Overview of Wavelet Analysis and Applications in Engineering

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Under the Wave off Kanagawa
Katsushika Hokusai (Japanese, Tokyo (Edo) 1760–1849)

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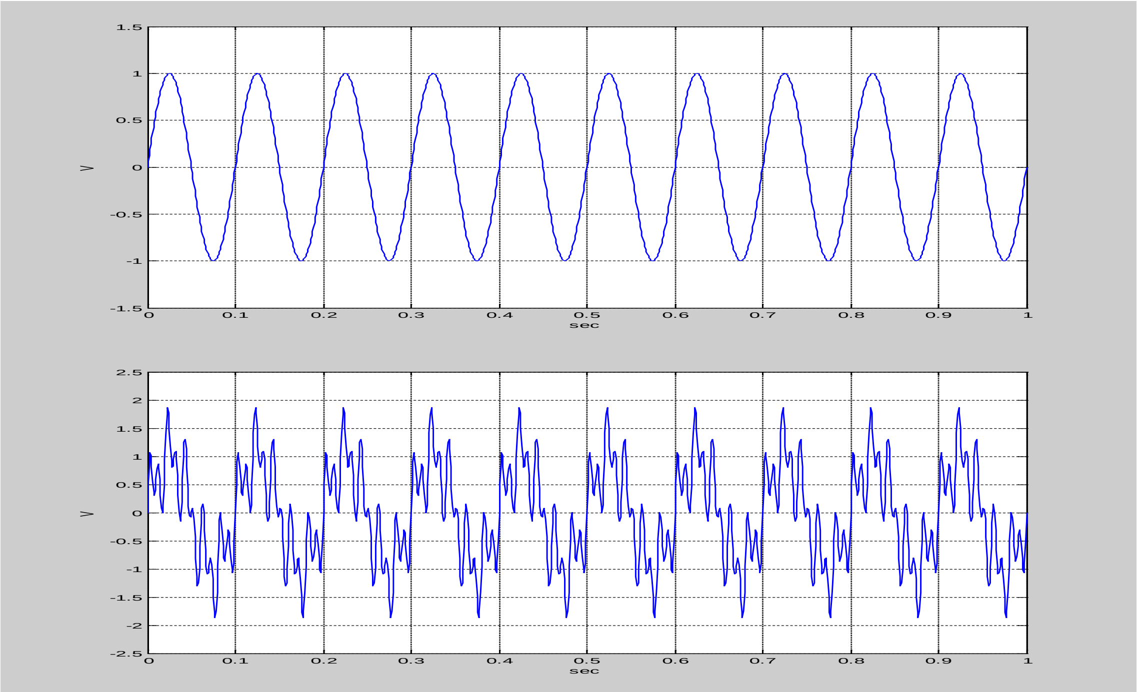
Overview

- **What are wavelets?**
  - Wavelets are mathematical functions that look like waves (small waves - *ondelette*)
  - They have varying frequency, limited duration and zero mean
- **What do they do?**
  - Wavelets analyze signals at different level of resolution
  - Provide an adaptive time-frequency representation
- **Why are wavelets needed?**
  - Fourier analysis is best for stationary signals
  - Most real signals are non-stationary
- **Fields of application are numerous**
  - Electrical Engineering, Civil Engineering, Mechanical Engineering, Computer Science, Communications, Physics, Geology, Astronomy, Music, …
  - Signal Processing, Image Analysis, Medical Imaging, Structural Health Monitoring, …
  - Denoising, Compression, Sparse Representation, System Identification, Clustering and Classification, Nonstationary analysis, Transient Detection, …

Introduction

- **Stationarity**

  A signal can be considered stationary if its characteristics are not changing with time. Classical analysis techniques assume the signal to be stationary and last from $-\infty$ to $+\infty$



$$f(t) = \sin(2\pi\ 10\ t)$$

$$f(t) = \sin(2\pi\ 10\ t) + \sin(2\pi\ 50\ t) + \sin(2\pi\ 100\ t)$$

How do we best represent a stationary signal ?

Introduction

- **Fourier Domain**

Jean-Baptiste Joseph Fourier (1768 – 1830)
https://en.wikipedia.org/wiki/Joseph_Fourier

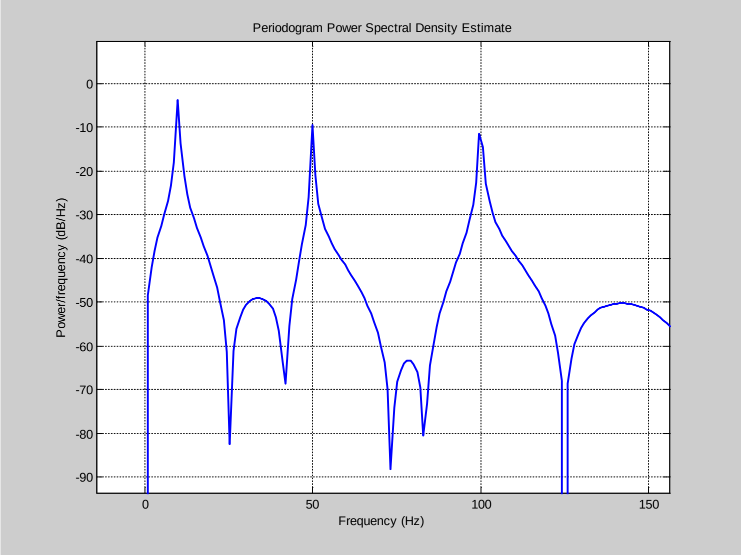
In 1807 Fourier claims that any periodic function can be represented as a series of harmonically related sinusoids

$$X(\omega) = \Im\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt$$

$$x(t) = \Im^{-1}\{X(\omega)\} = (2\pi)^{-1} \int_{-\infty}^{+\infty} X(\omega)e^{i\omega t}d\omega$$

Power estimate for each frequency

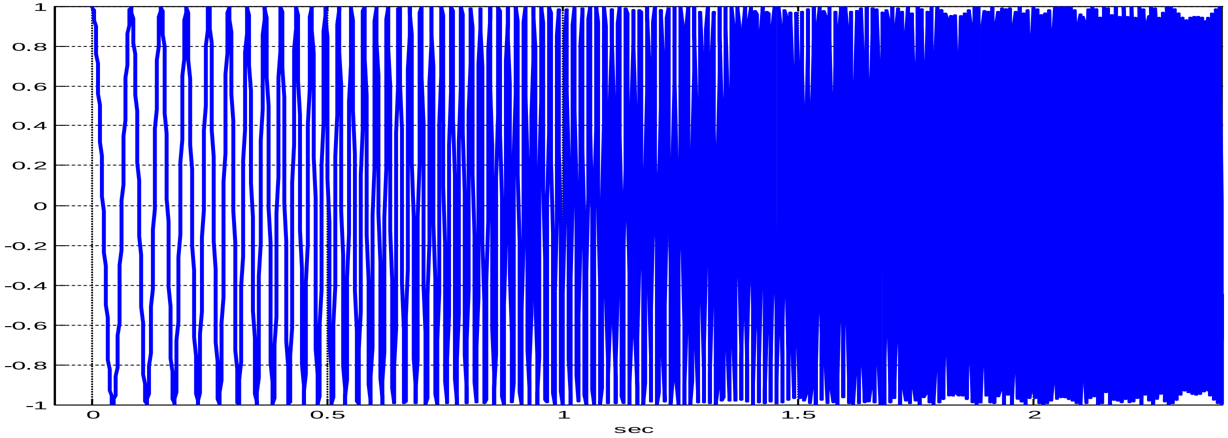
$$Periodogram = |X(\omega)X^{*}(\omega)|^{2}$$



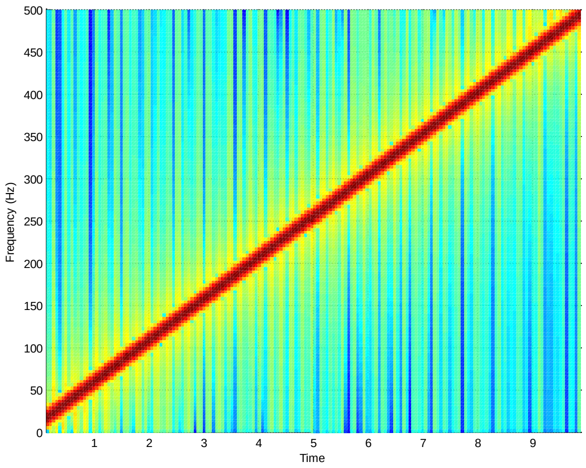
Introduction

- **Real Signals**

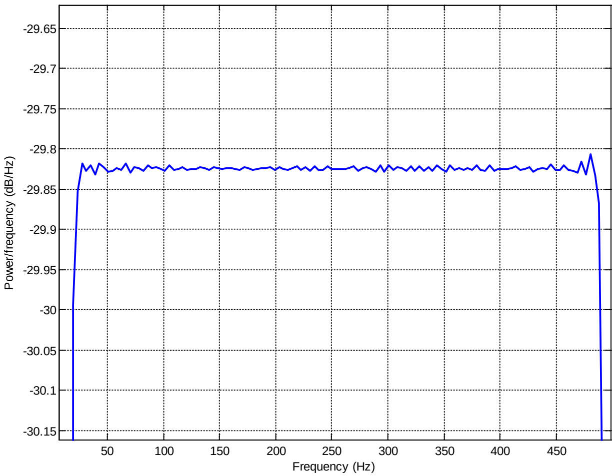
  Most real signals can be considered non-stationary, and have complex time-frequency characteristics



