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Outer space for untwisted automorphisms of right-angled Artin groups

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For a right-angled Artin group A_Γ, the untwisted outer automorphism group $U(A_\Gamma)$ is the subgroup of $\mathrm{Out}(A_\Gamma)$ generated by all of the Laurence–Servatius generators except twists (where a *twist* is an automorphism of the form $v \mapsto vw$ with $vw = wv$). We define a space Σ_Γ on which $U(A_\Gamma)$ acts properly and prove that Σ_Γ is contractible, providing a geometric model for $U(A_\Gamma)$ and its subgroups. We also propose a geometric model for all of $\mathrm{Out}(A_\Gamma)$, defined by allowing more general markings and metrics on points of Σ_Γ.

20F65; 20F28, 20F36

1 Introduction

A free group is defined by giving a set of generators with no relations; in particular, none of the generators commute. A free abelian group is defined by giving a set of generators which all commute, and no other relations. Finitely generated free and free abelian groups are examples of *right-angled Artin groups* (RAAGs for short): a general RAAG is defined by giving a finite set of generators, *some* of which commute, and no other relations. A convenient way of describing a RAAG is by drawing a graph Γ with one vertex for each generator and an edge between each pair of commuting generators; the resulting RAAG is denoted by A_Γ. RAAGs and their subgroups are important sources of examples and counterexamples in geometric group theory (see Charney [3] for a survey) and have recently played a key role in the solution of Thurston's conjectures on the structure of hyperbolic 3–manifolds (see Agol [1]).

Automorphism groups of RAAGs have received less attention, with the notable exception of $A_\Gamma = \mathbb{Z}^n$ and $A_\Gamma = F_n$. Since it is easy to determine the center of any A_Γ, the inner automorphisms of A_Γ are well understood, so it remains to study the outer automorphism group $\mathrm{Out}(A_\Gamma)$. The groups $\mathrm{Out}(F_n)$ and $\mathrm{Out}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Z})$ have been shown to have many features in common, and it is natural to ask whether these features are in fact shared by all of $\mathrm{Out}(A_\Gamma)$. On the other hand, there are important

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differences between $\mathrm{GL}(n,\mathbb{Z})$ and $\mathrm{Out}(F_n)$ — such as the fact that $\mathrm{Out}(F_n)$ is not linear! — so we are also interested in how the structure of Γ affects the group-theoretic properties of $\mathrm{Out}(A_\Gamma)$.

In previous work we have explored properties of $\mathrm{Out}(A_\Gamma)$ using inductive local-to-global ideas, based ultimately on the fact that an outer automorphism of A_Γ must send certain special subgroups $A_{\mathrm{st}[v]}$ to conjugates of themselves; see Bux, Charney and Vogtmann [2], Charney and Vogtmann [5; 6] and Charney, Crisp and Vogtmann [4]. In this paper we take a more uniformly global approach by introducing an "outer space" for any A_Γ, which should play the role of the symmetric space $Q_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ in the study of $\mathrm{SL}(n,\mathbb{Z})$ or of the outer space $\mathcal{O}_n$ in the study of $\mathrm{Out}(F_n)$, ie it should be a contractible finite-dimensional space with a proper action of $\mathrm{Out}(A_\Gamma)$.

Among the many possible ways of defining Q_n and $\mathcal{O}_n$ are as spaces of free cocompact actions (of $\mathbb{Z}^n$ on $\mathbb{R}^n$ or of F_n on simplicial trees) or as spaces of marked metric spaces (flat tori with fundamental group $\mathbb{Z}^n$ or metric graphs with fundamental group F_n, equipped with a homotopy equivalence called a *marking* to a fixed torus or graph). All A_Γ act freely and cocompactly on CAT(0) cube complexes, so it is natural to try to define outer space in general in the first way, as a space of actions. In [4] we were motivated by this idea but were unable to prove contractibility of any appropriate space of CAT(0) actions; instead we looked at local data one would obtain from such an action and defined a point of outer space to be such a data set (whether or not it actually comes from an action). This trick was successful for RAAGs defined by graphs which contain no triangles (called 2–dimensional RAAGs), but the methods do not generalize to higher dimension.

In this paper we take the second approach, defining a space whose points are marked metric spaces. For every RAAG A_Γ, there is a standard minimal nonpositively curved cube complex with fundamental group A_Γ, called the *Salvetti complex*. We build our space out of somewhat more general cube complexes called Γ*–complexes*; these are homotopy equivalent to Salvetti complexes via an elementary operation called *hyperplane collapse*. For $A_\Gamma = \mathbb{Z}^n$ the only Γ–complex is the Salvetti complex, which is an n–torus. For $A_\Gamma = F_n$ the Salvetti complex is a rose with n petals, Γ–complexes are graphs with no univalent or bivalent vertices or separating edges and hyperplane collapse amounts to collapsing a maximal tree.

We restrict attention to the subgroup $U(A_\Gamma)$ of $\mathrm{Out}(A_\Gamma)$ generated by inversions, graph automorphisms, partial conjugations and transvections of the form $v \mapsto vw$, where v and w are generators of A_Γ with $vw \neq wv$. By a theorem of Laurence and Servatius these generate $\mathrm{Out}(A_\Gamma)$ unless there are vertices v and w with $\mathrm{st}(v) \subseteq \mathrm{st}(w)$, in which case we must also add *twists* sending $v \mapsto vw = wv$. We refer to $U(A_\Gamma)$ as the *untwisted* subgroup of $\mathrm{Out}(A_\Gamma)$. One reason for interest in this subgroup, even when it is not equal to all of $\mathrm{Out}(A_\Gamma)$, is that for any Γ the kernel IA_Γ of the natural map $\mathrm{Out}(A_\Gamma) \to \mathrm{GL}(n, \mathbb{Z})$ is contained in $U(A_\Gamma)$.

We build a simplicial complex K_Γ on which $U(A_\Gamma)$ acts properly and cocompactly, so that K_Γ is quasi-isometric to $U(A_\Gamma)$ by the Schwarz–Milnor lemma. The vertices of K_Γ are marked Γ–complexes, where a marking is an "untwisted" homotopy equivalence to a fixed Salvetti complex. Two vertices are joined by an edge if one is obtained from the other by a hyperplane collapse, and the resulting graph is then completed to a flag complex to form K_Γ. Our main theorem is:

Theorem 6.24 *For any right-angled Artin group* A_Γ, *the complex* K_Γ *is contractible.*

As an immediate corollary we obtain:

Corollary *The dimension of* K_Γ *is an upper bound for the virtual cohomological dimension* (VCD) *of the untwisted subgroup* $U(A_\Gamma)$ *of* $\mathrm{Out}(A_\Gamma)$.

The dimension of K_Γ is always finite. In Section 5 we compute this dimension for a few examples of 2–dimensional RAAGs. Upper and lower bounds for the VCD of $\mathrm{Out}(A_\Gamma)$ when A_Γ is 2–dimensional were computed in Bux, Charney and Vogtmann [2] but these bounds seldom agree. Our computations of the dimension of K_Γ improve on the bounds given in [2] and, in some cases, determine the exact VCD of $U(A_\Gamma)$.

We can obtain a larger space Σ_Γ by equipping Γ–complexes with metrics in which the cubes are rectilinear parallelepipeds. The simplicial complex K_Γ then embeds as an equivariant deformation retract of Σ_Γ. For a free group, $U(A_\Gamma) = \mathrm{Out}(A_\Gamma)$, Σ_Γ is the usual (reduced) outer space $\mathcal{O}_n$ and K_Γ is the spine of outer space. For a free abelian group of rank n, $U(A_\Gamma)$ is the finite group of signed permutation matrices, Σ_Γ is the positive orthant in $\mathbb{R}^{n-1}$ and K_Γ is a single point.

To get a space on which the entire group $\mathrm{Out}(A_\Gamma)$ acts, we allow the markings to be arbitrary homotopy equivalences and the metrics on Γ–complexes to be "twisted" in certain restricted ways. This is discussed briefly in the last section of the paper.

Our description of marked Γ–complexes and proof of contractibility of K_Γ are modeled on Culler and Vogtmann's original proof [7] that outer space for a free group is contractible. The idea is that K_Γ is the union of the simplicial stars of the marked Salvettis, and we assemble all of K_Γ by attaching these stars one at a time, making sure that at each stage we are gluing along a contractible subcomplex. The order in which we attach the stars is determined by a Morse function which measures the lengths of conjugacy classes of A_Γ under the marking of the Salvetti. The proof that the subcomplexes along which we glue are contractible requires understanding how this Morse function changes under basic automorphisms; this depends on a generalization of the classical peak reduction algorithm for free groups. A version of peak reduction for RAAGs was established by M Day [8; 9]. We require a stronger version (see Theorems 6.18 and 6.19) and give an independent proof.

We will make use of the standard notions of nonpositively curved cube complexes and hyperplanes, and we refer the reader to Haglund and Wise [10] for these concepts.

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2 Whitehead automorphisms

In this section we recall some basic facts about right-angled Artin groups and their automorphisms. Fix a right-angled Artin group A_Γ with generating set $V = \text{vertices}(\Gamma)$.

2.1 Partial orders

Recall from [5] that the relation $\text{lk}(v) \subseteq \text{st}(w)$ for v and w vertices of Γ is denoted by $v \le w$. Vertices are called *equivalent* if $v \le w$ and $w \le v$, and we write $v \sim w$. (The justification for this notation is that $\le$ is a partial order on equivalence classes of vertices.) If v is adjacent to w then $v \le w$ if and only if $\text{st}(v) \subseteq \text{st}(w)$, and if v is not adjacent to w then $v \le w$ if and only if $\text{lk}(v) \subseteq \text{lk}(w)$.

When considered as elements of A_Γ, each element $v \in V$ has an inverse v^{-1}, and we will often work with the symmetric set $V^\pm = \{v, v^{-1} \mid v \in V\}$. For $x,\ y \in V^\pm$ we say $x \le y$ and so on if the corresponding vertices of Γ satisfy the relation.

2.2 Generators for $\text{Out}(A_\Gamma)$

Laurence [12] and Servatius [14] proved that the following simple types of automorphisms generate all of $\text{Aut}(A_\Gamma)$ (and hence their images generate $\text{Out}(A_\Gamma)$):

(1) An automorphism of the graph Γ permutes the vertices V and induces an automorphism of A_Γ, called a *graph automorphism*.

(2) If $v \in V$, the map sending $v \mapsto v^{-1}$ and fixing all other generators is an automorphism of A_Γ, called an *inversion*.

(3) If $v \leq w$, then the map sending $v \mapsto vw$ and fixing all other generators is an automorphism of A_Γ, called a *transvection*. If v is adjacent to w this is called a *twist*, and if not it is called a *fold*.

(4) If C is a component of $\Gamma \setminus \mathrm{st}(v)$, then the map sending $x \to vxv^{-1}$ for every vertex x of C and fixing all other generators is an automorphism of A_Γ, called a *partial conjugation*.

Twists play a specialized role in the study of $\mathrm{Out}(A_\Gamma)$, and we define $\mathrm{T}(A_\Gamma)$ to be the subgroup of $\mathrm{Out}(A_\Gamma)$ generated by these. We then define $\mathrm{U}(A_\Gamma)$ to be the subgroup generated by all other types of generators, ie graph automorphisms, inversions, partial conjugations and folds. In the terminology of [8], elements of $\mathrm{U}(A_\Gamma)$ are called *long-range* automorphisms. The subgroup $\mathrm{U}(A_\Gamma)$ will be the focus of the rest of the paper.

2.3 Γ–Whitehead partitions

There is a larger generating set for $\mathrm{U}(A_\Gamma)$ which is more natural for our purposes. This larger set includes simple combinations of folds and partial conjugations; it consists of automorphisms of A_Γ which are induced by Whitehead automorphisms of the free group $\mathrm{F}(V)$. We recall that a Whitehead automorphism of $\mathrm{F}(V)$ is determined by a pair (P, m), where $P \subset V^{\pm}$ has at least 2 elements, and $m \in P$ with $m^{-1} \notin P$. The automorphism $\varphi = (P, m)$ is given by

$$\varphi(v) = \begin{cases} m^{-1} & \text{if } v = m, \\ vm^{-1} & \text{if } v \in P \text{ and } v^{-1} \notin P, \\ mv & \text{if } v^{-1} \in P \text{ and } v \notin P, \\ mvm^{-1} & \text{if } v, v^{-1} \in P, \\ v & \text{otherwise.} \end{cases}$$

We remark that it is more usual to define a Whitehead automorphism with $\varphi(m) = m$, but, as we will see in Lemma 3.2 below, setting $\varphi(m) = m^{-1}$ corresponds more naturally with the geometric version of a Whitehead move. (This was originally observed by Hoare [11]). With this definition, φ is an involution, so $\varphi = \varphi^{-1}$. Replacing P by its complement P^* and m by m^{-1} changes φ by an inner automorphism (conjugation by m). Thus, (P, m) and (P^*, m^{-1}) determine the same outer automorphism.

Not every Whitehead automorphism of $\mathrm{F}(V)$ induces an automorphism of A_Γ, and even if it does the induced automorphism may be trivial (eg conjugating v by an adjacent w). Both of these problems are solved by the following definition:

Definition 2.1 Let $P \subset V^{\pm}$ have at least 2 elements, including some $m \in P$ with $m^{-1} \notin P$. Then (P, m) is a Γ–*Whitehead pair* if

(1) no element of P is adjacent to m;

(2) if $x \in P$ and $x^{-1} \notin P$, then $x \leq m$; and

(3) if $v, v^{-1} \in P$ then w, $w^{-1} \in P$ for all w in the same component as v of $\Gamma \setminus \mathrm{st}(m)$.

Lemma 2.2 *If (P, m) is a Γ–Whitehead pair, then the Whitehead automorphism of $\mathrm{F}(V)$ defined by (P, m) induces a nontrivial automorphism of A_Γ. The induced outer automorphism lies in $\mathrm{U}(\mathrm{A}_\Gamma)$.*

Proof Conditions (1)–(3) in the definition of a Γ–Whitehead pair guarantee that any relation $[v, w] = 1$ is preserved by the Whitehead automorphism (P, m). □

If (P, m) is a Γ–Whitehead pair, we define

$$\begin{aligned} \mathrm{double}(P) &= \{x \in P \mid x^{-1} \text{ is also in } P\}, \\ \mathrm{single}(P) &= \{x \in P \mid x^{-1} \text{ is not in P}\}, \\ \max(P) &= \{x \in \mathrm{single}(P) \mid x \sim m\}, \\ \mathrm{lk}(P) &= \mathrm{lk}(m)^{\pm}. \end{aligned}$$

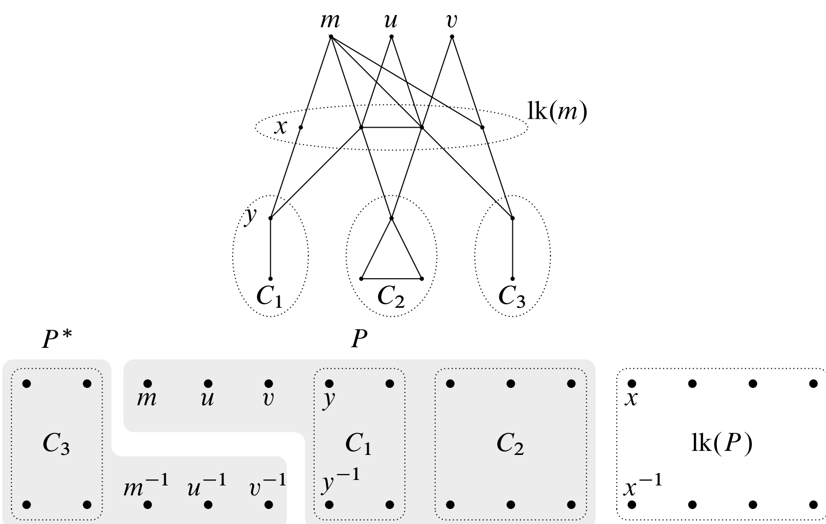


Figure 1: A graph Γ and a Γ–Whitehead partition

Remark 2.3 Here are some elementary observations about Γ–Whitehead pairs. Since all elements of $\max(P)$ are equivalent, the subcomplex $\mathrm{lk}(P)$ is independent of the choice of $m \in \max(P)$. For any $m' \in \max(P)$, the pair (P, m') is also a Γ–Whitehead pair. The set $\max(P)$ can be recovered from P without reference to m as the set of maximal elements in $\operatorname{single}(P)$. Since every $v \in \operatorname{single}(P)$ satisfies $v \leq m$ and no $v \in \operatorname{single}(P)$ is adjacent to m, no two elements of $\operatorname{single}(P)$ are adjacent to each other.

By Definition 2.1(1), $\operatorname{lk}(P)$ is disjoint from P. Let P^* be the complement of $P \cup \operatorname{lk}(P)$ in $V^{\pm}$, ie we have a partition of $V^{\pm}$ into three disjoint subsets

$$V^{\pm} = P + \operatorname{lk}(P) + P^*.$$

It is easy to verify that (P^*, v^{-1}) is also a Γ–Whitehead pair for any $v \in \max(P)$, that $\operatorname{lk}(P^*) = \operatorname{lk}(P)$ and that $\max(P^*) = \max(P)^{-1}$.

Definition 2.4 If (P, m) is a Γ–Whitehead pair, the triple $\boldsymbol{P} = \{P, \operatorname{lk}(P), P^*\}$ is called a Γ*–Whitehead partition of* $V^{\pm}$, and P and P^* are called the *sides* of $\boldsymbol{P}$.

We remark that a Γ–Whitehead partition is completely determined by either of its sides.

Notation We will often use $P^{\times}$ to denote a choice of side of $\boldsymbol{P}$. The following notation distinguishes vertices of Γ (as opposed to elements of $V^{\pm}$):

$$\begin{aligned}
\operatorname{lk}(\boldsymbol{P}) &= \{v \in V \mid v, v^{-1} \in \operatorname{lk}(P)\} = \{v \in V \mid v, v^{-1} \in \operatorname{lk}(P^*)\}, \\
\operatorname{single}(\boldsymbol{P}) &= \{v \in V \mid v \text{ and } v^{-1} \text{ are in different sides of } \boldsymbol{P}\}, \\
\operatorname{double}(\boldsymbol{P}) &= \{v \in V \mid v \text{ and } v^{-1} \text{ are both in the same side of } \boldsymbol{P}\}, \\
\max(\boldsymbol{P}) &= \{v \in V \mid v \text{ or } v^{-1} \text{ is in } \max(P)\}.
\end{aligned}$$

3 Γ–complexes

3.1 Blowing up a single Γ–Whitehead partition

We begin by recalling the construction of the Salvetti complex $\mathbb{S} = \mathbb{S}_{\Gamma}$. Let n be the cardinality of V and let $\mathbb{T}^n$ denote an n–torus with edges labeled $\{e_v \mid v \in V\}$. Then $\mathbb{S}$ is the subcomplex of $\mathbb{T}^n$ consisting of faces whose edges are labeled by mutually commuting sets of vertices. It is easily verified that $\mathbb{S}$ is locally CAT(0) (hence aspherical) and has fundamental group A_{Γ}. For any subset $U \subset V$, let $\mathbb{S}_U$ denote the subcomplex of $\mathbb{S}$ spanned by the edges labeled e_u for $u \in U$. Note that this is isomorphic to the Salvetti complex for the RAAG generated by U.

Let $\boldsymbol{P} = \{P, \operatorname{lk}(P), P^*\}$ be a Γ–Whitehead partition of $V^{\pm}$. We obtain a new cube complex $\mathbb{S}^{\boldsymbol{P}}$ by *blowing up* $\mathbb{S}$ as follows. Let D denote the vertices represented in $\operatorname{double}(P)$, D^* the vertices in $\operatorname{double}(P^*)$ and $L = \operatorname{lk}(\boldsymbol{P})$. To construct $\mathbb{S}^{\boldsymbol{P}}$:

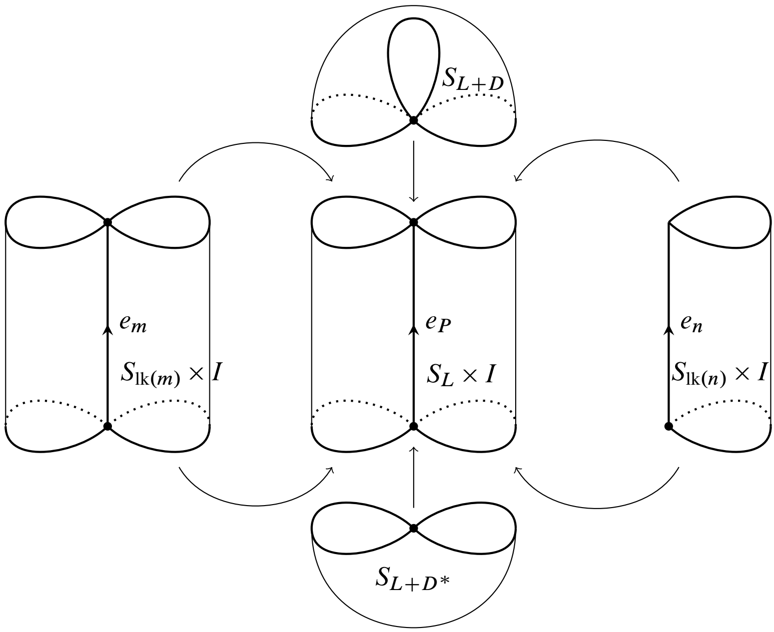


Figure 2: Blowup $\mathbb{S}^{\boldsymbol{P}}$ of $\mathbb{S}$

- Start with a copy of $\mathbb{S}_L \times [0,1]$. Label the (unique) 1–cell $e_{\boldsymbol{P}}$.
- Attach a copy of $\mathbb{S}_{L+D}$ by identifying the natural subcomplex $\mathbb{S}_L \subset \mathbb{S}_{L+D}$ with $\mathbb{S}_L \times \{1\}$.
- Attach a copy of $\mathbb{S}_{L+D^*}$ by identifying the natural subcomplex $\mathbb{S}_L \subset \mathbb{S}_{L+D^*}$ with $\mathbb{S}_L \times \{0\}$.
- For each $v \in \mathrm{single}(\boldsymbol{P})$, attach a copy of $\mathbb{S}_{\mathrm{lk}(v)} \times [0,1]$ at its ends using the natural inclusions $\mathbb{S}_{\mathrm{lk}(v)} \times \{1\} \subset \mathbb{S}_L \times \{1\}$ and $\mathbb{S}_{\mathrm{lk}(v)} \times \{0\} \subset \mathbb{S}_L \times \{0\}$. Label the edge of $\mathbb{S}_{\mathrm{lk}(v)} \times [0,1]$ with e_v. Orient it from 0 to 1 if $v \in P$ and from 1 to 0 if $v^{-1} \in P$.

