SUBCENTRIC LINKING SYSTEMS

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Abstract. We propose a definition of a linking system which is slightly more general than the one currently in the literature. Whereas the objects of linking systems in the current definition are always quasicentric, the objects of our linking systems only need to satisfy a weaker condition. This leads to the definition of subcentric subgroups of fusion systems. We prove that there is a unique linking system associated to each fusion system whose objects are the subcentric subgroups. Furthermore, the nerve of such a subcentric linking system is homotopy equivalent to the nerve of the centric linking system. The existence of subcentric linking systems seems to be of interest for a classification of fusion systems of characteristic p-type. The various results we prove about subcentric subgroups indicate furthermore that the concept is of interest for studying extensions of linking system and fusion systems.

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1. Introduction

Centric linking systems associated to fusion systems were introduced by Broto, Levi and Oliver [9] to be able to study p-completed classifying spaces of fusion systems. It presents a problem in many contexts that centric linking systems do not form a category in a meaningful way. Different notions of linking systems were introduced to allow a more flexible choice of objects making it at least in special cases possible to define maps between linking systems in a useful way. So Broto, Castellana, Grodal, Levi and Oliver [7] introduced *quasicentric linking systems* and, much later, Oliver [16] introduced a general notion of a *linking system* providing an axiomatic setup for the full subcategories of quasicentric linking systems studied before. *Transporter systems*, as defined by Oliver and Ventura [17] give an even more general framework. To summarize: Centric linking systems are full subcategories of quasicentric linking systems; centric and quasicentric linking systems are special cases of linking systems; every linking system is a transporter system. The main purpose of this paper is to suggest a new notion of a linking system, allowing a more flexible choice of objects than in the existing notion.

Throughout, p is a prime, S is a finite p-group, and $\mathcal{F}$ is a saturated fusion system on S.

Recall that a subgroup $Q \leq S$ is called *quasicentric* if, for any fully centralized $\mathcal{F}$-conjugate P of Q, $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_S(P))$. The set of quasicentric subgroups is denoted by $\mathcal{F}^q$. The objects of a linking system associated to $\mathcal{F}$ in the sense of Oliver are always quasicentric subgroups. The objects of linking systems in our new definition are *subcentric* subgroups as defined next.

Definition 1. A subgroup $Q \leq S$ is said to be *subcentric* in $\mathcal{F}$ if, for any fully normalized $\mathcal{F}$-conjugate P of Q, $O_p(N_{\mathcal{F}}(P))$ is centric in $\mathcal{F}$. Write $\mathcal{F}^s$ for the set of subcentric subgroups of $\mathcal{F}$.

Recall that $\mathcal{F}$ is called constrained if $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$. As we show in detail in Lemma 3.1, a subgroup $Q \leq S$ is subcentric if and only if for some (and thus for any) fully normalized $\mathcal{F}$-conjugate P of Q, $N_{\mathcal{F}}(P)$ is constrained. Similarly, Q is subcentric if and only if for some (and thus for any) fully centralized $\mathcal{F}$-conjugate P of Q, $C_{\mathcal{F}}(P)$ is constrained. It follows that every quasicentric subgroup is subcentric. Thus we have the following inclusions:

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$$\mathcal{F}^{cr} \subseteq \mathcal{F}^c \subseteq \mathcal{F}^q \subseteq \mathcal{F}^s$$

Even though originally the main motivation for the definition of linking systems came from homotopy theory, it can be hoped that linking systems are also useful from an algebraic point of view. Andrew Chermak [10] introduced with *localities* a concept which in a certain sense is equivalent to the concept of a transporter systems, but has a more group-like flavour. While he defined localities first in the context of his proof of the existence and uniqueness of centric linking systems, he is currently developing a local theory of localities; see [11]. Our results are mostly formulated in terms of localities. We say that a locality $(\mathcal{L}, \Delta, S)$ is a locality for $\mathcal{F}$, if $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. We now introduce the second main definition of this paper:

Definition 2.

- A finite group G is said to be of *characteristic* p if $C_G(O_p(G)) \leq O_p(G)$.
- A locality $(\mathcal{L}, \Delta, S)$ is called a *linking locality*, if $\mathcal{F}_S(\mathcal{L})^{rc} \subseteq \Delta$ and, for any $P \in \Delta$, the group $N_{\mathcal{L}}(P)$ is of characteristic p.
- Let $\mathcal{T}$ be a transporter system associated to $\mathcal{F}$. Then $\mathcal{T}$ is called a *linking system*, if $\mathcal{F}^{rc} \subseteq \Delta$ and $\mathrm{Aut}_{\mathcal{T}}(P)$ is a group of characteristic p for every object P of $\mathcal{T}$.
- A *subcentric linking locality* for $\mathcal{F}$ is a linking locality $(\mathcal{L}, \mathcal{F}^s, S)$ for $\mathcal{F}$. Similarly, we call a linking system $\mathcal{T}$ associated to $\mathcal{F}$ a *subcentric linking system* if $\mathrm{ob}(\mathcal{T}) = \mathcal{F}^s$.

Given a locality $(\mathcal{L}, \Delta, S)$, we denote the transporter system corresponding to $(\mathcal{L}, \Delta, S)$ by $\mathcal{T}(\mathcal{L}, \Delta)$; see [10, Remark 2.8(1)]. The objects of $\mathcal{T}(\mathcal{L}, \Delta)$ are the elements of Δ, and a morphism between objects $P, Q \in \Delta$ is a triple (f, P, Q) with $f \in \mathcal{L}$ and $P^f \leq Q$. Composition of morphisms corresponds to multiplication in the locality $\mathcal{L}$. Note that $\mathrm{Aut}_{\mathcal{T}(\mathcal{L},\Delta)}(P) \cong N_{\mathcal{L}}(P)$. In the following remark, we summarize some basic but important properties of linking systems and linking localities. Moreover, we explain the connection between our notion of a linking system and the one currently in the literature.

Remark 1. Let $(\mathcal{L}, \Delta, S)$ be a locality for $\mathcal{F}$, and let $\mathcal{T}$ be a transporter system associated to $\mathcal{F}$. Then the following hold.

(a) $\mathcal{T}(\mathcal{L}, \Delta)$ is a linking system if and only $(\mathcal{L}, \Delta, S)$ is a linking locality.

(b) If $(\mathcal{L}, \Delta, S)$ is a linking locality, then $\Delta \subseteq \mathcal{F}^s$. Moreover, for any $P \in \Delta \cap \mathcal{F}^f$, $N_{\mathcal{L}}(P)$ is a model for $N_{\mathcal{F}}(P)$. Similarly, if $\mathcal{T}$ is a linking system then $\mathrm{ob}(\mathcal{T}) \subseteq \mathcal{F}^s$ and for any $P \in \mathrm{ob}(\mathcal{T}) \cap \mathcal{F}^f$, the group $\mathrm{Aut}_{\mathcal{T}}(P)$ is isomorphic to a model for $N_{\mathcal{F}}(P)$.

(c) If $\Delta \subseteq \mathcal{F}^q$ then $C_{\mathcal{L}}(P) = C_S(P) \times O_{p'}(C_{\mathcal{L}}(P))$ for every $P \in \Delta$. So in this case, $(\mathcal{L}, \Delta, S)$ is a linking locality if and only if $C_{\mathcal{L}}(P)$ is a p-group for every $P \in \Delta$. If $\mathrm{ob}(\mathcal{T}) \subseteq \mathcal{F}^q$, then $\mathcal{T}$ is a linking system in the sense defined above if and only if it is a linking system in the sense of Oliver [16, Definition 3].

(d) If $\Delta \subseteq \mathcal{F}^c$ then $(\mathcal{L}, \Delta, S)$ is a linking locality if and only if $(\mathcal{L}, \Delta, S)$ is a Δ-linking system in the sense of Chermak [10], meaning $C_{\mathcal{L}}(P) \leq P$ for every $P \in \Delta$. If $\mathrm{ob}(\mathcal{T}) = \mathcal{F}^c$, then $\mathcal{T}$ is a linking system if and only if it is a centric linking system in the sense of [9, Definition 1.7].

Here a model for the fusion system $\mathcal{F}$ is a finite group G of characteristic p such that $S \in \mathrm{Syl}_p(G)$ and $\mathcal{F}_S(G) = \mathcal{F}$. As shown in [7], there exists a model for $\mathcal{F}$ if and only if $\mathcal{F}$ is constrained.

Given a set of subgroups $\Delta \subseteq \mathcal{F}^q$ closed under $\mathcal{F}$-conjugation and with respect to overgroups, it follows from the existence and uniqueness of centric linking systems combined with [7, Theorem A, Proposition 3.12] that there is a linking system with object set Δ associated to $\mathcal{F}$, and that such a linking system is unique up to isomorphism. Moreover, the nerve of the linking system does not depend on the object set Δ. In particular, quasicentric linking systems exist and are unique up to isomorphism, and the nerve of a quasicentric linking system is homotopy equivalent to the nerve of a centric linking system. Except for the statement about nerves, a formulation of these results and an algebraic proof using the methods in [10] was given by Andrew Chermak in unpublished notes before the idea to define subcentric subgroups arose. We similarly give a version for subcentric linking systems. We also include a statement about nerves, which follows from a result of Oliver and Ventura [17, Proposition 4.7] generalizing the arguments in [7]. The crucial property is that the radical objects of a linking system $\mathcal{T}$ (i.e. the objects P of $\mathcal{T}$ with $O_p(\mathrm{Aut}_{\mathcal{T}}(P)) \cong P$) are precisely the elements of $\mathcal{F}^{rc}$.

Theorem A. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S.*

(a) *Let $\mathcal{F}^{rc} \subseteq \Delta \subseteq \mathcal{F}^s$ such that Δ is closed under $\mathcal{F}$-conjugation and with respect to overgroups. Then there exists a linking locality for $\mathcal{F}$ with object set Δ, which is unique up to a rigid isomorphism. Similarly, there exists a linking system $\mathcal{T}$ for $\mathcal{F}$ whose set of objects is Δ, and such a linking system is unique up to an isomorphism of transporter systems. Moreover, the nerve $|\mathcal{T}|$ is homotopy equivalent to the nerve of a centric linking system associated to $\mathcal{F}$.*

(a) *The set $\mathcal{F}^s$ is closed under $\mathcal{F}$-conjugation and with respect to overgroups. In particular, there exists a subcentric linking locality associated to $\mathcal{F}$ which is unique up to a rigid isomorphism, and a subcentric linking system associated to $\mathcal{F}$ which is unique up to an isomorphism of transporter systems.*

Recall here that a *rigid isomorphism* between localities $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}^*, \Delta, S)$ with the same set of objects is a homomorphism $\mathcal{L} \to \mathcal{L}^*$ of partial groups which restricts to the identity on S.

We believe that the existence of subcentric linking localities and of subcentric linking systems is important, mainly for two reasons: Firstly, subcentric linking localities seem to provide a useful setup for a classification of fusion systems of characteristic p-type. Secondly, there is some evidence that the more flexible choice of objects facilitates the study of extensions and of "maps" between linking systems in the spirit of [8], [16], [17], [1]. We explain the first point before we state some results supporting our second point.

Recall that a finite group G is said to be of *characteristic p-type* (or of *local characteristic p*), if every p-local subgroup (i.e. every normalizer of a non-trivial p-subgroup) is of characteristic p. Similarly, the fusion system $\mathcal{F}$ is said to be of characteristic p-type if, for every non-trivial fully $\mathcal{F}$-normalized subgroup $P \leq S$, $N_{\mathcal{F}}(P)$ is constrained. Since a subgroup is subcentric if and only if the normalizer of every fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate is constrained, $\mathcal{F}$ is of characteristic p-type if and only if every non-trivial subgroup of S is subcentric. So provided $\mathcal{F}$ is of characteristic p-type and $(\mathcal{L}, \Delta, S)$ is a linking locality for $\mathcal{F}$, the normalizer $N_{\mathcal{L}}(P)$ of any non-trivial subgroup P of S is a finite group of characteristic p. Hence, "locally" it looks very much like a finite group of characteristic p-type. On the other hand, every group of characteristic p-type leads easily to a linking locality of this kind:

Example 1. Let G be a group of characteristic p-type and $S \in \mathrm{Syl}_p(G)$. Let Δ be the set of non-trivial subgroups of G. Write $\mathcal{L}_\Delta(G)$ for the partial group introduced in [10, Example/Lemma 2.10]. That is, $\mathcal{L}_\Delta(G)$ consists of all elements $g \in G$ with $S \cap S^g \neq 1$, and the product in $\mathcal{L}_\Delta(G)$ is the restriction of the product in G to all tuples $(g_1, \ldots, g_n)$ such that there exist elements $P_0, \ldots, P_n \in \Delta$ with $P_{i-1}^{g_i} = P_i$ for $i = 1, \ldots, n$. Then $N_{\mathcal{L}_\Delta(G)}(P) = N_G(P)$ is of characteristic p for all $P \in \Delta$. Hence, $(\mathcal{L}_\Delta(G), \Delta, S)$ is a subcentric linking locality for $\mathcal{F}_S(G)$

Previous treatments of fusion systems of characteristic p-type (as for example in [2], [4], [5] and [12]) have used the existence of models for normalizers of fully normalized subgroups. When looking at arbitrary subgroups of S, this involves moving on to a fully normalized $\mathcal{F}$-conjugate, which often complicates the arguments. Moreover, the connections between models of different p-local subsystems become more transparent in a subcentric linking locality, where these models can all be seen as subgroups and thus have a set-theoretic intersection. We thus believe that subcentric linking localities allow a much more canonical translation of the arguments used to classify groups of characteristic p-type. Building on the ongoing programme of Meierfrankenfeld, Stellmacher, Stroth (short: MSS-programme) to classify groups of local characteristic p, one can hope to achieve a classification of fusion systems of characteristic p-type. If one prefers a more group-like language and wants to avoid the language of fusion systems, in view of Example 1 a unifying approach to the classification of fusion systems of characteristic p-type and of groups of local characteristic p might be possible as follows: In a first step one classifies localities $(\mathcal{L},\Delta,S)$ where Δ is the set of non-trivial subgroups of S, and $N_{\mathcal{L}}(P)$ is of characteristic p for every $1\neq P\leq S$. Then one separately deduces a classification of fusion systems of characteristic p-type and of groups of local characteristic p from that. To implement in subcentric linking localities the amalgam method, which is widely used in the MSS-programme, one presumably needs to work with free amalgamated products of suitable subgroups of $\mathcal{L}$ amalgamated over their set-theoretic intersection in $\mathcal{L}$. A similar approach should be possible for groups and fusion systems of parabolic characteristic p; see Remark 8.1 for details.

We continue by listing some elementary properties of subcentric linking systems which we prove in Section 3. In particular, we work out some relations between the subcentric subgroups of $\mathcal{F}$ and subcentric subgroups of certain subsystems.

Proposition 1. *The following hold:*

(a) *Let $R\trianglelefteq\mathcal{F}$ and $P\leq S$. Then $PR\in\mathcal{F}^s$ if and only if $P\in\mathcal{F}^s$.*
(b) *Let $Z\leq Z(\mathcal{F})$ and $P\leq S$. Then $P\in\mathcal{F}^s$ if and only if PZ/Z is subcentric in $\mathcal{F}/Z$.*
(c) *If $Q\in\mathcal{F}^f$ and $P\in N_{\mathcal{F}}(Q)^s$ then $PQ\in\mathcal{F}^s$. More generally, if $Q\in\mathcal{F}$ and $K\trianglelefteq\mathrm{Aut}_{\mathcal{F}}(Q)$ such that Q is fully K-normalized, then $PQ\in\mathcal{F}^s$ for every $P\in N_{\mathcal{F}}^K(Q)^s$.*
(d) *For any $Q\in\mathcal{F}^f$, we have $\{P\in\mathcal{F}^s\colon P\leq N_S(Q)\}\subseteq N_{\mathcal{F}}(Q)^s$. More generally, for any $Q\leq S$ and any $K\trianglelefteq\mathrm{Aut}_{\mathcal{F}}(Q)$, $\{P\in\mathcal{F}^s\colon P\leq N_S^K(Q)\}\subseteq N_{\mathcal{F}}^K(Q)^s$.*

Property (b) holds accordingly for quasicentric subgroups as proved by Broto, Castellana, Grodal, Levi and Oliver in [8, Lemma 6.4(b)]. Building on that, the authors show that a quasicentric linking system for $\mathcal{F}/Z$ ($Z\leq Z(\mathcal{F})$) can be constructed as a "quotient" of a quasicentric linking system associated to $\mathcal{F}$. A similar construction can be carried out in the world of localities, both for quasicentric and subcentric linking systems. We provide details on that in Remark 8.3. Results corresponding to (c) and (d) are also true for centric and quasicentric subgroups. As we explain in more detail in Remark 8.4, property (c) leads to a kind of inclusion map from the subcentric linking system (respectively, the subcentric linking locality) of $N_{\mathcal{F}}^K(Q)$ to the subcentric linking system (respectively, the subcentric linking locality) of $\mathcal{F}$. We now turn attention to weakly normal subsystems.

