Robust Control Design for Vehicle Active Suspension Systems with Uncertainty

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Whilst registered as a candidate for the above degree, I have not been registered for any other research award. The results and conclusions embodied in this thesis are the work of the named candidate and have not been submitted for any other academic award.

This thesis is dedicated to the one who supported me all the moments, specially to my parents and my wife Qi Zhou, with love.

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Abstract

A vehicle active suspension system, in comparison with its counterparts, plays a crucial role in adequately guarantee the stability of the vehicle and improve the suspension performances. With a full understanding of the state of the art in vehicle control systems, this thesis identifies key issues in robust control design for active suspension systems with uncertainty, contributes to enhance the suspension performances via handling tradeoffs between ride comfort, road holding and suspension deflection. Priority of this thesis is to emphasize the contributions in handing actuator-related challenges and suspension model parameter uncertainty.

The challenges in suspension actuators are identified as time-varying actuator delay and actuator faults. Time-varying delay and its effects in suspension actuators are targeted and analyzed. By removing the assumptions from the state of the art methods, state-feedback and output-feedback controller design methods are proposed to design less conservative state-feedback and output-feedback controller existence conditions. It overcomes the challenges brought by generalized time-varying actuator delay. On the other hand, a novel fault-tolerant controller design algorithm is developed for active suspension systems with uncertainty of actuator faults. A continuous-time homogeneous Markov process is presented for modeling the actuator failure process. The fault-tolerant H_∞ controller is designed to guarantee asymptotic the stability, H_∞ performance, and the constrained performance with existing possible actuator failures.

It is evident that vehicle model parameter uncertainty is a vital factor affecting the performances of suspension control system. Con-

sequently, this thesis presents two robust control solutions to overcome suspension control challenges with nonlinear constraints. A novel fuzzy control design algorithm is presented for active suspension systems with uncertainty. By using the sector nonlinearity method, Takagi-Sugeno (T-S) fuzzy systems are used to model the system. Based on Lyapunov stability theory, a new reliable fuzzy controller is designed to improve suspension performances. A novel adaptive sliding mode controller design approach is also developed for nonlinear uncertain vehicle active suspension systems. An adaptive sliding mode controller is designed to guarantee the stability and improve the suspension performances.

In conclusion, novel control design algorithms are proposed for active suspension systems with uncertainty in order to guarantee and improve the suspension performance. Simulation results and comparison with the state of the art methods are provided to evaluate the effectiveness of the research contributions. The thesis shows insights into practical solutions to vehicle active suspension systems, it is expected that these algorithms will have significant potential in industrial applications and electric vehicles industry.

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Chapter 1

Introduction

It is evident that there is a constantly growing interest in providing acceptable system performances of vehicle suspension systems, especially in the past two decades (Ha\'c, 1992; Karnopp, 1983; Yamashita *et al.*, 1990; Zhu & Knospe, 2010) as vehicle suspension systems have many vital functions: for instance, to support the vehicle weight, to provide effective isolation of the chassis from road excitations, to keep tyre contact with the ground, and to maintain the wheels in appropriate position on the road surface. Vehicle suspension systems play an important role in guaranteeing the stability and improving suspension performances of vehicles. It is worth noting that the problem of control design for active suspension systems should be paid considerable attention. In addition, the vehicle suspension systems can provide as much comfort as possible for the passengers and ensure the other suspension performance by serving the basic function of isolating passengers from road-induced vibration and shocks (Montazeri-Gh & Soleymani, 2010; Rajamani & Hedrick, 1995). Hence, the control design problem of proper active suspension systems is always an important research topic for achieving the desired vehicle suspension performances.

1.1 A Brief Background

Several performance characteristics should be considered (Alleyne & Hedrick, 1995) and need to be optimized for designing a good performance suspension system. It is widely accepted that three main suspension performances should

1

be taken into account when designing a suspension controller, namely, ride comfort (i.e., directly related to acceleration sensed by passengers), road handling (i.e., associated with the contact forces of tyres and road surface), and suspension deflection (i.e., referred to the displacement between the sprung mass and unsprung mass) (Hrovat, 1997). However, it is difficult to minimize all three parameters simultaneously as these performances are often conflicting with each other (Chalasani, 1986). For example, the minimization of suspension travel cannot be accomplished simultaneously with the maximization of the ride comfort. In other words, enhancing ride comfort performance results in larger suspension stroke and smaller damping in the wheel-hop mode. Hence, how to derive an appropriate trade-off between these performances is the main task for successfully designing a vehicle suspension control system.

Recently, many researchers have paid considerable attention to develop different suspension systems to improve the suspension performance. Generally speaking, suspension systems can be categorized into the following several types: passive (Naudé & Snyman, 2003a,b), semi-active (Choi *et al.*, 2001; Poussot-Vassal *et al.*, 2008, 2010; Yao *et al.*, 2002), and active suspension systems (Cao *et al.*, 2008b; Ting *et al.*, 1995). The passive suspension system is a conventional system that contains non-controlled spring or damper devices being assumed to have almost linear characteristics, and its performance depends on the values of certain vehicle parameters (Naudé & Snyman, 2003a,b). The passive suspension system can not offer the control force and its performance is always limited. The improvement of ride comfort, road holding and suspension travel are effective only in a certain frequency range due to the intrinsic limitation of passive suspension system. However, the automobile industry adopts the device as it can offer high reliability without consuming energy.

The semi-active suspension system can supply controlled real-time dissipation of energy (Williams, 1994), which is implemented through a mechanical device called an active damper. In the semi-active suspension system, the active damper is used in parallel to a conventional spring. The disadvantage of the damper is that it is difficult to find proper device to generate a high force at low velocities and a low force at high velocities, and be able to move rapidly between the two.

The active suspension system is different from the conventional passive suspension system and the semi-active suspension system since it has the capability to provide energy to the system, as well as store and dissipate it by employing pneumatic or hydraulic actuators to create the desired force (Crolla & Abdel, 1991; Hać, 1992; Hrovat, 1997) . The actuator in the active suspension is placed in parallel with the damper and the spring. Due to the fact the actuator connects the unsprung mass to the body, it can control both the wheel hop motion, suspension travel and the body motion. Therefore, the active suspension now can improve suspension performances such as ride comfort, ride handling and suspension reflection simultaneously.

It is well known that an active suspension system is the effective way to improve suspension performance and has been intensively investigated (Alleyne & Hedrick, 1995; Cao *et al.*, 2010; Du *et al.*, 2005; Gao *et al.*, 2006; Ma & Chen, 2011; Yamashita *et al.*, 1994). In order to manage the trade-off between the conflicting performance, some active control approaches are presented based on various control techniques such as fuzzy logic and neural network control (Cao *et al.*, 2008a; Cherry & Jones, 1995), gain scheduling control (Sivrioglu & Cansever, 2009), linear optimal control (ElMadany & Abduljabbar, 1999), adaptive control (Fialho & Balas, 2002) and H_∞ control (Chen & Guo, 2005; Du & Zhang, 2007; Gao *et al.*, 2010a). It has been confirmed that H_∞ control method for active suspension systems is applicable to manage the trade-off and obtain a compromise performance in the references (Chen & Guo, 2005; Du & Zhang, 2007). Therefore, the H_∞ control design problems of active suspensions systems have been paid considerable attention and many results have been reported in the literature (Chen & Guo, 2005; Chen *et al.*, 2005; Du & Zhang, 2007; Du *et al.*, 2003). For instance, the Lyapunov-Krasovkii functional method (Chen, 2007; Goldhirsch *et al.*, 1987) and linear matrix inequality (LMI) approach (Boyd *et al.*, 1994; Gahinet *et al.*, 2002; Scherer *et al.*, 1997) have been employed to develop the H_∞ control results.

1.2 Problems and Challenges

In control design process of vehicle active suspension systems, the time delay of the system is an important problem needing careful treatment to avoid poor

performance or even possible instability of the closed-loop system. Time delay or transportation lag is a characteristic that commonly encounters in various engineering systems, such as pneumatic and hydraulic systems, chemical processes, long transmission lines, for instance. The systems using the electrohydraulic actuators to track the desired forces may be inevitably confronted with actuator delay. The presence of actuator delay, if not taken into account in the controller design process, may degrade the control performances and even cause instability in the resulting control systems. In view of this, more recently, many stability analysis and controller design schemes have been presented for linear systems with state delay or actuator delay (Han, 2005; He *et al.*, 2004; Li *et al.*, 2008; Mou *et al.*, 2008; Shi *et al.*, 2002; Wang *et al.*, 2010; Wu *et al.*, 2009, 2010; Zhang *et al.*, 2007). More recently, the problems of stability analysis and controller synthesis for quarter-vehicle, half-vehicle and seat active suspension suspension systems with actuator delay have been investigated in (Du & Zhang, 2007, 2008; Gao *et al.*, 2010b) respectively, among which there exist two main approaches dealing with the actuator delay problems. One is to design a controller using the integrated system model where the actuator dynamics are included (Thompson & Davis, 2001). The other is to design a controller by considering the actuator delay in the controller design process (Du & Zhang, 2007). However, it should be pointed out that the actuator delay in the existing vehicle suspension is constant delay. In practice, the more general time-varying delay should be considered in the control design for vehicle active suspension systems.

An active suspension system has the ability to enhance vehicle dynamics by relaxing external impact such as road surface on vehicle travel comfort. In terms of its control design, uncertainty of vehicle sprung and unsprung masses such as its loading conditions must be taken into account to meet vehicle travel performance criteria. For instance, the polytopic parameter uncertainties were employed to model the varying vehicle sprung or unsprung masses (Du *et al.*, 2008; Gao *et al.*, 2006, 2010a). The parameter-dependent controllers were proposed for the quarter-car suspension systems with sprung mass variation (Du *et al.*, 2008; Gao *et al.*, 2006). The parameter-independent sampled-data H_∞ controller design strategy was presented to handle both sprung and unsprung mass variations in a case study of a quarter-car suspension system (Gao *et al.*, 2010a). The

state of the art in suspension control design in these scenarios, however, could not provide feasible performance for half-vehicle active suspension systems with parameter uncertainties.

It should be pointed out however, that the aforementioned suspension control results are under a full reliability assumption that all control components of the systems are in ideal working conditions. Due to the growing complexity of automated control systems, various faults are likely to be encountered, especially faults from actuators and sensors Shi *et al.* (2003); Wang *et al.* (1999); Yang *et al.* (2002). During the past few decades, many researches have attempted to resolve the reliable and fault tolerant control problems for dynamic systems with uncertainty such as actuator and sensor faults, and a great number of theoretic results have also been presented (Mao *et al.*, 2010; Wang *et al.*, 2009b). For instance, the reliable H_∞ controller design problem was investigated at a context of linear systems (Yang *et al.*, 2001a), and a controller was designed to ensure the reliability of the control system, i.e., guaranteed asymptotic stability and H_∞ performance, under the assumption that all control components of sensors and actuators are operational. As a matter of fact, an active suspension system is different from its counterpart of a passive suspension system in that its actuator has the capability of adjusting the force to meet the criteria of the vehicle dynamics, such as guaranteeing the stability of the vehicle, securing passenger comfort and satisfying the suspension performance. However, when either the actuator or sensor faults occur in an active suspension system, the conventional controllers could not achieve better performance in comparison with the reliable and fault-tolerant controllers as discussed in (Zhao *et al.*, 2010c), where the considered actuator fault was described as a static behavior. It is practically reasonable to assume that the actuator fault should be regarded as the dynamic behavior in stochastic distributions (Dong *et al.*, 2010; Wang *et al.*, 2009b).

As mentioned in this section, we know that the effects of uncertainties in the active suspension control systems should be considered. The main uncertainties are derived from parameter uncertainties, actuator uncertainties, actuator faults and actuator nonlinear dynamics. For the existing active suspension control systems and control design methods, there still exist the following several challenges:

1. It is challenging to choose the proper active suspension systems, present the uncertainty from these kind systems and the controllers, and build the active suspension control systems including the suspension performances.

2. It is difficult to develop new stability analysis and controller synthesis approaches for the built control systems to improve the suspension performances.

3. It is challenging to propose simply, efficient and less conservative suspension performances analysis methods in the control design process.

4. It is difficult to develop a estimate platform for the control design results of the the vehicle active suspension systems which take into account the effect of parameter uncertainty.

1.3 Overview of Approaches and Contributions

To consider the proposed problems in section 1.2, the thesis makes four contributions to investigate these problems. The main contributions of this thesis can be summarized below:

1. To begin with, this thesis takes into account the more general actuator time-varying delay for vehicle active suspension systems and builds the corresponding vehicle active suspension dynamical models. By developing the advanced methods, the state-feedback and output-feedback controllers are designed to guarantee the stability and improve the suspension performances.

2. Secondly, this thesis models the actuator fault in a dynamic behavior, which is different from the existing static behavior. We establish vehicle active suspension control systems with actuator faults, which can be modeled by a continuous-time Markov process. Based on this model, a novel fault-tolerant H_∞ controller is designed such that the resulting control system is tolerant in that it guarantees asymptotic stability and H_∞ performance, and

simultaneously satisfies the constrained performances with existing possible actuator failures.

3. Thirdly, this thesis proposes a novel fuzzy control method for vehicle active suspension systems with uncertainty. The sector nonlinearity method is exploited to represent the uncertain systems for the control design objective. Linear matrix inequality (LMI)-based fuzzy H_∞ controller existence conditions are derived to guarantee the stability as well as improve the suspension performances.

4. Finally, this thesis investigates the problem of adaptive sliding mode control design for vehicle active suspension systems with uncertainty and nonlinear actuator dynamic. After considering the model uncertainty and the actuator nonlinearity, the nonlinear systems can be built with the constrained suspension performances. The adaptive sliding mode controller is designed to improve the suspension performances and guarantee the suspension constraints.

1.4 Outline of Thesis

To fulfil the proposed approaches, the thesis is organized as follows.

Chapter 2 presents an overview of H_∞ control method for vehicle active suspension control systems in response with the problems raised in practical implementations by actuator delay, actuator fault, actuator nonlinear and system uncertain. First, we briefly introduce quarter-, half- and full-vehicle active suspension models and build the state-space form system including the output performances (ride comfort) and the constrained performance (road holding and suspension reflection). Two types of uncertain models in the vehicle active suspension systems are constructed. This chapter reviews state-feedback control, output-feedback control, fault-tolerant control, fuzzy control and adaptive sliding mode control approaches for vehicle active suspension systems.

Chapter 3 is aimed at proposing state-feedback and output-feedback H_∞ controllers for the active suspension systems with actuator time-varying delay.

By constructing novel Lyapunov functional, some delay-dependent H_∞ performance analysis and controller design conditions are derived in the forms of LMIs based on new less conservative delay-dependent techniques. These presented controllers can guarantee the closed-loop systems stability and simultaneously satisfy the requested performance. Simulation results are provided to illustrate the effectiveness of the proposed method and point out that the control delay should be taken into account for the suspension systems when carrying out the active control problems.

Chapter 4 focuses on developing fault-tolerant H_∞ control strategy for vehicle active suspension systems with actuator faults. By modeling the actuator failure process as stochastic behavior via a continuous-time homogeneous Markov process, a novel fault-tolerant H_∞ controller is designed such that the resulting control system is tolerant in that it guarantees asymptotic stability and H_∞ performance, and simultaneously satisfies the constrained performance with existing possible actuator failures. A quarter-vehicle active suspension system is exploited to demonstrate the effectiveness of this control design method.

Chapter 5 is concerned with fuzzy control for vehicle active suspension systems with uncertainty. We first build the T-S fuzzy model to represent the uncertain active suspension systems with sprung and unsprung mass variations, actuator delay and actuator fault. Based on the T-S fuzzy model, we obtain LMI-based reliable fuzzy H_∞ controller existence conditions. Simulation results validate the effectiveness of the proposed approaches.

Chapter 6 investigates the problem of adaptive sliding mode control for nonlinear uncertain active suspension systems. The suspension performances are considered in the controller design process and the T-S fuzzy model approach is utilized to represent the nonlinear uncertain suspension system by T-S fuzzy system. The sliding mode controller is designed to ensure that the T-S fuzzy system is stable and improve the suspension performance. A half-vehicle model is employed to demonstrate the effectiveness of the presented method.

Chapter 7 presents some concluding remarks and future plans.

Chapter 2

Literature Review

2.1 Vehicle Suspension Modelling

As is well known, a suspension system is one of the crucial parts of a vehicle and plays an important role in modern vehicles for handling vehicle suspension performances, such as improving ride comfort of the vehicle. The major task of suspension system is twofold: one is to isolate the car body with its passenger from external disturbance inputs which mainly come from road irregularities, to improve riding quality, and the other one is to maintain a firm contact between the road and the tyres to provide guidance along the track, namely handling performances. A conventional suspension system consists of passive components, so the task of providing both ride comfort and good handling can lead to conflict these requirements. On one hand, a stiff suspension is necessary to support the weight and to follow the track. On the other hand, a soft suspension is required to isolate the disturbance from the road, which means that the ride comfort performance and the other suspension performances such as handling performance and suspension travel constraints are conflicted. Thus, many kinds of vehicle suspensions systems have been developed to improve both ride quality and handling performance.

An active suspension system can employ some pneumatic, magneto-rheological or hydraulic actuators to generate the force to control this suspension system. With the development of microprocessors and electronics, many researchers (Esmailzadeh & Bateni, 1992; Hrovat, 1987) have done some work on practical ap-

plications of active suspension systems since the middle of 1980s. In addition, related surveys on theories and applications of active suspension control systems have been presented by (Hrovat, 1997; Nagai, 1993; Sharp & Crolla, 1987). It should be noticed that ride comfort, road handling and suspension deflection are mainly used to evaluate the suspension performances. Generally speaking, ride comfort of the passengers is related to vehicle acceleration sensed, road handling is associated with the contact forces of tyres and road surface, and suspension deflection is referred to the displacement between the sprung mass and unsprung mass (Lai & Liao, 2002; Yamashita *et al.*, 1990). In the past two decades, a great number of research projects have been carried out, targeting the challenge of how to improve the vehicle suspension systems performances (Hrovat, 1997). Due to the inherent conflicting nature of the systems performance criteria, for instance, enhancing ride comfort needs larger suspension stroke and smaller damping of wheel-hop mode, and this results in a degradation in ride safety (Chen & Guo, 2005). The problem, hence, is still open for a better solution to be excavated. It is evident that considerable attention has been drawn to the problem of solving trade-off among the conflicting objectives (Gordon *et al.*, 1991). Three main types of suspension systems, namely, passive (Tamboli & Joshi, 1999), semi-active (Hrovat *et al.*, 1988; Shen *et al.*, 2006) and active suspension systems (Cao *et al.*, 2008b; Ting *et al.*, 1995) have been investigated to achieve the vehicle requirement performance and avoid the trade-off. For the proposed suspension system, it is widely accepted that active suspensions is the effective way to improve suspension performances due to its flexibility in dealing with the conflicting parameters. Further interested researchers have been reported to address the active suspension systems design problems (Alleyne & Hedrick, 1995; Gao *et al.*, 2006; Yamashita *et al.*, 1994). In this thesis, the quarter, half and full-vehicle active suspension systems are presented and reviewed in the following subsections.

2.1.1 A Quarter-vehicle Suspension Model

The generalized quarter-car suspension model is shown in Fig. 2.1, where z_s and z_u stand for the displacements of the sprung and unsprung masses respectively; z_r denotes the road displacement input; u is the active input of the suspension
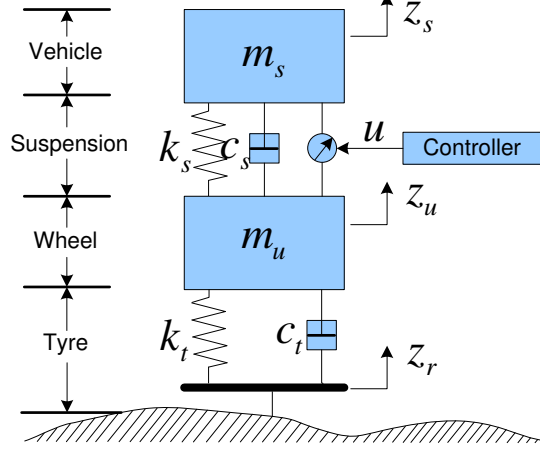
Figure 2.1: A quarter-car model

system; m_s is the sprung mass, which represents the car chassis; m_u is the unsprung mass, which represents mass of the wheel assembly; c_s and k_s are damping and stiffness of the suspension system, respectively; k_t and c_t stand for compressibility and damping of the pneumatic tyre, respectively. Then, we can obtain the equations of the motion as (Du *et al.*, 2008):

$$
\begin{aligned}
m_s\ddot{z}_s(t) &= c_s[\dot{z}_u(t) - \dot{z}_s(t)] + k_s[z_u(t) - z_s(t)] + u(t), \\
m_u\ddot{z}_u(t) &= c_s[\dot{z}_s(t) - \dot{z}_u(t)] + k_s[z_s(t) - z_u(t)] \\
&\quad + k_t[z_r(t) - z_u(t)] + c_t[\dot{z}_r(t) - \dot{z}_u(t)] - u(t).
\end{aligned}
\tag{2.1}
$$

Let us define the following state variables: $x_1(t) = z_s(t) - z_u(t)$ denotes the suspension deflection, $x_2(t) = z_u(t) - z_r(t)$ denotes the tire deflection, $x_3(t) = \dot{z}_s(t)$ denotes the sprung mass speed, $x_4(t) = \dot{z}_u(t)$ denotes the unsprung mass speed.

Then, we define the disturbance input $w(t) = \dot{z}_r(t)$ and the state vector as

$$
x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T,
$$

the dynamic equations in (2.1) can be expressed as the following state-space form:

$$
\dot{x}(t) = Ax(t) + B_1 w(t) + Bu(t),
\tag{2.2}
$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_u} & -\frac{k_t}{m_u} & \frac{c_s}{m_u} & -\frac{c_s+c_t}{m_u} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_s} \\ -\frac{1}{m_u} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \frac{c_t}{m_u} \end{bmatrix}.$$

The key suspension performances such as ride comfort, suspension deflection, and road holding are considered as the control design objectives for the vehicle active suspension system in this study. As in references (Du *et al.*, 2008; Gao *et al.*, 2010a), we regard that the seat suspension system is simplified and included in the quarter-car suspension systems. The sprung mass m_s consists of the mass of seat cushion. Due to the fact that ride comfort can be generally quantified by the body acceleration in the vertical direction, it is reasonable to choose body acceleration as the first control output, that is, $\ddot{z}_s(t)$.

When we design the controller for suspension systems, one of our main objectives is to minimize the vertical acceleration $\ddot{z}_s(t)$. Thus, we can apply the H_∞ norm to measure the performance, whose value actually generates an upper bound of the root mean square gain. Hence, our main goal is to minimize the H_∞ norm of the transfer function from the disturbance $w(t)$ to the control output $z_1(t) = \ddot{z}_s(t)$ in order to improve the vehicle ride comfort.

In addition, due to the mechanical structure, the suspension stroke should not exceed the allowable maximum, that is,

$$|z_s(t) - z_u(t)| \le z_{\max}, \tag{2.3}$$

where $z_{\max}$ is the maximum suspension deflection.

Moreover, in order to ensure a firm uninterrupted contact of the wheels with the road, the dynamic tyre load should not exceed the static tyre load:

$$k_t\left(z_u(t) - z_r(t)\right) < \left(m_s + m_u\right)g. \tag{2.4}$$

Based on the above conditions, therefore, we choose the body acceleration $\ddot{z}_s(t)$ as performance control output, and the suspension stroke $z_s(t) - z_u(t)$, relative dynamic tire load $k_t\left(z_u(t) - z_r(t)\right)/\left(m_s + m_u\right)g$ as constrained control output

$z_2(t)$, where $y(t)$ denotes the measured output vector. Then, the vehicle active suspension system can be described by the following state-space equations:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + Bu(t), \\
z_1(t) &= C_1 x(t) + D_1 u(t), \\
z_2(t) &= C_2 x(t),
\end{aligned}
\tag{2.5}
$$

where the matrices A, B_1 and B are defined in (2.2), and

$$
\begin{aligned}
C_1 &= \begin{bmatrix} -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \end{bmatrix}, \\
D_1 &= \frac{1}{m_s}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{k_t}{(m_s+m_u)g} & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

The quarter-vehicle model parameters are listed in Table 2.1 (Du *et al.*, 2008) for the following controller design.

Table 2.1: Systems parameter values for the quarter-vehicle suspension model

m_s	m_u	k_s	k_t	c_s	c_t
973kg	114kg	42720N/m	101115N/m	1095Ns/m	14.6Ns/m

2.1.2 A Half-vehicle Suspension Model

The addressed problem is formulated in terms of a half-vehicle model as shown in Fig. 2.2, in which $z_{sf}(t)$ is used to denote the front body displacement; $z_{sr}(t)$ stands for the rear body displacement; l_1 is the distance between the front axle and the centre of mass; l_2 is the distance between the rear axle and the centre of mass; $\varphi(t)$ is the pitch angle, and $z_c(t)$ is the displacement of the centre of mass; The mass of the car body is denoted by m_s, the unsprung masses on the front and rear wheels are denoted by m_{uf} and m_{ur}, the pitch moment of inertia about the center of mass is denoted by I_φ, the front and rear unsprung mass displacements are denoted by $z_{uf}(t)$ and $z_{ur}(t)$ respectively; $z_{rf}(t)$ and $z_{rr}(t)$ stand for the front and rear terrain height displacements, while c_{sf} and c_{sr} are the stiffness of the passive elements of the front and rear wheels respectively. k_{sf} and k_{sr} are the
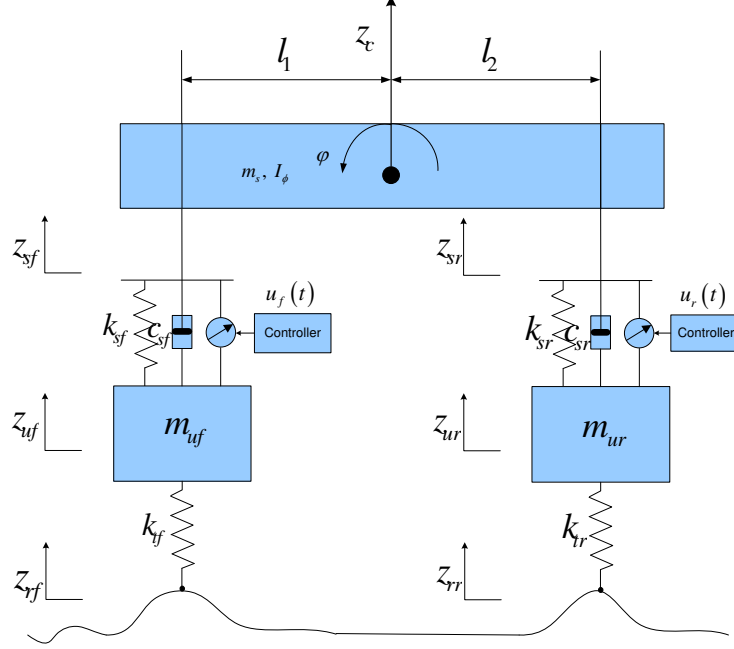
Figure 2.2: A half-vehicle model

front and rear tyre stiffness; $u_f(t)$ and $u_r(t)$ are the front and rear actuator force inputs respectively. With the assumption of a small pitch angle $\varphi(t)$ in (Du & Zhang, 2008), one can obtain:

$$
\begin{aligned}
z_{sf}(t) &= z_c(t) - l_1 \varphi(t), & (2.6)\\
z_{sr}(t) &= z_c(t) + l_2 \varphi(t). & (2.7)
\end{aligned}
$$

By using Newton's second law and the static equilibrium position as the origin for both the displacement of the mass center and the angular displacement of the car body, the motion equations for the half-car suspension model can be represented

as:

$$
\begin{aligned}
& m_s \ddot{z}_c(t) + k_{sf} \left[z_{sf}(t) - z_{uf}(t) \right] + c_{sf} \left[\dot{z}_{sf}(t) - \dot{z}_{uf}(t) \right] \\
& + k_{sr} \left[z_{sr}(t) - z_{ur}(t) \right] + c_{sr} \left[\dot{z}_{sr}(t) - \dot{z}_{ur}(t) \right] \\
= \; & u_f(t) + u_r(t), \\
& I_\varphi \ddot{\varphi}(t) - l_1 k_{sf} \left[z_{sf}(t) - z_{uf}(t) \right] - l_1 c_{sf} \left[\dot{z}_{sf}(t) - \dot{z}_{uf}(t) \right] \\
& + l_2 k_{sr} \left[z_{sr}(t) - z_{ur}(t) \right] + l_2 c_{sr} \left[\dot{z}_{sr}(t) - \dot{z}_{ur}(t) \right] \\
= \; & -l_1 u_f(t) + l_2 u_r(t), \\
& m_{uf} \ddot{z}_{uf}(t) - k_{sf} \left[z_{sf}(t) - z_{uf}(t) \right] - c_{sf} \left[\dot{z}_{sf}(t) - \dot{z}_{uf}(t) \right] + k_{tf} \left[z_{uf}(t) - z_{rf}(t) \right] \\
= \; & -u_f(t), \\
& m_{ur} \ddot{z}_{ur}(t) - k_{sr} \left[z_{sr}(t) - z_{ur}(t) \right] - c_{sr} \left[\dot{z}_{sr}(t) - \dot{z}_{ur}(t) \right] + k_{tr} \left[z_{ur}(t) - z_{rr}(t) \right] \\
= \; & -u_r(t). \tag{2.8}
\end{aligned}
$$

It can be seen from (2.6)–(2.8) that

$$
\begin{aligned}
\ddot{z}_{sf}(t) & = \ddot{z}_c(t) - l_1 \ddot{\varphi}(t) \\
& = a_1 \left\{ u_f(t) - k_{sf} \left[z_{sf}(t) - z_{uf}(t) \right] - c_{sf} \left[\dot{z}_{sf}(t) - \dot{z}_{uf}(t) \right] \right\} \\
& \quad + a_2 \left\{ u_r(t) - k_{sr} \left[z_{sr}(t) - z_{ur}(t) \right] - c_{sr} \left[\dot{z}_{sr}(t) - \dot{z}_{ur}(t) \right] \right\}, \\
\ddot{z}_{sr}(t) & = \ddot{z}_c(t) - l_2 \ddot{\varphi}(t) \\
& = a_2 \left\{ u_f(t) - k_{sf} \left[z_{sf}(t) - z_{uf}(t) \right] - c_{sf} \left[\dot{z}_{sf}(t) - \dot{z}_{uf}(t) \right] \right\} \\
& \quad + a_3 \left\{ u_r(t) - k_{sr} \left[z_{sr}(t) - z_{ur}(t) \right] - c_{sr} \left[\dot{z}_{sr}(t) - \dot{z}_{ur}(t) \right] \right\}, \quad (2.9)
\end{aligned}
$$

where

$$
a_1 = \frac{1}{m_s} + \frac{l_1^2}{I_\varphi}, \quad a_2 = \frac{1}{m_s} - \frac{l_1 l_2}{I_\varphi}, \quad a_3 = \frac{1}{m_s} + \frac{l_2^2}{I_\varphi}.
$$

To establish the state-space form, we define the following state variables: $x_1(t) = z_{sf}(t) - z_{uf}(t)$ is the suspension deflection of the front car body; $x_2(t) = z_{sr}(t) - z_{ur}(t)$ is the suspension deflection of the rear car body; $x_3(t) = z_{uf}(t) - z_{rf}(t)$ is the tyre deflection of the front car body; $x_4(t) = z_{ur}(t) - z_{rr}(t)$ is the tyre deflection of the rear car body; $x_5(t) = \dot{z}_{sf}(t)$ is the vertical velocity of the front car body; $x_6(t) = \dot{z}_{sr}(t)$ is the vertical velocity of the rear car body; $x_7(t) = \dot{z}_{uf}(t)$ is the vertical velocity of the front wheel; $x_8(t) = \dot{z}_{ur}(t)$ is the vertical velocity of

the rear wheel. After choosing the disturbance input $w\left(t\right) = \begin{bmatrix} \dot{z}_{rf}(t) \\ \dot{z}_{rr}(t) \end{bmatrix}$ and the variables as,

$$x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) & x_3^T(t) & x_4^T(t) & x_5^T(t) & x_6^T(t) & x_7^T(t) & x_8^T(t) \end{bmatrix}^T,$$
$$u(t) = \begin{bmatrix} u_f(t) \\ u_r(t) \end{bmatrix},$$

then we can express the dynamic equations in (2.8) and (2.9) as the following state-space form:

$$\dot{x}\left(t\right) = Ax\left(t\right) + Bu\left(t\right) + B_1 w\left(t\right), \tag{2.10}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -a_1 k_{sf} & -a_2 k_{sr} & 0 & 0 & -a_1 c_{sf} & -a_2 c_{sr} & a_1 c_{sf} & a_2 c_{sr} \\ -a_2 k_{sf} & -a_3 k_{sr} & 0 & 0 & -a_2 c_{sf} & -a_3 c_{sr} & a_2 c_{sf} & a_3 c_{sr} \\ \frac{k_{sf}}{m_{uf}} & 0 & -\frac{k_{tf}}{m_{uf}} & 0 & \frac{c_{sf}}{m_{uf}} & 0 & -\frac{c_{sf}}{m_{uf}} & 0 \\ 0 & \frac{k_{sr}}{m_{ur}} & 0 & -\frac{k_{tr}}{m_{ur}} & 0 & \frac{c_{sr}}{m_{ur}} & 0 & -\frac{c_{sr}}{m_{ur}} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & a_1 & a_2 & -\frac{1}{m_{uf}} & 0 \\ 0 & 0 & 0 & 0 & a_2 & a_3 & 0 & -\frac{1}{m_{ur}} \end{bmatrix}^T,$$

$$B_1 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \tag{2.11}$$

For the active suspension control design problem, it should be noticed that ride comfort, road holding ability and suspension deflection are three key performance characteristics to be considered. First of all, it is widely accepted that ride comfort is closely related to the vertical acceleration experienced by the car body. In this study, both the heave and the pitch accelerations are chosen as the performance output vector, namely,

$$z_1(t) = \begin{bmatrix} \ddot{z}_c\left(t\right) \\ \ddot{\varphi}\left(t\right) \end{bmatrix}.$$

To ensure that the designed controllers must have the capability of performing the suspension system, which is subject to the mechanical constraints of vehicle components and passenger comfort generation, the inequalities as below are

provided to guarantee the suspension deflection constraint

$$|z_{sf}(t) - z_{uf}(t)| \leq z_{f\,\max}, \quad |z_{sr}(t) - z_{ur}(t)| \leq z_{r\,\max}, \quad (2.12)$$

where $z_{f\,\max}$ and $z_{r\,\max}$ denote the maximum front and rear suspension deflection hard limits, respectively. Moreover, to ensure a firm uninterrupted contact of the wheels with the road, it is very reasonably to assume that the dynamic tire loads should not exceed the static tire loads for both the front and rear wheels:

$$|k_{sf}(z_{uf}(t) - z_{rf}(t))| \leq F_f, \quad |k_{sr}(z_{ur}(t) - z_{rr}(t))| \leq F_r, \quad (2.13)$$

where F_f and F_r stand for static tyre loads that can be calculated by

$$F_r(l_1 + l_2) = m_s g l_1 + m_{ur} g (l_1 + l_2), \quad (2.14)$$

$$F_f + F_r = (m_s + m_{uf} + m_{ur}) g. \quad (2.15)$$

The conditions in (2.12) and (2.13) are chosen as constraint output, the vehicle active suspension system can be rewritten as follows:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + Bu(t), \\
z_1(t) &= C_1 x(t) + D_1 u(t), \\
z_2(t) &= C_2 x(t),
\end{aligned}
\quad (2.16)
$$

where A, B_1 and B are defined in (2.10), and

$$
C_1 = \begin{bmatrix}
-\frac{k_{sf}}{m_s} & -\frac{k_{sr}}{m_s} & 0 & 0 & -\frac{c_{sf}}{m_s} & -\frac{c_{sr}}{m_s} & \frac{c_{sf}}{m_s} & \frac{c_{sr}}{m_s} \\
\frac{l_1 k_{sf}}{I_\varphi} & -\frac{l_2 k_{sr}}{I_\varphi} & 0 & 0 & \frac{l_1 c_{sf}}{I_\varphi} & -\frac{l_2 c_{sr}}{I_\varphi} & -\frac{l_1 c_{sf}}{I_\varphi} & \frac{l_2 c_{sr}}{I_\varphi}
\end{bmatrix},
$$

$$
D_1 = \begin{bmatrix}
\frac{1}{m_s} & \frac{1}{m_s} \\
-\frac{l_1}{I_\varphi} & \frac{l_2}{I_\varphi}
\end{bmatrix}, C_2 = \begin{bmatrix}
\frac{1}{z_{f\,\max}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{z_{r\,\max}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{k_{sf}}{F_f} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{k_{sr}}{F_r} & 0 & 0 & 0 & 0
\end{bmatrix}. (2.17)
$$

The half-vehicle model parameters are employed as shown in Table 2.2 for the case study (Du & Zhang, 2008).