Chirp signal with frequencies from 10Hz to 500Hz sampled at 1kHz

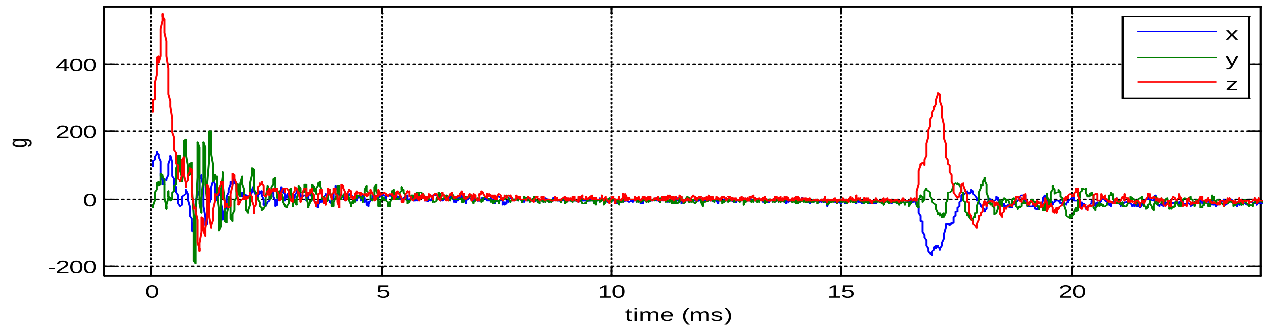


Spectrogram

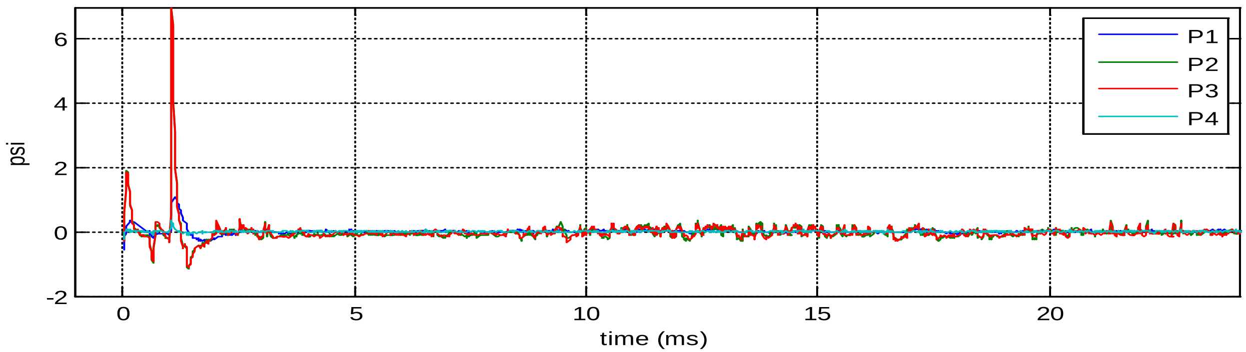


Power Spectral Density

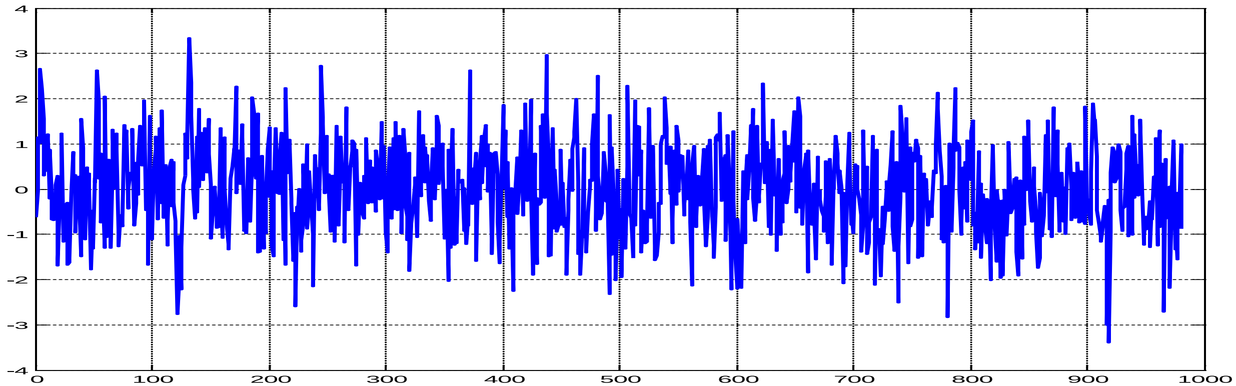
Examples of Real Signals



Vibration response from a tri-axial accelerometer to a violent impact



Sound pressure from 4 piezoelectric transducers recorded during an explosion



Voxel time series from a functional MRI scan of an anesthetized rat

Introduction

- **Shortcoming of Fourier Analysis**
  - The signal is represented globally
  - The time information is lost: which frequency appears when and where? (*localization problem*)
  - Most real signals have a frequency contents that changes with time
  - To properly describe a non-stationary evolving signal we need *time* and *frequency* localization

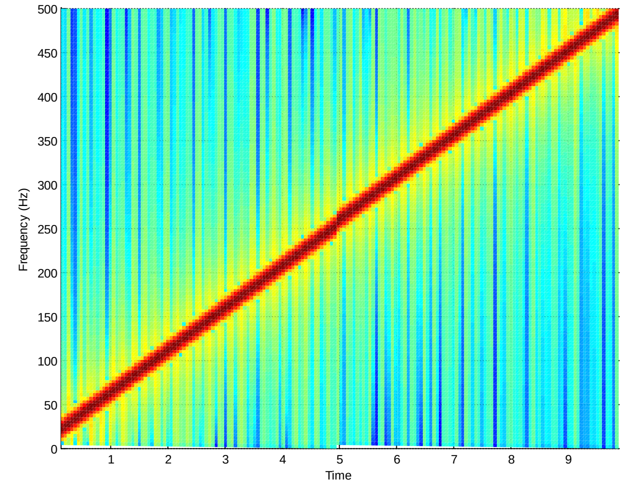
- **Windowed Fourier Transform**
  - Partially stationary condition can be obtained dividing the signals into short segments in which the signal can be assumed *quasi* stationary
  - In 1946 Dennis Gabor (Nobel in physics) decomposed a signal over a dictionary of elementary waveforms, called *time-frequency atoms*
  - A *windowed Fourier Transform* can be thought using a windowed Fourier dictionary, obtained by translating a time window $g(t)$ in time and frequency

Introduction

- **The Short Time Fourier Transform**

$$STFT(t,\omega) = \int_{-\infty}^{+\infty} x(t)h(t-\tau)e^{-i\omega\tau}d\tau = \langle x(t)h(t,f)\rangle$$

$$Spectrogram(t,\omega) = |STFT(t,\omega)|^2$$



- **How to select the proper window size?**
  - Short windows yields poor frequency resolution
  - Long window increase resolution but compromise assumption of stationarity

Dennis Gabor (1900 – 1979)
https://en.wikipedia.org/wiki/Dennis_Gabor

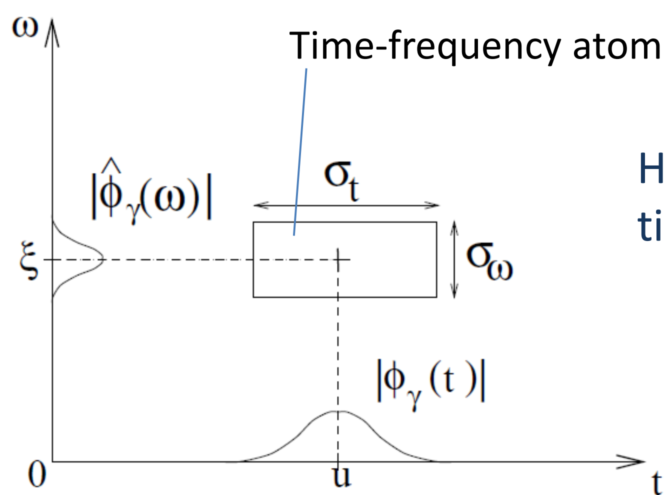
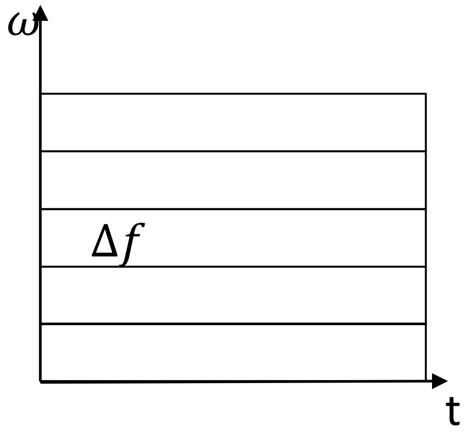


Fig. 1.3. A Wavelet Tour of Signal Processing, 3rd ed. Heisenberg box representing an atom ϕ_γ.

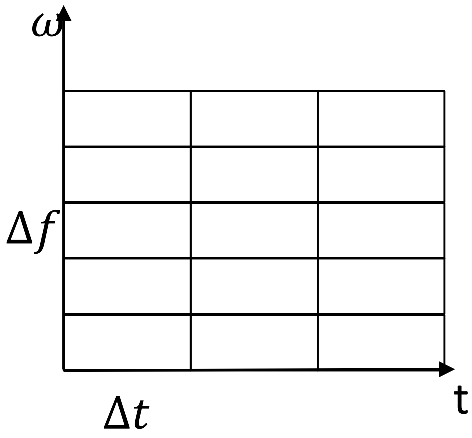
Heisenberg Uncertainty in the time-frequency plane

$$\sigma_\omega \sigma_t \geq \frac{1}{2}$$

Introduction



Fourier



Gabor

- A windowed Fourier Transform decomposed signal over a basis with constant time-frequency resolution
- Signal with localized features in time/frequency are not represented well
- Not adaptive to the signal

- **Solution** **Use atoms with different time-frequency resolution**

=

WAVELETS

Wavelets

- **What is a wavelet?**
  - A wavelet is a function $\psi(t) \in \mathbf{L}^2(\mathbb{R})$ with

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0$$

  - It has unit norm $\|\psi(t)\| = 1$
  - Dilations by s and translation by u generates a dictionary of time-frequency atoms

$$\left\{\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)\right\}_{u,s\epsilon\mathbb{R}}$$

  - The *wavelet transform* decomposes a signal over dilated and translated wavelets. For a time u and scale s, the continuous wavelet transform is defined as

$$Wf(u,s) = \langle f(t), \psi_{u,s}(t)\rangle = \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right)dt$$

Wavelets

- **Connection with frequency – wavelet and linear filtering**
  - We can think of $Wf(u,s)$ as a convolution

$$Wf(u,s) = \langle f(t), \psi_{u,s}(t) \rangle = f(t) \circledast \bar{\psi}_s(u)$$

with $\bar{\psi}_s(t) = \frac{1}{\sqrt{s}} \psi^* \left(-\frac{t}{s}\right)$

  - Now taking the Fourier Transform of $\bar{\psi}_s(t)$ we have

$$\Im\{\bar{\psi}_s(t)\} = \sqrt{s}\hat{\psi}^*(s\omega)$$

and

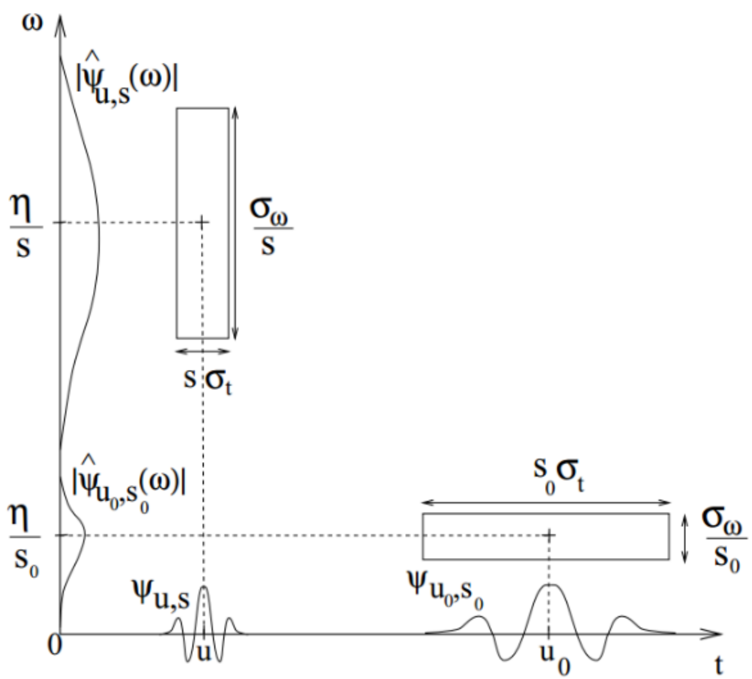
$$\hat{\psi}(\omega = 0) = \int_{-\infty}^{+\infty} \psi(t)dt = 0$$

  - Therefore, $\hat{\psi}$ is the transfer function of a bandpass filter, having zero energy for $\omega = 0$

Each coefficient of the wavelet transform is the result of a filtering operation between the function and the bandpass filter defined by the wavelet atom

