# An Optimal Evader Strategy in a Two-Pursuer One-Evader Problem

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*Abstract*— **We consider a relay pursuit-evasion problem with two pursuers and one evader. We reduce the problem to a one-pursuer/one-evader problem subject to a state constraint. A suboptimal control strategy for the evader to prolong capture is proposed and is compared to the optimal evading strategy. Extensions to the multiple-pursuer/one-evader case are also presented and evaluated via numerical simulations.**

## I. INTRODUCTION

There has been a lot of prior work on multiple pursuit evasion problems [1]–[13]. Many of these references focus on the pursuer's strategy given the evader's strategy [1]–[6]. References [7]–[13] use a differential game formulation, where the pursuers and the evaders know the strategies of each other and make decisions accordingly. The current paper addresses a pursuit problem of two pursuers and one evader, in the case when the two pursuers implement a fixed feedback strategy to capture the evader. It is assumed that the pursuers are faster than the evader, so capture is ensured. The objective of the evader is therefore to maximize the time-to-capture under the assumption that it knows the strategy of the pursuers.

Previous similar work on the multiple-pursuer/evader differential game include references [14]–[19], which focus on finding the evading strategy in a pursuit-evasion problem. Pshenichnyi is one of the pioneers in studying multiple pursuit-evasion problems. In [14] he provided conditions for a successful evasion of one evader from multiple pursuers. In [15], Chernousko showed that a maneuvering point evader can avoid exact capture by any number of point pursuers having lower speed. Follow-up work was presented by Zak [16], where he studied the evading strategy of an evader followed by many pursuers with geometric constraints in $\mathbb{R}^r$, where $r \geq 2$. Ibragimov et al. [6] dealt with the problem of evasion from many pursuers with simple motions and integral constraints. Chodun [19] extended the evading method of Zymowski [18] and presented an evading strategy in a multiple-pursuer/one-evader game.

These papers aim at generating (sub)optimal evading strategies for successful evasion. Our paper differs from these prior works in the sense that due to our problem assumptions capture is always guaranteed. The objective of the evader is therefore to extend the time of capture. This problem is motivated by the following situation: consider a decoy, whose speed is limited, entering a defense area guarded by multiple agents following a prescribed pursuing protocol. The decoy's objective is to avoid capture as long as possible so that it can "buy more time" for the other evader(s). The goal is to find the optimal evading strategy to maximize the time-of-capture.

We assume a simplified version of this problem, involving only two-pursuers and one evader. Both pursuers implement a *relay-pursuit strategy*, according to which only one pursuer is assigned to go after the target at every instant of time. The latter is the *active* pursuer, while the other pursuer is designated as the *inactive* pursuer. This strategy may be desirable in cases when the pursuing agents play a dual role, namely, both as pursuers and as guardians protecting some area of interest, or when the power or energy consumption of the pursuing agents needs to be taken into consideration. Relay-pursuit has been applied before to solve multiple pursuit-evasion problems in [20] and [21].

In our problem formulation, we also assume that the pursuers have a stroboscopic view of the evader's position, i.e., at every instant of time, the pursuers only know the current position of the evader but not its velocity. It is further assumed that the evader is slower than the two pursuers, but is aware of their strategies and current positions.

## II. PROBLEM FORMULATION AND ANALYSIS

### A. Problem Formulation

Given a distinct set of points in the plane, known as the *generators*, we associate with each of the point locations a region in the plane, such that each point in this region is closer (with respect to some distance metric) to its own generator than to any other generator. This yields a tessellation of the plane to a set of regions associated with the given generators. This tessellation is the *Voronoi diagram* (VD) generated by the given point set. The corresponding regions are the *Voronoi cells* of the tessellation [22] and the curves that partition these regions are the *Voronoi boundaries*.

Consider the problem with just two pursuers and one evader on the plane. Associated to the two pursuers is a VD whose boundary is the bisector between the locations of the two pursuers. Assuming that the pursuers use a relay-pursuit strategy [20], the pursuer whose Voronoi cell contains the evader is assigned as the *active* pursuer to chase the evader. The other pursuer stays at its original location and plays the role of a guard. The problem terminates when one of the pursuers captures the evader. Capture is achieved when the active pursuer enters a ball of radius $\epsilon > 0$ centered at the evader's current position. It is assumed that the pursuers are faster, but they only have accurate measurements of the current position of the evader at every instant of time. In the absence of any informative model of the future evader position, a reasonable approach for each active pursuer is to apply a pure pursuit strategy, according to which the pursuer's velocity vector points towards the current position

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of the target. The active pursuer switches when the evader enters the interior of the Voronoi cell of another pursuer.