Proposition 2. *Let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$ on T. Then the following hold:*

(a) *The set $\mathcal{E}^s$ is invariant under $\mathcal{F}$-conjugation.*
(b) *For every $P\in\mathcal{F}^s$ with $P\leq T$, $P\in\mathcal{E}^s$.*
(c) *If $\mathcal{E}$ is normal in $\mathcal{F}$ of index prime to p, then $\mathcal{E}^s=\mathcal{F}^s$.*
(d) *If $\mathcal{E}$ is normal in $\mathcal{F}$ of p-power index, then $\mathcal{E}^s=\{P\in\mathcal{F}^s\colon P\leq T\}$.*
(e) *If $R\trianglelefteq\mathcal{F}$ and $K\trianglelefteq\mathrm{Aut}_{\mathcal{F}}(R)$ then $N_{\mathcal{F}}^K(R)^s=\{P\in\mathcal{F}^s\colon P\leq N_S^K(R)\}$. In particular, $C_{\mathcal{F}}(R)^s=\{P\in\mathcal{F}^s\colon P\leq C_S(R)\}$.*

Corresponding statements to (a) and (b) are also true for centric and quasicentric subgroups. Property (c) is clearly also true if one considers centric subgroups rather than subcentric subgroups, and a statement corresponding to (d) is true for quasicentric subgroups by [8, Theorem 4.3]. It is shown in [8, Theorem 5.5] that, given a subsystem $\mathcal{E}$ of index prime to p, a centric linking system associated to $\mathcal{E}$ can be naturally constructed from the centric linking system associated to $\mathcal{F}$. Similarly, it is shown in [8, Theorem 4.4] that a quasicentric linking system of a subsystem of p-power index can be obtained from a quasicentric linking system associated to $\mathcal{F}$. Property (e) fails for centric and quasicentric subgroups as it is stated, but if $\mathrm{Inn}(R)\leq K$, it is true that every centric or quasicentric subgroup of $N_{\mathcal{F}}^K(R)$ which contains R is centric or quasicentric respectively, which is enough for many purposes. In [1, Definition 1.27], Andersen, Oliver and Ventura define normal linking systems. The results we just stated enable them to associate normal pairs of linking systems to $(\mathcal{E},\mathcal{F})$ if $\mathcal{E}$ is a weakly normal subsystem of $\mathcal{F}$ of index prime to p, or of p-power index, or if $\mathcal{E}=N_{\mathcal{F}}^K(R)$ for some normal subgroup $R\trianglelefteq\mathcal{F}$ and $\mathrm{Inn}(R)\leq K\trianglelefteq\mathrm{Aut}(Q)$; see [1, Proposition 1.31]. Andersen, Oliver and Ventura [1] define also the reduction of a fusion system $\mathcal{F}$. The reduction of $\mathcal{F}$ is taken by setting $\mathcal{F}_0:=C_{\mathcal{F}}(O_p(\mathcal{F}))/Z(O_p(\mathcal{F}))$ and then alternately taking $\mathcal{F}_i=O^p(\mathcal{F}_{i-1})$ and $\mathcal{F}_i=O^{p'}(\mathcal{F}_{i-1})$ until the process terminates. The connection between the linking system of the reduction and the linking system of $\mathcal{F}$ is complicated by the fact that one alternately needs to work with quasicentric and centric subgroups when alternating between taking subsystems of p-power index and index prime to p. We believe that it could be an advantage to work with subcentric subgroups instead, since Proposition 1(b) together with Proposition 2(c),(d),(e) gives a clear connection between the subcentric subgroups of $\mathcal{F}$ and the subcentric subgroups of the reduction of $\mathcal{F}$.

We now turn attention to arbitrary normal subsystems. It is work in progress of Andrew Chermak to show that, for linking localities with a delicate choice of subcentric objects, there is a one-to-one correspondence between subnormal subsystems of fusion systems and partial subnormal subgroups of these linking localities. We state and prove here two results which point already into that direction. For an arbitrary normal subsystem $\mathcal{E}$ of $\mathcal{F}$, we prove the following connection between the subcentric subgroups of $\mathcal{E}$ and the subcentric subgroups of $\mathcal{F}$.

Theorem B. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$. Then for every subcentric subgroup P of $\mathcal{E}$, $PC_S(\mathcal{E})$ is subcentric in $\mathcal{F}$.*

Here $C_S(\mathcal{E})$ is the subgroup introduced by Aschbacher [3, Chapter 6]. It is the largest subgroup X of S with $\mathcal{E}\subseteq C_{\mathcal{F}}(X)$. In the above mentioned work in progress, Andrew Chermak shows that every normal subsystem of a linking locality is realized by a partial normal subgroup. In this situation we prove:

Proposition 3. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ on T and let $(\mathcal{L},\Delta,S)$ be a linking locality for $\mathcal{F}$. Suppose there exists a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ such that $S\cap\mathcal{N}=T$ and $\mathcal{E}=\mathcal{F}_T(\mathcal{N})$. Then $C_S(\mathcal{E})=C_S(\mathcal{N})$.*

Assuming that $\mathcal{E}$ is realized by a partial normal subgroup $\mathcal{N}$ as above, Theorem B and Proposition 3 allow to get a sort of inclusion map from the linking system (respectively the linking locality) of $\mathcal{E}$ to the linking system (respectively the linking locality) of $\mathcal{F}$. We provide details on that in Remark 8.5.

The connection between the normal p-subgroups of a fusion system and between the normal p-subgroups of the linking locality is given in the following proposition, which is short and elementary to prove:

Proposition 4. *Let $(\mathcal{L},\Delta,S)$ be a linking locality for $\mathcal{F}$. Then a subgroup $Q\leq S$ is normal in $\mathcal{F}$ if and only if $\mathcal{L}=N_{\mathcal{L}}(Q)$. Similarly, $Q\leq Z(\mathcal{F})$ if and only if $\mathcal{L}=C_{\mathcal{L}}(R)$.*

Finally, a word about our proofs: Since there is some hope that the theory of fusion systems can be revisited using linking localities, we seek to keep the proofs of the results on subcentric subgroups of fusion systems as elementary as possible. In particular, we reprove some known results on constrained systems in Section 2 without using the theory of components of fusion systems. However, it should be pointed out that this theory and Aschbacher's version of the L-balance theorem for fusion systems is required for the proof of Theorem B.

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that subcentric linking systems should exist. He also pointed out that the nerve of a subcentric linking system would be homotopy equivalent to the nerve of a centric linking system. It was Andrew Chermak who suggested to me using the iterative procedure introduced in [10] to construct subcentric linking systems. He applied this method before in an unpublished preprint to prove similar results about localities whose object sets are quasicentric subgroups.

2. Groups of characteristic p and constrained fusion systems

Throughout, this text, we continue to assume that $\mathcal{F}$ is a saturated fusion system on a finite p-group S. Given a subsystem $\mathcal{E}$ of $\mathcal{F}$ we write $T = \mathcal{E} \cap S$ to express that $\mathcal{E}$ is a subsystem on $T \leq S$.

Recall that $\mathcal{F}$ is called *constrained* if $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$, and a finite group G is said to be of *characteristic* p if $C_G(O_p(G)) \leq O_p(G)$ (or equivalently, $C_G(O_p(G)) = Z(O_p(G))$). If G has characteristic p then $O_{p'}(G) = 1$ as $[O_p(G), O_{p'}(G)] \leq O_p(G) \cap O_{p'}(G) = 1$ and $C_G(O_p(G)) = Z(O_p(G))$ does not contain any non-trivial p'-elements. A finite group G is called a *model* for $\mathcal{F}$ if $S \in \mathrm{Syl}_p(G)$, $\mathcal{F} = \mathcal{F}_S(G)$ and G has characteristic p. The following lemma summarizes the connection between constrained fusion systems and groups of characteristic p which was basically established in [7].

Theorem 2.1. (a) *$\mathcal{F}$ is constrained if and only if there exists a model for $\mathcal{F}$. In this case, a model is unique up to an isomorphism which is the identity on S.*

(b) *If $\mathcal{F}$ is constrained and G is a model for $\mathcal{F}$ then a subgroup of S is normal in $\mathcal{F}$ if and only if it is normal in G. If $Q \leq S$ is normal and centric in $\mathcal{F}$, then in addition $C_G(Q) \leq Q$.*

Proof. If G is a model for $\mathcal{F}$ then clearly any normal subgroup of G is normal in $\mathcal{F}$, so in particular, $\mathcal{F}$ is constrained. Thus, (a) follows from [6, Theorem 5.10]. Let now $\mathcal{F}$ be constrained and G a model for $\mathcal{F}$. If Q is a normal centric subgroup of $\mathcal{F}$ then it follows again from [6, Theorem 5.10] that $Q \trianglelefteq G$ and $C_G(Q) \leq Q$. In particular, $O_p(\mathcal{F}) \trianglelefteq G$. So if $g \in G$ then $c_g|_{O_p(\mathcal{F})} \in \mathrm{Aut}_{\mathcal{F}}(O_p(\mathcal{F}))$ and thus $P^g = P$ for any normal subgroup P of $\mathcal{F}$. This shows that any normal subgroup of $\mathcal{F}$ is normal in G completing the proof. □

We continue by listing some properties of groups of characteristic p.

Lemma 2.2. *Let G be a finite group of characteristic p. Then the following hold:*

(a) *$N_G(P)$ and $C_G(P)$ have characteristic p for all non-trivial p-subgroups P of G.*
(b) *Every subnormal subgroup of G has characteristic p.*

Proof. By Part (c) of [15, Lemma 1.2], $N_G(P)$ has characteristic p and by Part (a) of the same lemma, (b) holds. As $C_G(P) \trianglelefteq N_G(P)$, it follows now that $C_G(P)$ has characteristic p. □

Lemma 2.3. *Let G be a finite group of characteristic p and $Z \leq Z(G) \cap O_p(G)$. Then G/Z has characteristic p.*

Proof. As Z is central, every p'-element of $C_G(O_p(G)/Z)$ centralizes $O_p(G)$. As $C_G(O_p(G)) \leq O_p(G)$, this implies that $C_G(O_p(G)/Z)$ is a p-group. As $C_G(O_p(G)/Z) \trianglelefteq G$, it follows $C_G(O_p(G)/Z) \leq O_p(G)$ and thus $C_{G/Z}(O_p(G)/Z) \leq O_p(G)/Z$. This shows that G/Z has characteristic p. □

Lemma 2.4. *Let G be a finite group with a normal p-subgroup P such that $\mathcal{F}_{C_S(P)}(C_G(P)) = \mathcal{F}_{C_S(P)}(C_S(P))$. Then $G = C_S(P) \times O_{p'}(C_G(P))$, and G has characteristic p if and only if $C_G(P)$ is a p-group.*

Proof. By the Theorem of Frobenius [14, Theorem 1.4], $C_G(P) = C_S(P)O_{p'}(C_G(P))$. If G has characteristic p, then $O_{p'}(C_G(P)) \leq O_{p'}(G) = 1$ and thus $C_G(P) = C_S(P)$ is a p-group. On the other hand, if $C_G(P)$ is a p-group then $C_G(P) \leq O_p(G)$ as $C_G(P) \trianglelefteq G$. Hence, as $P \leq O_p(G)$, $C_G(O_p(G)) \leq C_G(P) \leq O_p(G)$ and G has characteristic p. □

In the remainder of this section we explore some connections between $\mathcal{F}$ being constrained and certain subsystems or factor systems of $\mathcal{F}$ being constrained. We start with factor systems:

Lemma 2.5. *Let $Z \leq Z(\mathcal{F})$. Then $\mathcal{F}$ is constrained if and only if $\mathcal{F}/Z$ is constrained. Moreover, if G is a model for $\mathcal{F}$, then $Z \trianglelefteq G$ and G/Z is a model for $\mathcal{F}/Z$.*

Proof. Suppose first that $\mathcal{F}$ is constrained and that G is a model for $\mathcal{F}$. Note that, by Theorem 2.1(a), a model G always exists if $\mathcal{F}$ is constrained. By Theorem 2.1(b), Z is normal in G. So every $g \in G$ induces an $\mathcal{F}$-automorphism of Z which then has to be the identity, as $Z \leq Z(\mathcal{F})$. Hence, $Z \leq Z(G)$. Hence, G/Z has characteristic p by Lemma 2.3. By [6, Example II.5.6], $\mathcal{F}/Z = \mathcal{F}_{S/Z}(G/Z)$ and so G/Z is a model for $\mathcal{F}/Z$. Hence, by Theorem 2.1(a), $\mathcal{F}/Z$ is constrained. Assume now that $\mathcal{F}/Z$ is constrained and let $Z \leq Q \leq S$ with $Q/Z = O_p(\mathcal{F}/Z)$. Then $C_S(Q) \leq Q$ as $C_{S/Z}(Q/Z) \leq Q/Z$. So it is sufficient to show that Q is normal in $\mathcal{F}$. Observe that Q is strongly closed in $\mathcal{F}$, since Q/Z is strongly closed in $\mathcal{F}/Z$ and every morphism in $\mathcal{F}$ induces a morphism in $\mathcal{F}/Z$. By [6, Proposition I.4.5], a subgroup of a fusion system is normal if and only if it is strongly closed and contained in every centric radical subgroup. So Q/Z is contained in every element of $(\mathcal{F}/Z)^{rc}$ and it is sufficient to show that Q is contained in every element of $\mathcal{F}^{rc}$. As shown in [13, Proposition 3.1], we have $R/Z \in (\mathcal{F}/Z)^{rc}$ for every $R \in \mathcal{F}^{rc}$. So Q is contained in every element of $\mathcal{F}^{rc}$ as required. $\square$

We now turn attention to subsystems of $\mathcal{F}$, in particular to p-local subsystems and (weakly) normal subsystems.

Lemma 2.6. *Let $\mathcal{F}$ be constrained and $P \in \mathcal{F}^f$. Then $N_\mathcal{F}(P)$ and $C_\mathcal{F}(P)$ are constrained. Moreover, if G is a model for $\mathcal{F}$, then $N_G(P)$ is a model for $N_\mathcal{F}(P)$ and $C_G(P)$ is a model for $C_\mathcal{F}(P)$.*

Proof. Let $\mathcal{F}$ be a constrained fusion system on a finite p-group S and G a model for $\mathcal{F}$. Note that G always exists by Theorem 2.1(a). By [6, Proposition I.5.4], $N_S(P) \in \mathrm{Syl}_p(N_G(P))$, $C_S(P) \in \mathrm{Syl}_p(C_G(P))$, $N_\mathcal{F}(P) = \mathcal{F}_{N_S(P)}(N_G(P))$ and $C_\mathcal{F}(P) = \mathcal{F}_{C_S(P)}(C_G(P))$. By Lemma 2.2, $N_G(P)$ and $C_G(P)$ have characteristic p, so $N_G(P)$ is a model for $N_\mathcal{F}(P)$ and $C_G(P)$ is a model for $C_\mathcal{F}(P)$. In particular, by Theorem 2.1(a), $N_\mathcal{F}(P)$ and $C_\mathcal{F}(P)$ are constrained. $\square$

We continue with a general lemma needed afterwards to prove results about constrained fusion systems. It could be obtained as a consequence of [3, (7.4)] and the fact that for any $P \in \mathcal{F}$, $P \trianglelefteq \mathcal{F}$ if and only if $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$. We give however an elementary direct proof.