Wavelets

- **Time-Frequency Resolution**



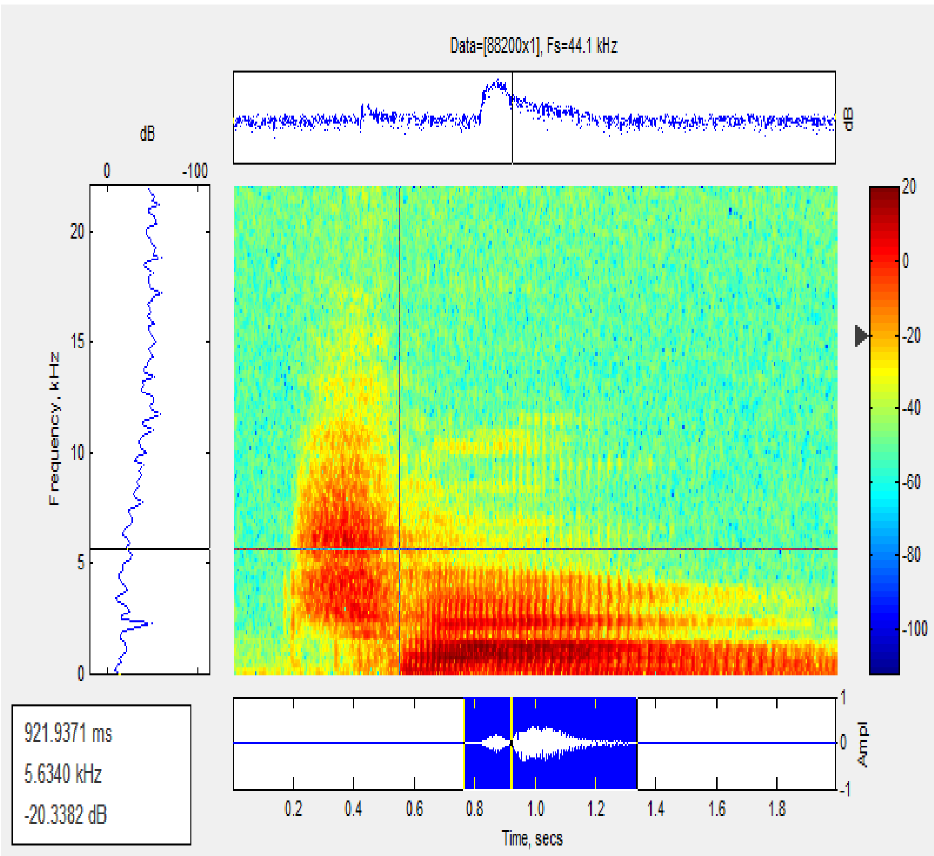
- A wavelet atom has time support centered at u and proportional to s
- The Heisenberg box of a wavelet atom is a rectangle centered at $(u, {}^{\eta}/_{s})$
- The area of the Heisenberg box remain the same, but its width and height changes

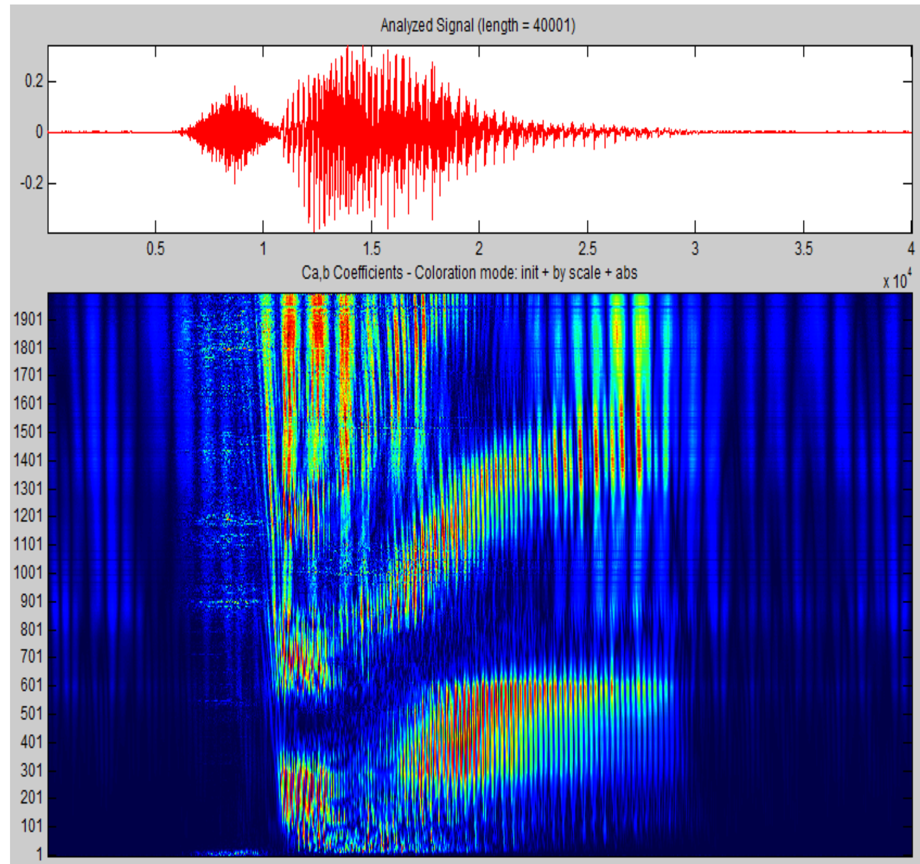
- The wavelet transform of a function $f(t)$ at any scale s and position u is the projection of f on the corresponding wavelet atom

This representation is highly redundant and not optimal for fast implementation

Wavelets

Example: the word “CIAO”





Wavelets

- **Discretization and Orthogonality**
  - It is possible to construct $\psi(t)$ such that the translation and dilation form an orthonormal basis for $\mathbf{L}^2(\mathbb{R})$
  - Discretization using dyadic tree: $s = 2^j \; ; \; u = 2^j k \;\; \forall j, k \in \mathbb{Z}^2$

$$\left\{ \psi_{j,k}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j k}{2^j} \right) \right\}_{j,k \epsilon \mathbb{Z}^2}$$

  - Wavelet expansion in the discrete domain

$$f(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \langle f(t), \psi_{j,k}(t) \rangle \cdot \psi_{j,k}(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} w_{j,k}(t) \cdot \psi_{j,k}(t)$$

  - $w_{j,k}(t)$ are the discrete wavelet transform coefficients
  - Computation of $w_{j,k}(t)$ can be done efficiently, $O(N)$

Wavelets

- **Multiresolution Approximation**
  - Partial sums defined as

$$d_j(t) = \sum_{k=-\infty}^{+\infty} \langle f(t), \psi_{j,k}(t) \rangle \cdot \psi_{j,k}(t)$$

    can be interpreted as the difference between two approximation of $f(t)$ at resolution 2^{-j} and 2^{-j-1}
  - The subspace spanned by $\psi_{j,k}(t)$ at resolution 2^j is indicated with $\mathbf{V}_j$
  - The signal belong to $\mathbf{V}_0$
  - The subspaces $\mathbf{V}_j$ are nested: $\forall\, j \in \mathbb{Z} \quad \mathbf{V}_j \subset \mathbf{V}_{j-1} \subset \mathbf{L}^2(\mathbb{R})$
  - $\mathbf{V}_j$ is always part of $\mathbf{L}^2(\mathbb{R})$: $\mathbf{V}_j \subset \mathbf{L}^2(\mathbb{R})$
  - The orthogonal projection of $f(t)$ in $\mathbf{V}_j$ is $f_j(t) = P_{\mathbf{V}_j} f \in \mathbf{V}_j$
  - $P_{\mathbf{V}_j} f$ is such that $\left\| f - P_{\mathbf{V}_j} f \right\|$ is minimized

Multiresolution Approximation

- Shift Invariant: $f(t) \in \mathbf{V}_j \Leftrightarrow f\left(t - 2^j k\right) \in \mathbf{V}_j \quad \forall\, j,k \in \mathbb{Z}^2$
- Causality: $\mathbf{V}_j \subset \mathbf{V}_{j-1} \subset \cdots \subset \mathbf{V}_1 \subset \mathbf{V}_0 \subset \mathbf{L}^2(\mathbb{R}),\ \forall\, j \in \mathbb{Z}$
- Dilation: $f(2t) \in \mathbf{V}_{j-1} \Leftrightarrow f(t) \in \mathbf{V}_j,\ \forall\, j \in \mathbb{Z}$
- Resolution: $\bigcap_{j\in\mathbb{Z}} \mathbf{V}_j = \{0\}$ *and* $\overline{\bigcup_{j\in\mathbb{Z}} \mathbf{V}_j} = \mathbf{L}^2(\mathbb{R})$.

- Scaling Function:
  $\exists \varphi(t) \in \mathbf{L}^2(\mathbb{R})$ *such that s.t. the family* $\{\varphi_{j,k}(t)\}_{k\in\mathbb{Z}} = \left\{2^{-j/2}\, \varphi\left(\frac{t-k}{2^j}\right)\right\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* $\mathbf{V}_j$.

- Complementary subspace of $\mathbf{V}_j$

$$\mathbf{W}_j = \mathbf{V}_{j-1} - \mathbf{V}_j$$

$$\mathbf{V}_{j-1} = \mathbf{V}_j \oplus \mathbf{W}_j$$

$$P_{\mathbf{V}_{j-1}} f = P_{\mathbf{V}_j} f + P_{\mathbf{W}_j} f$$

Fine approximation

Coarser approximation

Detail

Wavelets

- **Approximations**
  - $P_{\mathbf{V}_j} f$ can be expressed as

$$P_{\mathbf{V}_j} f = \sum_{k \in \mathbb{Z}} a_j(t) \cdot \varphi_{j,k}(t)$$

where $a_j(t) = \langle f(t), \varphi_{j,k}(t) \rangle$

- **Details**
  - $P_{\mathbf{W}_j} f$ can be expressed as

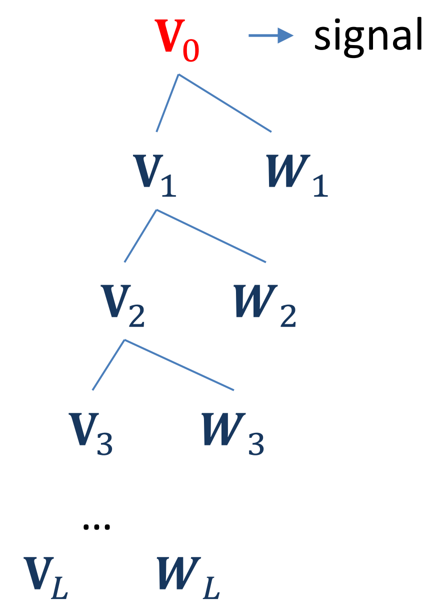
$$P_{\mathbf{W}_j} f = \sum_{k \in \mathbb{Z}} d_j(t) \cdot \psi_{j,k}(t)$$

where $d_j(t) = \langle f(t), \psi_{j,k}(t) \rangle$

$$\mathbf{L}^2(\mathbb{R}) \supset \mathbf{V}_0 = \mathbf{V}_L \oplus \mathbf{W}_L \oplus \mathbf{W}_{L-1} \oplus \mathbf{W}_{L-2} \oplus \ldots \mathbf{W}_1$$

$$f(t) = \sum_{k \in \mathbb{Z}} a_L(t)\, \varphi_{L,k}(t) + \sum_{j<L} \sum_{k \in \mathbb{Z}} d_j(t) \cdot \psi_{j,k}(t)$$

L depends on the signals and sets the coarsest scale →



Wavelets

- **Fast Orthogonal Wavelet Transform**
  - There is a connection between wavelets and filter banks

$$a_{j+1}(k) = \sum_{n\in\mathbb{Z}} h(n-2k)\, a_j(n) = a_j(k) * h(-2k) = a_j(k) * \overline{h(2k)} = H\, a_j$$

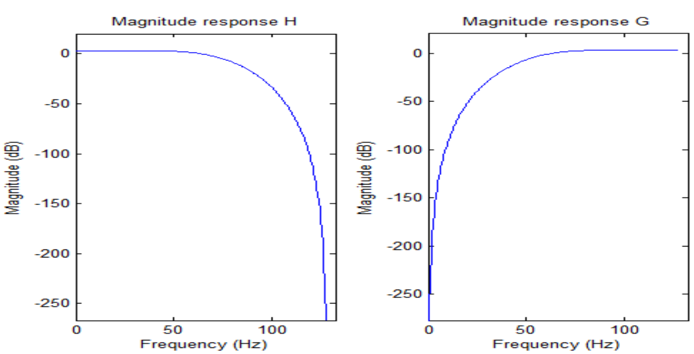
$$d_{j+1}(k) = \sum_{n\in\mathbb{Z}} g(n-2k)\, d_j(n) = d_j(k) * g(-2k) = d_j(k) * \overline{g(2k)} = G\, a_j$$