Due to the symmetry of the problem, when the evader resides on the Voronoi cell boundary, we can assign any one of the two pursuers to be the active pursuer. Therefore, throughout the pursuit process, we can fix one pursuer to be the active pursuer, while the inactive pursuer remains stationary on the plane, whose mere presence, however, imposes a state restriction, namely that the evader does not enter the interior of its corresponding Voronoi cell.

## B. State Equations

Without loss of generality, we assume that the inactive pursuer is located at the origin. Henceforth, the subscripts $P$, $E$ and $G$ will be used to denote the active pursuer, the evader, and the inactive pursuer, respectively. The equations of motion are

$$\dot{x}_P = u\cos\phi_P, \quad \dot{y}_P = u\sin\phi_P, \tag{1}$$

$$\dot{x}_E = v\cos\phi_E, \quad \dot{y}_E = v\sin\phi_E, \tag{2}$$

where $\mathsf{x}_P = (x_P, y_P)$ and $\mathsf{x}_E = (x_E, y_E)$ denote the position of the active pursuer and the target, respectively, $\phi_P \in [-\pi, \pi)$ denotes the control input of the active pursuer, $\phi_E \in [-\pi, \pi)$ denotes the control input of the evader, and $u$ and $v$ are the velocities (constant) of the pursuer and the evader, respectively, with $u > v$. The state of the system is $\mathsf{x} = [x_P, y_P, x_E, y_E]^\mathsf{T} \in \mathbb{R}^4$.

Since the active pursuer implements a pure pursuit strategy, it follows that

$$\cos\phi_P = \frac{x_E - x_P}{\|\mathsf{x}_E - \mathsf{x}_P\|}, \quad \sin\phi_P = \frac{y_E - y_P}{\|\mathsf{x}_E - \mathsf{x}_P\|}, \tag{3}$$

where $\|\mathsf{x}_E - \mathsf{x}_P\| = \sqrt{(x_E - x_P)^2 + (y_E - y_P)^2}$. Our goal is to find the optimal control of the evader $\phi_E \in [-\pi, \pi)$ to maximize the time-to-capture $t_c$ under the state constraint $\|\mathsf{x}_E - \mathsf{x}_P\| \le \|\mathsf{x}_E - \mathsf{x}_G\|$, where $\mathsf{x}_G = (0,0)$, or equivalently,

$$S(\mathsf{x}) = x_P(x_P - 2x_E) + y_P(y_P - 2y_E) \le 0. \tag{4}$$

## C. Problem Analysis

It is well known that in the one-pursuer/one-evader problem without state constraints, when the pursuer applies a pure pursuit strategy, the optimal strategy for the evader is to move away from the pursuer along their common line-of-sight (LoS) [23]. The LoS is defined as the line passing through the pursuer's and evader's instantaneous positions.

Given the problem formulation in Section II-A, if the evader never reaches the boundary of $\mathcal{X}$ before it is captured by moving along the LoS, the problem reduces to a one-pursuer/one-evader problem without state constraints. In this case, the time-to-capture is calculated by

$$t_c^* = \frac{\|\mathsf{x}_E(0) - \mathsf{x}_P(0)\|}{u - v}. \tag{5}$$

*Proposition 2.1:* Consider the pursuit-evasion problem stated in Section II-A. The evader will be captured before entering the Voronoi cell of the inactive pursuer while moving along the LoS and away from the active pursuer, if and only if the quadratic equation

$$at^2 + bt + c = 0, \tag{6}$$

where $a = u^2 - 2uv$, $b = 2[(ux_P(0) - vx_P(0) - ux_E(0))\cos\phi_E(0) + (uy_P(0) - vy_P(0) - uy_E(0))\sin\phi_E(0)]$ and $c = x_P(0)^2 + y_P(0)^2 - 2(x_E(0)x_P(0) + y_E(0)y_P(0))$, does not have a solution inside the interval $[0, t_c^*]$, and $\phi_E(0)$ is determined by the equations:

$$\cos\phi_E(0) = \frac{x_E(0) - x_P(0)}{\|\mathsf{x}_E(0) - \mathsf{x}_P(0)\|}, \tag{7a}$$

$$\sin\phi_E(0) = \frac{y_E(0) - y_P(0)}{\|\mathsf{x}_E(0) - \mathsf{x}_P(0)\|}. \tag{7b}$$