Figure 2 may help the reader visualize this construction.

Remark 3.1 If v and w commute, then $\mathbb{S}$ contains a corresponding torus $T(v,w)$, say with e_v as longitude and e_w as meridian. This torus "blows up" to the following subcomplex of $\mathbb{S}^{\boldsymbol{P}}$:

- If $v \in \mathrm{double}(P)$ then w must be in $\mathrm{lk}(P)$ or $\mathrm{double}(P)$. In either case, $T(v,w)$ gives rise to a torus attached at the vertex of $e_{\boldsymbol{P}}$ in $\mathbb{S}_L \times \{1\}$, with e_v as longitude and (the appropriate copy of) e_w as meridian. If $v \in \mathrm{double}(P^*)$ then the torus is attached at the vertex of $e_{\boldsymbol{P}}$ in $\mathbb{S}_L \times \{0\}$.

- If $v \in \mathrm{single}(\boldsymbol{P})$, the longitude of $T(v,w)$ is subdivided into two edges labeled $e_{\boldsymbol{P}}$ and e_v. The meridian loop $w \in \mathrm{lk}(v) \subset \mathrm{lk}(P)$ has two representatives, one at each end of $e_{\boldsymbol{P}}$, both labeled e_w.
- If v and w are both in $\mathrm{lk}(P)$, then T blows up to the product $T(v,w) \times e_{\boldsymbol{P}} \subseteq \mathbb{S}^{\boldsymbol{P}}$.

We note the following properties of the blowup:

(1) $\mathbb{S}^{\boldsymbol{P}}$ has exactly two vertices, which correspond to the two sides of $\boldsymbol{P}$.

(2) The edges emanating from the P–vertex are labeled by the elements of $P \cup \mathrm{lk}(P)$ plus one extra edge, labeled $e_{\boldsymbol{P}}$, and similarly for the P^*–vertex.

(3) Two edges at a vertex span a square if and only if they are labeled by commuting generators, or by $e_{\boldsymbol{P}}$ and an element of $\mathrm{lk}(P)$.

(4) The links of the vertices are flag, hence $\mathbb{S}^{\boldsymbol{P}}$ is nonpositively curved.

Note that collapsing the cylinder $\mathbb{S}_L \times [0,1]$ down to $\mathbb{S}_L \times \{0\}$ recovers $\mathbb{S}$; we call this the *canonical collapse* $c_{\boldsymbol{P}}$. For each $m \in \max(P)$, there is an isomorphism h_m of $\mathbb{S}^{\boldsymbol{P}}$ which interchanges $\mathbb{S}_L \times e_P$ and $\mathbb{S}_L \times e_m$. Let c_m denote the composite map $c_P \circ h_m \colon \mathbb{S}^{\boldsymbol{P}} \to \mathbb{S}$.

Lemma 3.2 *Let $c_{\boldsymbol{P}}^{-1}$ be a homotopy inverse of the canonical collapse. Then the composition $c_m \circ c_{\boldsymbol{P}}^{-1} \colon \mathbb{S} \to \mathbb{S}^{\boldsymbol{P}} \to \mathbb{S}$ induces the Whitehead automorphism (P,m) on $A_\Gamma = \pi_1(\mathbb{S})$.*

3.2 Compatible and commuting Γ–Whitehead partitions

It is possible to build a connected graph with a proper cocompact action of $U(A_\Gamma)$ using just Salvetti complexes and single blowups $\mathbb{S}^{\boldsymbol{P}}$, but to make a *contractible* complex we will need to do further blowing up to "fill in the holes" in this graph. To this end, we make the following definitions:

Definition 3.3 Let $\boldsymbol{P} = \{P, \mathrm{lk}(P), P^*\}$ and $\boldsymbol{Q} = \{Q, \mathrm{lk}(Q), Q^*\}$ be Γ–Whitehead partitions.

(1) We say $\boldsymbol{P}$ and $\boldsymbol{Q}$ *commute* if the equivalence classes of $\max(P)$ and $\max(Q)$ are distinct and commute in A_Γ.

(2) We say $\boldsymbol{P}$ and $\boldsymbol{Q}$ are *compatible* if either they commute or $P^\times \cap Q^\times$ is empty for (at least) one choice of sides $P^\times \in \{P, P^*\}$ and $Q^\times \in \{Q, Q^*\}$.

Lemma 3.4 *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be noncommuting compatible Γ–Whitehead partitions. If $P \cap Q = \varnothing$, then $P \cap \operatorname{lk}(Q) = \varnothing$, ie $P \subset Q^*$ and $Q \subset P^*$.*

Proof Suppose $u \in P \cap \operatorname{lk}(Q)$, and let $m \in \max(Q)$. Then $u \in \operatorname{lk}(Q) = \operatorname{lk}(m)$ implies that $m \in \operatorname{lk}(u)$. If $u \in \operatorname{single}(P)$ then $\operatorname{lk}(u) \subseteq \operatorname{lk}(P)$, so $m \in \operatorname{lk}(P)$, contradicting the assumption that $\boldsymbol{P}$ and $\boldsymbol{Q}$ do not commute. If $u \in \operatorname{double}(P)$ then the fact that u and m are connected by an edge implies that either $m \in \operatorname{lk}(P)$ or m lies in the same component of $\Gamma - \operatorname{lk}(P)$ as u. The former contradicts the assumption that $\boldsymbol{P}$ and $\boldsymbol{Q}$ do not commute, and the latter implies that $m \in P \cap Q$.

The last statement follows by symmetry. □

Remark 3.5 Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be noncommuting compatible Γ–Whitehead partitions and suppose that $m \in \max(P) \cap \max(Q)$. Then either $P^* \cap Q$ or $P \cap Q^*$ is empty, say $P^* \cap Q = \varnothing$. Then it follows from the lemma that $Q \subset P$ and, setting $R = (P \setminus Q) \cup \{m\}$, a straightforward exercise shows that (R, m) is also a Γ–Whitehead pair and the corresponding Whitehead automorphisms satisfy $(P, m) \circ (Q, m)^{-1} = (P, m) \circ (Q, m) = (R, m) \circ i_m$, where i_m is the inversion taking $m \mapsto m^{-1}$.

Lemma 3.6 *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be distinct compatible Γ–Whitehead partitions. If $\boldsymbol{P}$ and $\boldsymbol{Q}$ do not commute, then exactly one of the sets $P^\times \cap Q^\times$ is empty.*

Proof Since $\boldsymbol{P}$ and $\boldsymbol{Q}$ are compatible at least one of the sets $P^\times \cap Q^\times$ is empty, so without loss of generality we may assume $P \cap Q = \varnothing$. By the previous lemma, it follows that $P \cap Q^* = P$ and $P^* \cap Q = Q$.

Suppose $P^* \cap Q^* = \varnothing$. Then any $m \in \max(P)$ must have $m^{-1} \in Q$, so $\operatorname{lk}(P) \subseteq \operatorname{lk}(Q)$ and similarly $\operatorname{lk}(Q) \subseteq \operatorname{lk}(P)$. Thus $V^\pm = P \sqcup \operatorname{lk}(P) \sqcup Q$, ie $Q = P^*$ and $\boldsymbol{P} = \boldsymbol{Q}$, contradicting our hypothesis. □

It follows from the lemma that for noncommuting, compatible partitions $\boldsymbol{P}$ and $\boldsymbol{Q}$ with nonempty intersection $P^\times \cap Q^\times$, we can switch sides of either $\boldsymbol{P}$ or $\boldsymbol{Q}$, but not necessarily both, and still get a nonempty intersection.

3.3 Blowing up compatible collections of Γ–Whitehead partitions

Now let

$$\boldsymbol{\Pi} = \{\boldsymbol{P}_1, \dots, \boldsymbol{P}_k\}$$

be a set of pairwise compatible Γ–Whitehead partitions. We want to simultaneously blow up $\mathbb{S}$ along all of the partitions in $\boldsymbol{\Pi}$ to obtain a nonpositively curved cube complex $\mathbb{S}^{\boldsymbol{\Pi}}$.

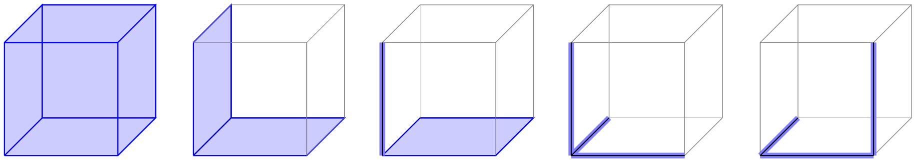


Figure 3: Possibilities for $\mathbb{E}_{\{\boldsymbol{P},\boldsymbol{Q},\boldsymbol{R}\}}$

The role played by the edge $e_{\boldsymbol{P}}$ in the single blowup $\mathbb{S}^{\boldsymbol{P}}$ will now be played by a cubical subcomplex of a k–dimensional cube, $[0,1]^k$. The vertices of this subcomplex will form the vertices of $\mathbb{S}^{\boldsymbol{\Pi}}$, and to describe them we make the following definition:

Definition 3.7 A *region* of $\boldsymbol{\Pi}$ is a choice of side $P_i^\times \in \{P_i, P_i^*\}$ for each i such that, for $i \neq j$, either $\boldsymbol{P}_i$ and $\boldsymbol{P}_j$ commute or $P_i^\times \cap P_j^\times \neq \varnothing$.

To each region $R = (P_1^\times, \dots, P_k^\times)$ of $\boldsymbol{\Pi}$ we associate a vertex $x_R = (a_1, \dots, a_k)$ of $[0,1]^k$ by the rule

$$a_i = \begin{cases} 0 & \text{if } P_i^\times = P_i, \\ 1 & \text{if } P_i^\times = P_i^*. \end{cases}$$

Let $\mathbb{E}_{\boldsymbol{\Pi}}$ denote the cubical subcomplex of $[0,1]^k$ spanned by the x_R, and label all edges parallel to the i^{th} basis vector with the label $e_{\boldsymbol{P}_i}$.

Example 3.8 Suppose $\boldsymbol{\Pi} = \{\boldsymbol{P}, \boldsymbol{Q}\}$. If $\boldsymbol{P}$ and $\boldsymbol{Q}$ commute, then $\mathbb{E}_{\boldsymbol{\Pi}}$ is the entire square $[0,1]^2$, with two (parallel) edges labeled $e_{\boldsymbol{P}}$ and the other two labeled $e_{\boldsymbol{Q}}$. If $\boldsymbol{P}$ and $\boldsymbol{Q}$ do not commute, then by Lemma 3.6 exactly three of (P, Q), (P^*, Q), (P, Q^*) and (P^*, Q^*) are regions, so that $\mathbb{E}_{\boldsymbol{\Pi}}$ consists of two adjacent edges of the square, one labeled $e_{\boldsymbol{P}}$ and one labeled $e_{\boldsymbol{Q}}$.

If $\boldsymbol{\Pi}$ contains three Γ–Whitehead partitions, the possibilities for $\mathbb{E}_{\boldsymbol{\Pi}}$ are illustrated in Figure 3.

The following lemma guarantees that every set of pairwise compatible Γ–Whitehead partitions has regions associated to it:

Lemma 3.9 *Let $\boldsymbol{\Pi} = \{\boldsymbol{P}_1, \dots, \boldsymbol{P}_k\}$ be a set of pairwise compatible Γ–Whitehead partitions, and $(P_1^\times, \dots, P_k^\times)$ a region for $\boldsymbol{\Pi}$. If a Γ–Whitehead partition $\boldsymbol{P}$ is compatible with each $\boldsymbol{P}_i$ then, for some choice of sides $P^\times \in \{P, P^*\}$, $(P_1^\times, \dots, P_k^\times, P^\times)$ is a region for $\boldsymbol{\Pi} \cup \{\boldsymbol{P}\}$.*

Proof If $\boldsymbol{P}$ commutes with $\boldsymbol{P}_i$ for all i, either choice of sides will do. So suppose $\boldsymbol{P}$ does not commute with $\boldsymbol{P}_i$ for some i. By Lemma 3.4, if $P_i^\times$ is not contained in either side of $\boldsymbol{P}$, then either choice of side works for this pair. If $P_i^\times$ is contained in one side of $\boldsymbol{P}$, we must choose $P^\times$ to be that side. Thus, to prove the lemma, we must show that, if $P_i^\times$ and $P_j^\times$ are each contained in a side of $\boldsymbol{P}$, then they are contained in the same side.

To see this, suppose that $P_i^\times \subset P$ and $P_j^\times \subset P^*$, so $P_i^\times \cap P_j^\times = \varnothing$. By assumption, the choice of sides for $\boldsymbol{P}_i$ and $\boldsymbol{P}_j$ defines a region, so they must commute. That is, $\max(\boldsymbol{P}_i)$ and $\max(\boldsymbol{P}_j)$ are adjacent in Γ. Let $v \in \max(\boldsymbol{P}_i^\times)$. If $v \in \mathrm{single}(\boldsymbol{P})$, then $\max(\boldsymbol{P}_j) \subseteq \mathrm{lk}(v) \subseteq \mathrm{lk}(\boldsymbol{P})$, contradicting the assumption that $\boldsymbol{P}$ does not commute with $\boldsymbol{P}_j$. Thus, the elements of $\max(\boldsymbol{P}_i)$ appear as doubles in P. Likewise, elements of $\max(\boldsymbol{P}_j)$ appear as doubles in P^*. But, since $\max(\boldsymbol{P}_i)$ and $\max(\boldsymbol{P}_j)$ are adjacent, they lie in the same component of $\Gamma \setminus \mathrm{st}(v)$ for $v \in \max(\boldsymbol{P})$, hence they must appear on the same side of $\boldsymbol{P}$. ◻

We continue building $\mathbb{S}^{\boldsymbol{\Pi}}$ by attaching edges to $\mathbb{E}_{\boldsymbol{\Pi}}$ for each element of V. We need the following lemmas in order to explain how this is done.

Associated to a region $R = (P_1^\times, \dots, P_k^\times)$ is a subset of $V^\pm$ defined by

$$I(R) = \overline{P}_1^\times \cap \dots \cap \overline{P}_k^\times$$

where $\overline{P}_i^\times = P_i^\times \cup \mathrm{lk}(P_i)$. As we will see below, the elements of $I(R)$ will correspond to the directed edges attached at the vertex x_R

Note that if switching sides of $\boldsymbol{P}_i$ and leaving all other $P_j^\times$ unchanged gives a valid region R_i, then there is an edge in $\mathbb{E}_{\boldsymbol{\Pi}}$ labeled $e_{\boldsymbol{P}_i}$ from x_R to x_{R_i}.

Lemma 3.10 *Let $\boldsymbol{\Pi} = \{\boldsymbol{P}_1, \dots, \boldsymbol{P}_k\}$ be a set of pairwise compatible Γ–Whitehead partitions. Then the regions of $\boldsymbol{\Pi}$ satisfy the following:*

(1) *For every element v in $V^\pm$, there exists a region R with $v \in I(R)$.*
(2) *If $I(R)$ contains v, then switching sides of every $\boldsymbol{P}_i$ for which v is a singleton gives a region R_v such that $I(R_v)$ contains v^{-1}. Moreover, if $w \in I(R)$ commutes with v, then $I(R_v)$ also contains w.*

Proof We proceed by induction on k. For $k = 1$ this is obvious. Assume $k > 1$. If $v \in \mathrm{lk}(P_i)$ for all i, then $v \in I(R)$ and $R_v = R$ for *every* region R, so we need only show that $\boldsymbol{\Pi}$ has at least one region, which follows from Lemma 3.9.

Now suppose that v is not in the link of at least one of the partitions. Say $v \notin \mathrm{lk}(P_i)$ for $1 \le i < r$. For these partitions, choose $P_i^\times$ to be the side of $\boldsymbol{P}_i$ containing v. For

$r \geq i \leq k$, choose any collection of sides $P_i^{\times}$ such that $(P_r^{\times}, \dots, P_k^{\times})$ is a region for the partition $(\boldsymbol{P}_r, \dots, \boldsymbol{P}_k)$. (Such a choice exists by induction.) We claim that $R = (P_1^{\times}, \dots, P_k^{\times})$ is a valid region for $\boldsymbol{\Pi}$. For this, we must verify that the chosen sides for any two noncommuting partitions $\boldsymbol{P}_i$ and $\boldsymbol{P}_j$ intersect. This is true by definition for $i,\ j \geq r$. For $i,\ j < r$, we have $v \in P_i^{\times} \cap P_j^{\times}$, so this intersection is nonempty, and, for $i < r \geq j$, we have $v \in P_i^{\times} \cap \mathrm{lk}(P_j)$ so, by Lemma 3.4, $\boldsymbol{P}_i$ and $\boldsymbol{P}_j$ commute. Thus R is a region.

If R_v is obtained from R by switching the sides of those $\boldsymbol{P}_i$ for which v is a singleton, then the same argument, with v replaced by v^{-1}, shows that R_v is a region. Moreover, if v is a singleton in $\boldsymbol{P}_i$, then $\mathrm{lk}(v) \subseteq \mathrm{lk}(P_i)$, hence $I(R) \cap \mathrm{lk}(v)^{\pm} = I(R_v) \cap \mathrm{lk}(v)^{\pm}$. □

We can now explain how to attach edges to $\mathbb{E}_{\boldsymbol{\Pi}}$. For each v in $I(R)$ we attach an edge labeled e_v joining x_R and x_{R_v}, where R_v is obtained as in the lemma. (Note that we may have $R = R_v$ if v is not a singleton in any $\boldsymbol{P}_i$, in which case we are attaching a loop.) The edge e_v is oriented from the region containing v^{-1} to the region containing v. The 1–skeleton of the resulting complex, including edges labeled e_v and $e_{\boldsymbol{P}_i}$, will be the entire 1–skeleton of $\mathbb{S}^{\boldsymbol{\Pi}}$, so we denote it by $(\mathbb{S}^{\boldsymbol{\Pi}})^{(1)}$.

Note that while a given label occurs at most once at each vertex, it does not determine a unique edge in $(\mathbb{S}^{\boldsymbol{\Pi}})^{(1)}$. For example, an edge labeled e_v will occur at every vertex x_R with $v \in I(R)$. Indeed, once the higher-dimensional cells are added, we will see that two edges have the same label if and only if they determine the same hyperplane in $\mathbb{S}^{\boldsymbol{\Pi}}$.

To complete the construction of $\mathbb{S}^{\boldsymbol{\Pi}}$ we need to add higher-dimensional cubes which capture the commutation relations in A_Γ. Define two edges to have *commuting labels* if their labels are one of the following:

(1) e_v and e_w with v and w distinct, commuting elements of V;

(2) e_v and $e_{\boldsymbol{P}_i}$ with $v \in \mathrm{lk}(\boldsymbol{P}_i)$;

(3) $e_{\boldsymbol{P}_i}$ and $e_{\boldsymbol{P}_j}$ with $\boldsymbol{P}_i$ and $\boldsymbol{P}_j$ distinct, commuting partitions.

Lemma 3.11 *Let e_a and e_b be edges at a vertex x_R with commuting labels. Then e_a and e_b belong to a 4–cycle in $(\mathbb{S}^{\boldsymbol{\Pi}})^{(1)}$ with opposite edges having the same label.*

Proof If the labels are both of the form $e_{\boldsymbol{P}_i}$ then they span a square in $\mathbb{E}_{\boldsymbol{\Pi}}$, and we are done.

If the labels are e_v and e_w, they terminate at x_{R_v} and x_{R_w}, respectively. It follows from Lemma 3.10(2) that there is an edge labeled e_v emanating from x_{R_w} and an edge labeled e_w emanating from x_{R_v}. These form a square with the vertex opposite $x_{\boldsymbol{R}}$ corresponding to $(R_v)_w = (R_w)_v$.

