$ is dimension of the output space and $q$ is the number of modes in the system. The MMF is found to be computationally simpler for large number of modes or large number of outputs. It is also worth noting that if the steady state covariance of the system is used in the MMF equations to find an approximate filter, the MMF is *always* computationally simpler than a filter bank. A possible scheme that can be employed is to begin with a filter bank algorithm such as the IMM and then switch to the MMF after convergence to steady state. The MMF can also be used adaptively in the case of fixed but unknown occupation probabilities. A detection scheme can be created based on the residuals and the probabilities can be estimated based on the mode detected.

## 9.2 Intermittent Signal Observations with Noise

This is a special case of the general problem treated above, and relevant in the sensor failure problem. Consider the system,

$$x_{k+1} = Fx_k + Gw_k \tag{179}$$

$$y_k = \theta_k H x_k + v_k. \tag{180}$$

The $\{\theta_k\}$ is a scalar Bernoulli process, with value space $\{0, 1\}$. Let $\pi_1(k) = \Pr\{\theta_k = 1\}$. Making this process nonstationary allows for a model incorporating faulty behavior of the sensor as function of time. In this case there are two modes for the system. In mode 1, the parameters are $\mathcal{S}_1 = \{F, G, H, Q = I, R > 0\}$, whereas the mode 0, the sensor failure mode, is characterized by $\mathcal{S}_0 = \{F, G, 0, Q = I, R\}$. The probabilistic description of the modes is given by $\pi(k) = [\pi_1(k), \pi_0(k)] = [\lambda_k, 1 - \lambda_k]$. Additionally, the two noise sequences $w_k^{(i)}$ and $v_k^{(i)}$ are uncorrelated for all time i.e., $C^{(i)} = 0$.

It follows from Theorem 9.1.1, with the averaged parameters $\overline{F} = F, \overline{H} = \lambda_k H, \overline{GG}' = GG', \overline{R} = R$, that the MMF is (with $Pr\{sensorfailure\} = 1 - \lambda_k$)

$$\begin{aligned}
\hat{x}_{k+1|k} &= F\hat{x}_{k|k-1} + K_k\left(y_k - \lambda_k H\hat{x}_{k|k-1}\right) \\
K_k &= \lambda_k F P_{k|k-1} H' \mathcal{R}_k^{-\epsilon} \\
\Pi_{k+1} &= F\Pi_k F' + GG' \\
P_{k+1|k} &= F P_{k|k-1} F' + GG' - K_k \mathcal{R}_k^{\epsilon} K_k' \\
\mathcal{R}_k^{\epsilon} &= \lambda_k^2 H P_{k|k-1} H' + R + \lambda_k(1-\lambda_k) H \Pi_k H'
\end{aligned} \tag{181}$$

### 9.2.1 Stability

The following two theorems characterize the stability/instability of this filter for the case of a stationary Bernoulli process.

**Theorem 9.2.1.** *The multi-mode filter (181) for the intermittent signal observations in the noise case is stable if the system (179-180) is asymptotically stable. The error covariance matrix $P_{k+1|k}$, given by the following recursion*

$$P_{k+1} = FP_kF' + GG' - \lambda^2 FP_kH' \left[\lambda^2 HP_kH' + R + \lambda(1-\lambda)H\Pi_kH'\right]^{-1} HP_kF' \tag{182}$$

*converges to a steady state $P$ which satisfies an algebraic Riccati equation,*

$$P = FPF' + GG' - FPH' \left[HPH' + R_{eq}\right]^{-1} HPF'$$

*where,*

$$R_{eq} = \frac{R + \lambda(1-\lambda)H\Pi H'}{\lambda^2}$$

*and $\Pi$ satisfies an algebraic Lyapunov equation,*

$$\Pi = F\Pi F' + GG'.$$

**Theorem 9.2.2.** *The multi-mode filter (181) for the intermittent signal observations in noise case is unstable if the system (179-180) is unstable, i.e., the error covariance matrix does not converge to a steady state.*