*Proof:* First note that for the evader to enter the Voronoi cell of the inactive pursuer, there must exist some time $0 \le \tau \le t_c^*$ such that

$$\|\mathsf{x}_E(\tau) - \mathsf{x}_P(\tau)\| = \|\mathsf{x}_E(\tau)\|. \tag{8}$$

Suppose that the evader moves along the LoS before capture occurs. Then $\phi_E(t) = \phi_E(0)$ for all $t \ge 0$, where $\phi_E(0)$ satisfies (7a) and (7b). If there does not exist a time $\tau \in [0, t_c^*]$ such that equation (8) is satisfied, then the evader will be captured without entering the Voronoi cell of the inactive pursuer. Otherwise, we can express the position of the active pursuer and the evader at time $\tau$ in terms of their initial conditions and plug these expressions in (8), we eventually obtain (6). If no solution of (6) lies in the time interval $[0, t_c^*]$, then the evader will be captured before entering the Voronoi cell of the inactive pursuer, thus completing the proof. ■

## D. The Region for Non-LoS Evasion

In order to find the explicit expression for the region in which the condition of Proposition 2.1 is not satisfied, and without loss of generality, let the initial position of the active pursuer be $P = (x_P(0), 0)$.

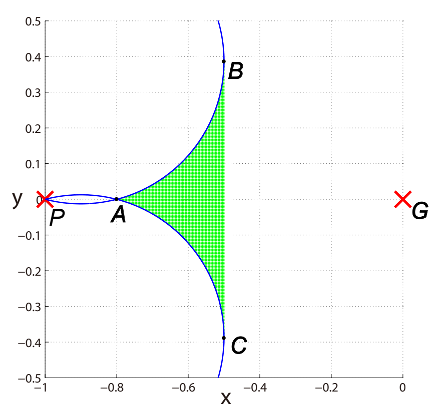


Fig. 1. Shaded region of the evader's initial positions such that the problem is not degenerate.

In Figure 1, the green region depicts the evader's initial positions for which the condition of Proposition 2.1 is not satisfied. That is, if the evader starts from a position inside the shaded region, it will not be able to move along the LoS throughout the pursuit without violating the state constraint.

This region has a triangle-like shape. We denote the three vertices as $A, B$ and $C$, where $A$ resides on the line segment between the active pursuer and the inactive pursuer, and $B, C$ are on the Voronoi boundary.

The curves $AB$ and $AC$ satisfy the equation

$$0 = (((v-u)x_P(0) + ux)(x - x_P(0)) + uy^2)^2 - u(2v-u)x_P(0)(2x - x_P(0))((x - x_P(0))^2 + y^2), \tag{9}$$

which is derived from $0 = b^2 - 4ac$, where $a, b$ and $c$ are defined in Proposition 2.1, by plugging in (7a), (7b) and the initial condition for the active pursuer. Equation (9) is plotted in Figure 1 in blue. The coordinates of the points $A, B$ and $C$ are given by $A = (x_P(0)v/u, 0)$, $B = (x_P(0)/2, \sqrt{v/(2u) - 1/4}\,|x_P(0)|)$, and $C = (x_P(0)/2, -\sqrt{v/(2u) - 1/4}\,|x_P(0)|)$.

Henceforth, let $\mathcal{D}$ denote the shaded region, shown in Figure 1, of the evader's initial conditions that lead to a non-degenerate solution of the problem.

## III. Problem Formulation and Analysis in the Reduced State Space

### A. The Reduced State Space

The equations (1)-(2) can be expressed in a three-dimensional space by fixing the origin of a new coordinate system to the active pursuer and by aligning the positive direction of the $x$-axis with the direction of the active pursuer's velocity vector [23]. Let us denote this new frame by $\mathcal{M}$. In the frame $\mathcal{M}$, the active pursuer is fixed at the origin, and the motion of the evader is restricted to lie on the $x$-axis due to the pure pursuit strategy of the active pursuer. The inactive pursuer, however, is no longer stationary in $\mathcal{M}$.