A similar argument applies for edges labeled e_v and $e_{\boldsymbol{P}_i}$. Since $v \in \mathrm{lk}(\boldsymbol{P}_i)$, switching the side of $\boldsymbol{P}_i$ does not affect v. So if R' is the result of this switch, then there is an edge labeled e_v emanating from $x_{\boldsymbol{R}'}$. The other end of this edge corresponds to a region R'_v which differs from R_v only on $\boldsymbol{P}_i$. Thus, the vertices R_v and R'_v are also connected by an edge labeled $e_{\boldsymbol{P}_i}$, completing the square. □

Corollary 3.12 *If a collection of edges $e_1, \dots e_m$ emanating from a vertex $x_{\boldsymbol{R}}$ have pairwise commuting labels, then these edges form a corner of the 1–skeleton of an m–cube in $(\mathbb{S}^{\boldsymbol{\Pi}})^{(1)}$ such that parallel edges have the same labels.*

Proof This follows from Lemma 3.11 by induction on m. □

It follows that we can glue an m–cube into $(\mathbb{S}^{\boldsymbol{\Pi}})^{(1)}$ whenever we have a set of m edges at a vertex with commuting labels. The resulting cube complex is $\mathbb{S}^{\boldsymbol{\Pi}}$. Note that the subcomplex of $\mathbb{S}^{\boldsymbol{\Pi}}$ spanned by the edges labeled $e_{\boldsymbol{P}_i}$ is precisely the complex $\mathbb{E}_{\boldsymbol{\Pi}}$ that we started with. By construction, the link of the vertex $x_{\boldsymbol{R}}$ has an $(m-1)$–simplex for each set of m mutually commuting edge labels emanating from $x_{\boldsymbol{R}}$, so, by Gromov's link condition, $\mathbb{S}^{\boldsymbol{\Pi}}$ is locally $\mathrm{CAT}(0)$.

Definition 3.13 A Γ–*complex* is any cube complex which is isomorphic to $\mathbb{S}^{\boldsymbol{\Pi}}$ for some compatible set $\boldsymbol{\Pi}$ of Γ–Whitehead partitions, including $\mathbb{S} = \mathbb{S}^{\varnothing}$. A Γ–complex isomorphic to $\mathbb{S}$ is called simply a *Salvetti.*

Theorem 3.14 *Let $\boldsymbol{\Pi}$ be a compatible set of Γ–Whitehead partitions. Then the Γ–complex $\mathbb{S}^{\boldsymbol{\Pi}}$ has the following properties:*

(1) *Any two vertices $x_{\boldsymbol{R}}$ and $x_{\boldsymbol{R}'}$ are connected by a path with labels in the set*
$$\{e_{\boldsymbol{P}_i} \mid R \text{ and } R' \text{ contain opposite sides of } \boldsymbol{P}_i\}.$$

(2) *Any two edges with the same label are dual to the same hyperplane.*

(3) *$\mathbb{S}^{\boldsymbol{\Pi}}$ is connected and locally* $\mathrm{CAT}(0)$.

Proof (1) Let R and R' be two regions of $\boldsymbol{\Pi}$ and (reordering if necessary) suppose they differ in the choice of sides of $\boldsymbol{P}_1, \dots, \boldsymbol{P}_l$ but agree on the remaining partitions. We will show by induction on l that they are connected by a path with labels in $\{e_{\boldsymbol{P}_i}\}_{i=1,\dots,l}$. For $l = 1$, two regions which differ on only one partition $\boldsymbol{P}$ are, by construction, connected by an edge labeled $e_{\boldsymbol{P}}$.

Suppose $l > 1$. For simplicity, write $R = (P_1, \dots, P_k)$. (It makes no difference which side we call P_i and which side we call P_i^*). Choose $i \le l$ such that P_i is minimal among the sets $P_1, \dots, P_l$. That is, P_i does not contain any of the other sets in this collection. Then it follows from Lemma 3.4 that switching P_i^* is also compatible with (ie commutes or intersects) the remaining P_j for $j \le l$. For $j > l$, P_j must also be compatible with P_i^* since they both appear in the region R'. Thus, setting $R_i = (P_1, \dots, P_i^*, \dots P_k)$ we obtain a valid region which differs from R' in $l-1$ places. The vertices corresponding to R and R_i are connected by an edge labeled $e_{\boldsymbol{P}_i}$ and, by induction, the vertices corresponding to R_i and R' are connected with labels in $\{e_{\boldsymbol{P}_j}\}_{1 \le j \le l,\, j \ne i}$.

(2) We consider two cases. Suppose R and R' are regions with $v \in I(R) \cap I(R')$, so there is an edge labeled e_v emanating from both vertices x_R and $x_{R'}$. Then, for any partition $\boldsymbol{P}_i$ with $v \notin \mathrm{lk}(\boldsymbol{P}_i)$, both R and R' must contain the (unique) side of $\boldsymbol{P}_i$ containing v. That is, R and R' differ only on partitions containing v in their link. By part (1), it follows that x_R and $x_{R'}$ are connected by a path labeled by $e_{\boldsymbol{P}_i}$ such that $e_{\boldsymbol{P}_i}$ commutes with e_v and hence these two edges span a 2–cube. Proceeding along this path gives a sequence of such cubes joining the e_v edges at x_R and $x_{R'}$. It follows that they are dual to the same hyperplane.

For two edges labeled by $e_{\boldsymbol{P}_i}$ consider the four vertices contained in these edges. Say the regions for these vertices are R, R_i and R', R'_i. If R and R' differ on some P_j then all possible combinations of P_i, P_i^* with P_j, P_j^* occur in these four regions. But this is possible only if $\boldsymbol{P}_i$ and $\boldsymbol{P}_j$ are commuting partitions. Thus, arguing as above, we can connect R to R' with a path labeled by $e_{\boldsymbol{P}_j}$'s which commute with $e_{\boldsymbol{P}_i}$ and conclude that there is a sequence of cubes between the $e_{\boldsymbol{P}_i}$–edges at x_R and $x_{R'}$.

(3) It follows from (1) that $\mathbb{S}^{\boldsymbol{\Pi}}$ is connected, and it was observed above that it is locally CAT(0) by construction. □

4 Collapsing along hyperplanes

In the case of a single blowup $\mathbb{S}^{\boldsymbol{P}}$, we observed in Lemma 3.2 that for any element $m \in \max(P)$ there is a subcomplex containing the edge e_m which can be collapsed to give a Salvetti, ie a cube complex isomorphic to $\mathbb{S}$. Furthermore, the map on $\mathbb{S}$ obtained by blowing up followed by this collapse corresponds to a Whitehead automorphism. In this section we identify all subcomplexes of general Γ–complexes which can be collapsed to give a Salvetti.

Let X be a nonpositively curved cubical complex and H a hyperplane of X. If e is an edge of X then e and H are said to be *dual* if e intersects H. The *carrier* $\kappa(H)$

of H is the subcomplex of X formed by the closures of the cubes of all dimensions that intersect H.

Definition 4.1 Let H be a hyperplane of X. We say H is a *carrier retract* if $\kappa(H)$ is isomorphic to $H \times [0,1]$; in particular, H is embedded in X and there are no identifications on the boundary of $\kappa(H)$. If H is a carrier retract, we define the *collapse of X along H* to be the cube complex formed by collapsing $\kappa(H)$ orthogonally onto H. Denote the resulting complex by X_H, and note that there is a canonical projection $X \twoheadrightarrow X_H$.

Example 4.2 In the blowup $\mathbb{S}^{\boldsymbol{P}}$, there is one hyperplane dual to $e_{\boldsymbol{P}}$ and one dual to e_v for each $v \in V$. The hyperplane dual to $e_{\boldsymbol{P}}$ is isomorphic to $\mathbb{S}_{\mathrm{lk}(\boldsymbol{P})}$, and for every $v \in V$ the hyperplane dual to e_v is isomorphic to $\mathbb{S}_{\mathrm{lk}(v)}$. The hyperplane dual to $e_{\boldsymbol{P}}$ is a carrier retract. The hyperplane dual to e_v is a carrier retract if and only if $v \in \mathrm{single}(\boldsymbol{P})$.

Definition 4.3 Let $\mathcal{H} = \{H_1, H_2, \dots, H_k\}$ be a set of hyperplanes. We say that $\mathcal{H}$ *has compatible carriers* if each H_i is a carrier retract and any loop in X consisting of edges dual to the H_i is null homotopic. Given such a set, define the *collapse of X along $\mathcal{H}$* to be the complex $X_{\mathcal{H}}$ obtained by collapsing each cube C in $\bigcup \kappa(H_i)$ to the intersection of the midplanes of C lying in some H_i.

The proof of the following lemma is an easy exercise.

Lemma 4.4 *Let $\mathcal{H} = \{H_1, H_2, \dots, H_k\}$ be a set of hyperplanes in X with compatible carriers. Let $Y = X_{H_1}$ and, for $i > 1$, let $\overline{H}_i$ denote the image of H_i in Y. Then*

1. *Y is a nonpositively curved cube complex.*
2. *$\overline{\mathcal{H}} = \{\overline{H}_2, \dots, \overline{H}_k\}$ has compatible carriers in Y.*
3. *$Y_{\overline{\mathcal{H}}} = X_{\mathcal{H}}$.*

In particular, it follows that collapsing along the H_i one at a time in any order results in the same space $X_{\mathcal{H}}$.

Lemma 4.5 *Let $\mathcal{H}$ be set of hyperplanes in X with compatible carriers and let $c\colon X \to X_{\mathcal{H}}$ be the projection map. Then:*

1. *c is a homotopy equivalence.*
2. *Distinct hyperplanes in X not contained in $\mathcal{H}$ map to distinct hyperplanes in $X_{\mathcal{H}}$.*

(3) *If g is a conjugacy class in $\pi_1(X)$ and p is a minimal length edge path in X representing g, then $c(p)$ is a minimal length edge path in $X_{\mathcal{H}}$ representing g.*

Proof By the previous lemma, it suffices to consider the case where $\mathcal{H}$ consists of a single hyperplane H which is a carrier retract. The first statement is clear: the homotopy equivalence between $\kappa(H)$ and H extends to a homotopy equivalence between X and X_H, since $\kappa(H)$ is a strong deformation retract of the open neighborhood consisting of points at distance less than $\frac{1}{2}$ from $\kappa(H)$.

For the second statement, recall that a hyperplane can be identified with an equivalence class of edges. Two edges in $\kappa(H)$ that become identified under the collapse c are parallel edges in some cube, hence they are already equivalent.

For the third statement, let g be a conjugacy class in $\pi_1(X)$ and let p be an edge path in X representing g. Lift p to a path $\widetilde{p}$ in the universal cover $\widetilde{X}$ and let $\widetilde{p}^\infty$ denote the union of the g^k–translates of $\widetilde{p}$ for $k \in \mathbb{Z}$. Since $\widetilde{X}$ is a CAT(0) cube complex, it follows from [15] that p is minimal if and only if $\widetilde{p}^\infty$ crosses no hyperplane of $\widetilde{X}$ more than once. The analogous statement holds for $c(p)$.

The universal cover of $X_{\mathcal{H}}$ is obtained from $\widetilde{X}$ by collapsing along all hyperplanes $\widetilde{\mathcal{H}}$ in the inverse image of $\mathcal{H}$. Let $\widetilde{c}\colon \widetilde{X} \to \widetilde{X}_{\widetilde{\mathcal{H}}}$ be the lift of c. Set $q = c(p)$, and define $\widetilde{q}^\infty$ as above. Then $\widetilde{c}(\widetilde{p}^\infty) = \widetilde{q}^\infty$, so by part (2) of the lemma, if $\widetilde{p}^\infty$ crosses each hyperplane at most once, the same holds for $\widetilde{q}^\infty$. Statement (3) follows. □

We are now ready to apply these observations to the hyperplanes of a blowup $\mathbb{S}^{\mathbf{\Pi}}$.

Theorem 4.6 *Let $\mathbf{\Pi} = \{\boldsymbol{P}_1, \dots, \boldsymbol{P}_k\}$ be a compatible set of Γ–Whitehead partitions for Γ. Then the set of hyperplanes $\mathcal{H} = \{H_1, \dots, H_k\}$ dual to the edges $e_{\boldsymbol{P}_i}$ has compatible carriers. For any subset $J \subseteq \{1, 2, \dots, k\}$, collapsing along the hyperplanes $\{H_i\}_{i \in J}$ gives a complex naturally isomorphic to the blowup of $\mathbb{S}$ by the partitions $\{\boldsymbol{P}_i\}_{i \notin J}$. In particular, collapsing along all of $\mathcal{H}$ gives a complex isomorphic to the Salvetti complex $\mathbb{S}$.*

Proof It is easy to check that the carrier of each H_i is isomorphic to $H_i \times [0,1]$. Thus, to show that $\mathcal{H}$ has compatible carriers, we need to show that any loop γ formed by edges labeled $e_{\boldsymbol{P}_i}$ is null homotopic. We will induct on the number of the $\boldsymbol{P}_i$ appearing in this loop. Say the loop involves only a single $\boldsymbol{P}_i$. Since no two edges with the same label occur at a vertex, the loop would have to consist of a single edge. But an $e_{\boldsymbol{P}_i}$–edge switches the side of $\boldsymbol{P}_i$, so it cannot be a loop.

Now suppose γ involves more than one $\boldsymbol{P}_i$. Orient γ and say the initial vertex is $x_{\boldsymbol{R}}$ with $R = (P_1, \dots, P_k)$. Once γ crosses an $e_{\boldsymbol{P}_i}$–edge, all of the regions it encounters will have P_i^* in the i^{th} position until it crosses another $e_{\boldsymbol{P}_i}$–edge. Thus, the labels must occur in pairs. Let α be a segment of γ joining two consecutive $e_{\boldsymbol{P}_i}$–edges, so γ decomposes as

$$\gamma = \gamma_1 \cdot e_1 \cdot \alpha \cdot e_2 \cdot \gamma_2,$$

where e_1 and e_2 are edges labeled by $e_{\boldsymbol{P}_i}$. By Theorem 3.14(2), there exists a path β in the carrier $\kappa(H_i)$ between the endpoints of α. Then $\alpha\beta^{-1}$ forms a loop not involving $\boldsymbol{P}_i$, so by induction α is homotopic to β. Moreover, the path $e_1\beta e_2$ can be slid across the hyperplane H_i to get a path β' with no edges labeled $e_{\boldsymbol{P}_i}$. Thus, γ is homotopic to $\gamma' = \gamma_1\beta'\gamma_2$. Repeating this process if necessary, we can get rid of all $e_{\boldsymbol{P}_i}$–edges in γ and apply induction to conclude that γ is null-homotopic.

For the second statement, it suffices to consider the case where J is a singleton, say $J = \{1\}$. So consider the space obtained from $\mathbb{S}^{\boldsymbol{\Pi}}$ by collapsing along H_1, the hyperplane dual to $e_{\boldsymbol{P}_1}$. This collapse has the effect of identifying two vertices whose labels differ only in the choice of side for $\boldsymbol{P}_1$. So, letting $\boldsymbol{\Pi}' = \{\boldsymbol{P}_2, \dots, \boldsymbol{P}_k\}$, we can map vertices of the quotient space injectively to vertices of $S^{\boldsymbol{\Pi}'}$ by forgetting $\boldsymbol{P}_1$. By Lemma 3.9, this map is also surjective. The construction of $S^{\boldsymbol{\Pi}'}$ depends only on the vertex labels, so it is now easy to verify that this bijection extends to an isomorphism of complexes. □

We will call any collapse along hyperplanes dual to $e_{\boldsymbol{P}_i}$–edges, where $i \in J \subseteq \{1, \dots, k\}$, a *canonical collapse* of $\mathbb{S}^{\boldsymbol{\Pi}}$. In particular, taking $J = \{1, \dots, k\}$, we have a canonical collapse from $\mathbb{S}^{\boldsymbol{\Pi}}$ down to the Salvetti complex $\mathbb{S}$. However, one can obtain a Salvetti complex by collapsing along many other sets of hyperplanes $\mathcal{H}$ in $\mathbb{S}^{\boldsymbol{\Pi}}$.

Example 4.7 Let $\boldsymbol{P} = \{P, P^*, \mathrm{lk}(P)\}$ be a single Γ–Whitehead partition and $\mathbb{S}^{\boldsymbol{P}}$ the associated Γ–complex. As we saw in Lemma 3.2, collapsing along the hyperplane dual to the edge labeled e_v for any $v \in \max(P)$ gives a complex isomorphic to $\mathbb{S}$.

Example 4.8 Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be compatible Γ–partitions with $\mathrm{lk}(\boldsymbol{P}) = \mathrm{lk}(\boldsymbol{Q}) = L$. In particular, $\boldsymbol{P}$ and $\boldsymbol{Q}$ do not commute, so $\mathbb{S}^{\{\boldsymbol{P},\boldsymbol{Q}\}}$ has one edge labeled $e_{\boldsymbol{P}}$ and one edge labeled $e_{\boldsymbol{Q}}$. Let Θ denote the graph formed by $e_{\boldsymbol{P}}$, $e_{\boldsymbol{Q}}$ and all e_v with $\mathrm{lk}(v) = L$. The hyperplane dual to each edge in Θ is isomorphic to the Salvetti complex $\mathbb{S}_L$. Thus the subcomplex spanned by the carriers of all of these hyperplanes decomposes as a product $\Theta \times \mathbb{S}_L$. Now take any maximal tree T in Θ and let $\mathcal{H}$ be the set of hyperplanes dual to the edges in T. Then collapsing $\mathbb{S}^{\{\boldsymbol{P},\boldsymbol{Q}\}}$ along $\mathcal{H}$ reduces Θ to a rose and reduces $\mathbb{S}^{\{\boldsymbol{P},\boldsymbol{Q}\}}$ to a complex isomorphic to $\mathbb{S}$.

Example 4.9 Example 4.8 generalizes to any set $\boldsymbol{\Pi}$ of compatible Γ–partitions which all have the same link L. Since no two elements of $\boldsymbol{\Pi}$ commute, the 1–skeleton of $\mathbb{S}^{\boldsymbol{\Pi}}$

has exactly one edge labeled $e_{\boldsymbol{P}}$ for each $\boldsymbol{P} \in \boldsymbol{\Pi}$ and exactly one labeled e_v for each v which is not in L. For every v the hyperplane dual to e_v is isomorphic to $\mathbb{S}_{\mathrm{lk}(v)}$, and the hyperplane dual to each $e_{\boldsymbol{P}}$ is isomorphic to $\mathbb{S}_L$. If $[m]$ denotes the set of all vertices $v \in \Gamma$ with $\mathrm{lk}(v) = L$, then the union of the carriers of the hyperplanes dual to the edges $e_{\boldsymbol{P}}$ for $\boldsymbol{P} \in \boldsymbol{\Pi}$ and e_v for $v \in [m]$ decomposes as a product $\Theta \times \mathbb{S}_L$. We will call Θ the *base graph* of $\boldsymbol{\Pi}$. Collapsing $\mathbb{S}^{\boldsymbol{\Pi}}$ along any set of hyperplanes dual to a maximal tree in the base graph reduces $\mathbb{S}^{\boldsymbol{\Pi}}$ to a complex isomorphic to the Salvetti complex $\mathbb{S}$ for Γ.

Note that a hyperplane H of $\mathbb{S}^{\boldsymbol{\Pi}}$ is a carrier retract if and only if the dual edge e to H is not a loop, ie if and only if $e = e_{\boldsymbol{P}}$ for some $\boldsymbol{P}$ or $e = e_v$ for v a singleton in some $\boldsymbol{P}$. A set $\mathcal{K} = \{H_e\}$ of hyperplanes has compatible carriers if and only if the dual edges form a forest in the 1–skeleton.

Now let $\boldsymbol{\Pi}$ be any set of compatible Γ–Whitehead partitions. Subdivide $\boldsymbol{\Pi}$ into subsets $\boldsymbol{\Pi} = \boldsymbol{\Pi}_1 \cup \cdots \cup \boldsymbol{\Pi}_s$, where each $\boldsymbol{\Pi}_i$ is a maximal collection of $\boldsymbol{P}_j$ having the same link, L_i. (For example, if A_Γ is a free group then all of the links are empty, hence $s = 1$ and $\boldsymbol{\Pi} = \boldsymbol{\Pi}_1$.) Consider the blowups $\mathbb{S}^{\boldsymbol{\Pi}_i}$. By the discussion above, each of these contains a subcomplex of the form $\Theta_i \times \mathbb{S}_{L_i}$ where Θ_i is the base graph of $\boldsymbol{\Pi}_i$. The edges of Θ_i are labeled by $e_{\boldsymbol{P}}$ and e_v with $\boldsymbol{P} \in \boldsymbol{\Pi}_i$ and $\mathrm{lk}(v) = L_i$. In particular, for $i \neq j$ the labels on the edges of Θ_i and Θ_j are disjoint.