Table 2.2: Systems parameter values for the half-vehicle suspension model

m_s	m_{uf}	k_{sf}	k_{tf}	c_{sf}	l_1
690 kg	40 kg	18000 N/m	200000 N/m	1000 Ns/m	1.3 m
I_φ	m_{ur}	k_{sr}	k_{tr}	c_{sr}	l_2
1222 kgm^2	45 kg	22000 N/m	200000 N/m	1000 Ns/m	1.5 m
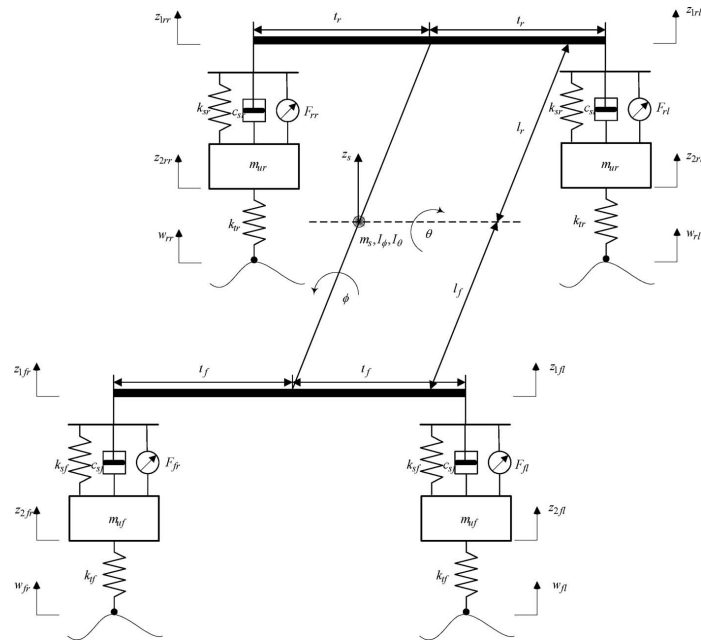
Figure 2.3: A full-vehicle model

2.1.3   A Full-vehicle Suspension Model

A full-car suspension model, as shown in Fig. 2.3 as (Du & Zhang, 2009), is considered in this subsection. This is a 7-DOF model where the sprung mass is assumed to be a rigid body with freedoms of motion in the vertical, pitch, and roll directions, and each unsprung mass has freedom of motion in the vertical direction. In Fig. 1, z_s is the vertical displacement at the center of gravity, θ and ϕ are the pitch and roll angles of the sprung mass, m_s, m_{uf}, and m_{ur} denote the sprung and unsprung masses, respectively, and I_θ and I_ϕ are pitch and roll moments of inertia. The front and rear displacements of the sprung mass on the left and right sides are denoted by z_{1fl}, z_{1fr}, z_{1rl}, and z_{1rr}. The front and rear displacements of the unsprung masses on the left and right sides are denoted by z_{2fl}, z_{2fr}, z_{2rl}, and z_{2rr}. The disturbances, which are caused by road irregularities, are denoted by w_{fl}, w_{fr}, w_{rl}, and w_{rr}. The front and rear suspension stiffnesses and the front and rear tyre stiffnesses are denoted by k_{sf}, k_{sr}, and k_{tf}, k_{tr}, respectively. The front and rear suspension damping coefficients are c_{sf} and c_{sr} . Four actuators are placed between the sprung mass and the unsprung masses to generate pushing forces, denoted by F_{fl}, F_{fr}, F_{rl}, and F_{rr}.

Assuming that the pitch angle θ and the roll angle ϕ are small enough, the following linear approximations are applied

$$
\begin{cases}
z_{1fl}\left(t\right) = z_s\left(t\right) + l_f\theta\left(t\right) + t_f\phi\left(t\right), \\
z_{1fr}\left(t\right) = z_s\left(t\right) + l_f\theta\left(t\right) - t_f\phi\left(t\right), \\
z_{1rl}\left(t\right) = z_s\left(t\right) - l_r\theta\left(t\right) + t_r\phi\left(t\right), \\
z_{1rr}\left(t\right) = z_s\left(t\right) - l_r\theta\left(t\right) - t_r\phi\left(t\right),
\end{cases}
\tag{2.18}
$$

and a kinematic relationship between $x_s\left(t\right)$ and $q(t)$ can be established as

$$
x_s\left(t\right) = L^T q(t), \tag{2.19}
$$

where

$$
\begin{aligned}
L &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ l_f & l_f & -l_r & -l_r \\ t_f & -t_f & t_r & -t_r \end{bmatrix}, \\
q(t) &= \begin{bmatrix} z_s\left(t\right) & \theta\left(t\right) & \phi\left(t\right) \end{bmatrix}^T, \\
x_s\left(t\right) &= \begin{bmatrix} z_{1fl}\left(t\right) & z_{1fr}\left(t\right) & z_{1rl}\left(t\right) & z_{1rr}\left(t\right) \end{bmatrix}^T.
\end{aligned}
$$

In terms of mass, damping, and stiffness matrices, the motion equations of the full-car suspension model can been formalized as

$$
\begin{aligned}
M_s \ddot{q}(t) &= LB_s(\dot{x}_u(t) - \dot{x}_s(t)) + LK_s(\dot{x}_u(t) - \dot{x}_s(t)) - LF(t), \\
M_u \ddot{x}_u(t) &= B_s(\dot{x}_s(t) - \dot{x}_u(t)) + K_s(\dot{x}_s(t) - \dot{x}_u(t)) + K_t(w(t) - x_u(t)) + F(t),
\end{aligned}
$$
$$ (2.20) $$

where

$$
\begin{aligned}
x_u(t) &= \begin{bmatrix} z_{2fl}(t) & z_{2fr}(t) & z_{2rl}(t) & z_{2rr}(t) \end{bmatrix}, \\
w(t) &= \begin{bmatrix} w_{fl}(t) & w_{fr}(t) & w_{rl}(t) & w_{rr}(t) \end{bmatrix}, \\
F(t) &= \begin{bmatrix} F_{fl}(t) & F_{fr}(t) & F_{rl}(t) & F_{rr}(t) \end{bmatrix},
\end{aligned}
$$

and the matrices are given as

$$
\begin{aligned}
M_s &= \operatorname{diag}\{ m_s \quad I_\theta \quad I_\phi \}, M_u = \operatorname{diag}\{ m_{uf} \quad m_{uf} \quad m_{ur} \quad m_{ur} \}, \\
B_s &= \operatorname{diag}\{ c_{sf} \quad c_{sf} \quad c_{sr} \quad c_{sr} \}, K_s = \operatorname{diag}\{ k_{sf} \quad k_{sf} \quad k_{sr} \quad k_{sr} \}, \\
K_t &= \operatorname{diag}\{ k_{tf} \quad k_{tf} \quad k_{tr} \quad k_{tr} \}.
\end{aligned}
$$

After substitute (2.19) into (2.20), one can have

$$ M_m \ddot{z}_m(t) + B_m \dot{z}_m(t) + K_m z_m(t) = K_{mt} w(t) + L_m F(t), \qquad (2.21) $$

where

$$
\begin{aligned}
z_m &= \begin{bmatrix} q(t) \\ x_u(t) \end{bmatrix}, M_m = \begin{bmatrix} M_s & 0 \\ 0 & M_u \end{bmatrix}, B_m = \begin{bmatrix} LB_s L^T & -LB_s \\ -B_s L^T & B_s \end{bmatrix}, \\
K_m &= \begin{bmatrix} LK_s L^T & -LK_s \\ -K_s L^T & K_s + K_t \end{bmatrix}, K_{mt} = \begin{bmatrix} 0 \\ K_t \end{bmatrix}, L_m = \begin{bmatrix} -L \\ I \end{bmatrix}.
\end{aligned}
$$

By setting $x_1(t) = z_m(t)$, $x_2(t) = \dot{z}_m(t)$ and $u(t) = F(t)$, we can develop the state-space form:

$$ \dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t), $$

where

$$
\begin{aligned}
x(t) &= \begin{bmatrix} z_m(t) \\ \dot{z}_m(t) \end{bmatrix}, A = \begin{bmatrix} 0 & I \\ -M_m^{-1} K_m & -M_m^{-1} B_m \end{bmatrix}, \\
B &= \begin{bmatrix} 0 \\ -M_m^{-1} L_m \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ -M_m^{-1} K_{mt} \end{bmatrix},
\end{aligned}
$$

The full-vehicle model parameters are employed as shown in Table 2.3 for the case study (Du & Zhang, 2009).

Table 2.3: Systems parameter values for the full-vehicle suspension model

m_s	m_{uf}	k_{sf}	k_{tf}
1400 kg	40 kg	23500 N/m	190000 N/m
c_{sf}	l_f	t_f	I_θ
1000 Ns/m	0.96 m	0.71 m	2100 kgm^2
m_{ur}	k_{sr}	k_{tr}	c_{sr}
40 kg	25500 N/m	190000 N/m	1100 Ns/m
l_r	t_r	I_ϕ	
1.44m	1.44 m	460 kgm^2	

2.1.4 Parameter Uncertainty Models in Vehicle Active Suspension Systems

When modelling the suspension systems, many uncertain factors such as the inaccuracies of model parameters and the errors of sensors and actuators should be considered as these uncertain factors degrade the vibration attenuation performance and safety during the driving process. In addition, the suspension model becomes an uncertain model because of vehicle inertial properties in the modelling process. Furthermore, an active suspension system has the ability to enhance vehicle dynamics by relaxing external impact such as road surface on vehicle travel comfort. In terms of its control design, uncertainty of vehicle sprung and unsprung masses such as its loading conditions should be taken into account to meet vehicle travel performance criteria. The following two main parameter uncertainties forms are used to present the vehicle active suspension systems with uncertainty.

2.1.4.1 Norm-bounded Parameter Uncertainty

Considering the following quarter-vehicle suspension systems with norm-bounded parameter uncertainties:

$$\dot{x}(t) = (A + \Delta A)\, x(t) + (B_1 + \Delta B_1)\, w(t) + (B + \Delta B)\, u(t), \qquad (2.22)$$

where the matrices ΔA, ΔB and ΔB_1 are unknown matrices representing time-varying parameter uncertainties, which are assumed to be of the following form:

$$\begin{bmatrix} \Delta A & \Delta B & \Delta B_1 \end{bmatrix} = L_1 F(t) \begin{bmatrix} E_A & E_B & E_{B_1} \end{bmatrix}, \tag{2.23}$$

where L_1, E_A, E_B and E_{B_1} are known constant real matrices of appropriate dimensions, and $F(t)$ is an unknown matrix function with Lebesgue-measurable elements satisfying $F^T(t) F(t) \leq I$.

The authors in (Zhao *et al.*, 2010a) investigated the control design problem for the semi-active seat suspension systems with norm-bounded parameter uncertainties. The delay-range-dependent Lyapunov function has been constructed to derive the existence conditions of the desired state-feedback controller.

On the other hand, for the active suspension system, actuator uncertainties exist in many cases, which can be caused by many factors such as buffer size in digital systems and aging of controller devices for active suspension systems. Taking advantage of the fact that the non-fragile feedback controller is insensitive to gain changes of feedback control, we construct the following state feedback controller,

$$u(t) = (K + \Delta K(t))x(t), \tag{2.24}$$

where K is to be designed for the objective of non-fragile control problem. In this paper, the controller uncertainty is considered in the following form:

$$\Delta K(t) = HF(t)E, \tag{2.25}$$

where H and E are known constant matrices with appropriate dimensions, and $F(t)$ is unknown matrix functions with the property $F^T(t) F(t) \leq I$.

An actuator uncertainty-existing non-fragile H_∞ controller was designed for a quarter-car active suspension system, providing the existence conditions for guaranteeing the systems controller performance in (Du *et al.*, 2003). Regarding the existing controller uncertainties, it is evident that the phenomenon exists in a stochastic way subject to contextual constraints.

2.1.4.2 Polytopic Parameter Uncertainty

By considering the modeling uncertainty, in this chapter we replace the matrices A, B, B_1, C_1, D_1 and C_2 with the matrices $A(\lambda)$, $B(\lambda)$, $B_1(\lambda)$, $C_1(\lambda)$, $D_1(\lambda)$ and $C_2(\lambda)$ in (2.5) as λ is used to denote uncertain parameter.

It is assumed that λ varies in a polytope of vertices $\lambda_1, \lambda_2, \ldots, \lambda_r$, i.e., $\lambda \in \Psi := Co\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, where the symbol Co denotes the convex hull, and thus we have

$$\Phi \triangleq (A(\lambda), \quad B(\lambda), \quad B_1(\lambda), \quad C_1(\lambda), \quad C_2(\lambda), \quad D_1(\lambda)) \in \Psi,$$

where Ψ is a given convex bounded polyhedral domain described by r vertices:

$$\Psi \triangleq \left\{ \Phi \,\middle|\, \Phi = \sum_{i=1}^{r} \lambda_i \Phi_i; \quad \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \geq 0 \right\} \tag{2.26}$$

with $\Phi_i \triangleq (A_i, \quad B_i, \quad B_{1i}, \quad C_{1i}, \quad C_{2i}, \quad D_{1i})$ denoting the vertices of the polytope. The uncertain suspension systems with actuator delay can be expressed as:

$$
\begin{aligned}
\dot{x}(t) &= A(\lambda)x(t) + B(\lambda)u(t) + B_1(\lambda)w(t), \\
z_1(t) &= C_1(\lambda)x(t) + D_1(\lambda)u(t), \\
z_2(t) &= C_2(\lambda)x(t).
\end{aligned}
\tag{2.27}
$$

In terms of its control design, uncertainty of vehicle sprung and unsprung masses such as its loading conditions should be taken into account to meet vehicle travel performance criteria. The polytopic parameter uncertainties was employed to model the varying vehicle sprung or unsprung masses in the quarter-vehicle suspension systems (Du *et al.*, 2008; Gao *et al.*, 2006, 2010a). The parameter-dependent controllers was proposed for the quarter-car suspension systems with sprung mass variation (Du *et al.*, 2008). The parameter-independent sampled-data H_∞ controller design strategy was presented to handle both sprung and unsprung mass variations in a case study of a quarter-car suspension system (Gao *et al.*, 2010a).

2.2 Review of H_∞ Control for Vehicle Active Suspension Systems

With the development of H_∞ control theory (Kwakernaak, 1993) and linear matrix inequality (LMI) Toolbox (Boyd *et al.*, 1994; Gahinet *et al.*, 2002), LMI-based H_∞ control methods have been extensively investigated in the past decades (Gahinet & Apkarian, 1994; Scherer *et al.*, 1997; Xie, 1996). We introduce the disturbance attenuation control theory first and then review the corresponding H_∞ control approaches for active suspensions systems.

The H_∞ control problem is one of disturbance rejection. Specifically, it consists of minimizing the closed-loop root-mean-square (RMS) gain from the disturbance w to the output z in the control loop of Figure 2.4. This can be interpreted as minimizing the effect of the worst-case disturbance w on the output z. The H_∞ norm of a stable transfer function $\|T_{zw}\|_\infty$ is its largest output RMS gain, i.e.,

$$\|T_{zw}\|_\infty = \sup_{w \in L_2} \frac{\|z\|_\infty}{\|w\|_\infty} \tag{2.28}$$

where L_2 is the space of signals with finite energy.
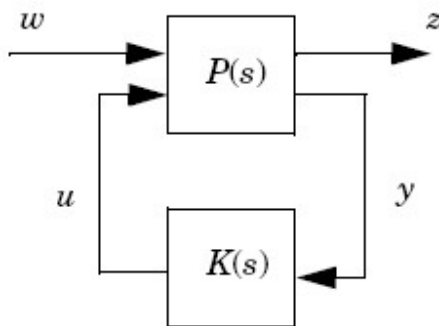
Figure 2.4: H_∞ control

A wide spectrum of active suspension control methods, very recently, have been proposed to address the trade-off between conflicting performance by utilizing different control techniques such as fuzzy logic and neural network control (Al-Holou *et al.*, 2002), gain scheduling control (Sivrioglu & Cansever, 2009), linear optimal control (ElMadany & Abduljabbar, 1999), adaptive control (Fialho

& Balas, 2002) and H_∞ control (Chen & Guo, 2005; Gao *et al.*, 2010a). Among the existing methods, it is evident that H_∞ control strategy for active suspension systems can lead to feasible solutions to manage the trade-off by compromising the requirements being achieved for the better combination performances (Chen & Guo, 2005). Therefore, there is a growing interest in employing the strategy to overcome the problem, some research work has been reported in the literature (Chen & Guo, 2005).

During the last few years, active H_∞ control strategies for vehicle suspensions were intensively investigated in the context of robustness and disturbance attenuation (Park & Kim, 1999; Tuan *et al.*, 2001; Yamashita *et al.*, 1994). It can be observed from these approaches that a uniform point is that all requirements, including those associated with hard constraints, are weighted and formulated in a single objective functional, which is minimized to find an optimal controller. When road conditions are unavailable or seriously bad, the weights are fixed and may be chosen such that the hard constraints are satisfied, i.e., the suspension stroke limitation is not exceeded, and the wheels have a firm contact to road, which may make the controller not obtain the best of suspension stroke to enhance ride comfort (Fialho & Balas, 2002; Lin & Kanellakopoulos, 1997). (Lin & Kanellakopoulos, 1997) utilized the backstepping technique and (Fialho & Balas, 2002) used the linear parameter-varying technique and backstepping method to study the active control design problems. These two papers showed that good road holding and limiting suspension stroke within bounds are naturally time-domain hard constraints that require variables to be within given bounds, a minimum is not necessary here. Therefore, formulating all different requirements in a single objective functional and minimizing the single objective function may result in conservatism. Moreover, the authors (Hrovat, 1997) provided a detailed discussion on weights and achievable performance for a quarter-car suspension model and showed that it is very difficult to choose appropriate and possibly frequency-dependent weights to manages the trade-off between conflicting requirements in single objective approaches.

2.2.1 State-feedback Control Method

Many researchers investigated the H_∞ control design problem for active suspension systems under the assumption that all state variables are measurable in (Chen & Guo, 2005; Du & Zhang, 2007; Du *et al.*, 2008; Gao *et al.*, 2010a), where the state-feedback control method was exploited to consider this problem. The authors in (Chen & Guo, 2005) proposed a constrained H_∞ control scheme for active suspensions with output and control constraints. In this reference, the authors considered the suspension performance ride comfort and time-domain constraints good road holding, suspension stroke limitation and avoidance of actuator saturation. By using LMI approach, a state-feedback controller is designed to improve suspension performance. The authors in (Gao *et al.*, 2010a) developed parameter-independent sampled-data H_∞ controller design strategy for a quarter-car suspension system by using state-feedback method. (Du & Zhang, 2007; Du *et al.*, 2008) investigate the state-feedback control problem for the quarter-vehicle active suspension systems with actuator delay.

2.2.2 Output-feedback Control Method

With the different road situations and loads, the state information may be unmeasurable. The state feedback H_∞ controller design methods for vehicle active suspension systems are not feasible. Recently, H_∞ output feedback controller design results for the active suspension systems have been reported in (Hayakawa *et al.*, 1999; Sun & Chen, 2003; Thompson & Davis, 1988; Wang & Wilson, 2001). In (Sun & Chen, 2003), the authors considered the output-feedback control for half-vehicle suspension systems via LMI optimization method. In (Wang & Wilson, 2001), the authors exploited the LMI method to solve the output-feedback control problem for the active suspension systems, in which the pole placement problem was considered. In addition, this paper (Wang & Wilson, 2001) applied multi-objective control framework to the vehicle active suspension.

2.2.3   Multi-objective Control Method

Recently, multi-objective control methods (Scherer, 2000) for active suspension systems have been presented in (Chen & Guo, 2001, 2005; Chen *et al.*, 2003; Gao *et al.*, 2006; Wang & Wilson, 2001). In these proposed approaches, the performance is used to measure ride comfort so that more general road disturbances than white noise can be considered and can be minimized to enhance ride comfort. In addition, the other suspension performance can be guaranteed by using hard constraints, in which the concept of reachable sets is defined by a quadratic storage function in a state-pace ellipsoid. These works made the contributions to enhance ride comfort under the hard constraints that keep the time-domain variables within bounds.

2.3   Control Design for Actuator Imperfect Information

2.3.1   Actuator Delay

It is well-known that actuator delays are often encountered in many control systems due to the electrical and electromagnetic characteristics of the actuators and transmission of the measurement information. The systems using electro-hydraulic actuators to track the desired forces may be inevitably involved with actuator delay. The presence of actuator delay, if not taken into account in the controller deign, may degrade the control performances and even cause instability in the resulting control systems. In view of this, the researchers have paid increasing attention to the problems of stability analysis and controller synthesis for the active suspension systems with actuator delay (Du & Zhang, 2007, 2008; Du *et al.*, 2008).

The authors in (Du & Zhang, 2007, 2008; Du *et al.*, 2008) considered the actuator time delay in the controller design process in order to design a controller that can stabilize the system and guarantee the closed-loop performance in spite of the existence of time delay. In detail, by using the Moon's inequality method,

existence conditions of H_∞ controller for active suspension systems with quarter-car and half-car were developed in (Du & Zhang, 2007, 2008). The parameter-dependent H_∞ control problem has been investigated in (Du $et\ al.$, 2008) for active suspension systems that consider both vehicle inertial parameter variations and actuator time delays. It should be pointed out that the actuator delays considered in (Du & Zhang, 2007, 2008; Du $et\ al.$, 2008) are constant delays, which can be covered by time-varying delays. On the other hand, with the different road situations and loads, the state information may be unmeasurable. The state feedback H_∞ controller design methods for vehicle active suspension systems with actuator delay are not feasible. There are few results on output feedback H_∞ control for vehicle active suspension systems with actuator delay.

2.3.2 Actuator Fault

With the growing complexity of automated control systems and actuators, various faults are likely to be encountered, especially actuator and sensor faults (Chen & Liu, 2004; Jiang $et\ al.$, 2006; Liao $et\ al.$, 2002; Selmic $et\ al.$, 2006; Shi $et\ al.$, 2003; Veillette $et\ al.$, 2002; Wang $et\ al.$, 1999; Yang $et\ al.$, 2001b, 2002; Zhang $et\ al.$, 2004). Therefore, it is important to design a fault-tolerant controller such that the system is stable and the performance of the closed-loop system can be guaranteed in the presences of sensor and actuator faults, which motivates the interests in the fault tolerant control system design. The objective of the fault-tolerant controller is to prevent the faults in the control loop from causing an overall system failure.

During the past few decades, many researchers have paid considerable attention to the reliable and fault tolerant control problems for dynamic systems and a great number of theoretic results have also been presented, see e.g. (Dong $et\ al.$, 2010; Ma $et\ al.$, 2010; Mao $et\ al.$, 2010; Wang & Qiao, 2004; Wang $et\ al.$, 2009b; Yang $et\ al.$, 2009; Zuo $et\ al.$, 2010). For example, (Yang $et\ al.$, 2001b) investigated fault-tolerant H_∞ controller design problem for linear systems, and the fault-tolerant controller was designed such that the resulting control systems are tolerant in that they provide guaranteed asymptotic stability and H_∞ performance when all control components (i.e., sensors and actuators) are operational

and when some control components experience failures. In addition, Wang and his group in (Dong *et al.*, 2010; Wang *et al.*, 2009b) dealt with the fault-tolerant control problem for the systems with sensor faults being modelled by the probabilistic distributions. In particular, the reliable H_∞ control problem of seat suspension systems with actuator faults is handled in (Zhao *et al.*, 2010c), where the considered actuator fault was described to be static behavior.

2.4 Fuzzy Control of Vehicle Active Suspension Systems

Since fuzzy sets were proposed by Zadeh (Zadeh, 1965), fuzzy logic control has developed into a conspicuous and successful branch of automation and control theory. During the last two decades, it has been well known that the T-S fuzzy model is very effective in representing complex nonlinear systems (Feng, 2006; Lin *et al.*, 2007; Sugeno, 1985; Tanaka & Wang, 2001). These kinds of systems are described as a weighted sum of some simple linear subsystems, and thus are easily analyzable. Consequently, over the past decades, there have been a great number of significant results on the stability analysis and controller synthesis problems for T-S fuzzy systems and various techniques have been obtained during the past decades (Cao & Frank, 2002; Chen *et al.*, 2008; Dong *et al.*, 2009; Dong & Yang, 2008; Gao *et al.*, 2009; Lam & Narimani, 2009; Nguang & Shi, 2003; Wang *et al.*, 2004; Wu & Li, 2007; Xu & Lam, 2005; Zhang & Xu, 2009; Zhou *et al.*, 2005).

Over the past years, some works about the fuzzy controller design for suspension systems have been reported, for example, (Al-Holou *et al.*, 2002; Cao *et al.*, 2010; Du & Zhang, 2009; Huang & Lin, 2003a; Kuo & Li, 1999; Rao & Prahlad, 1997; Yagiz *et al.*, 2008). In (Rao & Prahlad, 1997), a fuzzy-logic-based controller for vehicle-active suspension was designed to reduce the vehicle vibration and disturbance and to enhance comfort in riding faced with uncertain road terrains. The authors in (Kuo & Li, 1999) proposed a genetic-algorithm-based fuzzy proportional-plus-integral-proportional-plus-derivative (PI/PD) controller for an automotive active suspension system. With the different road conditions, the

fuzzy PI- and PD-type controllers with genetic-algorithm were designed. In (Al-Holou *et al.*, 2002), the authors designed a robust intelligent nonlinear controller for active suspension systems based on a comprehensive and realistic nonlinear model. In detail, the authors mixed sliding mode control, fuzzy logic control and neural network control methodologies to deal with complex uncertain suspension systems. In (Al-Holou *et al.*, 2002), in order to enhance the ride and comfort, a sliding mode neural network inference fuzzy logic controller was designed for automotive suspension systems. The authors in (Cao *et al.*, 2010) proposed a novel interval type-2 fuzzy controller to resolve nonlinear control problems of vehicle active suspension systems. By considering the Takagi-Sugeno (T-S) fuzzy model, interval type-2 fuzzy reasoning, the Wu-Mendel uncertainty bound method, and optimization algorithms together, the authors in (Cao *et al.*, 2010) constructed the switching routes between generated linear model control surfaces. In (Yagiz *et al.*, 2008), a robust fuzzy sliding-mode controller were proposed for a nonlinear half-car active suspensions system. The sliding-mode control method was combined with a single-input-single-output fuzzy logic controller to improve its performance. In (Lian *et al.*, 2005), the authors designed a self-organizing fuzzy controller for an active suspension system to evaluate its control performance. The authors improved self-organizing fuzzy-control approach to improve the control performance of the system, while reduce the time consumed to establish a suitable fuzzy rule table, and support practically convenient fuzzy-controller applications in an active suspension control system. The authors in (Du & Zhang, 2009) presented T-S model-based fuzzy control design approach for electrohydraulic active suspension systems considering nonlinear dynamics of the actuator, sprung mass variation, and constraints on the control input. The authors used the T-S fuzzy model to represent the nonlinear uncertain electro-hydraulic suspension and applied parallel distributed compensation method to build the fuzzy controller. The sufficient conditions for the existence of fuzzy controller were obtained in terms of LMIs.

2.5 Adaptive Sliding Mode Control of Vehicle Active Suspension Systems

It has been widely accepted that sliding mode control method is an effective robust control strategy for the nonlinear systems and can be successfully applied to a wide variety of practical engineering systems such as robot manipulators (Feng *et al.*, 2002), aircrafts (Jafarov & Tasaltin, 2000), underwater vehicles (Healey & Lienard, 1993) and suspension systems (Chen & Huang, 2008; Kim & Ro, 1998; Sam *et al.*, 2004; Yagiz & Yuksek, 2001; Yoshimura *et al.*, 2001). The main idea of sliding mode control is to utilize a discontinuous control to force the system state trajectories to some predefined sliding surfaces on which the system has desired properties such as stability, disturbance rejection capability, and tracking ability.

Recently, sliding mode control has received attention since it has various attractive features such as fast response, good transient performance, order-reduction and so on (Edwards & Spurgeon, 1998; Feng *et al.*, 2009; Ho & Niu, 2007; Niu *et al.*, 2005, 2007; Utkin, 1993; Wang *et al.*, 2009a; Yu & Kaynak, 2009). The authors in (Wu *et al.*, 2006) designed an adaptive sliding mode controller for uncertain nonlinear state-delayed systems under H_∞ performance. Recently, the sliding mode controller design problems have been extensively investigated for nonlinear suspension systems in (Chen & Huang, 2008, 2005, 2006; Kim & Ro, 1998; Sam *et al.*, 2004; Yagiz & Yuksek, 2001; Yoshimura *et al.*, 2001). Kim & Ro (1998) investigated the sliding mode control for a quarter-vehicle nonlinear active suspension system. The authors considered the presence of non-linearities such as a hardening spring, a quadratic damping force and the tyre lift-off phenomenon in the suspension system. A sliding mode controller was designed to improve the suspension performances. In this study, a linear seven degrees of freedom vehicle model is used in order to design and check the performance of sliding mode controlled active suspensions. Force actuators are mounted as parallel to the four suspensions and a non-chattering control is realized. Sliding mode control is preferred because of its robust character since any change in vehicle parameters should not affect the performance of the active suspensions. Improvement in ride comfort is aimed by decreasing the amplitudes of motions of vehicle body. In

(Yagiz & Yuksek, 2001), sliding mode controller was designed to check the performance of a linear seven degrees of freedom vehicle model In (Chen & Huang, 2005), an adaptive sliding controller was designed for a non-autonomous quarter-car suspension system with time-varying loadings. The Lyapunov direct method was be used in (Chen & Huang, 2005) to find adaptive laws for updating coefficients in the approximating series and to prove stability of the closed-loop system. The authors in (Chen & Huang, 2006) used adaptive sliding mode control method to deal with the active control for the nonlinear quarter-car active suspension systems with hydraulic actuator where was assumed to have some time-varying uncertainties with unknown bounds.

In the past few years, the authors in (Huang & Chen, 2006; Huang & Lin, 2003b; Lin *et al.*, 2009; Yagiz *et al.*, 2008) considered the fuzzy sliding mode control design problems for the suspension systems. In (Huang & Lin, 2003b), the authors proposed an adaptive fuzzy sliding mode controller to suppress the sprung mass position oscillation in the nonlinear suspension systems with hydraulic actuator. The intelligent control strategy mixed an adaptive rule with fuzzy and sliding mode control algorithms and had online learning capability to handle the system time-varying and nonlinear uncertainty behaviors, and adjust the control rules parameters. Based on the results and methods proposed in (Huang & Lin, 2003b), the authors in (Huang & Chen, 2006) further investigated adaptive fuzzy sliding controller for the nonlinear suspension systems with hydraulic actuator. This control method used the functional approximation technique to establish the unknown function for releasing the model-based requirement. Furthermore, a fuzzy scheme with online learning ability was introduced to compensate the functional approximation error for improving the control performance and reducing the implementation difficulty. In (Yagiz *et al.*, 2008), a robust fuzzy sliding-mode controller was designed for a nonlinear half-car active suspensions system with nonlinear spring and piecewise linear damper with dry friction. This control method mixed adaptive sliding mode controller and a single-input-single-output fuzzy logic controller to improve the suspension performances. (Lin *et al.*, 2009) designed a fuzzy sliding mode controller to control a nonlinear active suspension system and evaluated its control performance.

2.6 Summary

Robust control approaches are required due to the real-time, external disturbance and uncertain properties of vehicle active suspension systems. This chapter provided an account of state of the art of robust control of active suspension systems with uncertainty. Through the above observation, this thesis will propose research methods and focus on how to close the gaps in current researches. Research challenges are identified and enumerated as below.

1. This thesis takes into account the actuator time-varying delay for vehicle active suspension systems. The resulting control systems model is more general than the existing ones. For the measurable state variable, a novel state-feedback robust controller is designed to guarantee the stability of the systems and improve suspension performances for uncertain vehicle active suspension systems with actuator time-varying delay. For the unmeasurable state variable, this thesis construct a new type dynamic output-feedback controller and synthesis the controller design for vehicle active suspension systems with actuator time-varying delay via new techniques.

2. This thesis is concerned with fault-tolerant H_∞ controller design for vehicle active suspension systems with actuator faults. In this thesis, we regard the actuator failure process as stochastic behavior, which can be modeled by a continuous-time homogeneous Markov process. By using stochastic stability theory, a fault-tolerant H_∞ controller is designed such that the resulting control system is tolerant in the sense that it guarantees asymptotic stability and H_∞ performance, and simultaneously satisfies the constraint performance with possible actuator failures.

3. This thesis presents a new approach to design fuzzy control for vehicle active suspension systems with uncertainty. By building T-S fuzzy model to represent the uncertain active suspension systems, the LMI-based reliable fuzzy controller conditions are derived to ensure that the resulting T-S fuzzy system is asymptotically stable with H_∞ performance, and satisfy the constraint performance simultaneously.

4. This thesis focuses on the problem of adaptive sliding mode H_∞ control for a nonlinear uncertain active suspension system under the framework of multi-objective control. We model the corresponding nonlinear uncertain system by considering the variations of the sprung mass, the front and rear unsprung masses, the nonlinear actuator dynamics and the suspension performances. This control design process is different from the existing sliding mode control methods as the suspension performances are considered and the T-S fuzzy model approach is utilized to represent the nonlinear uncertain suspension system by T-S fuzzy system. A novel adaptive sliding mode controller is designed for the resulting closed-loop systems.

Chapter 3

Robust H_∞ Control for Active Suspensions Systems with Actuator Time-varying Delay

3.1 Introduction

It is well-known that actuator delays are often encountered in many control systems due to the electrical and electromagnetic characteristics of the actuators and transmission of the measurement information. The systems using electrohydraulic actuators to track the desired forces may be inevitably involved with input delay. The presence of input delay, if not taken into account in the controller deign, may degrade the control performances and even cause instability in the resulting control systems. In view of this, many stability analysis and controller design schemes have been presented for linear systems with delay or input delay (Han, 2005; He *et al.*, 2004; Li *et al.*, 2008; Mou *et al.*, 2008; Shi *et al.*, 2002; Wang *et al.*, 2010; Wu *et al.*, 2009; Zhang *et al.*, 2007). More recently, the problems of stability analysis and controller synthesis for the active suspension systems with quarter model and half model, and seat suspension systems with input delay have been investigated in (Du & Zhang, 2007, 2008; Gao *et al.*, 2010b) respectively, among which there exist two main approaches to deal with the input problem. One is to design a controller using the integrated system model where the actuator dynamics are included (Thompson & Davis, 2001). The other is to

design a controller by considering the actuator time delay in the controller design process (Du & Zhang, 2007).

Over the past years, the authors in (Chen *et al.*, 2005; Du & Zhang, 2008; Giua *et al.*, 2000; Lin & Huang, 2003; Wu *et al.*, 2005) investigated the control problems via different methods for a half-car active suspension system. Specially, the authors in (Du & Zhang, 2008) investigated H_∞ controller design for the suspension model with input time delay. And the input delay was assumed to be uncertain time invariant within a known constant upper bound. Based on the Moon's inequality method, delay-dependent H_∞ control conditions have been developed in terms of linear matrix inequalities (LMIs) (Molina-Cristobal *et al.*, 2006). However, the parameter uncertainties, which are caused by vehicle load vibration and affect the performances of the control systems, have not been considered in (Du & Zhang, 2008). In addition, in practice, it should be noted that the actuator delays are always time-varying and belong to a given interval, and the lower bound of the delays may not be zero. To the best of author's knowledge, few results on robust H_∞ control of uncertain half-vehicle active suspension system with actuator time-varying delay have been developed, which motivated this presence.

3.2 State-feedback Control for Active Suspensions Systems with Actuator Time-varying Delay

3.2.1 Problem Formulation

For uncertain half-car active suspension systems with time-varying actuator delay, this section firstly sets up the model for the control design aim. By considering the modeling uncertainty, in this chapter we replace the matrices A, B, B_1, C_1, D_1 and C_2 with the matrices $A(\lambda)$, $B(\lambda)$, $B_1(\lambda)$, $C_1(\lambda)$, $D_1(\lambda)$ and $C_2(\lambda)$ in (2.11) and (2.17) as λ is used to denote uncertain parameter.

It is assumed that λ varies in a polytope of vertices $\lambda_1, \lambda_2, \ldots, \lambda_r$, i.e., $\lambda \in \Psi := Co\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, where the symbol Co denotes the convex hull, and thus

we have

$$\Phi \triangleq (A(\lambda), \quad B(\lambda), \quad B_1(\lambda), \quad C_1(\lambda), \quad C_2(\lambda), \quad D_1(\lambda)) \in \Psi,$$

where Ψ is a given convex bounded polyhedral domain described by r vertices:

$$\Psi \triangleq \left\{ \Phi \,\middle|\, \Phi = \sum_{i=1}^{r} \lambda_i \Phi_i; \quad \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \geq 0 \right\} \tag{3.1}$$

with $\Phi_i \triangleq (A_i, \quad B_i, \quad B_{1i}, \quad C_{1i}, \quad C_{2i}, \quad D_{1i})$ denoting the vertices of the polytope. The uncertain suspension systems with actuator delay can be expressed as:

$$\begin{aligned}
\dot{x}(t) &= A(\lambda)x(t) + B(\lambda)u\left(t - d(t)\right) + B_1(\lambda)w(t), \\
z_1(t) &= C_1(\lambda)x(t) + D_1(\lambda)u\left(t - d(t)\right), \\
z_2(t) &= C_2(\lambda)x(t).
\end{aligned} \tag{3.2}$$

$d(t)$ is time-varying delay which does not require to know its derivative information, and $d(t)$ satisfies $\eta_m \leq d(t) \leq \eta_M$. It is assumed that the state variables are on-line measurable. Then, we consider the state feedback controller as

$$u(t) = Kx(t). \tag{3.3}$$

Under the controller (3.3), the system in (3.2) can be transformed into the following system:

$$\begin{aligned}
\dot{x}(t) &= A(\lambda)x(t) + B(\lambda)Kx\left(t - d(t)\right) + B_1(\lambda)w(t), \\
z_1(t) &= C_1(\lambda)x(t) + D_1(\lambda)Kx\left(t - d(t)\right), \\
z_2(t) &= C_2(\lambda)x(t).
\end{aligned} \tag{3.4}$$

It is assumed that $w \in L_2[0, \infty)$, and without loss of generality, we have $\|w\|_2^2 \leq w_{\max} < \infty$. Then, the objective of this section is to determine a controller gain K such that

(1) the closed-loop system is asymptotically stable;

(2) under zero initial condition, the closed-loop system guarantees that $\|z_1\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0, \infty)$, where $\gamma > 0$ is a prescribed scalar; the following control output constraints are guaranteed:

$$|\{z_2(t)\}_q| \leq \{z_{2,\max}\}_q, \quad q = 1, 2, 3, 4, \quad t > 0, \tag{3.5}$$

where $z_{2,\max} = [z_{f\max}, z_{r\max}, 1, 1]^T$.

3.2.2 Robust H_∞ Controller Design

In this subsection, for a given control gain matrix K, we will develop robust H_∞ performance analysis criterion by constructing the novel Lyapunov functional and using some advanced techniques. Then, based on the presented condition, the existence condition of H_∞ controller design will be given. First of all, we have the following proposition.