Let $\theta = \phi_E - \phi_P$ be the angle between the active pursuer's and evader's velocity vectors. Then $\dot{\phi}_P$ is given by

$$\dot{\phi}_P = \frac{v \sin\theta}{\|\mathrm{x}_E - \mathrm{x}_P\|}. \tag{10}$$

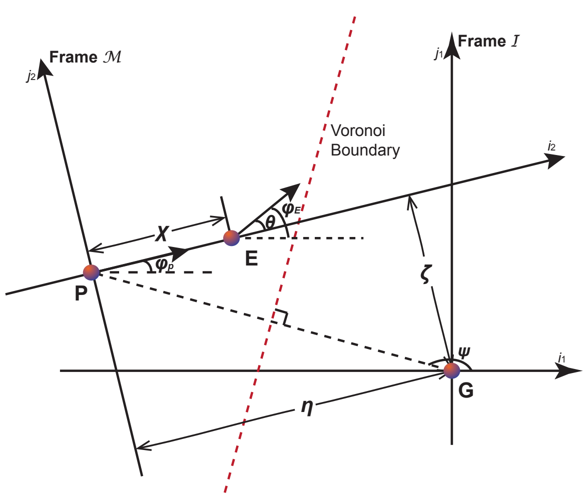


Fig. 2. Geometry of evader (E), active (P) pursuer, and inactive (G) pursuer in the inertial ($\mathcal{I}$) and the active pursuer frames ($\mathcal{M}$).

Let the coordinates of the inactive pursuer and the evader be, respectively, $\mathrm{x}_G = (\eta, \zeta)$ and $\mathrm{x}_E = (\chi, 0)$ in the frame $\mathcal{M}$, as shown in Figure 2. The equations of motion in the frame $\mathcal{M}$ are then given by

$$\dot{\chi} = -u + v\cos\theta, \tag{11}$$

$$\dot{\eta} = -u + v\frac{\zeta}{\chi}\sin\theta, \tag{12}$$

$$\dot{\zeta} = -v\frac{\eta}{\chi}\sin\theta, \tag{13}$$

while the constraint (4) is given by

$$\mathcal{S}(\xi) = \frac{1}{2}(\chi^2 - (\eta - \chi)^2 - \zeta^2) \le 0, \tag{14}$$

where $\xi = [\chi, \eta, \zeta]^\intercal \in \mathbb{R}^3$. The boundary conditions in frame $\mathcal{M}$ are given by

$$\begin{aligned} &\chi(0) = \|\mathrm{x}_E(0) - \mathrm{x}_P(0)\|, \quad \chi(t_f) = \epsilon, \\ &\eta(0) = \|\mathrm{x}_P(0)\|\cos(\pi + \psi(0) - \phi_P(0)), \quad \eta(t_f) \text{ free}, \\ &\zeta(0) = \|\mathrm{x}_P(0)\|\sin(\pi + \psi(0) - \phi_P(0)), \quad \zeta(t_f) \text{ free}, \end{aligned} \tag{15}$$

where $\psi(0)$ is the initial value of the angle $\psi = \operatorname{atan}(y_P, x_P)$, as shown in Figure 2. Given (11)-(13), the constraint (14) and boundary conditions (15), our goal is to find the optimal control $\theta$ to maximize $t_c$.

Let $\mathcal{E}$ denote the region $\mathcal{D}$ in the frame $\mathcal{M}$, and let $\mathcal{F} = \{\xi \in \mathbb{R}^3 : (\chi\cos\nu, \chi\sin\nu) \in \mathcal{E}\}$ denote the set of initial conditions in the reduced space leading to a non-degenerate solution.

### B. Optimal Evading Strategy

To solve the previous problem, we utilize optimal control theory. To this end, and noticing that $\dot{\mathcal{S}}(\xi) = -\chi u + \eta v\cos\theta + \zeta v\sin\theta$, let the Hamiltonian be

$$H = -1 + \lambda_1(-u + v\cos\theta) + \lambda_2\left(-u + v\frac{\zeta}{\chi}\sin\theta\right) + \lambda_3\left(-v\frac{\eta}{\chi}\sin\theta\right) + \mu(-\chi u + \eta v\cos\theta + \zeta v\sin\theta),$$

where $\lambda_1, \lambda_2, \lambda_3$ and $\mu$ are co-state variables. The co-states $\lambda_1, \lambda_2$ and $\lambda_3$ evolve according to