Definition 4.10 Let $\boldsymbol{\Pi}$ be a set of compatible Γ–Whitehead partitions. If all of the partitions in $\boldsymbol{\Pi}$ have the same link, call a set of hyperplanes $\mathcal{H}$ in $\mathbb{S}^{\boldsymbol{\Pi}}$ *treelike* if the edges dual to $\mathcal{H}$ form a maximal tree in the base graph Θ. More generally, if $\boldsymbol{\Pi} = \boldsymbol{\Pi}_1 \cup \cdots \cup \boldsymbol{\Pi}_s$ is the decomposition into partitions with the same link, call $\mathcal{H}$ *treelike* if $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_s$, where $\mathcal{H}_i$ is the lift of a treelike set of hyperplanes in $\mathbb{S}^{\boldsymbol{\Pi}_i}$, that is, the edges dual to $\mathcal{H}_i$ form a maximal tree in Θ_i.

For example, the set of hyperplanes dual to the edges labeled $e_{\boldsymbol{P}_i}$ for $1 \le i \le k$ is always treelike.

Theorem 4.11 *Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{S}^{\boldsymbol{\Pi}}$. Then $\mathcal{H}$ is treelike if and only if it has compatible carriers and the collapse of $\mathbb{S}^{\boldsymbol{\Pi}}$ along $\mathcal{H}$ is isomorphic to the Salvetti complex $\mathbb{S}$.*

Proof Suppose $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_s$ is treelike. We proceed by induction on s; the case $s = 1$ was done in Example 4.9. For $s > 1$, reordering if necessary we may assume that the link L_1 is maximal among the L_i. Suppose v is a vertex of Γ with $\mathrm{lk}(v) = L_1$. If v is a singleton in some $\boldsymbol{P} \in \boldsymbol{\Pi}$, then $v \le \max(\boldsymbol{P})$ implies that $L_1 \subseteq \mathrm{lk}(\boldsymbol{P})$, so by our maximality assumption $L_1 = \mathrm{lk}(\boldsymbol{P})$ and hence $\boldsymbol{P} \in \boldsymbol{\Pi}_1$. It follows that edges labeled e_v connect vertices of $\mathbb{S}^{\boldsymbol{\Pi}}$ which differ only on partitions in $\boldsymbol{\Pi}_1$, and hence the graph Θ_1 lifts isomorphically to a graph with the same labels in $\mathbb{S}^{\boldsymbol{\Pi}}$. The carriers of the hyperplanes dual to this graph in $\mathbb{S}^{\boldsymbol{\Pi}}$ span a subcomplex $Y = H \times \Theta_1 \cong \mathbb{S}_{L_1} \times \Theta_1$.

Since $\mathcal{H}$ is treelike, $\mathcal{H}_1$ consists of hyperplanes in Y dual to some maximal tree in Θ_1. In particular, $\mathcal{H}_1$ has compatible carriers and the resulting collapse reduces the subcomplex Y to the product of H with a rose and leaves everything else unchanged. The resulting complex is thus isomorphic to $\mathbb{S}^{\boldsymbol{\Pi}'}$, where $\boldsymbol{\Pi}' = \boldsymbol{\Pi} \setminus \boldsymbol{\Pi}_1$. Let $\rho\colon\ \mathbb{S}^{\boldsymbol{\Pi}} \to S^{\boldsymbol{\Pi}'}$ be the collapsing map followed by this isomorphism.

The image of the hyperplanes $\mathcal{H}' = \mathcal{H} \setminus \mathcal{H}_1$ in $\mathbb{S}^{\boldsymbol{\Pi}'}$ is treelike (since the canonical projection from $\mathbb{S}^{\boldsymbol{\Pi}}$ to $\mathbb{S}^{\boldsymbol{\Pi}_i}$ factors through $\mathbb{S}^{\boldsymbol{\Pi}'}$) so, by induction, it has compatible carriers and the resulting space is isomorphic to the Salvetti complex $\mathbb{S}$. It now follows that the original set $\mathcal{H}$ has compatible carriers in $\mathbb{S}^{\boldsymbol{\Pi}}$ since, if p is a loop of edges dual to $\mathcal{H}$, then its image in $S^{\boldsymbol{\Pi}'}$ is a loop dual to $\mathcal{H}'$. This loop must be null-homotopic, hence the same holds for p.

Conversely, suppose $\mathcal{H}$ is a set of hyperplanes in $\mathbb{S}^{\boldsymbol{\Pi}}$ which has compatible carriers and collapses $\mathbb{S}^{\boldsymbol{\Pi}}$ down to $\mathbb{S}$. We again proceed by induction on s. The case $s = 1$ is discussed in Example 4.9, where it is observed that, since $\mathcal{H}$ has compatible carriers, the dual edges $\{e_i\}$ form a forest in the 1–skeleton of $\mathbb{S}^{\boldsymbol{\Pi}}$. Since collapsing along $\mathcal{H}$ reduces $\mathbb{S}^{\boldsymbol{\Pi}}$ to a complex isomorphic to $\mathbb{S}$, which has only one vertex, these edges must form a maximal tree T in the 1–skeleton. Since edges of T are not loops, they correspond to the $\boldsymbol{P}_i$ or to singletons in the $\boldsymbol{P}_i$, so the hyperplane dual to each edge in T is isomorphic to a subcomplex of $\mathbb{S}_L$. An edge of T cannot correspond to a nonmaximal singleton v, since then $\mathrm{lk}(v)$ would be a proper subcomplex of L, the carrier of $\mathcal{H}$ would have fewer cubes than the carrier of $\{H_{\boldsymbol{P}}\}_{\boldsymbol{P}\in\boldsymbol{\Pi}}$, and collapsing along $\mathcal{H}$ and along $\{H_{\boldsymbol{P}}\}_{\boldsymbol{P}\in\boldsymbol{\Pi}}$ would not result in isomorphic complexes. Therefore, T is a maximal tree in the base graph, ie $\mathcal{H}$ is treelike.

Now suppose $s > 1$. As above, assume that the link L_1 is maximal, so the graph Θ_1 may be viewed as a subcomplex of $\mathbb{S}^{\boldsymbol{\Pi}}$. Let $\mathcal{H}_1$ be the set of hyperplanes in $\mathcal{H}$ dual to some edge of Θ_1. We claim that these edges form a maximal tree in Θ_1. Let $Z = (\mathbb{S}^{\boldsymbol{\Pi}})_{\mathcal{H}}$ be the collapse of $\mathbb{S}^{\boldsymbol{\Pi}}$ along $\mathcal{H}$ and $c_{\mathcal{H}}\colon \mathbb{S}^{\boldsymbol{\Pi}} \to Z$ the collapsing map. Let $c\colon \mathbb{S}^{\boldsymbol{\Pi}} \to \mathbb{S}$ be the canonical collapse. By assumption, Z is isomorphic to the Salvetti complex $\mathbb{S}$, so the image of Θ_1 under both c and $c_{\mathcal{H}}$ is a rose. The former generates a free subgroup of the fundamental group A_Γ (namely the subgroup generated by the vertices of Γ with link equal to L_1). Since both collapsing maps are homotopy equivalences, the same must be true of the latter. It follows that the edges dual to $\mathcal{H}_1$ must form a maximal tree in Θ_1.

Now set $X = (\mathbb{S}^{\mathbf{\Pi}})_{\mathcal{H}_1}$ and $\mathbf{\Pi}' = \mathbf{\Pi} \setminus \mathbf{\Pi}_1$. Then the images of Θ_1 in $S^{\mathbf{\Pi}'}$ and in X are isomorphic roses and this isomorphism extends to an isomorphism of the whole complex $S^{\mathbf{\Pi}'} \cong X$. Collapsing X along the image $\mathcal{H}'$ of $\mathcal{H} \setminus \mathcal{H}_1$ gives Z, hence, by induction, it is treelike (viewed as hyperplanes in $S^{\mathbf{\Pi}'}$). We conclude that the original set of hyperplanes $\mathcal{H}$ was treelike in $\mathbb{S}^{\mathbf{\Pi}}$. □

Theorem 4.12 *Let $\mathcal{H}$ and $\mathcal{K}$ be two treelike sets of hyperplanes in $\mathbb{S}^{\mathbf{\Pi}}$. Given any $K \in \mathcal{K}$, there exists $H \in \mathcal{H}$ such that the set of hyperplanes obtained from $\mathcal{H}$ by replacing H by K is again treelike.*

Proof Note that the label dual to a hyperplane in $\mathcal{H}$ (or $\mathcal{K}$) appears in one and only one of the graphs Θ_i, since each Θ_i corresponds to a different link. Say $K \in \mathcal{K}$ is dual to an edge e in Θ_i. Let T_i be the maximal tree in Θ_i formed by edges dual to $\mathcal{H}$. If e lies in T_i, then K also lies in $\mathcal{H}$ and we can take $H = K$. If not, let e' be an edge in T_i on the minimal path between the two vertices of e. Then, replacing e' by e gives another maximal tree T_i' in Θ_i and has no effect on the remaining Θ_j. Thus, replacing the hyperplane H dual to e' by K gives another treelike set. □

Corollary 4.13 *Let $\mathbf{\Pi} = \{\boldsymbol{P}_1, \dots, \boldsymbol{P}_k\}$ be a compatible collection of Γ–Whitehead partitions, let H_i be the hyperplane in $\mathbb{S}^{\mathbf{\Pi}}$ dual to the edges $e_{\boldsymbol{P}_i}$, and let $\mathcal{K}$ be another a treelike set of hyperplanes in $\mathbb{S}^{\mathbf{\Pi}}$. Then the automorphism of A_Γ induced by $\mathbb{S} \leftarrow \mathbb{S}^{\mathbf{\Pi}} \to (\mathbb{S}^{\mathbf{\Pi}})_{\mathcal{K}} \cong \mathbb{S}$ is an element of $\mathrm{U}(\mathrm{A}_\Gamma)$.*

Proof By Theorem 4.12, we can order the elements of $\mathcal{H}$ as $\{H_1, \dots, H_k\}$, so that for each $i = 0, \dots, k$ the set $\mathcal{H}_i = \{H_1, \dots, H_i, K_{i+1}, \dots, K_k\}$ is treelike. If we now set

$$\widehat{\mathcal{H}}_i = \{H_1, \dots, H_{i-1}, K_{i+1}, \dots, K_k\} \quad \text{for } i = 1, \dots, k,$$

then the blowup-collapse $\mathbb{S} \leftarrow \mathbb{S}^{\mathbf{\Pi}} \to (\mathbb{S}^{\mathbf{\Pi}})_{\mathcal{K}}$ factors into the sequence of single blowup-collapses

$$\mathbb{S} = (\mathbb{S}^{\mathbf{\Pi}})_{\mathcal{H}_k} \leftarrow (\mathbb{S}^{\mathbf{\Pi}})_{\widehat{\mathcal{H}}_k} \to (\mathbb{S}^{\mathbf{\Pi}})_{\mathcal{H}_{k-1}} \leftarrow \cdots \to (\mathbb{S}^{\mathbf{\Pi}})_{\mathcal{H}_1} \leftarrow (\mathbb{S}^{\mathbf{\Pi}})_{\widehat{\mathcal{H}}_1} \to (\mathbb{S}^{\mathbf{\Pi}})_{\mathcal{H}_0} = (\mathbb{S}^{\mathbf{\Pi}})_{\mathcal{K}}.$$

The statement of the corollary now follows from Lemmas 2.2 and 3.2. □

4.1 Construction of K_Γ

We are now ready to define the simplicial complex K_Γ as the geometric realization of a partially ordered set of Γ–complexes.

Definition 4.14 A *marked* Γ*–complex* σ is a pair $\sigma = (X, \alpha)$, where:

(1) X is a cube complex isomorphic to $\mathbb{S}^{\mathbf{\Pi}}$ for some (possibly empty) compatible set $\mathbf{\Pi}$ of Γ–Whitehead partitions.

(2) $\alpha\colon X \to \mathbb{S}$ is a homotopy equivalence and the composition $\mathbb{S} \xrightarrow{c_{\mathbf{\Pi}}^{-1}} \mathbb{S}^{\mathbf{\Pi}} \cong X \xrightarrow{\alpha} \mathbb{S}$ induces an element of $\mathrm{U}(\mathrm{A}_\Gamma)$.

Two marked Γ–complexes $\sigma = (X, \alpha)$ and $\sigma' = (X', \alpha')$ are *equivalent* if there is an isomorphism of cube complexes $h\colon X \to X'$ with $\alpha' \circ h \simeq \alpha$. If X is isomorphic to $\mathbb{S}$, the equivalence class of (X, α) is called a *marked Salvetti*.

Note that the second condition in the definition of a marked Γ–complex is independent of the choice of isomorphism $X \cong \mathbb{S}^{\mathbf{\Pi}}$ or collapse $c_{\mathbf{\Pi}}$ by Corollary 4.13.

Examples 4.15 (1) For a Γ–Whitehead pair (P, m) we observed, in the discussion preceding Lemma 3.2, that the collapsing maps c_P and c_m on $\mathbb{S}^{\boldsymbol{P}}$ differ by the isomorphism that interchanges the hyperplanes dual to e_P and e_m. It follows that $(\mathbb{S}^{\boldsymbol{P}}, c_P) \sim (\mathbb{S}^{\boldsymbol{P}}, c_m)$.

(2) If $\varphi \in \mathrm{Out}(\mathrm{A}_\Gamma)$ is a product of symmetries and inversions, then it can be represented by an isomorphism $\widehat{\varphi}\colon \mathbb{S} \to \mathbb{S}$, hence $(\mathbb{S}, \mathrm{id}) \sim (\mathbb{S}, \widehat{\varphi})$.

We now define a partial ordering on the set of marked Γ–complexes. If $\sigma = (X, \alpha)$, $\mathcal{H}$ is a set of hyperplanes of X contained in some treelike set, and $c\colon X \to X_{\mathcal{H}}$ is the collapsing map, we denote by $\sigma_{\mathcal{H}}$ the marked Γ–complex $(X_{\mathcal{H}}, \alpha \circ c^{-1})$. For two marked Γ–complexes σ and σ', define

$$\sigma' < \sigma \quad \text{if } \sigma' = \sigma_{\mathcal{H}} \text{ for some } \mathcal{H}.$$

Definition 4.16 The Γ*–spine* K_Γ is the simplicial complex associated to the partially ordered set of equivalence classes of marked Γ–complexes.

We can identify $\mathrm{Out}(\mathrm{A}_\Gamma)$ with the group of homotopy classes of maps $\mathbb{S} \to \mathbb{S}$. Using this identification, we define a left action of $\mathrm{U}(\mathrm{A}_\Gamma)$ on K_Γ by $\varphi \cdot (X, \alpha) = (X, \varphi \circ \alpha)$.

Proposition 4.17 *The action of* $\mathrm{U}(\mathrm{A}_\Gamma)$ *on* K_Γ *is proper.*

Proof Since each marked blowup (X, α) can be collapsed to finitely many marked Salvettis, it suffices to prove that the stabilizer of some (hence any) marked Salvetti is finite. This is true for $(\mathbb{S}, \mathrm{id})$ since any isomorphism $\mathbb{S} \to \mathbb{S}$ takes the one skeleton to the one skeleton, hence induces a permutation on $V^{\pm}$. Thus $(\mathbb{S}, \mathrm{id}) \sim (\mathbb{S}, \alpha)$ if and only if α lies in the (finite) group generated by graph symmetries and inversions. □

Let (P,m) be a Γ–Whitehead pair and α the corresponding Whitehead automorphism. By Lemma 3.2, α is realized by the blowup-collapse, $\alpha = c_m \circ c_{\boldsymbol{P}}^{-1} \colon \mathbb{S} \to \mathbb{S}^{\boldsymbol{P}} \to \mathbb{S}$. If we start at the Salvetti $(\mathbb{S}, \mathrm{id})$, this gives a path in K_Γ which ends at $(\mathbb{S}, \alpha)$:

$$(1) \qquad (\mathbb{S}, \mathrm{id}) < (\mathbb{S}^{\boldsymbol{P}}, c_{\boldsymbol{P}}) \sim (\mathbb{S}^{\boldsymbol{P}}, c_m) = (\mathbb{S}^{\boldsymbol{P}}, \alpha \circ c_{\boldsymbol{P}}) > (\mathbb{S}, \alpha).$$

More generally, for any $\varphi \in \mathrm{U}(\mathrm{A}_\Gamma)$, we can translate this path by φ to obtain a path from $(\mathbb{S}, \varphi)$ to $(\mathbb{S}, \varphi \circ \alpha)$.

Definition 4.18 If $\sigma = (\mathbb{S}, \varphi)$, we call the φ–translate of path (1) above the *Whitehead move* at σ associated to $(\boldsymbol{P}, m)$, and write $\sigma_m^{\boldsymbol{P}} = (\mathbb{S}, \varphi \circ \alpha)$.

Using this terminology, Corollary 4.13 can be restated in the following useful form:

Corollary 4.19 (factorization lemma) *Let $\sigma = (\mathbb{S}, \alpha)$ be a marked Salvetti, $\boldsymbol{\Pi} = \{\boldsymbol{P}_1, \dots, \boldsymbol{P}_k\}$ a compatible collection of Γ–Whitehead partitions, $\sigma^{\boldsymbol{\Pi}} = (\mathbb{S}^{\boldsymbol{\Pi}}, c_{\boldsymbol{\Pi}} \circ \alpha)$ be the blowup of σ with respect to $\boldsymbol{\Pi}$, and $\mathcal{H}$ a treelike set of hyperplanes in $\mathbb{S}^{\boldsymbol{\Pi}}$. Then with a suitable ordering of the elements of $\mathcal{H}$ there is a chain $\sigma = \sigma_0, \sigma_1, \dots, \sigma_k = \sigma_{\mathcal{H}}^{\boldsymbol{\Pi}}$ such that each σ_i is connected to σ_{i-1} by a Whitehead move.*

Our goal is show that K_Γ is contractible. As a first step we have the following:

Proposition 4.20 K_Γ *is connected.*

Proof By definition, every vertex of K_Γ lies in the star of some marked Salvetti. It is straightforward to verify that the subgroup generated by Γ–Whitehead automorphisms is normal in $\mathrm{U}(\mathrm{A}_\Gamma)$, hence any $\varphi \in \mathrm{U}(\mathrm{A}_\Gamma)$ can be factored as a product $\varphi = \varphi_1 \circ \varphi_2$, where φ_1 is a product of symmetries and inversions and φ_2 is a product of Γ–Whitehead automorphisms. It follows from Examples 4.15 and Corollary 4.19 that $(\mathbb{S}, \mathrm{id}) = (\mathbb{S}, \varphi_1)$ is connected by a path in K_Γ to $(\mathbb{S}, \varphi_1 \circ \varphi_2) = (\mathbb{S}, \varphi)$. □

5 Examples

In this section we pause to compute a few examples of the complexes K_Γ. These complexes quickly become very complicated and difficult to describe, but invariants such as dimension are relatively easy to compute.

In the next section we will show that K_Γ is contractible. Since the action is proper, the dimension of K_Γ gives an upper bound for the virtual cohomological dimension of $U(A_\Gamma)$. We remark that the kernel IA_Γ of the natural map $\mathrm{Out}(A_\Gamma) \to \mathrm{GL}(n,\mathbb{Z})$ is a torsion-free subgroup of $U(A_\Gamma)$, so acts freely on K_Γ, and the dimension of K_Γ is an upper bound for its cohomological dimension.

Since there are only finitely many ways to partition a finite set, and maximal simplices of K_Γ correspond to maximal sets of pairwise-compatible Γ–partitions, K_Γ is always finite-dimensional. In any explicit example the dimension of K_Γ can be computed precisely by finding a maximal set of pairwise-compatible Γ–partitions.

5.1 3–vertex graphs

We first consider graphs with 3 vertices.

For any n, if Γ has n vertices and no edges, then A_Γ is the free group of rank n, the Salvetti complex is a rose with n petals and Γ–complexes are graphs of rank n without univalent or bivalent vertices and with no separating edges; in other words, K_Γ is the spine K_n of (reduced) outer space. For $n = 2$ this can be identified with the usual tree for $\mathrm{SL}(2,\mathbb{Z})$, but for $n = 3$ this is already quite a complicated space; for example, the link of a rose is 2–dimensional and homotopy equivalent to a wedge of eleven 2–spheres.

If Γ is a complete graph on n vertices then A_Γ is the free abelian group of rank n and there are no Γ–Whitehead partitions, so there are no Γ–complexes other than the Salvetti (which is a torus) and K_Γ is a single point.

If Γ has 3 vertices $\{a,b,c\}$ and two edges, one joining a to c and one joining b to c, then A_Γ is the product $F_2 \times \mathbb{Z} = \langle a,b\rangle \times \langle c\rangle$ and the Salvetti is the product of a rose with a circle. Recall that a Γ–Whitehead partition $\{P, P^*, \mathrm{lk}(P)\}$ is completely determined by giving either of its sides P or P^*. The only Γ–Whitehead partitions in this case are those determined by $P_1 = \{a,b\}$ and $P_2 = \{a,b^{-1}\}$, so the only Γ complexes are products of a circle with a rank 2 graph (with no separating edges), and K_Γ can be identified with the spine K_2 of outer space for the rank 2 free group $\langle a,b\rangle$.

