REALIZING HOMOLOGY CLASSES UP TO COBORDISM

MARK GRANT, ANDRÁS SZŰCS, AND TAMÁS TERPAI

Abstract. It is known that neither immersions nor maps with a fixed finite set of multisingularities are enough to realize all mod 2 homology classes in manifolds. In this paper we define the notion of realizing a homology class up to cobordism; it is shown that for realization in this weaker sense immersions are sufficient, but maps with a fixed finite set of multisingularities are still insufficient.

1. Introduction

In 1949 Steenrod [4] posed the following question: given a homology class h of a space X, does there exist a closed manifold V and a continuous map $f : V \to X$ such that $f_*[V] = h$, where $[V]$ is the fundamental class of V? Thom's famous result answers the question affirmatively if h is a $\mathbb{Z}_2$-homology class, and shows that for integral homology the answer in general is negative. It is a natural further question whether f can be chosen to be "nice" if X itself is a smooth manifold. For example, can it be always an embedding or an immersion? If not, then can f be chosen to have only mild singularities?

For embeddings Thom himself gave some necessary and sufficient conditions. From these conditions it is not hard to deduce that there are $\mathbb{Z}_2$-homology classes of codimension 2 not realizable by embeddings in some manifolds.

In [5] it was shown that for any $k > 1$ there is a manifold M (of dimension approximately $4k$) and a cohomology class $\alpha \in H^k(M; \mathbb{Z}_2)$ such that the Poincaré dual of α cannot be realized by an immersion. Moreover it was shown there that for any $k > 1$ singular maps of finite complexity (see Section 3 for the precise definition) are insufficient to realize all codimension k homology classes in manifolds.

Therefore in order to obtain positive answers it is natural to relax the notion of "realization of a homology class". The relaxed version we use will be "realization up to cobordism". For this purpose we define the cobordism group of pairs (M^n, α) where M^n is a closed smooth n-manifold and $\alpha \in H^k(M; \mathbb{Z}_2)$ for a fixed k.

Definition: Given two pairs (M^n, α) and (N^n, β) we say that they are *cobordant* if there is a pair (W^{n+1}, γ) such that W^{n+1} is a compact $(n+1)$-manifold with boundary $\partial W = M \sqcup N$ and $\gamma \in H^k(W; \mathbb{Z}_2)$ is a cohomology class such that $\gamma|_M = \alpha$ and $\gamma|_N = \beta$.

Remark: The obtained group of pairs is clearly isomorphic to $\mathfrak{N}_n(K(\mathbb{Z}_2, k))$, the nth bordism group of the Eilenberg-MacLane space $K(\mathbb{Z}_2, k)$.

Definition: Let $\mathcal{F}$ be a class of smooth maps (for example, embeddings, immersions, or singular maps of some given complexity). We say that a pair (M, α)

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is $\mathcal{F}$*-realizable* if there exist a closed manifold V and a map $f : V \to M$ such that $f \in \mathcal{F}$ and $f_*[V]$ is Poincaré dual to α. We say that (M, α) is $\mathcal{F}$*-realizable up to cobordism* if there is an $\mathcal{F}$-realizable pair (N, β) cobordant to (M, α).

We show that this relaxation allows to give a positive answer in the case of immersions but for singular maps of finite complexity the answer remains negative.

2. Realization by immersions

Theorem 1. *Any pair* (M, α) *is realizable by immersions up to cobordism.*

For conciseness, (co)homology coefficients $\mathbb{Z}_2$ will be omitted and K will stand for $K(\mathbb{Z}_2, k)$.

In what follows, $MO(k)$ denotes as usual the Thom space of the universal vector bundle over $BO(k)$, and for any space X we denote by ΓX the space $\Omega^\infty S^\infty X = \lim_{N\to\infty} \Omega^N S^N X$. Recall that $\Gamma MO(k)$ is the classifying space of codimension k immersions, in particular, the group of cobordism classes of codimension $k > 0$ immersions into a fixed closed manifold P (where cobordisms are codimension k immersions into $P \times [0, 1]$) is isomorphic to the group of homotopy classes $[P, \Gamma MO(k)]$.