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial \chi} = \frac{(\lambda_2\zeta - \lambda_3\eta)v\sin\theta}{\chi^2} + \mu u, \tag{16}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial \eta} = \frac{\lambda_3 v\sin\theta}{\chi} - \mu v\cos\theta, \tag{17}$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial \zeta} = -\frac{\lambda_2 v\sin\theta}{\chi} - \mu v\sin\theta, \tag{18}$$

The multiplier $\mu$ satisfies the Kuhn-Tucker condition and the complementary slackness condition, that is, $\mu = 0$, if $\mathcal{S} \neq 0$, and $\mu \ge 0$, if $\mathcal{S} = 0$. The transversality conditions for this optimal control problem are

$$\lambda_2(t_c) = 0, \qquad \lambda_3(t_c) = 0, \qquad H(t_c) = 0. \tag{19}$$

Furthermore, since $H$ does not depend explicitly on time, the optimal Hamiltonian $H^*(t) = 0$ for all $t \in [0, t_c]$.

By Pontryagin's Minimum Principle, the optimal control $\theta^*$ is computed from $H_\theta = 0$, which yields

$$\tan\theta^* = \frac{\lambda_2\zeta - \lambda_3\eta + \mu\zeta\chi}{(\lambda_1 + \mu\eta)\chi}. \tag{20}$$

*Theorem 3.1:* Consider the optimal control problem (11)-(15). Assume that $u > v > u/2$ and assume that the initial conditions are such that $\xi(0) \in \mathcal{F}$. Then the optimal control of the evader is given as follows:

$$\tan\theta^*(t) = \begin{cases} \dfrac{\lambda_2\zeta - \lambda_3\eta}{\lambda_1\chi}, & t \in [0, \tau_1], \\ \dfrac{q - \sigma p\sqrt{p^2+q^2-1}}{p + \sigma q\sqrt{p^2+q^2-1}}, & t \in [\tau_1, \tau_2], \\ 0, & t \in [\tau_2, t_c], \end{cases}$$

where $p = v\eta/(u\chi), q = v\zeta/(u\chi), \sigma = \text{sgn}(q)$. Furthermore, $\tau_2$ satisfies the switching condition:

$$v\eta(\tau_2) - u\chi(\tau_2) = 0. \tag{21}$$

*Proof:* If the initial condition of the evader is not on the boundary of the Voronoi cell of the inactive pursuer, it follows that the control is given by (20) with $\mu = 0$, and hence

$$\tan\theta^*(t) = \frac{\lambda_2\zeta - \lambda_3\eta}{\lambda_1\chi}, \quad \text{for } t \in [0, \tau_1]. \tag{22}$$

The evader will follow this strategy till some time $\tau_1 > 0$ when it will hit the boundary of $\mathcal{X}$ and then will stay on the boundary, defined by $\mathcal{S}(\xi) = 0$, for $t \in [\tau_1, \tau_2]$. For the evader to stay on the boundary of $\mathcal{X}$, one easily computes

$$\dot{\mathcal{S}}(\xi) = -\chi u + \eta v\cos\theta + \zeta v\sin\theta = 0, \tag{23}$$

and thus $p\cos\theta + q\sin\theta = 1$, where $p = v\eta/(u\chi)$ and $q = v\zeta/(u\chi)$. It follows that

$$\cos\theta = \frac{p \pm q\sqrt{p^2+q^2-1}}{p^2+q^2}, \quad \sin\theta = \frac{q \mp p\sqrt{p^2+q^2-1}}{p^2+q^2}.$$

During the time when the evader moves on the boundary, we want to keep $\cos\theta$ positive and as large as possible without violating the boundary condition. It follows that when the evader moves along the boundary, the control to use is

$$\cos\theta = \frac{p + \sigma q\sqrt{p^2+q^2-1}}{p^2+q^2}, \quad \sin\theta = \frac{q - \sigma p\sqrt{p^2+q^2-1}}{p^2+q^2},$$

where $\sigma = \text{sgn}(q)$. It follows that

$$\tan\theta^*(t) = \frac{q - \sigma p\sqrt{p^2+q^2-1}}{p + \sigma q\sqrt{p^2+q^2-1}}, \quad \text{for } t \in [\tau_1, \tau_2]. \tag{24}$$