Proposition 3.1 Consider the closed-loop system in (3.4). For given scalars $\gamma > 0$, $\eta_m > 0$, $\eta_M > 0$ and a matrix K, the closed-loop system (3.4) is robustly asymptotically stable with an H_∞ disturbance attenuation level γ, if there exist matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $S_1 > 0$, $S_2 > 0$, X_j, Y_j, and M_j $(j = 1, 2, \ldots, r)$ with appropriate dimensions such that the following LMIs hold:

$$\Sigma_{\varsigma ii} < 0, \qquad \varsigma = 1, 2, \tag{3.6}$$

$$\Sigma_{\varsigma ij} + \Sigma_{\varsigma ji} < 0, \quad i < j, \quad i, j = 1, 2, \ldots, r, \tag{3.7}$$

where

$$\Sigma_{1ij} = \begin{bmatrix} \Omega_{ij} & \sqrt{\frac{\eta_m}{m}} X_j & \sqrt{\eta_M - \eta_m} Y_j \\ * & -S_1 & 0 \\ * & * & -S_2 \end{bmatrix}, \tag{3.8}$$

$$\Sigma_{2ij} = \begin{bmatrix} \Omega_{ij} & \sqrt{\frac{\eta_m}{m}} X_j & \sqrt{\eta_M - \eta_m} N_j \\ * & -S_1 & 0 \\ * & * & -S_2 \end{bmatrix}, \tag{3.9}$$

$$\Omega_{ij} = \Theta_{ij} + W_{z_{1i}}^{\mathbf{T}} W_{z_{1i}} - \gamma^2 W_w^{\mathbf{T}} W_w,$$

with m is the number of delay partitioning,

$$
\begin{aligned}
\Theta_{ij} &= \operatorname{sym}\{W_{A_i}^{\mathbf{T}}PW_B + Z_jW_Z\} + W_{Q_1}^{\mathbf{T}}\check{Q}_1W_{Q_1} + W_{Q_2}^{\mathbf{T}}\check{Q}_2W_{Q_2} \\
&\quad + W_{A_i}^{\mathbf{T}}\left[\frac{\eta_m}{m}S_1 + (\eta_M - \eta_m)S_2\right]W_{A_i}, \\
W_{A_i} &= \begin{bmatrix} A_i & 0_{n,(m+1)n} & B_iK & B_{1i} \end{bmatrix}, \quad W_B = \begin{bmatrix} I_n & 0_{n,(m+2)n+1} \end{bmatrix}, \\
Z_j &= \begin{bmatrix} X_j & Y_j & M_j \end{bmatrix}, \quad W_{z_{1i}} = \begin{bmatrix} C_{1i} & 0_{1,(m+1)n} & D_{1i}K & 0 \end{bmatrix}, \\
W_w &= \begin{bmatrix} 0_{1,(m+3)n} & 1 \end{bmatrix}, \quad W_Z = \begin{bmatrix} I_n & -I_n & 0_{n,(m+1)n+1} \\ 0_{n,mn} & I_n & 0_n & -I_n & 0_{n,1} \\ 0_{n,(m+1)n} & -I_n & I_n & 0_{n,1} \end{bmatrix}, \\
\check{Q}_1 &= \begin{bmatrix} Q_1 & 0 \\ * & -Q_1 \end{bmatrix}, \quad W_{Q_1} = \begin{bmatrix} I_{mn} & 0_{mn,3n+1} \\ 0_{mn,n} & I_{mn} & 0_{mn,2n+1} \end{bmatrix}, \\
\check{Q}_2 &= \begin{bmatrix} Q_2 & 0 \\ * & -Q_2 \end{bmatrix}, \quad W_{Q_2} = \begin{bmatrix} 0_{n,mn} & I_n & 0_{n,2n+1} \\ 0_{n,(m+1)n} & I_n & 0_{n,n+1} \end{bmatrix}.
\end{aligned}
$$

Proof. To obtain a less conservative criterion, we first represent the time delay $d(t)$ as two parts: constant part η_m and time-varying part $\eta(t)$, that is,

$$
d(t) = \eta_m + \eta(t), \tag{3.10}
$$

where

$$
0 \leq \eta(t) \leq \eta_M - \eta_m.
$$

Then, it can be observed from (3.4) that

$$
\begin{aligned}
\dot{x}(t) &= A(\lambda)x(t) + B_1(\lambda)w(t) + B(\lambda)Kx(t - \eta_m - \eta(t)), \\
z_1(t) &= C_1(\lambda)x(t) + D_1(\lambda)Kx(t - \eta_m - \eta(t)), \\
z_2(t) &= C_2(\lambda)x(t). \tag{3.11}
\end{aligned}
$$

The following proof is divided into twofold: we first show that system (3.4) is robustly asymptotically stable with $w(t) = 0$ and then H_∞ performance index is satisfied. Now, consider the Lyapunov-Krasovskii functional as follows:

$$
V(t) = V_1(t) + V_2(t) + V_3(t), \tag{3.12}
$$

where

$$
\begin{aligned}
V_1(t) &= x^{\mathbf{T}}(t) P x(t), \\
V_2(t) &= \int_{t-\frac{\eta_m}{m}}^{t} \Upsilon^{\mathbf{T}}(s) Q_1 \Upsilon(s) \, \mathrm{d}s + \int_{t-\eta_M}^{t-\eta_m} x^{\mathbf{T}}(s) Q_2 x(s) \, \mathrm{d}s, \\
V_3(t) &= \int_{-\frac{\eta_m}{m}}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathbf{T}}(s) S_1 \dot{x}(s) \, \mathrm{d}s \mathrm{d}\theta \\
&\quad + \int_{-\eta_M}^{-\eta_m} \int_{t+\theta}^{t} \dot{x}^{\mathbf{T}}(s) S_2 \dot{x}(s) \, \mathrm{d}s \mathrm{d}\theta
\end{aligned}
$$

with

$$
\Upsilon^{\mathbf{T}}(s) = \begin{bmatrix} x^{\mathbf{T}}(s) & x^{\mathbf{T}}\left(s - \frac{1}{m}\eta_m\right) & x^{\mathbf{T}}\left(s - \frac{2}{m}\eta_m\right) & \cdots & x^{\mathbf{T}}\left(s - \frac{m-1}{m}\eta_m\right) \end{bmatrix}.
$$

(3.13)

Then, the derivative of $V(t)$ along the solution of system in (3.4) is given by

$$
\begin{aligned}
\dot{V}_1(t) &= 2x^{\mathbf{T}}(t) P \dot{x}(t), \\
\dot{V}_2(t) &= \Upsilon^{\mathbf{T}}(t) Q_1 \Upsilon(t) - \Upsilon^{\mathbf{T}}\left(t - \frac{\eta_m}{m}\right) Q_1 \Upsilon\left(t - \frac{\eta_m}{m}\right) \\
&\quad + x^{\mathbf{T}}(t - \eta_m) Q_2 x(t - \eta_m) - x^{\mathbf{T}}(t - \eta_M) Q_2 x(t - \eta_M), \\
\dot{V}_3(t) &= \dot{x}^{\mathbf{T}}(t) \left[\frac{\eta_m}{m} S_1 + (\eta_M - \eta_m) S_2\right] \dot{x}(t) \\
&\quad - \int_{t-\frac{\eta_m}{m}}^{t} \dot{x}^{\mathbf{T}}(s) S_1 \dot{x}(s) \, \mathrm{d}s - \int_{t-\eta_M}^{t-\eta_m} \dot{x}^{\mathbf{T}}(s) S_2 \dot{x}(s) \, \mathrm{d}s \\
&= \dot{x}^{\mathbf{T}}(t) \left[\frac{\eta_m}{m} S_1 + (\eta_M - \eta_m) S_2\right] \dot{x}(t) \\
&\quad - \int_{t-\frac{\eta_m}{m}}^{t} \dot{x}^{\mathbf{T}}(s) S_1 \dot{x}(s) \, \mathrm{d}s - \int_{t-\eta_m-\eta(t)}^{t-\eta_m} \dot{x}^{\mathbf{T}}(s) S_2 \dot{x}(s) \, \mathrm{d}s \\
&\quad - \int_{t-\eta_M}^{t-\eta_m-\eta(t)} \dot{x}^{\mathbf{T}}(s) S_2 \dot{x}(s) \, \mathrm{d}s.
\end{aligned}
$$

(3.14)

For any appropriately dimensioned matrices $\hat{X}(\lambda) = \sum_{i=1}^{r} \lambda_i \hat{X}_i$, $\hat{Y}(\lambda) = \sum_{i=1}^{r} \lambda_i \hat{Y}_i$,

and $\hat{M}(\lambda) = \sum\limits_{i=1}^{r} \lambda_i \hat{M}_i$, the following equations can be easily derived:

$$\Pi_1 = 2\xi^{\mathbf{T}}(t)\hat{X}(\lambda)\left(x(t) - x\left(t - \frac{\eta_m}{m}\right) - \int_{t-\frac{\eta_m}{m}}^{t} \dot{x}(s)\,\mathrm{d}s\right) = 0,$$

$$\Pi_2 = 2\xi^{\mathbf{T}}(t)\hat{Y}(\lambda)\left(x(t-\eta_m) - x(t - \eta_m - \eta(t)) - \int_{t-\eta_m-\eta(t)}^{t-\eta_m} \dot{x}(s)\,\mathrm{d}s\right) = 0,$$

$$\Pi_3 = 2\xi^{\mathbf{T}}(t)\hat{M}(\lambda)\left(x(t-\eta_m-\eta(t)) - x(t - \eta_M) - \int_{t-\eta_M}^{t-\eta_m-\eta(t)} \dot{x}(s)\,\mathrm{d}s\right) = 0,$$

where

$$\xi^{\mathbf{T}}(t) = \begin{bmatrix} \Upsilon^{\mathbf{T}}(t) & x^{\mathbf{T}}(t-\eta_m) & x^{\mathbf{T}}(t-\eta_M) & x^{\mathbf{T}}(t-\eta_m-\eta(t)) \end{bmatrix}. \qquad (3.15)$$

To develop the final result, adding Π_1, Π_2 and Π_3 to the right hand side of (3.14) and carrying out some computations, then we have

$$\begin{aligned}
\dot{V}(t) \leq\ & \xi^{\mathbf{T}}(t)\left[\bar{\Theta}(\lambda) + \frac{\eta_m}{m}\hat{X}(\lambda)S_1^{-1}\hat{X}^{\mathbf{T}}(\lambda) + \eta(t)\hat{Y}(\lambda)S_2^{-1}\hat{Y}^{\mathbf{T}}(\lambda)\right.\\
& \left. + (\eta_M - \eta_m - \eta(t))\hat{M}(\lambda)S_2^{-1}\hat{M}^{\mathbf{T}}(\lambda)\right]\xi(t)\\
& - \int_{t-\frac{\eta_m}{m}}^{t}\left[\xi^{\mathbf{T}}(t)\hat{X}(\lambda) + \dot{x}^{\mathbf{T}}(s)S_1\right]S_1^{-1}\left[\hat{X}^{\mathbf{T}}(\lambda)\xi(t) + S_1\dot{x}(s)\right]\mathrm{d}s\\
& - \int_{t-\eta_m-\eta(t)}^{t-\eta_m}\left[\xi^{\mathbf{T}}(t)\hat{Y}(\lambda) + \dot{x}^{\mathbf{T}}(s)S_2\right]S_2^{-1}\left[\hat{Y}^{\mathbf{T}}(\lambda)\xi(t) + S_2\dot{x}(s)\right]\mathrm{d}s\\
& - \int_{t-\eta_M}^{t-\eta_m-\eta(t)}\left[\xi^{\mathbf{T}}(t)\hat{M}(\lambda) + \dot{x}^{\mathbf{T}}(s)S_2\right]S_2^{-1}\left[\hat{M}^{\mathbf{T}}(\lambda)\xi(t) + S_2\dot{x}(s)\right]\mathrm{d}s\\
\leq\ & \xi^{\mathbf{T}}(t)\left[\bar{\Theta}(\lambda) + \frac{\eta_m}{m}\hat{X}(\lambda)S_1^{-1}\hat{X}^{\mathbf{T}}(\lambda) + \eta(t)\hat{Y}(\lambda)S_2^{-1}\hat{Y}^{\mathbf{T}}(\lambda)\right.\\
& \left. + (\eta_M - \eta_m - \eta(t))\hat{M}(\lambda)S_2^{-1}\hat{M}^{\mathbf{T}}(\lambda)\right]\xi(t)\\
=\ & \xi^{\mathbf{T}}(t)\left[\frac{\eta(t)}{\eta_M - \eta_m}\left(\bar{\Theta}(\lambda) + \frac{\eta_m}{m}\hat{X}(\lambda)S_1^{-1}\hat{X}^{\mathbf{T}}(\lambda)\right.\right.\\
& \left. + (\eta_M - \eta_m)\hat{Y}(\lambda)S_2^{-1}\hat{Y}^{\mathbf{T}}(\lambda)\right)\\
& + \frac{\eta_M - \eta_m - \eta(t)}{\eta_M - \eta_m}\left(\bar{\Theta}(\lambda) + \frac{\eta_m}{m}\hat{X}(\lambda)S_1^{-1}\hat{X}^{\mathbf{T}}(\lambda)\right.\\
& \left.\left. + (\eta_M - \eta_m)\hat{M}(\lambda)S_2^{-1}\hat{M}^{\mathbf{T}}(\lambda)\right)\right]\xi(t),
\end{aligned}$$

where

$$\bar{\Theta}\left(\lambda\right) = \text{sym}\{\bar{W}_A^{\mathbf{T}}P\bar{W}_B + \hat{Z}\left(\lambda\right)\bar{W}_Z\} + \bar{W}_{Q_1}^{\mathbf{T}}\check{Q}_1\bar{W}_{Q_1}$$
$$+\bar{W}_{Q_2}^{\mathbf{T}}\check{Q}_2\bar{W}_{Q_2} + \bar{W}_A^{\mathbf{T}}\left[\frac{\eta_m}{m}S_1 + \left(\eta_M - \eta_m\right)S_2\right]\bar{W}_A,$$

with

$$\bar{W}_A = \begin{bmatrix} A\left(\lambda\right) & 0_{n,(m+1)n} & B_2\left(\lambda\right)K \end{bmatrix}, \quad \bar{W}_B = \begin{bmatrix} I_n & 0_{n,(m+2)n} \end{bmatrix},$$

$$\hat{Z}\left(\lambda\right) = \begin{bmatrix} \hat{X}\left(\lambda\right) & \hat{Y}\left(\lambda\right) & \hat{M}\left(\lambda\right) \end{bmatrix}, \quad \bar{W}_Z = \begin{bmatrix} I_n & -I_n & 0_{n,(m+1)n} \\ 0_{n,mn} & I_n & 0_n & -I_n \\ 0_{n,(m+1)n} & -I_n & I_n \end{bmatrix},$$

$$\bar{W}_{Q_1} = \begin{bmatrix} I_{mn} & 0_{mn,3n} \\ 0_{mn,n} & I_{mn} & 0_{mn,2n} \end{bmatrix}, \quad \bar{W}_{Q_2} = \begin{bmatrix} 0_{n,mn} & I_n & 0_{n,2n} \\ 0_{n,(m+1)n} & I_n & 0_{n,n} \end{bmatrix}.$$

On the other hand, (3.6)–(3.7) imply

$$\Sigma_1\left(\lambda\right) = \sum_{i=1}^{r}\lambda_i^2\Sigma_{1ii} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\lambda_i\lambda_j\left(\Sigma_{1ij} + \Sigma_{1ji}\right), \qquad (3.16)$$

$$\Sigma_2\left(\lambda\right) = \sum_{i=1}^{r}\lambda_i^2\Sigma_{2ii} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\lambda_i\lambda_j\left(\Sigma_{2ij} + \Sigma_{2ji}\right), \qquad (3.17)$$

where

$$\Sigma_1\left(\lambda\right) = \begin{bmatrix} \bar{\Theta}\left(\lambda\right) & \sqrt{\frac{\eta_m}{m}}\hat{X}\left(\lambda\right) & \sqrt{\eta_M - \eta_m}\hat{Y}\left(\lambda\right) \\ * & -S_1 & 0 \\ * & * & -S_2 \end{bmatrix} < 0, \qquad (3.18)$$

$$\Sigma_2\left(\lambda\right) = \begin{bmatrix} \bar{\Theta}\left(\lambda\right) & \sqrt{\frac{\eta_m}{m}}\hat{X}\left(\lambda\right) & \sqrt{\eta_M - \eta_m}\hat{M}\left(\lambda\right) \\ * & -S_1 & 0 \\ * & * & -S_2 \end{bmatrix} < 0. \qquad (3.19)$$

Applying Schur complement to (3.18) and (3.19), it yields

$$\bar{\Theta}\left(\lambda\right) + \frac{\eta_m}{m}\hat{X}\left(\lambda\right)S_1^{-1}\hat{X}^{\mathbf{T}}\left(\lambda\right) + \left(\eta_M - \eta_m\right)\hat{Y}\left(\lambda\right)S_2^{-1}\hat{Y}^{\mathbf{T}}\left(\lambda\right) < 0,$$
$$\bar{\Theta}\left(\lambda\right) + \frac{\eta_m}{m}\hat{X}\left(\lambda\right)S_1^{-1}\hat{X}^{\mathbf{T}}\left(\lambda\right) + \left(\eta_M - \eta_m\right)\hat{M}\left(\lambda\right)S_2^{-1}\hat{M}^{\mathbf{T}}\left(\lambda\right) < 0,$$

which mean $\dot{V}\left(t\right) < 0$, then system in (3.4) is robustly asymptotically stable for all uncertain parameter satisfying (3.1).

In the following part, we shall establish the H_∞ performance of the system in (3.4) under zero initial conditions. First of all, we also define the Lyapunov functional as in (3.12). Then, by following the same line as in the above proof, the time derivative of $V(t)$ is given by:

$$
\begin{aligned}
\dot{V}(t) &\leq \bar{\xi}^{\mathbf{T}}(t) \Big[\Theta(\lambda) + \frac{\eta_m}{m} X(\lambda) S_1^{-1} X(\lambda)^{\mathbf{T}} + \eta(t) Y(\lambda) S_2^{-1} Y^{\mathbf{T}}(\lambda) \\
&\quad + (\eta_M - \eta_m - \eta(t)) M(\lambda)(S_2)^{-1} M^{\mathbf{T}}(\lambda) \Big] \bar{\xi}(t). \\
&\quad - \int_{t-\frac{\eta_m}{m}}^{t} \Big[\bar{\xi}^{\mathbf{T}}(t) X(\lambda) + \dot{x}^{\mathbf{T}}(s) S_1 \Big] S_1^{-1} \Big[X^{\mathbf{T}}(\lambda) \bar{\xi}(t) + S_1 \dot{x}(s) \Big] \mathrm{d}s \\
&\quad - \int_{t-\eta_m-\eta(t)}^{t-\eta_m} \Big[\bar{\xi}^{\mathbf{T}}(t) Y(\lambda) + \dot{x}^{\mathbf{T}}(s) S_2 \Big] S_2^{-1} \Big[Y^{\mathbf{T}}(\lambda) \bar{\xi}(t) + S_2 \dot{x}(s) \Big] \mathrm{d}s \\
&\quad - \int_{t-\eta_M}^{t-\eta_m-\eta(t)} \Big[\bar{\xi}^{\mathbf{T}}(t) M(\lambda) + \dot{x}^{\mathbf{T}}(s) S_2 \Big] S_2^{-1} \Big[M^{\mathbf{T}}(\lambda) \bar{\xi}(t) + S_2 \dot{x}(s) \Big] \mathrm{d}s \\
&\leq \bar{\xi}^{\mathbf{T}}(t) \Big[\Theta(\lambda) + \frac{\eta_m}{m} X(\lambda) S_1^{-1} X^{\mathbf{T}}(\lambda) + \eta(t) Y(\lambda) S_2^{-1} Y^{\mathbf{T}}(\lambda) \\
&\quad + (\eta_M - \eta_m - \eta(t)) M(\lambda) S_2^{-1} M^{\mathbf{T}}(\lambda) \Big] \bar{\xi}(t) \\
&= \bar{\xi}^{\mathbf{T}}(t) \Big[\frac{\eta(t)}{\eta_M - \eta_m} \Big(\Theta(\lambda) + \frac{\eta_m}{m} X(\lambda) S_1^{-1} X^{\mathbf{T}}(\lambda) \\
&\quad + (\eta_M - \eta_m) Y(\lambda) S_2^{-1} Y^{\mathbf{T}}(\lambda) \Big) \\
&\quad + \frac{\eta_M - \eta_m - \eta(t)}{\eta_M - \eta_m} \Big(\Theta(\lambda) + \frac{\eta_m}{m} X(\lambda) S_1^{-1} X^{\mathbf{T}}(\lambda) \\
&\quad + (\eta_M - \eta_m) M(\lambda) S_2^{-1} M^{\mathbf{T}}(\lambda) \Big) \Big] \bar{\xi}(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta(\lambda) &= \mathrm{sym}\{ W_A^{\mathbf{T}} P W_B + Z(\lambda) W_Z \} + W_{Q_1}^{\mathbf{T}} \check{Q}_1 W_{Q_1} \\
&\quad + W_{Q_2}^{\mathbf{T}} \check{Q}_2 W_{Q_2} + W_A^{\mathbf{T}} \Big[\frac{\eta_m}{m} S_1 + (\eta_M - \eta_m) S_2 \Big] W_A
\end{aligned}
$$

with

$$
\begin{aligned}
W_A &= \begin{bmatrix} A(\lambda) & 0_{n,(m+1)n} & B(\lambda) K & B_1(\lambda) \end{bmatrix}, \\
Z(\lambda) &= \begin{bmatrix} X(\lambda) & Y(\lambda) & M(\lambda) \end{bmatrix}, \\
\bar{\xi}^{\mathbf{T}}(t) &= \begin{bmatrix} \Upsilon^{\mathbf{T}}(t) & x^{\mathbf{T}}(t-\eta_m) & x^{\mathbf{T}}(t-\eta_M) & x^{\mathbf{T}}(t-\eta_m-\eta(t)) & w^{\mathbf{T}}(t) \end{bmatrix}.
\end{aligned}
$$

Thus, we have

$$
\dot{V}(t) + z_1^T(t)z_1(t) - \gamma^2 w^T(t)w(t)
$$

$$
\leq \quad \bar{\xi}^{\mathbf{T}}(t) \Big[\Theta(\lambda) + W_{z_1}^{\mathbf{T}}W_{z_1} - \gamma^2 W_w^{\mathbf{T}}W_w + \frac{\eta_m}{m}X(\lambda)S_1^{-1}X^{\mathbf{T}}(\lambda)
$$

$$
+\eta(t)Y(\lambda)S_2^{-1}Y^{\mathbf{T}}(\lambda) + (\eta_M - \eta_m - \eta(t))M(\lambda)S_2^{-1}M^{\mathbf{T}}(\lambda) \Big] \bar{\xi}(t)
$$

$$
= \quad \bar{\xi}^{\mathbf{T}}(t) \Big[\frac{\eta(t)}{\eta_M - \eta_m} \Big(\Theta(\lambda) + W_{z_1}^{\mathbf{T}}W_{z_1} - \gamma^2 W_w^{\mathbf{T}}W_w + \frac{\eta_m}{m}X(\lambda)S_1^{-1}X^{\mathbf{T}}(\lambda)
$$

$$
+ (\eta_M - \eta_m)Y(\lambda)S_2^{-1}Y^{\mathbf{T}}(\lambda) \Big) + \frac{\eta_M - \eta_m - \eta(t)}{\eta_M - \eta_m} \Big(\Theta(\lambda) + W_{z_1}^{\mathbf{T}}W_{z_1}
$$

$$
-\gamma^2 W_w^{\mathbf{T}}W_w + \frac{\eta_m}{m}X(\lambda)S_1^{-1}X^{\mathbf{T}}(\lambda) + (\eta_M - \eta_m)M(\lambda)S_2^{-1}M^{\mathbf{T}}(\lambda) \Big) \Big] \bar{\xi}(t),
$$

where

$$
W_{z_1} = \begin{bmatrix} C_1(\lambda) & 0_{1,(m+1)n} & D_1(\lambda)K & 0 \end{bmatrix}.
$$

By using Schur complement to (3.6)–(3.7) and the above method, it can be easily seen that

$$
\dot{V}(t) + z_1^{\mathbf{T}}(t)z_1(t) - \gamma^2 w^{\mathbf{T}}(t)w(t) < 0, \tag{3.20}
$$

for all nonzero $w \in L_2[0, \infty)$. Under zero initial conditions, we have $V(0) = 0$ and $V(\infty) \geq 0$. Integrating both sides of (3.20) yields $\|z_1\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0, \infty)$, and the H_∞ performance is established. The proof is completed. ∎

Remark 3.1 Note that Proposition 1 presents a new delay-range-dependent H_∞ performance analysis condition for the active suspension system in (3.4) by exploiting a novel Lyapunov-Krasovskii functional in (3.12) constructed based on the delay partitioning idea (Mou *et al.*, 2008), which may bring much less conservative results. In addition, it is also worth noticing that, the term

$$
\Theta(\lambda) + W_{z_1}^{\mathbf{T}}W_{z_1} - \gamma^2 W_w^{\mathbf{T}}W_w + \frac{\eta_m}{m}X(\lambda)S_1^{-1}X^{\mathbf{T}}(\lambda)
$$

$$
+\eta(t)Y(\lambda)S_2^{-1}Y^{\mathbf{T}}(\lambda) + (\eta_M - \eta_m - \eta(t))M(\lambda)S_2^{-1}M^{\mathbf{T}}(\lambda)
$$

is divided into two parts, which has been proved very effective to reduce the conservativeness of the delay-dependent or delay-range-dependent results.

In the following discussion, we will show that the hard constraints in (3.5) are guaranteed. Inequality (3.20) guarantees $\dot{V}(t) - \gamma^2 w^T(t)w(t) < 0$. Integrating

both sides of the above inequality from zero to any $t > 0$, we obtain

$$V(t) - V(0) < \gamma^2 \int_0^t w^T(s)w(s)\mathrm{d}s < \gamma^2 \|w\|_2^2.$$

From the definition in (3.12), we know that $x^T(t)Px(t) < \rho$ with $\rho = \gamma^2 w_{\max} + V(0)$. Similar to (Gao *et al.*, 2010a), the following inequality holds

$$
\begin{aligned}
\max_{t>0} |\{z_2(t)\}_q|^2 &= \max_{t>0} \left\| x^T(t)\{C_{2i}\}_q^T\{C_{2i}\}_q x(t) \right\|_2 \\
&= \max_{t>0} \left\| x^T(t)P^{\frac{1}{2}}P^{-\frac{1}{2}}\{C_{2i}\}_q^T\{C_{2i}\}_q P^{-\frac{1}{2}}P^{\frac{1}{2}}x(t) \right\|_2 \\
&< \rho \cdot \theta_{\max}(P^{-\frac{1}{2}}\{C_{2i}\}_q^T\{C_{2i}\}_q P^{-\frac{1}{2}}), \\
&\quad i = 1, 2, \cdots, r, \quad q = 1, 2, 3, 4,
\end{aligned}
$$

where $\theta_{\max}(\cdot)$ represents the maximal eigenvalue. From the above inequality, we know that the constraint in (3.5) is guaranteed, if

$$\rho \cdot P^{-\frac{1}{2}}\{C_{2i}\}_q^T\{C_{2i}\}_q P^{-\frac{1}{2}} < \{z_{2,\max}\}_q^2 I, \quad i = 1, 2, \cdots, r, \quad q = 1, 2, 3, 4, \quad (3.21)$$

which is equivalent to the following

$$\Sigma_{3qi} = \begin{bmatrix} -\{z_{2,\max}\}_q^2 P & \sqrt{\rho}\{C_{2i}\}_q^T \\ * & -I \end{bmatrix} < 0, \quad i = 1, 2, \cdots, r, \quad q = 1, 2, 3, 4. \tag{3.22}$$

Subsequently, robust H_∞ performance analysis criterion subjecting to output constraints in (3.5) for the closed-loop system in (3.4) is presented in the following theorem.

Theorem 3.1 Consider the closed-loop system in (3.4). For given scalars $\gamma > 0$, $\eta_m > 0$, $\eta_M > 0$ and a matrix K, if there exist matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $S_1 > 0$, $S_2 > 0$, X_j, Y_j and M_j $(j = 1, 2, \ldots, r)$ with appropriate dimensions such that the following LMIs hold:

$$
\begin{aligned}
\Sigma_{\varsigma ii} &< 0, \quad \varsigma = 1, 2, \quad i < j, & (3.23) \\
\Sigma_{\varsigma ij} + \Sigma_{\varsigma ji} &< 0, \quad i, j = 1, 2, \ldots, r, & (3.24) \\
\Sigma_{3qi} &< 0, \quad q = 1, 2, 3, 4, & (3.25)
\end{aligned}
$$

where Σ_{1ij}, Σ_{2ij} and Σ_{3qi} are defined in (3.8)–(3.9) and (3.22), respectively. Then

(1) the closed-loop system is robustly asymptotically stable;

(2) under zero initial condition, the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to output constraint (3.5) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$, where $T_{z_1 w}$ denotes the closed-loop transfer function from the road disturbance $w(t)$ to the control output $z_1(t)$.

Based on robust H_∞ performance analysis condition proposed in Theorem 3.1, robust H_∞ controller existence condition for the active suspension system in (3.4) is developed in the following theorem.

Theorem 3.2 Consider the active suspension system in (3.4). For given scalars $\gamma > 0$, $\eta_m > 0$, $\eta_M > 0$, if there exist matrices $\bar{P} > 0$, $\bar{Q}_1 > 0$, $\bar{Q}_2 > 0$, $\bar{S}_1 > 0$, $\bar{S}_2 > 0$, and $\bar{K}$, $\bar{X}$, $\bar{Y}$ and $\bar{M}$ with appropriate dimensions satisfying the following LMIs:

$$
\begin{aligned}
\bar{\Sigma}_{\varsigma ii} &< 0, \quad \varsigma = 1, 2, \quad i < j, &(3.26)\\
\bar{\Sigma}_{\varsigma ij} + \bar{\Sigma}_{\varsigma ji} &< 0, \quad i, j = 1, 2, \ldots, r, &(3.27)\\
\bar{\Sigma}_{3qi} &< 0, \quad q = 1, 2, 3, 4, &(3.28)
\end{aligned}
$$

where

$$
\bar{\Sigma}_{1ij} = \begin{bmatrix}
\bar{\Omega}_{ij} & \sqrt{\frac{\eta_m}{m}}\bar{X}_j & \sqrt{\eta_M - \eta_m}\bar{Y}_j & \bar{W}_{A_i}^{\mathbf{T}} A_I & \bar{W}_{z_{1i}}^{\mathbf{T}} \\
* & \bar{S}_1 - 2\bar{P} & 0 & 0 & 0 \\
* & * & \bar{S}_2 - 2\bar{P} & 0 & 0 \\
* & * & * & \Upsilon_1 & 0 \\
* & * & * & * & -1
\end{bmatrix}, \quad (3.29)
$$

$$
\bar{\Sigma}_{2ij} = \begin{bmatrix}
\bar{\Omega}_{ij} & \sqrt{\frac{\eta_m}{m}}\bar{X}_j & \sqrt{\eta_M - \eta_m}\bar{M}_j & \bar{W}_{A_i}^{\mathbf{T}} A_I & \bar{W}_{z_{1i}}^{\mathbf{T}} \\
* & \bar{S}_1 - 2\bar{P} & 0 & 0 & 0 \\
* & * & \bar{S}_2 - 2\bar{P} & 0 & 0 \\
* & * & * & \Upsilon_1 & 0 \\
* & * & * & * & -1
\end{bmatrix}, \quad (3.30)
$$

$$
\bar{\Sigma}_{3i} = \begin{bmatrix}
-\{z_{2,\max}\}_q^2 \bar{P} & \sqrt{\rho}\bar{P}\{C_{2i}\}_q^T \\
* & -I
\end{bmatrix}, \quad \bar{\Omega}_{ij} = \bar{\Theta}_{ij} - \gamma^2 W_w^{\mathbf{T}} W_w \quad (3.31)
$$

with

$$
\begin{aligned}
\bar{\Theta}_{ij} &= \text{sym}\{\bar{W}_{A_i}^{\mathbf{T}} W_B + \bar{Z}_j W_Z\} + W_{Q_1}^{\mathbf{T}} \hat{Q}_1 W_{Q_1} + W_{Q_2}^{\mathbf{T}} \hat{Q}_2 W_{Q_2} \\
\bar{W}_{A_i} &= \begin{bmatrix} A_i \bar{P} & 0_{n,(m+1)n} & B_i \bar{K} & B_{1i} \end{bmatrix}, \qquad \bar{Z}_j = \begin{bmatrix} \bar{X}_j & \bar{Y}_j & \bar{M}_j \end{bmatrix}, \\
\bar{W}_{z_{1i}} &= \begin{bmatrix} C_{1i}\bar{P} & 0_{1,(m+1)n} & D_{1i}\bar{K} & 0 \end{bmatrix}, \quad W_w = \begin{bmatrix} 0_{1,(m+3)n} & 1 \end{bmatrix}, \\
A_I &= \begin{bmatrix} \sqrt{\frac{\eta_m}{m}} & \sqrt{\eta_M - \eta_m} \end{bmatrix}, \quad \Upsilon_1 = \text{diag}\left\{ -\bar{S}_1, -\bar{S}_2 \right\}, \\
\hat{Q}_1 &= \begin{bmatrix} \bar{Q}_1 & 0 \\ * & -\bar{Q}_1 \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} \bar{Q}_2 & 0 \\ * & -\bar{Q}_2 \end{bmatrix},
\end{aligned}
$$

W_Z, W_{Q_1} and W_{Q_2} are defined in Proposition 3.1. Then a stabilizing controller in the form of (3.3) exists, such that

(1) the closed-loop system is robustly asymptotically stable;

(2) under zero initial condition, the performance $\|T_{z_1w}\|_\infty < \gamma$ is minimized subject to output constraint (3.5) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$.

Moreover, if inequalities (3.26)–(3.28) have a feasible solution, then the control gain K in (3.3) is given by $K = \bar{K}\bar{P}^{-1}$.

Proof. For $\bar{S}_p^{-1} > 0$ $(p = 1, 2)$, it is easy to see that

$$
\left(\bar{S}_p - \bar{P}\right) \bar{S}_p^{-1} \left(\bar{S}_p - \bar{P}\right) \geq 0,
$$

which is equivalent to

$$
-\bar{P}\bar{S}_p^{-1}\bar{P} \leq \bar{S}_p - 2\bar{P}.
$$

Therefore, it follows from (3.26) and (3.27)

$$
\begin{aligned}
\hat{\Sigma}_{\varsigma ii} &< 0, \quad \varsigma = 1, 2, \quad i < j, & (3.32) \\
\hat{\Sigma}_{\varsigma ij} + \hat{\Sigma}_{\varsigma ji} &< 0, \quad i, j = 1, 2, \dots, r, & (3.33)
\end{aligned}
$$

where

$$
\hat{\Sigma}_{1ij} = \begin{bmatrix}
\bar{\Omega}_{ij} & \sqrt{\frac{\eta_m}{m}}\bar{X}_j & \sqrt{\eta_M - \eta_m}\bar{Y}_j & \bar{W}_{A_i}A_I & \bar{W}_{z_{1i}}^{\mathbf{T}} \\
* & -\bar{P}\bar{S}_1^{-1}\bar{P} & 0 & 0 & 0 \\
* & * & -\bar{P}\bar{S}_2^{-1}\bar{P} & 0 & 0 \\
* & * & * & \Upsilon_1 & 0 \\
* & * & * & * & -1
\end{bmatrix}, \qquad (3.34)
$$

$$
\hat{\Sigma}_{2ij} = \begin{bmatrix}
\bar{\Omega}_{ij} & \sqrt{\frac{\eta_m}{m}}\bar{X}_j & \sqrt{\eta_M - \eta_m}\bar{M}_j & \bar{W}_{A_i}A_I & \bar{W}_{z_{1i}}^{\mathbf{T}} \\
* & -\bar{P}\bar{S}_1^{-1}\bar{P} & 0 & 0 & 0 \\
* & * & -\bar{P}\bar{S}_2^{-1}\bar{P} & 0 & 0 \\
* & * & * & \Upsilon_1 & 0 \\
* & * & * & * & -1
\end{bmatrix}, \qquad (3.35)
$$

Now, introduce the following matrices

$$
J = \mathrm{diag}\left\{J_1, J_2, J_3, 1\right\},
$$

where

$$
\begin{aligned}
J_1 &= \mathrm{diag}\left\{\bar{P}^{-1}, \bar{P}^{-1}, \ldots, \bar{P}^{-1}, 1\right\} \in \mathbb{R}^{((m+3)n+1)\times((m+3)n+1)}, \\
J_2 &= \mathrm{diag}\left\{\bar{P}^{-1}, \bar{P}^{-1}\right\} \in \mathbb{R}^{2n\times 2n}, \\
J_3 &= \mathrm{diag}\left\{\bar{S}_1^{-1}, \bar{S}_2^{-1}\right\} \in \mathbb{R}^{2n\times 2n},
\end{aligned}
$$

After setting

$$
\begin{aligned}
P &= \bar{P}^{-1}, \quad S_p = \bar{S}_p^{-1} > 0, \quad (p = 1, 2) \\
Q_1 &= \bar{P}^{-1}\bar{Q}_1\bar{P}^{-1}, \quad Q_2 = \bar{P}^{-1}\bar{Q}_2\bar{P}^{-1},
\end{aligned}
$$

$$
\begin{bmatrix} X(\lambda) & Y(\lambda) & M(\lambda) \end{bmatrix} = J_1 \begin{bmatrix} \bar{X}(\lambda) & \bar{Y}(\lambda) & \bar{M}(\lambda) \end{bmatrix} J_4,
$$

where

$$
J_4 = \mathrm{diag}\left\{\bar{P}^{-1}, \bar{P}^{-1}, \bar{P}^{-1}\right\} \in \mathbb{R}^{3n\times 3n},
$$

Pre- and post multiplying (3.32) and (3.33) by $J^{\mathbf{T}}$ and J, one can see that conditions in (3.23) and (3.24) hold by using Schur complement. On the other hand, (3.28) is equivalent to (3.25) by performing a simple congruence transformation with $\mathrm{diag}\left\{\bar{P}^{-1}, I\right\}$. Therefore, all the conditions in Theorem 1 are satisfied. The proof is completed. ∎

Remark 3.2 This chapter is the first attempt to investigate the problem of robust H_∞ control of the uncertain suspension system for a half-car model with input time-varying delay. It is reasonably assumed that the vehicle front sprung mass and rear unsprung mass are varying due to vehicle load variation and the fact that the parameter uncertainties can be modeled by polytopic uncertainties type. Latter, the effectiveness of the proposed method will be validated by providing a design example.