Lemma 2.7. *Let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$. Then $O_p(\mathcal{E})$ is normal in $\mathcal{F}$.*

Proof. Let $T = \mathcal{E} \cap S$. As $\mathcal{E}$ is normal in $\mathcal{F}$, every element of $\mathrm{Aut}_\mathcal{F}(T)$ induces an automorphism of $\mathcal{E}$. Thus $O_p(\mathcal{E})$ is $\mathrm{Aut}_\mathcal{F}(T)$-invariant. Since $O_p(\mathcal{E})$ is normal and thus strongly closed in $\mathcal{E}$, it follows now from the Frattini condition as stated in [6, Definition I.6.1] that $O_p(\mathcal{E})$ is strongly closed in $\mathcal{F}$. Hence, by [6, Theorem I.4.5], it is sufficient to prove that $O_p(\mathcal{E})$ is contained in any element of $\mathcal{F}^{rc}$. Let $R \in \mathcal{F}^{rc}$ and set $R_0 := R \cap T$. Recall that T is strongly closed and so R_0 is $\mathrm{Aut}_\mathcal{F}(R)$ invariant. As $O_p(\mathcal{E})$ is normal in $\mathcal{E}$, $\mathrm{Aut}_{O_p(\mathcal{E})}(R_0) \trianglelefteq \mathrm{Aut}_\mathcal{E}(R_0)$. Thus, $\mathrm{Aut}_{O_p(\mathcal{E})}(R_0) \leq O_p(\mathrm{Aut}_\mathcal{E}(R_0)) \leq O_p(\mathrm{Aut}_\mathcal{F}(R_0))$ since $\mathrm{Aut}_\mathcal{E}(R_0) \trianglelefteq \mathrm{Aut}_\mathcal{F}(R_0)$. It follows that the restriction of every element of $X := \langle \mathrm{Aut}_{O_p(\mathcal{E})}(R)^{\mathrm{Aut}_\mathcal{F}(R)} \rangle$ to R_0 lies in $O_p(\mathrm{Aut}_\mathcal{F}(R_0))$. Hence, $[R_0, O^p(X)] = 1$. Since $[R, N_{O_p(\mathcal{E})}(R)] \leq [R, N_T(R)] \leq T \cap R = R_0$, we have $[R, X] \leq R_0$. Thus, $O^p(X) = 1$ meaning that X is a normal p-subgroup of $\mathrm{Aut}_\mathcal{F}(R)$. Consequently, as R is centric radical, $\mathrm{Aut}_{O_p(\mathcal{E})}(R) \leq X \leq \mathrm{Inn}(R)$ and $O_p(\mathcal{E}) \leq R$. $\square$

Lemma 2.8. *Let $Q \in \mathcal{F}^f$. Then $N_\mathcal{F}(Q)$ is constrained if and only if $C_\mathcal{F}(Q)$ is constrained.*

Proof. If $N_\mathcal{F}(Q)$ is constrained, then it follows from Lemma 2.6 applied to $N_\mathcal{F}(Q)$ in place of $\mathcal{F}$ that $C_\mathcal{F}(Q)$ is constrained. Assume now $C_\mathcal{F}(Q)$ is constrained. By [1, 1.25], $C_\mathcal{F}(Q)$ is weakly normal in $N_{\mathcal{F}}(Q)$). It follows now from Lemma 2.7 that $R := QO_p(C_{\mathcal{F}}(Q)) \trianglelefteq N_{\mathcal{F}}(Q)$. Moreover, $C_{N_S(Q)}(R) = C_{C_S(Q)}(O_p(C_{\mathcal{F}}(Q))) \leq O_p(C_{\mathcal{F}}(Q)) \leq R$ as $C_{\mathcal{F}}(Q)$ is constrained. Thus, $N_{\mathcal{F}}(Q)$ is constrained. □

The reader is referred to [6, Section I.7] for definitions and properties of subsystems of index prime to p and of subsystems of p-power index.

Lemma 2.9. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ of index prime to p. Then $\mathcal{E}$ is constrained if and only if $\mathcal{F}$ is constrained.*

Proof. Clearly, $O_p(\mathcal{F})$ is normal in $\mathcal{E}$, so $O_p(\mathcal{E}) = O_p(\mathcal{F})$ by Lemma 2.7. As $\mathcal{E} \cap S = S$, it follows that $\mathcal{E}$ is a constrained if and only if $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$, which is the case if and only if $\mathcal{F}$ is constrained. □

Lemma 2.10. *Let $\mathcal{F}$ be constrained and let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$. Then $\mathcal{E}$ is constrained.*

Proof. By Lemma 2.9, $\mathcal{E}$ is constrained if and only if $O^{p'}(\mathcal{E})$ is constrained. By a theorem of Craven [6, Theorem I.7.8], $O^{p'}(\mathcal{E})$ is normal in $\mathcal{F}$. So replacing $\mathcal{E}$ by $O^{p'}(\mathcal{E})$, we may assume that $\mathcal{E}$ is normal in $\mathcal{F}$. Let G be a model for $\mathcal{F}$, which exist by Theorem 2.1(a). By [6, Lemma II.7.4], there exists a normal subgroup N of G such that $T := \mathcal{E} \cap S = N \cap S \in \mathrm{Syl}_p(N)$ and $\mathcal{E} = \mathcal{F}_T(N)$. By Lemma 2.2(b), N is of characteristic p and thus $\mathcal{E}$ is constrained by Theorem 2.1(a). □

The following lemma is a version of [15, Lemma 1.3] for fusion systems, except that we do not require the subsystem $\mathcal{E}$ to be normal in $\mathcal{F}$. A different proof could be given using the theory of components of fusion systems as developed in [2], but we prefer to keep the proof as elementary as possible.

Lemma 2.11. *Let $\mathcal{E}$ be a subsystem of $\mathcal{F}$ of p-power index. Then $\mathcal{E}$ is constrained if and only if $\mathcal{F}$ is constrained.*

Proof. Let $T = \mathcal{E} \cap S$. Let $T = T_0 \trianglelefteq T_1 \trianglelefteq \dots T_n = S$ be a chain of subgroups such that $|T_i/T_{i-1}| = p$ for $i = 1, \dots, n$. By [6, Theorem I.7.4], there is a unique subsystem $\mathcal{F}_{T_i} = \langle \mathrm{Inn}(T_i), \mathrm{Aut}_{\mathcal{F}}(P) \colon P \leq T_i \rangle$ of $\mathcal{F}$ of p-power index on T_i for every $i = 1, \dots, n$. In particular, $\mathcal{F}_T = \mathcal{F}_{T_0} = \mathcal{E}$. Again by [6, Theorem I.7.4], $\mathcal{F}_{T_{i-1}}$ is a normal subsystem of $\mathcal{F}_{T_i}$ of p-power index for $i = 1, \dots, n$. Hence, we can reduce to the case that $|S : T| = p$ and $\mathcal{E}$ is normal in $\mathcal{F}$. By Lemma 2.7, $Q := O_p(\mathcal{E})$ is normal in $\mathcal{F}$. It is sufficient to show that $P := QC_S(Q)$ is normal in $\mathcal{F}$. As $\mathcal{E}$ is constrained, $C_T(Q) \leq Q$ and thus $|P : Q| \leq |S : T| = p$. As Q is normal in $\mathcal{F}$, P is weakly closed in $\mathcal{F}$. We prove now that P is strongly closed. Let $X \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(X, S)$. If $X \leq Q$ then $X\varphi \leq Q \leq P$. If $X \not\leq Q$ then $P = QX$ as $|P : Q| \leq p$. Since $Q \trianglelefteq \mathcal{F}$, φ extends in this case to an element $\mathrm{Hom}_{\mathcal{F}}(P, S)$. As P is weakly closed in $\mathcal{F}$, it follows $X\varphi \leq P$. So P is strongly closed. By [6, Proposition I.4.6], there exists a series $1 = P_0 \leq P_1 \leq \dots P_n = Q$ of subgroups strongly closed in $\mathcal{F}$ such that $[P_i, Q] \leq P_{i-1}$ for $i = 1, \dots, n$. Since $P = QC_S(P) \leq QC_S(P_i)$, it follows $[P_i, P] \leq P_{i-1}$ for $i = 1, \dots, n$. As $|P : Q| \leq p$, we have $[P, P] \leq Q$. Hence, $P \trianglelefteq \mathcal{F}$ by [6, Proposition I.4.6]. As $C_S(P) \leq C_S(Q) \leq P$, it follows that $\mathcal{F}$ is constrained. □

3. Properties of subcentric subgroups

Lemma 3.1. *For any $Q \in \mathcal{F}$, the following conditions are equivalent:*

- (a1) *The subgroup Q is subcentric in $\mathcal{F}$.*
- (a2) *For some fully normalized $\mathcal{F}$-conjugate P of Q, $O_p(N_{\mathcal{F}}(P))$ is centric in $\mathcal{F}$.*
- (b1) *For any fully normalized $\mathcal{F}$-conjugate P of Q, $N_{\mathcal{F}}(P)$ is constrained.*
- (b2) *For some fully normalized $\mathcal{F}$-conjugate P of Q, $N_{\mathcal{F}}(P)$ is constrained.*
- (c1) *For any fully centralized $\mathcal{F}$-conjugate P of $\mathcal{F}$, $C_{\mathcal{F}}(P)$ is constrained.*
- (c2) *For some fully centralized $\mathcal{F}$-conjugate P of $\mathcal{F}$, $C_{\mathcal{F}}(P)$ is constrained.*

Proof. If $P, P^* \in Q^{\mathcal{F}}$ are both fully normalized, then it follows from [6, Lemma I.2.6(c)] that there exists an isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(P^*))$ such that $P\varphi = P^*$. It is straightforward to check that any such φ induces an isomorphism from $N_{\mathcal{F}}(P)$ to $N_{\mathcal{F}}(P^*)$ and thus $N_{\mathcal{F}}(P)$ is constrained if and only if $N_{\mathcal{F}}(P^*)$ is constrained. Moreover, $O_p(N_{\mathcal{F}}(P))\varphi = O_p(N_{\mathcal{F}}(P^*))$. Thus, conditions (b1) and (b2) are equivalent, and conditions (a1) and (a2) are equivalent. Similarly, if $P, P^* \in Q^{\mathcal{F}}$ are both fully centralized in $\mathcal{F}$, then by the extension axiom, there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(C_S(P)P, C_S(P^*)P^*)$ with $P\varphi = P^*$ and $\varphi|_{C_S(P)}$ induces and isomorphism from $C_{\mathcal{F}}(P)$ to $C_{\mathcal{F}}(P^*)$. This proves that conditions (c1) and (c2) are equivalent. Let now $P \in Q^{\mathcal{F}}$ be fully normalized. By Lemma 2.8, $N_{\mathcal{F}}(P)$ is constrained if and only if $C_{\mathcal{F}}(P)$ is constrained. Since every fully normalized subgroup if fully centralized, this shows that (b2) implies (c2) and that (c1) implies (b1). Set now $R := O_p(N_{\mathcal{F}}(P))$. If Q is subcentric, then $C_{N_S(P)}(R) = C_S(R) \leq R$ and so $N_{\mathcal{F}}(P)$ is constrained. Hence, (a1) implies (b1). Assume now $N_{\mathcal{F}}(P)$ is constrained. By [6, Lemma I.2.6(c)], there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(R), S)$ such that $R\varphi \in \mathcal{F}^f$. As $N_S(P) \leq N_S(R)$ and P is fully normalized, it follows $N_S(P)\varphi = N_S(P\varphi)$ and $P\varphi \in \mathcal{F}^f$. Again, $\varphi|_{N_S(P)}$ induces an isomorphism from $N_{\mathcal{F}}(P)$ to $N_{\mathcal{F}}(P\varphi)$ and thus $R\varphi = O_p(N_{\mathcal{F}}(P\varphi))$ and $N_{\mathcal{F}}(P\varphi)$ is constrained. Hence, $C_S(R\varphi) = C_{N_S(P\varphi)}(R\varphi) \leq R\varphi$. So $R\varphi$ and thus R is centric as $R\varphi$ is fully normalized. Hence, (b2) implies (a2). □

Proposition 3.2. *The set $\mathcal{F}^s$ of subcentric subgroups of $\mathcal{F}$ is closed under taking $\mathcal{F}$-conjugates and overgroups.*

Proof. Note first that the set of subcentric subgroups is by definition closed under $\mathcal{F}$-conjugation. Let $Q \in \mathcal{F}^s$ and R an overgroup of Q. We need to show that R is subcentric. By induction on the length of a subnormal series of Q in R, we reduce to the case that $Q \trianglelefteq R$. Since every $\mathcal{F}$-conjugate of Q is subcentric, and any $\mathcal{F}$-conjugate of R contains an $\mathcal{F}$-conjugate of Q, we can and will furthermore assume from now on that $R \in \mathcal{F}^f$. Replacing Q be a suitable conjugate of Q in $N_{\mathcal{F}}(R)$ we will also assume that $Q \in N_{\mathcal{F}}(R)^f$. By [6, I.2.6(c)], there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), S)$ such that $Q\alpha \in \mathcal{F}^f$. Then by [2, (2.2)(1),(2)], $(N_S(Q) \cap N_S(R))\alpha = N_S(Q\alpha) \cap N_S(R\alpha)$, $R\alpha \in N_{\mathcal{F}}(Q\alpha)^f$, and α induces an isomorphism from $\mathcal{N}_1 := N_{N_{\mathcal{F}}(R)}(Q)$ to $\mathcal{N}_2 := N_{N_{\mathcal{F}}(Q\alpha)}(R\alpha)$. As Q is subcentric and $Q\alpha \in \mathcal{F}^f$, $N_{\mathcal{F}}(Q\alpha)$ is constrained. Therefore, by Lemma 2.6 applied with $N_{\mathcal{F}}(Q\alpha)$ in place of $\mathcal{F}$, $C_{\mathcal{N}_2}(R\alpha) = C_{N_{\mathcal{F}}(Q\alpha)}(R\alpha)$ is constrained. So since α induces an isomorphism $\mathcal{N}_1 \to \mathcal{N}_2$, $C_{\mathcal{F}}(R) = C_{\mathcal{N}_1}(R)$ is constrained. Now by Lemma 3.1, R is subcentric. □

Lemma 3.3. *Let $R \trianglelefteq \mathcal{F}$ and $P \in \mathcal{F}$. Then $RP \in \mathcal{F}^s$ if and only if $P \in \mathcal{F}^s$.*

Proof. If $P \in \mathcal{F}^s$ then by Proposition 3.2, $PR \in \mathcal{F}^s$. From now on we assume that $RP \in \mathcal{F}^s$ and want to show that $P \in \mathcal{F}^s$. Since $\mathcal{F}^s$ is closed under $\mathcal{F}$-conjugation, we can assume without loss of generality that $RP \in \mathcal{F}^f$. As $RP \in \mathcal{F}^s$ this means that $N_{\mathcal{F}}(RP)$ is a constrained fusion system. If Q is a fully normalized $\mathcal{F}$-conjugate of P then an isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ extends to $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QR, S)$ with $R\hat{\varphi} = R$. Hence, as $QR = PR \in \mathcal{F}^f$, there exists by [6, I.2.6(c)] a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(QR), S)$ such that $(Q\alpha)R = (QR)\alpha = PR$. As $N_S(Q) \leq N_S(QR)$ and Q is fully normalized, it follows that $Q\alpha$ is fully normalized. So replacing P by $Q\alpha$, we may assume that P is fully normalized in $\mathcal{F}$. Then P is also fully normalized in $N_{\mathcal{F}}(PR)$ and thus $N_{N_{\mathcal{F}}(PR)}(P)$ is constrained by Lemma 2.6. One easily observes that $N_{\mathcal{F}}(P) = N_{N_{\mathcal{F}}(PR)}(P)$, as R is normal in $\mathcal{F}$. So $N_{\mathcal{F}}(P)$ is constrained and P is subcentric by Lemma 3.1. □