The optimal value of the multiplier $\mu^*$ for $t \in [\tau_1, \tau_2]$ can be immediately computed from (20) as follows

$$\mu^* = \frac{\lambda_1\chi\tan\theta^* + \lambda_3\eta - \lambda_2\zeta}{\chi(\zeta - \eta\tan\theta^*)}, \tag{25}$$

with boundary condition $\mu(\tau_2^-) = \mu(\tau_2^+) = 0$. After the evader leaves the constraint at time $t = \tau_2$, and prior to capture, $\mu^* = 0$ and thus, equations (17) and (18) can be rewritten as

$$\begin{bmatrix} \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} = \frac{v\sin\theta^*}{\chi} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_3 \end{bmatrix}, \tag{26}$$

whose solution subject to the boundary conditions (19) is given by $\lambda_2(t) = \lambda_3(t) = 0$ for all $t \in [\tau_2, t_c]$. Therefore, when $t \in [\tau_2, t_c]$, we have from (20) that $\tan\theta^* = (\lambda_2\zeta - \lambda_3\eta)/\lambda_1\chi = 0$, and hence $\theta^* = 0$ for all $t \in [\tau_2, t_c]$. By imposing the Erdmann's corner conditions at the entry and exit points from the state constraint [24] after some tedious, but rather straightforward calculations, one obtains that $\theta^*(\tau_1^-) = \theta^*(\tau_1^+)$ and $\theta^*(\tau_2^-) = \theta^*(\tau_2^+)$. Hence the control is continuous at $\tau_1$ and $\tau_2$. Since $\theta^*(\tau_2^+) = 0$, it follows that $\theta^*(\tau_2^-) = 0$ and hence the evader will leave the boundary when the evader's velocity is parallel to the current LoS. Since after the switching at $t = \tau_2$ we have that $\theta^*(\tau_2) = 0$, it follows that $\sin\theta^*(\tau_2) = 0$ and hence the switching condition to leave the boundary follows immediately from (23) and is given by (21). This completes the proof. ■

## IV. A Suboptimal Evading Strategy

Summarizing the previous analysis, we conclude that the optimal trajectory of the evader involves three periods: first, the evader moves in the Voronoi cell of the active pursuer in a way such that the optimal conditions (transversality condition, Erdmann corner condition, etc.) are satisfied before she hits the boundary, then the evader moves along the boundary until the switching condition (21) is satisfied, finally the evader moves along the LoS till capture occurs.

Given the analysis of the previous section, one can compute numerically the evader's optimal trajectory. Note, however, that the second and third period strategies can be easily implemented without resorting to the solution of a two-point boundary value problem. The following result follows immediately from the previous observations.

*Corollary 4.1:* Consider the optimal control problem (11)-(15) and let $u > v > u/2$. Assume that the initial condition is such that $\mathcal{S}(\xi(0)) = 0$. Then the optimal control of the evader is given by

$$\tan\theta^*(t) = \begin{cases} (24), & t \in [0, \tau], \\ 0, & t \in [\tau, t_c], \end{cases}$$

where $\tau$ satisfies the switching condition (21).

When the evader starts from a general position, a suboptimal control scheme for the evader that can be computed analytically is proposed as follows: The evader first moves along the LoS away from the pursuer before it hits the boundary. Then she moves on the boundary until the switching condition (21) is satisfied. Afterwards, the evader resumes moving along the LoS until capture occurs. Simulation results in Section VI-A show that the relative differences between the time-to-capture by applying this strategy and the optimal one are always less than 5%, given different speed ratio and initial positions of the evader and the pursuer.

## V. Generalization in the Multiple-Pursuer/One-Evader Problem

The main idea of the proposed suboptimal evader control strategy is that the evader moves along the LoS whenever it is able to, otherwise it will move along the Voronoi cell boundary. We can generalize this idea and propose a suboptimal evading strategy for the multiple-pursuer/one-evader relay pursuit problem. In this case, there will be one active pursuer and multiple inactive pursuers that play the role of guards. The objective of the evader is to maximize the instantaneous velocity component along the LoS at every instant of time without violating the state constraints imposed by the fact that the evader never enters the interior of the Voronoi cells of an inactive pursuer.