Remark 3.3 In the further work, we will consider the fuzzy controller for half (full)-vehicle suspension systems with nonlinear uncertainty and focus on the efficient computing for real-time control. In detail, by using the fuzzy approximation method, we will build half (full)-vehicle suspension systems based on the T-S fuzzy model. By constructing quadratic Lyapunov functions and piecewise quadratic Lyapunov functions, the performance analysis and stability analysis condition will be derived. Based on these conditions, the controller design criteria will be presented in terms of LMIs, which can be checked efficiently by using the standard software. Furthermore, we will develop the novel fuzzy backstepping control to handle the control design problems for the nonlinear systems and design controller for four wheel steering integrated control system of vehicle by using the LMI method and backstepping control approach.

Note that LMIs condition in Theorem 3.2 is not only over the matrix variables, but also over the scalar γ. This implies that the scalar γ can be included as an optimization variable to obtain a lower bound of the guaranteed H_∞ disturbance attention level. Based on the condition, robust H_∞ controller can be obtained with the minimal γ by solving the following convex optimization problem:

$$\min \gamma \quad \text{s.t.} \quad (3.26)\text{-}(3.28)$$

$$\bar{P} > 0, \bar{Q}_1 > 0, \bar{Q}_2 > 0, \bar{S}_1 > 0, \bar{S}_2 > 0, \bar{K}, \bar{X}, \bar{Y}, \bar{M}. \tag{3.36}$$

3.2.3 Case Study

In this subsection, a design example is given to demonstrate the effectiveness of the proposed robust H_∞ controller design method. By using the parameters listed in Table 2.2 and (2.14)–(2.15), we can have $F_r = 3580.5$ N and $F_f = 4014.5$ N. Now, we consider the constrained robust H_∞ controller design problem for the

suspension system with input time-varying delay. It is assumed that the time-varying input delay is $d(t) = 1 + 4|\sin(t)|$ ms and satisfies $\eta_m = 1$ ms and $\eta_M = 5$ ms, the maximum allowable suspension strokes $z_{\max f} = z_{\max r} = 0.08$ m. Here, we choose $\rho = 1$ as discussed in (Chen & Guo, 2005). For $m = 1$, by solving the convex optimization problem formulated in (3.36), the minimum guaranteed closed-loop H_∞ performance index obtained is $\gamma_{\min m1} = 7.1277$ and admissible control gain matrix is given as

$$K_{m1} = 10^5 \times \begin{bmatrix} -0.0212 & 0.1362 & 0.7830 & 1.3296 \\ 0.0755 & -0.0604 & 0.2015 & 1.2005 \\ -0.0563 & 0.0047 & 0.0224 & 0.0173 \\ 0.0202 & -0.0529 & 0.0143 & 0.0324 \end{bmatrix}. \qquad (3.37)$$

In order to show the more detailed information of the proposed method, for $m = 2, 3, 4$, by solving the convex optimization problem formulated in (3.36), the minimum guaranteed closed-loop H_∞ performance indexes obtained are $\gamma_{\min m2} = 7.0117$, $\gamma_{\min m3} = 6.9876$, $\gamma_{\min m4} = 6.9832$ and admissible control gain matrices are given as

$$K_{m2} = 10^5 \times \begin{bmatrix} -0.0293 & 0.1360 & 0.7271 & 1.3739 \\ 0.0801 & -0.0654 & 0.1636 & 1.1478 \\ -0.0572 & 0.0025 & 0.0205 & 0.0170 \\ 0.0212 & -0.0522 & 0.0143 & 0.0306 \end{bmatrix}, \qquad (3.38)$$

$$K_{m3} = 10^5 \times \begin{bmatrix} -0.0229 & 0.1094 & 0.6961 & 1.3333 \\ 0.0751 & -0.0451 & 0.1221 & 1.0852 \\ -0.0558 & 0.0000 & 0.0195 & 0.0160 \\ 0.0203 & -0.049 & 0.0137 & 0.0294 \end{bmatrix}, \qquad (3.39)$$

$$K_{m4} = 10^5 \times \begin{bmatrix} -0.0213 & 0.1000 & 0.6620 & 1.2861 \\ 0.0708 & -0.0378 & 0.0904 & 1.0212 \\ -0.0550 & -0.0007 & 0.0186 & 0.0152 \\ 0.0198 & -0.0482 & 0.0131 & 0.0283 \end{bmatrix}. \qquad (3.40)$$

From the above computational closed-loop performance $\gamma_{\min m1}$, $\gamma_{\min m2}$, $\gamma_{\min m3}$ and $\gamma_{\min m4}$, it can be observed that the closed-loop performance $\gamma_{\min}$ is reduced when the partition number m increases. After the delay is partitioned, though

the improvement slows down as the partition number m increases. However, it should be noted that although conservatism is reduced as the fractioning becomes thinner, the computational complexity will be weighting. In order to obtain tradeoff between the closed-loop performance and the computational complexity, we choose $m = 4$ here.

To check the effectiveness of the proposed controllers in (3.37)–(3.40), we would like to have the desired controller to satisfy: 1) the first control output $z_1(t)$ including the heave acceleration $\ddot{z}_c(t)$ and the pitch acceleration $\ddot{\varphi}(t)$ is as small as possible; 2) the suspension deflection is below the maximum allowable suspension strokes $z_{f\max} = 0.08$ m and $z_{r\max} = 0.08$ m; 3) the controlled output defined in (3.5) satisfy $z_2(t)_3 < 1$ and $z_2(t)_4 < 1$. In order to evaluate the suspension characteristics with respect to ride comfort, vehicle handling, and working space of the suspension, the variability of the road profiles is taken into account. In the context of active suspension performance, road disturbances can be generally assumed as shocks. Shocks are discrete events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pothole on an otherwise smooth road surface. As the reference (Du *et al.*, 2008), this case of road profile is considered first to reveal the transient response characteristic, which is given by

$$
z_{rf}(t) = \begin{cases} \frac{A}{2}(1 - \cos(\frac{2\pi V}{L}t)), & \text{if } 0 \le t \le \frac{L}{V}, \\ 0, & \text{if } t > \frac{L}{V}, \end{cases} \tag{3.41}
$$

where A and L are the height and the length of the bump. Assume $A = 0.1$ m , $L = 2$ m and the vehicle forward velocity as $V = 10$ km/h. In this section, we assume that the road condition $z_{rr}(t)$ for the rear wheel is the same as the front wheel but with a time delay of $(l_1 + l_2)/V$. Fig. 3.1 illustrates the corresponding ground velocities for the front and rear wheels.

Fig. 3.2–3.4 plot bump responses of the heave accelerations and the pitch acceleration, the front and rear suspension deflections, the front and rear tire deflection constrains of the open- ($u(t) = 0$, passive mode) and closed-loop systems for the different control gain matrices K_{m1}, K_{m2}, K_{m3}, and K_{m4} respectively. It can be seen from Fig. 3.2 that the values of the heave accelerations and the pitch acceleration in closed-loop systems are much less than the the open-loop
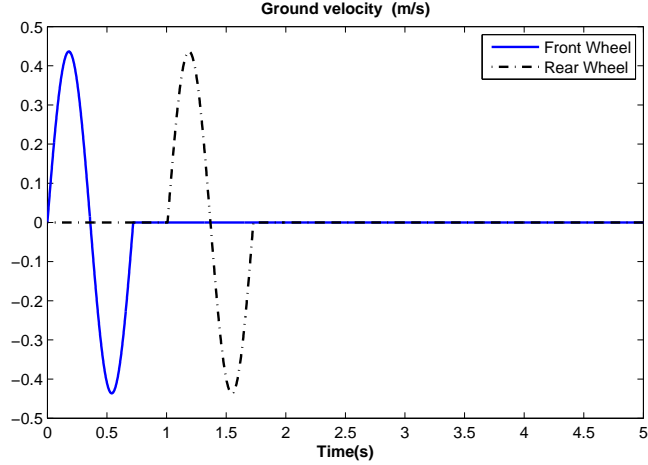
Figure 3.1: Bump inputs from ground

systems and an improved ride comfort has been achieved by using the different controllers. Moreover, from Fig. 3.3, we can see that the suspension strokes constraints are guaranteed. Fig. 3.4 illustrates that the relation dynamic front tire load $k_{tf}x_3(t)/F_f$ and rear tire load $k_{tr}x_4(t)/F_r$ are all below 1. In all, Fig. 3.2–3.4 show that the closed-loop system is asymptotically stable with the guaranteed output constraints.

In the following discussion, we consider the problem of robust H_∞ controller design for a half-car model uncertain suspension system. In this chapter, we assume that the front unsprung m_{uf} and the rear unsprung m_{ur} contain uncertainties, which may be caused by vehicle load variation and are expressed as $m_{uf} = (40 + \lambda_{muf})$ kg, $m_{ur} = (45 + \lambda_{mur})$ kg, where λ_{muf} and λ_{mur} satisfy $|\lambda_{muf}| \leq \bar{\lambda}_{muf}$ and $|\lambda_{mur}| \leq \bar{\lambda}_{mur}$. It is assumed that $\bar{\lambda}_{muf} = 4$ and $\bar{\lambda}_{mur} = 4.5$. Then, by using these m_{uf} and m_{ur}, and the half-car model parameters listed in Table I, we can represent suspension system by a four-vertex polytopic system. It is apparently seen that the dimension of LMIs in (3.29)–(3.30) increases with partition number m increases, which means that the computational complexity increases accordingly. Here, we only consider the case for robust controller existence condition in Theorem 3.2 when $m = 1$. For $m = 1$, by solving the convex optimization problem formulated in (3.36), we know that the minimum guaranteed closed-loop H_∞ performance index is $\gamma_{\min} = 7.9560$ and admissible control
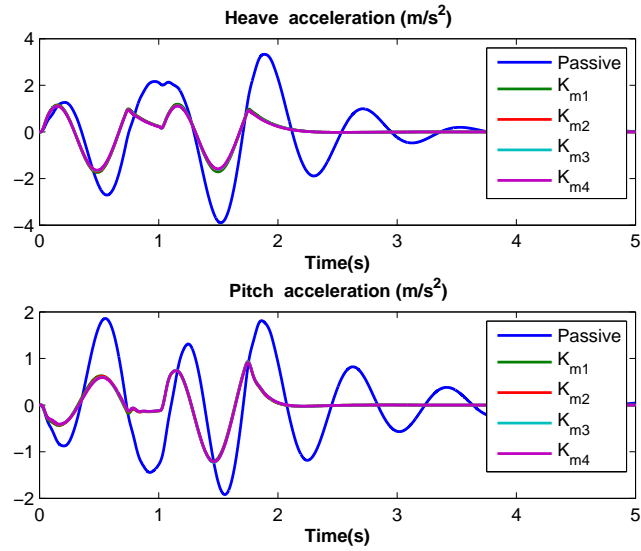
Figure 3.2: Bump responses of the heave accelerations and the pitch acceleration
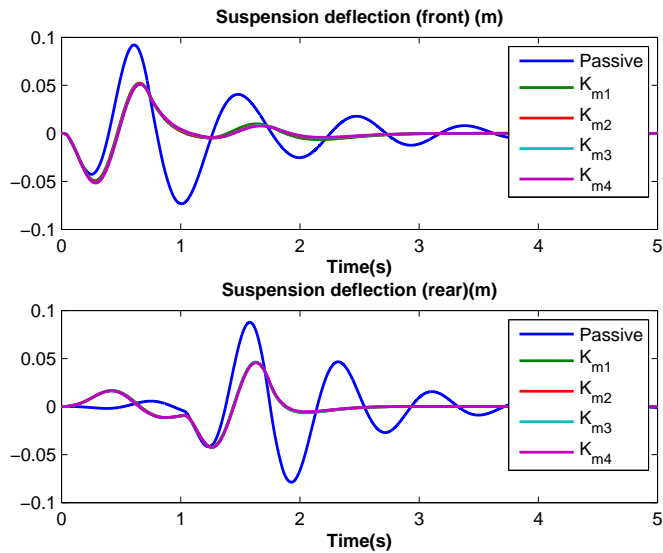
Figure 3.3: Bump responses of the front and rear suspension deflections
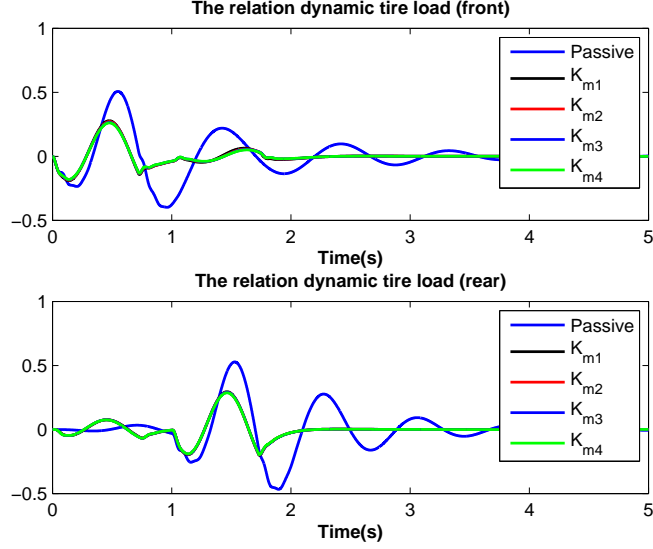
Figure 3.4: Bump responses of tire deflection constraints

gain matrix is given as

$$K_1 = 10^5 \times \begin{bmatrix} 0.0678 & 0.0146 & 0.8839 & 0.9241 \\ 0.0102 & 0.0573 & 0.0638 & 1.1549 \\ -0.0397 & -0.0025 & 0.0326 & 0.0100 \\ 0.0079 & -0.0395 & 0.0061 & 0.0393 \end{bmatrix}. \tag{3.42}$$

Then, we assume that $\bar{\lambda}_{muf} = 8$ and $\bar{\lambda}_{mur} = 9$. By solving the convex optimization problem formulated in (3.36), it is found that the minimum guaranteed closed-loop H_∞ performance index is $\gamma_{\min} = 8.8371$ and admissible control gain matrix is expressed as

$$K_2 = 10^4 \times \begin{bmatrix} 0.8139 & 0.1133 & 6.7534 & 6.8900 \\ -0.0220 & 0.7336 & 0.4975 & 9.1088 \\ -0.3248 & -0.0140 & 0.3468 & 0.0720 \\ 0.0430 & -0.3388 & 0.0353 & 0.4051 \end{bmatrix}. \tag{3.43}$$

Next, we will focus on the performance analysis of the closed-loop suspension system with different parameter uncertainties under the proposed corresponding controllers K_1 and K_2. Fig. 3.5 and 3.8 show bump responses of the heave accelerations and the pitch acceleration of the closed-loop systems for $\bar{\lambda}_{muf} = 4, -4,$

$\bar{\lambda}_{mur} = 4.5, -4.5$, $\bar{\lambda}_{muf} = 8, -8$, $\bar{\lambda}_{mur} = 9, -9$. Fig. 3.6 and 3.9 demonstrate the front and rear suspension deflections of the closed-loop systems in time domain, which means that the requirement constrains are satisfied. Fig. 3.7 and 3.10 depict the bump responses of the relation dynamic front and rear tyre deflection constrains. In all, Fig. 3.5–3.10 show that the closed-loop system are robustly asymptotically stable and have a much better performance than the open-loop system.
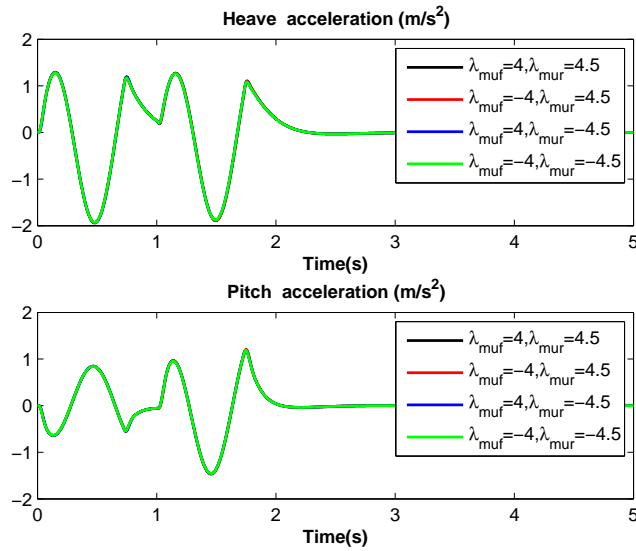
Figure 3.5: Bump responses of the heave accelerations and the pitch acceleration

Remark 3.4: It is assumed that the two actuator forces have the same time delay as (Du & Zhang, 2008), which investigated the H_∞ controller design problem for half-vehicle active suspension systems with actuator delay. However, it should be noticed that the half-vehicle suspension model does not involve with model uncertainties and the input time-varying delay. Our half-car suspension system model is more general than the one in (Du & Zhang, 2008) and makes more sense for the control design of the active suspension systems.
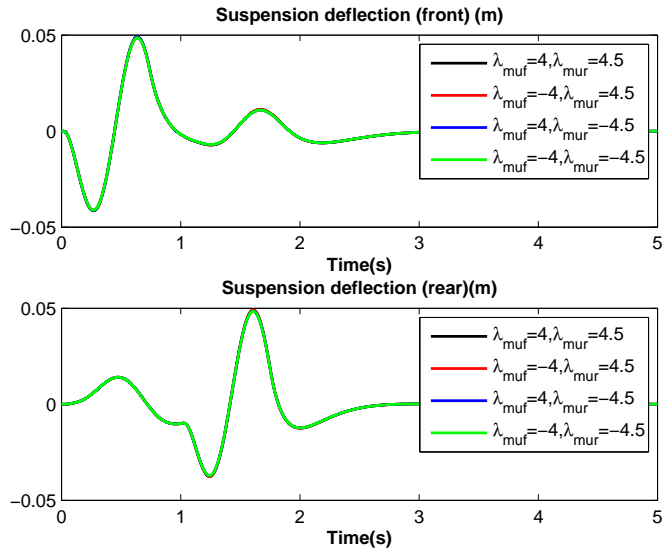
Figure 3.6: Bump responses of the front and rear suspension deflections
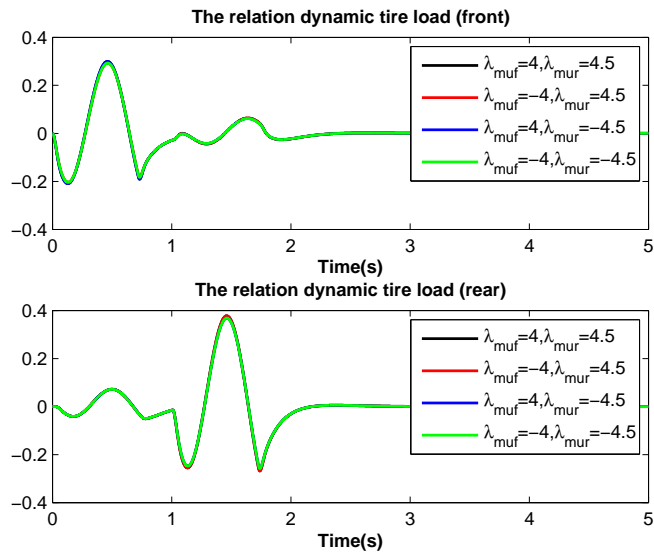
Figure 3.7: Bump responses of tire deflection constraints
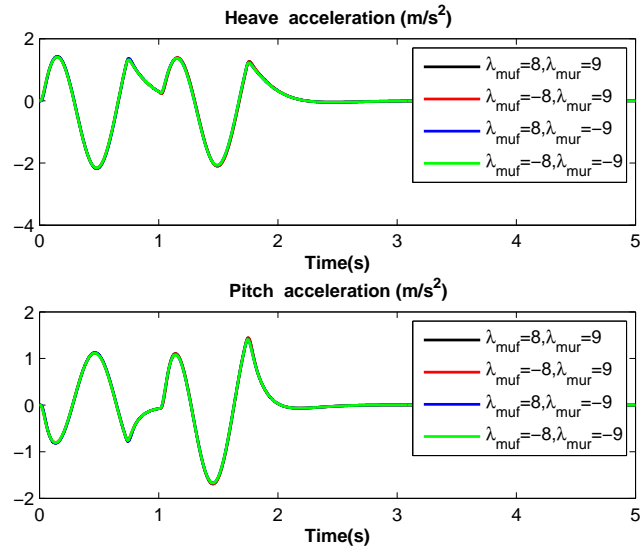
Figure 3.8: Bump responses of the heave accelerations and the pitch acceleration
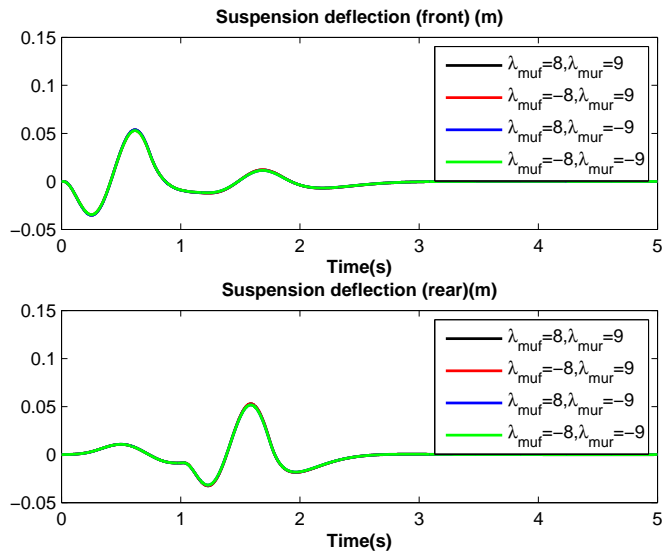
Figure 3.9: Bump responses of the front and rear suspension deflections
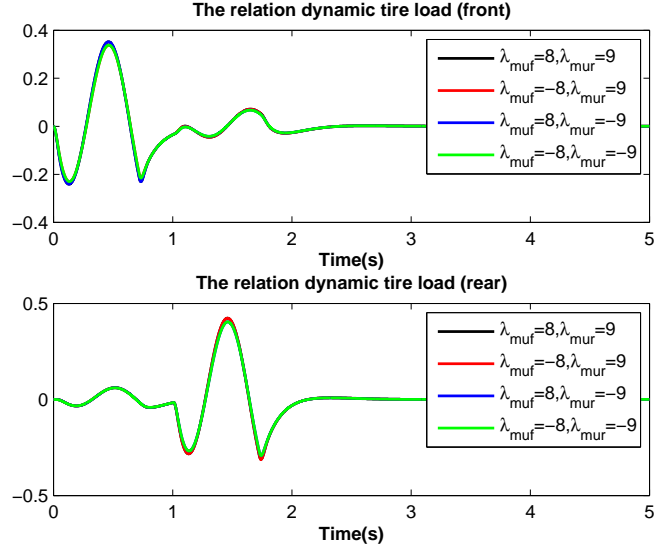
Figure 3.10: Bump responses of the tire deflection constraints

3.3 Output-feedback Control for Active Suspensions Systems with Actuator Time-varying Delay

3.3.1 Problem Formulation

The authors in (Du & Zhang, 2007; Du *et al.*, 2008) investigated the constrained H_∞ control scheme for active suspensions with actuator delay by using state feedback method under the assumption that the state vectors are all on-line measurable. However, when the state variables of the suspension systems are not measurable, the above mentioned methods are not feasible for these kinds of the systems with actuator delay. On the other hand, there exist some results on controller design for the suspension systems by using output-feedback control approach, e.g., Akbari & Lohmann (2008); Hayakawa *et al.* (2002); Wang & Wilson (2001), however, which are not feasible for the control design of suspension systems with actuator delay. Lack of effective research results motivates this study in investigating dynamic output-feedback H_∞ controller design for active suspension systems with actuator delay.

3.3 Output-feedback Control for Active Suspensions Systems with Actuator Time-varying Delay

Consider active vehicle suspension system in (2.5) with actuator time-varying delay,

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + Bu\left(t - d\left(t\right)\right), \\
z_1(t) &= C_1 x(t) + D_1 u\left(t - d\left(t\right)\right), \\
z_2(t) &= C_2 x(t), \\
y(t) &= Cx\left(t\right),
\end{aligned}
\tag{3.44}
$$

where A, B_1, B, C_1, D_1 are defined in Chapter 2 (2.2) and (2.5), and

$$
C = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix},
$$

$d\left(t\right)$ is a known time-varying delay and satisfies $0 < d\left(t\right) \le \bar{d}$, $\dot{d}\left(t\right) \le \mu$.

The active control force provided for the active suspension system should be confined to a certain range prescribed by limited power of the actuator:

$$
|u(t)| \le u_{\max},
\tag{3.45}
$$

where $u_{\max}$ is defined as the maximum possible actuator control force.

We design the dynamic output-feedback controller for the system in (3.44). First of all, the full order dynamic controller of the following form is constructed as:

$$
\begin{cases}
\dot{\hat{x}}(t) = A_c \hat{x}(t) + A_{cd}\hat{x}(t - d\left(t\right)) + B_c y\left(t\right), \\
u(t) = C_c \hat{x}(t),
\end{cases}
\tag{3.46}
$$

where $\hat{x}\left(t\right) \in \mathbb{R}^n$ is the state vector of the dynamic controller; A_c, A_{cd}, B_c, and C_c are appropriately dimensioned controller matrices to be determined. It is worth mentioning that in our approach, the introduction of the term $A_{cd}\hat{x}(t - d\left(t\right))$ is essential to make the controller synthesis feasible. Applying this controller in (3.46) to the system in (3.44) yields the following closed-loop system

$$
\begin{aligned}
\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\bar{x}(t - d\left(t\right)) + \bar{B}_1 w(t), \\
z_1(t) &= \bar{C}_1 \bar{x}(t) + \bar{D}_1 \bar{x}(t - d\left(t\right)), \\
z_2(t) &= \bar{C}_2 \bar{x}(t),
\end{aligned}
\tag{3.47}
$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \bar{A} = \begin{bmatrix} A & 0 \\ B_c C & A_c \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 & BC_c \\ 0 & A_{cd} \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \bar{C}_1 = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \bar{D}_1 = \begin{bmatrix} 0 & D_1 C_c \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} C_2 & 0 \end{bmatrix}.$$

It is assumed that $w \in L_2[0, \infty)$, without loss of generality, we have $\|w\|_2^2 \leq w_{\max} < \infty$. Then, the objective of this chapter is to determine the controller in (3.46) such that: (i) the closed-loop system is asymptotically stable; (ii) under zero initial condition, the closed-loop system guarantees that $\|z_1\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0, \infty)$, where $\gamma > 0$ is a prescribed scalar; (iii) the following control output constraints are guaranteed:

$$|\{z_2(t)\}_q| \leq \{z_{2,\max}\}_q, \quad q = 1, 2, \quad t > 0, \tag{3.48}$$

where $z_{2,\max} = \begin{bmatrix} z_{\max} & 1 \end{bmatrix}^T$; (iv) the following maximum possible actuator control force constraint is guaranteed:

$$|u(t)| \leq u_{\max}, \tag{3.49}$$

where $u(t) = \bar{C}_c \bar{x}(t)$ with $\bar{C}_c = \begin{bmatrix} 0 & C_c \end{bmatrix}$.

We formulate the multiple requirements in a unified framework, based on which the controller design can be cast into a multi-objective minimization problem, which will be solved by using output-feedback approach in the next section.

3.3.2 Output-feedback H_∞ Controller Design

In this subsection, we will solve the problem of dynamic output-feedback H_∞ controller design for the active suspension systems with control delay in (3.44). First, we develop H_∞ performance analysis condition for the system in (3.47). More specifically, for known controller gain matrices A_c, A_{cd}, B_c and C_c in (3.46), Theorem 1 presents the conditions under which the closed-loop systems in (3.47) is asymptotically stable with an H_∞ disturbance attention level and satisfies the output constrains in (3.48) and maximum actuator control force constraint in (3.49).

Theorem 3.3 Consider the closed-loop system in (3.47). For given scalars $\bar{d} > 0$, μ $q = 1, 2$ and controller matrices A_c, A_{cd}, B_c and C_c, if there exist matrices $P > 0$, $Q > 0$, $R > 0$, N_p, and M_p $(p = 1, 2, 3, 4)$ with appropriate dimensions such that the following LMIs hold:

$$
\begin{bmatrix}
\Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\
* & -R & 0 & 0 \\
* & * & -I & 0 \\
* & * & * & -R
\end{bmatrix}
< 0,
\tag{3.50}
$$

$$
\begin{bmatrix}
\Theta_{11} & \Theta_{22} & \Theta_{13} & \Theta_{14} \\
* & -R & 0 & 0 \\
* & * & -I & 0 \\
* & * & * & -R
\end{bmatrix}
< 0,
\tag{3.51}
$$

$$
\begin{bmatrix}
-\{z_{2,\max}\}_q^2 P & \sqrt{\rho}\{\bar{C}_2\}_q^T \\
* & -I
\end{bmatrix}
< 0,
\tag{3.52}
$$

$$
\begin{bmatrix}
-u_{\max}^2 P & \sqrt{\rho}\bar{C}_c^T \\
* & -I
\end{bmatrix}
< 0,
\tag{3.53}
$$

where

$$
\Theta_{11} =
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\
* & \Psi_{22} & \Psi_{23} & \Psi_{24} \\
* & * & \Psi_{33} & -N_4^T \\
* & * & * & -\gamma^2 I
\end{bmatrix},
\Theta_{12} =
\begin{bmatrix}
\sqrt{\bar{d}}M_1 \\
\sqrt{\bar{d}}M_2 \\
\sqrt{\bar{d}}M_3 \\
\sqrt{\bar{d}}M_4
\end{bmatrix},
$$

$$
\Theta_{22} =
\begin{bmatrix}
\sqrt{\bar{d}}N_1 \\
\sqrt{\bar{d}}N_2 \\
\sqrt{\bar{d}}N_3 \\
\sqrt{\bar{d}}N_4
\end{bmatrix},
\Theta_{13} =
\begin{bmatrix}
\bar{C}_1^T \\
\bar{D}_1^T \\
0 \\
0
\end{bmatrix},
\Theta_{14} =
\begin{bmatrix}
\sqrt{\bar{d}}\bar{A}^T R \\
\sqrt{\bar{d}}\bar{B}^T R \\
0 \\
\sqrt{\bar{d}}\bar{B}_1^T R
\end{bmatrix},
$$

$$
\Psi_{11} = P\bar{A} + \bar{A}^T P + Q + S + M_1 + M_1^T,
$$

$$
\Psi_{12} = P\bar{B} - M_1 + M_2^T + N_1,
$$

$$
\Psi_{13} = M_3^T - N_1, \Psi_{14} = P\bar{B}_1 + M_4^T,
$$

$$
\Psi_{22} = -(1 - \mu)S + N_2 + N_2^T - M_2 - M_2^T,
$$

$$
\Psi_{23} = N_3^T - N_2 - M_3^T, \Psi_{24} = N_4^T - M_4^T,
$$

$$
\Psi_{33} = -Q - N_3 - N_3^T.
$$

Then we know: (i) the closed-loop system is asymptotically stable for the delay $d(t)$ satisfying $0 < d(t) \leq \bar{d}$, $\dot{d}(t) \leq \mu$; (ii) under zero initial condition, the

performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to the output constraints in (3.48) and maximum possible actuator control force constraint in (3.49) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$, where $T_{z_1 w}$ denotes the closed-loop transfer function from the road disturbance $w(t)$ to the control output $z_1(t)$.

Proof. Considering the Lyapunov-Krasovskii functional as follows:

$$
\begin{aligned}
V(t) &= \bar{x}^T(t) P \bar{x}(t) + \int_{t-\bar{d}}^{t} \bar{x}^T(s) Q \bar{x}(s) \, \mathrm{d}s \\
&\quad + \int_{t-d(t)}^{t} \bar{x}^T(s) S \bar{x}(s) \, \mathrm{d}s + \int_{-\bar{d}}^{0} \int_{t+\alpha}^{t} \dot{\bar{x}}^T(s) R \dot{\bar{x}}(s) \, \mathrm{d}s \mathrm{d}\alpha.
\end{aligned}
\tag{3.54}
$$

We can obtain the derivative of $V(t)$ from the solution of system (3.47) as

$$
\begin{aligned}
\dot{V}(t) &\leq 2\bar{x}^T(t) P \dot{\bar{x}}(t) + \bar{x}^T(t)(Q+S)\bar{x}(t) - \bar{x}^T(t-\bar{d}) Q \bar{x}(t-\bar{d}) \\
&\quad + \bar{d} \dot{\bar{x}}^T(t) R \dot{\bar{x}}(t) - (1-\mu)\bar{x}^T(t-d(t)) S \bar{x}(t-d(t)) \\
&\quad - \int_{t-d(t)}^{t} \dot{\bar{x}}^T(s) R \dot{\bar{x}}(s) \, \mathrm{d}s - \int_{t-\bar{d}}^{t-d(t)} \dot{\bar{x}}^T(s) R \dot{\bar{x}}(s) \, \mathrm{d}s.
\end{aligned}
\tag{3.55}
$$

For any appropriately dimensioned matrices $\hat{M}$ and $\hat{N}$, the following equalities hold directly according to Newton-Leibniz formula:

$$
\delta_1(t) = 2\xi^T(t) \hat{M} \left(\bar{x}(t) - \bar{x}(t-d(t)) - \int_{t-d(t)}^{t} \dot{\bar{x}}(s) \, \mathrm{d}s \right) = 0,
$$

$$
\delta_2(t) = 2\xi^T(t) \hat{N} \left(\bar{x}(t-d(t)) - \bar{x}(t-\bar{d}) - \int_{t-\bar{d}}^{t-d(t)} \dot{\bar{x}}(s) \, \mathrm{d}s \right) = 0,
$$

where

$$
\begin{aligned}
\hat{M} &= \begin{bmatrix} M_1^T & M_2^T & M_3^T \end{bmatrix}^T, \hat{N} = \begin{bmatrix} N_1^T & N_2^T & N_3^T \end{bmatrix}^T, \\
\xi^T(t) &= \begin{bmatrix} \bar{x}^T(t) & \bar{x}^T(t-d(t)) & \bar{x}^T(t-\bar{d}) \end{bmatrix}.
\end{aligned}
$$

Adding $\delta_1(t)$ and $\delta_2(t)$ into the right hand side of (3.54) and after some simply

calculation, the following inequalities are true:

$$
\begin{aligned}
\dot{V}(t) \;\leq\;& \xi^T(t)\left[\Psi + d(t)\,\hat{M}R^{-1}\hat{M}^T + \left(\bar{d} - d(t)\right)\hat{N}R^{-1}\hat{N}^T\right]\xi(t) \\
& - \int_{t-d(t)}^{t}\left[\xi^T(t)\,\hat{M} + \dot{x}^T(s)\,R\right]R^{-1}\left[\hat{M}^T\xi(t) + R\dot{x}(s)\right]\mathrm{d}s \\
& - \int_{t-\bar{d}}^{t-d(t)}\left[\xi^T(t)\,\hat{N} + \dot{x}^T(s)\,R\right]R^{-1}\left[\hat{N}^T\xi(t) + R\dot{x}(s)\right]\mathrm{d}s \\
\leq\;& \xi^T(t)\left[\Psi + d(t)\,\hat{M}R^{-1}\hat{M}^T + \left(\bar{d} - d(t)\right)\hat{N}R^{-1}\hat{N}^T\right]\xi(t) \\
=\;& \xi^T(t)\left[\frac{d(t)}{\bar{d}}\left(\Psi + \bar{d}\hat{M}R^{-1}\hat{M}^T\right) + \frac{\bar{d} - d(t)}{\bar{d}}\left(\Psi + \bar{d}\hat{N}R^{-1}\hat{N}^T\right)\right]\xi(t),
\end{aligned}
$$

where

$$
\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \\ 0 \end{bmatrix}\bar{d}R\begin{bmatrix} \bar{A}^T \\ \bar{B}^T \\ 0 \end{bmatrix}^T .
$$

On the other hand, according to (3.50)–(3.51) and Schur complement, it is seen that

$$
\Psi + \bar{d}\hat{M}R^{-1}\hat{M}^T < 0, \quad \Psi + \bar{d}\hat{N}R^{-1}\hat{N}^T < 0,
$$

which implies $\dot{V}(t) < 0$, then system in (3.47) is asymptotically stable. Next, we establish the H_∞ performance of the system in (3.47) under zero initial conditions. Firstly, we define the Lyapunov functional as in (3.54). Then, by following the same line as in the above proof, we obtain

$$
\begin{aligned}
& \dot{V}(t) + z_1^T(t)z_1(t) - \gamma^2 w^T(t)w(t) \\
\leq\;& \bar{\xi}^T(t)\left[\Psi + d(t)\,MR^{-1}M^T + \left(\bar{d} - d(t)\right)NR^{-1}N^T\right]\bar{\xi}(t) \\
=\;& \bar{\xi}^T(t)\left[\frac{d(t)}{\bar{d}}\left(\Psi + \bar{d}MR^{-1}M^T\right) + \frac{\bar{d} - d(t)}{\bar{d}}\left(\Psi + \bar{d}NR^{-1}N^T\right)\right]\bar{\xi}(t),
\end{aligned}
$$

where

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ * & * & \Psi_{33} & -N_4^T \\ * & * & * & -\gamma^2 I \end{bmatrix}$$
$$+ \begin{bmatrix} \bar{C}_1^T \\ \bar{D}_1^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \bar{C}_1^T \\ \bar{D}_1^T \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \\ 0 \\ \bar{B}_1^T \end{bmatrix} \bar{d}R \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \\ 0 \\ \bar{B}_1^T \end{bmatrix}^T,$$

$$M = \begin{bmatrix} M_1^T & M_2^T & M_3^T & M_4^T \end{bmatrix}^T, N = \begin{bmatrix} N_1^T & N_2^T & N_3^T & N_4^T \end{bmatrix}^T,$$

$$\bar{\xi}^T(t) = \begin{bmatrix} \bar{x}^T(t) & \bar{x}^T(t - d(t)) & \bar{x}^T(t - \bar{d}) & w^T(t) \end{bmatrix}.$$

By using Schur complement to (3.50)–(3.51) and the above method, we develop

$$\dot{V}(t) + z_1^T(t)z_1(t) - \gamma^2 w^T(t)w(t) < 0, \tag{3.56}$$

for all nonzero $w \in L_2[0, \infty)$. Under zero initial conditions, we have $V(0) = 0$ and $V(\infty) \geq 0$. Integrating both sides of (3.56) yields $\|z_1\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0, \infty)$, and then the H_∞ performance is established. Then, we will show that the hard constraints in (3.48)–(3.49) are guaranteed. Inequality (3.56) guarantees $\dot{V}(t) - \gamma^2 w^T(t)w(t) < 0$. Integrating both sides of the above inequality from zero to any $t > 0$, we obtain

$$V(t) - V(0) < \gamma^2 \int_0^t w^T(\tau)w(\tau)\mathrm{d}\tau < \gamma^2 \|w\|_2^2.$$

From the definition of the Lyapunov functional in (3.54), we know that $\bar{x}^T(t)P\bar{x}(t) < \rho$ with $\rho = \gamma^2 w_{\max} + V(0)$. Similar to (Gao *et al.*, 2010a), the following inequalities hold

$$\max_{t>0} |\{z_2(t)\}_q|^2$$
$$= \max_{t>0} \left\| \bar{x}^T(t)\{\bar{C}_2\}_q^T\{\bar{C}_2\}_q\bar{x}(t) \right\|_2$$
$$= \max_{t>0} \left\| \bar{x}^T(t)P^{\frac{1}{2}}P^{-\frac{1}{2}}\{\bar{C}_2\}_q^T\{\bar{C}_2\}_q P^{-\frac{1}{2}}P^{\frac{1}{2}}\bar{x}(t) \right\|_2$$
$$< \rho \cdot \theta_{\max}\left(P^{-\frac{1}{2}}\{\bar{C}_2\}_q^T\{\bar{C}_2\}_q P^{-\frac{1}{2}}\right), \quad q = 1, 2,$$

and

$$
\begin{aligned}
& \max_{t>0} |u(t)|^2 \\
= {} & \max_{t>0} \left\| \bar{x}^T(t) \bar{C}_c^T \bar{C}_c \bar{x}(t) \right\|_2 \\
= {} & \max_{t>0} \left\| \bar{x}^T(t) P^{\frac{1}{2}} P^{-\frac{1}{2}} \bar{C}_c^T \bar{C}_c P^{-\frac{1}{2}} P^{\frac{1}{2}} \bar{x}(t) \right\|_2 \\
< {} & \rho \cdot \theta_{\max} \left(P \bar{C}_c^T \bar{C}_c P^{-\frac{1}{2}} \right),
\end{aligned}
$$

where $\theta_{\max}(\cdot)$ represents maximal eigenvalue. The constraints in (3.48) can be guaranteed, if

$$
\begin{aligned}
\rho \cdot P^{-\frac{1}{2}} \left\{ \bar{C}_2 \right\}_q^T \left\{ \bar{C}_2 \right\}_q P^{-\frac{1}{2}} &< \{z_{2,\max}\}_q^2 I, \quad q = 1, 2, \\
\rho \cdot P^{-\frac{1}{2}} \bar{C}_c^T \bar{C}_c P^{-\frac{1}{2}} &< u_{\max}^2 I,
\end{aligned}
$$

which can be guaranteed by the feasibility of (3.52) and (3.53). The proof is completed. ∎

Remark 3.5 Theorem 3.3 presents H_∞ performance analysis for the suspension system with control delay in system (3.47). When there is no input delay in the quarter-car model, the vehicle suspension system can be described by the following state-space equations:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + B_1 w(t), \\
z_1(t) &= C_1 x(t) + D_1 u(t), \\
z_2(t) &= C_2 x(t), \\
y(t) &= C x(t),
\end{aligned} \tag{3.57}
$$

where A, B_1, B, C_1, C_1, D_1 and C_2 are defined in Chapter 2 (2.2) and (2.5), respectively. We consider the following dynamics controller for the system in (3.57):

$$
\begin{cases}
\dot{\hat{x}}(t) = A_c \hat{x}(t) + B_c y(t), \\
u(t) = C_c \hat{x}(t),
\end{cases} \tag{3.58}
$$

where $\hat{x}(t) \in R^n$ is the state vector of the dynamics controller, A_c, B_c, and C_c are appropriately dimensioned controller matrices to be determined. Applying

the dynamics controller (3.58) to the system in (3.57) leads to the closed-loop system

$$
\begin{aligned}
\dot{\bar{x}}(t) &= \tilde{A}\bar{x}(t) + \tilde{B}_1 w(t), \\
z_1(t) &= \tilde{C}_1\bar{x}(t), \\
z_2(t) &= \tilde{C}_2\bar{x}(t),
\end{aligned}
\tag{3.59}
$$

where

$$
\begin{aligned}
\tilde{A} &= \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\
\tilde{C}_1 &= \begin{bmatrix} C_1 & D_1C_c \end{bmatrix}, \quad \tilde{C}_2 = \begin{bmatrix} C_2 & 0 \end{bmatrix}.
\end{aligned}
$$

Then, we have the following corollary, which can be proved by following arguments similar to the proof of Theorem 3.3.