Lemma 3.4. *Let $Z \leq Z(\mathcal{F})$ and $P \leq S$. Then $P \in \mathcal{F}^s$ if and only if PZ/Z is subcentric in $\mathcal{F}/Z$.*

Proof. By Lemma 3.3, we may assume that $Z \leq P$. Moreover, we can assume $P \in \mathcal{F}^f$. Since $Z \leq P$, we have $Z \leq Q$ and $N_{S/Z}(Q/Z) = N_S(Q)/Z$ for every $Q \in P^{\mathcal{F}}$. Hence, P/Z is fully normalized in $\mathcal{F}/Z$. Clearly, $N_{\mathcal{F}}(Q)/Z = N_{\mathcal{F}/Z}(Q/Z)$ and $Z \leq Z(N_{\mathcal{F}}(Q))$. Therefore, by Lemma 2.5, $N_{\mathcal{F}}(Q)$ is constrained if and only if $N_{\mathcal{F}/Z}(Q/Z)$ is constrained. The assertion follows now from Lemma 3.1. □

Lemma 3.5. *Let $\tilde{\mathcal{F}}$ be a saturated fusion system on a p-group $\tilde{S}$ and $\alpha : S \to \tilde{S}$ a group isomorphism which induces an isomorphism of fusion systems $\mathcal{F} \to \tilde{\mathcal{F}}$. Then $\tilde{\mathcal{F}}^s = \{P\alpha \colon P \in \mathcal{F}^s\}$.*

Proof. Note that $N_S(Q)\alpha = N_{\tilde{S}}(Q\alpha)$ for any $Q \leq S$. Moreover, for $P \leq S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, $P\psi\alpha = P\alpha(\alpha^{-1}\psi\alpha) \in (P\alpha)^{\tilde{\mathcal{F}}}$, since $\alpha^{-1}\psi\alpha$ is a morphism in $\tilde{\mathcal{F}}$ as α induces an isomorphism of fusion systems. Hence, $\{Q\alpha : Q \in P^{\mathcal{F}}\} = (P\alpha)^{\tilde{\mathcal{F}}}$ and $Q \in P^{\mathcal{F}}$ is fully $\mathcal{F}$-normalized if and only if $Q\alpha$ is fully $\tilde{\mathcal{F}}$-normalized. Let now $Q \in P^{\mathcal{F}}$ be fully $\mathcal{F}$-normalized. Then $\alpha|_{N_S(Q)} : N_S(Q) \to N_{\tilde{S}}(Q\alpha)$ induces an isomorphism from $N_{\mathcal{F}}(Q)$ to $N_{\tilde{\mathcal{F}}}(Q\alpha)$. In particular, $N_{\mathcal{F}}(Q)$ is constrained if and only if $N_{\tilde{\mathcal{F}}}(Q\alpha)$ is constrained. Hence, by Lemma 3.1 $P \in \mathcal{F}^s$ if and only if $P\alpha \in \tilde{\mathcal{F}}^s$. $\square$

Lemma 3.6. *Let $\mathcal{E}$ be weakly normal in $\mathcal{F}$, $P \in \mathcal{E}^s$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Then $P\varphi \in \mathcal{E}^s$.*

Proof. Let $T = \mathcal{E} \cap S$. Note that $P\varphi \leq T$ as T is strongly closed. By the Frattini condition [6, Definition I.6.1], there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,T)$ such that $\varphi = \varphi_0\alpha$. As φ_0 is a morphism in $\mathcal{E}$, $P\varphi_0 \in \mathcal{E}^s$. As $\mathcal{E}$ is normal in $\mathcal{F}$, α induces and automorphism of $\mathcal{E}$. Hence, by Lemma 3.5 applied with $\mathcal{E}$ in the role of $\mathcal{F}$ and $\tilde{\mathcal{F}}$, $P\varphi = (P\varphi_0)\alpha \in \mathcal{E}^s$. $\square$

Before we continue proving properties of subcentric subgroups we need two general lemmas.

Lemma 3.7. *Let $\mathcal{E}$ be an $\mathcal{F}$-invariant subsystem of $\mathcal{F}$ and $T = \mathcal{E} \cap S$. Let $P \in \mathcal{E}^f$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_T(P),S)$. Then $P\alpha \in \mathcal{E}^f$, $N_T(P)\alpha = N_T(P\alpha)$ and α induces an isomorphism from $N_{\mathcal{E}}(P)$ to $N_{\mathcal{E}}(P\alpha)$.*

Proof. By the Frattini condition [6, Definition I.6.1] there are $\alpha_0 \in \mathrm{Hom}_{\mathcal{E}}(N_T(P),T)$ and $\beta \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\alpha = \alpha_0\beta$. Clearly, $N_T(P)\alpha_0 \leq N_T(P\alpha_0)$ because T is strongly closed in $\mathcal{F}$. As $P \in \mathcal{E}^f$, it follows $N_T(P)\alpha_0 = N_T(P\alpha_0)$. Since β is an automorphism of T, $N_T(P\alpha_0)\beta = N_T(P\alpha_0\beta) = N_T(P\alpha)$. Hence, $N_T(P)\alpha = N_T(P)\alpha_0\beta = N_T(P\alpha_0)\beta = N_T(P\alpha)$. Since $\mathcal{E}$ is $\mathcal{F}$-invariant, it is now straightforward to check that α induces an isomorphism from $N_{\mathcal{E}}(P)$ to $N_{\mathcal{E}}(P\alpha)$. $\square$

Lemma 3.8. *Let $\mathcal{E}$ be an $\mathcal{F}$-invariant subsystem of $\mathcal{F}$, $T = \mathcal{E} \cap S$, and $P \leq T$. If $P \in \mathcal{F}^f$ then $P \in \mathcal{E}^f$.*

Proof. Suppose $P \in \mathcal{F}^f$ and choose a fully $\mathcal{E}$-normalized $\mathcal{E}$-conjugate Q of P. By [6, I.2.6(c)], there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q),S)$ such that $Q\alpha = P$. Applying Lemma 3.7 with Q in place of P yields then $|N_T(Q)| = |N_T(P)|$ and thus $P \in \mathcal{E}^f$. $\square$

Lemma 3.9. *Let $\mathcal{E}$ be a weakly normal subsystem of $\mathcal{F}$ on $T \leq S$. Then $P \in \mathcal{E}^s$ for any $P \in \mathcal{F}^s$ with $P \leq T$.*

Proof. By Lemma 3.6, we may replace P by any $\mathcal{F}$-conjugate of P and can thus assume that $P \in \mathcal{F}^f$. Then by Lemma 3.8, $P \in \mathcal{E}^f$. So $N_{\mathcal{F}}(P)$ and $N_{\mathcal{E}}(P)$ are saturated. It is now easy to see that $N_{\mathcal{E}}(P)$ is weakly normal in $N_{\mathcal{F}}(P)$. Since $P \in \mathcal{F}^s$, $N_{\mathcal{F}}(P)$ is constrained by Lemma 3.1. Hence, by Lemma 2.10, $N_{\mathcal{E}}(P)$ is constrained and $P \in \mathcal{E}^s$ again by Lemma 3.1. $\square$

Lemma 3.10. *Let $Q \in \mathcal{F}$ and $K \leq \mathrm{Aut}(Q)$. If Q is fully K-normalized and $N_{\mathcal{F}}^K(Q)$ is constrained then Q is subcentric.*

Proof. Since Q is fully K-normalized, Q is fully centralized by [6, Proposition I.5.2]. In particular, $C_{\mathcal{F}}(Q)$ is saturated. Now it is straightforward to check that $C_{\mathcal{F}}(Q)$ is weakly normal in $N_{\mathcal{F}}^K(Q)$. If $N_{\mathcal{F}}^K(Q)$ is constrained, it follows therefore from Lemma 2.10 that $C_{\mathcal{F}}(Q)$ is constrained and Q is subcentric by Lemma 3.1. $\square$

Lemma 3.11. *Let R be a subgroup of S normal in $\mathcal{F}$ and $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(R)$. Then $N_{\mathcal{F}}^K(R)^s = \{P \in \mathcal{F}^s \colon P \leq N_S^K(R)\}$. In particular, $C_{\mathcal{F}}(R)^s = \{P \in \mathcal{F}^s \colon P \leq C_S(R)\}$.*

Proof. It is straightforward to check that $N_{\mathcal{F}}^K(R)$ is weakly normal in $\mathcal{F}$, the proof is the same as the one of [1, Proposition 1.25(c)]. Hence, by Lemma 3.9, every $P \in \mathcal{F}^s$ with $P \leq N_S^K(R)$ is a member of $N_{\mathcal{F}}^K(R)^s$. Let now $P \in N_{\mathcal{F}}^K(R)^s$. We want to show that $P \in \mathcal{F}^s$. For that we may assume $P \in N_{\mathcal{F}}^K(R)^f$. Set

$$\tilde{K} := \{\alpha \in \mathrm{Aut}(RP)\colon \alpha|_R \in K,\ P\alpha = P\}.$$

Choose $\psi \in \mathrm{Hom}_{\mathcal{F}}(RP, S)$ such that $|N_S^{\tilde{K}^\psi}((RP)\psi)|$ is maximal. Then $(RP)\psi$ is fully $\tilde{K}^\psi$-normalized. Hence, by [6, I.5.2], there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S^{\tilde{K}}(PR)R, S)$ and $\chi \in \mathrm{Aut}_{\mathcal{F}}^{\tilde{K}^\psi}((PR)\psi)$ such that $\varphi|_{PR} = \psi\chi$. Note that $N_S(P) \cap N_S^K(R) = N_S^{\tilde{K}}(PR)$. By Lemma 3.7 applied with $N_{\mathcal{F}}^K(R)$ in place of $\mathcal{E}$, $N_{N_{\mathcal{F}}^K(R)}(P) \cong N_{N_{\mathcal{F}}^K(R)}(P\varphi)$. So as P is subcentric in $N_{\mathcal{F}}^K(R)$, it follows using Lemma 3.1 that $N_{N_{\mathcal{F}}^K(R)}(P\varphi)$ is constrained. As $\chi \in \mathrm{Aut}_{\mathcal{F}}^{\tilde{K}^\psi}((PR)\psi) \leq \tilde{K}^\psi$ and $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(R)$, we have

$$\tilde{K}^\psi = \tilde{K}^\varphi = \{\alpha \in \mathrm{Aut}(R(P\varphi))\colon \alpha|_R \in K,\ (P\varphi)\alpha = P\varphi\}.$$

Hence, $N_{\mathcal{F}}^{\tilde{K}^\psi}(R(P\varphi)) = N_{N_{\mathcal{F}}^K(R)}(P\varphi)$ is constrained. As $R(P\varphi)$ is fully $\tilde{K}^\psi$-normalized, it follows from Lemma 3.10 that $R(P\varphi) \in \mathcal{F}^s$. Now by Lemma 3.3, $P\varphi$ and thus P is subcentric in $\mathcal{F}$ proving the assertion. □

Lemma 3.12. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ of index prime to p. Then $\mathcal{E}^s = \mathcal{F}^s$.*

Proof. By Lemma 3.9, we only need to prove that $\mathcal{E}^s \subseteq \mathcal{F}^s$. By Lemma 3.6, it is sufficient to prove $\mathcal{E}^s \cap \mathcal{F}^f \subseteq \mathcal{F}^s$. Let $P \in \mathcal{E}^s \cap \mathcal{F}^f$. By Lemma 3.8, $P \in \mathcal{E}^f$. Thus $N_{\mathcal{F}}(P)$ and $N_{\mathcal{E}}(P)$ are saturated subsystems and one sees easily that $N_{\mathcal{E}}(P)$ is a normal subsystem of $N_{\mathcal{F}}(P)$. As they are both fusion systems on $N_S(P)$, it follows that $N_{\mathcal{E}}(P)$ is a normal subsystem of $N_{\mathcal{F}}(P)$ of index prime to p. As $P \in \mathcal{E}^s$, $N_{\mathcal{E}}(P)$ is constrained by Lemma 3.1. Hence, by Lemma 2.9, $N_{\mathcal{F}}(P)$ is constrained and $P \in \mathcal{F}^s$ again by Lemma 3.1. □

Lemma 3.13. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ of p-power index and $T = \mathcal{E} \cap S$. Then $\mathcal{E}^s = \{P \in \mathcal{F}^s\colon P \leq T\}$.*

Proof. By Lemma 3.9, it remains only to prove that $\mathcal{E}^s \subseteq \mathcal{F}^s$. By Lemma 3.6, it is sufficient to prove $\mathcal{E}^s \cap \mathcal{F}^f \subseteq \mathcal{F}^s$. Let $P \in \mathcal{E}^s \cap \mathcal{F}^f$. By Lemma 3.8, $P \in \mathcal{E}^f$. Hence, $N_{\mathcal{F}}(P)$ and $N_{\mathcal{E}}(P)$ are saturated. It follows directly from the definition of the hyperfocal subgroup that $\mathfrak{hyp}(N_{\mathcal{F}}(P)) \leq \mathfrak{hyp}(\mathcal{F}) \leq T$ and thus $\mathfrak{hyp}(N_{\mathcal{F}}(P)) \leq N_T(P)$. For any $R \leq N_T(P)$, a p'-element $\alpha \in \mathrm{Aut}_{N_{\mathcal{F}}(P)}(R)$ extends to a p'-element $\hat{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(PR)$ normalizing R. As $\mathcal{E}$ is a subsystem of $\mathcal{F}$ of p-power index, $\hat{\alpha} \in O^p(\mathrm{Aut}_{\mathcal{F}}(PR)) \leq \mathrm{Aut}_{\mathcal{E}}(PR)$. Hence, α extends to an element of $\mathrm{Aut}_{\mathcal{E}}(PR)$ normalizing R, which means $\alpha \in \mathrm{Aut}_{N_{\mathcal{E}}(P)}(R)$. This shows that $N_{\mathcal{E}}(P)$ is a subsystem of $N_{\mathcal{F}}(P)$ of p-power index. As $P \in \mathcal{E}^s$, $N_{\mathcal{E}}(P)$ is constrained by Lemma 3.1. Hence, by Lemma 2.11, it follows that $N_{\mathcal{F}}(P)$ is constrained and $P \in \mathcal{F}^s$ by Lemma 3.1. □

Lemma 3.14. *Let $Q \in \mathcal{F}^f$ and $P \in N_{\mathcal{F}}(Q)^s$. Then $PQ \in \mathcal{F}^s$.*

Proof. By Lemma 3.3, $PQ \in N_{\mathcal{F}}(Q)^s$. Thus, we can assume from now on $Q \leq P$ and need to show that $P \in \mathcal{F}^s$. Replacing P by a fully $N_{\mathcal{F}}(Q)$-normalized $N_{\mathcal{F}}(Q)$-conjugate, we can furthermore assume $P \in N_{\mathcal{F}}(Q)^f$. Then by Lemma 3.1, $C_{\mathcal{F}}(P) = C_{N_{\mathcal{F}}(Q)}(P)$ is constrained. So by the same lemma, it is sufficient to prove that P is fully centralized in $\mathcal{F}$. By [6, I.2.6(c)], there exists a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ such that $P\alpha \in \mathcal{F}^f$. In particular, $P\alpha$ is fully centralized. By [2, (2.2)(1),(2)], $(N_S(P) \cap N_S(Q))\alpha = N_S(P\alpha) \cap N_S(Q\alpha)$. Hence, as $C_S(P) \leq N_S(P) \cap N_S(Q)$ and $C_S(P\alpha) \leq N_S(P\alpha) \cap N_S(Q\alpha)$, it follows $C_S(P)\alpha = C_S(P\alpha)$. So P is fully centralized as $P\alpha$ is fully centralized. This completes the proof. □

Lemma 3.15. *Let $Q \in \mathcal{F}^f$ and $P \in \mathcal{F}^s$ with $P \leq N_S(Q)$. Then $P \in N_{\mathcal{F}}(Q)^s$.*