Specifically, suppose that, at some time, the evader reaches the intersection of three adjacent Voronoi cells. One of the Voronoi cells is generated by the active pursuer $\mathrm{x}_P$. Let the generators of the other two Voronoi cells be $\mathrm{x}_{G1}$ and $\mathrm{x}_{G2}$. Let $S_1 = \|\mathrm{x}_E - \mathrm{x}_P\|^2 - \|\mathrm{x}_E - \mathrm{x}_{G1}\|^2$, $S_2 = \|\mathrm{x}_E - \mathrm{x}_P\|^2 - \|\mathrm{x}_E - \mathrm{x}_{G2}\|^2$. The velocity $\tilde{v}_E$ of the evader to stay on the intersection of three adjacent Voronoi cells can be found by solving for $\dot{S}_1 = 0$ and $\dot{S}_2 = 0$. After some calculations, one obtains

$$\tilde{v}_E = \begin{bmatrix} \dot{x}_E \\ \dot{y}_E \end{bmatrix} = \begin{bmatrix} (s_2 - s_1)/(r_1 s_2 - r_2 s_1) \\ (r_2 - r_1)/(r_2 s_1 - r_1 s_2) \end{bmatrix}, \qquad (27)$$

where

$$r_1 = -\frac{x_P - x_{G1}}{u\|\mathrm{x}_E - \mathrm{x}_P\|}, \qquad s_1 = -\frac{y_P - y_{G1}}{u\|\mathrm{x}_E - \mathrm{x}_P\|}, \qquad (28)$$

$$r_2 = -\frac{x_P - x_{G2}}{u\|\mathrm{x}_E - \mathrm{x}_P\|}, \qquad s_2 = -\frac{y_P - y_{G2}}{u\|\mathrm{x}_E - \mathrm{x}_P\|}. \qquad (29)$$

Hence, $\tilde{v} = \|\tilde{v}_E\|$ is the speed of the evader to stay on the intersection of three Voronoi cells. Suppose that the evader hits the intersection at some time $\tau$ during the process, if it happens that $\tilde{v}(\tau) < v$, then the evader will move with speed $\tilde{v}$ for some time before she can freely move with maximum speed.

## VI. Simulation Results

### A. Comparison Between the Proposed Suboptimal and Optimal Strategy

In this section we compare the optimal strategy and the suboptimal strategy for the evader proposed in Section IV. We set $u = 1.0, v = 0.7$. Let the capture radius be $\epsilon = 0.001$, fixed throughout the simulations. Given the initial conditions of the evader $\mathrm{x}_E(0) = (-1.1, 0.6)$, the active pursuer $\mathrm{x}_P(0) = (-2, 1)$, and the inactive pursuer $\mathrm{x}_G(0) = (0, 0)$, the numerical result generated by GPOPS is shown in Figure 3 and the optimal time-to-capture is $t_c = 3.2033$. The evader's trajectory generated by applying the evading strategy proposed in Section IV as well as the active pursuer's trajectory are shown in Figure 4 and are represented by a red and a green line, respectively. With this approach, the time-to-capture is $t'_c = 3.1216$. In this simulation, the relative time difference between the optimal control and the suboptimal control is $\Delta = (t_c - t'_c)/t_c = 2.55\%$.

In Table I we present additional results under different initial positions and/or different evader speeds. As shown from this table, the relative difference in terms of time-to-capture between the optimal control and the suboptimal control is, in general, quite small.

### B. Case of Three Pursuers

We have implemented the generalization of the suboptimal evading strategy to a three-pursuer/one-evader relay pursuit problem and compared the result with the optimal control of the evader generated by GPOPS. The initial conditions of the evader and the active pursuer are $\mathrm{x}_E(0) = (-1.1, 0.6)$

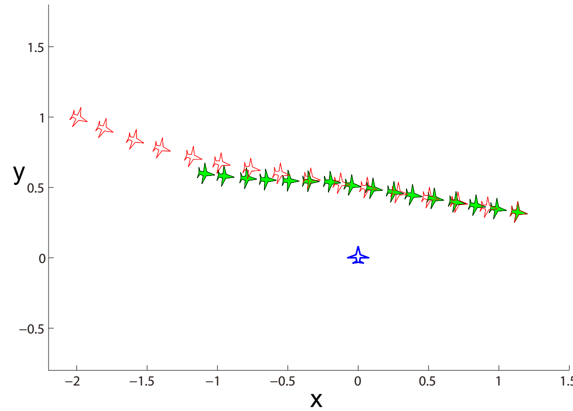


Fig. 3. Trajectory of the active pursuer in red and optimal trajectory of the evader in green.