Corollary 3.1 Consider the closed-loop system in (3.47). Given scalar $q = 1, 2$, and controller matrices A_c, B_c and C_c, the closed-loop system (3.47) is asymptotically stable with an H_∞ disturbance attenuation level γ, if there exists a matrix $P > 0$ with appropriate dimension such that the following LMIs hold:

$$
\begin{bmatrix} P\tilde{A} + \tilde{A}^T P & P\tilde{B}_1 & \tilde{C}_1^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{3.60}
$$

$$
\begin{bmatrix} -\{z_{2,\max}\}_q^2 P & \sqrt{\rho}\left\{\tilde{C}_2\right\}_q^T \\ * & -I \end{bmatrix} < 0, \tag{3.61}
$$

$$
\begin{bmatrix} -u_{\max}^2 P & \sqrt{\rho}\bar{C}_c^T \\ * & -I \end{bmatrix} < 0, \tag{3.62}
$$

Then (i) the closed-loop system is asymptotically stable; (ii) under zero initial condition, the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to output constraints in (3.48) and maximum possible actuator control force constraint in (3.49) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$.

Proof: The proof of Corollary 3.1 can be easily completed by choosing the Lyapunov functional $V(t) = \bar{x}^T(t) P\bar{x}(t)$ and following the similar line of Theorem 3.3 and the method in (Chen & Guo, 2005). Therefore, it is omitted.

In the sequel, we will solve the controller matrices A_c, A_{cd}, B_c and C_c in (3.46).To solve the controller synthesis problem, we will transform the conditions in (3.50)–(3.52) into tractable conditions. Based on dynamic output-feedback H_∞ performance analysis condition in Theorem 3.3, the controller existence condition for the suspension system in (3.47) is presented in the following theorem.

Theorem 3.4 Consider the suspension system in (3.47). Given scalars $\bar{d} > 0$, μ, $\theta_R > 0$, $q = 1, 2$, if there exist matrices $\bar{Q} = \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} > 0$, $\bar{S} = \begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} > 0$, $\bar{R} = \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} > 0$, $\mathcal{R} > 0$, $\mathcal{S} > 0$, $\mathcal{A}$, $\mathcal{A}_d$, $\mathcal{B}$, $\mathcal{C}$, $\bar{M}_p = \begin{bmatrix} M_{p1} & M_{p2} \\ M_{p3} & M_{p4} \end{bmatrix}$ and $\bar{N}_p = \begin{bmatrix} N_{p1} & N_{p2} \\ N_{p3} & N_{p4} \end{bmatrix}$ $(p = 1, 2, 3, 4)$ with appropriate dimensions such as the following LMIs hold:

$$\begin{bmatrix} \bar{\Theta}_{11} & \bar{\Theta}_{12} & \bar{\Theta}_{13} & \bar{\Theta}_{14} \\ * & -\bar{R} & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & \bar{\Theta}_{44} \end{bmatrix} < 0, \tag{3.63}$$

$$\begin{bmatrix} \bar{\Theta}_{11} & \bar{\Theta}_{22} & \bar{\Theta}_{13} & \bar{\Theta}_{14} \\ * & -\bar{R} & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & \bar{\Theta}_{44} \end{bmatrix} < 0, \tag{3.64}$$

$$\begin{bmatrix} -\{z_{2,\max}\}_q^2 \mathcal{R} & -\{z_{2,\max}\}_q^2 I & \mathcal{R}\{C_2\}_q^T \\ * & -\{z_{2,\max}\}_q^2 \mathcal{S} & \{C_2\}_q^T \\ * & * & -I \end{bmatrix} < 0, \tag{3.65}$$

$$\begin{bmatrix} -u_{\max}^2 \mathcal{R} & -u_{\max}^2 I & \mathcal{C}^T \\ * & -u_{\max}^2 \mathcal{S} & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{3.66}$$

$$\begin{bmatrix} \mathcal{R} & I \\ I & \mathcal{S} \end{bmatrix} > 0 \tag{3.67}$$

where

$$
\bar{\Theta}_{11} = \begin{bmatrix} \bar{\Psi}_{11} & \bar{\Psi}_{12} & \bar{\Psi}_{13} & \bar{\Psi}_{14} \\ * & \bar{\Psi}_{22} & \bar{\Psi}_{23} & \bar{\Psi}_{24} \\ * & * & \bar{\Psi}_{33} & -\bar{N}_4^T \\ * & * & * & -\gamma^2 I \end{bmatrix}, \bar{\Theta}_{12} = \begin{bmatrix} \sqrt{\bar{d}}\bar{M}_1 \\ \sqrt{\bar{d}}\bar{M}_2 \\ \sqrt{\bar{d}}\bar{M}_3 \\ \sqrt{\bar{d}}\bar{M}_4 \end{bmatrix},
$$

$$
\bar{\Theta}_{22} = \begin{bmatrix} \sqrt{\bar{d}}\bar{N}_1 \\ \sqrt{\bar{d}}\bar{N}_2 \\ \sqrt{\bar{d}}\bar{N}_3 \\ \sqrt{\bar{d}}\bar{N}_4 \end{bmatrix}, \bar{\Theta}_{13} = \begin{bmatrix} \bar{\Psi}_{16} \\ \bar{\Psi}_{26} \\ 0 \\ 0 \end{bmatrix}, \bar{\Theta}_{14} = \begin{bmatrix} \bar{\Psi}_{17} \\ \bar{\Psi}_{27} \\ 0 \\ \bar{\Psi}_{47} \end{bmatrix}
$$

$$
\bar{\Psi}_{11} = \begin{bmatrix} A\mathcal{R} + \mathcal{R}A^T + M_{11} + M_{11}^T + Q_1 + S_1 \\ * \end{bmatrix}
$$
$$
\begin{bmatrix} A + \mathcal{A}^T + M_{12} + M_{13}^T + Q_2 + S_2 \\ \mathcal{S}A + A^T\mathcal{S} + \mathcal{B}C + C^T\mathcal{B}^T + M_{14} + M_{14}^T + Q_3 + S_3 \end{bmatrix}
$$

$$
\bar{\Psi}_{12} = \begin{bmatrix} \mathcal{B}C + M_{21}^T - M_{11} + N_{11} & M_{23}^T - M_{12} + N_{12} \\ \mathcal{A}_d + M_{22}^T - M_{13} + N_{13} & M_{24}^T - M_{14} + N_{14} \end{bmatrix},
$$

$$
\bar{\Psi}_{13} = \begin{bmatrix} M_{31}^T - N_{11} & M_{33}^T - N_{12} \\ M_{32}^T - N_{13} & M_{34}^T - N_{14} \end{bmatrix}, \bar{\Psi}_{14} = \begin{bmatrix} B_1 + M_{41}^T \\ \mathcal{S}B_1 + M_{42}^T \end{bmatrix},
$$

$$
\bar{\Psi}_{16} = \begin{bmatrix} \mathcal{R}C_1^T \\ C_1^T \end{bmatrix}, \bar{\Psi}_{17} = \begin{bmatrix} \sqrt{\bar{d}}\mathcal{R}A^T & \sqrt{\bar{d}}\mathcal{A}^T \\ \sqrt{\bar{d}}A^T & \sqrt{\bar{d}}A^T\mathcal{S} + \sqrt{\bar{d}}C^T\mathcal{B}^T \end{bmatrix},
$$

$$
\bar{\Psi}_{24} = \begin{bmatrix} N_{41}^T - M_{41}^T \\ N_{42}^T - M_{42}^T \end{bmatrix}, \bar{\Psi}_{26} = \begin{bmatrix} \mathcal{C}D_1^T \\ 0 \end{bmatrix},
$$

$$
\bar{\Psi}_{23} = \begin{bmatrix} N_{31}^T - N_{21} - M_{31}^T & N_{33}^T - N_{22} - M_{33}^T \\ N_{32}^T - N_{23} - M_{32}^T & N_{34}^T - N_{24} - M_{34}^T \end{bmatrix},
$$

$$
\bar{\Psi}_{22} = \begin{bmatrix} N_{21} + N_{21}^T - M_{21} - M_{21}^T - (1-\mu)S_1 \\ * \end{bmatrix}
$$
$$
\begin{bmatrix} N_{22} + N_{23}^T - M_{22} - M_{23}^T - (1-\mu)S_2 \\ N_{24} + N_{24}^T - M_{24} - M_{24}^T - (1-\mu)S_3 \end{bmatrix},
$$

$$
\bar{\Psi}_{33} = \begin{bmatrix} -Q_1 - N_{31} - N_{31}^T & -Q_2 - N_{32} - N_{33}^T \\ * & -Q_3 - N_{34} - N_{34}^T \end{bmatrix},
$$

$$
\bar{\Psi}_{27} = \begin{bmatrix} \sqrt{\bar{d}}\mathcal{C}^T B^T & \sqrt{\bar{d}}\mathcal{A}_d^T \\ 0 & 0 \end{bmatrix}, \bar{\Psi}_{47} = \begin{bmatrix} \sqrt{\bar{d}}B_1^T & \sqrt{\bar{d}}B_1^T\mathcal{S} \end{bmatrix},
$$

$$
\bar{\Theta}_{44} = \begin{bmatrix} \theta_R^2 R_1 - 2\theta_R\mathcal{R} & \theta_R^2 R_2 - 2\theta_R I \\ * & \theta_R^2 R_3 - 2\theta_R\mathcal{S} \end{bmatrix}.
$$

Then, there exists a dynamic controller such that the closed-loop system in (3.47) is asymptotically stable. In this case, a desired output-feedback controller

is given in the form of (3.46) with parameters as follows:

$$A_c = \mathcal{N}^{-1} \left(\mathcal{A} - \mathcal{N}B_c C \mathcal{R} - \mathcal{S}A\mathcal{R} \right) \mathcal{M}^{-T}, \tag{3.68}$$

$$A_{cd} = \mathcal{N}^{-1} \left(\mathcal{A}_d - \mathcal{S}BC_c \mathcal{M}^T \right) \mathcal{M}^{-T}, \tag{3.69}$$

$$B_c = \mathcal{N}^{-1}\mathcal{B}, \tag{3.70}$$

$$C_c = \mathcal{C}\mathcal{M}^{-T}, \tag{3.71}$$

where $\mathcal{N}$ and $\mathcal{M}$ are any nonsingular matrices satisfying

$$\mathcal{M}\mathcal{N}^T = I - \mathcal{R}\mathcal{S}. \tag{3.72}$$

Then a controller in the form of (3.46) exists, such that: (i) the closed-loop system is asymptotically stable for the delay $d(t)$ satisfying $0 < d(t) \leq \bar{d}$, $\dot{d}(t) \leq \mu$; (ii) under zero initial condition, the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to output constraint (3.48) and maximum possible actuator control force constraint in (3.49) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$.

Proof: First, by using the method proposed in (Scherer *et al.*, 1997), we partition P and its inverse as

$$P = \begin{bmatrix} \mathcal{S} & \mathcal{N} \\ \mathcal{N}^T & \mathcal{Y} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \mathcal{R} & \mathcal{M} \\ \mathcal{M}^T & \mathcal{T} \end{bmatrix}.$$

Note that the equality $PP^{-1} = I$ leads to (3.72) holds. In fact, from $\bar{\Theta}_{44}$ in Theorem 3.4 it can be easily seen that

$$\begin{bmatrix} -\mathcal{R} & -I \\ -I & -\mathcal{S} \end{bmatrix} < 0,$$

by the Schur complement formula, which implies that $\mathcal{R} - \mathcal{S}^{-1} > 0$, therefore $I - \mathcal{R}\mathcal{S}$ is nonsingular. This ensures that there always exist nonsingular matrices $\mathcal{N}$ and $\mathcal{M}$ such that (3.72) is satisfied. Setting

$$\Phi_1 = \begin{bmatrix} \mathcal{R} & I \\ \mathcal{M}^T & 0 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} I & \mathcal{S} \\ 0 & \mathcal{N}^T \end{bmatrix}. \tag{3.73}$$

Then, we conclude form (3.73) that

$$P\Phi_1 = \Phi_2. \tag{3.74}$$

It follows that

$$\Phi_1^T P \Phi_1 = \Phi_1^T \Phi_2 = \begin{bmatrix} \mathcal{R} & I \\ I & \mathcal{S} \end{bmatrix},$$

which implies that the matrices Φ_1 and Φ_2 in (3.74) are square invertible. It is found that the matrix P can be constructed as $P = \Phi_2 \Phi_1^{-1}$ and it concludes from (3.67) and (3.74) that $P > 0$. Due to the nonsingular matrices $\mathcal{M}$ and $\mathcal{N}$, the controller matrices A_c, A_{cd}, B_c and C_c can be then obtained by solving equations (3.68)–(3.71). Then, we perform congruence transformations to (3.63)–(3.64) by $\text{diag}\{\Phi_1^{-1}, \Phi_1^{-1}, \Phi_1^{-1}, I, \Phi_1^{-1}, I, \Phi_1^{-1}\}$ and obtain the following inequalities,

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \hat{\Theta}_{14} \\ * & -R & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & \theta_R^2 R - 2\theta_R P \end{bmatrix} < 0, \tag{3.75}$$

$$\begin{bmatrix} \Theta_{11} & \Theta_{22} & \Theta_{13} & \hat{\Theta}_{14} \\ * & -R & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & \theta_R^2 R - 2\theta_R P \end{bmatrix} < 0, \tag{3.76}$$

where

$$Q = \Phi_1^{-T} \bar{Q} \Phi_1^{-1}, S = \Phi_1^{-T} \bar{S} \Phi_1^{-1}, R = \Phi_1^{-T} \bar{R} \Phi_1^{-1},$$

$$\hat{\Theta}_{14}^T = \begin{bmatrix} \sqrt{\bar{d}} P \bar{A} & \sqrt{\bar{d}} P \bar{B} & 0 & \sqrt{\bar{d}} P \bar{B}_1 \end{bmatrix},$$

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} \Phi_1^{-T} & 0 & 0 & 0 \\ 0 & \Phi_1^{-T} & 0 & 0 \\ 0 & 0 & \Phi_1^{-T} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \bar{M}_1 \\ \bar{M}_2 \\ \bar{M}_3 \\ \bar{M}_4 \end{bmatrix} \Phi_1^{-1},$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} \Phi_1^{-T} & 0 & 0 & 0 \\ 0 & \Phi_1^{-T} & 0 & 0 \\ 0 & 0 & \Phi_1^{-T} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \\ \bar{N}_4 \end{bmatrix} \Phi_1^{-1}.$$

For $\theta_R > 0$ and $R^{-1} > 0$, from

$$(\theta_R R - P) R^{-1} (\theta_R R - P) \geq 0,$$

we can conclude that

$$-P R^{-1} P \leq \theta_R^2 R - 2\theta_R P.$$

After replacing the term $\theta_R^2 R - 2\theta_R P$ in (3.75)–(3.76) with $-PR^{-1}P$ and performing congruence transformations by diag$\{I, I, I, I, I, I, P^{-1}R\}$, we know that conditions in (3.50) and (3.51) hold. On the other hand, (3.65) is equivalent to (3.52) by performing a simple congruence transformation with diag$\{\Phi_1^{-1}, I\}$. Therefore, all the conditions in Theorem 3.3 are satisfied. The proof is completed. ∎

Remark 3.6 When the actuator delay $d(t)$ is known but it is not differentiable, namely the delay $d(t)$ satisfies $0 < d(t) \leq \bar{d}$. By setting $S = 0$ in the LMIs-based conditions in Theorems 3.3–3.4, we can also develop the dynamic output-feedback controller in (3.46) for the systems in (3.44) with the actuator delay $d(t)$ satisfies $0 < d(t) \leq \bar{d}$.

Remark 3.7 To avoid bringing much conservativeness, we have introduced a scalar θ_R in the proof of Theorem 2, when enlarging the term $-PR^{-1}P$, that is,

$$-PR^{-1}P \leq \theta_R^2 R - 2\theta_R P.$$

When $\theta_R = 1$, the term reduces the one $-PR^{-1}P \leq R - 2P$, which has been used in many existing references for controller design problem for linear time-delay systems. In other words, the term $-PR^{-1}P$ has been handled with less conservativeness. This scalar θ_R must be given before solving the LMIs in Theorem 2, and the value of θ_R affects the feasibility of those related LMIs. In this chapter, we select the design parameter θ_R randomly in the allowable area, because our main aim is to design the dynamic output-feedback controller for active suspension systems with actuator delay. We can change the design parameter θ_R to solve the controller. How to choose the scalar θ_R is still an open problem, which has been targeted in our further work.

Similar to Corollary 3.1 and the proof of Theorem 3.3, the output-feedback controller synthesis condition for suspension system in (3.59) is given in the corollary as follows,

Corollary 3.2 Consider the active suspension system in (3.47). For given scalar $q = 1, 2$, if there exist matrices $\mathcal{R} > 0$, $\mathcal{S} > 0$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, with appropriate

dimensions such as the following LMIs hold:

$$
\begin{bmatrix}
\Lambda_1 & \Lambda_2 & \Lambda_3 \\
* & -\gamma^2 I & 0 \\
* & * & -I
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
-\{z_{2,\max}\}_q^2 \mathcal{R} & -\{z_{2,\max}\}_q^2 I & \mathcal{R}\{C_2\}_q^T \\
* & -\{z_{2,\max}\}_q^2 \mathcal{S} & \{C_2\}_q^T \\
* & * & -I
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
-u_{\max}^2 \mathcal{R} & -u_{\max}^2 I & \mathcal{C}^T \\
* & -u_{\max}^2 \mathcal{S} & 0 \\
* & * & -I
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
\mathcal{R} & I \\
I & \mathcal{S}
\end{bmatrix} > 0,
$$

where

$$
\Lambda_1 = \begin{bmatrix}
A\mathcal{R} + \mathcal{R}A^T + B\mathcal{C} + \mathcal{C}^T B^T & A + \mathcal{A}^T \\
* & \mathcal{S}A + A^T\mathcal{S} + \mathcal{B}C + C^T\mathcal{B}^T
\end{bmatrix},
$$

$$
\Lambda_2 = \begin{bmatrix}
B_1 \\
\mathcal{S}B_1
\end{bmatrix}, \Lambda_3 = \begin{bmatrix}
\mathcal{R}C_1^T + \mathcal{C}^T D_1^T \\
C_1^T
\end{bmatrix}.
$$

In this case, a desired dynamic output-feedback controller is given in the form of (3.58) with parameters as follows:

$$
A_c = \mathcal{N}^{-1}\left(\mathcal{A} - \mathcal{S}BC_c\mathcal{M}^T - \mathcal{N}B_cC\mathcal{R} - \mathcal{S}A\mathcal{R}\right)\mathcal{M}^{-T},
$$

$$
B_c = \mathcal{N}^{-1}\mathcal{B},
$$

$$
C_c = \mathcal{C}\mathcal{M}^{-T},
$$

where $\mathcal{N}$ and $\mathcal{M}$ are any nonsingular matrices satisfying

$$
\mathcal{M}\mathcal{N}^T = I - \mathcal{R}\mathcal{S}.
$$

such that: (i) the closed-loop system is asymptotically stable; (ii) under zero initial condition, the performance $\|T_{z_1w}\|_\infty < \gamma$ is minimized subject to output constraint (3.48) and maximum possible actuator control force constraint in (3.49).

Remark 3.8 It is evident that actuator delay is a crucial issue in vehicle suspension control systems as addressed in (Du & Zhang, 2007; Du *et al.*, 2008;

Gao *et al.*, 2010b), there exists a long-standing gap between control theory and its application in vehicle suspension systems. The main motivation of the chapter is to propose the approach and prove it theoretically. Our our future work is focused on identifying practical ways in implementing piratical scientific findings into our vehicle platform collaborated with Portean Electric Ltd.

3.3.3 Case Study

A design example is given to illustrate the effectiveness of the proposed controller design method. For the quarter-car suspension system, it is assumed that the maximum allowable suspension stroke is $z_{max} = 0.035$ m and the maximum possible actuator control force is $u_{max} = 2000N$ in this simulation part. Firstly, for $\rho = 1$, the dynamic output-feedback controller I in (3.58) for the active suspension systems without control delay in (3.57) can be derived listed as

$$
A_c = 10^6 \times \begin{bmatrix} -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ 0.0001 & -0.0001 & -0.0000 & 0.0000 \\ -0.2228 & -0.0072 & -0.0020 & 0.0000 \\ 1.2113 & 0.0502 & 0.0108 & -0.0001 \end{bmatrix},
$$

$$
B_c = 10^5 \times \begin{bmatrix} 0.0000 \\ -0.0001 \\ 0.3203 \\ -1.7482 \end{bmatrix}, \quad C_c = 10^3 \times \begin{bmatrix} -1.1222 \\ 1.4638 \\ 0.0637 \\ -0.0541 \end{bmatrix}^T. \quad (3.77)
$$

In addition, it can be found that the minimum guaranteed closed-loop H_∞ performance index γ_{min} is 9.6468.

Secondly, we propose the dynamic output-feedback controller in form of (3.46) for the active suspension systems with control delay in (3.44). For $\rho = 1$ and $\theta_R = 1$, by solving the conditions in Theorem 3.4, we can obtain the minimum guaranteed closed-loop H_∞ performance indexes γ_{min}, which are listed in Table 3.1, and the corresponding dynamic output-feedback controllers for different given upper bound $\bar{d}$ of actuator delay $d(t)$. In this chapter, we only give the dynamic output-feedback controller in (3.78) which is listed as follows for upper bound

$\bar{d} = 20$ ms due to limited space:

$$
A_c = 10^3 \times \begin{bmatrix} 0.0162 & -0.0228 & -0.0000 & -0.0000 \\ 0.1000 & -0.0550 & -0.0000 & 0.0000 \\ -4.0905 & 1.3781 & -0.0875 & 0.0003 \\ 6.8083 & -0.5845 & 0.2276 & -0.0276 \end{bmatrix},
$$

$$
A_{cd} = \begin{bmatrix} -22.5703 & 38.0368 & 1.4950 & -0.7548 \\ -3.6072 & 6.0785 & 0.2389 & -0.1206 \\ 216.7591 & -365.0326 & -14.3937 & 7.2016 \\ -92.6428 & 168.0367 & 8.6785 & -3.3059 \end{bmatrix},
$$

$$
B_c = 10^5 \times \begin{bmatrix} 0.0006 \\ -0.0207 \\ 0.9207 \\ -1.6905 \end{bmatrix}, \quad C_c = 10^3 \times \begin{bmatrix} -0.5971 \\ 1.0063 \\ 0.0396 \\ -0.0200 \end{bmatrix}^T. \quad (3.78)
$$

Table 3.1: Computational results

$\bar{d}$	5 ms	10 ms	15 ms	20 ms	25 ms
$\gamma_{\min}$	9.8577	10.2824	11.1341	12.6120	19.8366

According to ISO 2361 (Sun *et al.*, 2011), it is the fact that improving ride comfort is equivalent to minimizing the vertical acceleration of a vehicle system in the frequency range from 4Hz to 8Hz. Thus, we first focus on the frequency responses from the ground velocity to body vertical acceleration for the open- ($u(t) = 0$, passive mode) and closed-loop (active mode) systems by using the dynamic output-feedback controllers in (3.77) and (3.78) for the active suspension systems in (3.57) and with control delay (3.44) respectively. From Fig. 3.11-3.12, we can see that the desired controller in (3.58) with the parameters in (3.77) and the controller in (3.46) with the parameters in (3.78) can yield less value of H_∞ norm over the frequency range of 4Hz-8Hz.

Furthermore, when we consider the control design problem for the active suspension systems with control delay, in which the upper bounds $\bar{d}$ of the delay $d(t)$ are 1,5,10 and 20 ms, Fig. 3.13 shows that the frequency responses for the open- and closed-loop systems with control delay under different controllers, e.g., dynamic output-feedback controller I (without the delay case) and the corresponding
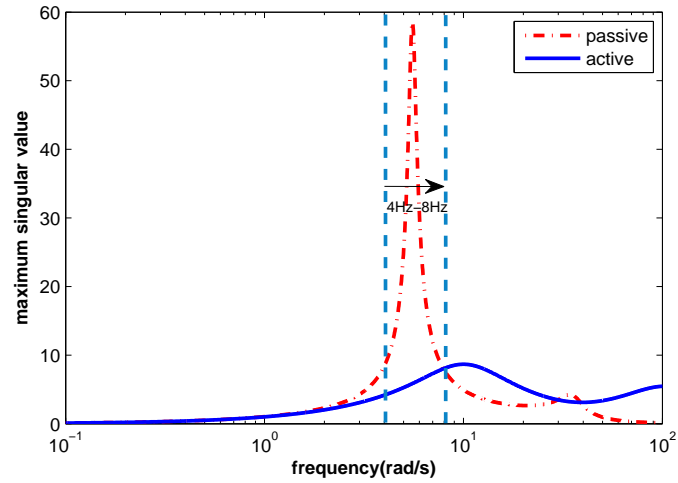
Figure 3.11: Frequency responses for the open- and closed-loop systems without control delay
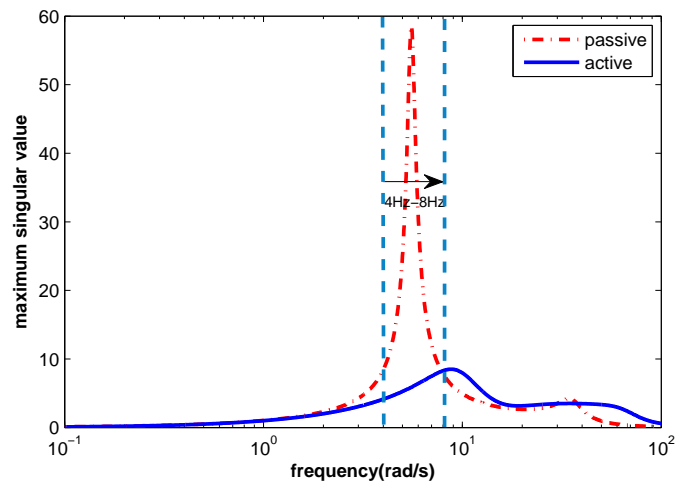
Figure 3.12: Frequency responses for the open- and closed-loop systems with control delay ($\bar{d} = 20$ ms)

(a)

(b)

(c)

(d)
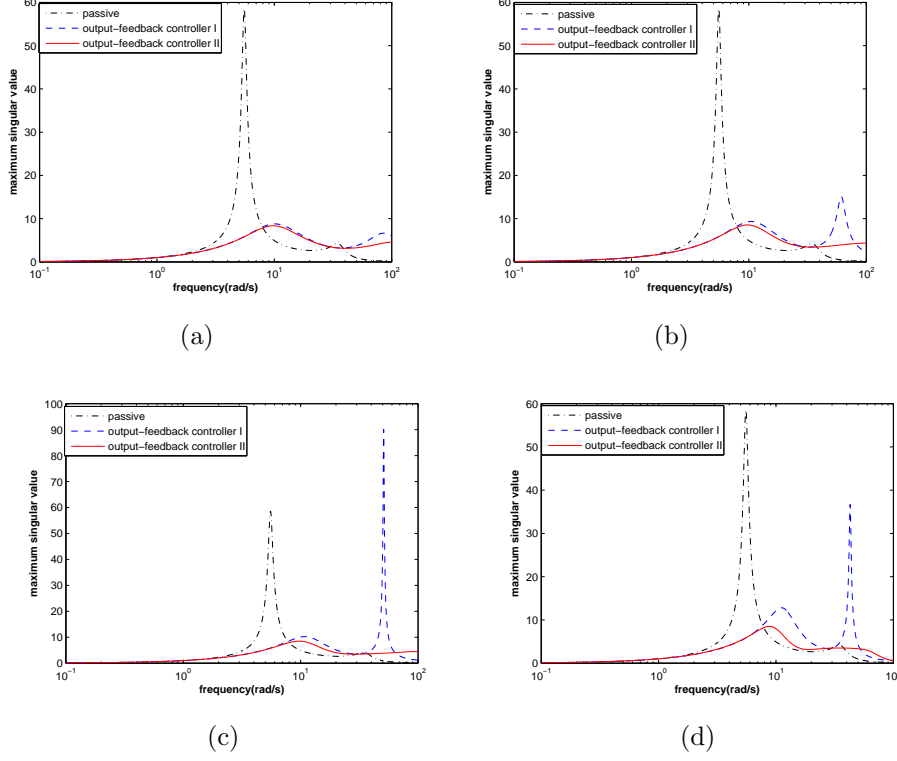
Figure 3.13: Frequency responses for the open- and closed-loop systems with control delay (a) $\bar{d} = 1$ ms (b) $\bar{d} = 5$ ms (c) $\bar{d} = 10$ ms (d) $\bar{d} = 20$ ms

dynamic output-feedback controller II which can be calculated according to Theorem 3.4. In Fig. 3.13, the black dash-dot line denotes the frequency response for the open-loop systems, the blue dash line and the red solid line denote the frequency response for the closed-loop systems with control delay under different controllers respectively. It can be observed that the controllers derived from actuator delayed active suspension systems can obtain better performance than the one for active suspension systems without delay.

3.3.3.1 Bump Response

The desired controller in (3.46) with the parameters in (3.78) can be designed such that: 1) the sprung mass acceleration $z_1(t)$ is as small as possible; 2) the suspension deflection is below the maximum allowable suspension stroke $z_{\max} = 0.035$ m; 3) the controlled output defined in satisfy $z_2(t)_2 < 1$; 4) the force of the

actuator is below the maximum bound $u_{\max} = 2000$ N. In order to evaluate the suspension characteristics with respect to ride comfort, vehicle handling, working space of the suspension and actuator force constraints, the variability of the road profiles is taken into account. In the context of active suspension performance, road disturbances can be generally assumed as shocks or vibrations, in which shocks are discrete events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pothole on an otherwise smooth road surface. In this study, this case of road profile is considered to reveal the transient response characteristic, which is given by

$$z_r(t) = \begin{cases} \frac{A}{2}(1 - \cos(\frac{2\pi V}{L}t)), & \text{if } 0 \le t \le \frac{L}{V}, \\ 0, & \text{if } t > \frac{L}{V}, \end{cases} \tag{3.79}$$

where A and L are the height and the length of the bump. Assume $A = 60$ mm, $L = 5$ m and the vehicle forward velocity as $V = 25$ (km/h).

Figs. 3.14–3.21 demonstrate the responses of body vertical accelerations, suspension deflections, dynamic tire load constraints and actuator forces for the open- and closed-loop system with actuator delays under the bump disturbance and different controllers, respectively. Among them, it can be seen from these figures that the designed controller II for the active suspension systems with control delay can achieve less value of the maximum body acceleration, compared with the passive systems and the controller I for the active suspension systems without delay, which clearly demonstrates that an improved ride comfort is achieved.

Moreover, all Figs. 3.14–3.21 present that the suspension stroke constraint $z_{\max} < 0.035$ is guaranteed, while they illustrate the relative dynamic tire load $k_t x_2(t)/(m_s + m_u)g < 1$ is also ensured the force of the actuator is below and the maximum bound $u_{\max} = 2000$ N by using the output-feedback controller II. In particular, it is apparent that the controller II can achieve better performance compared with the closed-loop system under the controller I when the upper bound $\bar{d}$ of the delay $d(t)$ is bigger, which means that the control delay is significant to be considered in the control design process of active suspension systems.