Proof. By Lemma 3.2, $PQ \in \mathcal{F}^s$. Moreover, by Lemma 3.3, $P \in N_{\mathcal{F}}(Q)^s$ if $PQ \in N_{\mathcal{F}}(Q)^s$. Hence, replacing P by PQ, we may assume $Q \leq P$. Moreover, replacing P by a fully $N_{\mathcal{F}}(Q)$-normalized $N_{\mathcal{F}}(Q)$-conjugate, we may assume $P \in N_{\mathcal{F}}(Q)^f$. By [6, I.2.6(c)], we can pick a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ such that $P\alpha \in \mathcal{F}^f$. By [2, (2.2)(1),(2)], $(N_S(P) \cap N_S(Q))\alpha = N_S(P\alpha) \cap N_S(Q\alpha)$, $Q\alpha \in N_{\mathcal{F}}(P\alpha)^f$ and $\alpha|_{N_S(P)\cap N_S(Q)}$ induces an isomorphism from $N_{N_{\mathcal{F}}(Q)}(P)$ to $N_{N_{\mathcal{F}}(P\alpha)}(Q\alpha)$. As $P\alpha \in \mathcal{F}^{sf}$, $N_{\mathcal{F}}(P\alpha)$ is constrained by Lemma 3.1. Hence, $N_{N_{\mathcal{F}}(P\alpha)}(Q\alpha)$ is constrained by Lemma 2.6. Thus, $N_{N_{\mathcal{F}}(Q)}(P)$ is constrained and $P \in N_{\mathcal{F}}(Q)^s$ by Lemma 3.1. $\square$

Lemma 3.16. *Let $Q \in \mathcal{F}$ and $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$ such that Q is fully K-normalized. Then $PQ \in \mathcal{F}^s$ for every $P \in N_{\mathcal{F}}^K(Q)^s$. Moreover, $\{P \in \mathcal{F}^s\colon P \leq N_S^K(Q)\} \subseteq N_{\mathcal{F}}^K(Q)^s$.*

Proof. By [6, I.2.6(c)], there exists a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), S)$ such that $Q\alpha \in \mathcal{F}^f$. Notice that for any $s \in N_S^K(Q)$ and any $x \in Q$, $(x\alpha)^{s\alpha} = (x^s)\alpha = (x\alpha)\alpha^{-1}c_s\alpha = (x\alpha)(c_s|_Q)^\alpha$. So as $c_x|_Q \in K$, $c_{s\alpha}|_{Q\alpha} = (c_x|_Q)^\alpha \in K^\alpha$. Hence, $N_S^K(Q)\alpha \leq N_S^{K^\alpha}(Q\alpha)$. As Q is fully K-normalized, it follows that $Q\alpha$ is fully K^α-normalized and $N_S^K(Q)\alpha = N_S^{K^\alpha}(Q\alpha)$. It is straightforward to check that α induces an isomorphism $N_{\mathcal{F}}^K(Q)$ to $N_{\mathcal{F}}^{K^\alpha}(Q\alpha)$ and thus, by Lemma 3.5, given $P \leq N_S^K(Q)$, we have $P\alpha \in N_{\mathcal{F}}^{K^\alpha}(Q\alpha)^s$ if and only if $P \in N_{\mathcal{F}}^K(Q)^s$. Moreover, $PQ \in \mathcal{F}^s$ if and only if $(P\alpha)(Q\alpha) = (PQ)\alpha \in \mathcal{F}^s$. Hence, as $\mathcal{F}^s$ is invariant under $\mathcal{F}$-conjugation, replacing Q by $Q\alpha$, we may assume that $Q \in \mathcal{F}^f$. Then $N_{\mathcal{F}}(Q)$ is saturated and as $N_{\mathcal{F}}^K(Q) = N_{N_{\mathcal{F}}(Q)}^K(Q)$, it follows from Lemma 3.11 that $N_{\mathcal{F}}^K(Q)^s = \{P \in N_{\mathcal{F}}(Q)^s\colon P \leq N_S^K(Q)\}$. So if $P \in N_{\mathcal{F}}^K(Q)^s$ then $P \in N_{\mathcal{F}}(Q)^s$ and $PQ \in \mathcal{F}^s$ by Lemma 3.14. If $P \leq N_S^K(Q)$ with $P \in \mathcal{F}^s$, then $P \in N_{\mathcal{F}}(Q)^s$ by Lemma 3.15, and then $P \in N_{\mathcal{F}}^K(Q)^s$ by the property stated before. This proves the assertion. $\square$

Lemma 3.17. *Let $Q \in \mathcal{F}^{fs}$ such that $Q = O_p(N_{\mathcal{F}}(Q))$. Then $Q \in \mathcal{F}^{frc}$.*

Proof. As $Q \in \mathcal{F}^{fs}$, $N_{\mathcal{F}}(Q)$ is constrained and so $Q = O_p(N_{\mathcal{F}}(Q)) \in \mathcal{F}^c$. By Theorem 2.1, there exists a model G for $N_{\mathcal{F}}(Q)$ and $O_p(G) = O_p(N_{\mathcal{F}}(Q)) = Q$. Note $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{N_{\mathcal{F}}(Q)}(Q) \cong G/C_G(Q) = G/Z(Q)$. Then $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) \cong O_p(G/Z(Q)) = Q/Z(Q) \cong \mathrm{Inn}(Q)$ and so Q is radical. $\square$

Proof of Proposition 1. This follows from Lemma 3.3, Lemma 3.4 and Lemma 3.10. Compare also Lemma 3.14 and Lemma 3.15. $\square$

Proof of Proposition 2. The proposition follows from Lemma 3.6, Lemma 3.9, Lemma 3.12, Lemma 3.13 and Lemma 3.11. $\square$

4. The proof of Theorem B

Throughout this section, let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ and $T = \mathcal{E} \cap S$. The subgroup $C_S(\mathcal{E})$ was introduced in [3, Chapter 6]. We will use throughout the following characterization of $C_S(\mathcal{E})$: The subgroup $C_S(\mathcal{E})$ is the largest subgroup X of $C_S(T)$ such that $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$.

Lemma 4.1. *The subsystem $\mathcal{E}$ is also a normal subsystem of $N_{\mathcal{F}}(C_S(\mathcal{E}))$.*

Proof. Recall $\mathcal{E} \subseteq C_{\mathcal{F}}(\mathcal{E}) \subseteq N_{\mathcal{F}}(\mathcal{E})$. It is straightforward to see that $\mathcal{E}$ is weakly normal in $N_{\mathcal{F}}(C_S(\mathcal{E}))$. Set $T = \mathcal{E} \cap S$. As $\mathcal{E} \trianglelefteq \mathcal{F}$, every element $\varphi \in \mathrm{Aut}_{\mathcal{E}}(T)$ extends to $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[C_S(T), \overline{\varphi}] \leq T$. Since $C_S(\mathcal{E}) \leq C_S(T)$ and $C_S(\mathcal{E})$ is strongly closed in $\mathcal{F}$ by [3, (6.7)(2)], we have $C_S(\mathcal{E})\overline{\varphi} = C_S(\mathcal{E})$. Hence, $\overline{\varphi} \in \mathrm{Aut}_{N_{\mathcal{F}}(C_S(\mathcal{E}))}(TC_S(T))$. This shows the assertion. $\square$

Lemma 4.2. *Let $Q \in \mathcal{F}^f$ such that $Q = (Q \cap T)C_S(\mathcal{E})$. Then $Q \cap T \in \mathcal{E}^f$.*

Proof. Set $P := Q \cap T$. Let $\alpha_0 \in \mathrm{Hom}_{\mathcal{E}}(P, T)$ such that $P\alpha_0$ is fully $\mathcal{E}$-normalized. By the characterization of $C_S(\mathcal{E})$, α_0 extends to $\alpha \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ such that α fixes every element of $C_S(\mathcal{E})$. In particular, $C_S(\mathcal{E})\alpha = C_S(\mathcal{E})$ and $Q\alpha = (P\alpha)C_S(\mathcal{E})$. Moreover, $P\alpha = (Q\cap T)\alpha \leq Q\alpha \cap T$ and $(Q\alpha \cap T)(\alpha|_Q)^{-1} \leq Q \cap T$, so $Q\alpha \cap T = (Q \cap T)\alpha = P\alpha$. Hence, $Q\alpha \cap T = P\alpha$.

As $Q = PC_S(\mathcal{E})$ and $C_S(\mathcal{E}) \leq C_S(T)$, we have $N_T(P) \leq N_T(Q)$. As $P = Q \cap T$, $N_T(Q) \leq N_T(P)$. Hence, $N_T(P) = N_T(Q)$. Similarly, $N_T(Q\alpha) = N_T(P\alpha)$. By [6, Lemma I.2.6(c)], there exists $\beta \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q\alpha), S)$ such that $Q\alpha\beta = Q$. For such β, we have $N_T(Q\alpha)\beta \leq N_T(Q)$ and thus $|N_T(P\alpha_0)| = |N_T(P\alpha)| = |N_T(Q\alpha)| \leq |N_T(Q)| = |N_T(P)|$. Hence, $P \in \mathcal{E}^f$ as $P\alpha_0 \in \mathcal{E}^f$. $\square$

Lemma 4.3. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ on T. Let $Q \in \mathcal{F}^f$ such that $Q = (Q \cap T)C_S(\mathcal{E})$. Then $N_{\mathcal{E}}(Q \cap T)$ is weakly normal in $N_{\mathcal{F}}(Q)$.*

Proof. Set $P := Q \cap T$. By Lemma 4.2, $P \in \mathcal{E}^f$. By assumption $Q \in \mathcal{F}^f$, so both $N_{\mathcal{E}}(P)$ and $N_{\mathcal{F}}(Q)$ are saturated. Every morphism $\alpha \in \mathrm{Hom}_{N_{\mathcal{E}}(P)}(A, B)$ $(A, B \leq N_T(P))$ extends to an element of $\mathrm{Hom}_{\mathcal{E}}(AP, BP)$ normalizing P, which then by definition of $C_S(\mathcal{E})$ extends to $\overline{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(APC_S(\mathcal{E}), BPC_S(\mathcal{E}))$ centralizing $C_S(\mathcal{E})$. As $Q = PC_S(\mathcal{E})$, it follows $Q\overline{\alpha} = Q$ and so α is a morphism in $N_{\mathcal{F}}(Q)$. This shows that $N_{\mathcal{E}}(P)$ is a subsystem of $N_{\mathcal{F}}(Q)$. Hence, it remains to prove only that $N_{\mathcal{E}}(P)$ is invariant in $N_{\mathcal{F}}(P)$. We prove the strong invariance condition as stated in [6, Proposition 6.4(d)]. Let $A \leq B \leq N_T(P)$, $\varphi \in \mathrm{Hom}_{N_{\mathcal{E}}(P)}(A, B)$ and $\psi \in \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(B, N_T(P))$. We need to prove that $(\psi|_A)^{-1}\varphi\psi \in \mathrm{Hom}_{N_{\mathcal{E}}(P)}(A\psi, B\psi)$. By definition of the normalizer subsystems, φ extends to $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{E}}(AP, BP)$ and ψ extends to $\overline{\psi} \in \mathrm{Hom}_{\mathcal{F}}(BQ, N_T(P)Q)$ with $Q\overline{\psi} = Q$. As T is strongly closed and, by assumption, $P = Q \cap T$, we have $P\overline{\psi} = P$ and thus $\hat{\psi} := \overline{\psi}|_{BP} \in \mathrm{Hom}_{\mathcal{F}}(BP, N_T(P))$. Since the strong invariance condition holds for $(\mathcal{E}, \mathcal{F})$, we have that $(\hat{\psi}|_{AP})^{-1}\overline{\varphi}\hat{\psi}$ is a morphism in $\mathcal{E}$. Moreover, $P(\hat{\psi}|_{AP})^{-1}\overline{\varphi}\hat{\psi} = P$ and $(\hat{\psi}|_{AP})^{-1}\overline{\varphi}\hat{\psi}$ extends $(\psi|_A)^{-1}\varphi\psi$, so $(\psi|_A)^{-1}\varphi\psi$ is a morphism in $N_{\mathcal{E}}(P)$ as required. $\square$

Lemma 4.4. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ and $\mathcal{C}$ a component of $\mathcal{F}$. Then $\mathcal{C} \subseteq \mathcal{E}$ or $\mathcal{C} \cap S \leq C_S(\mathcal{E})$.*

Proof. By the construction of central products in [3, Chapter 2], if $\mathcal{F}$ is the central product of two subsystems $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\mathcal{F}_1 \cap S \leq C_S(\mathcal{F}_2)$. Hence, the assertion follows from [3, (9.13)]. $\square$

Proof of Theorem B. Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ on $T \leq S$. Let $P \in \mathcal{E}^s$ and set $Q := PC_S(\mathcal{E})$.

Step 1: We show that it is enough to prove the assertion in the case that $Q \in \mathcal{F}^f$ and $P = Q \cap T$. For that take $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ such that $Q\varphi$ is fully $\mathcal{F}$-normalized. Then by Lemma 3.6, $P\varphi \in \mathcal{E}^s$. Moreover, as $C_S(\mathcal{E})$ is strongly closed by [3, (6.7)(2)], $C_S(\mathcal{E})\varphi = C_S(\mathcal{E})$ and thus $Q\varphi = (P\varphi)C_S(\mathcal{E})$. So replacing (P, Q) by $(P\varphi, Q\varphi)$, we may assume that Q is fully $\mathcal{F}$-normalized. Note also that $P \leq Q \cap T$, so by Proposition 3.2, $Q \cap T$ is subcentric in $\mathcal{E}$. Moreover, $Q = (Q \cap T)C_S(\mathcal{E})$. Hence, replacing P with $Q \cap T$, we may assume that $P = Q \cap T$.

From now on we assume that $Q \in \mathcal{F}^f$ and $P = Q \cap T$.

Step 2: We show that $E(N_{\mathcal{F}}(C_S(\mathcal{E}))) \subseteq \mathcal{E}$. Let $\mathcal{C}$ be a component of $N_{\mathcal{F}}(C_S(\mathcal{E}))$. Then by [3, (9.6)], $\mathcal{C} \subseteq C_{\mathcal{F}}(C_S(\mathcal{E}))$. By Lemma 4.1, $\mathcal{E}$ is normal in $N_{\mathcal{F}}(C_S(\mathcal{E}))$. Therefore, by Lemma 4.4, $\mathcal{C} \subseteq \mathcal{E}$ or $C := \mathcal{C} \cap S \leq C_S(\mathcal{E})$. Assume $C \leq C_S(\mathcal{E})$. As $\mathcal{C} \subseteq C_{\mathcal{F}}(C_S(\mathcal{E}))$ this means that C is abelian, contradicting [3, (9.1)(2)] and the fact that $\mathcal{C}$ is quasisimple. This proves $\mathcal{C} \subseteq \mathcal{E}$ and, as $\mathcal{C}$ was arbitrary, $E(N_{\mathcal{F}}(C_S(\mathcal{E}))) \subseteq \mathcal{E}$.