**PROOF.** The Riccati recursion for the prediction error covariance matrix is given below where the notation $P_k$ has been used instead of $P_{k|k-1}$ to save space.

$$\begin{aligned} P_{k+1} &= FP_kF' + GG' - \lambda^2 FP_kH' \left[\lambda^2 HP_kH' + R + \lambda(1-\lambda)H\Pi_kH'\right]^{-1} HP_kF' \\ &= GG' + F\left\{P_k - P_k\lambda H' \left[\lambda HP_kH'\lambda + R + \lambda(1-\lambda)H\Pi_kH'\right]^{-1} \lambda HP_k\right\} F' \\ &= F\left\{P_k^{-1} + \lambda H' \left[R + \lambda(1-\lambda)H\Pi_kH'\right]^{-1} H\lambda\right\}^{-1} F' + GG' \end{aligned}$$

where the last form is obtained by using Woodbury's lemma on the expression enclosed in the curly brackets. Since $(F, H)$ is observable and $F$ is unstable, the term $H\Pi_kH'$ is unbounded. This results in the term $[R + \lambda(1-\lambda)H\Pi_kH']^{-1}$ approaching 0 as $k$ approaches infinity. Thus, in the long run the Riccati equation behaves as the Lyapunov equation and $P_k$ is unbounded for the unstable system. □

### 9.2.2 Bounds on $P$

An upper bound and a lower bound for the error covariance matrix $P_{k+1|k}$, hereafter denoted by $P_{k+1}$, can be easily found using Riccati equation comparison theorems due to Wimmer and Pavon ([79]).

**Theorem 9.2.3.** *The error covariance matrix $P_k$, given in (182), is bounded below for all time by the following Riccati recursion,*

$$L_{k+1} = FL_kF' + GG' - FL_kH' \left(HL_kH' + R\right)^{-1} HL_kF'$$

*provided $L_0 \leq P_0$.*

**PROOF.** It follows directly from Wimmer and Pavon's comparison theorem if the following required condition is satisfied for all time,

$$\begin{bmatrix} GG' & F \\ F' & -H'\tilde{R}_k^{-1}H \end{bmatrix} \geq \begin{bmatrix} GG' & F \\ F' & -H'R^{-1}H \end{bmatrix}$$

where $\tilde{R}_k = \lambda_k^{-2}\left[R + \lambda_k(1-\lambda_k)H\Pi_k H'\right]$. Since the matrices $R$ and $\tilde{R}_k$ are symmetric, this condition is equivalent to $\tilde{R}_k \geq R$. Given that $\lambda_k \leq 1$ and $\Pi_k \geq 0$ for all $k$, the inequality $\tilde{R}_k \geq R$ is true for all time and thus the bound holds. □

**Theorem 9.2.4.** *The error covariance matrix $P_k$, given in (182), is bounded above for all time by the following Riccati recursion,*

$$U_{k+1} = FU_kF' + GG' - FU_kH'\left(HUL_kH' + R_{eq}\right)^{-1} HU_kF'$$

*where $R_{eq} = \lambda_k^{-2}\left[R + \lambda_k(1-\lambda_k)H\Pi H'\right]$ and $\Pi = F\Pi F' + GG'$ and provided $U_0 \geq P_0$.*

**PROOF.** Again this follows directly from Wimmer and Pavon's comparison theorem if the following required condition is satisfied for all time,

$$\begin{bmatrix} GG' & F \\ F' & -H'R_{eq}^{-1}H \end{bmatrix} \geq \begin{bmatrix} GG' & F \\ F' & -H'\tilde{R}_k^{-1}H \end{bmatrix}$$

where $\tilde{R}_k = \lambda_k^{-2}\left[R + \lambda_k(1-\lambda_k)H\Pi_k H'\right]$. Since the matrices $R_{eq}$ and $\tilde{R}_k$ are symmetric, the above condition is equivalent to $R_{eq} \geq \tilde{R}_k$. And this inequality is true for all $k$, since $\Pi_k \leq \Pi$ for all $k$. Therefore, the upper bound holds. □