If Γ has just a single edge from a to b, then A_Γ is the free product $\mathbb{Z}^2 at \mathbb{Z} = \langle a,b\rangle * \langle c\rangle$, and the Salvetti is a torus wedged with a circle. The only Γ–Whitehead partitions are those with sides $P_1 = \{a,c\}$, $P_2 = \{b,c\}$, $P_3 = \{a,c^{-1}\}$ and $P_4\{b,c^{-1}\}$. The simplicial star in K_Γ of a Salvetti is a square with corners $S^{\mathbf{\Pi}}$, where $\mathbf{\Pi}$ is $\{P_1,P_2\}$, $\{P_2,P_3\}$, $\{P_3,P_4\}$ or $\{P_4,P_1\}$. The Γ–complex associated to each corner is a torus cut into four squares, with an arc joining the two vertices. There are exactly four hyperplanes in this Γ–complex which can be collapsed, so four stars of Salvettis fit together at each corner, and K_Γ is a plane tiled by these stars. The group $U(A_\Gamma)$ is isomorphic to $\mathbb{Z}^2$ and acts on K_Γ by translations.

5.2 Trees

Recall that a RAAG A_Γ is called 2–*dimensional* if Γ has no triangles, ie if the associated Salvetti complex is 2–dimensional. In [2] we computed upper and lower bounds for the virtual cohomological dimension of $\mathrm{Out}(A_\Gamma)$ for 2–dimensional RAAGs. For Γ a tree, these bounds agree, and we get

$$\mathrm{VCD}(\mathrm{Out}(A_\Gamma)) = e + 2\ell - 3,$$

where e is the number of edges and ℓ is the number of leaves.

In the case that Γ is a tree, the only twists, $v \mapsto vw$, occur when v is a leaf attached at w. It follows that the twist subgroup $T(A_\Gamma)$ is free abelian of rank ℓ. It is easily checked that, in this case, $T(A_\Gamma)$ is a normal subgroup of $U(A_\Gamma)$ and $T(A_\Gamma) \cap U(A_\Gamma) = \{1\}$, so $\mathrm{Out}(A_\Gamma)$ is the semidirect product of these two subgroups. Since the cohomological dimension of $T(A_\Gamma) = \ell$, we conclude that $\mathrm{VCD}(U(A_\Gamma)) \geq e + \ell - 3$. In many (but not all) examples, the dimension of K_Γ is exactly $e + \ell - 3$ and hence $\mathrm{VCD}(U(A_\Gamma)) = \dim(K_\Gamma) = e + \ell - 3$.

Consider, for example, the case in which Γ is a line made up of e edges. Label the vertices $v_0, v_1, \dots, v_e$. Note that the generators of $U(A_\Gamma)$ are the partial conjugations by v_i for $2 \leq i \leq e-2$ and the folds $v_0 \mapsto v_0 v_2^{\pm}$ and $v_e \mapsto v_e v_{e-2}^{\pm}$. In particular, there are no Γ–Whitehead partitions with v_0, v_1, v_{e-1} or v_e as maximal element.

For $2 \leq i \leq e-2$, let (R_i, v_i) be the Γ–Whitehead pair corresponding to the partial conjugation by v_i of all vertices to the right of v_i, that is, $R_i = \{v_i, v_{i+2}^{\pm 1}, \dots v_e^{\pm 1}\}$. Similarly, let (L_i, v_i) be the pair corresponding to the partial conjugation by v_i of all vertices to the left of v_i. Let $P_0 = \{v_0, v_2\}$ and $P_e = \{v_e, v_{e-2}\}$, so (P_0, v_2) and (P_e, v_{e-2}) are the Γ–Whitehead pairs corresponding to folds onto v_0 and v_e. Then a typical maximal set of compatible partitions is of the form

$$P_0,\ L_2, \dots,\ L_j,\ R_{j+1}, \dots,\ R_{e-2},\ P_e.$$

Thus, the dimension of K_Γ is $e - 1 = e + \ell - 3$.

On the other hand, there exist examples of trees for which this equality does not hold. We leave it as an exercise for the reader to show that, for the tree Γ pictured in Figure 4, $\dim(K_\Gamma) = e + \ell - 1 = 10$. Thus, for this tree, we can only conclude that $8 \leq \mathrm{VCD}(U(A_\Gamma)) \leq 10$.

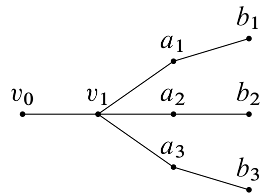


Figure 4: An example for which $\dim(\mathrm{K}_\Gamma) \neq e + \ell - 3$

5.3 Strings of diamonds

Let Γ be a "string of d diamonds", with $d \geq 3$ (see Figure 5).

This defines a 2–dimensional $RAAG$ which has no twist automorphisms, so $\mathrm{U}(\mathrm{A}_\Gamma) = \mathrm{Out}(\mathrm{A}_\Gamma)$. The upper and lower bounds from [2] in this case are

$$3d + 1 \leq \mathrm{VCD}(\mathrm{Out}(\mathrm{A}_\Gamma)) \leq 6d - 4.$$

We calculate the dimension of K_Γ by finding the maximal possible size of a collection of compatible Γ–Whitehead partitions.

Let (P, m) be a Γ–Whitehead pair with associated partition $\{P, P^*, \mathrm{lk}(m)\}$.

- There are no such pairs with $m = c_0$ or $m = c_d$.

- Suppose $m = c_1$. Removing the star of c_1 separates Γ into a left and right component, $L_1 = \{c_0\}$ and R_1. Since $c_0 \leq c_1$, either c_0 or c_0^{-1} can be a singleton. The possibilities for P are $\{c_1, c_0\}$, $\{c_1^{-1}, c_0^{-1}\}$, $\{c_1, c_0^{-1}\}$, $\{c_1^{-1}, c_0\}$, $\{c_1, c_0, c_0^{-1}\}$ and $\{c_1^{-1}, c_0, c_0^{-1}\}$. At most two of these are pairwise compatible, eg $\{c_1, c_0\}$ and $\{c_1^{-1}, c_0^{-1}\}$. The case $m = c_{d-1}$ is symmetric.

- Suppose $m = c_i$ with $2 \leq i \leq d-2$. Removing the star of c_i again separates Γ into two components, L_i and R_i. Now there are no other possible singletons, so $P = \{c_i, L_i^{\pm}\}$ or $P = \{c_i, R_i^{\pm}\}$ (which are not compatible.)

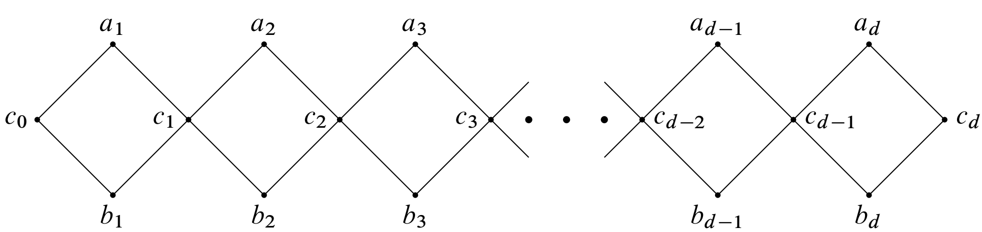


Figure 5: A string of d diamonds

- If $m = a_1$, the star separates Γ into $\{b_1\}$ and another component C; then b_1 can be a singleton, and the possibilities for P are $\{a_1, b_1\}$, $\{a_1, b_1^{-1}\}$, $\{a_1, b_1, b_1^{-1}\}$, $\{a_1, C^{\pm}\}$, $\{a_1, b_1, C^{\pm}\}$ and $\{a_1, b_1^{-1}, C^{\pm}\}$. At most two of these are pairwise compatible, eg $\{a_1, b_1\}$ and $\{a_1, b_1, C^{\pm}\} = \{a_1^{-1}, b_1^{-1}\}^*$. The case of $m = a_d$ is similar.

- If $m = a_i$ for $2 \le i \le d-1$, the star separates Γ into three components, K_i, $\{b_i\}$ and S_i. Possibilities for P include $\{a_i, b_i\}$, $\{a_i, b_i, K_i^{\pm}\}$ and $\{a_i, b_i, K_i^{\pm}, S_i^{\pm}\} = \{a_i^{-1}, b_i^{-1}\}^*$, which is a maximal pairwise compatible collection with $m = a_i$. The possibilities for $m = b_i$ are symmetric.

One can check that the following sets determine a collection of pairwise compatible Γ–Whitehead partitions:

(1) $\{c_0, c_1\}$, $\{c_0^{-1}, c_1^{-1}\}$, $\{c_{d-1}, c_d\}$ and $\{c_{d-1}^{-1}, c_d^{-1}\}$.

(2) $\{c_i, L_i^{\pm}\}$ for $2 \le i \le d-2$.

(3) $\{a_i, b_i\}$ and $\{a_i^{-1}, b_i^{-1}\}$ for $1 \le i \le d$.

(4) $\{a_i, b_i, K_i\}$ for $2 \le i \le d-1$.

By the remarks above, this is a maximal collection. There are $4+(d-3)+2d+(d-2) = 4d-1$ of them, so $\dim(\mathrm{K}_\Gamma) = 4d-1$.

5.4 Butterflies

Let Γ be the "butterfly" graph shown in Figure 6 (which is a string of two diamonds if $n = 2$). Again A_Γ is 2–dimensional with no twists, so $\mathrm{U}(\mathrm{A}_\Gamma) = \mathrm{Out}(\mathrm{A}_\Gamma)$. The bounds on the VCD from [2] in this case are

$$4n-1 \le \mathrm{VCD}(\mathrm{Out}(\mathrm{A}_\Gamma)) \le 4n.$$

The maximal size of a compatible set of Γ–Whitehead partitions is $4n-1$, realized for example by the partitions determined by the following sets P:

(1) $\{c_1, c_0\}$, $\{c_1^{-1}, c_0^{-1}\}$ and $\{c_1, c_0, c_2\}$.

(2) $\{a_1, a_2, \dots a_i\}$ and $\{a_1^{-1}, a_2^{-1}, \dots a_i^{-1}\}$ for $2 \le i \le n$.

(3) $\{b_1, b_2, \dots b_i\}$ and $\{b_1^{-1}, b_2^{-1}, \dots b_i^{-1}\}$ for $2 \le i \le n$.

Thus the dimension of K_Γ is $4n-1$. Since this matches the lower bound, this is equal to the VCD of $\mathrm{Out}(\mathrm{A}_\Gamma)$.

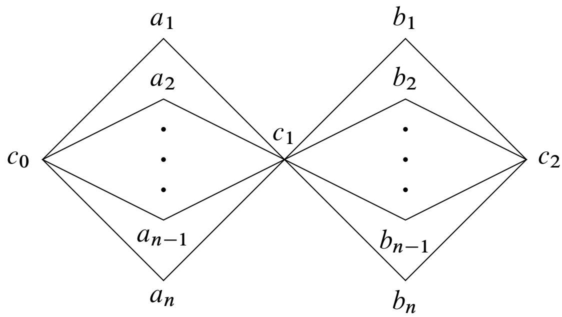


Figure 6: Butterfly graph

6 Contractibility of the Γ–spine

Our strategy for showing that K_Γ is contractible is to view it as the union of stars of marked Salvettis. We first define a norm which totally orders the marked Salvettis. We then construct the space by starting with the star of the marked Salvetti of minimal norm, then attaching the stars of the rest of the marked Salvettis in increasing order. We check at each stage that we are attaching along a contractible subcomplex.

6.1 The norm of a marked Salvetti

The norm is defined using lengths of conjugacy classes of elements of A_Γ, and we begin with some observations about these lengths. Let $\sigma = (\mathbb{S}, \alpha)$ be a marked Salvetti. For any conjugacy class g, define $\ell_\sigma(g)$ to be the minimal length of a word $\boldsymbol{w}$ in the free group $F(V)$ representing an element of the conjugacy class of $\alpha^{-1}(g)$ in A_Γ. In particular, if $\sigma = (\mathbb{S}, \mathrm{id})$, then $\ell_\sigma(g)$ is the minimal word length of an element of g.

Normal form for elements of A_Γ (see eg [3]) implies that $\ell_\sigma(g)$ is well-defined. Since the vertices V of Γ can be identified with the edges in the 1–skeleton of $\mathbb{S}$, $\ell_\sigma(g)$ can also be thought of as the length of a minimal edge-path in the 1–skeleton of $\mathbb{S}$ representing $\alpha^{-1}(g)$. If α is an isometry of $\mathbb{S}$, then this is the same as the length of g, reflecting the fact that $\sigma = (\mathbb{S}, \alpha)$ is equal to $(\mathbb{S}, \mathrm{id})$ as a marked Salvetti.

Let $\mathcal{G} = (g_1, g_2, \dots)$ be a list of all conjugacy classes in A_Γ and let $\mathcal{G}_0$ be the set of conjugacy classes which can be represented by words of length at most 2.

Definition 6.1 For a marked Salvetti $\sigma = (X, \alpha)$, the *norm*

$$\|\sigma\| = (\|\sigma\|_0, \|\sigma\|_1, \|\sigma\|_2, \dots) \in \mathbb{Z} \times \mathbb{Z}^{\mathcal{G}}$$

is defined by

$$\|\sigma\|_0 = \sum_{g\in\mathcal{G}_0} \ell_\sigma(g), \qquad \|\sigma\|_i = \ell_\sigma(g_i) \quad \text{for } i \geq 1.$$

We consider $\mathbb{Z}\times\mathbb{Z}^{\mathcal{G}}$ as an ordered abelian group, with the lexicographical ordering. Write $\mathbf{0}=(0,0,\dots)$ for the identity element. We say $x=(x_0,x_1,\dots)\in\mathbb{Z}\times\mathbb{Z}^{\mathcal{G}}$ is *negative* if $x<\mathbf{0}$ and *strongly negative* if its first coordinate x_0 is negative.

Lemma 6.2 *The marked Salvetti* $(\mathbb{S},\mathrm{id})$ *is the unique marked Salvetti of minimal norm. Indeed, for any other marked Salvetti* σ*, we have* $\|(\mathbb{S},\mathrm{id})\|_0 < \|\sigma\|_0$.

Proof An easy calculation shows that, if $|V|=m$, the first coordinate $\|(\mathbb{S},\mathrm{id})\|_0$ is equal to $2m^2+2m$, and that is the minimal possible value. It suffices to show that the first coordinate of the norm of any other marked Salvetti is strictly larger.

Suppose $\sigma=(\mathbb{S},\alpha)$ is another marked Salvetti complex with $\|\sigma\|_0=2m^2+2m$. Then α must permute the conjugacy classes in $\mathcal{G}_0$, since otherwise $\ell_\sigma(g)>2$ for some $g\in\mathcal{G}_0$. In fact, a stronger statement holds: α must permute the conjugacy classes of $V^{\pm}\subset\mathcal{G}_0$, since, if $\ell_\sigma(v)=2$ for some $v\in V$, then $\ell_\sigma(v^2)=4$. Thus, α induces a permutation of the directed edges of $\mathbb{S}$. Moreover, if two edges of $\mathbb{S}$ span a cube, then their images must also span a cube, since, if v, $w\in V$ do not commute, then neither do any conjugates of v and w. Thus, after composing with an isometry of $\mathbb{S}$ we may assume α takes every element of V to a conjugate of itself.

Let $V=\{v_1,\dots v_m\}$ and choose an automorphism $\alpha_1\in\mathrm{Aut}(\mathrm{A}_\Gamma)$ representing α such that $\alpha_1(v_1)=v_1$. Say $\alpha_1^{-1}(v_2)=av_2a^{-1}$, where a is of minimal length (ie av_2a^{-1} is a reduced word). Then $\ell_\sigma(v_1v_2)=2$ implies that the cyclic reduction of $v_1av_2a^{-1}$ is a word of length 2. The only way this can happen is if a lies in the centralizer of v_1. Thus, we can compose α_1 with conjugation by a^{-1} to get a new representative α_2 which acts as the identity on both v_1 and v_2.

Now repeat with v_3. Say $\alpha_2^{-1}(v_3)=bv_3b^{-1}$. Arguing as above, b must lie in the intersection of the centralizers $C(v_1)\cap C(v_2)$, so composing α_2 with conjugation by b^{-1} gives a representative for α which acts as the identity on v_1, v_2 and v_3. Continuing in this manner, we see that α has a representative which is the identity on all of V, that is, α is homotopic to the identity map. □

Corollary 6.3 *Given a marked Salvetti* $\sigma=(\mathbb{S},\alpha)$*, there is a finite set of conjugacy classes* $\mathcal{G}_\sigma\subset\mathrm{A}_\Gamma$ *such that* σ *is uniquely determined by* $\sum_{g\in\mathcal{G}_\sigma}\ell_\sigma(g)$.

Proof Replace $\mathcal{G}_0$ in the proof of Lemma 6.2 by the set of g with $\ell_\sigma(g)\leq 2$. □

In particular, no two marked Salvettis have the same norm. In Section 6.5 we will show that the norm induces a well-ordering of the set of marked Salvettis but we need some preparation first.

6.2 Effect of a Whitehead move on the norm

We want to construct K_Γ by adding stars of marked Salvettis to the star of $(\mathbb{S}, \mathrm{id})$ in order of increasing norm. Since adjacent stars are connected by Γ–Whitehead moves, we will need to understand how lengths of conjugacy classes change under these moves. We will do this first by using the geometric interpretation of $\ell_\sigma(g)$.

Let $\sigma = (\mathbb{S}, \alpha)$ be a marked Salvetti, $\boldsymbol{P}$ a Γ–Whitehead partition of $V^\pm$ and $v \in \max(P)$. The length $\ell_\sigma(g)$ is the length of the shortest edge path in the 1–skeleton of $\mathbb{S}$ representing the free homotopy class of $\alpha^{-1}(g)$. To understand what happens to this length under the Whitehead automorphism $(\boldsymbol{P}, v)$ we will find a minimal length edge path in $\mathbb{S}^{\boldsymbol{P}}$ representing $\alpha^{-1}(g)$ and apply Lemma 4.5.

Recall that $\mathbb{S}^{\boldsymbol{P}}$ has exactly two vertices, corresponding to the two sides of $\boldsymbol{P}$, and an edge labeled $e_{\boldsymbol{P}}$ between them. For $u \in \mathrm{lk}(\boldsymbol{P})$, there is a loop labeled e_u at each vertex. For $u \notin \mathrm{lk}(\boldsymbol{P})$, there is a unique edge labeled e_u with initial vertex corresponding to the side of $\boldsymbol{P}$ containing u and terminal vertex corresponding to the side containing u^{-1}. Let $\boldsymbol{w}$ be a cyclically reduced word for $\alpha^{-1}(g)$. Identifying directed edges of $\mathbb{S}$ with $V^\pm$, let p be the edge path in $\mathbb{S}$ labeled by $\boldsymbol{w}$. We can lift p to a loop $\tilde{p}$ in $\mathbb{S}^{\boldsymbol{P}}$ as follows. If the support of $\boldsymbol{w}$ lies entirely in $\mathrm{lk}(\boldsymbol{P})$, then p lifts to an edge path $\tilde{p}$ of the same length at either vertex. Otherwise, cyclically permuting $\boldsymbol{w}$ if necessary, let $\boldsymbol{w} = u_1 \cdots u_k$, where $u_i \in V^\pm$ and $u_1 \notin \mathrm{lk}(\boldsymbol{P})$. Then u_1 corresponds to a unique directed edge e_1 in $\mathbb{S}^{\boldsymbol{P}}$. If u_1 and u_2^{-1} both lie in $\bar{P} = P \cup \mathrm{lk}(P)$ or both in $\bar{P}^* = P^* \cup \mathrm{lk}(P)$, then u_2 lifts to a directed edge e_2 whose initial vertex equals the terminal vertex of e_1. Hence, u_1u_2 lifts to the path e_1e_2. If not, insert the edge $e_{\boldsymbol{P}}$ (appropriately oriented) to get a path $e_1e_{\boldsymbol{P}}e_2$ which projects to u_1u_2. Now repeat this process with each u_i to obtain the loop $\tilde{p}$.

It is easy to see that $\tilde{p}$ is a minimal length lift of p. To see that it is a minimal length representative for $\alpha^{-1}(g)$, note that any other minimal word $\boldsymbol{w}'$ for $\alpha^{-1}(g)$ can be obtained from $\boldsymbol{w}$ by interchanging commuting pairs u_iu_{i+1}. But, for such a pair, the edges e_{u_i} and $e_{u_{i+1}}$ span a square in $\mathbb{S}^{\boldsymbol{P}}$, so they can be traversed in either order without crossing $e_{\boldsymbol{P}}$. It follows that the length of $\tilde{p}$ is independent of choice of $\boldsymbol{w}$.

To keep track of the lengths of these paths, we introduce some new notation. Set

- $|\boldsymbol{P}|_{\boldsymbol{w}}$ to be the number of times $\tilde{p}$ traverses the edge $e_{\boldsymbol{P}}$, or equivalently, the number of (cyclically) adjacent letters u_iu_{i+1} in $\boldsymbol{w}$ such that u_i and u_{i+1}^{-1} do not both lie in $\bar{P}$ or both in $\bar{P}^*$;

- $|v|_{\boldsymbol{w}}$ to be the number of occurrences of v or v^{-1} in $\boldsymbol{w}$.