Step 3: We complete the proof by showing that Q is subcentric in $\mathcal{F}$. Suppose this is not true. As we assume that Q is fully normalized, this means that $N_{\mathcal{F}}(Q)$ is not constrained. Thus, by [3, (14.2)], $E(N_{\mathcal{F}}(Q)) \neq 1$. As $C_S(\mathcal{E})$ is strongly closed and contained in Q, $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}}(C_S(\mathcal{E}))}(Q)$. Since Q is fully normalized in $\mathcal{F}$ and $C_S(\mathcal{E}) \trianglelefteq S$, $N_{\mathcal{F}}(C_S(\mathcal{E}))$ is saturated and Q is fully normalized in $N_{\mathcal{F}}(C_S(\mathcal{E}))$. Thus, by Aschbacher's version of the L-balance Theorem for fusion systems [3, Theorem 7], $E(N_{\mathcal{F}}(Q)) = E(N_{N_{\mathcal{F}}(C_S(\mathcal{E}))}(Q)) \subseteq E(N_{\mathcal{F}}(C_S(\mathcal{E})))$. So by Step 2, $E(N_{\mathcal{F}}(Q)) \subseteq \mathcal{E}$. Let $\mathcal{D}$ be a component of $N_{\mathcal{F}}(Q)$ and $D = S \cap \mathcal{D}$. By Step 1 and Lemma 4.2, P is fully $\mathcal{E}$-normalized and, by Lemma 4.3, $N_{\mathcal{E}}(P)$ is weakly normal in $N_{\mathcal{F}}(Q)$. It follows from Lemma 2.7 that $O_p(N_{\mathcal{E}}(P))$ is normal in $N_{\mathcal{F}}(Q)$. Thus, by [3, (9.6)], $\mathcal{D} \subseteq C_{N_{\mathcal{F}}(Q)}(O_p(N_{\mathcal{E}}(P)))$ and so $[D, O_p(N_{\mathcal{E}}(P))] = 1$. As $E(N_{\mathcal{F}}(Q)) \subseteq \mathcal{E}$, we have $D \leq T$. Hence, $D \leq C_T(O_p(N_{\mathcal{E}}(P))) = Z(O_p(N_{\mathcal{E}}(P)))$ as P is subcentric and fully normalized in $\mathcal{E}$. Thus, D is abelian, again contradicting [3, (9.1)(2)] and the fact that $\mathcal{D}$ is quasisimple. $\square$

5. Properties of linking localities

Lemma 5.1. *Let $(\mathcal{L}, \Delta, S)$ be a locality for $\mathcal{F}$. Then $N_\mathcal{F}(P) = \mathcal{F}_{N_S(P)}(N_\mathcal{L}(P))$ and $C_\mathcal{F}(P) = \mathcal{F}_{C_S(P)}(C_\mathcal{L}(P))$.*

Proof. Clearly $\mathcal{F}_{N_S(P)}(N_\mathcal{L}(P)) \subseteq N_\mathcal{F}(P)$. To show the converse inclusion let $\varphi \in \mathrm{Hom}_{N_\mathcal{F}(P)}(A, B)$. We may assume without loss of generality that $P \leq A \cap B$ so that $P\varphi = P$. As $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$, $\varphi = c_f$ for some $f \in \mathcal{L}$. Then $P \leq \mathbf{D}(f)$ and $P^f = P\varphi = P$, so $f \in N_\mathcal{L}(P)$ and φ is morphism in $\mathcal{F}_{N_S(P)}(N_\mathcal{L}(P))$. This proves $N_\mathcal{F}(P) = \mathcal{F}_{N_S(P)}(N_\mathcal{L}(P))$. Similarly, $C_\mathcal{F}(P) = \mathcal{F}_{C_S(P)}(C_\mathcal{L}(P))$. $\square$

Lemma 5.2. *Let $(\mathcal{L}, \Delta, S)$ be a linking locality for $\mathcal{F}$. If $P \in \Delta \cap \mathcal{F}^f$ then $N_\mathcal{L}(P)$ is a model for $N_\mathcal{F}(P)$. In particular, $\Delta \subseteq \mathcal{F}^s$.*

Proof. Clearly, $N_\mathcal{L}(P)$ is a subgroup of $\mathcal{L}$ and by [10, Proposition 2.18(c)], $N_S(P) \in \mathrm{Syl}_p(N_\mathcal{L}(P))$. Hence, it follows from Lemma 5.1 that $N_\mathcal{L}(P)$ is a model for $N_\mathcal{F}(P)$ for any $P \in \mathcal{F}^f$. In particular, by Theorem 2.1(a), $N_\mathcal{F}(P)$ is constrained for every $P \in \Delta \cap \mathcal{F}^f$. Hence, $\Delta \subseteq \mathcal{F}^s$ by Lemma 3.1. $\square$

If $(\mathcal{L}, \Delta, S)$ is a locality, define $P \in \Delta$ to be *$\mathcal{L}$-radical* if $O_p(N_\mathcal{L}(P)) = P$.

Lemma 5.3. *Let $(\mathcal{L}, \Delta, S)$ be a linking locality for $\mathcal{F}$ and $P \in \Delta$. Then P is $\mathcal{L}$-radical if and only if $P \in \mathcal{F}^{cr}$.*

Proof. It follows from [10, Lemma 2.7] that the set of $\mathcal{L}$-radical subgroups is closed under $\mathcal{F}$-conjugation. The set $\mathcal{F}^{cr}$ is closed under $\mathcal{F}$-conjugation as well. Hence, we may assume that $P \in \mathcal{F}^f$. Then by Lemma 5.2, $G := N_\mathcal{L}(P)$ is a model for $N_\mathcal{F}(P)$. Note $G/C_G(P) \cong \mathrm{Aut}_\mathcal{F}(P)$. Hence, if $C_G(P) = Z(P)$ then $O_p(\mathrm{Aut}_\mathcal{F}(P)) = \mathrm{Inn}(P)$ if and only if $P = O_p(G)$. If $P \in \mathcal{F}^{rc}$ then $P \in N_\mathcal{F}(P)^c$ and so by Theorem 2.1(b), $C_G(P) = Z(P)$. Hence, $P = O_p(G)$ and P is $\mathcal{L}$-radical by what we just stated. On the other hand, assuming that P is $\mathcal{L}$-radical, $C_G(P) = Z(P)$ as G has characteristic p. So again by what we stated before, $P \in \mathcal{F}^r$. Moreover, $C_S(P) = C_{N_S(P)}(P) \leq N_G(P) \leq P$. So $P \in \mathcal{F}^c$ as $P \in \mathcal{F}^f$. This proves the assertion. $\square$

Proof of Proposition 4. Clearly, $Q \trianglelefteq \mathcal{F}$ if $\mathcal{L} = N_\mathcal{L}(Q)$ and $Q \leq Z(\mathcal{F})$ if $\mathcal{L} = C_\mathcal{L}(Q)$. Moreover, if $Q \leq Z(\mathcal{F})$ and $\mathcal{L} = N_\mathcal{L}(Q)$ then clearly, $\mathcal{L} = C_\mathcal{L}(Q)$. Hence, it is sufficient to prove that $\mathcal{L} = N_\mathcal{L}(Q)$ if $Q \trianglelefteq \mathcal{F}$. Assume now $Q \trianglelefteq \mathcal{F}$ and $\mathcal{L} \neq N_\mathcal{L}(Q)$. Choose $f \in \mathcal{L} \backslash N_\mathcal{L}(Q)$ such that $|S_f|$ is maximal. Since $Q \trianglelefteq \mathcal{F}$ it follows $Q \not\leq S_f$. In particular, $S_f < S$ and thus $S_f^f < N_S(S_f^f)$. By [6, Lemma I.2.6(c)], there exists $h \in \mathcal{L}$ such that $N_S(S_f^f) \leq S_h$ and $S_f^{fh} \in \mathcal{F}^f$. Then $(f, h, h^{-1}) \in \mathbf{D}$ via S_f. By the maximality of $|S_f|$, $h \in N_\mathcal{L}(Q)$. So if $fh \in N_\mathcal{L}(Q)$ then $f = (fh)h^{-1} \in N_\mathcal{L}(Q)$ as $N_\mathcal{L}(Q)$ is a partial subgroup of $\mathcal{L}$. Hence, $fh \notin N_\mathcal{L}(Q)$ and by the maximality of $|S_f|$, $S_f = S_{fh}$. So replacing f by fh, we may assume that $S_f^f \in \mathcal{F}^f$.

Since $c_f : S_f \to S_f^f$ is a morphism in $\mathcal{F}$ and $Q \trianglelefteq \mathcal{F}$, there exists $g \in \mathcal{L}$ such that $S_f Q \leq S_g$, $c_g|_{S_f} = c_f$ and $Q^g = Q$. Then $(f^{-1}, g) \in \mathbf{D}$ via S_f^f and $f^{-1}g \in C_\mathcal{L}(S_f^f) \subseteq N_\mathcal{L}(S_f^f)$. Since $Q \not\leq S_f$, $S_f \notin \mathcal{F}^{cr}$ by [6, Proposition I.4.5]. So $S_f^f \notin \mathcal{F}^{cr}$ and thus, by Lemma 5.3, $S_f^f < R := O_p(N_\mathcal{L}(S_f^f))$. As $S_f^f \in \mathcal{F}^f$, it follows from [10, Proposition 2.18(c)] that $R \leq S$. So the maximality of $|S_f| = |S_f^f|$ yields $f^{-1}g \in N_\mathcal{L}(Q)$. As $(f^{-1}, g, g^{-1}) \in \mathbf{D}$ via S_f^f, it follows $f^{-1} = (f^{-1}g)g \in N_\mathcal{L}(Q)$. This yields a contradiction to $f \notin N_\mathcal{L}(Q)$. $\square$

Lemma 5.4. *Let $(\mathcal{L}, \Delta, S)$ be a locality for $\mathcal{F}$.*

(a) *If $P \in \Delta \cap \mathcal{F}^{fc}$, then $N_\mathcal{L}(P)$ is a model for $N_\mathcal{F}(P)$ if and only if $C_\mathcal{L}(Q) \leq Q$ for any $Q \in P^\mathcal{F}$.*

(b) *If $P \in \Delta \cap \mathcal{F}^{fq}$, then $C_{\mathcal{L}}(P) = C_S(P) \times O_{p'}(C_{\mathcal{L}}(P))$. Moreover, $N_{\mathcal{L}}(P)$ is a model for $N_{\mathcal{F}}(P)$ if and only if $C_{\mathcal{L}}(Q)$ is a p-group for any $Q \in P^{\mathcal{F}}$.*

Proof. Let $P \in \Delta \cap \mathcal{F}^f$. Then by Lemma 5.1 and [10, Proposition 2.18(c)], $G := N_{\mathcal{L}}(P)$ is a finite group with $N_S(P) \in \mathrm{Syl}_p(G)$, $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(G)$ and $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_G(P))$. Note that every $\mathcal{F}$-morphism between elements of Δ can be realized as a conjugation map by an element of $\mathcal{L}$. So by [10, Lemma 2.8(b)], $C_{\mathcal{L}}(P) \cong C_{\mathcal{L}}(Q)$ for any $Q \in P^{\mathcal{F}}$. Hence, $C_{\mathcal{L}}(P) \leq P$ if and only if $C_{\mathcal{L}}(Q) \leq Q$ for any $Q \in P^{\mathcal{F}}$, and $C_{\mathcal{L}}(P)$ is a p-subgroup if and only if $C_{\mathcal{L}}(Q)$ is a p-subgroup for any $Q \in P^{\mathcal{F}}$. If P is $\mathcal{F}$-centric then P is also centric in $N_{\mathcal{F}}(P)$. Hence, in this case by Theorem 2.1(b), G is a model for $N_{\mathcal{F}}(P)$ if and only if $C_G(P) \leq P$. If P is quasicentric then $C_{C_S(P)}(C_S(P)) = C_{\mathcal{F}}(P) = C_{C_S(P)}(C_G(P))$. So by Lemma 2.4, $C_{\mathcal{L}}(P) = C_G(P) = C_S(P) \times O_{p'}(C_{\mathcal{L}}(P))$ and G is a model for $N_{\mathcal{F}}(P)$ if and only if $C_{\mathcal{L}}(P) = C_G(P)$ is a p-group. This proves (b). $\square$

Proof of Remark 1. We use that, by [10, Proposition A.13, Lemma A.14], every transporter system $(\mathcal{T}, \epsilon, \rho)$ is isomorphic one realized by a locality $(\mathcal{L}, \Delta, S)$ with $\Delta = \mathrm{ob}(\mathcal{T})$ in the sense explained in [10, Definition A.2]. Assume $(\mathcal{T}, \epsilon, \rho)$ is of this form. It follows directly from the construction that $N_{\mathcal{L}}(P) \cong \mathrm{Aut}_{\mathcal{T}}(P)$ and $E(P) = \ker(\rho_{P,P} \colon \mathrm{Aut}_{\mathcal{T}}(P) \to \mathrm{Aut}_{\mathcal{F}}(P)) \cong C_{\mathcal{L}}(P)$ for every $P \in \Delta$. In particular, $N_{\mathcal{L}}(P)$ is of characteristic p if and only if $\mathrm{Aut}_{\mathcal{T}}(P)$ is of characteristic p, proving (a). Property (b) follows now from Lemma 5.2. Comparing the definitions of a transporter system in [17, Definition 3.1] and of a linking system in the sense of Oliver [16, Definition 3], one sees that the transporter system $(\mathcal{T}, \epsilon, \rho)$ is a linking system in the sense of Oliver if and only if $\mathcal{F}^{rc} \subseteq \mathrm{ob}(\mathcal{T})$ and $E(P)$ is a p-group for every $P \in \Delta = \mathrm{ob}(\mathcal{T})$. Moreover, if $\mathcal{T}$ is a linking system in the sense of Oliver, then $\mathcal{T}$ is a centric linking system if and only if $\Delta = \mathcal{F}^c$. Hence, it suffices to show the claims about $(\mathcal{L}, \Delta, S)$ stated in (c) and (d). These properties follow however from Lemma 5.4. $\square$

6. Construction of linking localities

Lemma 6.1. *Let Δ be a set of subcentric subgroups, which is closed under $\mathcal{F}$-conjugation and with respect overgroups. Suppose $(\mathcal{L}, \Delta, S)$ is a linking locality for $\mathcal{F}$. Let $T \in \mathcal{F}^f$ such that any proper overgroup of T is in Δ and $O_p(N_{\mathcal{F}}(T)) \in \Delta$. Then $N_{\mathcal{L}}(T)$ is a subgroup of $\mathcal{L}$ which is a model for $N_{\mathcal{F}}(T)$.*

Proof. As every proper overgroup of T is in Δ, $\Delta_T := \{N_P(T) \colon T \leq P \in \Delta\} = \{P \in \Delta \colon T \leq P \leq N_S(T)\} \subseteq \Delta$. By [10, Lemma 2.19(c)] and Lemma 5.1, $(N_{\mathcal{L}}(T), \Delta_T, N_S(T))$ is a locality for $N_{\mathcal{F}}(T)$. If $P \in \Delta_T$ then $N_{\mathcal{L}}(P)$ is a group of characteristic p, as $P \in \Delta$ and $(\mathcal{L}, \Delta, S)$ is a linking locality. In particular, $N_{N_{\mathcal{L}}(T)}(P) = N_{N_{\mathcal{L}}(P)}(T)$ is a group of characteristic p by Lemma 2.2(a). Hence, $(N_{\mathcal{L}}(T), \Delta_T, N_S(T))$ is a linking locality for $N_{\mathcal{F}}(T)$. Hence, by Proposition 4, we have $N_{\mathcal{L}}(T) = N_{N_{\mathcal{L}}(T)}(Q)$ for $Q = O_p(N_{\mathcal{F}}(T))$. As $Q \in \Delta$ by assumption, $Q \in \Delta_T$ and so $N_{\mathcal{L}}(T)$ is a linking locality for $N_{\mathcal{F}}(T)$ with a normal object. Thus, $N_{\mathcal{L}}(T)$ is a group of characteristic p and, using Lemma 5.2 with $N_{\mathcal{L}}(T)$ and Q in place of $\mathcal{L}$ and P, we conclude that $N_{\mathcal{L}}(T)$ is a model for $N_{\mathcal{F}}(T)$. $\square$

Theorem 6.2. *Let Δ and Δ^+ be collections of subgroups of S which are both closed under $\mathcal{F}$-conjugation and with respect to overgroups. Suppose that $\mathcal{F}^{cr} \subseteq \Delta \subseteq \Delta^+ \subseteq \mathcal{F}^s$, and let $(\mathcal{L}, \Delta, S)$ be a linking locality over $\mathcal{F}$.*

(a) *There exists a linking locality $(\mathcal{L}^+, \Delta^+, S)$ such that $\mathcal{L}$ is the restriction $\mathcal{L}^+|_{\Delta}$ of $\mathcal{L}^+$ to Δ and $\mathcal{F}_S(\mathcal{L}^+) = \mathcal{F}$. The inclusion of nerves $|\mathcal{T}(\mathcal{L}, \Delta)| \subseteq |\mathcal{T}(\mathcal{L}^+, \Delta^+)|$ is a homotopy equivalence.*