### 9.2.3 Numerical Example

The performance of the MMF, for the intermittent signal in observations case with stationary Bernoulli process, is compared numerically to the IMM algorithm, the averaged filter, the approximate MMF and the exact knowledge case. The averaged filter is a Kalman filter for the averaged case, specifically $\overline{H} = \lambda H$. The exact knowledge case has perfect knowledge of the mode of operation at all times and employs the respective Kalman filter. The approximate MMF uses the system's steady state covariance ($\Pi$) to make the computations simpler. The performance metric used is the ratio of the euclidean norm of the error between the actual state and the estimated one to the norm of the error for the exact knowledge case. For each experiment, 1000 statistical trials are performed and the figures depict the computed ratios using statistical mean of the norm. Figure 26 compares the performance of the different filters for the system with the parameters $Q = 1$, $R = 25$, Sensor failure probability = 0.3 and

$$F = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.7 & 0.7 \\ 0 & -0.7 & 0.7 \end{bmatrix}, \quad G = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, \quad H = [1, 0, 0].$$

The error is highest for the averaged filter followed by the MMF and then the IMM algorithm. It is also worth noting that the difference in the error norms is small and the approximate MMF, which is computationally simpler, performs as well as the full MMF.

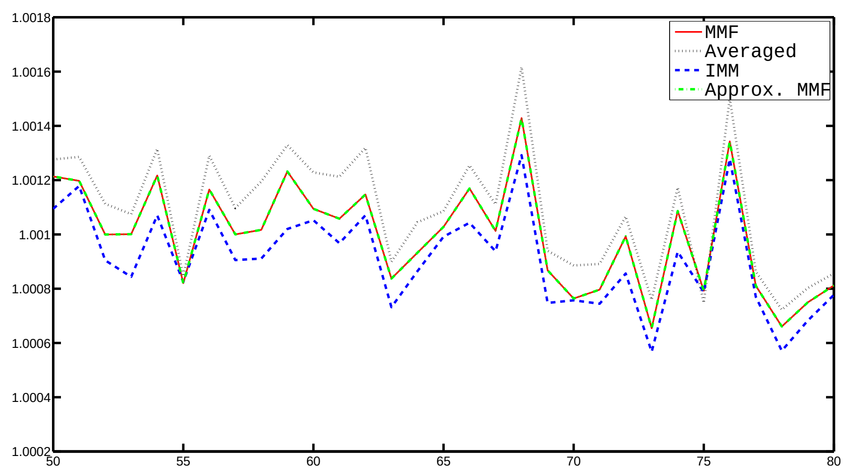



**Figure 26. Performance comparison between different filters for a typical stable case**

Figure 27 illustrates the behavior of the filters when the system is close to instability i.e. one of the eigen values of the system is moved to 0.99 while keeping all the other parameters the same as the previous case. The trend observed in the previous case continues but now the difference in the scale of the norm between the MMF and averaged filter is much larger. The difference between the results of MMF and the IMM algorithm is slightly higher as well.