Reconstruction

$$a_j(k) = \sum_{n\in\mathbb{Z}} h(k-2n)\, a_{j+1}(n) + \sum_{n\in\mathbb{Z}} g(k-2n)\, d_{j+1}(n)$$

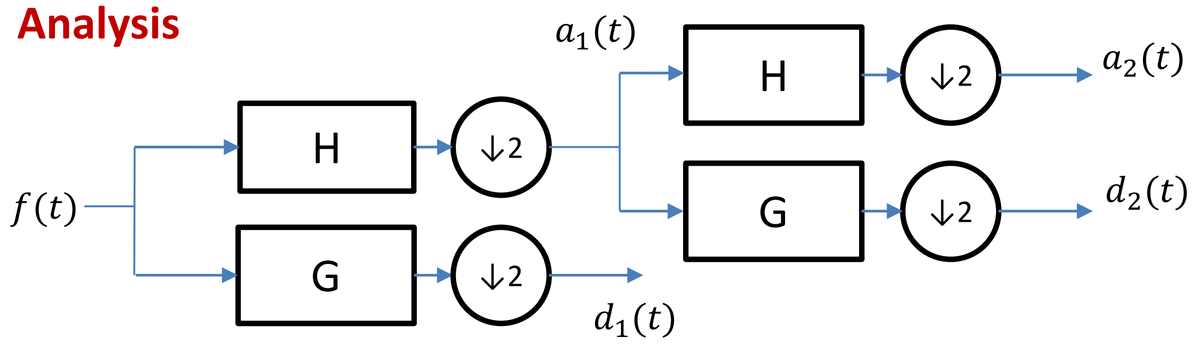
H and G are a pair of conjugated mirror filters

$$g(k) = (-1)^k h(N-k)$$

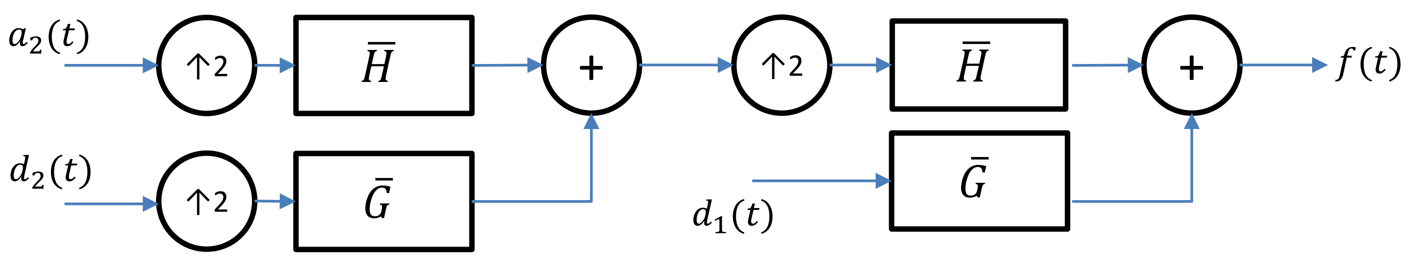


Wavelets

Analysis



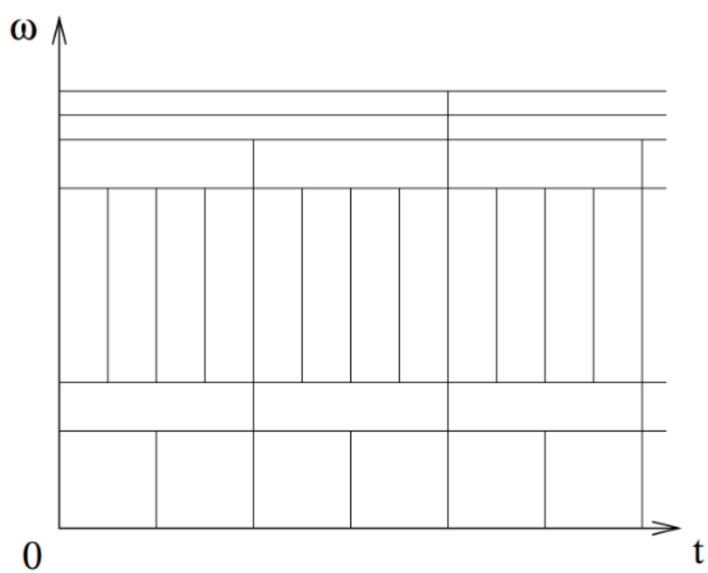
Synthesis

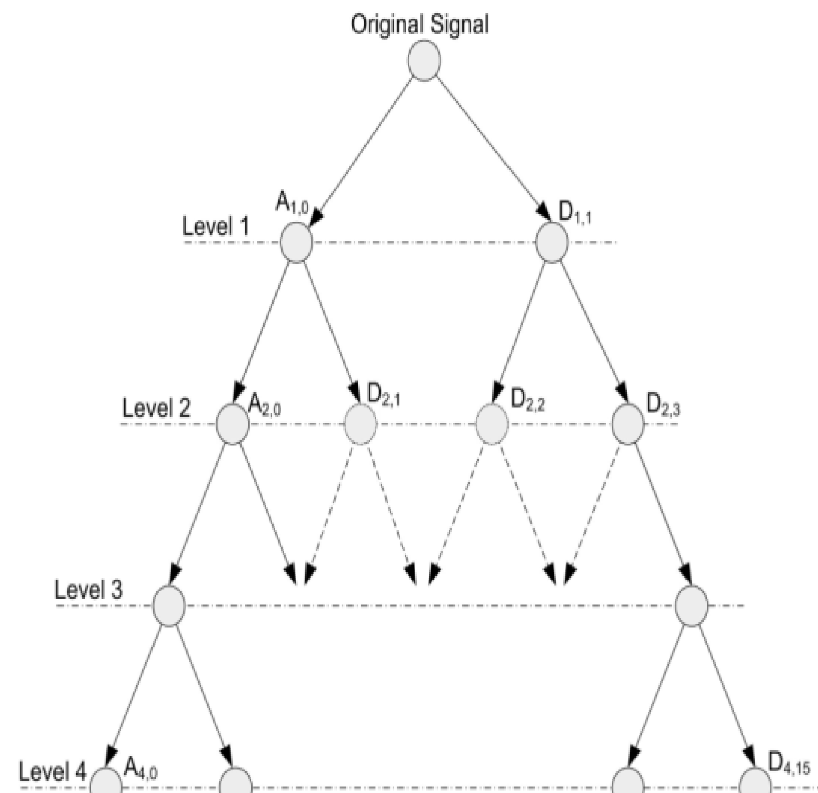


Wavelets

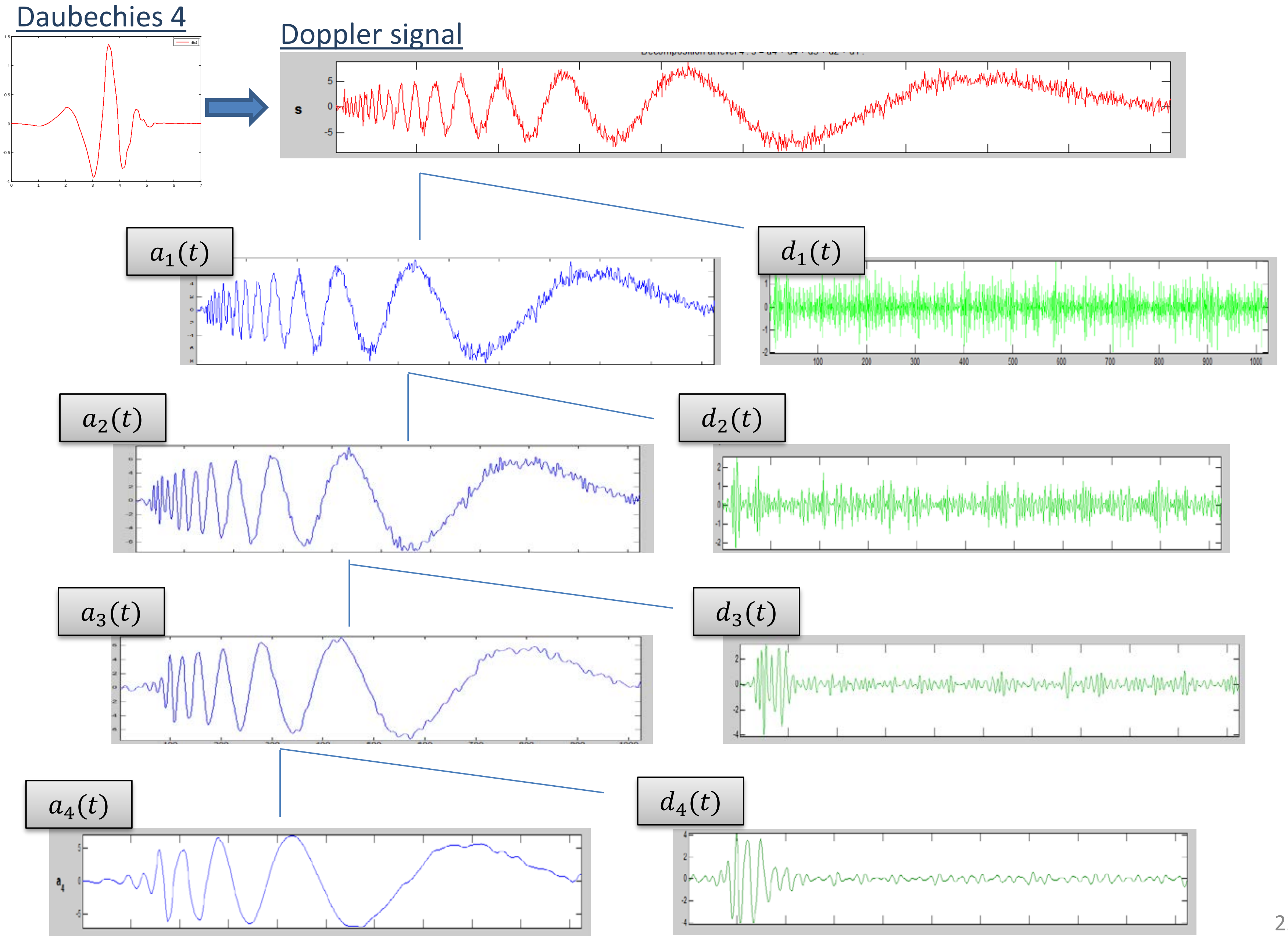
- **Wavelet Packets**
  - Wavelet packets are an extension of traditional wavelets
  - The details are also decomposed using H and G
  - They provide a complete tiling of the time-frequency plane

$$
\begin{cases}
\vartheta_{j+1}^{2n}(t) = 2^{j/2} \sum_{\ell \in \mathbb{Z}} c(\ell)\, \vartheta_j^n\left(2^{j/2} t - k\right) \\
\vartheta_{j+1}^{2n+1}(t) = 2^{j/2} \sum_{\ell \in \mathbb{Z}} d(\ell)\, \vartheta_j^n\left(2^{j/2} t - k\right)
\end{cases}
$$