(a)   (b)
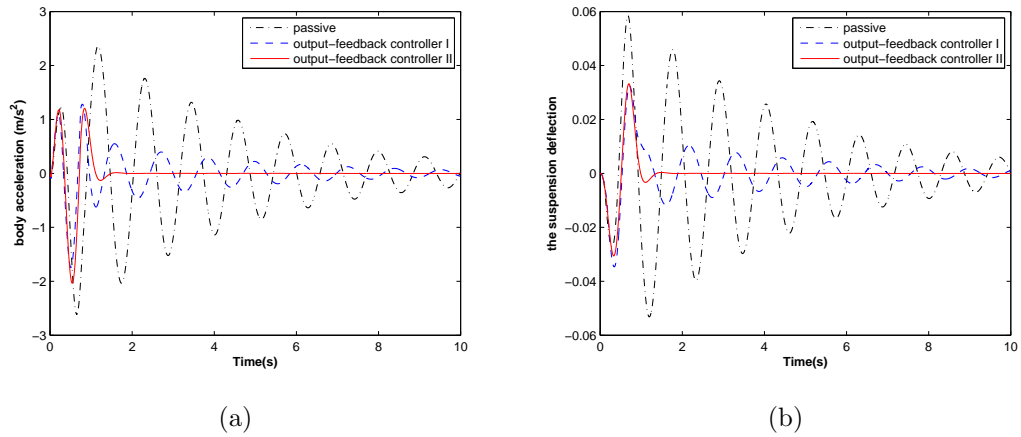
Figure 3.14: Body acceleration and suspension deflection responses of the open- and closed-loop systems with control delay ($\bar{d} = 1$ ms)

(a)   (b)
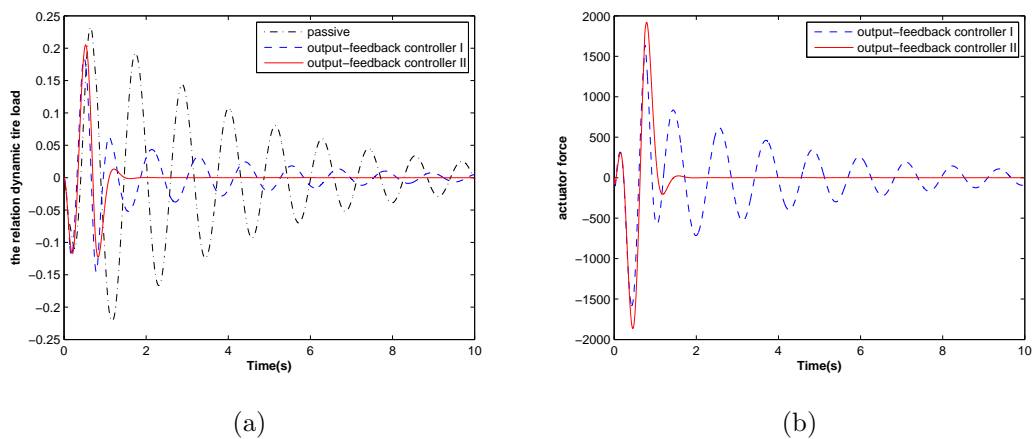
Figure 3.15: Tire stroke constrains and actuator force responses of the open- and closed-loop systems with control delay ($\bar{d} = 1$ ms)

(a) (b)
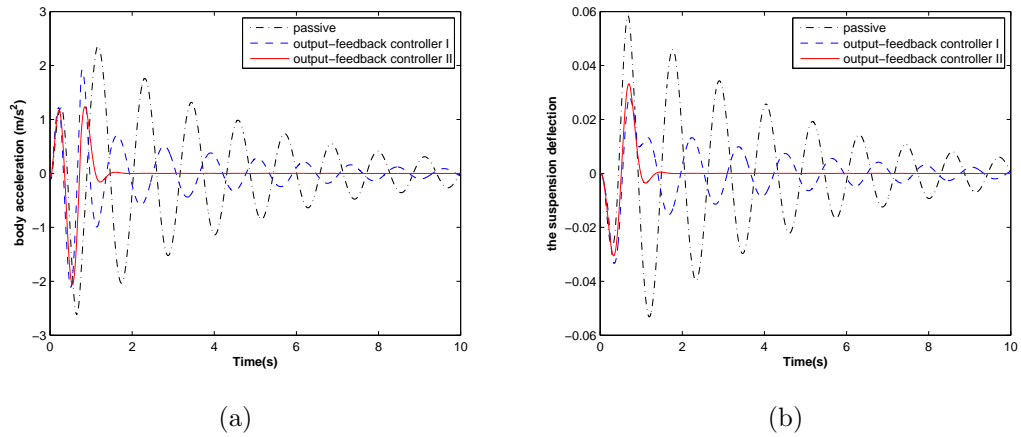
Figure 3.16: Body acceleration and suspension deflection responses of the open- and closed-loop systems with control delay ($\bar{d} = 5$ ms)

(a) (b)
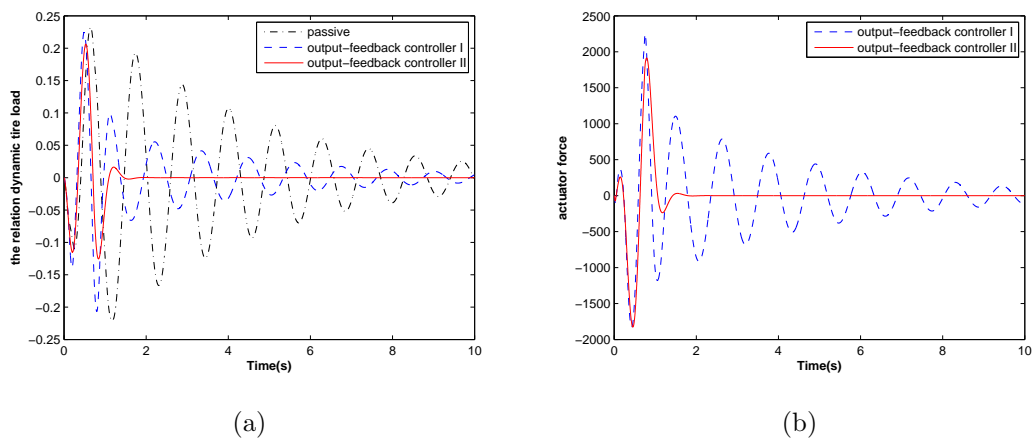
Figure 3.17: Tire stroke constrains and actuator force responses of the open- and closed-loop systems with control delay ($\bar{d} = 1$ ms)
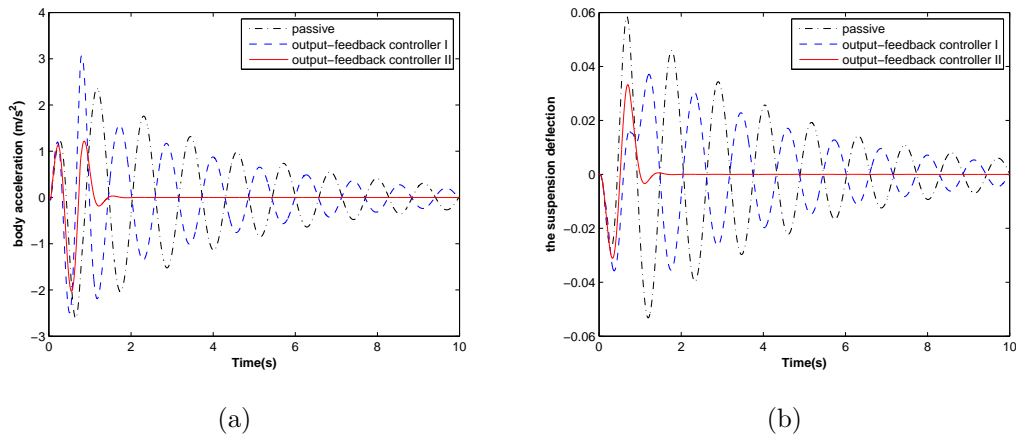
Figure 3.18: Body acceleration and suspension deflection responses of the open- and closed-loop systems with control delay ($\bar{d} = 10$ ms)
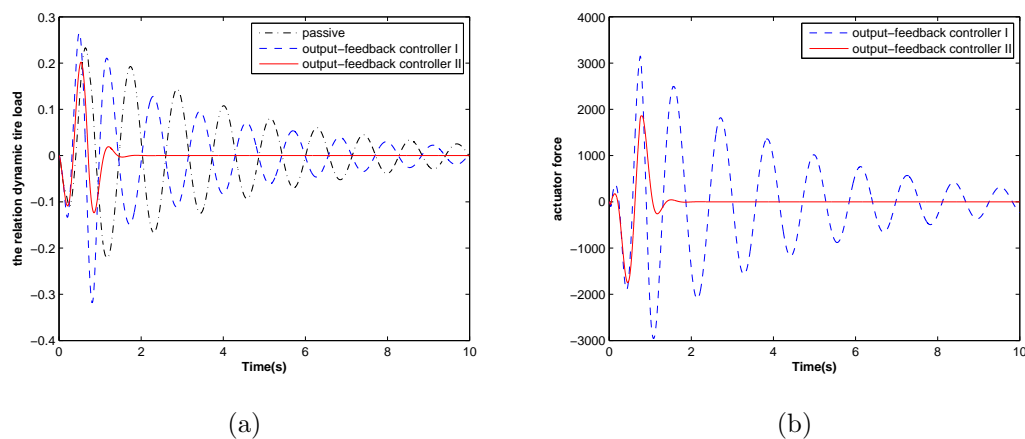
Figure 3.19: Tire stroke constrains and actuator force responses of the open- and closed-loop systems with control delay ($\bar{d} = 10$ ms)

80

(a)

(b)
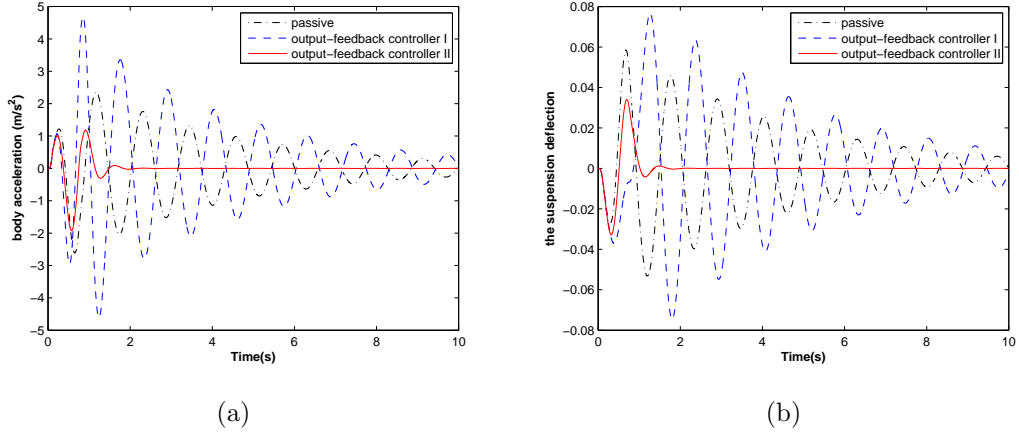
Figure 3.20: Body acceleration and suspension deflection responses of the open- and closed-loop systems with control delay ($\bar{d} = 20$ ms)

(a)

(b)
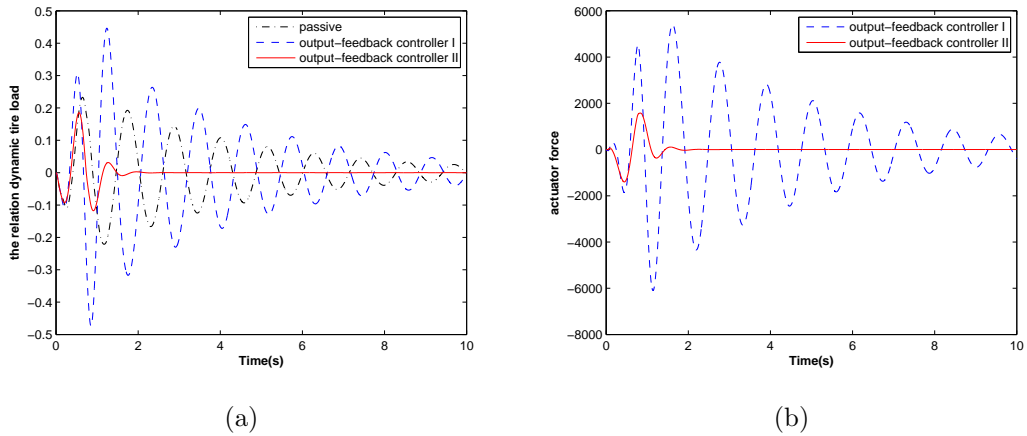
Figure 3.21: Tire stroke constrains and actuator force responses of the open- and closed-loop systems with control delay ($\bar{d} = 20$ ms)

3.3.3.2 Random Response

In the context of active suspension performance, road disturbances can also be generally assumed as random vibrations (Sun *et al.*, 2011), which are consistent and typically specified as random process with a given ground displacement power spectral density (PSD) of

$$G_q(n) = G_q(n_0) \left(\frac{n}{n_0}\right)^{-c}, \tag{3.80}$$

where n_0 denotes the spatial frequency and n_0 is the reference spatial frequency of $n_0 = 0.1$ (1/m); $G_q(n_0)$ is used to stand for the road roughness coefficient; $c = 2$ is the road roughness constant. Related to the time frequency f, we have $f = nV$ with V for the vehicle forward velocity. Based on (3.80), we can obtain the PSD ground displacement:

$$G_q(f) = G_q(n_0) n_0^{-2} \frac{V}{f^2}. \tag{3.81}$$

Accordingly, PSD ground velocity is given by

$$G_{\dot{q}}(f) = (2\pi f)^2 G_q(f) = 4\pi G_q(n_0) n_0^2 V, \tag{3.82}$$

which is only related with the vehicle forward velocity. When the vehicle forward velocity is fixed, the ground velocity can be viewed as a white-noise signal. To check the PSD body acceleration, which can be calculated by

$$G_{z_1}(f) = |G(j\omega)| G_{\dot{q}}(f) = |G(j\omega)| 4\pi G_q(n_0) n_0^2 V, \tag{3.83}$$

we choose the four difference road roughness $G_q(n_0) = 16 \times 10^{-6}$ m³, 64×10^{-6} m³, 256×10^{-6} m³ and 1024×10^{-6} m³, which are corresponded to B Grade (Good), C Grade (Average), D Grade (Poor) and E Grade (Very Poor) for the vehicle forward velocity $V = 25$ (km/h), respectively. Fig. 3.22 shows PSD body acceleration for the four different type of road disturbances. In order to show the advantages of the proposed output-feedback controller for the active suspension systems with control delay under the white noise disturbance, for $\bar{d} = 20$ ms, Fig. 3.23–3.26 demonstrate the responses of body vertical accelerations, suspension strokes, and the dynamic tire load constraints for the closed-loop system under the

four types of different road disturbances by using the output-feedback controllers I in (3.77) and II in (3.78), respectively. It can be observed that the improved suspension performance has been achieved, satisfying the required suspension deflection , dynamic tire load and maximum actuator force constraints by using the designed controller in (3.78) compared with the controller I which can be solved without taking into account control delay.
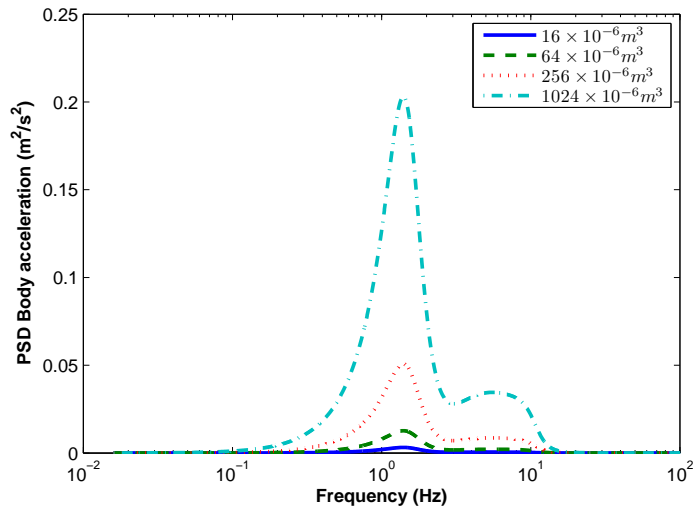
Figure 3.22: The power spectral density of body acceleration

To further evaluate the suspension system performance, the root mean square (RMS) values of the body acceleration are exploited to demonstrate the effectiveness of the proposed control design method. RMS are strictly related to the ride comfort, which are often employed to quantify the amount of acceleration transmitted to the vehicle body. The RMS value of variable $x(t)$ is calculated as $\text{RMS}_x = \sqrt{(1/T) \int_0^T x^T(t)x(t)\mathrm{d}t}$. In this study, we choose $T = 100$ s to calculate the RMS values of the body acceleration, suspension stroke and relative dynamics tire load for different road roughness coefficient $G_q(n_0)$, which are listed in Tables 3.2–3.4 for different upper bound $\bar{d}$=20 ms by using the output-feedback controllers I in (3.77) and II in (3.78), respectively. It can be observed that these tables indicate that the improvement in ride comfort and the satisfaction of hard constraints can be achieved for the different load conditions.

(a)

(b)

(c)

(d)
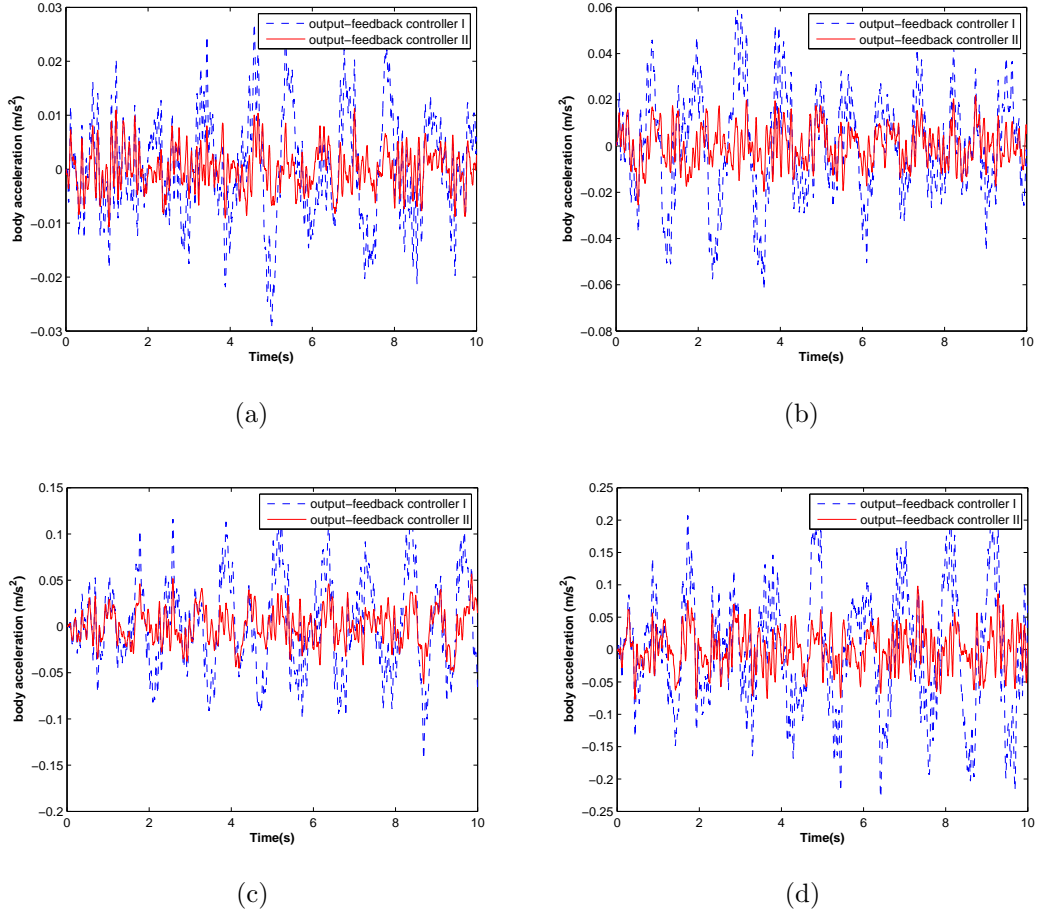
Figure 3.23: Random responses of body acceleration for the closed-loop systems under white noise disturbance by using controllers I and II. (a) B Grade Good (b) C Grade Average (c) D Grade Poor (d) E Grade Very Poor

Table 3.2: RMS body acceleration

Parameter	Controller I	Controller II
$G_q(n_0) = 16 \times 10^{-6}$ m^3	0.0117	0.0041
$G_q(n_0) = 64 \times 10^{-6}$ m^3	0.0235	0.0081
$G_q(n_0) = 256 \times 10^{-6}$ m^3	0.0447	0.0164
$G_q(n_0) = 1024 \times 10^{-6}$ m^3	0.0974	0.0331

(a)

(b)

(c)

(d)
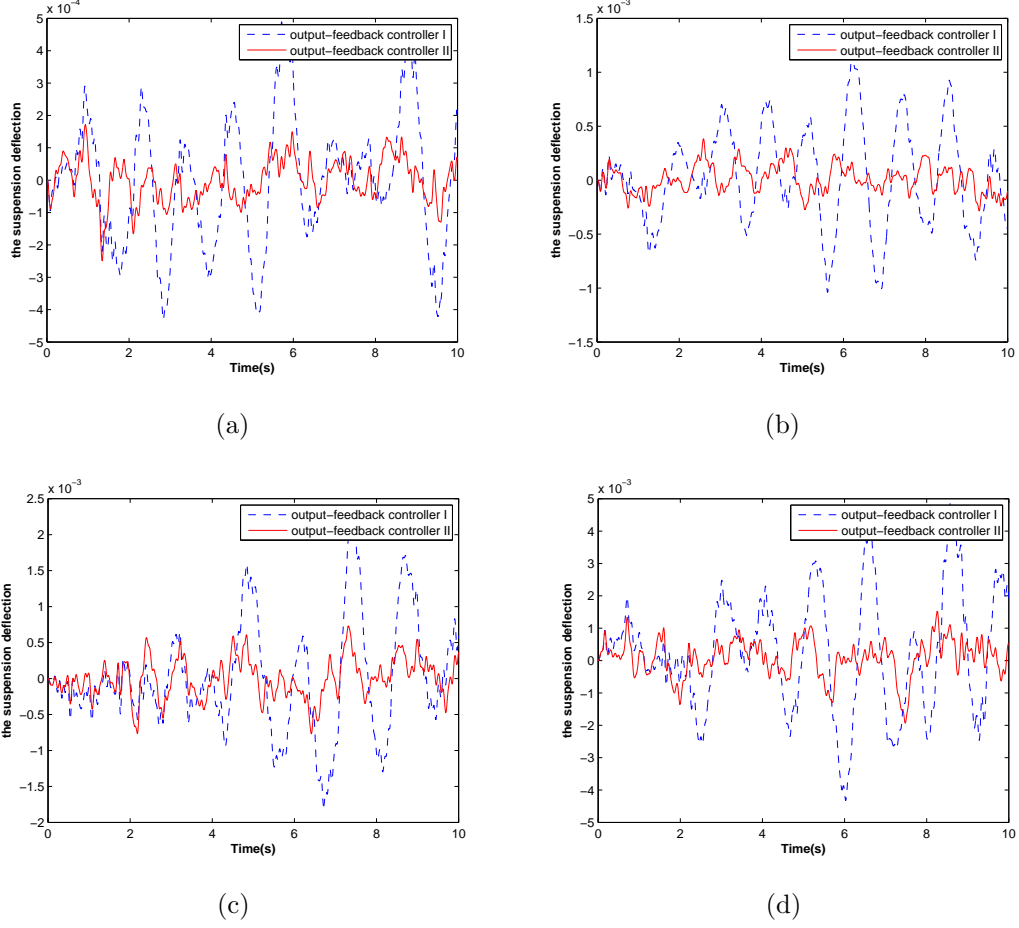
Figure 3.24: Random responses of suspension deflection for the closed-loop systems under white noise disturbance by using controllers I and II. (a) B Grade Good (b) C Grade Average (c) D Grade Poor (d) E Grade Very Poor

Table 3.3: RMS suspension stroke

Parameter	Controller I	Controller II
$G_q(n_0) = 16 \times 10^{-6}$ m^3	2.1484×10^{-4}	6.9215×10^{-5}
$G_q(n_0) = 64 \times 10^{-6}$ m^3	4.4955×10^{-4}	1.3132×10^{-4}
$G_q(n_0) = 256 \times 10^{-6}$ m^3	8.2051×10^{-4}	2.6750×10^{-4}
$G_q(n_0) = 1024 \times 10^{-6}$ m^3	0.0019	5.4571×10^{-4}
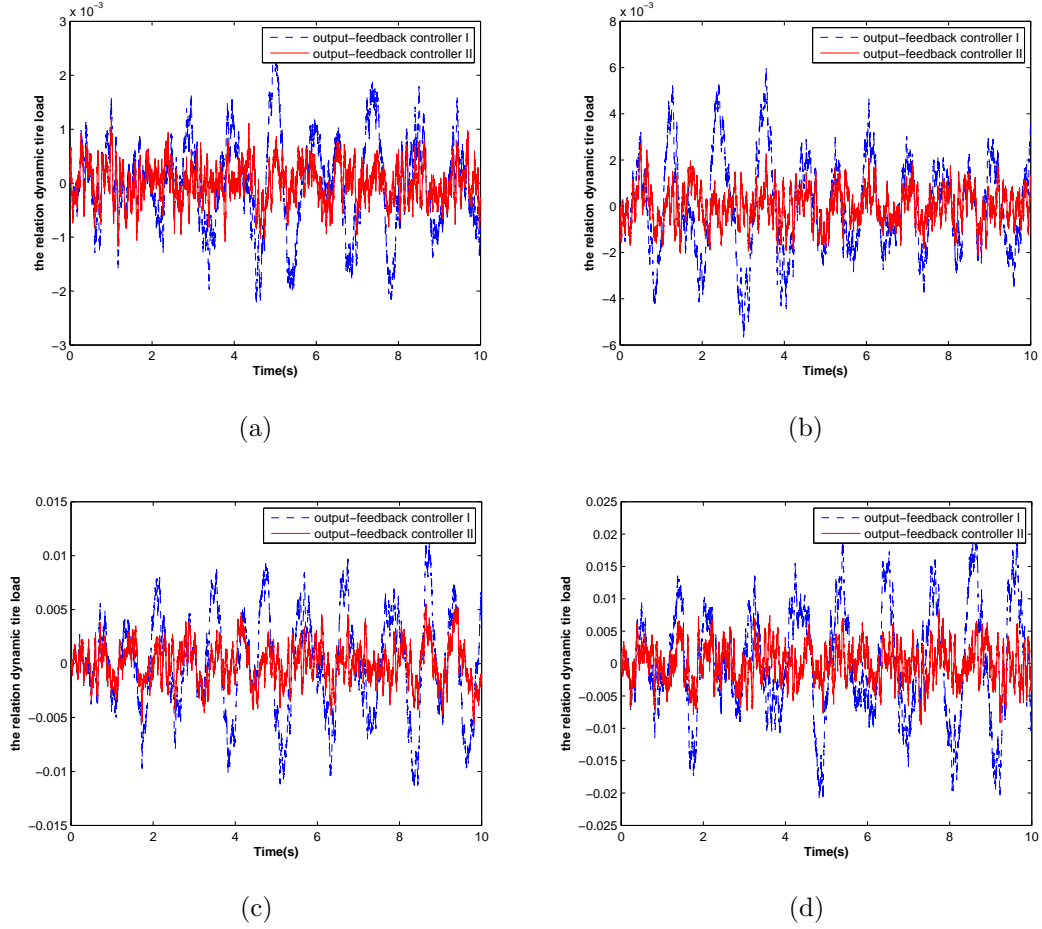
Figure 3.25: Random responses of tire stroke constrains for the closed-loop systems under white noise disturbance by using controllers I and II. (a) B Grade Good (b) C Grade Average (c) D Grade Poor (d) E Grade Very Poor

Table 3.4: RMS relative dynamics tire load

Parameter	Controller I	Controller II
$G_q(n_0) = 16 \times 10^{-6}$ m^3	0.0011	3.8624×10^{-4}
$G_q(n_0) = 64 \times 10^{-6}$ m^3	0.0022	7.6119×10^{-4}
$G_q(n_0) = 256 \times 10^{-6}$ m^3	0.0041	0.0015
$G_q(n_0) = 1024 \times 10^{-6}$ m^3	0.0090	0.0031
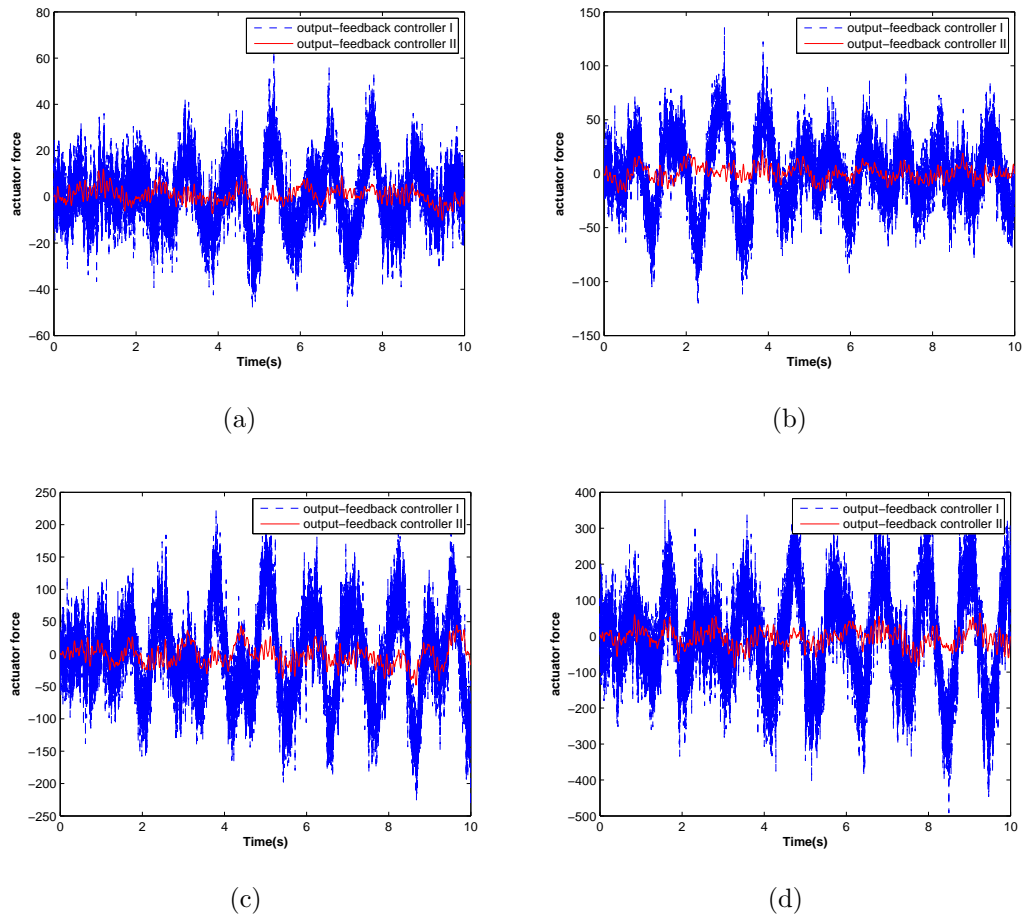
Figure 3.26: Random responses of actuator force for the closed-loop systems under white noise disturbance by using controllers I and II. (a) B Grade Good (b) C Grade Average (c) D Grade Poor (d) E Grade Very Poor

3.4 Summary

In this chapter, firstly, the uncertain half-vehicle active suspension system has been modelled and then a novel robust controller for the system with actuator delay has been proposed. The delay was assumed to be interval time-varying delay. The uncertainties of the systems are caused by the vehicle load variation and can be modeled by polytopic uncertainties type. A sufficient condition for the existence of robust H_∞ controller has been proposed to ensure robust asymptotical stability of the closed-loop system with a prescribed level of disturbance attenuation and also satisfy the desirable output constraint performance. The condition has converted into convex optimization problem. A practical design example has been given to illustrate the effectiveness of the proposed approach.

Secondly, for the state signals are unmeasurable, a novel output-feedback H_∞ controller design method has been presented for a class of active quarter-vehicle suspension systems with actuator time-varying delay. The dynamic system has been established when taking into account the required performance, such as ride comfort, road holding, and suspension deflection, as control objectives. A new dynamic output-feedback H_∞ controller has been designed to guarantee asymptotic stability of the closed-loop system with H_∞ disturbance attenuation level and meanwhile satisfy the required output constraints. The existence condition of admissible controller has been expressed as convex optimization problems. Finally, we have provided a quarter-vehicle model to demonstrate the effectiveness of the proposed method.

Chapter 4

Fault-Tolerant H_∞ Control for Vehicle Active Suspension Systems with Actuator Fault

4.1 Introduction

With the growing complexity of automated control systems and actuators, various faults are likely to be encountered, especially actuator and sensor faults Chen & Liu (2004); Liao *et al.* (2002); Shi *et al.* (2003); Wang *et al.* (1999); Yang *et al.* (2001b, 2002); Zhang *et al.* (2004). Therefore, it is important to design a fault-tolerant controller such that the system stability and performance of the closed-loop system can tolerate both sensor and actuator faults, which motivates the interests in the fault tolerant control system design, and the objective is to prevent the faults in the control loop from causing an overall system failure. During the past few decades, many researchers have paid considerable attention to reliable and fault tolerant control problems for dynamic systems and a great number of theoretic results have been presented, e.g. Dong *et al.* (2010); Ma *et al.* (2010); Mao *et al.* (2010); Wang & Qiao (2004); Wang *et al.* (2009b); Yang *et al.* (2009); Zuo *et al.* (2010). For example, the authors in Yang *et al.* (2001b) investigated reliable H_∞ controller design problem for linear systems, and the reliable controller was designed such that the resulting control systems are reliable in that they provide guaranteed asymptotic stability and H_∞ performance when

all control components (i.e., sensors and actuators) are operational and when some control components experience failures. Recently, Wang and his group in Dong *et al.* (2010); Wang *et al.* (2009b) dealt with the reliable control problem for the systems with sensor faults being modeled by the probabilistic distributions. The reliable H_∞ control problem of seat suspension systems with actuator faults were handled in Zhao *et al.* (2010c), where the considered actuator fault was described to be static behavior. It is practically reasonable to assume that the actuator fault should be treated as the dynamic behavior in stochastic distributions Dong *et al.* (2010); Wang *et al.* (2009b). To the authors' best knowledge, few results on fault-tolerant H_∞ control for active suspension systems with dynamic actuator fault behaviors have been developed, which motivates this study.

This chapter is concerned with the problem of fault-tolerant H_∞ control for a class of quarter-car active suspension systems with actuator faults. When taking into account suspension systems performance such as ride comfort, road holding, suspension deflection and maximum actuator force limitation, we establish a corresponding state-space system in terms of control strategy. Actuator faults are considered in the controller design process. It is assumed that actuator failure process is a stochastic behavior, which can be modeled by a continuous-time homogeneous Markov process. The fault-tolerant H_∞ controller is designed such that the resulting control system is tolerant in that it guarantees asymptotic stability and H_∞ performance, and simultaneously satisfy the constraint performance with existing possible actuator failures. Furthermore, the existence conditions of admissible controller are derived in terms of LMIs. Finally, a quarter-car model is exploited to demonstrate the effectiveness of the proposed method. The remainder of this chapter is organized as follows.

The problem to be addressed is formulated in Section 4.2. Section 4.3 presents the proposed new fault-tolerant H_∞ controller design method. An example is provided to evaluate the proposed method in Section 4.4, and finally we conclude the chapter in Section 4.5.

4.2 Problem Formulation

Consider the active vehicle suspension model in the state-space form (2.5)

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + Bu(t), \\
z_1(t) &= C_1 x(t) + D_1 u(t), \\
z_2(t) &= C_2 x(t),
\end{aligned}
\tag{4.1}
$$

where the matrices A, B, B_1, C_1, D_1 and C_2 are defined in Chapter 1 (2.5). We consider ride comfort performance, road holding and suspension deflection output constraints in this system. In practice, the active control force provided for the active suspension system should be confined to a certain range prescribed by limited power of the actuator:

$$
|u(t)| \leq u_{\max},
\tag{4.2}
$$

where $u_{\max}$ is defined as the maximum possible actuator control force.

Consider the following actuator failure model, in which the actuator suffers from failures, $u^f(t)$ is employed to describe the control signal sent from the actuator.

$$
u^f(t) = m_{ar_t}(t) K_{ar_t} x(t),
\tag{4.3}
$$

where $\{r_t, \ t \geq 0\}$ is a homogeneous finite-state Markovian process with right continuous trajectories, which takes value in a finite state space $S = \{1, 2, \cdots, s\}$ with generator $\Xi = \{\pi_{ij}\}$, $i, j \in S$, and has the mode transition probabilities

$$
\Pr\left(r_{t+\triangle t} = j \mid r_t = i\right) = \left\{
\begin{array}{ll}
\pi_{ij}\triangle t + o\left(\triangle t\right), & i \neq j, \\
1 + \pi_{ii}\triangle t + o\left(\triangle t\right), & i = j,
\end{array}
\right.
\tag{4.4}
$$

where $\triangle t > 0$, and $\lim_{\triangle t \to 0}\left(\frac{o(\triangle t)}{\triangle t}\right) = 0$. $\pi_{ij} \geq 0$ $(i, j \in S, \ i \neq j)$ denotes the switching rate from i th fault mode to j th fault mode, and $\pi_{ii} = -\sum_{j=1, j \neq i}^{s} \pi_{ij}$ for all $i \in S$. K_{ar_t} is the actuator fault-tolerant feedback control gain matrix to be determined; $m_{ar_t}(t)$ is used to represent the possible fault of the corresponding actuator $u^f(t)$. $\check{m}_{ar_t} \leq m_{ar_t}(t) \leq \hat{m}_{ar_t}$, where $\check{m}_{ar_t}$ and $\hat{m}_{ar_t}$ are constant scalars and used to prescribe lower and upper bounds of the actuator faults. Three following cases corresponding to three different actuator conditions are considered:

1) $\check{m}_{ar_t} = \hat{m}_{ar_t} = 0$, then $m_{ar_t}(t) = 0$, which implies that the corresponding actuator $u^f(t)$ is failed completely.

2) $\check{m}_{ar_t} = \hat{m}_{ar_t} = 1$, thus we obtain $m_{ai}(t) = 1$, which represents the case of no fault in the very actuator $u^f(t)$.

3) $0 < \check{m}_{ar_t} < \hat{m}_{ar_t} < 1$, which means that there exists partial fault in the corresponding actuator $u^f(t)$.

Remark 4.1 Fault free, partial fault and complete fault are three modes of the corresponding actuator $u^f(t)$. It is essential in terms of both theoretical development and practical implementation to model a controller as dynamic behavior rather than static behavior. It is evident that the existing fault models only consider one kind of the fault modes, especially as static behavior, a special case of our fault modes Chen & Liu (2004); Liao *et al.* (2002); Shi *et al.* (2003); Yang *et al.* (2001b, 2002); Zhao *et al.* (2010c); Zuo *et al.* (2010). It is novel and timely that the proposed fault model covers existing unrealistic actuator failure models.

Remark 4.2 The state of the art represent actuator fault modes in terms of Bernoulli distribution, which is a more general way to deal with the reliable control problem Dong *et al.* (2010); Wang *et al.* (2009b). Since there exist three or more different modes in the actuator fault, it is convincing that the behavior of an actuator fault be modeled in a linear-time homogeneous Markov process, that is to say, the actuator failure model is governed by a continues-time homogeneous Markov chain.