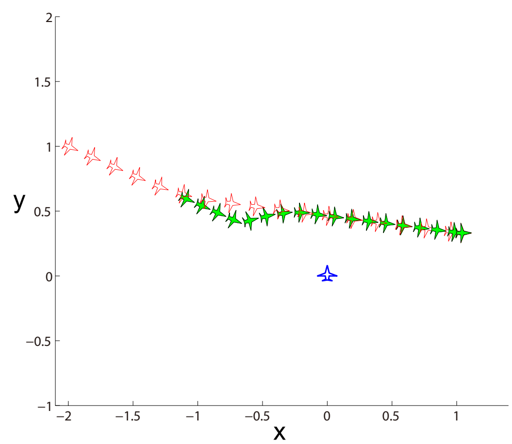


Fig. 4. Trajectory of the active pursuer in red and trajectory of the evader generated by applying the proposed evading strategy in green.

and $\mathrm{x}_P(0) = (-2, 1)$, respectively. The two inactive pursuers are located at $\mathrm{x}_{G1}(0) = (0, 0)$, $\mathrm{x}_{G2}(0) = (-0.2, 1)$. The numerical result generated by GPOPS is shown in Figure 5. The corresponding optimal time-to-capture is $t_c = 3.1530$. The evader's trajectory generated by applying the evading strategy we proposed in Section IV, and the active pursuer's trajectory are presented in Figure 6 in red and green lines, respectively. As seen from Figures 5 and 6 the trajectories are very similar. The time-to-capture is $t'_c = 3.1366$. The relative time difference this time is $\Delta = 0.52\%$.

## VII. Conclusions

This paper deals with the optimal control of the evader in a two-pursuer/one-evader relay-pursuit problem. We provide the conditions for the evader to reach and stay on the boundary of the Voronoi diagram formed by the two pursuers, and derive the optimal control strategy of the evader to maximize capture time. We also propose a suboptimal, yet

TABLE I

RELATIVE TIME DIFFERENCE UNDER VARYING EVADER SPEEDS AND INITIAL CONDITIONS

| $[x_P\ y_P\ x_E\ y_E]$ \ $v$ | 0.7 | 0.8 | 0.9 |
|---|---|---|---|
| [-2, 1, -1.1, 0.6] | 2.55% | 2.72% | 3.13% |
| [-2, 0, -1.1, 0.1] | 1.70% | 0.41% | 1.27% |

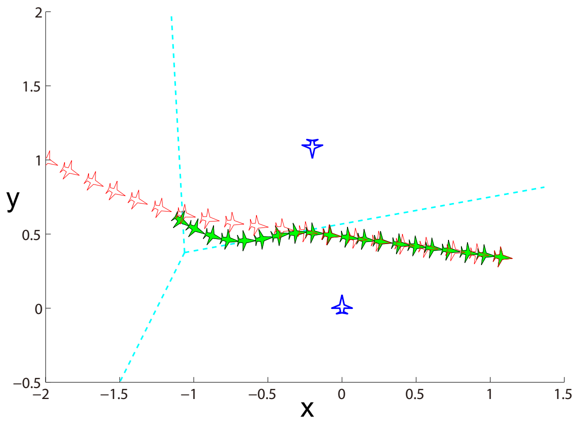


Fig. 5. Trajectory of the active pursuer (in red) and optimal trajectory of the evader (in green) obtained from GPOPS. The cyan line represents the Voronoi diagram of the three pursuers with initial conditions.

practical, control strategy for the evader that does not require the solution of the corresponding two-point boundary-value optimal control problem. We generalize this idea and apply it to the multiple-pursuer/one evader relay pursuit problem. Simulation results support the theoretical claims.

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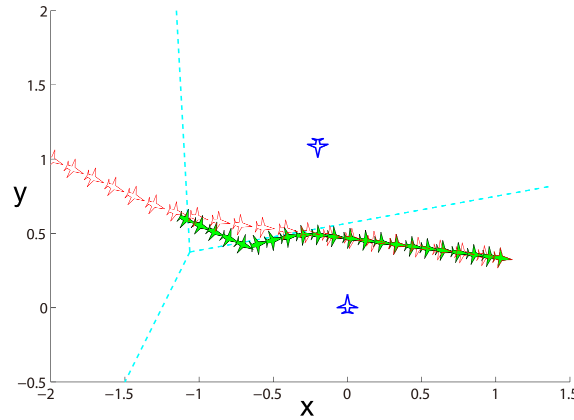


Fig. 6. Trajectory of the active pursuer (in red) and trajectory of the evader (in green) generated by applying the proposed suboptimal evading strategy.

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