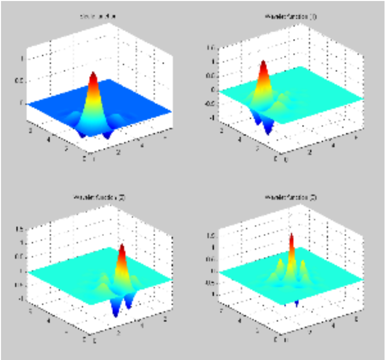


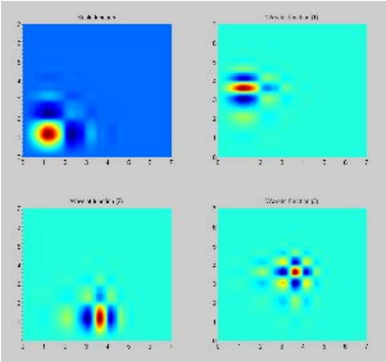


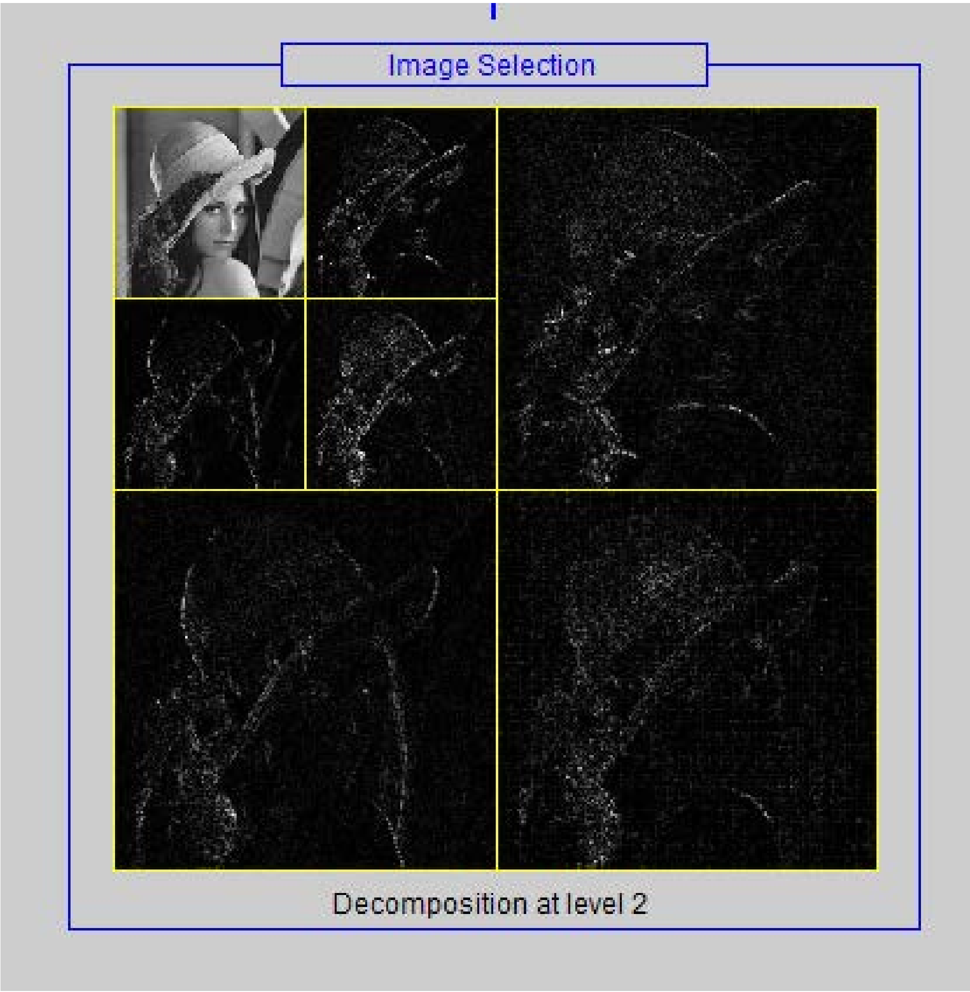
Example 1D



Lena image decomposition



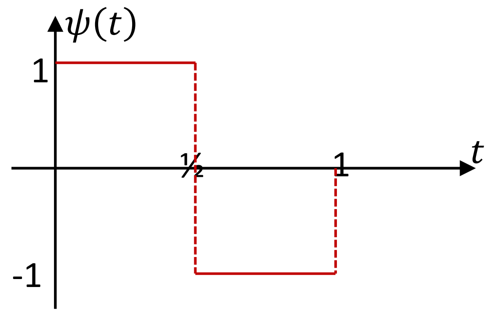




Wavelets

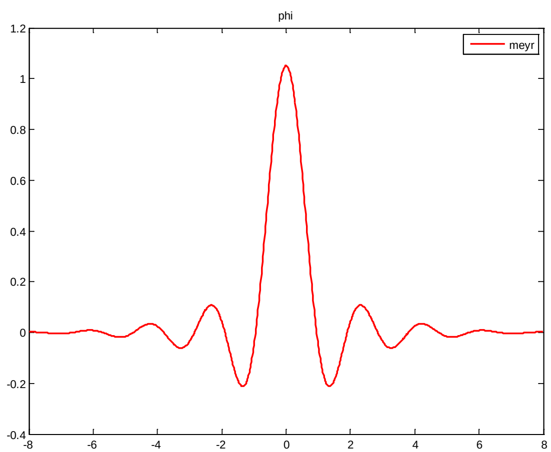
- **1st wavelet: HAAR**
  - In 1910 Haar constructed a piecewise constant function:

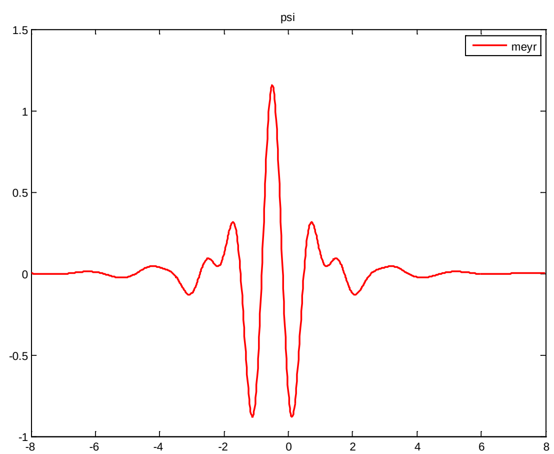
$$\psi(t) = \begin{cases} 1 & t \in [0, 1/2) \\ -1 & t \in [1/2, 1) \\ 0 & otherwise \end{cases}$$



Alfréd Haar (1885 – 1933)
http://gtwavelet.bme.gatech.edu/images/haar.html

- **Meyer wavelet**





Frequency band-limited function with smooth Fourier transform.

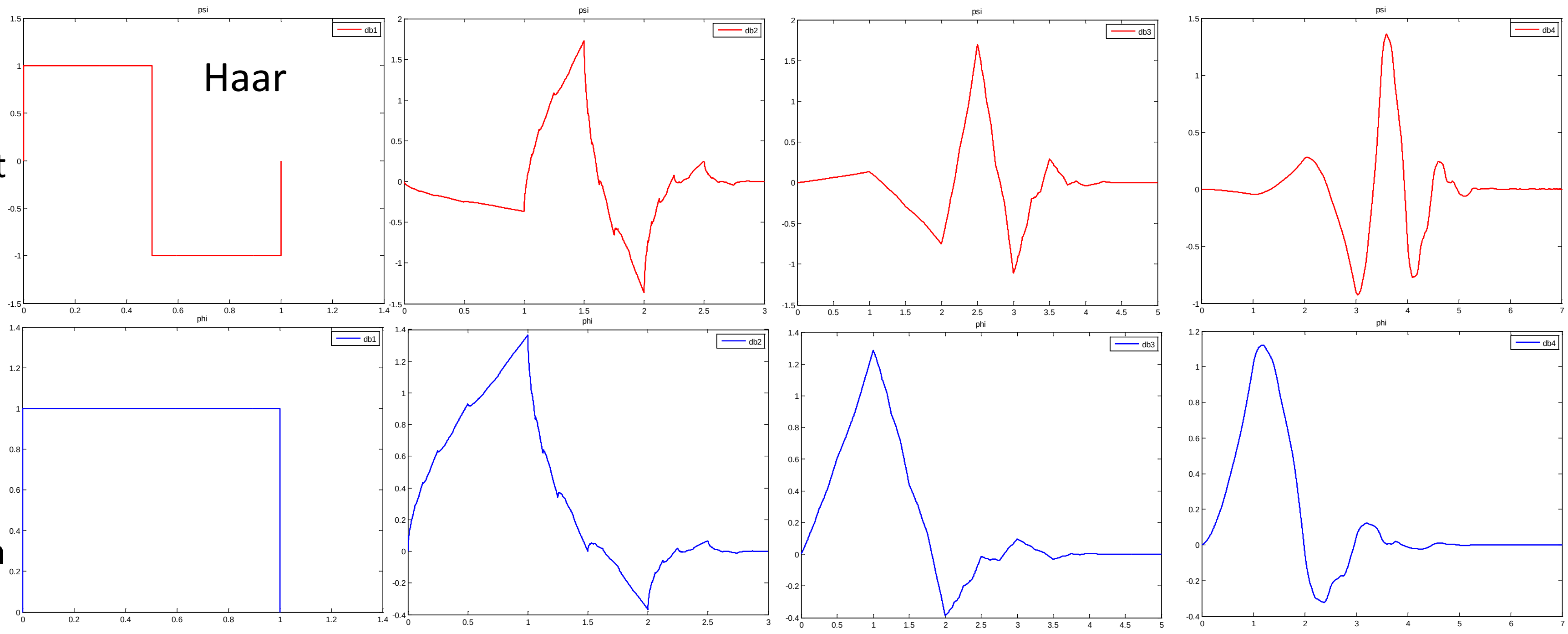
Wavelets

- **Daubechies Wavelet Family**

Daubechies wavelets have finite support of minimum size for any given order.
Daubechies's are orthogonal wavelets

Wavelet

Scaling function



p=1

Support = 2p-1 = 1

p=2

Support = 3

p=3

Support = 5

p=4

Support = 7

Applications

- **Denoising**

Noise model

$f(t) = g(t) \oplus w(t)$

Denoising

Estimate $\hat{f}(t)$ *such that* $\|\hat{f}(t) - g(t)\|$ is minimized

$$f(t) = \sum_{j=-\infty}^{+\infty} \sum_{|\langle f,\psi_{j,k}\rangle| > THR} \langle f(t), \psi_{j,k}(t)\rangle \cdot \psi_{j,k}(t)$$