It is well-known that $\Gamma MO(k)$ is stably equivalent to a bouquet that contains $MO(k)$ (i.e. there is a space Y such that $\Gamma MO(k) \underset{stably}{\cong} MO(k) \vee Y$). Hence $H^*(MO(k))$ embeds naturally into $H^*(\Gamma MO(k))$. In particular the Thom class $u_k \in H^k(MO(k))$ can be considered (uniquely, since Y is known to be $2k-1$-connected) as a cohomology class of $\Gamma MO(k)$. Denote by u the corresponding map into K, that is, $u : \Gamma MO(k) \to K$ has the property that $u^*(\iota_k) = u_k$, where $\iota_k l \in H^k(K)$ is the fundamental class.

Alternatively, we may use the universal property of the functor Γ that is as follows ([2, p. 39.], [6, pp.42–43.]): for any map $f : X \to Y$ from a compactly generated Hausdorff space X to an infinite loop space Y there is a homotopically unique extension $\hat{f} : \Gamma X \to Y$ that is an infinite loop map. Applying this property to u_k yields the map u.

For any P the map $u_*^P : [P, \Gamma MO(k)] \to [P, K]$ induced by u associates to (a cobordism class of) an immersion the Poincaré dual of the homology class represented by the immersion.

This shows that Theorem 1 has the following equivalent reformulation:

Theorem 1′. *The map* $u : \Gamma MO(k) \to K$ *induces an epimorphism of the bordism groups in any dimension. That is, for any* n

$$u_* : \mathfrak{N}_n(\Gamma MO(k)) \to \mathfrak{N}_n(K)$$

is onto.

Proof: It is well-known ([3]) that there is an isomorphism $H_*(X; \mathbb{Z}_2) \otimes \mathfrak{N}_* \to \mathfrak{N}_*(X)$, natural in X, defined by taking a representative $[\hat{\alpha} : M_\alpha \to X] \in \mathfrak{N}_*(X)$ for all elements α of a basis of $H_*(X)$ and mapping $\sum_j \alpha_j \otimes [N_j]$ to $\sum_j [\hat{\alpha}_j \circ pr_j : M_{\alpha_j} \times N_j \to X]$, where $pr_j : M_{\alpha_j} \times N_j \to M_{\alpha_j}$ is the projection. Hence a map induces epimorphism of the (unoriented) bordism groups if and only if it does so in the $\mathbb{Z}_2$-homology groups.

For later use, recall that for any space X the ring $H_*(\Gamma X)$ is a polynomial ring (multiplication being the Pontryagin product) in variables x_λ, $y_{I,\lambda}$, where $\{x_\lambda\}_\lambda$ is a homogeneous basis of $H_*(X)$ and $y_{I,\lambda}$ are further variables defined using Kudo-Araki operations as $y_{I,\lambda} = Q^I x_\lambda$ (their precise description will be unimportant in our argument).

In order to show that

$$u_* : H_*(\Gamma MO(k)) \to H_*(K)$$

is onto it is enough to show that the composition

$$\varphi \stackrel{\text{def}}{:} \overline{H}_*(MO(k)) \stackrel{(u_k)_*}{\to} \overline{H}_*(K) \stackrel{p}{\to} Q(H_*(K)) = \overline{H}_*(K)/\mu\left(\overline{H}_*(K) \otimes \overline{H}_*(K)\right)$$

is onto, where $\mu : H_*(K) \otimes H_*(K) \to H_*(K)$ is the multiplication map and p is the natural projection onto the quotient group of indecomposables. Indeed, assume that φ is onto and for all j choose elements in $H_j(K)$ such that they form a (linear) basis in $\overline{H}_j(K)/\mu\left(\overline{H}_j(K) \otimes \overline{H}_j(K)\right)$. It is easy to see by induction on j that the chosen elements generate $\overline{H}_*(K)$ multiplicatively and hence the subring of $H_*(\Gamma MO(k))$ generated by the preimages of these elements is mapped onto the entire $H_*(K)$ (here we use that u_* is a ring homomorphism, since u is an infinite loop map).

Hence to prove Theorem 1 we have to show that $\varphi : H_*(MO(k)) \to QH_*(K)$ is onto. This is equivalent to the dual homomorphism φ^* being injective. By [7, Proposition 3.10.], the dual of $QH_*(K)$ is $PH^*(K)$, the submodule of primitive elements of the Hopf algebra $H^*(K)$. This latter group is known to be

$$PH^*(K) = \mathbb{Z}_2 \left\langle Sq^I \iota_k : I \text{ admissible of excess } e(I) \le k \right\rangle,$$

the vector space over $\mathbb{Z}_2$ freely generated by the $Sq^I \iota_k$ (see eg. [1, p. 23.]). The dual of $H_*(MO(k))$ is $H^*(MO(k))$ and can be identified with the ideal generated by w_k in $\mathbb{Z}_2[w_1, \dots, w_k]$ (w_k corresponds to the Thom class u_k). The map φ^* maps ι_k to u_k and then to w_k, and commutes with the action of the Steenrod algebra, allowing to calculate the image of φ^*.