Lemma 6.4 *Let $\sigma = (\mathbb{S}, \alpha)$ be a marked Salvetti, let $\varphi = (\boldsymbol{P}, v)$ be a Whitehead automorphism, let g be a conjugacy class in A_Γ and let $\boldsymbol{w}$ be a minimal length word representing $\alpha^{-1}(g)$. Then*

$$\ell_{\sigma_v^{\boldsymbol{P}}}(g) = \ell_\sigma(g) + |\boldsymbol{P}|_{\boldsymbol{w}} - |v|_{\boldsymbol{w}}.$$

More generally, if σ' is obtained from σ by blowing up a compatible collection of Γ–Whitehead partitions $\boldsymbol{\Pi} = \{\boldsymbol{P}_1, \dots, \boldsymbol{P}_k\}$ and collapsing a treelike set of hyperplanes $\mathcal{H} = \{H_1, \dots, H_k\}$ dual to edges labeled e_{v_i}, then

$$\ell_{\sigma'}(g) = \ell_\sigma(g) + \sum_{i=1}^{k} |\boldsymbol{P}_i|_{\boldsymbol{w}} - \sum_{i=1}^{k} |v_i|_{\boldsymbol{w}}.$$

Proof First consider the blowup-collapse for a single Whitehead pair $(\boldsymbol{P}, v)$. Let p and $\tilde{p}$ be as above. By construction, $\ell_\sigma(g) = \text{length}(p) = \text{length}(\tilde{p}) - |\boldsymbol{P}|_{\boldsymbol{w}}$. Collapsing the hyperplane in $\mathbb{S}^{\boldsymbol{P}}$ dual to the edge labeled e_v gives the marked Salvetti $\sigma_v^{\boldsymbol{P}} = (\mathbb{S}, \varphi\alpha)$. Let p' be the image of $\tilde{p}$ under this collapse. By Lemma 4.5, p' is a minimal length representative for $(\varphi\alpha)^{-1}(g)$. Hence, $\ell_{\sigma_v^{\boldsymbol{P}}}(g) = \text{length}(p') = \text{length}(\tilde{p}) - |v|_{\boldsymbol{w}}$. This proves the first statement.

For the second statement, let $c\colon \mathbb{S}^{\boldsymbol{\Pi}} \to \mathbb{S}$ be the canonical projection and let $c_{\mathcal{H}}$ be the collapsing map onto $\mathbb{S}^{\boldsymbol{\Pi}}_{\mathcal{H}}$. Let $\tilde{p}$ be a minimal edge path in $\mathbb{S}^{\boldsymbol{\Pi}}$ representing $\alpha^{-1}(g)$. (Here we identify the fundamental group of $\mathbb{S}^{\boldsymbol{\Pi}}$ and $\mathbb{S}$ via c.) Let $p = c(\tilde{p})$ and $p' = c_{\mathcal{H}}(\tilde{p})$. Then, by Lemma 4.5, p and p' are minimal paths in their homotopy class. In particular, p corresponds to a minimal word $\boldsymbol{w}$ representing $\alpha^{-1}(g)$, so the number of edges of p (and hence also of $\tilde{p}$) labeled e_{v_i} equals $|v_i|_{\boldsymbol{w}}$.

Collapsing $\mathbb{S}^{\boldsymbol{\Pi}}$ to a single blowup $\mathbb{S}^{\boldsymbol{P}_i}$ maps $\tilde{p}$ to a minimal lift p_i of p, hence, by the discussion above, the number of edges of p_i (and hence also of $\tilde{p}$) labeled $e_{\boldsymbol{P}_i}$ equals $|\boldsymbol{P}_i|_{\boldsymbol{w}}$. It now follows that

$$\ell_\sigma(g) = \text{length}(p) = \text{length}(\tilde{p}) - \sum |\boldsymbol{P}_i|_{\boldsymbol{w}},$$

$$\ell_{\sigma'}(g) = \text{length}(p') = \text{length}(\tilde{p}) - \sum |v_i|_{\boldsymbol{w}}. \qquad \square$$

Remark 6.5 The hypothesis that every hyperplane in $\mathcal{H}$ be dual to an edge labeled e_{v_i} is crucial in this lemma. In general, a treelike set $\mathcal{H} = \{H_1, \dots, H_k\}$ in $\mathbb{S}^{\boldsymbol{\Pi}}$ may contain hyperplanes dual to edges labeled $e_{\boldsymbol{P}_i}$. Collapsing these hyperplanes first to get a smaller Γ–complex, we see that $\sigma' = \sigma^{\boldsymbol{\Pi}}_{\mathcal{H}}$ can be obtained from σ by a blowup-collapse satisfying the conditions of the lemma.

We put all of this information together for a marked Salvetti $\sigma = (\mathbb{S}, \alpha)$ and a Whitehead automorphism $\varphi = (P, v)$ by defining absolute values $|P|_\sigma$ and $|v|_\sigma$ in $\mathbb{Z} \times \mathbb{Z}^{\mathcal{G}}$ coordinatewise, ie

$$|P|_\sigma = (|P|_0, |P|_{\boldsymbol{w}_1}, |P|_{\boldsymbol{w}_2}, \dots),$$

where

- $\boldsymbol{w}_i$ is a minimal length word representing the conjugacy class $\alpha^{-1}(g_i)$, and
- $|P|_0 = \sum_{\boldsymbol{w} \in \mathcal{W}_0} |P|_{\boldsymbol{w}}$ for a set of words $\mathcal{W}_0$ representing the $\alpha^{-1}(g)$ for $g \in \mathcal{G}_0$.

Similarly, define

$$|v|_\sigma = (|v|_0, |v|_{\boldsymbol{w}_1}, |v|_{\boldsymbol{w}_2}, \dots),$$

where $|v|_0 = \sum_{\boldsymbol{w} \in \mathcal{W}_0} |v|_{\boldsymbol{w}}$.

Lemma 6.4 can now be restated as:

Corollary 6.6 *Let $\boldsymbol{\Pi}$ and $\mathcal{H}$ be as in Lemma 6.4. Then*

$$\|\sigma_{\mathcal{H}}^{\boldsymbol{\Pi}}\| = \|\sigma\| + \sum |\boldsymbol{P}_i|_\sigma - \sum |v_i|_\sigma.$$

Definition 6.7 A Γ–Whitehead partition $\boldsymbol{P}$ is *reductive* for a marked Salvetti σ if for some $v \in \max(P)$ the Whitehead automorphism $\varphi = (P, v)$ reduces $\|\sigma\|$, that is, $\|\sigma_v^{\boldsymbol{P}}\| < \|\sigma\|$, or, equivalently, $|\boldsymbol{P}_\sigma| < |v_\sigma|$. It is *strongly reductive* if the first coordinate $\|\sigma_v^{\boldsymbol{P}}\|_0$ is less than $\|\sigma\|_0$.

By Corollary 6.3, σ and $\sigma_v^{\boldsymbol{P}}$ cannot have the same norm, since they are different marked Salvettis.

Corollary 6.8 *Let $\boldsymbol{\Pi}$ and $\mathcal{H}$ be as in Lemma 6.4. If $\|\sigma_{\mathcal{H}}^{\boldsymbol{\Pi}}\| < \|\sigma\|$, then some $\boldsymbol{P}_i \in \boldsymbol{\Pi}$ is reductive for σ. If $\sigma_{\mathcal{H}}^{\boldsymbol{\Pi}}$ is strongly reductive, then so is some $\boldsymbol{P}_i$.*

Proof By Theorem 4.12, the elements of $\mathcal{H}$ can ordered so that, if e_{v_i} is the edge dual to H_i, then (P_i, v_i) is a Γ–Whitehead pair. If $\|\sigma_{\mathcal{H}}^{\boldsymbol{\Pi}}\| < \|\sigma\|$, then, by the previous corollary, $|\boldsymbol{P}_i|_\sigma - |v_i|_\sigma < \boldsymbol{0}$ for some i, so $\|\sigma_{v_i}^{\boldsymbol{P}_i}\| < \|\sigma\|$. The same argument applied to the first coordinate of the norm shows that if $\sigma_{\mathcal{H}}^{\boldsymbol{\Pi}}$ is strongly reductive, then so is some $\sigma_{v_i}^{\boldsymbol{P}_i}$. □

6.3 Star graphs

There is a convenient combinatorial way to keep track of computations such as those we did in Section 6.2 using a diagram called a *star graph*. Star graphs have been extensively used to study free groups and their automorphisms (see eg [11]). The star graph $\mathfrak{g}(\boldsymbol{w})$ of a cyclically reduced word $\boldsymbol{w}$ in a free group $F(V)$ is defined by taking a vertex for each element of $V^{\pm}$ and an edge from x to y for every occurrence of xy^{-1} as a (cyclic) subword of $\boldsymbol{w}$. If we consider $F(V)$ as a right-angled Artin group on the discrete graph Γ, then $|v|_{\boldsymbol{w}}$ (as defined in the previous section) is equal to the valence of a vertex v in $\mathfrak{g}(\boldsymbol{w})$, and for any partition $\boldsymbol{P}$ of $V^{\pm}$, $|\boldsymbol{P}|_{\boldsymbol{w}}$ is equal to the number of edges in the star graph with one vertex in P and one vertex in P^*. Since the star graph $\mathfrak{g}(\boldsymbol{w})$ depends only on $\boldsymbol{w}$ in the case of a free group, it can be used to compute $|\boldsymbol{P}|_{\boldsymbol{w}}$ for any $\boldsymbol{P}$.

We would like to imitate this construction for more general A_Γ, but, for a conjugacy class g of A_Γ and Γ–Whitehead partition $\boldsymbol{P}$, to compute $|\boldsymbol{P}|_{\boldsymbol{w}}$ we need to count how many times a minimal path $\tilde{p}$ in the Γ–complex crosses an edge labeled $e_{\boldsymbol{P}}$. This involves counting not only when $\boldsymbol{w}$ crosses from P to P^*, but when it is forced to cross from $\bar{P} = P \cup \operatorname{lk}(P)$ to $\bar{P}^* = P^* \cup \operatorname{lk}(P^*)$. Since our star graphs must take into account the link of $\boldsymbol{P}$, they cannot be defined independently of the partition.

Consequently, for a Γ–Whitehead partition $\boldsymbol{P} = \{P, \operatorname{lk}(P), P^*\}$ and a cyclically reduced word $\boldsymbol{w} = u_1 \cdots u_k$, we define the star graph $\mathfrak{g}_{\boldsymbol{P}}(\boldsymbol{w})$ as follows. The vertices of $\mathfrak{g}_{\boldsymbol{P}}(\boldsymbol{w})$ are the elements of the disjoint union of $\bar{P} = P \cup \operatorname{lk}(P)$ and $\bar{P}^* = P \cup \operatorname{lk}(P)$, ie we have two copies of $\operatorname{lk}(P)$ instead of one. View $\boldsymbol{w}$ as a cyclic word and set $u_{k+1} = u_1$. Beginning with $i = 1$ draw an edge from u_i to u_{i+1}^{-1} staying within $\bar{P}$ or $\bar{P}^*$ whenever possible. If every u_i lies in $\operatorname{lk}(P)$, then the star graph can be drawn entirely in $\bar{P}$ (or in $\bar{P}^*$). Otherwise, we may cyclically permute $\boldsymbol{w}$ so that u_1 does not lie in $\operatorname{lk}(P)$, in which case there is no choice of where to start. See Figure 7 for an example.

The quantities $|v|_{\boldsymbol{w}}$ for $v \notin \operatorname{lk}(P)$ and $|\boldsymbol{P}|_{\boldsymbol{w}}$ can now be read off the star graph $\mathfrak{g}_{\boldsymbol{P}}(\boldsymbol{w})$. Namely, $|v|_{\boldsymbol{w}}$ is equal to the valence of the vertex v, while $|\boldsymbol{P}|_{\boldsymbol{w}}$ equals the number of edges with one vertex in $\bar{P}$ and one vertex in $\bar{P}^*$.

We will need to compare star graphs for the same word with respect to different partitions, but the graph we have constructed depends on the partition $\boldsymbol{P}$, not just on the word $\boldsymbol{w}$. To solve this problem, we will need to consider slightly more general decompositions of $V^{\pm}$ and a more general definition of a star graph.

Fix a symmetric subset $L \subset V^{\pm}$ and a decomposition $A_1 + \cdots + A_k$ of the complement L^c. The *star graph* $\mathfrak{g}^{L}_{A_1,\dots,A_k}(\boldsymbol{w})$ is constructed as follows. Take a copy L_i of L

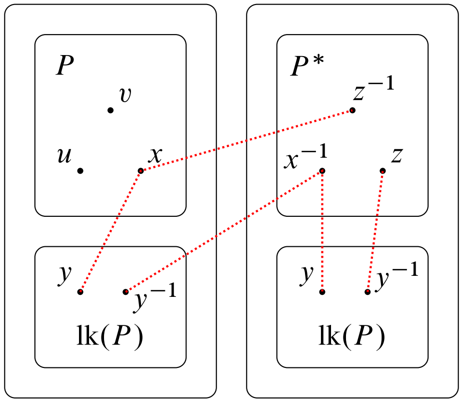


Figure 7: $\mathfrak{g}_{\boldsymbol{P}}(xy^{-1}xzy) = \mathfrak{g}_{P,P*}^{\mathrm{lk}(P)}(xy^{-1}xzy)$

for each A_i and let $\bar{A}_i = A_i \cup L_i$. The vertices of $\mathfrak{g}^L_{A_1,\dots,A_k}(\boldsymbol{w})$ are the elements of the (disjoint) union of the $\bar{A}_i$. We draw the star graph $\mathfrak{g}^L_{A_1,\dots,A_k}(\boldsymbol{w})$ by first drawing circles to isolate each $\bar{A}_i$. The idea is then to draw the edges of $\mathfrak{g}^L_{A_1,\dots,A_k}(\boldsymbol{w})$ in order, avoiding crossing circles whenever possible.

More precisely, we proceed as follows. If all letters of $\boldsymbol{w}$ are in L, we will draw the entire star graph with vertices in L_1. Otherwise, list all of the 2–letter subwords xy of $\boldsymbol{w}$ in order (cyclically), starting at a letter $x \in L^c$. Since $\{A_i\}$ partitions L^c, x lies in a unique A_i. If $y \in L^c$, there are unique vertices labeled y and y^{-1}, so we have no choice: we draw an edge from x to y^{-1} and start the next edge at y. If $y \in L$, draw an edge from x to the copy of $y^{-1} \in L_i$, and start the next edge at $y \in L_i$. We continue in this way, remaining inside each $\bar{A}_j$–circle as long as possible. Note that if $\boldsymbol{P}$ is a Γ–Whitehead partition and $L = \mathrm{lk}(\boldsymbol{P})$, then $\mathfrak{g}^L_{P,P*}(\boldsymbol{w})$ is precisely the graph $\mathfrak{g}_{\boldsymbol{P}}(\boldsymbol{w})$ constructed above.

If $v \in A_i$, the valence of v in $\mathfrak{g}^L_{A_1,\dots,A_k}(\boldsymbol{w})$ is equal to the number of occurrences of v or v^{-1} in $\boldsymbol{w}$, and if $v \in L$, then the number of such occurrences is equal to the sum of the valences of the copies of v in the L_i.

6.4 Counting lemmas

This section contains several elementary counting lemmas related to star graphs which are at the heart of the proofs in the next section.

As above, let L be a symmetric subset of $V^{\pm}$. For a subset $A \subset L^c$, let

$$A^* = A^c \setminus L, \quad \bar{A} = A \cup L_A, \quad \bar{A}^* = A^* \cup L_{A^*},$$

where L_A and L_{A^*} are copies of L.

Definition 6.9 For a cyclically reduced word in $\boldsymbol{w} \in F(V)$ and disjoint subsets A and B of L^c, define the *dot product* $(A.B)^L_{\boldsymbol{w}}$ to be the number of edges of $\mathfrak{g}^L_{A,B,(A+B)^*}(\boldsymbol{w})$ with one vertex in $\bar{A}$ and one vertex in $\bar{B}$.

The dot product $(A.B)^L_{\boldsymbol{w}}$ can also be described as the number of cyclic subwords of $\boldsymbol{w}$ of the form $a\boldsymbol{u}b^{-1}$ or $b\boldsymbol{u}a^{-1}$ for $a \in A$, $b \in B$ and $\boldsymbol{u}$ a word in L. If $B = A^*$, then $A + B = L^c$, so $(A+B)^* = \varnothing$. In this case, no edge of the star graph enters the $\overline{(A+B)^*}$–circle, so for the purposes of our computations, we can identify $\mathfrak{g}^L_{A,B,(A+B)^*}(\boldsymbol{w})$ with $\mathfrak{g}^L_{A,A^*}(\boldsymbol{w})$.

Definition 6.10 For a cyclically reduced word $\boldsymbol{w}$ and a subset $A \subset L^c$, define the *absolute value* of A by $|A|^L_{\boldsymbol{w}} = (A.A^*)^L_{\boldsymbol{w}}$, that is, the number of edges of $\mathfrak{g}^L_{A,A^*}(\boldsymbol{w})$ with one vertex in $\bar{A}$ and one vertex in $\bar{A}^*$.

Example 6.11 Let $\boldsymbol{P} = \{P, P^*, \mathrm{lk}(P)\}$ be a Γ–Whitehead partition. Then

$$(P.P^*)^{\mathrm{lk}(P)}_{\boldsymbol{w}} = |P|^{\mathrm{lk}(P)}_{\boldsymbol{w}} = |P^*|^{\mathrm{lk}(P)}_{\boldsymbol{w}} = |\boldsymbol{P}|_{\boldsymbol{w}}.$$

Our justification for calling $(A.B)^L_{\boldsymbol{w}}$ a "dot product" rests partly on the observation that $(A.B)^L_{\boldsymbol{w}} = (B.A)^L_{\boldsymbol{w}}$. We also have the following linearity relation:

Lemma 6.12 *Let $L \subset V^{\pm}$ be a symmetric subset, and let A, B and C be disjoint subsets of L^c. Then $(A.(B+C))^L_{\boldsymbol{w}} = (A.B)^L_{\boldsymbol{w}} + (A.C)^L_{\boldsymbol{w}}$.*

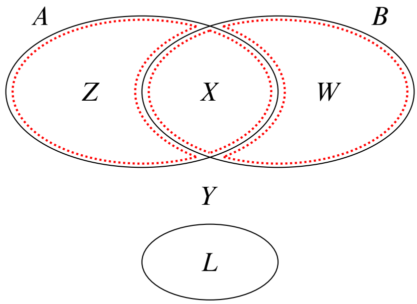
Proof The star graph $\mathfrak{g}^L_{A,B+C,(A+B+C)^*}(\boldsymbol{w})$ is obtained from $\mathfrak{g}^L_{A,B,C,(A+B+C)^*}(\boldsymbol{w})$ by identifying the vertices of L_B and L_C. Thus the number of edges between $\bar{A}$ and $\overline{(B+C)}$ is equal to the number between $\bar{A}$ and $\bar{B}$ plus the number between $\bar{A}$ and $\bar{C}$, as desired. □

In the following three lemmas we fix $\boldsymbol{w}$ and omit it from the notation for simplicity.

Lemma 6.13 *Let L be a symmetric subset of $V^{\pm}$. For any subsets A and B of L^c,*

$$|A \cap B^*|^L + |A^* \cap B|^L = |A|^L + |B|^L - 2((A \cap B).(A^* \cap B^*))^L.$$

Proof We set notation according to the figure below:



Thus $X = A \cap B$, $Z = A \cap B^*$, $W = A^* \cap B$ and $Y = A^* \cap B^*$ and we are trying to show

$$|Z|^L + |W|^L = |A|^L + |B|^L - 2(X.Y)^L.$$

We calculate

$$\begin{aligned}
|A|^L &= ((X+Z).(Y+W))^L = (X.Y)^L + (X.W)^L + (Z.Y)^L + (Z.W)^L,\\
|B|^L &= ((X+W).(Y+Z))^L = (X.Y)^L + (Y.W)^L + (Z.X)^L + (Z.W)^L,\\
|Z|^L &= (Z.(X+Y+W))^L = (Z.X)^L + (Z.Y)^L + (Z.W)^L,\\
|W|^L &= (W.(X+Z+Y)^L = (W.X)^L + (W.Z)^L + (W.Y)^L.
\end{aligned}$$

The result follows. □

Lemma 6.14 *Let $L_0 \subset L$ be symmetric subsets of $V^{\pm}$ and let $A \subset C \subset L^c$. Then $|A|^{L_0} - |A|^L \le |C|^{L_0} - |C|^L$.*

Proof $|A|^L$ counts the number of subwords of the form $a.\boldsymbol{u}.b^{-1}$ or $b.\boldsymbol{u}.a^{-1}$, with $a \in A$, $b \in (A+L)^c$ and $\boldsymbol{u}$ a (possibly empty) word in elements of L. Notice that each such subword also contributes exactly one to $|A|^{L_0}$.