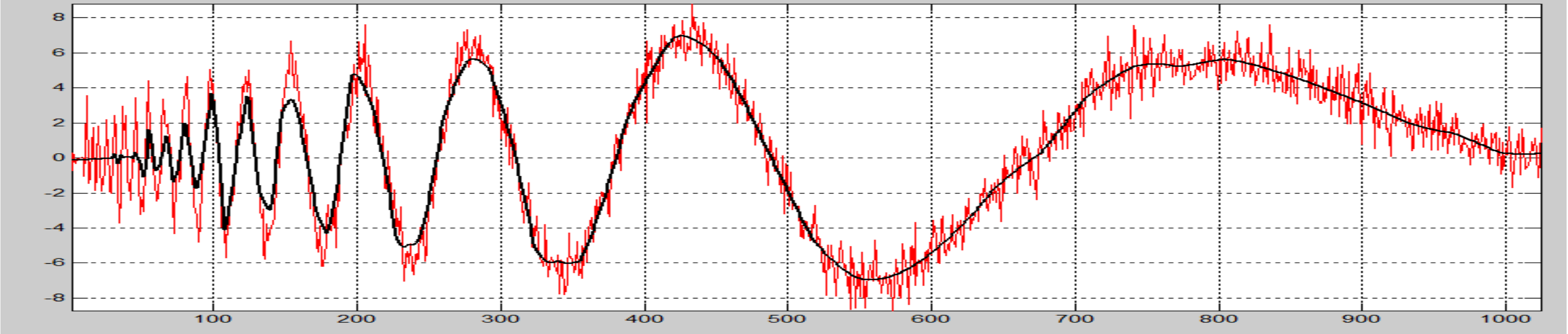
Global Threshold

$$THR = \sqrt{2\log(N)}\sigma$$

Good Threshold

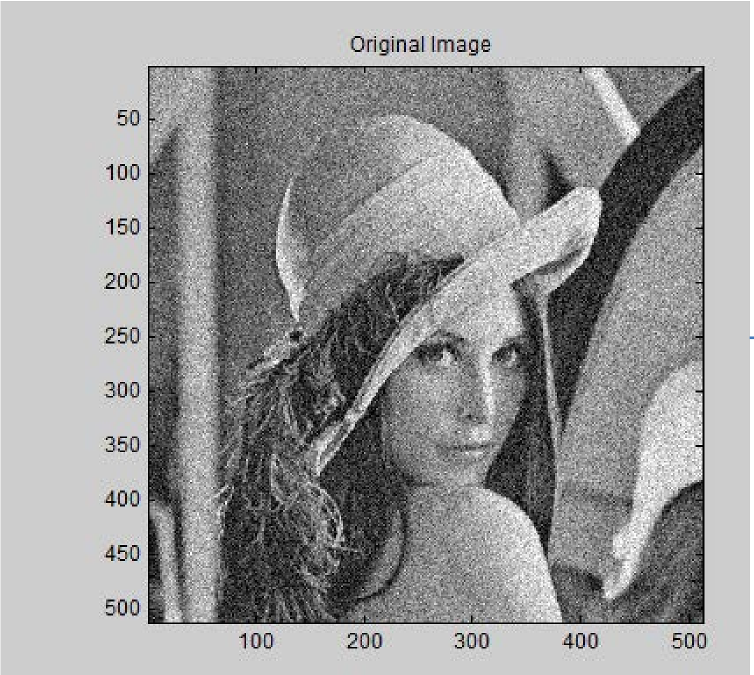
$$THR \approx 3\sigma$$

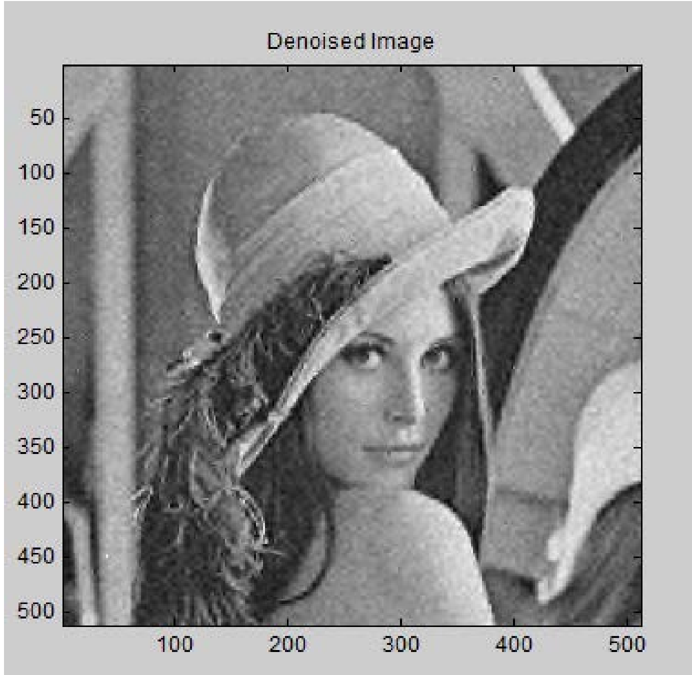
1D Signal Denoising

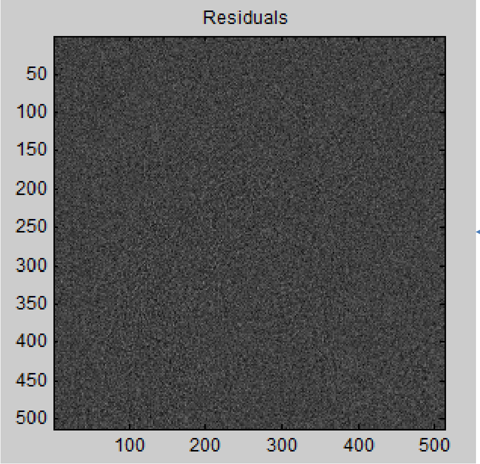


Applications

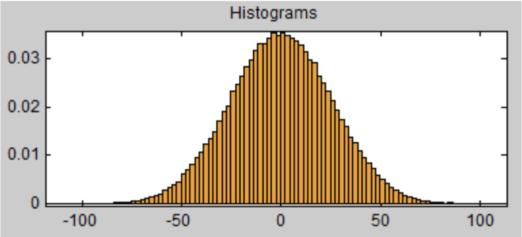
2D Signal Denoising







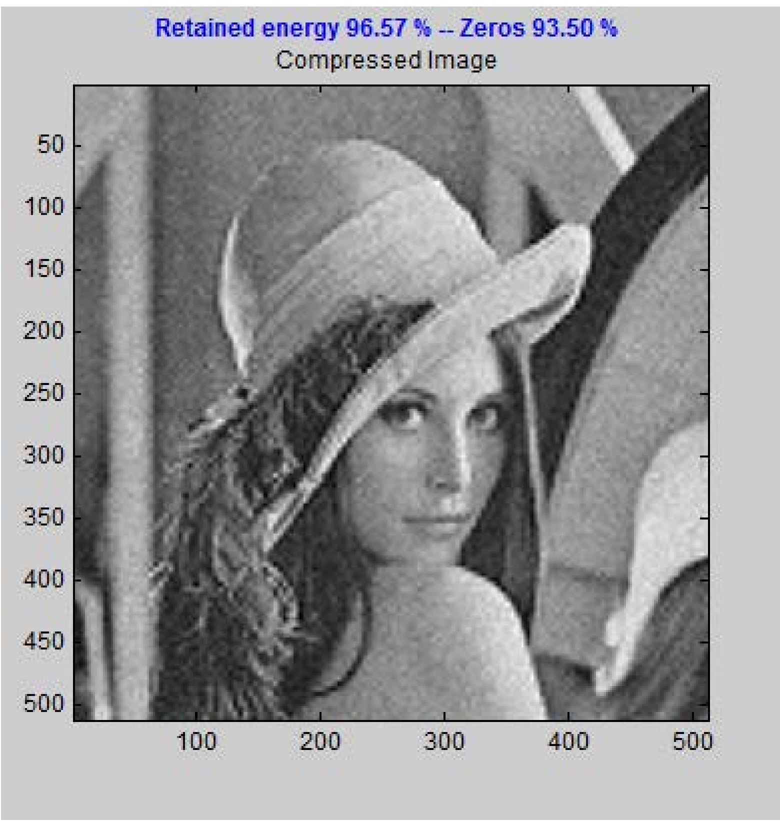
Residuals



Applications

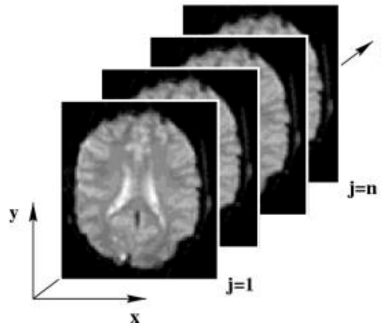
- **Compression**

Retain only the coefficients that carry information in the transform domain

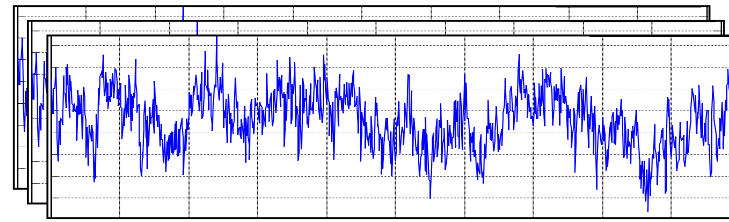


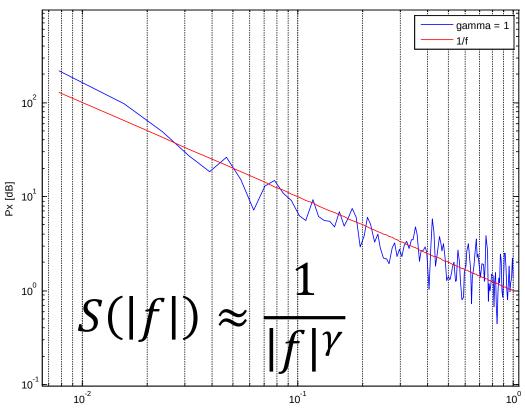
Application - Neuroimaging

- **Clustering of functional MRI data**

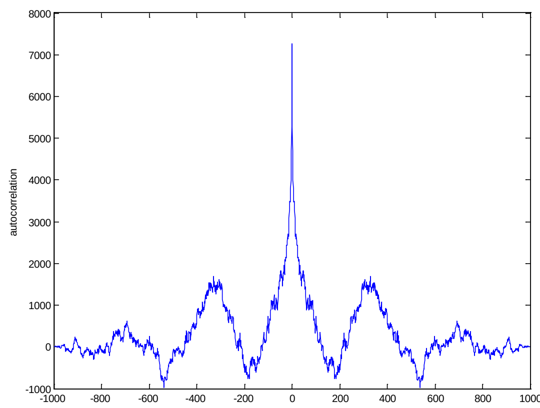


Voxels Structure

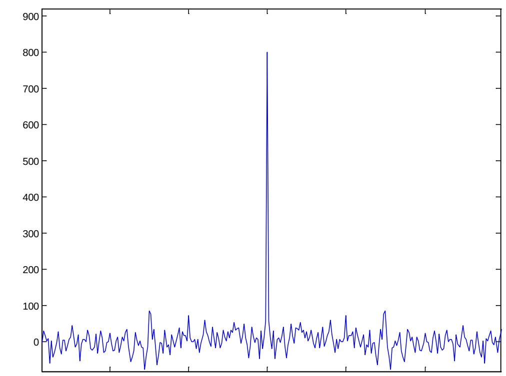




$$S(|f|) \approx \frac{1}{|f|^{\gamma}}$$

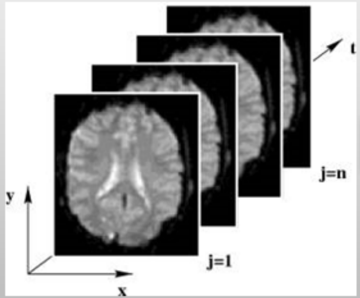


Wavelet Transform

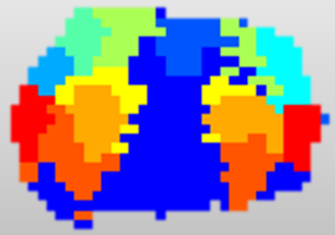


- Neurological time series are characterized by long range autocorrelation functions and often exhibit fractal properties in the time domain
- Wavelets represent a natural basis for the analysis tool for $1/f$ – type processes

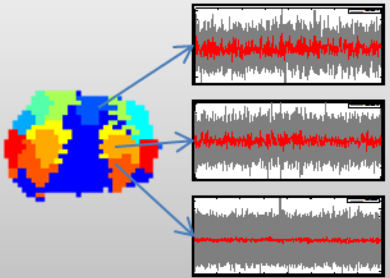
Application - Neuroimaging



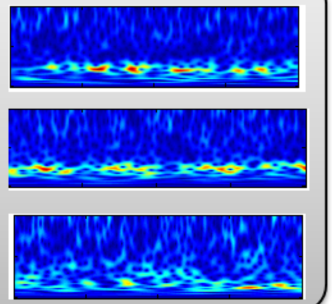
BOLD Signal



Functional Clustering



Representative Signal Extraction



Time-frequency Dynamics

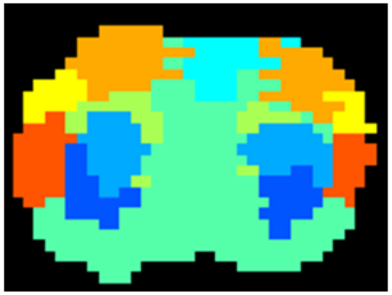
Feature Extraction

$$F_i \equiv \begin{cases} w_j\ (k) = \langle vox^i(t), \psi_j\ (2^j t - k)\rangle \\ w_j^p(k) = \left\langle vox^i(t), \psi_j^p(2^j t - k)\right\rangle \end{cases}$$