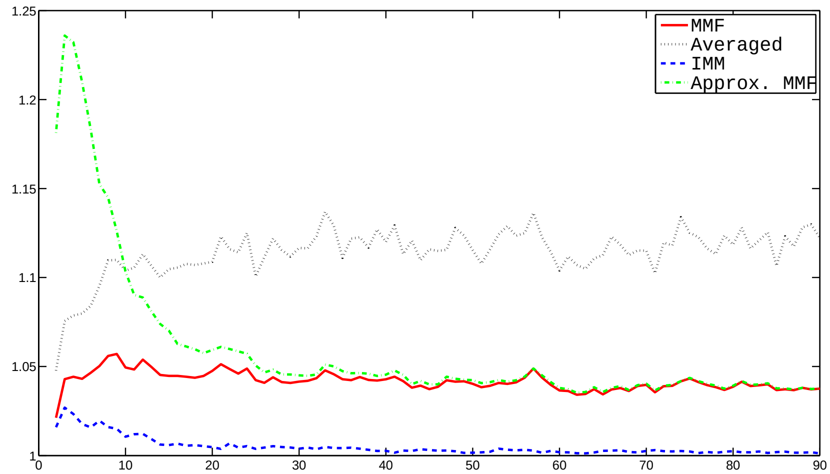



**Figure 27. Performance comparison between different filters close to instability**

# CHAPTER 10

# FUTURE WORK

The problem of state estimation for a multimode system with purely random switching was solved in Chapter 9. The work can be extended by now considering a Markov process. The stability of the resultant filter has not been analyzed in general. Therefore, the Riccati like equation obtained for the error covariance matrix needs to be analyzed. In the special case of intermittent signals in noise, it was shown that the filter is only stable if the underlying system is. This leads to the question as to how the filter could be made stable by perhaps considering nonlinear variants. The problem of mode identification is also an interesting one.

The dual of the intermittent estimation problem is that of intermittent control. This problem has received widespread attention in recent years due to the wide applications and occurrence of networked systems. In designing estimators and controllers for networked system, three possible communication network related events need to be considered: packet losses, packet delays, and quantization effects. These three problems have been considered separately and in combination of varying degrees by various researchers in a number of publications with varying success. Thus, there are still open problems in the area and room for improvement in the schemes already proposed.

# APPENDIX A

# SOME RESULTS ABOUT MATRICES

## A.1 The Matrix Exponential

The matrix exponential of a square matrix $A$ is defined as,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2} A^2 + \cdots$$

This infinite sum is well-defined and converges for all matrices $A$, i.e. the series obtained for each of the entries of $e^A$ converges. This is a special case of Theorem 5.6.15 in [80], but a simple proof is presented here.

The $(i, j)$th entry of the matrix $e^A$ is given by the series $\sum_{k=0}^{\infty} \frac{1}{k!} \left(A^k\right)_{ij}$, where $\left(A^k\right)_{ij}$ is the $ij$th entry of the matrix $A^k$. It is now shown that this sum is finite as follows.

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(A^k\right)_{ij} \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left\|A^k\right\|_F \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|A\|_F^k = e^{\|A\|_F},$$

where $\|\cdot\|_F$ is the Frobenius norm of the matrix. Since, each of the entries is given by a convergent series, the matrix series is convergent, and the matrix exponential is defined for all matrices. Furthermore, this shows that $e^{At}$ is an analytic function, since $At$ is also a matrix and so the series converges. This means that each of the entries of $e^{At}$ is an analytic function.

## A.2 Schur Complement

An LDU decomposition of a partitioned matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is derived here, where one of diagonal entries is the Schur complement of the block $A$.

$$MU = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ O & I \end{bmatrix} = \begin{bmatrix} A & O \\ C & D - CA^{-1}B \end{bmatrix}$$

$$LMU = \begin{bmatrix} I & O \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & O \\ C & D - CA^{-1}B \end{bmatrix} = \begin{bmatrix} A & O \\ O & D - CA^{-1}B \end{bmatrix}$$

$$\Rightarrow M = \begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & O \\ O & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ O & I \end{bmatrix}$$

# APPENDIX B

# SOME RESULTS FROM FUNCTIONAL ANALYSIS

## B.1 Gâteaux Derivative

In this section, some ideas related to differentiability of functionals will be reviewed. The content of this section has been gathered up in most part from [81], [82] and [83].