(b) *If $(\widetilde{\mathcal{L}}^+, \Delta^+, S)$ is another linking locality for $\mathcal{F}$ with object set Δ^+ and $\beta : \mathcal{L} \to \widetilde{\mathcal{L}}^+|_{\Delta}$ is a rigid isomorphism, then β extends to a rigid isomorphism $\mathcal{L}^+ \to \widetilde{\mathcal{L}}^+$. So in particular, $\mathcal{L}^+$ is unique up to an isomorphism that restricts to the identity on $\mathcal{L}$.*

(c) *If $\Delta^+ \backslash \Delta$ is a single $\mathcal{F}$-conjugacy class then $N_{\mathcal{L}}(R) = N_{\mathcal{L}^+}(R)$ for every $R \in \Delta^+ \backslash \Delta$ which is fully $\mathcal{F}$-normalized.*

Proof of Theorem 6.2. We may assume $\Delta \neq \Delta^+$. Choose $T \in \Delta^+ \backslash \Delta$ such that T is maximal with respect to inclusion. Since Δ^+ is closed under taking overgroups, it follows that every proper overgroup of T is in Δ. Therefore, as Δ is closed under $\mathcal{F}$-conjugation, every proper overgroup of an $\mathcal{F}$-conjugate of T is in Δ. Hence, $\Delta \cup T^{\mathcal{F}}$ is closed under taking overgroups. By construction, this set is closed under taking $\mathcal{F}$-conjugates. Furthermore, $\Delta \cup T^f \subseteq \Delta^+$, as Δ^+ is closed under taking $\mathcal{F}$-conjugates. Now by induction on $|\Delta^+ \backslash \Delta|$, we may assume $\Delta^+ = \Delta \cup T^{\mathcal{F}}$. Replacing T by a suitable $\mathcal{F}$-conjugate, we may assume $T \in \mathcal{F}^f$. As $\mathcal{F}^{cr} \subseteq \Delta$ and $T \notin \Delta$, $T \notin \mathcal{F}^{rc}$. Then by Lemma 3.17, $T < O_p(N_{\mathcal{F}}(T))$ and thus $O_p(N_{\mathcal{F}}(T)) \in \Delta$, as every proper overgroup of T is in Δ. Hence, by Lemma 6.1, $M := N_{\mathcal{L}}(T)$ is a subgroup of $\mathcal{L}$ which is a model for $N_{\mathcal{F}}(T)$. Now clearly properties (1)-(4) of [10, Hypothesis 5.3] hold. By Theorem 2.1(b), $O_p(M) = O_p(N_{\mathcal{F}}(T)) \in \Delta$. So setting $\Delta_T := \{P \in \Delta \colon T \trianglelefteq P\}$, the locality $\mathcal{L}_{\Delta_T}(M)$ introduced in [10, Example/Lemma 2.10] is just the group M and $\lambda = \mathrm{id}_M$ can be considered as a rigid isomorphism $N_{\mathcal{L}}(T) \to \mathcal{L}_{\Delta_T}(M)$. So Hypothesis 5.3 in [10] is fulfilled. So by [10, Theorem 5.14], there exists a locality $(\mathcal{L}^+, \Delta^+, S)$ such that $\mathcal{L}$ is the restriction $\mathcal{L}^+|_\Delta$ of $\mathcal{L}^+$ to Δ and $\mathcal{F}_S(\mathcal{L}^+) = \mathcal{F}$. Furthermore, $\mathcal{L}^+$ can be taken to be the locality $\mathcal{L}^+(\lambda)$ constructed in [10]. So the first part of (a) holds. To prove (b) let $(\widetilde{\mathcal{L}}^+, \Delta^+, S)$ be another linking locality for $\mathcal{F}$ with object set Δ^+ and let $\beta : \mathcal{L} \to \widetilde{\mathcal{L}}^+|_\Delta$ be a rigid isomorphism. Then $\widetilde{\mathcal{L}} := \widetilde{\mathcal{L}}^+|_\Delta$ is a linking locality as well and has thus the same properties we proved above for $\mathcal{L}$. In particular, $N_{\widetilde{\mathcal{L}}}(T)$ is a subgroup of $\widetilde{\mathcal{L}}$ which is a model for $N_{\mathcal{F}}(T)$. Then $\beta_T = \beta|_M \colon M \to N_{\widetilde{\mathcal{L}}}(T)$ will be an isomorphism of groups which restricts to the identity on $N_S(T)$, as β is a rigid isomorphism. As $(\widetilde{\mathcal{L}}^+, \Delta^+, S)$ is a linking locality and $T \in \Delta^+ \cap \mathcal{F}^f$, $N_{\widetilde{\mathcal{L}}^+}(T)$ is a model for $N_{\mathcal{F}}(T)$ by Lemma 5.2. Clearly, $N_{\widetilde{\mathcal{L}}}(T) \subseteq N_{\widetilde{\mathcal{L}}^+}(T)$ and thus $N_{\widetilde{\mathcal{L}}}(T) = N_{\widetilde{\mathcal{L}}^+}(T)$ by Theorem 2.1(a). Hence, β_T is also a group isomorphism $M \to N_{\widetilde{\mathcal{L}}^+}(T)$ which restrict to the identity on $N_S(T)$. So by [10, Theorem 5.15(a)] applied with $\widetilde{\mathcal{L}}^+$ in place of $\mathcal{L}^*$ and β_T in place of μ, there exists a rigid isomorphism $\beta^+ \colon \mathcal{L}^+(\lambda) \to \widetilde{\mathcal{L}}^+$ which restricts to the identity on $\mathcal{L}$. This proves (b). Since our choice of T was arbitrary, the arguments above give that $N_{\mathcal{L}}(R)$ is a model for $N_{\mathcal{F}}(R)$ for any $R \in T^{\mathcal{F}} \cap \mathcal{F}^f$ and thus $N_{\mathcal{L}^+}(T) = N_{\mathcal{L}}(R)$. This proves (c).

It remains to prove the statement in part (a) about the nerves of the transporter systems. Note that $\mathcal{T}(\mathcal{L}, \Delta)$ is the full subcategory of $\mathcal{T}(\mathcal{L}^+, \Delta^+)$ with object set Δ. As $\mathrm{Aut}_{\mathcal{T}(\mathcal{L}^+,\Delta^+)}(P) \cong N_{\mathcal{L}^+}(P)$ for every $P \in \Delta^+$, P is $\mathcal{T}(\mathcal{L}^+, \Delta^+)$-radical in the sense defined in [17, p. 1015], if and only if P is $\mathcal{L}^+$-radical in the sense defined above. Hence, by Lemma 5.3, the $\mathcal{T}(\mathcal{L}^+, \Delta)$-radical elements of Δ^+ are precisely the elements of $\mathcal{F}^{rc}$. As by assumption, $\mathcal{F}^{cr} \subseteq \Delta$, it follows $\mathcal{T}(\mathcal{L}^+, \Delta^+)^r \subseteq \mathcal{T}(\mathcal{L}, \Delta)$. Hence, by [17, Lemma 4.8], the inclusion of nerves $|\mathcal{T}(\mathcal{L}, \Delta)| \subseteq |\mathcal{T}(\mathcal{L}^+, \Delta^+)|$ is a homotopy equivalence. $\square$

Proof of Theorem A. By Lemma 3.2, the set $\mathcal{F}^s$ is closed under taking $\mathcal{F}$-conjugates and overgroups. Hence, it is sufficient to prove (a). Let Δ_0 be the set of overgroups of the elements of $\mathcal{F}^{rc}$ in S. Then Δ_0 is closed under taking $\mathcal{F}$-conjugates, as $\mathcal{F}^{rc}$ is closed under taking $\mathcal{F}$-conjugates.

Step 1: We show that, up to a rigid isomorphism, there exists a unique linking locality $(\mathcal{L}_0, \Delta_0, S)$ for $\mathcal{F}$ and the nerve of $\mathcal{T}(\mathcal{L}_0, \Delta)$ is homotopy equivalent to the nerve of a centric linking system. Note first that by Remark 1, a centric linking system for $\mathcal{F}$ in the sense of [10] is a linking locality $(\mathcal{L}^*, \Delta^*, S)$ for $\mathcal{F}$ with $\Delta^* = \mathcal{F}^c$. Furthermore, by [10], a centric linking system $\mathcal{L}^*$ for $\mathcal{F}$ exists and is unique up to a rigid isomorphism. Then clearly, $\mathcal{L}_0 := \mathcal{L}^*|_{\Delta_0}$ is a linking locality. Suppose we are given another linking locality $(\widetilde{\mathcal{L}}_0, \Delta_0, S)$ for $\mathcal{F}$. Then by Theorem 6.2, there exists a centric linking system $\widetilde{\mathcal{L}}^*$ for $\mathcal{F}$ with $\widetilde{\mathcal{L}}^*|_{\Delta_0} = \widetilde{\mathcal{L}}_0$. Moreover, $|\mathcal{T}(\widetilde{\mathcal{L}}^*, \Delta)| \simeq |\mathcal{T}(\widetilde{\mathcal{L}}_0, \Delta_0)|$. Since centric linking systems are unique up to a rigid isomorphism, there exists then a rigid isomorphism $\lambda : \mathcal{L}^* \to \widetilde{\mathcal{L}}^*$. Clearly, λ restricts to a rigid isomorphism $\mathcal{L}_0 \to \widetilde{\mathcal{L}}_0$. By [10, Proposition A.3(b)], every rigid isomorphism of localities leads to an isomorphism between the corresponding transporter systems. In particular, $|\mathcal{T}(\mathcal{L}^*, \Delta^*)| \simeq |\mathcal{T}(\widetilde{\mathcal{L}}^*, \Delta)| \simeq |\mathcal{T}(\widetilde{\mathcal{L}}_0, \Delta_0)| \simeq |\mathcal{T}(\mathcal{L}_0, \Delta_0)|$.

Step 2: We complete the proof by showing that, up to a rigid isomorphism, there exists a unique linking locality $(\mathcal{L},\Delta,S)$ and $|\mathcal{T}(\mathcal{L},\Delta)|$ is homotopy equivalent to the nerve of a centric linking system. Note that $\mathcal{F}^{rc}\subseteq\Delta_0\subseteq\Delta\subseteq\mathcal{F}^s$. By Step 1 there is a linking locality $(\mathcal{L}_0,\Delta_0,S)$ which is unique up to rigid isomorphism and $|\mathcal{T}(\mathcal{L}_0,\Delta_0)|$ is homotopy equivalent to the nerve of a centric linking system. By Theorem 6.2, there exists a linking locality $(\mathcal{L},\Delta,S)$ for $\mathcal{F}$ with $\mathcal{L}|_{\Delta_0}=\mathcal{L}_0$ and $|\mathcal{T}(\mathcal{L},\Delta)|\simeq|\mathcal{T}(\mathcal{L}_0,\Delta_0)|$ is homotopy equivalent to the nerve of a centric linking system. Moreover, for every linking locality $(\widetilde{\mathcal{L}},\Delta,S)$, any rigid isomorphism $\mathcal{L}_0\to\widetilde{\mathcal{L}}|_{\Delta_0}$ extends to a rigid isomorphism $\mathcal{L}\to\widetilde{\mathcal{L}}$. Let $(\widetilde{\mathcal{L}},\Delta,S)$ be a linking locality. Note that $(\widetilde{\mathcal{L}}|_{\Delta_0},\Delta_0,S)$ is a linking locality. So by the uniqueness of $\mathcal{L}_0$, there exists a rigid isomorphism $\gamma:\mathcal{L}_0\to\widetilde{\mathcal{L}}|_{\Delta_0}$. This extends to a rigid isomorphism $\mathcal{L}\to\widetilde{\mathcal{L}}$ proving that $\mathcal{L}$ is unique up to a rigid isomorphism. □

7. Partial normal subgroups

Lemma 7.1. *Let $(\mathcal{L},\Delta,S)$ be a locality and $\mathcal{N}$ a partial normal subgroup of $\mathcal{L}$. Let $Q\in\Delta$. Then there exists $x\in\mathcal{N}$ such that $N_S(Q)\leq S_x$ and $N_T(Q^x)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q^x))$.*

Proof. By [6, I.2.6(c)], there exists $g\in\mathcal{L}$ such that $N_S(Q)\leq S_g$ and $Q^g\in\mathcal{F}^f$. Then by [10, Proposition 2.18(c)], $N_S(Q^g)\in\mathrm{Syl}_p(N_{\mathcal{L}}(Q^g))$. As $N_{\mathcal{L}}(Q^g)$ is a subgroup of $\mathcal{L}$ with normal subgroup $N_{\mathcal{N}}(Q^g)$, it follows $N_T(Q^g)=N_S(Q^g)\cap N_{\mathcal{N}}(Q^g)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q^g))$. Take $f\in\mathcal{L}$ and $R\in\Delta$ such that $(g,S_g)\uparrow(f,R)$ and f is $\uparrow$-maximal, where the relation $\uparrow$ is defined as in [10, Definition 4.3]. Then by [10, Proposition 4.5, Lemma 4.6], there exists $x\in\mathcal{N}$ such that $g=xf$, $S_g\leq S_{(x,f)}$ and $N_T(Q^x)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q^x))$. Then $N_S(Q)\leq S_g=S_{(x,f)}\leq S_x$ and the assertion holds. □

Proposition 7.2. *Suppose $(\mathcal{L},\Delta,S)$ is a linking locality for $\mathcal{F}$. Let $\mathcal{N}$ be a partial normal subgroup of $\mathcal{L}$ and $T=\mathcal{N}\cap S$. Assume that R is a subgroup of $C_S(T)$ which is weakly closed in $\mathcal{F}$. Then the following are equivalent:*

1. *$N_{\mathcal{N}}(T)\subseteq C_{\mathcal{L}}(R\cap T)$.*
2. *$N_{\mathcal{N}}(T)\subseteq C_{\mathcal{L}}(R)$.*
3. *$\mathcal{N}\subseteq C_{\mathcal{L}}(R)$.*

Proof. Set $T:=\mathcal{N}\cap S$. Assume first that (1) holds. By [11, Lemma 3.5], $N_{\mathcal{N}}(T)\subseteq N_{\mathcal{L}}(TC_S(T))$. In particular, $N_{\mathcal{N}}(T)$ is a subgroup of $\mathcal{L}$. As R is weakly closed in $\mathcal{F}$, R is normal in $N_{\mathcal{L}}(TC_S(T))$. Clearly, $[C_S(T),N_{\mathcal{N}}(T)]\leq\mathcal{N}\cap(TC_S(T))=T$ and thus $[R,N_{\mathcal{N}}(T)]\leq R\cap T$. So by (1), $[R,O^p(N_{\mathcal{N}}(T))]=1$. As $R\leq C_S(T)$ and $T\in\mathrm{Syl}_p(N_{\mathcal{N}}(T))$, it follows $N_{\mathcal{N}}(T)=O^p(N_{\mathcal{N}}(T))T\subseteq C_{\mathcal{L}}(R)$ and (2) holds. Clearly, (3) implies (1), so it remains only to prove that (2) implies (3).

Suppose (2) holds and that $\mathcal{N}\not\subseteq C_{\mathcal{L}}(R)$. Choose $n\in\mathcal{N}$ such that $n\notin C_{\mathcal{L}}(R)$ and $P:=S_n$ is of maximal order subject to this property. We proceed in two steps.