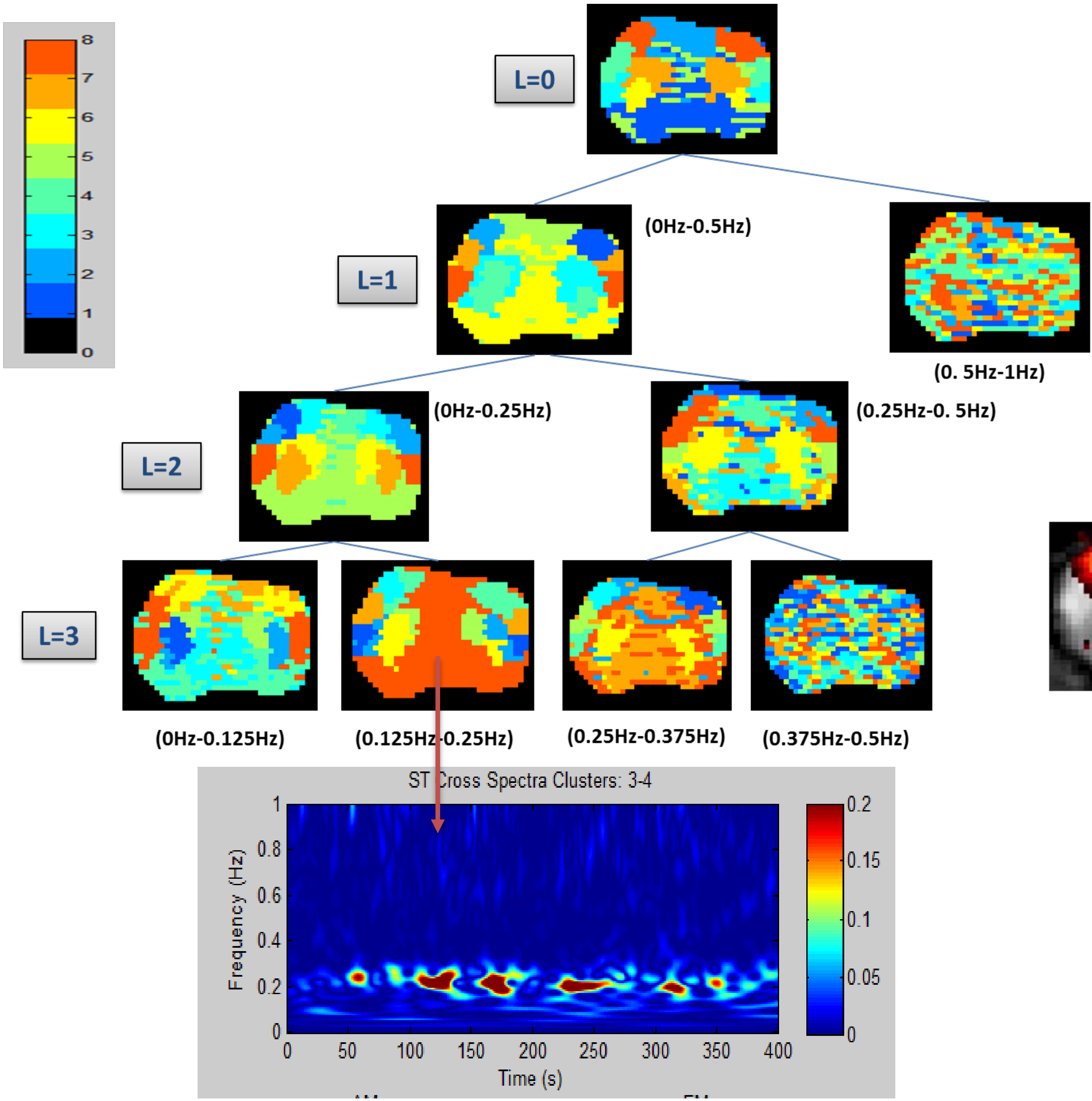
$i = 1, \dots, N$

Agglomerative clustering using Ward's method

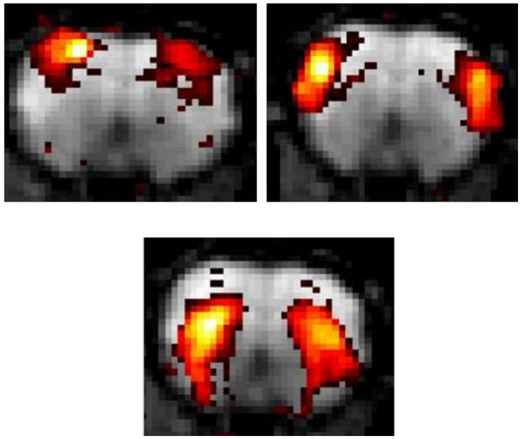
Minimizes the total within cluster variance



Application - Neuroimaging



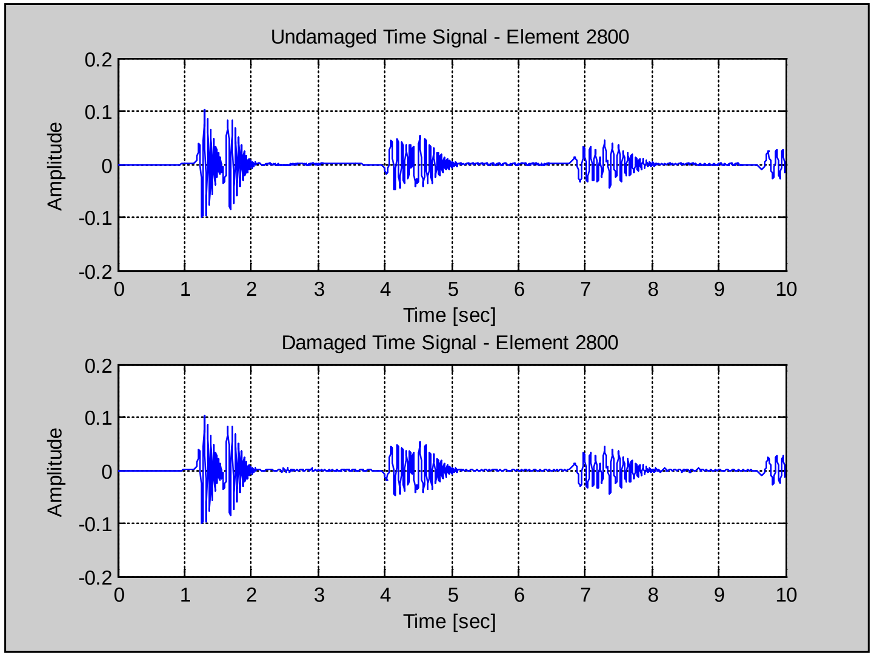
SeedBased



Application – Structural Health Monitoring

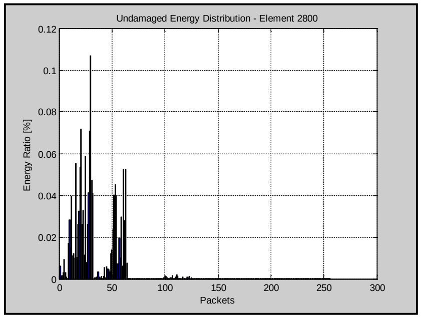
Structural damage

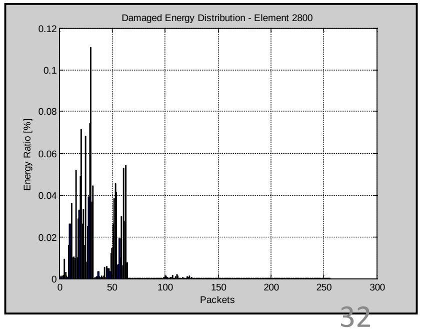
- Excessive stress
- Traffic and natural induced vibration
- Time factor: age of the structure
- Cracks generate interfaces that originate reflection boundaries in the structure



Wavelet Packet Energy

$$E^{\%}_{j,n} = \frac{\int f_j^n(t)^2\,dt}{\int f(t)^2\,dt}$$





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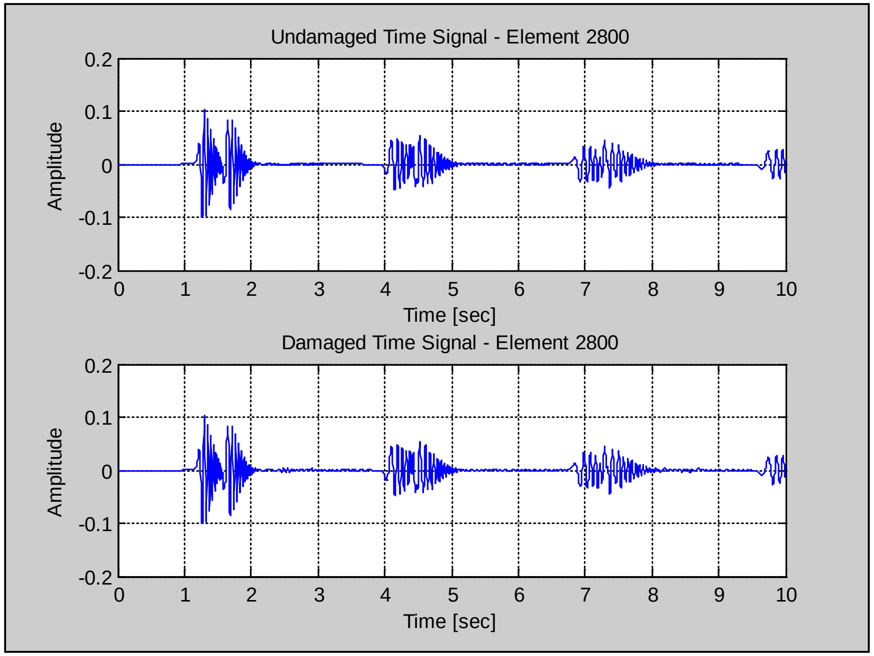