Finally, we need to show that the set $\left\{Sq^I(w_k) : I \text{ is admissible with } e(I) \le k\right\}$ is linearly independent in the ideal $(w_k) \subset \mathbb{Z}_2[w_1, \dots, w_k]$. This is the immediate consequence of [8, Remark 2.4.] that shows that the Steenrod algebra acts freely unstably on w_k in $H^*(BO(k))$, and this finishes the proof of Theorem 1. □

3. Non-realizability up to cobordism by singular maps of finite complexity

Recall some definitions from singularity theory that are necessary for the formulation of Theorem 2.

Definition: Fix a natural number $k \ge 1$ and consider equivalence classes of germs $\eta : (\mathbb{R}^{n-k}, 0) \to (\mathbb{R}^n, 0)$, $n \ge k$, up to left-right equivalence and stabilization, that is, we consider η to be equivalent to $\eta \times id_{\mathbb{R}^1} : (\mathbb{R}^{n-k+1}, 0) \to (\mathbb{R}^{n+1}, 0)$. An equivalence class is called a (codimension k) *local singularity* (even if its rank is maximal).

Definition: A *multisingularity* is a finite multiset (set with elements equipped with multiplicities) of local singularities.

Definition: Let $f : M \to N$ be a smooth map such that for any $y \in N$ the preimage $f^{-1}(y)$ is a finite set. For $y \in N$ and $f^{-1}(y) = \{x_1, \dots, x_m\}$ let $\left[f_{x_j}\right]$ denote the local singularity class of the germ f at x_j. The multiset $\{[f_{x_1}], \dots, [f_{x_m}]\}$ is called the *multisingularity of f at y*.

Definition: Let τ be a set of multisingularities. The map f is said to be a *τ-map* if its multisingularity at any point $y \in N$ belongs to τ.

Theorem 2. *Let τ be any finite set of multisingularities of codimension $k > 1$ stable maps and let $\mathcal{F}$ be the class of τ-maps. Then the class $\mathcal{F}$ is insufficient for realizing up to cobordism all codimension k homology classes in manifolds. That is, for any $k > 1$ there is a pair (M, α) with M a smooth manifold and $\alpha \in H^k(M)$ such that (M, α) is not $\mathcal{F}$-realizable up to cobordism.*

Proof: The proof given in [5, Theorem 1.3.] for non-realizability of homologies by τ-maps also proves the stronger statement of Theorem 2. In that proof there was a classifying space X_τ for τ-maps (analogously to $\Gamma MO(k)$ being the classifying space for immersions). X_τ has a single nonzero element in its first nontrivial (reduced) cohomology group, $H^k(X_\tau)$, which can be called the Thom class $u_\tau : X_\tau \to K$. If any pair (M, α) could be realizable up to cobordism by τ-maps, then the map u_τ would induce an epimorphism $(u_\tau)_* : \mathfrak{N}_*(X_\tau) \to \mathfrak{N}_*(K)$ between the unoriented bordism groups or, equivalently, between the homology groups (using the same argument as in the proof of Theorem 1). But [5] shows that for any sufficiently high dimension j (under the assumption that $k > 1$) we have $\dim_{\mathbb{Z}_2} H_j(X_\tau) < \dim_{\mathbb{Z}_2} H_j(K)$, hence $(u_\tau)_* : H_j(X_\tau) \to H_j(K)$ cannot be surjective. □

Remark: In particular, embeddings or immersions with self-intersection multiplicity bounded by a fixed number are insufficient for realizing all homology classes in manifolds even up to cobordism.

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Institute of Mathematics, University of Aberdeen, Fraser Noble Building, Aberdeen AB24 3UE

E-mail address: mark.grant@abdn.ac.uk

ELTE Analysis Department, 1117 Budapest, Pázmány Péter sétány 1/C, Hungary

E-mail address: szucs@math.elte.hu

Alfréd Rényi Institute of Mathematics, 1053 Budapest, Reáltanoda u. 13-15., Hungary

E-mail address: terpai@math.elte.hu