Step 1: We show that $N_{\mathcal{N}}(Q)\subseteq C_{\mathcal{L}}(R)$ for all $Q\in\Delta$ with $|Q|\geq|P|$ and $N_T(Q)\in\mathrm{Syl}_p(N_{\mathcal{N}}(Q))$. Assuming this is wrong we choose a counterexample Q. Then $|Q|=|P|$ because of the maximality of P. Set $G:=N_{\mathcal{L}}(Q)$ and notice that $N:=N_{\mathcal{N}}(Q)$ is a normal subgroup of G. As $N_T(Q)\in\mathrm{Syl}_p(N)$, we have $O_p(N)\leq N_T(Q)$. As $N\leq N_{\mathcal{N}}(QO_p(N))$, the maximality of $|Q|=|P|$ yields $O_p(N)\leq Q$. As $Q_0:=Q\cap T=Q\cap\mathcal{N}\trianglelefteq N$, it follows $Q_0=O_p(N)$. Since $(\mathcal{L},\Delta,S)$ is a linking locality, $G=N_{\mathcal{L}}(Q)$ is of characteristic p. So by Lemma 2.2(b), N has characteristic p and thus $C_N(Q_0)\leq Q_0$. Hence, $[N_{C_S(Q_0)}(Q),N]\leq C_N(Q_0)\leq Q_0$ and $QN_{C_S(Q_0)}(Q)$ is normalized by N. The maximality of $|Q|=|P|$ yields now $N_{C_S(Q_0)}(Q)\leq Q$. As $QC_S(Q_0)$ is a p-group, this implies $C_S(Q_0)\leq Q$. In particular, $R\leq C_S(T)\leq C_S(Q_0)\leq Q$. As R is weakly closed in $\mathcal{F}$, it follows that $R\trianglelefteq G$. By assumption $[R,T]=1$ and $N_T(Q)\in\mathrm{Syl}_p(N)$ yielding $[R,O^{p'}(N)]=[R,\langle N_T(Q)^N\rangle]=1$. If $T\leq Q$ then, as T is strongly closed, $N\leq N_{\mathcal{N}}(T)\subseteq C_{\mathcal{L}}(R)$, contradicting (2). Thus $T\not\leq Q$ and, as TQ is a p-group, we have $N_T(Q)\not\leq Q$. Thus, by the maximality of $|Q|$, $N_N(N_T(Q))\subseteq$

$N_{\mathcal{N}}(N_T(Q)Q) \subseteq C_{\mathcal{L}}(R)$. By a Frattini argument, $N = O^{p'}(N)N_N(N_T(Q)) \leq C_G(R) \subseteq C_{\mathcal{L}}(R)$. This contradicts our assumption and thus completes Step 1.

Step 2: We derive the final contradiction. By Lemma 7.1, there exists $x \in \mathcal{N}$ such that $N_S(P^n) \leq S_x$ and $N_T(P^{nx}) \in \mathrm{Syl}_p(N_{\mathcal{N}}(P^{nx}))$. If $T \leq P$ then $n \in N_{\mathcal{N}}(T) \subseteq C_{\mathcal{L}}(R)$ contradicting the choice of n. Hence, $T \not\leq P$ and $T \not\leq P^n$. In particular, $N_S(P^n) \not\leq P^n$ and the maximality of $|P| = |P^n|$ yields that $x \in C_{\mathcal{L}}(R)$. By [10, Lemma 2.7], conjugation with nx induces a group isomorphism from $N_{\mathcal{L}}(P)$ to $N_{\mathcal{L}}(P^{nx})$ and so $N_T(P)^{nx}$ is a p-subgroup of $N_{\mathcal{N}}(P^{nx})$. As $N_T(P^{nx}) \in \mathrm{Syl}_p(N_{\mathcal{N}}(P^{nx}))$, there exists $y \in N_{\mathcal{N}}(P^{nx})$ such that $N_T(P)^{nxy} = (N_T(P)^{nx})^y \leq N_T(P^{nx})$. As $T \not\leq P$, $N_T(P) \not\leq P$. Moreover, $N_T(P)P \leq S_{nxy}$. Hence, the maximality of $|P|$ yields $nxy \in C_{\mathcal{L}}(R)$. By Step 1, $y \in N_{\mathcal{N}}(P^{nx}) \subseteq C_{\mathcal{L}}(R)$. Similar as in the proof of [10, Lemma 2.19(a)], one sees that $C_{\mathcal{L}}(R)$ is a partial subgroup of $\mathcal{L}$. As $(n, x, y, y^{-1}) \in \mathbf{D}$ via P, it follows that $nx = (nx)(yy^{-1}) = (nxy)y^{-1} \in C_{\mathcal{L}}(R)$. Similarly, as $x \in C_{\mathcal{L}}(R)$ and $(n, x, x^{-1}) \in \mathbf{D}$ via P, $n = n(xx^{-1}) = (nx)x^{-1} \in C_{\mathcal{L}}(R)$. This contradicts the choice of n and gives thus the final contradiction. □

Proof of Proposition 3. Clearly, $C_S(\mathcal{N}) \subseteq C_S(\mathcal{E})$. So it is sufficient to show that $R := C_S(\mathcal{E})$ is contained in $C_S(\mathcal{N})$, or equivalently, $\mathcal{N} \subseteq C_{\mathcal{L}}(R)$. By [3, (6.7)(1)], R is strongly closed in $\mathcal{F}$ and thus weakly closed in $\mathcal{F}$. Furthermore, $R \leq C_S(T)$. As $\mathcal{E} \subseteq C_{\mathcal{F}}(R)$, $c_n|_{R \cap T}$ is the identity for every $n \in N_{\mathcal{N}}(T)$, i.e. $N_{\mathcal{N}}(T) \subseteq C_{\mathcal{L}}(R)$. Hence, by Lemma 7.2, $R \leq C_S(\mathcal{N})$. □

Remark 7.3. Our arguments show actually that in the situation of Proposition 3, the subgroup $C_S(\mathcal{E}) = C_S(\mathcal{N})$ is the largest subgroup of $C_S(T)$ weakly closed in $\mathcal{F}$ such that $N_{\mathcal{N}}(T) \subseteq C_{\mathcal{L}}(R \cap T)$. Similarly, it is the largest subgroup of $C_S(T)$ strongly closed in $\mathcal{F}$ such that $N_{\mathcal{N}}(T) \subseteq C_{\mathcal{L}}(R \cap T)$.

8. Final Remarks

Remark 8.1. Many results in the MSS-programme are proved not only for groups of local characteristic p, but more generally for groups of *parabolic characteristic* p. These are finite groups where every p-local subgroup containing a Sylow p-subgroup is of characteristic p. We say similarly that $\mathcal{F}$ is of parabolic characteristic p if the normalizer of every normal subgroup of S is constrained. In a unifying approach to classify groups and fusion systems of parabolic characteristic p, one would classify linking localities $(\mathcal{L}, \Delta, S)$ such that every normal subgroup of S is in Δ. It should be pointed out though that, to our knowledge, for a group of parabolic characteristic p, the subcentric linking locality cannot be so easily constructed directly from the group anymore. Generally, given a finite group G with $S \in \mathrm{Syl}_p(G)$, and a set Δ of subcentric subgroups closed under $\mathcal{F}_S(G)$-conjugation and with respect to overgroups, the locality $(\mathcal{L}_\Delta(G), \Delta, S)$ is not necessarily a linking locality. However, if $\Delta \subseteq \mathcal{F}_S(G)^q$, then for any $P \in \Delta$, the centralizer $C_{\mathcal{L}}(P)$ splits as the direct product of a p-group with the p'-group $O_{p'}(C_{\mathcal{L}}(P))$ by Lemma 5.4(c). So the linking locality for $\mathcal{F}_S(G)$ with object set Δ can be obtained from $\mathcal{L}_\Delta(G)$ by "factoring out" the p'-elements in the centralizers of elements of Δ as rigorously developed in [10, Theorem 4.8]. So one can always construct the centric or quasicentric linking locality of $\mathcal{F}_S(G)$ from the group G, and then expand it to a subcentric linking locality via Theorem 6.2. If G is of parabolic characteristic p then, for any normal subgroup P of S, the group $O_p(N_G(P))$ is centric in $\mathcal{F}_S(G)$, so $N_G(P) \leq N_G(O_p(N_G(P)))$ can be seen inside the centric linking locality. Similarly, this holds for any other subgroup $P \leq S$ whose normalizer in G is of characteristic p, provided $O_p(N_G(P)) \leq S$. Thus, still a lot of local information about G can be deduced from the subcentric linking locality for $\mathcal{F}_S(G)$.

Remark 8.2. Let $(\mathcal{L}, \Delta, S)$ be a linking locality for $\mathcal{F}$ and $R \trianglelefteq \mathcal{F}$. By Proposition 4, $\mathcal{L} = N_{\mathcal{L}}(R)$. In particular, R is a partial normal subgroup of $\mathcal{L}$ and we can form the quotient locality $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ as introduced in [11] with $\overline{\mathcal{L}} = \mathcal{L}/R$, $\overline{S} = S/R$ and $\overline{\Delta} = \{\overline{P} \colon P \in \Delta\}$. We have then $\mathcal{F}/R = \mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})$.

Proof. For $g \in \mathcal{L}$, we have $R \leq S_g$ and the map $\overline{S_g} \to \overline{S}$ induced by $c_g \colon S_g \to S$ is the same as the map $\overline{S_g} \to \overline{S}$ induced by $\overline{g}$, since the natural projection $\mathcal{L} \to \overline{\mathcal{L}}$ is a homomorphism of partial groups. On the other hand, if $\overline{f} \in \overline{\mathcal{L}}$, then we can choose the preimage $f \in \mathcal{L}$ to be $\uparrow$-maximal with respect to the partial normal subgroup R. As seen in the proof of [11, Proposition 4.2], then $\overline{S_f} = \overline{S}_{\overline{f}}$, and again the map $c_{\overline{f}} \colon \overline{S_f} \to \overline{S}$ is the same as the one induced by the map $c_f \colon S_f \to S$. □

Note that, in general, the locality $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ above does not need to be a linking locality. This is however the case for central subgroups.

Remark 8.3. Let $(\mathcal{L}, \Delta, S)$ be a linking locality for $\mathcal{F}$ with $\Delta = \mathcal{F}^s$ or $\Delta = \mathcal{F}^q$, and $Z \leq Z(\mathcal{F})$. By Proposition 4, $\mathcal{L} = C_{\mathcal{L}}(Z)$. Form the quotient locality $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ for $\mathcal{F}/Z$ as in Remark 8.2. Then $\overline{\Delta} = (\mathcal{F}/R)^s$ if $\Delta = \mathcal{F}^s$ and $\overline{\Delta} = (\mathcal{F}/R)^q$ if $\Delta = \mathcal{F}^q$ by Proposition 1(b) and [8, Lemma 6.4(b)]. Moreover, for $P \in \Delta$, $N_{\overline{\mathcal{L}}}(\overline{P}) \cong N_{\mathcal{L}}(P)/Z$ is of characteristic p by Lemma 2.3. So $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a linking locality for $\mathcal{F}/Z$.

Remark 8.4. Let $(\mathcal{L}, \Delta, S)$ be a subcentric linking locality for $\mathcal{F}$. Fix $Q \in \mathcal{F}$ and $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$ such that Q is fully K-normalized. Set $N_{\mathcal{L}}^K(Q) := \{f \in N_{\mathcal{L}}(Q) \colon c_f|_Q \in K\}$. Let $\Delta_0 := N_{\mathcal{F}}^K(Q)^s$ and $\mathcal{L}_0 := \{f \in N_{\mathcal{L}}^K(Q) \colon S_f \cap N_S^K(Q) \in \Delta_0\}$. Let $\mathbf{D}_0$ be the set of words $(f_1, \ldots, f_n)$ in $\mathcal{L}_0$ such that there exists $P_0, P_1, \ldots, P_n \in \Delta_0$ with $P_{i-1}^{f_i} = P_i$ for $i = 1, \ldots, n$. By Proposition 1(c), $P_i Q \in \mathcal{F}^f$ for $i = 0, \ldots, n$, so $(f_1, \ldots, f_n) \in \mathbf{D}$ via $P_0Q, P_1Q, \ldots, P_nQ$. Hence, $\mathbf{D}_0 \subseteq \mathbf{D}$. Then we can turn $\mathcal{L}_0$ into a partial group where the product is the restriction of the product in $\mathcal{L}$ to $\mathbf{D}_0$. From the way we constructed $\mathcal{L}_0$, it is immediate that $(\mathcal{L}_0, \Delta_0, N_S^K(Q))$ is a locality. Moreover, the inclusion map $\beta \colon \mathcal{L}_0 \to \mathcal{L}$ is a homomorphism of partial groups as $\mathbf{D}_0 \subseteq \mathbf{D}$. One easily sees that $N_{\mathcal{F}}^K(Q) = \mathcal{F}_{N_S^K(Q)}(N_{\mathcal{L}}^K(Q))$ and then $N_{\mathcal{F}}^K(Q) = \mathcal{F}_{N_S^K(Q)}(\mathcal{L}_0)$ by Alperin's fusion theorem. Since $\mathcal{L}$ is a linking locality, for every $P \in \Delta_0$, $G := N_{\mathcal{L}}(PQ)$ is a group of characteristic p. Hence, $N_{\mathcal{L}_0}(P) = N_{\mathcal{L}}^K(Q) \cap N_{\mathcal{L}}(P) = N_G(P) \cap N_G^K(Q) \trianglelefteq N_G(P) \cap N_G(Q) = N_{N_G(Q)}(P)$ is of characteristic p by Lemma 2.2. Hence, $(\mathcal{L}_0, \Delta_0, N_S^K(Q))$ is a linking locality for $N_{\mathcal{F}}^K(Q)$ and the inclusion map $\beta \colon \mathcal{L}_0 \to \mathcal{L}$ can be thought of as an inclusion map of linking localities. It induces a functor $\mathcal{T}(\mathcal{L}_0, \Delta_0) \to \mathcal{T}(\mathcal{L}, \Delta)$ between the linking systems which sends an object $P \in \Delta_0$ to $PQ \in \Delta$, and a morphism (f, P_1, P_2) to (f, P_1Q, P_2Q). A similar construction works for centric and quasicentric linking localities.

Remark 8.5. Assume the hypothesis of Proposition 3 and assume $\Delta_0 = \mathcal{F}^s$. Set $\Delta_0 := \mathcal{E}^s$ and $\mathcal{N}_0 := \{f \in \mathcal{N} \colon S_f \cap T \in \Delta_0\}$. By Theorem B and Proposition 3, we have $PC_S(\mathcal{N}) \in \Delta$ for all $P \in \Delta_0$. Let $\mathbf{D}_0$ be the set of words $(f_1, \ldots, f_n)$ in $\mathcal{N}_0$ such that there exist $P_0, \ldots, P_n \in \Delta_0$ with $P_{i-1}^{f_i} = P_i$ for $i = 1, \ldots, n$. As $P_i C_S(\mathcal{N}) \in \Delta$ for $i = 0, \ldots, n$ and each f_i centralizes $C_S(\mathcal{N})$, it follows that $(f_1, \ldots, f_n) \in \mathbf{D}$ via $P_0 C_S(\mathcal{N}), \ldots, P_n C_S(\mathcal{N})$. So $\mathbf{D}_0 \subseteq \mathbf{D}$. Let $\mathcal{N}_0$ be the partial group whose product is the restriction of the product on $\mathcal{L}$ to $\mathbf{D}_0$. Then $(\mathcal{N}_0, \Delta_0, T)$ is a locality. By Alperin's fusion theorem, $\mathcal{F}_T(\mathcal{N}_0) = \mathcal{F}_T(\mathcal{N}) = \mathcal{E}$. Moreover, as $\mathcal{L}$ is a linking locality, for any $P \in \Delta_0$, $N_{\mathcal{L}}(PC_S(\mathcal{N}))$ is of characteristic p and thus $N_{\mathcal{N}_0}(P) = N_{\mathcal{N}}(P) = \mathcal{N} \cap N_{N_{\mathcal{L}}(PC_S(\mathcal{N}))}(P) \trianglelefteq N_{N_{\mathcal{L}}(PC_S(\mathcal{N}))}(P)$ is of characteristic p by Lemma 2.2. Therefore, $(\mathcal{N}_0, \Delta_0, T)$ is a linking locality for $\mathcal{E}$. As $\mathbf{D}_0 \subseteq \mathbf{D}$, the inclusion map $\mathcal{N}_0 \to \mathcal{L}$ is a homomorphism of partial groups. It induces a functor $\mathcal{T}(\mathcal{N}_0, \Delta_0) \to \mathcal{T}(\mathcal{L}, \Delta)$ between the linking systems. This functor sends an object $P \in \Delta_0$ to $PC_S(\mathcal{N})$ and a morphism (f, P_1, P_2) to $(f, P_1C_S(\mathcal{N}), P_2C_S(\mathcal{N}))$. By iterating this procedure, one can similarly obtain subnormal inclusions of linking localities and linking systems.

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