Furthermore, the closed-loop system with the reliable controller (4.3) can be written as:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B m_{ar_t}(t) K_{ar_t} x(t), \\
z_1(t) &= C_1 x(t) + D_1 m_{ar_t}(t) K_{ar_t} x(t), \\
z_2(t) &= C_2 x(t).
\end{aligned}
\tag{4.5}
$$

Without loss of generality, It is assumed, $w \in L_2[0, \infty)$, and $\|w\|_2^2 \leq w_{\max} < \infty$. Then, the objective of this chapter is to design a controller gain matrix K_{r_t} such that:

(1) the closed-loop system is asymptotically stable;

(2) under zero initial condition, the closed-loop system guarantees that $\|z_1\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0,\infty)$, where $\gamma > 0$ is a prescribed scalar;

(3) the following control output constraints are guaranteed:

$$|\{z_2(t)\}_q| \leq 1, \quad q = 1, 2, \tag{4.6}$$

(4) the following maximum possible actuator control force constraint is guaranteed:

$$|u(t)| \leq u_{\max}. \tag{4.7}$$

In the above proposed control strategy, the multiple requirements are formulated in a unified framework, based on which the controller design is cast into a multiple-objective minimization problem.

4.3 Fault-Tolerant Controller Design

A fault-tolerant state-feedback controller is designed in this section such that the closed-loop system in (4.5) is asymptotically stable and can also ensure a prescribed gain from disturbance $w(t)$ to performance output $z_1(t)$ while keeping the output and maximum control force constraints in (4.6)–(4.7) satisfied. The following lemma is firstly given for further controller design.

Lemma 4.1 (Yang *et al.* (2001b)) For any scalar $\varepsilon > 0$, vectors x and y, the following inequality holds

$$x^T y + y^T x \leq \varepsilon x^T x + \varepsilon^{-1} y^T y. \tag{4.8}$$

The following scalars are also introduced for further modelling,

$$
\begin{aligned}
\hat{M}_{a0r_t} &= \frac{\check{m}_{ar_t} + \hat{m}_{ar_t}}{2}, \\
\check{M}_{a0r_t} &= \frac{\hat{m}_{ar_t} - \check{m}_{ar_t}}{2},
\end{aligned}
$$

and rewrite m_{ar_t} as follows

$$m_{ar_t}(t) = \hat{M}_{a0r_t} + \Delta_{a0r_t},$$

where

$$|\Delta_{ar_t}| \leq \frac{\hat{m}_{ar_t} - \check{m}_{ar_t}}{2}.$$

For notational simplicity, we set $r_t = i$, $i \in S$.

The H_∞ performance analysis criterion will be given in the following theorem.

Theorem 4.1 For the closed-loop system in (4.5), given matrix K_{ai} and positive constant ρ, if there exist matrix $P_i > 0$, and scalars ε_{aqi} $(i = 1, 2, \ldots, s, q = 1, 2)$ satisfying

$$
\begin{bmatrix}
\Theta_i & P_i B_1 & C_1^T + K_{ai}^T \hat{M}_{a0i} D_1^T & \varepsilon_{a1i} K_{ai}^T & P_i B \check{M}_{a0i} \\
* & -\gamma^2 & 0 & 0 & 0 \\
* & * & -I & 0 & D_1 \check{M}_{a0i} \\
* & * & * & -\varepsilon_{a1i} I & 0 \\
* & * & * & * & -\varepsilon_{a1i} I
\end{bmatrix} < 0, \qquad (4.9)
$$

$$
\begin{bmatrix}
-u_{\max}^2 P_i & \sqrt{\rho} K_{ai}^T \hat{M}_{a0i} & \varepsilon_{a2i} \sqrt{\rho} K_{ai}^T & 0 \\
* & -I & 0 & \check{M}_{a0i} \\
* & * & -\varepsilon_{a2i} I & 0 \\
* & * & * & -\varepsilon_{a2i} I
\end{bmatrix} < 0, \qquad (4.10)
$$

$$
\begin{bmatrix}
-P_i & \sqrt{\rho} \{C_2\}_q^T \\
* & -I
\end{bmatrix} < 0, \qquad (4.11)
$$

where

$$
\Theta_i = \mathrm{sym}\left(P_i \left(A + B \hat{M}_{a0i} K_{ai} \right) \right) + \sum_{j=1}^{s} \pi_{ij} P_j. \qquad (4.12)
$$

Then,

(1) the closed-loop system is asymptotically stable;

(2) the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to output constraints in (4.6) and maximum possible actuator control force constraint in (4.7) with the disturbance energy under the bound $w_{\max} = (\rho - V_{r_0}(0))/\gamma^2$, where $T_{z_1 w}$ denotes the closed-loop transfer function from the road disturbance $w(t)$ to the control output $z_1(t)$, $V_{r_0}(0)$ is defined as $V_{r_t}(t) = x^T(t) P_{r_t} x(t)$ with $r_t = r_0$ and $t = 0$.

Proof. To begin with, we establish the asymptotic stability of the system in (4.5) with $w(t) = 0$. Defining a Lyapunov-Krasovskii functional candidate for system (4.5) as:

$$
V_{r_t}(t) = x^T(t) P_{r_t} x(t) \qquad (4.13)
$$

where $P_{r_t} > 0$ is to be determined. Let $\Im$ be the weak infinitesimal operator, one has

$$
\Im V_{r_t}(t) = x^T(t) \left(\mathrm{sym}(P_i (A + B M_{ai} K_{ai})) + \sum_{j=1}^{s} \pi_{ij} P_j \right) x(t).
$$

By Lemma 4.3, for positive constant $\varepsilon_{a1i} > 0$, it holds that

$$
\begin{aligned}
&\text{sym}(P_i \left(A + BM_{ai}K_{ai} \right)) \\
=\ &\text{sym}\left(P_i \left(A + B\hat{M}_{a0i}K_{ai} \right)\right) + \text{sym}\left(P_i B \Delta_{a0i} K_{ai} \right) \\
\leq\ &\text{sym}\left(P_i \left(A + B\hat{M}_{a0i}K_{ai} \right)\right) + \varepsilon_{a1i}^{-1} P_i B \Delta_{a0i} \Delta_{a0i} B^T P_i^T + \varepsilon_{a1i} K_{ai}^T K_{ai} \\
\leq\ &\text{sym}\left(P_i \left(A + B\hat{M}_{a0i}K_{ai} \right)\right) + \varepsilon_{a1i}^{-1} P_i B \check{M}_{a0i}^2 B^T P_i^T + \varepsilon_{a1i} K_{ai}^T K_{ai}.
\end{aligned}
$$

According to Schur complement and the inequality (4.9) in Theorem 4.1 guarantees

$$
\text{sym}(P_i \left(A + BM_{ai}K_{ai} \right)) + \sum_{j=1}^{s} \pi_{ij} P_j < 0, \tag{4.14}
$$

which further leads to $\Im V_{r_t}(t) < 0$ for any $x(t) \neq 0$. Therefore, we conclude that the system in (4.5) with $w(t) = 0$ and actuator fault in (4.3) is asymptotically stable.

Next, we will investigate the H_∞ performance of the active suspension system under zero initial condition. Consider the following index:

$$
J \triangleq \mathcal{E} \int_0^\infty \left[z_1^T(t) z_1(t) - \gamma^2 w^T(t) w(t) \right] \mathrm{d}t. \tag{4.15}
$$

Then, by Dynkin' formula, it can be seen that

$$
J \leq \mathcal{E} \int_0^\infty \left[z_1^T(t) z_1(t) - \gamma^2 w^T(t) w(t) + \Im V_{r_t}(t) \right] \mathrm{d}t. \tag{4.16}
$$

According to some algebraic manipulations and Schur complement, it is not difficult to obtain

$$
z_1^T(t) z_1(t) - \gamma^2 w^T(t) w(t) + \Im V_{r_t}(t) = \bar{\xi}^T(t) \Pi_i \bar{\xi}(t), \tag{4.17}
$$

where

$$
\begin{aligned}
\bar{\xi}(t) &= \left[\begin{array}{cc} x^T(t) & w^T(t) \end{array}\right]^T, \quad \Pi_i = \left[\begin{array}{cc} \bar{\Theta}_i & P_i B_1 \\ * & -\gamma^2 \end{array}\right], \\
\bar{\Theta}_i &= \text{sym}\left(P_i \left(A + BM_{ai}K_{ai} \right)\right) + \sum_{j=1}^{s} \pi_{ij} P_j \\
&\quad + \left(C_1 + D_1 M_{ai}K_{ai} \right)^T \left(C_1 + D_1 M_{ai}K_{ai} \right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\bar{\Pi}_i &= \begin{bmatrix} \mathrm{sym}(P_i\,(A + BM_{ai}K_{ai})) + \sum\limits_{j=1}^{s} \pi_{ij}P_j & P_iB_1 & C_1^T + K_{ai}^T\hat{M}_{ai}D_1^T \\ * & -\gamma^2 & 0 \\ * & * & -I \end{bmatrix} \\
&= \begin{bmatrix} \Theta_i & P_iB_1 & C_1^T + K_{ai}^T\hat{M}_{a0i}D_1^T \\ * & -\gamma^2 & 0 \\ * & * & -I \end{bmatrix} \\
&\quad + \begin{bmatrix} \mathrm{sym}\,(P_iB\Delta_{a0i}K_{ai}) & 0 & K_{ai}^T\Delta_{a0i}D_1^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \Theta_i & P_iB_1 & C_1^T + K_{ai}^T\hat{M}_{a0i}D_1^T \\ * & -\gamma^2 & 0 \\ * & * & -I \end{bmatrix} + \mathrm{sym}\left(\begin{bmatrix} P_iB \\ 0 \\ D_1 \end{bmatrix} \Delta_{a0i} \begin{bmatrix} K_{ai} & 0 & 0 \end{bmatrix} \right) \\
&\leq \begin{bmatrix} \Theta_i & P_iB_1 & C_1^T + K_{ai}^T\hat{M}_{a0i}D_1^T \\ * & -\gamma^2 & 0 \\ * & * & -I \end{bmatrix} + \varepsilon_{a1i}^{-1} \begin{bmatrix} P_iB \\ 0 \\ D_1 \end{bmatrix} \check{M}_{a0i}^2 \begin{bmatrix} P_iB \\ 0 \\ D_1 \end{bmatrix}^T \\
&\quad + \varepsilon_{a1i} \begin{bmatrix} K_{ai} & 0 & 0 \end{bmatrix}^T \begin{bmatrix} K_{ai} & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

By using Schur complement to (4.9), $\bar{\Pi}_i < 0$, which implies $\Pi_i < 0$. Thus, if (4.9) holds, i.e. $\Pi_i < 0$, we have

$$
z_1^T(t)z_1(t) - \gamma^2 w^T(t)w(t) + \Im V_{r_t}(t) < 0 \tag{4.18}
$$

for any non-zero $w \in L_2[0, \infty)$, which indicates $J < 0$. Hence $\|z_1\|_2 < \gamma \|w\|_2$ is guaranteed for any non-zero $w \in L_2[0, \infty)$.

In the following part, we will consider the problems of the output constraints. From (4.18), it can be seen that

$$
\mathcal{E}\Im V_{r_t}(t) - \gamma^2 w^T(t)w(t) < 0. \tag{4.19}
$$

After integrating both sides of the above inequality from zero to any $t > 0$, we obtain

$$
\mathcal{E}V_{r_t}(t) - V_{r_0}(0) < \gamma^2 \int_0^t w^T(\tau)w(\tau)\mathrm{d}\tau < \gamma^2 \|w\|_2^2. \tag{4.20}
$$

From the definition of the Lyapunov functional in (4.13), we obtain $x^T(t)P_i x(t) < \rho$, with $\rho = \gamma^2 w_{\max} + V_{r_0}(0)$. Consider

$$
\begin{aligned}
\max_{t>0} |\{z_2(t)\}_q|^2 &= \max_{t>0} \left\| x^T(t)\{C_2\}_q^T\{C_2\}_q x(t) \right\|_2 \\
&= \max_{t>0} \left\| x^T(t)P_i^{\frac{1}{2}}P_i^{-\frac{1}{2}}\{C_2\}_q^T\{C_2\}_q P_i^{-\frac{1}{2}}P_i^{\frac{1}{2}}x(t) \right\|_2 \\
&< \rho \cdot \theta_{\max}(P_i^{-\frac{1}{2}}\{C_2\}_q^T\{C_2\}_q P_i^{-\frac{1}{2}}), \quad q = 1, 2, \\
\max_{t>0} \left| u^f(t) \right|^2 &= \max_{t>0} \left\| x^T(t)K_{ai}^T M_{ai} M_{ai} K_{ai} x(t) \right\|_2 \\
&< \rho \cdot \theta_{\max}(P_i^{-\frac{1}{2}}K_{ai}^T M_{ai} M_{ai} K_{ai} P_i^{-\frac{1}{2}}),
\end{aligned}
$$

where $\theta_{\max}(\cdot)$ represents maximal eigenvalue. From the above inequalities and (4.6)–(4.7), we know that the constraints is guaranteed, if

$$
\rho P_i^{-\frac{1}{2}}\{C_2\}_q^T\{C_2\}_q P_i^{-\frac{1}{2}} < I, \quad q = 1, 2, \tag{4.21}
$$

$$
\rho P_i^{-\frac{1}{2}}K_{ai}^T M_{ai} M_{ai} K_{ai} P_i^{-\frac{1}{2}} < u_{\max}^2 I. \tag{4.22}
$$

On the other hand, it can be seen that

$$
\begin{aligned}
&\begin{bmatrix} -u_{\max}^2 P_i & \sqrt{\rho}K_{ai}^T M_{ai} \\ * & -I \end{bmatrix} \\
=\ &\begin{bmatrix} -u_{\max}^2 P_i & \sqrt{\rho}K_{ai}^T \hat{M}_{a0i} \\ * & -I \end{bmatrix} + \mathrm{sym}\left(\begin{bmatrix} 0 \\ I \end{bmatrix} \Delta_{a0i} \begin{bmatrix} \sqrt{\rho}K_{ai} & 0 \end{bmatrix} \right) \\
\leq\ &\begin{bmatrix} -u_{\max}^2 P_i & \sqrt{\rho}K_{ai}^T \hat{M}_{a0i} \\ * & -I \end{bmatrix} + \varepsilon_{a2i}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} \check{M}_{a0i}^2 \begin{bmatrix} 0 \\ I \end{bmatrix}^T \\
&+ \varepsilon_{a2i}\begin{bmatrix} \sqrt{\rho}K_{ai} & 0 \end{bmatrix}^T \begin{bmatrix} \sqrt{\rho}K_{ai} & 0 \end{bmatrix}.
\end{aligned}
$$

By Schur complement, (4.21)–(4.22) are equivalent to (4.10)–(4.11) in Theorem 4.3, and the proof is completed. ∎

Remark 4.3 In case of no failure in the actuator, that is, $m_{ar_t} = I$, $r_t \in S$. By using a state feedback controller $u(t) = K_s x(t)$, it can be observed from (4.5) that

$$
\begin{aligned}
\dot{x}(t) &= (A + BK_s)x(t) + B_1 w(t), \\
z_1(t) &= (C_1 + D_1 K_s)x(t), \\
z_2(t) &= C_2 x(t). \tag{4.23}
\end{aligned}
$$

By choosing the corresponding Lyapunov functional $V(t) = x^T(t) P x(t)$ and following the same line as the proof of Theorem 1, we can derive the following corollary.

Corollary 4.1 For the closed-loop system in (4.23), given matrix K_s, positive constant ρ and $q = 1, 2$, if there exists a matrix $P > 0$ such that the following LMIs hold

$$\begin{bmatrix} \operatorname{sym}\left(P\left(A + BK_s\right)\right) & PB_1 & C_1^T + K_s^T D_1^T \\ * & -\gamma^2 & 0 \\ * & * & -I \end{bmatrix} < 0, \quad (4.24)$$

$$\begin{bmatrix} -u_{\max}^2 P & \sqrt{\rho} K_s^T \\ * & -I \end{bmatrix} < 0, \quad (4.25)$$

$$\begin{bmatrix} -P & \sqrt{\rho}\left\{C_2\right\}_q^T \\ * & -I \end{bmatrix} < 0, \quad (4.26)$$

Then,

(1) the closed-loop system is asymptotically stable;

(2) the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to output constraints (4.6) and maximum possible actuator control force constraint in (4.7) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$.

Based on the proposed criterion in Theorem 1, the existence condition of fault-tolerant H_∞ controller is presented in the following theorem.

Theorem 4.2 For the closed-loop system in (4.5) and given positive constant ρ, if there exist matrices $\bar{P}_i > 0$ and $\bar{K}_{ai}$ and scalars $\bar{\varepsilon}_{aqi}$ $(i = 1, 2, \ldots, s, q = 1, 2)$ such as the following LMIs hold:

$$\begin{bmatrix} \bar{\Theta}_i & B_1 & \Upsilon_i & \bar{K}_{ai}^T & \bar{\varepsilon}_{a1i} B \check{M}_{a0i} & \Omega_{1i} \\ * & -\gamma^2 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & \bar{\varepsilon}_{a1i} D_1 \check{M}_{a0i} & 0 \\ * & * & * & -\bar{\varepsilon}_{a1i} I & 0 & 0 \\ * & * & * & * & -\bar{\varepsilon}_{a1i} I & 0 \\ * & * & * & * & * & -\Omega_{2i} \end{bmatrix} < 0, \quad (4.27)$$

$$\begin{bmatrix} -u_{\max}^2 \bar{P}_i & \sqrt{\rho} \bar{K}_{ai}^T \hat{M}_{a0i} & \sqrt{\rho} \bar{K}_{ai}^T & 0 \\ * & -I & 0 & \bar{\varepsilon}_{a2i} \check{M}_{a0i} \\ * & * & -\bar{\varepsilon}_{a2i} I & 0 \\ * & * & * & -\bar{\varepsilon}_{a2i} I \end{bmatrix} < 0, \quad (4.28)$$

$$\begin{bmatrix} -\bar{P}_i & \sqrt{\rho} \bar{P}_i \left\{C_2\right\}_q^T \\ * & -I \end{bmatrix} < 0, \quad (4.29)$$

where

$$
\begin{aligned}
\bar{\Theta}_i &= \operatorname{sym}\left(A\bar{P}_i + B\hat{M}_{a0i}\bar{K}_{ai}\right) + \pi_{ii}\bar{P}_i, \\
\Upsilon_i &= \bar{P}_i C_1^T + \bar{K}_{ai}^T \hat{M}_{a0i} D_1^T, \\
\Omega_{1i} &= \left[\sqrt{\pi_{i1}}\bar{P}_i, \ldots, \sqrt{\pi_{ii-1}}\bar{P}_i, \sqrt{\pi_{ii+1}}\bar{P}_i, \ldots, \sqrt{\pi_{is}}\bar{P}_i\right], \\
\Omega_{2i} &= \operatorname{diag}\left\{\bar{P}_1, \ldots, \bar{P}_{i-1}, \bar{P}_{i+1}, \ldots, \bar{P}_s\right\}.
\end{aligned}
$$

Then, under the fault-tolerant controller (4.3), we have

(1) the closed-loop system is asymptotically stable;

(2) the performance $\|T_{z_1w}\|_\infty < \gamma$ is minimized subject to output constraint in (4.6) and maximum possible actuator control force constraint in (4.7) with the disturbance energy under the bound $w_{\max} = (\rho - V_{r_0}(0))/\gamma^2$.

Moreover, if inequalities (4.27)–(4.29) have a feasible solution, then the controller in (4.3) is given by $u^f(t) = m_{ai}\bar{K}_{ai}\bar{P}_i^{-1}x(t)$.

Proof. For $K = \bar{K}_{ai}\bar{P}_i^{-1}$, defining some following variables:

$$
P_i = \bar{P}_i^{-1}, \quad \varepsilon_{aqi} = \bar{\varepsilon}_{aqi}^{-1}, \quad q = 1, 2.
$$

According to Schur complement, (4.27)–(4.29) are equivalent to the following inequalities

$$
\begin{bmatrix}
\hat{\Theta}_i & B_1^T & P_i^{-1}C_1^T + P_i^{-1}K_{ai}^T\hat{M}_{a0i}^T D_1^T & \varepsilon_{aqi}P_i^{-1}K_{ai}^T & B\check{M}_{a0i} \\
* & -\gamma^2 & 0 & 0 & 0 \\
* & * & -I & 0 & D_1\check{M}_{a0i} \\
* & * & * & -\varepsilon_{aqi}I & 0 \\
* & * & * & * & -\varepsilon_{aqi}I
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
-u_{\max}^2 P_i^{-1} & \sqrt{\rho}P_i^{-1}K_{ai}^T\hat{M}_{a0i}^T & \sqrt{\rho}\varepsilon_{a2i}P_i^{-1}K_{ai}^T & 0 \\
* & -I & 0 & \check{M}_{a0i} \\
* & * & -\varepsilon_{a2i}I & 0 \\
* & * & * & -\varepsilon_{a2i}I
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
-P_i^{-1} & \sqrt{\rho}P_i^{-1}\{C_{2i}\}_q^T \\
* & -I
\end{bmatrix} < 0,
$$

which are equivalent to (4.9)–(4.11) by performing congruence transformations with

$$
\operatorname{diag}\{P_i, I, I, I, I\}, \quad \operatorname{diag}\{P_i, I, I, I\}, \quad \operatorname{diag}\{P_i, I\},
$$

where

$$\hat{\Theta}_i = \text{sym}\left(AP_i^{-1} + BM_{a0}K_{ar_t}P_i^{-1}\right) + P_i^{-1}\left(\sum_{j=1}^{s}\pi_{ij}P_j\right)P_i^{-1}.$$

respectively. Therefore, all the conditions in Theorem 1 are satisfied. The proof is completed.   ■

Remark 4.4 The proposed fault-tolerant control design method not only applies to the active suspension systems with actuator faults but also can solve the stochastic systems Wu & Ho (2009) and fuzzy systems with actuator faults Wu (2004) and so on. In addition, the problem of fault-tolerant H_∞ control of the closed-loop system in (2.5) with sensor faults is also solved by following the similar line as the proof of Theorem 4.3 and 4.3. In this chapter, we devote to designing the actuator controller in (4.3).

Remark 4.5 More recently, more attention has been paid to the stability analysis and controller synthesis for continuous- and discrete-time Markovian jump systems with uncertain and partly unknown transition probabilities, e.g., Xiong *et al.* (2005); Zhang & Boukas (2009); Zhang *et al.* (2008). It should be mentioned that the condition in Theorem 4.3 and 4.3 here can be extended to the case of uncertain and partly unknown transition probabilities by following the similar methods in Xiong *et al.* (2005); Zhang & Boukas (2009); Zhang *et al.* (2008).

Similar to Corollary 1 and Theorem 2, the following corollary can be obtained.

Corollary 4.2 For the closed-loop system in (4.23), given positive constant ρ and $q = 1, 2$, if there exist matrices $\bar{P} > 0$ and $\bar{K}_s$ such as the following LMIs hold:

$$\begin{bmatrix} \text{sym}(A\bar{P} + B\bar{K}_s) & B_1 & \bar{P}C_1^T + \bar{K}_s^T D_1^T \\ * & -\gamma^2 & 0 \\ * & * & -I \end{bmatrix} < 0, \qquad (4.30)$$

$$\begin{bmatrix} -u_{\max}^2\bar{P} & \sqrt{\rho}\bar{K}_s^T \\ * & -I \end{bmatrix} < 0, \qquad (4.31)$$

$$\begin{bmatrix} -\bar{P} & \sqrt{\rho}\bar{P}\{C_2\}_q^T \\ * & -I \end{bmatrix} < 0. \qquad (4.32)$$

Then a state feedback controller in the form of (4.3) exists, such that

(1) the closed-loop system is asymptotically stable;

(2) the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to output constraint (4.6) and maximum possible actuator control force constraint in (4.7) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$.

Moreover, if inequalities (4.30)–(4.32) have a feasible solution, then the controller in (4.3) is given by $u(t) = \bar{K}_s \bar{P}^{-1} x(t)$.

Remark 4.6 In this chapter, we only consider the active suspension linear systems with actuator faults and develop the novel fault-tolerant controller design algorithm for the systems. If both unmodelled dynamics and parametric uncertainties (mass, damping coefficient, stiffness) are considered here, then we know that the control design process will contain both parametric uncertainties, which can be modeled by norm-bounded uncertainties Li *et al.* (2009, 2011); Zhao *et al.* (2010b) or polytopic type uncertainties Gao *et al.* (2010a). For the uncertain active suspension systems with actuator faults, the fault-tolerant controller design results are also derived by using the methods proposed in this chapter and the authors' previous papers Gao *et al.* (2010a); Li *et al.* (2009, 2011); Zhao *et al.* (2010b).

Remark 4.7 For the active suspension system with both uncertain and nonlinear dynamic characteristics, we will build the nonlinear suspensions systems with actuator faults and consider the fault-tolerant design strategy by utilizing the methods proposed in this chapter and Ma & Yang (2011).

4.4 Case Study

It is assumed that the maximum allowable suspension stroke is $z_{\max} = 0.08$ m and $\rho = 1$. Firstly, we consider the state-feedback controller $u(t) = K_s x(t)$ for the active suspension systems in (4.23) without actuator faults. By using the convex optimization, it is found that the minimum guaranteed closed-loop H_∞ performance index γ_{min} is 8.1706 and the H_∞ controller gain matrix

$$K_s = 10^3 \times \left[\begin{array}{cccc} -0.0261 & 3.8886 & -5.2620 & -0.2003 \end{array} \right]. \qquad (4.33)$$

We first illustrate the effectiveness of the proposed standard state-feedback controller $u(t) = K_s x(t)$ for the no actuator faults active suspension systems in

(4.23). It is expected that the desired the H_∞ controller in $u(t) = K_s x(t)$ can be designed such that: 1) the sprung mass acceleration $z_1(t)$ is as small as possible; 2) the suspension deflection is below the maximum allowable suspension stroke $z_{\max} = 0.08$ m, which means that $x_1(t)/z_{\max}$ below 1; 3) the relative dynamic tire load $k_t x_2(t)/(m_s + m_u)g < 1$; 4) the force of the actuator is below the maximum bound $u_{\max} = 1500$ N.

In order to evaluate the suspension characteristics with respect to the above four performance requirements, we apply the following road disturbance signal to clarify the effectiveness of reliable controller design method. In the context of active suspension performance, road disturbances can be generally represented as shocks. Shocks are discrete events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pothole on an otherwise smooth road surface. In this work, this case of road profile is considered to reveal the transient response characteristic:

$$z_r(t) = \begin{cases} \frac{A}{2}(1 - \cos(\frac{2\pi V}{L}t)), & \text{if } 0 \leq t \leq \frac{L}{V}, \\ 0, & \text{if } t > \frac{L}{V}, \end{cases} \tag{4.34}$$

where A and L are the height and the length of the bump. We assume $A = 60$ mm, $L = 5$ m and the vehicle forward velocity as $V = 30$ (km/h).

Fig. 4.1 plots the responses of body vertical accelerations, suspension stroke constrait and tire deflection constraint for the open- (passive) and closed-loop (active) systems and standard actuator force under the designed standard state-feedback H_∞ controller $u(t) = K_s x(t)$ in (4.33) respectively. It can be seen from Fig. 4.1 that the designed controller can achieve less value of the maximum body acceleration for the active suspension system without actuator faults than the passive system, and passenger acceleration in the active suspension system is reduced significantly, which guarantees better ride comfort. In addition, it can be observed from Fig. 4.1 that the suspension deflection constraint $x_1(t)/z_{\max} < 1$ is guaranteed. And Fig. 4.1 indicates the relation dynamic tire load $k_t x_2(t)/(m_s + m_u)g$ is below 1, which implies the road holding capability is ensured by the desired controller. It is clear that the force of the actuator is below the maximum bound $u_{\max}$ as showed in Fig. 4.1. The figure confirms that the designed standard state-feedback H_∞ controller can achieve much better ride comfort and road

handling, and guarantee constrain suspension deflection and maximum actuator force limitation for the active suspension system without actuator faults.
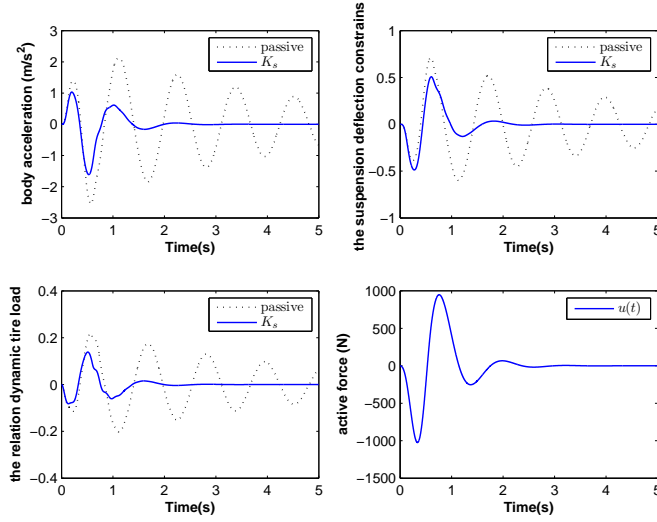
Figure 4.1: Bump responses of vertical body accelerations, suspension deflection constraints and tire stroke constraints for the open- and closed-loop systems and the active force.

Next, for the following three possible actuator faults modes, namely,

(1) Faulty model I: there is a loss of effectiveness in the actuator, $\check{m}_{a1} = 0.1$, $\hat{m}_{a1} = 0.9$, which implies $\hat{M}_{a01} = 0.5$ and $\check{M}_{a01} = 0.4$;

(2) Faulty model II: there is a loss of effectiveness in the actuator, $\check{m}_{a2} = 0.2$, $\hat{m}_{a2} = 0.8$, which implies $\hat{M}_{a02} = 0.5$ and $\check{M}_{a02} = 0.3$;

(3) Faulty model III: there is a loss of effectiveness in the actuator, $\check{m}_{a3} = 0.3$, $\hat{m}_{a3} = 0.7$, which implies $\hat{M}_{a03} = 0.5$ and $\check{M}_{a03} = 0.2$;

The transition probability matrix (TPM) is,

$$\Xi = \begin{bmatrix} -2 & 1 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -3 \end{bmatrix}.$$

By solving the conditions in Theorem 2 via the convex optimization method, it can be found that the minimum guaranteed closed-loop H_∞ performance index

γ_{min} is 20.0930, and the corresponding fault-tolerant H_∞ controller gain matrices are

$$K_{a1} = 10^3 \times \left[\begin{array}{cccc} -2.1322 & 4.6901 & -8.8826 & -0.3219 \end{array} \right], \qquad (4.35)$$

$$K_{a2} = 10^3 \times \left[\begin{array}{cccc} -1.5474 & 5.0070 & -8.8373 & -0.3167 \end{array} \right], \qquad (4.36)$$

$$K_{a3} = 10^3 \times \left[\begin{array}{cccc} -1.5480 & 5.0148 & -9.1485 & -0.3288 \end{array} \right]. \qquad (4.37)$$

The Markovian jump mode r_t is depicted in Fig. 4.2 under the initial mode $r_0 = 1$. In Fig. 4.3–4.6, the bump responses of open and closed-loop systems with standard state controller K_s and fault-tolerant controller K_{ai} ($i = 1, 2, 3$) with 30%, 40%, 50% and 60% actuator thrust loss are illustrated, respectively. From Fig. 4.3–4.6, we know that the less value of the maximum body acceleration for the active suspension system is achieved, the suspension deflection constrain $x_1/z_{\max} < 1$ is guaranteed, the relation dynamic tire load $k_t x_2(t)/(m_s + m_u)g$ is below 1 and the force of the actuator is below the maximum bound $u_{\max}$ by using standard state controller K_s and fault-tolerant controller K_{ai} ($i = 1, 2, 3$) respectively. However, it is shown that the fault-tolerant controller K_{ai} ($i = 1, 2, 3$) is capable to provide a much more steady control force in fault condition than conventional controller K_s.
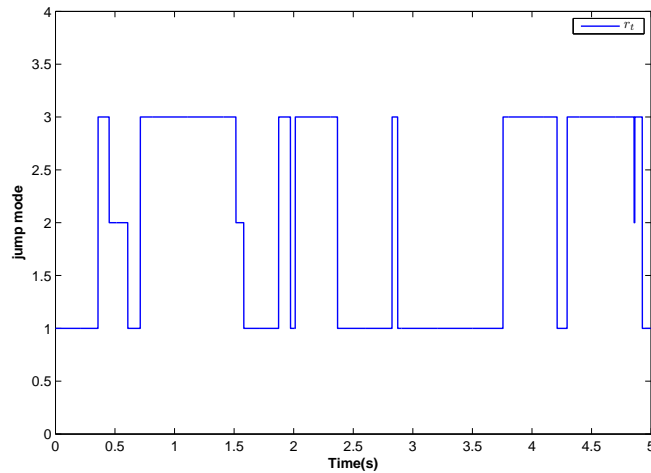
Figure 4.2: Markovian jump mode
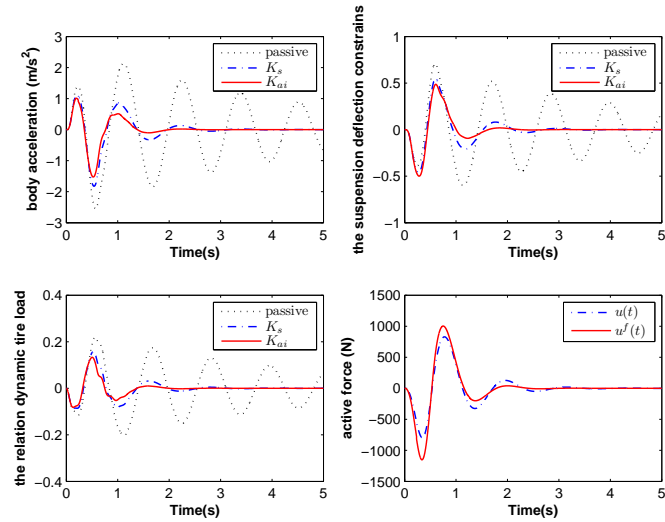
Figure 4.3: Bump responses of vertical body accelerations, suspension deflection constraints, tire stroke constraints and the active force with 30% actuator thrust loss.
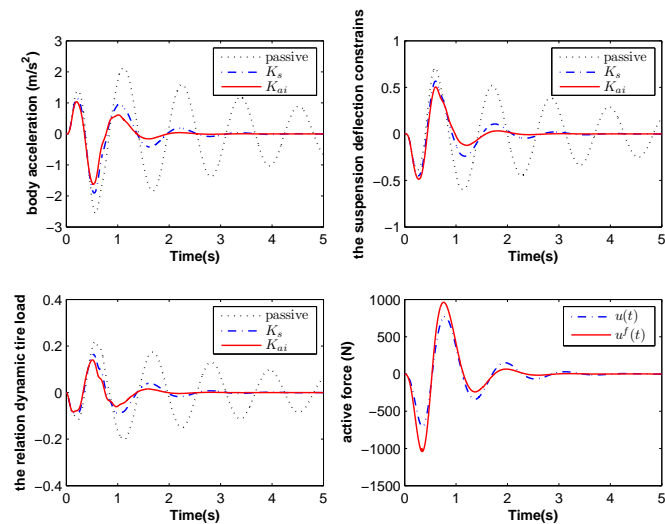
Figure 4.4: Bump responses of vertical body accelerations, suspension deflection constraints, tire stroke constraints and the active force with 40% actuator thrust loss.
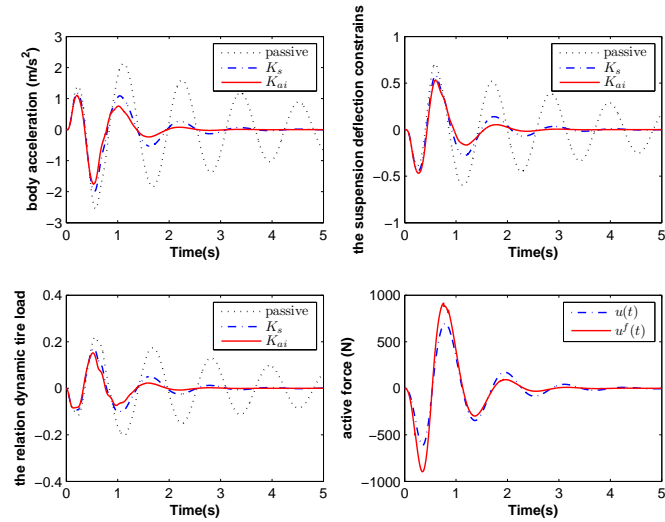
Figure 4.5: Bump responses of vertical body accelerations, suspension deflection constraints, tire stroke constraints and the active force with 50% actuator thrust loss.
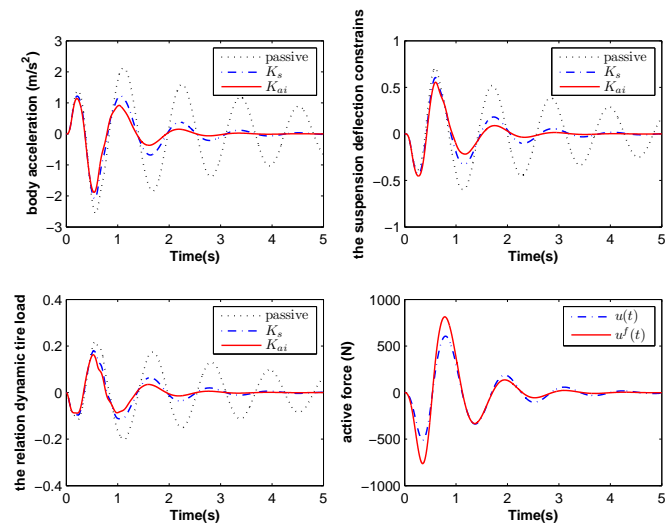
Figure 4.6: Bump responses of vertical body accelerations, suspension deflection constraints, tire stroke constraints and the active force with 60% actuator thrust loss.