Let $\mathcal{D}$ be the set of all words in elements of L that use at least one letter which is not in L_0. The only other contributions to $|A|^{L_0}$ come from subwords of the form $a'.\boldsymbol{u}.a^{-1}$ for a, $a' \in A$ and $\boldsymbol{u} \in \mathcal{D}$; each of these subwords contributes 2. Thus the difference $|A|^{L_0} - |A|^L$ is the number of subwords $a'.\boldsymbol{u}.a^{-1}$ with $\boldsymbol{u} \in \mathcal{D}$.

We now do the same computation for C. Since $A \subset C$, there are at least as many words of the form $c'.\boldsymbol{u}.c^{-1}$ with $\boldsymbol{u} \in \mathcal{D}$ as words $a'.\boldsymbol{u}.a^{-1}$, and the lemma is proved. □

Lemma 6.15 *Let L_1 and L_2 be a symmetric subsets of $V^{\pm}$ and let $L = L_1 \cup L_2$. Then, for any subsets A and B of L^c,*

$$|A \cap B^*|^{L_1} + |A^* \cap B|^{L_2} \le |A|^{L_1} + |B|^{L_2}.$$

Proof By Lemma 6.14 applied to $(A \cap B^*) \subset A \subset L^c$, we have

$$|A \cap B^*|^{L_1} + |A|^L \le |A \cap B^*|^L + |A|^{L_1}.$$

By Lemma 6.14 applied to $(A^* \cap B) \subset B \subset L^c$,

$$|A^* \cap B|^{L_2} + |B|^L \le |A^* \cap B|^L + |B|^{L_2}.$$

Adding these and applying Lemma 6.13 gives

$$\begin{aligned} |A \cap B^*|^{L_1} + |A^* \cap B|^{L_2} + |A|^L + |B|^L &\le |A \cap B^*|^L + |A^* \cap B|^L + |A|^{L_1} + |B|^{L_2} \\ &\le |A|^L + |B|^L + |A|^{L_1} + |B|^{L_2}. \end{aligned}$$ □

6.5 Reductive Γ–Whitehead partitions

Recall that a Γ–Whitehead partition $\boldsymbol{P}$ of a marked Salvetti σ is *reductive* if for some $v \in \max(P)$ the Whitehead automorphism $\varphi = (P, v)$ reduces the norm of σ, ie $\|\sigma_v^{\boldsymbol{P}}\| < \|\sigma\|$, and *strongly reductive* if (P, v) reduces the first coordinate of the norm, ie $\|\sigma_v^{\boldsymbol{P}}\|_0 < \|\sigma\|_0$.

The strategy of our proof will require us to find reductive Γ–Whitehead partitions which are compatible with each other, so our next task is to determine where we can look for such partitions. We first consider a pair of noncompatible partitions, and show how to find a partition which is compatible with both of them.

In our definition of Γ–Whitehead partition, we did not allow P to be a singleton. For convenience, we now define a *degenerate* Γ*–Whitehead partition* to be one of the form $\boldsymbol{P} = (P, P^*, \mathrm{lk}(P))$ with $P = \{v\}$. In this case, the associated Whitehead automorphism (P, v) is the inversion ι_v and $|\boldsymbol{P}|_\sigma = |v|_\sigma$ for every σ. In particular, a reductive Γ–Whitehead partition cannot be degenerate.

Suppose $\boldsymbol{P}$ and $\boldsymbol{Q}$ are Γ–Whitehead partitions which are not compatible, ie they do not commute and each of the sets $P \cap Q$, $P^* \cap Q$, $P \cap Q^*$ and $P^* \cap Q^*$ is nonempty. We will refer to these four intersections as *quadrants*. Two quadrants are *opposite* if one is obtained from the other by switching sides of both $\boldsymbol{P}$ and $\boldsymbol{Q}$.

Lemma 6.16 *For any noncompatible partitions $\boldsymbol{P}$ and $\boldsymbol{Q}$, there is a pair of opposite quadrants, each of which defines a (possibly degenerate) Γ–Whitehead partition with maximal element in $\{v^{\pm}, w^{\pm}\}$.*

Proof Let $v \in \max(P)$ and $w \in \max(Q)$. Recall that $x \in \mathrm{double}(\boldsymbol{Q})$ means that x and x^{-1} both lie on the same side of $\boldsymbol{Q}$ and $x \in \mathrm{single}(\boldsymbol{Q})$ means that x and x^{-1} lie on opposite sides of $\boldsymbol{Q}$. We divide the proof into three cases.

Case 1 **($v \in$ double(Q) and $w \in$ double(P))** In this case, some quadrant contains an element of both $\{v^{\pm}\}$ and $\{w^{\pm}\}$. Without loss of generality, we may assume that $v, w \in P \cap Q$. We claim that, in this case, $(P \cap Q^*, w^{-1})$ is a Γ–Whitehead pair.

Let C_v denote the component of $\Gamma \setminus \mathrm{st}(w)$ which contains v. Then $v \in Q$ implies $C_v \subset Q$. Moreover, we have:

(1) If $\mathrm{lk}(x) \subseteq \mathrm{lk}(v)$ then either $\mathrm{lk}(x) \subseteq \mathrm{lk}(w)$ or $x \in C_v \subset Q$.

(2) Every component C of $\Gamma \setminus \mathrm{st}(w)$ with $C \neq C_v$ lies entirely in some component of $\Gamma \setminus \mathrm{st}(v)$.

The first property follows from the fact that if $\mathrm{lk}(x) \not\subseteq \mathrm{lk}(w)$ then x is connected to v via some vertex not in $\mathrm{lk}(w)$. Hence, x and v lie in the same component of $\Gamma \setminus \mathrm{st}(w)$. The second property follows from the fact that in order for $\mathrm{st}(v)$ to disconnect C, this C must intersect $\mathrm{st}(v)$ and hence it must contain v.

We can now verify that $(P \cap Q^*, w^{-1})$ is Γ–Whitehead: If $x \in \mathrm{single}(P \cap Q^*)$ then either $x \in \mathrm{single}(Q^*)$, so $\mathrm{lk}(x) \subseteq \mathrm{lk}(w)$, or $x \in \mathrm{single}(P) \cap Q^*$, so $\mathrm{lk}(x) \subseteq \mathrm{lk}(v)$ and $x \notin Q$. By (1), it follows that $\mathrm{lk}(x) \subseteq \mathrm{lk}(w)$. If $x \in \mathrm{double}(P \cap Q^*) = \mathrm{double}(P) \cap \mathrm{double}(Q^*)$, then by (2) so is the component of x in $\Gamma \setminus \mathrm{st}(w)$. This proves that $(P \cap Q^*, w^{-1})$ is a Γ–Whitehead pair. By symmetry, $(P^* \cap Q, v^{-1})$ is also a Γ–Whitehead pair.

Case 2 **($v \in$ double(Q) and $w \in$ single(P))** In this case, w and w^{-1} lie in opposite quadrants while v and v^{-1} lie in adjacent quadrants. It follows that some quadrant contains an element of both $\{v, v^{-1}\}$ and $\{w, w^{-1}\}$. Without loss of generality, we may assume that $v^{-1}, w \in P^* \cap Q$. We claim that $P^* \cap Q$ and $P \cap Q^*$ are Γ–Whitehead.

First consider $P \cap Q^*$. Let C_v be as above. By assumption, $C_v \subset Q$. Thus, the same argument as in case (1) applies to show that $(P \cap Q^*, w^{-1})$ is a Γ–Whitehead pair.

Next consider $P^* \cap Q$. Since $w \in \mathrm{single}(P)$, we have $w \leq v$ so any $x \in \mathrm{single}(P^* \cap Q)$ satisfies $x \leq v$. For $\mathrm{double}(P^* \cap Q)$, note that $\mathrm{lk}(w) \subseteq \mathrm{lk}(v)$ implies that every component of $\Gamma \setminus \mathrm{st}(v)$ (other than the singleton $\{w\}$) is contained is some component of $\Gamma \setminus \mathrm{st}(w)$. It follows that $\mathrm{double}(P^* \cap Q)$ is a union of components of $\Gamma \setminus \mathrm{st}(v)$. Hence, $(P^* \cap Q, v^{-1})$ is a Γ–Whitehead pair.

Case 3 **($v \in$ single(Q) and $w \in$ single(P))** This is only possible if $\mathrm{lk}(v) = \mathrm{lk}(w)$. Since v is a singleton in both partitions, v and v^{-1} lie in opposite quadrants. Say $v \in P \cap Q$ and $v^{-1} \in P^* \cap Q^*$. Then it is easy to see that $(P \cap Q, v)$ and $(P^* \cap Q^*, v^{-1})$ are Γ–Whitehead pairs. Likewise, the opposite quadrants containing w and w^{-1} also give Γ–Whitehead pairs. $\square$

We next need to add the condition that our Γ–Whitehead partitions be reductive in certain situations. Let σ be a marked Salvetti and $\boldsymbol{P}$ a Γ–Whitehead partition. For the purpose of this discussion, we introduce a weaker notion of reductively: we say that $\boldsymbol{P}$ is 0–*reductive* for σ if $\|\sigma_v^{\boldsymbol{P}}\|_0 \le \|\sigma\|_0$ for some $v \in \max(P)$.

Fix $\sigma = (\mathbb{S}, \alpha)$ and let $\mathcal{W}_0$ be a set of cyclically reduced words representing the set $\{\alpha^{-1}(g) \mid g \in \mathcal{G}_0\}$. Write $|\boldsymbol{P}|_0 = \sum_{\boldsymbol{w} \in \mathcal{W}_0} |\boldsymbol{P}|_{\boldsymbol{w}}$ and $|v|_0 = \sum_{\boldsymbol{w} \in \mathcal{W}_0} |v|_{\boldsymbol{w}}$. Then $\boldsymbol{P}$ is:

- 0–reductive if $\|\sigma_v^{\boldsymbol{P}}\|_0 - \|\sigma\|_0 = |\boldsymbol{P}|_0 - |v|_0 \le 0$.
- reductive if $\|\sigma_v^{\boldsymbol{P}}\| - \|\sigma\| = |\boldsymbol{P}|_\sigma - |v|_\sigma < \boldsymbol{0} \in \mathbb{Z} \times \mathbb{Z}^{\mathcal{G}}$.
- strongly reductive if $\|\sigma_v^{\boldsymbol{P}}\|_0 - \|\sigma\|_0 = |\boldsymbol{P}|_0 - |w|_0 < 0$.

In particular, strongly reductive $\Longrightarrow$ reductive $\Longrightarrow$ 0–reductive, but none of the converses hold.

Lemma 6.17 (Higgins–Lyndon lemma) *Let σ be a marked Salvetti and let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be noncompatible Γ–Whitehead partitions. If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are both σ–reductive then at least one of the quadrants $P \cap Q^*$, $P^* \cap Q$, $P \cap Q$ or $P^* \cap Q^*$ determines a σ–reductive Γ–Whitehead partition which is compatible with both $\boldsymbol{P}$ and $\boldsymbol{Q}$. If $\boldsymbol{P}$ is strongly reductive and $\boldsymbol{Q}$ is 0–reductive, then one of the quadrants is strongly reductive.*

Proof Let $\sigma = (\mathbb{S}, \alpha)$. By hypothesis, we can choose $v \in \max(P)$ and $w \in \max(Q)$ such that either $|\boldsymbol{P}|_\sigma - |v|_\sigma < \boldsymbol{0}$ and $|\boldsymbol{Q}|_\sigma - |w|_\sigma < \boldsymbol{0}$ (case 1), or $|\boldsymbol{P}|_0 - |v|_0 < 0$ and $|\boldsymbol{Q}|_0 - |w|_0 \le 0$ (case 2).

Suppose first that there is exactly one quadrant which contains none of $\{v, v^{-1}, w, w^{-1}\}$. By changing sides of $\boldsymbol{P}$ and $\boldsymbol{Q}$ if necessary, we may assume this is $P \cap Q$. Then both $(P \cap Q^*, v)$ and $(P^* \cap Q, w)$ are Γ–Whitehead by Lemma 6.16. Since P and $P^* \cap Q$ are disjoint and v and w don't commute, Lemma 3.4 shows that $P \cap \operatorname{lk}(Q) = \varnothing$; similarly, $Q \cap \operatorname{lk}(P) = \varnothing$. Thus P and Q are both in the complement of $L = \operatorname{lk}(P) \cup \operatorname{lk}(Q)$. So, by Lemma 6.15, for every cyclically reduced word $\boldsymbol{w}$ we have

$$|P \cap Q^*|_{\boldsymbol{w}}^{\operatorname{lk}(v)} + |P^* \cap Q|_{\boldsymbol{w}}^{\operatorname{lk}(w)} \le |P|_{\boldsymbol{w}}^{\operatorname{lk}(v)} + |Q|_{\boldsymbol{w}}^{\operatorname{lk}(w)}.$$

Letting $\boldsymbol{P} \cap \boldsymbol{Q}^*$ denote the Γ–Whitehead partition determined by $P \cap Q^*$ and $\boldsymbol{Q} \cap \boldsymbol{P}^*$ the Γ–Whitehead partition determined by $Q \cap P^*$, it follows that

$$|\boldsymbol{P} \cap \boldsymbol{Q}^*|_{\boldsymbol{w}} - |v|_{\boldsymbol{w}} + |\boldsymbol{P}^* \cap \boldsymbol{Q}|_{\boldsymbol{w}} - |w|_{\boldsymbol{w}} \le |\boldsymbol{P}|_{\boldsymbol{w}} - |v|_{\boldsymbol{w}} + |\boldsymbol{Q}|_{\boldsymbol{w}} - |w|_{\boldsymbol{w}}.$$

In case 1, it follows that

$$|\boldsymbol{P} \cap \boldsymbol{Q}^*|_\sigma - |v|_\sigma + |\boldsymbol{P}^* \cap \boldsymbol{Q}|_\sigma - |w|_\sigma \le |\boldsymbol{P}|_\sigma - |v|_\sigma + |\boldsymbol{Q}|_\sigma - |w|_\sigma < \boldsymbol{0},$$

so at least one of $(P \cap Q^*, v)$ or $(P^* \cap Q, w)$ is reductive. In case 2,

$$|\boldsymbol{P} \cap \boldsymbol{Q}^*|_0 - |v|_0 + |\boldsymbol{P}^* \cap \boldsymbol{Q}|_0 - |w|_0 \le |\boldsymbol{P}|_0 - |v|_0 + |\boldsymbol{Q}|_0 - |w|_0 < 0,$$

so one of $(P \cap Q^*, v)$ or $(P^* \cap Q, w)$ is strongly reductive.

Next suppose each quadrant contains an element of $\{v, v^{-1}, w, w^{-1}\}$ and say $v \in P \cap Q$. This forces $w \in Q \cap P^*$, $w^{-1} \in P \cap Q^*$ and $v^{-1} \in P^* \cap Q^*$. Since v is a singleton in Q and w is a singleton in P, we have $\mathrm{lk}(v) = \mathrm{lk}(w) = L$ and, by Lemma 6.16, all four quadrants are Γ–Whitehead. Recall that $|P|^L_{\boldsymbol{w}} = |P^*|^L_{\boldsymbol{w}}$, so applying Lemma 6.15 to both pairs of opposite quadrants gives

$$\begin{aligned}(|P \cap Q^*|^L_{\boldsymbol{w}} - |w|_{\boldsymbol{w}} + |P^* \cap Q|^L_{\boldsymbol{w}} - |w|_{\boldsymbol{w}}) + (|P^* \cap Q^*|^L_{\boldsymbol{w}} - |v|_{\boldsymbol{w}} + |P \cap Q|^L_{\boldsymbol{w}} - |v|_{\boldsymbol{w}})& \\ \le (|P|^L_{\boldsymbol{w}} + |Q|^L_{\boldsymbol{w}} - 2|w|_{\boldsymbol{w}}) + (|P^*|^L_{\boldsymbol{w}} + |Q|^L_{\boldsymbol{w}} - 2|v|_{\boldsymbol{w}})& \\ = 2(|P|^L_{\boldsymbol{w}} - |v|_{\boldsymbol{w}}) + 2(|Q|^L_{\boldsymbol{w}} - |w|_{\boldsymbol{w}}).&\end{aligned}$$

In case 1 we obtain

$$\begin{aligned}(|\boldsymbol{P} \cap \boldsymbol{Q}^*|_\sigma - |w|_\sigma) + (|\boldsymbol{P}^* \cap \boldsymbol{Q}|_\sigma - |w|_\sigma) + (|\boldsymbol{P}^* \cap \boldsymbol{Q}^*|_\sigma - |v|_\sigma) + (|\boldsymbol{P} \cap \boldsymbol{Q}|_\sigma - |v|_\sigma)& \\ \le 2(|\boldsymbol{P}|_\sigma - |v|_\sigma) + 2(|\boldsymbol{Q}|_\sigma - |w|_\sigma)& \\ < \boldsymbol{0},&\end{aligned}$$

so at least one of the quadrants is reductive. In case 2 we have

$$\begin{aligned}(|\boldsymbol{P} \cap \boldsymbol{Q}^*|_0 - |w|_0) + (|\boldsymbol{P}^* \cap \boldsymbol{Q}|_0 - |w|_0) + (|\boldsymbol{P}^* \cap \boldsymbol{Q}^*|_0 - |v|_0) + (|\boldsymbol{P} \cap \boldsymbol{Q}|_0 - |v|_0)& \\ \le 2(|\boldsymbol{P}|_0 - |v|_0) + 2(|\boldsymbol{Q}|_0 - |w|_0)& \\ < 0,&\end{aligned}$$

so one of the quadrants is strongly reductive.

The last possibility is that only two quadrants contain elements of $\{v, v^{-1}, w, w^{-1}\}$. In this case, we may assume $v, w \in P \cap Q$ and $v^{-1}, w^{-1} \in P^* \cap Q^*$. Here again $\mathrm{lk}(v) = \mathrm{lk}(w) = L$, and $(P \cap Q, v)$ and $(P^* \cap Q^*, w^{-1})$ are both Γ–Whitehead by Lemma 6.16. Applying Lemma 6.15 gives

$$(|\boldsymbol{P} \cap \boldsymbol{Q}|_{\boldsymbol{w}} - |v|_{\boldsymbol{w}}) + (|\boldsymbol{P}^* \cap \boldsymbol{Q}^*|_{\boldsymbol{w}} - |w|_{\boldsymbol{w}}) \le (|\boldsymbol{P}|_{\boldsymbol{w}} - |v|_{\boldsymbol{w}}) + (|\boldsymbol{Q}|_{\boldsymbol{w}} - |w|_{\boldsymbol{w}})$$

and arguing as above we conclude that one of these quadrants is reductive (case 1) or strongly reductive (case 2).

Finally, note that the requirement that the chosen quadrant define a partition compatible with both $\boldsymbol{P}$ and $\boldsymbol{Q}$ is immediate from the fact that every quadrant is contained in one side of $\boldsymbol{P}$ and one side of $\boldsymbol{Q}$. □

We have shown that any two marked Salvettis σ and σ' can be joined by a path in K_Γ consisting of a sequence of Whitehead moves. We call such a path a Γ–Whitehead path.

Theorem 6.18 (peak reduction) *Let (P, v) and (Q, w) be two reductive Γ–Whitehead moves from σ. Then there is a Γ–Whitehead path from $\sigma_v^{\boldsymbol{P}}$ to $\sigma_w^{\boldsymbol{Q}}$ which passes only through marked Salvettis τ with $\|\tau\| < \|\sigma\|$.*

Proof First observe that in the case where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are compatible and $v = w$, it follows from Remark 3.5 that $\sigma_v^{\boldsymbol{P}}$ and $\sigma_w^{\boldsymbol{Q}}$ differ by a single Γ–Whitehead move, so there is nothing to prove.

Consider the case where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are compatible and $v \neq w$. If the edges e_v and e_w in $S^{\boldsymbol{P},\boldsymbol{Q}}$ do not join the same two vertices, then the hyperplanes $\mathcal{H} = \{H_v, H_w\}$ through these edges form a treelike set in $S^{\boldsymbol{P},\boldsymbol{Q}}$. In this case, setting $\tau = \sigma_{\mathcal{H}}^{\boldsymbol{P},\boldsymbol{Q}}$, we obtain a Γ–Whitehead path $\sigma_v^{\boldsymbol{P}} \to \tau \to \sigma_w^{\boldsymbol{Q}}$. Since (P, v) and (Q, W) are both reductive, Corollary 6.6 gives

$$\|\tau\| = \|\sigma\| + (|\boldsymbol{P}|_\sigma - |v|_\sigma) + (|\boldsymbol{Q}|_\sigma - |w|_\sigma) < \|\sigma\|.$$

The only situation in which e_v and e_w can join the same pair of vertices is if v and w are singletons in both partitions, say $v, w \in P \cap Q$ and $v^{-1}, w^{-1} \in P^* \cap Q^*$. In this case, (P, w) and (Q, v) are also Γ–Whitehead pairs. Suppose $|v|_\sigma \leq |w|_\sigma$. Then, by Corollary 6.6,

$$\|\sigma_w^{\boldsymbol{P}}\| = \|\sigma\| + (|\boldsymbol{P}|_\sigma - |w|_\sigma) \leq \|\sigma\| + (|\boldsymbol{P}|_\sigma - |v|_\sigma) < \|\sigma\|,$$

so $\sigma_v^{\boldsymbol{P}} \to \sigma_w^{\boldsymbol{P}} \to \sigma_w^{\boldsymbol{Q}}$ is the desired path. If $|v|_\sigma > |w|_\sigma$, use $\sigma_v^{\boldsymbol{P}} \to \sigma_v^{\boldsymbol{Q}} \to \sigma_w^{\boldsymbol{Q}}$ instead.