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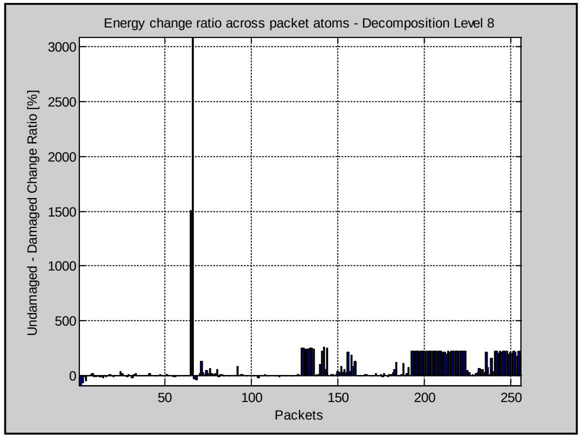
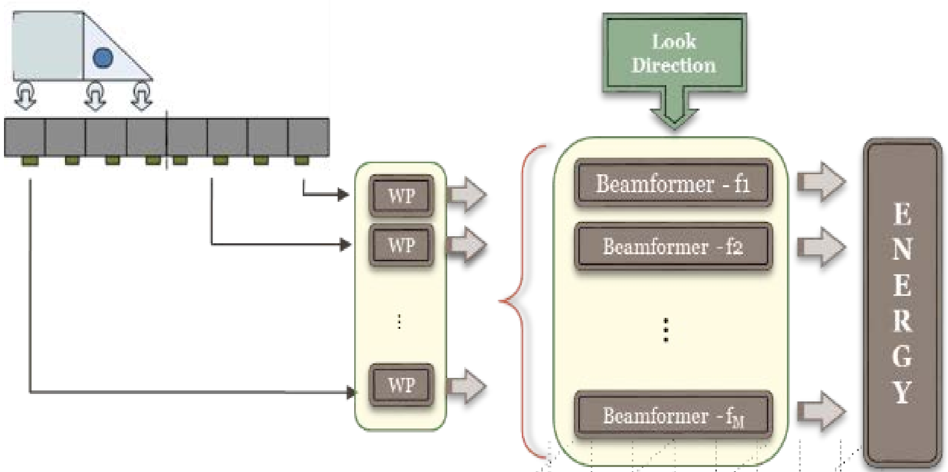
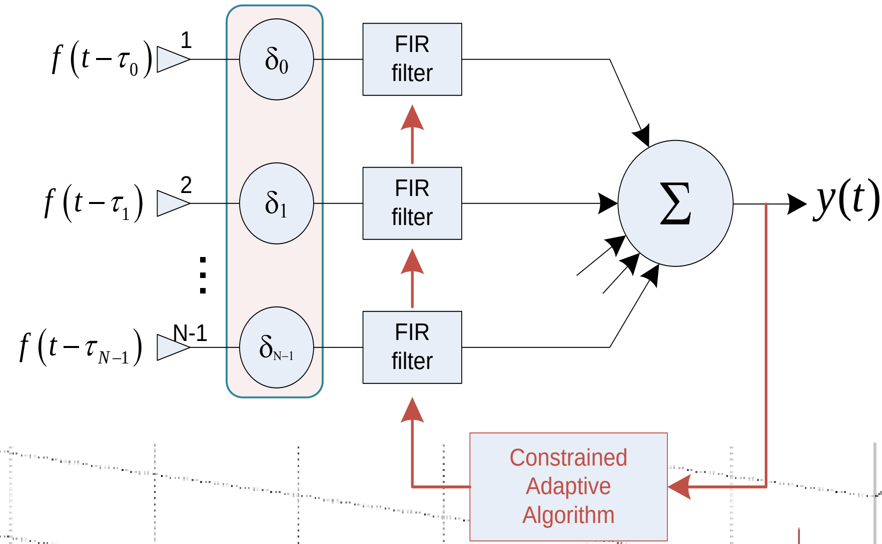
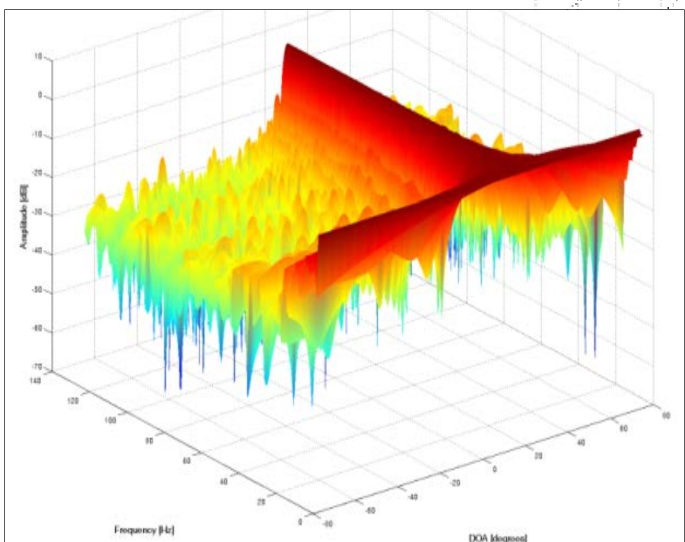
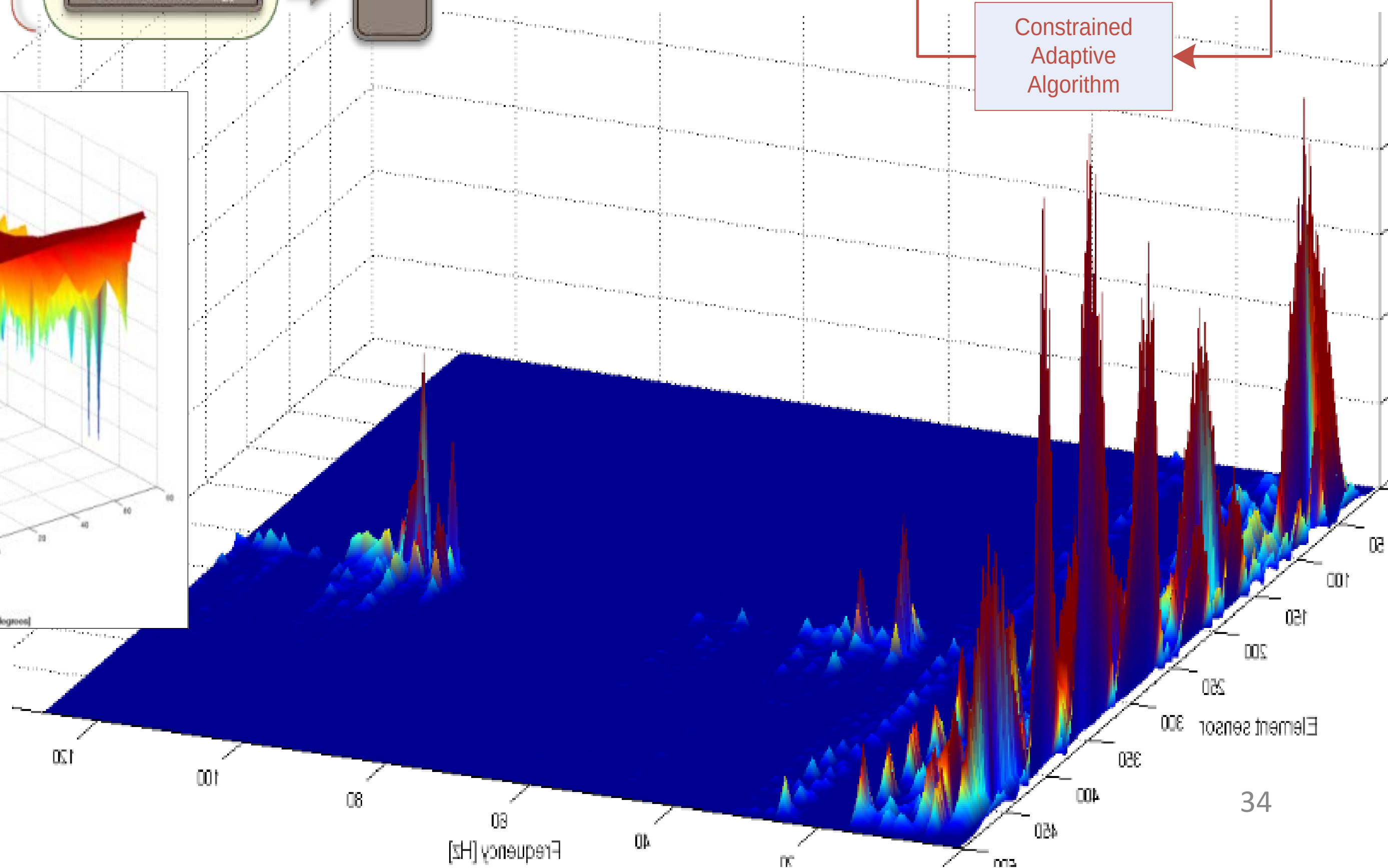


Application – Structural Health Monitoring







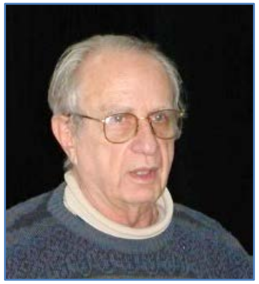


Conclusion - a little bit of history…

- 1807- J.-B. Fourier
  "any periodic function can be represented as a series of harmonically related sinusoids"
- 1909 – Alfred Haar
  First simplest orthogonal wavelet
- 1946 – Dennis Gabor
  Windowed Fourier Transform and time-frequency atoms
- 1970s – Jean Morlet's problem
  Application of variable length window to variable signals in geophysics leads to wavelets
- 1980s – Alexander Grossman
  Formalization of wavelet transform
- 1985 – Yeves Meyer
  Orthogonal wavelet basis function
- 1980s – Ingrid Daubechies
  Discretization of the wavelet transform; Wavelet frames; Compactly supported wavelets
- 1980s – Stephan Mallat
  Multiresolution Approximations; Discrete wavelet Transform; Cascade algorithm;
- 1980s – Martin Vetterli
  Wavelets and Filter banks; Perfect reconstruction; Subband coding; Multidimensional filter banks
- 1996 – Coifman, Meyer, and Wickerhauser
  Wavelet Packets

Conclusion - a little bit of history…

J.-B. Fourier



Alex Grossman

Ingrid Daubechies

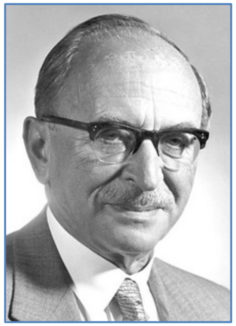
Stephan Mallat

Alfred Haar

Yeves Meyer

Martin Vetterli

Ronald Coifman

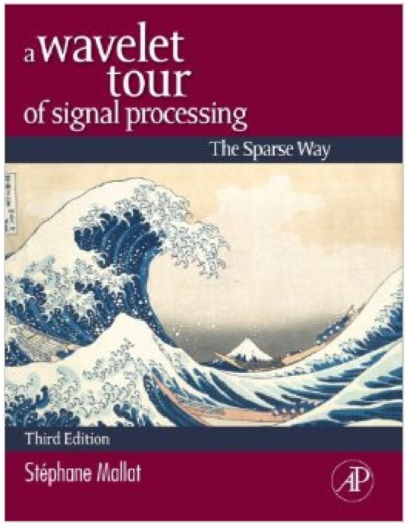


Dennis Gabor

Jean Morlet

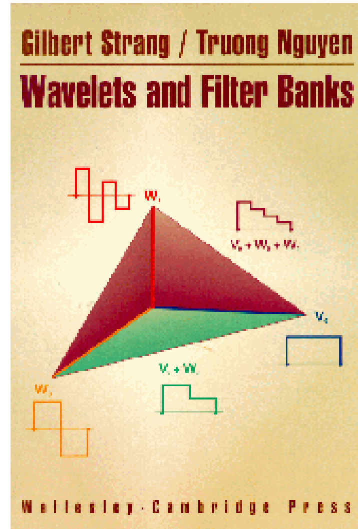
Mladen Victor Wickerhauser

Resources - books



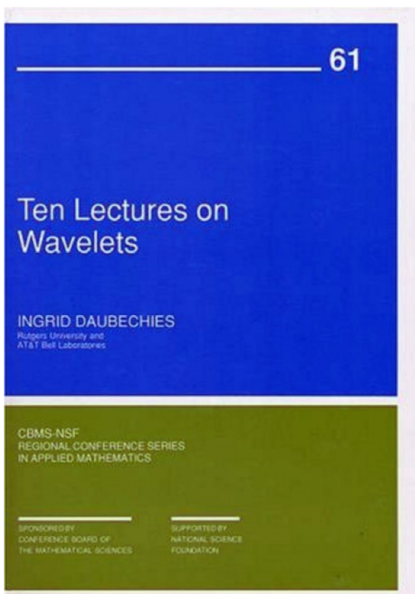
S. Mallat
“A Wavelet Tour of Signal Processing”

Academic Press



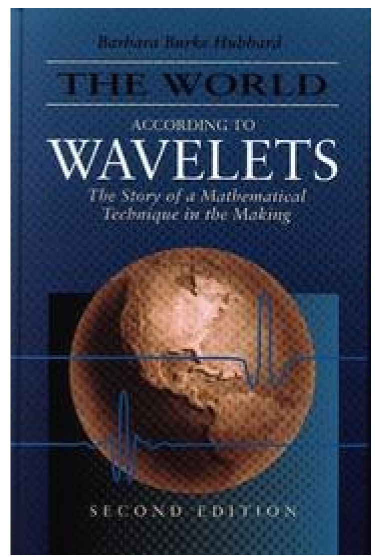
G. Strang and T. Nguyen
“Wavelets and Filter Banks”

Wellesley-Cambridge Press



I. Daubechies
“Ten Lectures on Wavelets”

SIAM



B.B. Hubbard
“The World According to Wavelets”

CRC Press