**Definition B.1.1.** *Consider a function $F : X \to Y$, where $X$ and $Y$ are topological vector spaces. Let $U \subset X$ be an open set. Then, $F$ is said to have the* **Gâteaux differential** *or the* **Gâteaux variation** *$\delta F(u; h)$ at $u \in U$ in the direction $h \in X$ if the following limit exists for all $h \in X$.*

$$\delta F(u; h) = \lim_{t \to 0} \frac{F(u + th) - F(u)}{t} = \frac{d}{dt} F(u + th)\Big|_{t=0}$$

*If the limit exists for each $h \in X$ then the function $F$ is said to be Gâteaux differentiable.*

This variation as we know it today was studied by Gâteaux in the early twentieth century. But the concept of variation for certain special functionals has its roots in the work of Lagrange who introduced it in the context of calculus of variations.

Since, the limit of a function is unique if it exists, the Gâteaux variation is unique. The ideal of Gâteaux variation generalizes the concept of directional derivative from multivariable calculus to infinite dimensions. It is clear from the definition that if $F$ has Gâteaux variation at $u$, then $\delta F(u; 0) = 0$. Moreover, the variation is homogeneous in the second argument, i.e.

$$\delta F(u; ch) = c\delta F(u; h)$$

for any $c \in \mathbb{R}$ since

$$\delta F(u; ch) = \frac{d}{dt} F(u + cth)\Big|_{t=0} = c\frac{d}{d\sigma} F(u + \sigma h)\Big|_{t=0} = c\delta F(u; h).$$

The Gâteaux variation does not necessarily have to satisfy the additivity property and so it is not necessarily linear in the second argument. If the variation is linear in the second argument then it is called the **Gâteaux derivative** and we get the following identity:

$$F(u + th) = F(u) + \delta F(u; th) + \epsilon(th),$$

where $\epsilon(th)/t \to 0$ as $t \to 0$.

## B.2 Convergence of inner products over an uncountable index set

**Lemma B.2.1.** *Let $\langle \cdot, \cdot \rangle : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ be an inner product, where $\mathbb{E}$ is a vector space. Let $\{f_t\}$ and $\{g_t\}$ be two convergent sequences of functions with $\lim_{t\to a} f_t = f$ and $\lim_{t\to a} g_t = g$, where $t$ is a real variable. Then,*

$$\lim_{t\to a} \langle f_t, g_t \rangle = \left\langle \lim_{t\to a} f_t, \lim_{t\to a} g_t \right\rangle.$$

**PROOF.** Given any $\epsilon > 0$, let

$$\eta_1 = \begin{cases} \epsilon/3 & \text{if } \|g\| = 0 \\ \min\left(\frac{\epsilon}{3}, \frac{\epsilon}{3\|g\|}\right) & \text{otherwise} \end{cases} \quad \text{and} \quad \eta_2 = \begin{cases} \epsilon/3 & \text{if } \|f\| = 0 \\ \min\left(\frac{\epsilon}{3}, \frac{\epsilon}{3\|f\|}\right) & \text{otherwise} \end{cases}.$$

Since the series are convergent, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $\|f_t - f\| < \eta_1$ for all $|t - a| < \delta_1$ and $\|g_t - g\| < \eta_2$ for all $|t - a| < \delta_2$. Then,

$$\begin{aligned} |\langle f_t, g_t \rangle - \langle f, g \rangle| &= |\langle f_t - f, g \rangle - \langle f_t, g_t - g \rangle| \\ &= |\langle f_t - f, g \rangle + \langle f_t - f, g_t - g \rangle + \langle f, g_t - g \rangle| \\ &\leq |\langle f_t - f, g \rangle| + |\langle f_t - f, g_t - g \rangle| + |\langle f, g_t - g \rangle| \\ &\leq \|f_t - f\|\,\|g\| + \|f_t - f\|\,\|g_t - g\| + \|f\|\,\|g_t - g\|. \end{aligned}$$

Choose $\delta = \min(\delta_1, \delta_2)$. Then for $|t - a| < \delta$ we have,

$$|\langle f_t, g_t \rangle - \langle f, g \rangle| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

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