Next, suppose $\boldsymbol{P}$ and $\boldsymbol{Q}$ are not compatible. Apply the Higgins–Lyndon lemma to find $\boldsymbol{R}$ compatible with both $\boldsymbol{P}$ and $\boldsymbol{Q}$, with $\boldsymbol{R}$ reductive, ie $\|\sigma_u^{\boldsymbol{R}}\| < \|\sigma\|$. Then by the discussion above, there are Γ–Whitehead paths from $\sigma_v^{\boldsymbol{P}}$ to $\sigma_u^{\boldsymbol{R}}$ and from $\sigma_u^{\boldsymbol{R}}$ to $\sigma_w^{\boldsymbol{Q}}$ satisfying the required condition. □

Theorem 6.19 (strong peak reduction) *Let (P, v) and (Q, w) be two Γ–Whitehead partitions such that $\|\sigma_v^{\boldsymbol{P}}\|_0 < \|\sigma\|_0$ and $\|\sigma_v^{\boldsymbol{Q}}\|_0 \leq \|\sigma\|_0$. Then there is a Γ–Whitehead path $\sigma_v^{\boldsymbol{P}} \to \sigma_w^{\boldsymbol{Q}}$ which passes only through marked Salvettis τ with $\|\tau\|_0 < \|\sigma\|_0$.*

Proof Let $\mathcal{W}_0$ be a set of cyclically reduced words representing $\{\alpha^{-1}(g) \mid g \in \mathcal{G}_0\}$. Write $|\boldsymbol{P}|_0 = \sum_{\boldsymbol{w} \in \mathcal{W}_0} |\boldsymbol{P}|_{\boldsymbol{w}}$ and $|v|_0 = \sum_{\boldsymbol{w} \in \mathcal{W}_0} |v|_{\boldsymbol{w}}$. Then

$$\|\sigma_v^{\boldsymbol{P}}\|_0 - \|\sigma\|_0 = |\boldsymbol{P}|_0 - |v|_0 < 0,$$
$$\|\sigma_v^{\boldsymbol{Q}}\|_0 - \|\sigma\|_0 = |\boldsymbol{Q}|_0 - |w|_0 \leq 0.$$

We now proceed as in the proof of the previous theorem. In the case where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are compatible and e_v and e_w join different vertices in $S^{\boldsymbol{P},\boldsymbol{Q}}$, set $\tau = \sigma_{\mathcal{H}}^{\boldsymbol{P},\boldsymbol{Q}}$ and note that

$$\|\tau\|_0 = \|\sigma\|_0 + (|\boldsymbol{P}|_0 - |v|_0) + (|\boldsymbol{Q}|_0 - |w|_0) < \|\sigma\|_0.$$

If e_v and e_w join the same vertices and $|v|_0 \le |w|_0$, then

$$\|\sigma_w^{\boldsymbol{P}}\|_0 = \|\sigma\|_0 + (|\boldsymbol{P}|_0 - |w|_{\mathbf{0}}) \le \|\sigma\|_0 + (|\boldsymbol{P}|_0 - |v|_0) < \|\sigma\|_0.$$

If e_v and e_w join the same vertices and $|v|_0 > |w|_0$, then

$$\|\sigma_v^{\boldsymbol{Q}}\|_0 = \|\sigma\|_0 + (|\boldsymbol{Q}|_0 - |v|_{\mathbf{0}}) < \|\sigma\|_0 + (|\boldsymbol{Q}|_0 - |w|_0) \le \|\sigma\|_0.$$

Hence, either $\sigma_v^{\boldsymbol{P}} \to \sigma_w^{\boldsymbol{P}} \to \sigma_w^{\boldsymbol{Q}}$ or $\sigma_v^{\boldsymbol{P}} \to \sigma_v^{\boldsymbol{Q}} \to \sigma_w^{\boldsymbol{Q}}$ gives the desired path. □

Corollary 6.20 *If $\|\sigma\|$ is not minimal, then there is a strongly reductive Γ–Whitehead move from σ.*

Proof As observed in the proof of Proposition 4.20, there is a path, $\sigma = \sigma_1, \sigma_2, \dots,$ $\sigma_k = (\mathbb{S}, \mathrm{id})$ of Γ–Whitehead moves from σ to the unique marked Salvetti with minimal 0–norm, $\|(\mathbb{S}, \mathrm{id})\|_0$. Consider the sequence of 0–norms $\|\sigma\|_0, \|\sigma_2\|_0, \dots, \|\sigma_k\|_0$. Using Theorem 6.19, we can reduce peaks in this sequence to obtain a Γ–Whitehead path $\sigma = \tau_0, \tau_1, \dots, \tau_j = (\mathbb{S}, \mathrm{id})$ which begins downward, that is, with $\|\sigma\|_0 > \|\tau_1\|_0$. □

Corollary 6.21 *Let $N_0 = \|(\mathbb{S}, \mathrm{id})\|_0$. For any $N \ge N_0$, there are finitely many marked Salvettis σ with $\|\sigma\|_0 \le N$.*

Proof We observed in Lemma 6.2 that $(\mathbb{S}, \mathrm{id})$ is the unique marked Salvetti with minimal 0–norm. By Corollary 6.20, if $\|\sigma\|_0 \le N$, there is a Γ–Whitehead path of length at most $N - N_0$ to $(\mathbb{S}, \mathrm{id})$. Since the number of Whitehead moves at any marked Salvetti is bounded, the corollary follows. □

Proposition 6.22 *The set of marked Salvettis is well-ordered with respect to the norm $\|\ \|$.*

Proof This follows immediately from Corollary 6.21. Let $N = \|\sigma\|_0$. Since $\|\tau\| < \|\sigma\|$ implies $\|\tau\|_0 \le \|\sigma\|_0$, there are only finitely many such τ. Hence there can be no infinite decreasing chain of marked Salvettis. □

For the proof of contractibility, we will also need the following stronger form of Lemma 6.17.

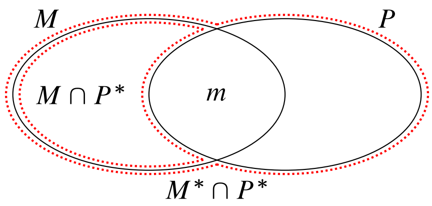


Figure 8: Setup for the pushing lemma

Lemma 6.23 (pushing lemma) *Fix a marked Salvetti σ. Suppose that (M, m) is a reductive Γ–Whitehead pair such that, at σ,*

(1) $\mathrm{lk}(\boldsymbol{M})$ *is maximal among links of reductive Γ–Whitehead partitions, and*

(2) (M, m) *is maximally reductive among Γ–Whitehead pairs (Q, w) with* $\mathrm{lk}(\boldsymbol{Q}) = \mathrm{lk}(\boldsymbol{M})$.

Let $\boldsymbol{P}$ be a reductive Γ–Whitehead partition that is not compatible with $\boldsymbol{M}$. Choose sides so that $m \in M \cap P$. Then at least one of $M \cap P^*$ or $M^* \cap P^*$ determines a reductive Γ–Whitehead partition whose link is equal to $\mathrm{lk}(\boldsymbol{P})$ (see Figure 8).

Proof $\boldsymbol{P}$ is reductive, so $|P|_\sigma - |v|_\sigma < \boldsymbol{0}$ for some $v \in \max(\boldsymbol{P})$. We will apply Lemma 6.16 to $\boldsymbol{P}$ and $\boldsymbol{M}$.

We first consider the case where $m \in \mathrm{single}(\boldsymbol{P})$ and $v \in \mathrm{single}(\boldsymbol{M})$. This corresponds to case (3) in the proof of Lemma 6.16. In this case, $\mathrm{lk}(v) = \mathrm{lk}(m)$ and the lemma produces opposite quadrants X and Y such that (X, v) and (Y, v^{-1}) are Γ–Whitehead pairs, as well as opposite quadrants X' and Y' such that (X', m) and (Y', m^{-1}) are Γ–Whitehead pairs. Applying Lemma 6.15 to both pairs of opposite quadrants gives

$$|\boldsymbol{X}|_\sigma + |\boldsymbol{Y}|_\sigma + |\boldsymbol{X}'|_\sigma + |\boldsymbol{Y}'|_\sigma \le 2(|\boldsymbol{P}|_\sigma + |\boldsymbol{M}|_\sigma).$$

Hence,

$$\begin{aligned}(|\boldsymbol{X}|_\sigma - |v|_\sigma) + (|\boldsymbol{Y}|_\sigma - |v|_\sigma) + (|\boldsymbol{X}'|_\sigma - |m|_\sigma) + (|\boldsymbol{Y}'|_\sigma - |m|_\sigma)& \\ \le 2(|\boldsymbol{P}|_\sigma - |v|_\sigma) + 2(|\boldsymbol{M}|_\sigma - |m|_\sigma)& \\ < 2(|\boldsymbol{M}|_\sigma - |m|_\sigma).&\end{aligned}$$

Two of these quadrants, say Y and Y', lie in P^*. By hypothesis (2), (M, m) is maximally reductive, so

$$2(|\boldsymbol{M}|_\sigma - |m|_\sigma) \le (|\boldsymbol{X}|_\sigma - |v|_\sigma) + (|\boldsymbol{X}'|_\sigma - |m|_\sigma)$$

and we conclude that

$$(|\boldsymbol{Y}|_\sigma - |v|_\sigma) + (|\boldsymbol{Y}'|_\sigma - |m|_\sigma) < \mathbf{0}.$$

Thus one of the pairs (Y, v^{-1}) or (Y', m^{-1}) is reductive and satisfies the requirements of the lemma.

In all other cases, Lemma 6.16 gives an opposite pair of quadrants X and Y, with maximal elements $x \in \{v^\pm\}$ and $y \in \{m^\pm\}$, that define (possibly degenerate) Γ–Whitehead partitions. By Lemma 6.15 we have

$$(|\boldsymbol{X}|_\sigma - |v|_\sigma) + (|\boldsymbol{Y}|_\sigma - |m|_\sigma) \leq (|\boldsymbol{P}|_\sigma - |v|_\sigma) + (|\boldsymbol{M}|_\sigma - |m|_\sigma) < (|\boldsymbol{M}|_\sigma - |m|_\sigma).$$

By hypothesis (2), $|\boldsymbol{M}|_\sigma - |m|_\sigma \leq |\boldsymbol{Y}|_\sigma - |m|_\sigma$, so we conclude that (X, x) is reductive. If $X \subset P^*$, we are done. If $X \subset P$, then $Y \subset P^*$, so P^* contains $y \in \{m^\pm\}$ and we must have $m \in \mathrm{single}(P)$. This implies that $\mathrm{lk}(m) \subseteq \mathrm{lk}(v)$, so, by hypothesis (1), $\mathrm{lk}(m) = \mathrm{lk}(v)$. It then follows from hypothesis (2) that $|\boldsymbol{M}|_\sigma - |m|_\sigma \leq |\boldsymbol{X}|_\sigma - |v|_\sigma$, and we conclude that (Y, y) is also reductive. □

6.6 Contractibility of the Γ–spine K_Γ

In this section we prove our main theorem.

Theorem 6.24 *For any right-angled Artin group A_Γ, the Γ–spine K_Γ is contractible.*

The proof will make frequent use the following lemma, which is standard in the topology of posets and dates back to work of Quillen [13].

Lemma 6.25 (poset lemma) *Let X be a poset and $f\colon X \to X$ a poset map with the property that $x \leq f(x)$ for all $x \in X$ (or $x \geq f(x)$ for all $x \in X$). Then f induces a deformation retraction from the geometric realization of X to the geometric realization of the image $f(X)$.*

Proof of Theorem 6.24 We view the spine K_Γ as the union of stars of marked Salvettis. By Lemma 6.2 there is a unique marked Salvetti $(\mathbb{S}, \mathrm{id})$ of minimal norm, and we start with its (contractible) star. We build the entire spine by gluing on stars of marked Salvettis in increasing order.

When we add a marked Salvetti, we need to check that we are gluing along something contractible. So fix a marked Salvetti σ, and let $K_{<\sigma}$ be the union of stars of marked Salvettis τ with $\|\tau\| < \|\sigma\|$. The intersection $\mathrm{st}(\sigma) \cap K_{<\sigma}$ consists of marked blowups $\sigma^{\boldsymbol{\Pi}}$ which can be collapsed to a marked Salvetti of smaller norm; here

$\mathbf{\Pi} = \{\boldsymbol{P}_1, \dots \boldsymbol{P}_k\}$ is a set of compatible Γ–Whitehead partitions, which we will refer to as an "ideal forest". We can identify $\mathrm{st}(\sigma) \cap K_{<\sigma}$ with the geometric realization of the poset of such ideal forests, ordered by inclusion. To prove that $\mathrm{st}(\sigma) \cap K_{<\sigma}$ is contractible, we will repeatedly apply the poset lemma to retract this poset to a single point.

First note that by Corollary 6.20, $\mathrm{st}(\sigma) \cap K_{<\sigma}$ is nonempty. By Corollary 6.8, if $\mathbf{\Pi}$ is in $\mathrm{st}(\sigma) \cap K_{<\sigma}$, then $\mathbf{\Pi}$ contains a σ–reductive Γ–Whitehead partition. Therefore, the map that throws out the nonreductive Γ–Whitehead partitions from each $\mathbf{\Pi}$ in $\mathrm{st}(\sigma) \cap K_{<\sigma}$ is a poset map, and we can use the poset lemma to retract $\mathrm{st}(\sigma) \cap K_{<\sigma}$ to its image, which is the subposet $\mathcal{R}$ spanned by ideal forests all of whose Γ–Whitehead partitions are reductive.

Choose a reductive pair (M, m) satisfying the maximality conditions of Lemma 6.23. We will ultimately retract $\mathcal{R}$ to the ideal forest consisting of the single partition $\{\boldsymbol{M}\}$.

If all of the ideal forests in $\mathcal{R}$ are compatible with $\boldsymbol{M}$, then we can contract $\mathcal{R}$ to $\{\boldsymbol{M}\}$ via the poset maps $\mathbf{\Pi} \to \mathbf{\Pi} \cup \{\boldsymbol{M}\} \to \{\boldsymbol{M}\}$. If not, choose a reductive Γ–Whitehead partition $\boldsymbol{P}$ such that

(1) $\boldsymbol{P}$ and $\boldsymbol{M}$ are not compatible, and

(2) the side P containing m is maximal among sides of all such partitions, ie if $\boldsymbol{Q}$ is a reductive Γ–Whitehead partition with $P \subset Q$, then $\boldsymbol{Q}$ is compatible with $\boldsymbol{M}$.

Note that if $m \in P \subset Q$, then $m \notin \mathrm{lk}(\boldsymbol{Q})$, so $\boldsymbol{M}$ and $\boldsymbol{Q}$ do not commute. It follows that the only way they can be compatible is if $M \subset Q$. Thus, the second condition can be restated as: $(2')$ if $P \subset Q$, then $M \subset Q$.

By the pushing lemma, one of $M \cap P^*$ or $M^* \cap P^*$ determines a reductive Γ–Whitehead partition whose link is equal to $\mathrm{lk}(\boldsymbol{P})$; call it $\boldsymbol{P}_0$. We claim that

$$\mathbf{\Pi} \mapsto \begin{cases} \mathbf{\Pi} \cup \{\boldsymbol{P}_0\} & \text{if } \boldsymbol{P} \in \mathbf{\Pi}, \\ \mathbf{\Pi} & \text{if } \boldsymbol{P} \notin \mathbf{\Pi}, \end{cases}$$

is a poset map from $\mathcal{R}$ to itself. If $\boldsymbol{P} \in \mathbf{\Pi}$ then $\boldsymbol{Q} \in \mathbf{\Pi}$ implies that $\boldsymbol{Q}$ is compatible with $\boldsymbol{P}$, so we have to check that any such $\boldsymbol{Q}$ is also compatible with $\boldsymbol{P}_0$.

If $\boldsymbol{Q}$ commutes with $\boldsymbol{P}$, then it also commutes with $\boldsymbol{P}_0$, since they have the same link. Otherwise, compatibility implies that some side Q of $\boldsymbol{Q}$ is either contained in P or contains P. If $Q \subset P$ then $Q \cap P^* = \varnothing$, so $\boldsymbol{Q}$ is compatible with both $M \cap P^*$ and $M^* \cap P^*$. If $P \subset Q$ then, by $(2')$, $M \cup P \subset Q$. It follows that $M \cap P^* \subset Q$ and $Q^* \subset (M^* \cap P^*)$, so both of these quadrants are again compatible with $\boldsymbol{Q}$. This proves the claim.

This map clearly satisfies the hypotheses of the poset lemma, so $\mathcal{R}$ retracts to the image, in which every ideal forest that contains $\boldsymbol{P}$ also contains $\boldsymbol{P_0}$. Now map this image to itself by the poset map which throws $\boldsymbol{P}$ out of every $\boldsymbol{\Pi}$ that contains it. The poset lemma applies again, and the image is now the subcomplex of $\mathcal{R}$ spanned by all reductive ideal forests which do not contain $\boldsymbol{P}$.

Repeat this process until every Γ–Whitehead partition that is not compatible with $\boldsymbol{M}$ has been eliminated. The resulting poset can be retracted to the single point $\{\boldsymbol{M}\}$ as described above. □

7 Outer space for $\mathrm{Out}(\mathrm{A}_\Gamma)$

In this final section we briefly discuss the problem of finding an outer space for the entire group $\mathrm{Out}(\mathrm{A}_\Gamma)$. If there are no vertices v and w in Γ with $\mathrm{st}(v) \subseteq \mathrm{st}(w)$, then $\mathrm{U}(\mathrm{A}_\Gamma) = \mathrm{Out}(\mathrm{A}_\Gamma)$ and the spine K_Γ is quasi-isometric to $\mathrm{Out}(\mathrm{A}_\Gamma)$, so nothing more needs to be done.

In particular, if A_Γ is a free group then K_Γ is the spine of Culler and Vogtmann's outer space $\mathcal{O}_n$. The complete space $\mathcal{O}_n$ is obtained by giving edges of marked graphs positive real lengths, then taking projective classes of the resulting metric graphs. The spine K_n naturally embeds into $\mathcal{O}_n$, by sending vertices of K_n to metric graphs with all edges of equal length. We can do the same thing for general K_Γ by considering the cubes in a Γ–complex to be arbitrary rectilinear parallelepipeds, ie giving each set of edges dual to a hyperplane the same positive real length, then taking the projective class of the resulting metric space. We call the space we obtain in this way Σ_Γ, and K_Γ embeds into it, as in the free group case, as an equivariant deformation retract.

The opposite extreme from the free group is the free abelian group $\mathbb{Z}^n$. In this case $\mathrm{Out}(\mathrm{A}_\Gamma) \cong \mathrm{GL}(n,\mathbb{Z})$ acts properly on the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, which we can regard as a space of projective classes of flat tori. For general Γ, abelianization $\mathrm{A}_\Gamma \to \mathbb{Z}^n$ induces a map $\mathrm{Out}(\mathrm{A}_\Gamma) \to \mathrm{GL}(n,\mathbb{Z})$ which sends the twist subgroup T_Γ injectively into $\mathrm{SL}(n,\mathbb{Z})$. There is a natural contractible subspace $\mathbb{D}_\Gamma$ of the symmetric space on which T_Γ acts; this space $\mathbb{D}_\Gamma$ can be regarded as a space of projective classes of certain "flat Salvettis", where we regard the Salvetti as a subcomplex of a flat n–torus and take the induced path metric. This gives us a restricted class of allowable metrics on Salvetti complexes, which can be extended to a restricted class of allowable metrics on Γ–complexes. Cubes are no longer necessarily rectilinear; some are allowed to become parallelepipeds.

We can then define an outer space $\mathcal{O}(\mathrm{A}_\Gamma)$ on which all of $\mathrm{Out}(\mathrm{A}_\Gamma)$ acts as follows: A point in $\mathcal{O}(\mathrm{A}_\Gamma)$ is an equivalence class of triples (X, d, h), where

- X is a Γ–complex;
- d is an allowable metric on X;
- $h\colon X \to \mathbb{S}$ is an arbitrary homotopy equivalence;
- $(X, d, h) \sim (X', d', h')$ if there is an isometry $f\colon X \to X'$ with $h' \circ f \simeq h$.

The action of $\varphi \in \mathrm{Out}(A_\Gamma)$ on $\mathcal{O}(A_\Gamma)$ is by composition: realize φ by $f\colon \mathbb{S} \to \mathbb{S}$; then $\varphi(X, d, h) = (X, d, f \circ h)$.

Both Σ_Γ and $\mathbb{D}_\Gamma$ embed into this space $\mathcal{O}(A_\Gamma)$. In many cases it is easy to reduce the contractibility of $\mathcal{O}(A_\Gamma)$ to the contractibility of Σ_Γ, proved in this paper. However the general case seems to be quite subtle and not at all short, so this will form the contents of a second paper.

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