Then, we will consider the following three more general actuator fault modes:

(1) Normal model I: the actuator is normal, $\check{m}_{a1} = \hat{m}_{a1} = 1$, which implies $\hat{M}_{a01} = 1$ and $\check{M}_{a01} = 0$;

(2) Faulty model II: there is a loss of effectiveness in the actuator, $\check{m}_{a2} = 0.5$, $\hat{m}_{a2} = 1$, which implies $\hat{M}_{a02} = 0.75$ and $\check{M}_{a02} = 0.25$;

(3) Faulty model III: the actuator is in outage, $\check{m}_{a3} = \hat{m}_{a3} = 0$; which implies $\hat{M}_{a03} = 0$ and $\check{M}_{a03} = 0$;

By solving the conditions in Theorem 2 via the convex optimization method, it can be found that the minimum guaranteed closed-loop H_∞ performance index γ_{min} is 18.3676, and the corresponding fault-tolerant H_∞ controller gain matrices are

$$K_{a1} = 10^3 \times \begin{bmatrix} 1.2252 & 4.0273 & -4.5832 & -0.1761 \end{bmatrix}, \qquad (4.38)$$

$$K_{a2} = 10^3 \times \begin{bmatrix} 0.5093 & 4.4873 & -5.5834 & -0.2072 \end{bmatrix}, \qquad (4.39)$$

$$K_{a3} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (4.40)$$

The case study is aimed at understanding its transition states of the minimum H_∞ performance index $\gamma_{\min}$ when the faulty model II and transition probability matrix change, respectively. Table 4.1 shows the minimum H_∞ performance index $\gamma_{\min}$ and fault-tolerant control gain matrices K_{a1} and K_{a2} for the above same faulty modes and different transition probability matrix. Moreover, Table 4.2 lists the corresponding results on the minimum H_∞ performance index $\gamma_{\min}$ and fault-tolerant control gain matrices K_{a1} and K_{a2} for the different $\check{m}_{a2}$ in Faulty model II. It can be observed from Table 4.2 that the minimum H_∞ performance index $\gamma_{\min}$ is lower when the lower bound of the Faulty model II is larger.

Fig. 4.7 depicts the responses of body vertical accelerations, suspension stroke constraints, tire deflection constraints and fault actuator force for the passive and active systems under the designed reliable control gain matrices in (4.38)–(4.40), respectively. These figures further confirm that the designed fault-tolerant controller can achieve much better ride comfort and road handling, guarantee constrained suspension deflection and maximum actuator force limitation for the active suspension system with actuator faults.

Table 4.1: Results for different transition probability matrix

TPM	$\gamma_{\min}$	$K_{a1}, \quad K_{a2}, \quad K_{a3} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$			
2Ξ	15.4913	$K_{a1} = 10^3 \times$	$-0.2740 \quad 2.9383 \quad -5.1585 \quad -0.1949$		
		$K_{a2} = 10^3 \times$	$-0.8371 \quad 3.4344 \quad -5.9945 \quad -0.2216$		
3Ξ	14.5720	$K_{a1} = 10^3 \times$	$-0.7681 \quad 2.6952 \quad -5.4381 \quad -0.2087$		
		$K_{a2} = 10^3 \times$	$-1.3088 \quad 3.1531 \quad -6.1703 \quad -0.2290$		
4Ξ	14.2389	$K_{a1} = 10^3 \times$	$-1.0071 \quad 2.5834 \quad -5.5965 \quad -0.2181$		
		$K_{a2} = 10^3 \times$	$-1.4220 \quad 3.0723 \quad -6.2208 \quad -0.2294$		
5Ξ	13.9964	$K_{a1} = 10^3 \times$	$-1.0935 \quad 2.6323 \quad -5.7193 \quad -0.2261$		
		$K_{a2} = 10^3 \times$	$-1.5451 \quad 3.0399 \quad -6.2817 \quad -0.2311$		

Table 4.2: Results for different $\breve{m}_{a2}$ in Faulty model II

$\breve{m}_{a2}$	$\gamma_{\min}$	$K_{a1}, \quad K_{a2}, \quad K_{a3} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$			
0	25.5957	$K_{a1} = 10^3 \times$	$-0.0896 \quad 2.9064 \quad -5.0931 \quad -0.1867$		
		$K_{a2} = 10^3 \times$	$2.2750 \quad 3.2372 \quad -5.2869 \quad -0.1239$		
0.3	19.7250	$K_{a1} = 10^3 \times$	$0.8132 \quad 3.7004 \quad -4.7272 \quad -0.1806$		
		$K_{a2} = 10^3 \times$	$0.4649 \quad 5.0513 \quad -6.1855 \quad -0.2255$		
0.7	17.5179	$K_{a1} = 10^3 \times$	$1.2824 \quad 3.9139 \quad -4.4280 \quad -0.1736$		
		$K_{a2} = 10^3 \times$	$0.5957 \quad 4.0174 \quad -5.1142 \quad -0.1944$		
0.9	16.8738	$K_{a1} = 10^3 \times$	$1.7276 \quad 4.4055 \quad -4.3263 \quad -0.1743$		
		$K_{a2} = 10^3 \times$	$0.7915 \quad 3.8986 \quad -4.7897 \quad -0.1919$		
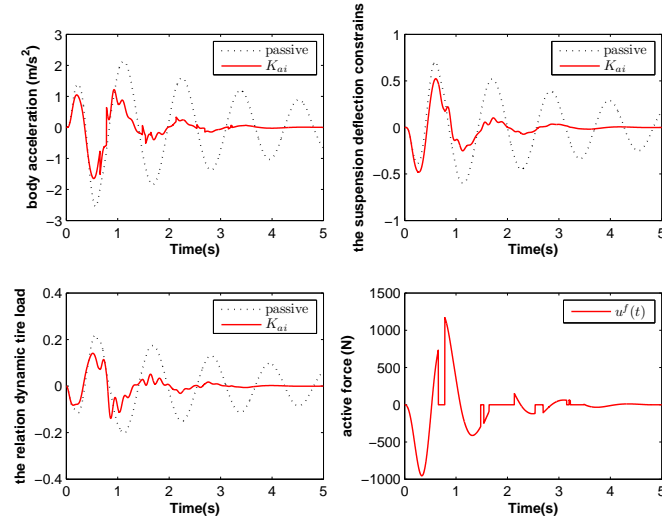
Figure 4.7: Bump responses of vertical body accelerations, suspension deflection constraints, tire stroke constraints and the active force for open and closed-loop systems

4.5   Summary

This chapter has designed fault-tolerant controller for a class of quarter-car active suspension systems subject to actuator faults. The state-space system has been established based on the suspension system performance such as ride comfort, road holding, suspension deflection and maximum actuator force limitation. Actuator failure process within the suspension system has been regarded as stochastic behavior and modeled as a continuous-time homogeneous Markov process. LMI-based conditions have been formulated for the existence of admissible fault-tolerant H_∞ controller, which ensures the closed-loop to be asymptotically stable with a prescribed H_∞ disturbance attenuation level, and simultaneously satisfy the constraint performance in spite of the possible actuator faults, and the existence conditions of admissible controller have been resolved. A quarter-car suspension model has been provided to evaluate the effectiveness of the proposed reliable controller design approach. It is worth noting that the idea behind this chapter could be used to deal with more complex suspension systems, such as half-car and full-car suspension systems.

Chapter 5

Fuzzy Control for Vehicle Active Suspension Systems with Uncertainty

5.1 Introduction

An active suspension system has the ability to enhance vehicle dynamics by relaxing external impact such as road surface on vehicle travel comfort. In terms of its control design, uncertainty of vehicle sprung and unsprung masses such as its loading conditions should be taken into account to meet vehicle travel performance criteria. For instance, the polytopic parameter uncertainties was employed to model the varying vehicle sprung or unsprung masses (Du *et al.*, 2008; Gao *et al.*, 2006, 2010a). The parameter-dependent controllers was proposed for the quarter-car suspension systems with sprung mass variation (Du *et al.*, 2008). The parameter-independent sampled-data H_∞ controller design strategy was presented to handle both sprung and unsprung mass variations in a case study of a quarter-car suspension system (Gao *et al.*, 2010a). The state of the art in suspension control design in these scenarios, however, could not provide feasible performance for uncertain active suspension systems with actuator delay and fault. Clearly, there is a requirement for a new controller design method which has the capability of satisfying the control condition. On the other hand, since fuzzy sets were proposed by Zadeh (Zadeh, 1965), fuzzy logic control has been

developed into a conspicuous and successful branch of automation and control theory. The T-S fuzzy model has been proved as an effective theoretical method and practical tool for representing complex nonlinear systems and applications (Feng, 2006; Lin *et al.*, 2007; Sugeno, 1985; Tanaka & Wang, 2001).

T-S fuzzy model based systems are described as a weighted sum of some simple linear subsystems, and thus are easily analyzable, the success on control analysis and synthesis problems have been also demonstrated by various techniques (Lam & Narimani, 2010; Nguang & Shi, 2003; Zhang *et al.*, 2010). Recently, research has been conducted to challenge the reliability of the continuous-time T-S fuzzy systems (Chen & Liu, 2004; Nguang *et al.*, 2007; Wang *et al.*, 2007; Wu & Zhang, 2006). However, in the context of vehicle suspension control design, there are few results on reliable fuzzy H_∞ controller design for T-S fuzzy systems with both actuator delay and fault. On the other hand, fuzzy controller design had been investigated for suspension systems in the past years, for example, (Cao *et al.*, 2010; Du & Zhang, 2009; Huang & Lin, 2003a). In particular, a T-S model-based fuzzy control design approach was presented for electrohydraulic active suspension systems with input constraints (Du & Zhang, 2009). It is evident, however, there are few results on fuzzy H_∞ controller design for uncertain active suspension systems with actuator delay and fault.

This chapter is concerned with the problem of reliable fuzzy H_∞ control for uncertain active suspension systems with actuator delay and fault based on the T-S fuzzy model approach. The vehicle dynamic system is established by the fact that vehicle sprung and unsprung mass variations, the actuator delay and fault have been taken into account the suspension performances. The parallel-distributed compensation (PDC) scheme is, then, used to develop reliable fuzzy H_∞ performance analysis condition for the proposed T-S fuzzy system, and the reliable fuzzy H_∞ controller is designed to guarantee the systems asymptotic stability and H_∞ performance, simultaneously satisfying the constraint performances. Furthermore, LMI-based condition of reliable fuzzy H_∞ controller design is derived. Finally, the proposed method is evaluated on a quarter-car suspension model. Simulation results demonstrate that the designed reliable fuzzy H_∞ controller has robust capability of guaranteeing better suspension performance with

uncertainty of the sprung and unsprung mass variations, the actuator delay and fault.

The reminder of this chapter is organized as follows. The problem to be addressed is formulated in Section 5.2. Section 5.3 presents the results of reliable fuzzy H_∞ controller design and Section 5.4 provides fuzzy H_∞ controller design scheme. Simulation results are provided to evaluate the proposed method in Section 5.5, and finally the chapter is concluded in Section 5.6.

5.2 Problem Formulation

As pointed out in (Du & Zhang, 2009; Du *et al.*, 2008; Gao *et al.*, 2010a), with the different loading conditions, the vehicle sprung and unsprung masses vary in the given ranges. Note that the suspension system in (2.5) is a model with uncertainty as the sprung mass m_s and the unsprung mass m_u vary in the given ranges, in which m_s and m_u denote $m_s(t)$ and $m_u(t)$ respectively. In the meantime, the actuator delay and fault should be taken into account since the suspension performance could be affected by these factors. It leads to the system as:

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B(t)u_f(t - d(t)), \\
z_1(t) &= C_1(t)x(t) + D_1(t)u_f(t - d(t)), \\
z_2(t) &= C_2(t)x(t), \\
x(t) &= \phi(t), \quad t \in \left[-\bar{d}, 0\right],
\end{aligned}
\tag{5.1}
$$

where $\phi(t)$ is a vector-valued initial continuous function defined on $t \in \left[-\bar{d}, 0\right]$. $d(t)$ denotes the time-varying delay satisfying

$$
0 \leq d(t) \leq \bar{d}, \quad d(t) \leq \mu.
\tag{5.2}
$$

Considering the fault channel from controller to actuator,

$$
u^f(t) = m_a u(t),
\tag{5.3}
$$

m_a is used to represent the possible fault of the corresponding actuator $u^f(t)$. $\check{m}_a \leq m_a \leq \hat{m}_a$, where $\check{m}_a$ and $\hat{m}_a$ are constant scalars and used to constrain lower

and upper bounds of the actuator faults. Three following cases are considered corresponding to three different actuator conditions:

1) $\check{m}_a = \hat{m}_a = 0$, then $m_a = 0$, which implies that the corresponding actuator $u^f(t)$ is completely failed.

2) $\check{m}_a = \hat{m}_a = 1$, thus we obtain $m_a = 1$, which represents the case of no fault in the actuator $u^f(t)$.

3) $0 < \check{m}_a < \hat{m}_a < 1$, which means that there exists partial fault in the corresponding actuator $u^f(t)$.

The sprung mass $m_s(t)$ and the unsprung mass $m_u(t)$ are uncertainties, which vary in a given range, i.e. $m_s(t) \in [m_{s\min}, m_{s\max}]$ and $m_u(t) \in [m_{u\min}, m_{u\max}]$. It deliver that the uncertainty scenarios of the mass $m_s(t)$ is bounded by its minimum value $m_{s\min}$ and its maximum value $m_{s\max}$. In addition, the mass $m_u(t)$ is bounded by its minimum value $m_{u\min}$ and its maximum value $m_{u\max}$.

When considering the time-varying uncertainty, actuator delay and faults in the suspension model, it is very difficult to design the controller directly for this kind system to improve the suspension performances. In this chapter, the fuzzy reliable control method is presented to handle this issue. Firstly, we obtain the values of $\frac{1}{m_s(t)}$ and $\frac{1}{m_u(t)}$ from $m_s(t) \in [m_{s\min}, m_{s\max}]$ and $m_u(t) \in [m_{u\min}, m_{u\max}]$. Then we have

$$
\begin{aligned}
\max\frac{1}{m_s(t)} &= \frac{1}{m_{s\min}} =: \hat{m}_s, \quad \min\frac{1}{m_s(t)} = \frac{1}{m_{s\max}} =: \check{m}_s, \\
\max\frac{1}{m_u(t)} &= \frac{1}{m_{u\min}} =: \hat{m}_u, \quad \min\frac{1}{m_u(t)} = \frac{1}{m_{u\max}} =: \check{m}_u.
\end{aligned}
$$

The sector nonlinear method (Tanaka & Wang, 2001) is employed to represent $\frac{1}{m_s(t)}$ and $\frac{1}{m_u(t)}$ by,

$$
\begin{aligned}
\frac{1}{m_s(t)} &= M_1(\xi_1(t))\,\hat{m}_s + M_2(\xi_1(t))\,\check{m}_s, \\
\frac{1}{m_u(t)} &= N_1(\xi_2(t))\,\hat{m}_u + N_2(\xi_2(t))\,\check{m}_u,
\end{aligned}
$$

where $\xi_1(t) = \frac{1}{m_s(t)}$ and $\xi_2(t) = \frac{1}{m_u(t)}$ are premise variables,

$$
\begin{aligned}
M_1(\xi_1(t)) + M_2(\xi_1(t)) &= 1, \\
N_1(\xi_2(t)) + N_2(\xi_2(t)) &= 1.
\end{aligned}
$$

The membership functions $M_1\left(\xi_1\left(t\right)\right)$, $M_2\left(\xi_1\left(t\right)\right)$, $N_1\left(\xi_2\left(t\right)\right)$ and $N_2\left(\xi_2\left(t\right)\right)$ can be calculated as

$$
\begin{aligned}
M_1\left(\xi_1\left(t\right)\right) &= \frac{\frac{1}{m_s(t)} - \check{m}_s}{\hat{m}_s - \check{m}_s}, \quad M_2\left(\xi_1\left(t\right)\right) = \frac{\hat{m}_s - \frac{1}{m_s(t)}}{\hat{m}_s - \check{m}_s}, \\
N_1\left(\xi_2\left(t\right)\right) &= \frac{\frac{1}{m_u(t)} - \check{m}_u}{\hat{m}_u - \check{m}_u}, \quad N_2\left(\xi_2\left(t\right)\right) = \frac{\hat{m}_u - \frac{1}{m_u(t)}}{\hat{m}_u - \check{m}_u}.
\end{aligned}
$$

The membership functions are labeled as **Heavy**, **Light**, **Heavy** and **Light** as shown in Fig. 5.1. In addition, Table 5.1 lists the fuzzy rules for the systems in (5.1).

Table 5.1: List of Fuzzy Rules

Rule No.	Premise variables	
	$\xi_1\left(t\right)$	$\xi_2\left(t\right)$
1	**Heavy**	**Heavy**
2	**Heavy**	**Light**
3	**Light**	**Heavy**
4	**Light**	**Light**

Then, the system with uncertainty in (5.1) is represented by the following fuzzy model:
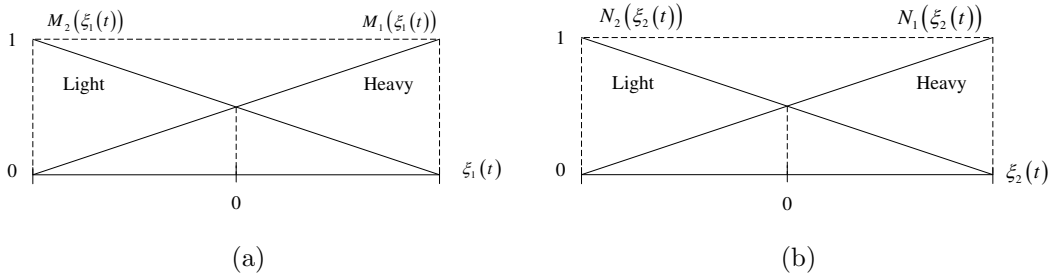
Figure 5.1: (a) Membership functions $M_1\left(\xi_1\left(t\right)\right)$ and $M_2\left(\xi_1\left(t\right)\right)$ (b) Membership functions $N_1\left(\xi_2\left(t\right)\right)$ and $N_2\left(\xi_2\left(t\right)\right)$

Model Rule 1: **IF** $\xi_1\left(t\right)$ is **Heavy** and $\xi_2\left(t\right)$ is **Heavy**,

THEN

$$
\begin{aligned}
\dot{x}\left(t\right) &= A_1 x\left(t\right) + B_1 u_f\left(t - d\left(t\right)\right) + B_{11} w\left(t\right), \\
z_1\left(t\right) &= C_{11} x\left(t\right) + D_{11} u_f\left(t - d\left(t\right)\right), \\
z_2\left(t\right) &= C_{21} x\left(t\right),
\end{aligned}
$$

matrices A_1, B_1, B_{11}, C_{11}, D_{11} and C_{21} are obtained by replacing $\frac{1}{m_s(t)}$ and $\frac{1}{m_u(t)}$ with $\hat{m}_s$ and $\hat{m}_u$ respectively in matrices $A\left(t\right)$, $B\left(t\right)$, $B_1\left(t\right)$, $C_1\left(t\right)$, $D_1\left(t\right)$ and $C_2\left(t\right)$.

Model Rule 2: **IF** $\xi_1\left(t\right)$ is **Heavy** and $\xi_2\left(t\right)$ is **Light**,

THEN

$$
\begin{aligned}
\dot{x}\left(t\right) &= A_2 x\left(t\right) + B_2 u_f\left(t - d\left(t\right)\right) + B_{12} w\left(t\right), \\
z_1\left(t\right) &= C_{12} x\left(t\right) + D_{12} u_f\left(t - d\left(t\right)\right), \\
z_2\left(t\right) &= C_{22} x\left(t\right),
\end{aligned}
$$

matrices A_2, B_2, B_{12}, C_{12}, D_{12} and C_{22} are obtained by replacing $\frac{1}{m_s(t)}$ and $\frac{1}{m_u(t)}$ with $\hat{m}_s$ and $\check{m}_u$ in matrices $A\left(t\right)$, $B\left(t\right)$, $B_1\left(t\right)$, $C_1\left(t\right)$, $D_1\left(t\right)$ and $C_2\left(t\right)$ respectively.

Model Rule 3: **IF** $\xi_1\left(t\right)$ is **Light** and $\xi_2\left(t\right)$ is **Heavy**,

THEN

$$
\begin{aligned}
\dot{x}\left(t\right) &= A_3 x\left(t\right) + B_3 u_f\left(t - d\left(t\right)\right) + B_{13} w\left(t\right), \\
z_1\left(t\right) &= C_{13} x\left(t\right) + D_{13} u_f\left(t - d\left(t\right)\right), \\
z_2\left(t\right) &= C_{23} x\left(t\right),
\end{aligned}
$$

matrices A_3, B_3, B_{13}, C_{13}, D_{13} and C_{23} are obtained by replacing $\frac{1}{m_s(t)}$ and $\frac{1}{m_u(t)}$ with $\check{m}_s$ and $\hat{m}_u$ in matrices $A\left(t\right)$, $B\left(t\right)$, $B_1\left(t\right)$, $C_1\left(t\right)$, $D_1\left(t\right)$ and $C_2\left(t\right)$ respectively.

Model Rule 4: **IF** $\xi_1\left(t\right)$ is **Light** and $\xi_2\left(t\right)$ is **Light**,

THEN

$$
\begin{aligned}
\dot{x}\left(t\right) &= A_4 x\left(t\right) + B_4 u_f\left(t - d\left(t\right)\right) + B_{14} w\left(t\right), \\
z_1\left(t\right) &= C_{14} x\left(t\right) + D_{14} u_f\left(t - d\left(t\right)\right), \\
z_2\left(t\right) &= C_{24} x\left(t\right),
\end{aligned}
$$

matrices A_4, B_4, B_{14}, C_{14}, D_{14} and C_{24} are obtained by replacing $\frac{1}{m_s(t)}$ and $\frac{1}{m_u(t)}$ with $\check{m}_s$ and $\check{m}_u$ in matrices $A\left(t\right)$, $B\left(t\right)$, $B_1\left(t\right)$, $C_1\left(t\right)$, $D_1\left(t\right)$ and $C_2\left(t\right)$ respectively.

Fuzzy blending allows to infer the overall fuzzy model as follows:

$$
\begin{aligned}
\dot{x}\left(t\right) &= \sum_{i=1}^{4} h_i\left(\xi\left(t\right)\right)\left[A_i x\left(t\right) + B_i u_f\left(t - d\left(t\right)\right) + B_{1i} w\left(t\right)\right], \\
z_1\left(t\right) &= \sum_{i=1}^{4} h_i\left(\xi\left(t\right)\right)\left[C_{1i} x\left(t\right) + D_{1i} u_f\left(t - d\left(t\right)\right)\right], \\
z_2\left(t\right) &= \sum_{i=1}^{4} h_i\left(\xi\left(t\right)\right) C_{2i} x\left(t\right),
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
h_1\left(\xi\left(t\right)\right) &= M_1\left(\xi_1\left(t\right)\right) \times N_1\left(\xi_2\left(t\right)\right), \\
h_2\left(\xi\left(t\right)\right) &= M_1\left(\xi_1\left(t\right)\right) \times N_2\left(\xi_2\left(t\right)\right), \\
h_3\left(\xi\left(t\right)\right) &= M_2\left(\xi_1\left(t\right)\right) \times N_1\left(\xi_2\left(t\right)\right), \\
h_4\left(\xi\left(t\right)\right) &= M_2\left(\xi_1\left(t\right)\right) \times N_2\left(\xi_2\left(t\right)\right).
\end{aligned}
$$

It is apparent that the fuzzy weighting functions $h_i(\xi\left(t\right))$ satisfy $h_i\left(\xi\left(t\right)\right) \geq 0$, $\sum_{i=1}^{4} h_i\left(\xi\left(t\right)\right) = 1$. In order to design a fuzzy reliable controllers, PDC is adapted and the following fuzzy controller is obtained:

Control Rule 1: IF $\xi_1\left(t\right)$ is **Heavy** and $\xi_2\left(t\right)$ is **Heavy**,
 THEN $u(t) = K_{a1}x(t)$.

Control Rule 2: IF $\xi_1\left(t\right)$ is **Heavy** and $\xi_2\left(t\right)$ is **Light**,
 THEN $u(t) = K_{a2}x(t)$.

Control Rule 3: IF $\xi_1\left(t\right)$ is **Light** and $\xi_2\left(t\right)$ is **Heavy**,
 THEN $u\left(t\right) = K_{a3}x(t)$.

Control Rule 4: IF $\xi_1\left(t\right)$ is **Light** and $\xi_2\left(t\right)$ is **Light**,
 THEN $u\left(t\right) = K_{a4}x(t)$.

Hence, the overall fuzzy control law is represented by

$$
u(t) = \sum_{j=1}^{4} h_j(\xi\left(t\right)) K_{aj} x(t)
\tag{5.5}
$$

where K_{aj} $(j = 1, 2, 3, 4)$ are the local control gains and

$$u(t - d(t)) = \sum_{j=1}^{4} h_j(\xi(t - d(t))) K_{aj} x(t - d(t)).$$

Therefore, in this chapter, we assume that $h_j(\xi(t - d(t)))$ is well defined for $t \in [-\bar{d}, 0]$, and $h_j(\xi(t - d(t))) \geq 0$, $(j = 1, 2, 3, 4)$ $\sum_{j=1}^{4} h_j(\xi(t - d(t))) = 1$. For simplicity, the following notations will be used:

$$h_i =: h_i(\xi(t)), \quad h_j^d =: h_j(\xi(t - d(t))).$$

Applying the fuzzy controller (5.5) to system (5.4) yields the closed-loop system:

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{4} \sum_{j=1}^{4} h_i h_j^d \left[A_i x(t) + B_i m_a(t) K_{aj} x(t - d(t)) \right. \\
&\quad \left. + B_{1i} w(t) \right], \\
z_1(t) &= \sum_{i=1}^{4} \sum_{j=1}^{4} h_i h_j^d \left[C_{1i} x(t) + D_{1i} m_a K_{aj} x(t - d(t)) \right], \\
z_2(t) &= \sum_{i=1}^{4} h_i C_{2i} x(t). \tag{5.6}
\end{aligned}
$$

The T-S fuzzy system in (5.6) is established based on the practically measurable sprung $m_s(t)$ and unsprung $m_u(t)$. The sector nonlinearity method (Tanaka & Wang, 2001) is employed to analyze the variation of the sprung $m_s(t)$ and unsprung $m_u(t)$ and presents the T-S fuzzy system in (5.6).

Without loss of generality, it is assumed, $w \in L_2[0, \infty)$, and $\|w\|_2^2 \leq w_{\max} < \infty$. The objective in this subsection is to design the feedback gain matrices K_{aj} $(j = 1, 2, 3, 4)$ such that the following requirements are satisfied:

(1) the closed-loop system is asymptotically stable;

(2) under zero initial condition, the closed-loop system guarantees that $\|z_1\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0, \infty)$, where $\gamma > 0$ is a prescribed scalar;

(3) the following control output constraints are guaranteed:

$$|\{z_2(t)\}_q| \leq 1, \quad q = 1, 2. \tag{5.7}$$

5.3   Reliable Fuzzy Controller Design

In this section, reliable fuzzy H_∞ state-feedback controller is derived for the active suspension system with actuator delay and fault. It ensures that the closed-loop system in (5.6) is asymptotically stable, and it also guarantees a prescribed gain from disturbance $w(t)$ to performance output $z_1(t)$, under the condition that the suspension stroke and tire deflection constraints are satisfied. First, the following lemma is presented,

Lemma 5.1 (Yang *et al.*, 2001a) For a time-varying diagonal matrix $\Phi(t) = \text{diag}\{\sigma_1(t), \sigma_2(t), \cdots, \sigma_p(t)\}$ and two matrices R and S with appropriate dimensions, if $|\Phi(t)| \leq V$, where $V > 0$ is a known diagonal matrix, then for any scalar $\varepsilon > 0$, it is true that

$$R\Phi S + S^T \Phi^T R^T \leq \varepsilon R V R^T + \varepsilon^{-1} S^T V S.$$

Next, the following scalars is introduced which will be used in the later development in this chapter. $M_{a0} = (\check{m}_a + \hat{m}_a)/2$, $L_a = [m_a - M_{a0}]/m_{a0}$ and $J_a = (\hat{m}_a - \check{m}_a)/(\hat{m}_a + \check{m}_a)$. Thus, one has $m_a = M_{a0}(I + L_a)$ and $L_a^T L_a \leq J_a^T J_a \leq I$. Then, it leads to the following theorem.

Theorem 5.1 Consider the closed-loop system in (5.6). For given scalars $\bar{d} > 0$, μ and matrices K_{aj}, if there exist matrices $P > 0$, $Q > 0$, $S > 0$, $R > 0$, N_j, and M_j with appropriate dimensions and positive scalars $\varepsilon_{1ij} > 0$ and

$\varepsilon_{2ij} > 0$ $(i, j = 1, 2, 3, 4)$ such that the following LMIs hold for $q = 1, 2$:

$$
\begin{bmatrix}
\Phi_{11}^{ij} & \sqrt{\bar{d}}M & \Phi_{13}^{ij} & \Phi_{14}^{ij} & \Phi_{15}^{ij} & \Phi_{16}^{1ij} \\
0 & -R & 0 & 0 & 0 & 0 \\
0 & 0 & -I & 0 & D_{1i} & 0 \\
0 & 0 & 0 & -R & \sqrt{\bar{d}}RB_i & 0 \\
0 & 0 & 0 & 0 & -\varepsilon_{1ij}J_a^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & -\varepsilon_{1ij}J_a^{-1}
\end{bmatrix} < 0, \qquad (5.8)
$$

$$
\begin{bmatrix}
\Phi_{11}^{ij} & \sqrt{\bar{d}}N & \Phi_{13}^{ij} & \Phi_{14}^{ij} & \Phi_{15}^{ij} & \Phi_{16}^{2ij} \\
0 & -R & 0 & 0 & 0 & 0 \\
0 & 0 & -I & 0 & D_{1i} & 0 \\
0 & 0 & 0 & -R & \sqrt{\bar{d}}RB_i & 0 \\
0 & 0 & 0 & 0 & -\varepsilon_{2ij}J_a^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & -\varepsilon_{2ij}J_a^{-1}
\end{bmatrix} < 0, \qquad (5.9)
$$

$$
\begin{bmatrix}
-P & \sqrt{\rho}\,\{C_{2i}\}_q^T \\
* & -I
\end{bmatrix} < 0, \qquad (5.10)
$$

where

$$
\Phi_{11}^{ij} = \Xi_{11}^{ij} + \mathrm{sym}\,(\Xi_2), \quad \Xi_{11}^{ij} = \begin{bmatrix} \Theta_{11}^{ij} & \Theta_{12}^{ij} \\ * & -\gamma^2 I \end{bmatrix},
$$

$$
\Theta_{11}^{ij} = \begin{bmatrix}
\mathrm{sym}\,(PA_i) + Q + S & PB_iM_{a0}K_{aj} & 0 \\
* & -(1-\mu)S & 0 \\
* & * & -Q
\end{bmatrix},
$$

$$
\Theta_{12}^{ij} = \begin{bmatrix} PB_{1i} \\ 0 \\ 0 \end{bmatrix}, \Xi_2 = \begin{bmatrix} M & N-M & -N & 0 \end{bmatrix},
$$

$$
\Phi_{13}^{ij} = \begin{bmatrix} C_{1i} & D_{1i}M_{a0}K_{aj} & 0 & 0 \end{bmatrix}^T, \quad \Phi_{15}^{ij} = \begin{bmatrix} B_i^TP & 0 & 0 & 0 \end{bmatrix}^T,
$$

$$
\Phi_{14}^{ij} = \begin{bmatrix} \sqrt{\bar{d}}RA_i & \sqrt{\bar{d}}RB_iM_{a0}K_{aj} & 0 & \sqrt{\bar{d}}RB_{1i} \end{bmatrix}^T,
$$

$$
\Phi_{16}^{1ij} = \begin{bmatrix} 0 & \varepsilon_{1ij}M_{a0}K_{aj} & 0 & 0 \end{bmatrix}^T, \quad \Phi_{16}^{2ij} = \begin{bmatrix} 0 & \varepsilon_{2ij}M_{a0}K_{aj} & 0 & 0 \end{bmatrix}^T,
$$

$$
M = \begin{bmatrix} M_1^T & M_2^T & M_3^T & M_4^T \end{bmatrix}^T, \quad N = \begin{bmatrix} N_1^T & N_2^T & N_3^T & N_4^T \end{bmatrix}^T.
$$

Furthermore,

(1) the closed-loop system is robustly asymptotically stable;

(2) the performance $\|T_{z_1w}\|_\infty < \gamma$ is minimized subject to output constraints (5.7) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$, where T_{z_1w} denotes the closed-loop transfer function from the road disturbance $w(t)$ to the control output $z_1(t)$.

Proof. Considering the Lyapunov-Krasovskii functional as follows:

$$
\begin{aligned}
V(t) \;=\;& x^T(t)\,Px(t) + \int_{t-\bar{d}}^{t} x^T(s)\,Qx(s)\,\mathrm{d}s \\
&+ \int_{t-d(t)}^{t} x^T(s)\,Sx(s)\,\mathrm{d}s + \int_{-\bar{d}}^{0}\int_{t+\alpha}^{t} \dot{x}^T(s)\,R\dot{x}(s)\,\mathrm{d}s\mathrm{d}\alpha. \quad (5.11)
\end{aligned}
$$

The derivative of $V(t)$ along the solution of system (5.6) is expressed as

$$
\begin{aligned}
\dot{V}(t) \;\leq\;& 2x^T(t)\,P\dot{x}(t) + x^T(t)\,(Q+S)\,x(t) \\
&- x^T(t-\bar{d})\,Qx(t-\bar{d}) + \bar{d}\dot{x}^T(t)\,S\dot{x}(t) \\
&- (1-\mu)\,x^T(t-d(t))\,Qx(t-d(t)) \\
&- \int_{t-d(t)}^{t} \dot{x}^T(s)\,R\dot{x}(s)\,\mathrm{d}s - \int_{t-\bar{d}}^{t-d(t)} \dot{x}^T(s)\,R\dot{x}(s)\,\mathrm{d}s. \quad (5.12)
\end{aligned}
$$

To develop H_∞ performance analysis criterion, the system (5.6) is stable with $w(t) = 0$; then the H_∞ performance index is satisfied. For any appropriately dimensioned matrices $\hat{M}$ and $\hat{N}$, the following equalities hold directly according to Newton-Leibniz formula:

$$
\begin{aligned}
\eta_1(t) \;=\;& 2\xi^T(t)\,\hat{M}\left(x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)\,\mathrm{d}s \right) = 0, \\
\eta_2(t) \;=\;& 2\xi^T(t)\,\hat{N}\left(x(t-d(t)) - x(t-\bar{d}) - \int_{t-\bar{d}}^{t-d(t)} \dot{x}(s)\,\mathrm{d}s \right) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\xi^T(t) \;=\;& \left[\begin{array}{ccc} x^T(t) & x^T(t-d(t)) & x^T(t-\bar{d}) \end{array}\right], \\
\hat{M} \;=\;& \left[\begin{array}{ccc} M_1^T & M_2^T & M_3^T \end{array}\right]^T, \hat{N} = \left[\begin{array}{ccc} N_1^T & N_2^T & N_3^T \end{array}\right]^T.
\end{aligned}
$$

Adding $\eta_1(t)$ and $\eta_2(t)$ into the right hand side of (5.12), the following in-

equalities is obtained:

$$
\begin{aligned}
\dot{V}(t) \;\leq\; & \sum_{i=1}^{4}\sum_{j=1}^{4} h_i h_j^d \xi^T(t)\left[\hat{\bar{\Xi}}_{ij} + d(t)\hat{M}R^{-1}\hat{M}^T + \left(\bar{d}-d(t)\right)\hat{N}R^{-1}\hat{N}^T\right]\xi(t) \\
& - \int_{t-d(t)}^{t}\left[\xi^T(t)\hat{M} + \dot{x}^T(s)R\right]R^{-1}\left[\hat{M}^T\xi(t) + R\dot{x}(s)\right]\mathrm{d}s \\
& - \int_{t-\bar{d}}^{t-d(t)}\left[\xi^T(t)\hat{N} + \dot{x}^T(s)R\right]R^{-1}\left[\hat{N}^T\xi(t) + R\dot{x}(s)\right]\mathrm{d}s \\
\leq\; & \sum_{i=1}^{4}\sum_{j=1}^{4} h_i h_j^d \xi^T(t)\left[\hat{\bar{\Xi}}_{ij} + d(t)\hat{M}R^{-1}\hat{M}^T + \left(\bar{d}-d(t)\right)\hat{N}R^{-1}\hat{N}^T\right]\xi(t) \\
=\; & \sum_{i=1}^{4}\sum_{j=1}^{4} h_i h_j^d \xi^T(t)\left[\frac{d(t)}{\bar{d}}\left(\hat{\bar{\Xi}}_{ij} + \bar{d}\hat{M}R^{-1}\hat{M}^T\right)\right. \\
& \left. + \frac{\bar{d}-d(t)}{\bar{d}}\left(\hat{\bar{\Xi}}_{ij} + \bar{d}\hat{N}R^{-1}\hat{N}^T\right)\right]\xi(t),
\end{aligned}
$$

where

$$
\hat{\bar{\Xi}}_{ij} = \hat{\Theta}_{11}^{ij} + \mathrm{sym}\left(\hat{\Pi}_2\right) + \Upsilon\bar{d}R\Upsilon^T,
$$

and

$$
\hat{\Pi}_2 = \begin{bmatrix} \hat{M} & \hat{N}-\hat{M} & -\hat{N} \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} A_i & B_i m_a K_{aj} & 0 \end{bmatrix}^T,
$$

where the matrix $\hat{\Theta}_{11}^{ij}$ is the matrix Θ_{11}^{ij}, where the term $PB_i M_{a0}K_{aj}$ is replaced by $PB_i m_a K_{aj}$. It is found that

$$
\begin{aligned}
\tilde{\bar{\Xi}}_{ij}^{1} \;=\; & \begin{bmatrix} \hat{\Theta}_{11}^{ij} + \mathrm{sym}\left(\hat{\Pi}_2\right) & \sqrt{\bar{d}}\hat{M} & \sqrt{\bar{d}}\Upsilon R \\ * & -R & 0 \\ * & * & -R \end{bmatrix} \\
\leq\; & \begin{bmatrix} \Theta_{11}^{ij} & \sqrt{\bar{d}}\hat{M} & \tilde{\Phi}_{14}^{ij} \\ * & -R & 0 \\ * & * & -R \end{bmatrix} + \varepsilon_{1ij}^{-1}\Lambda^T J_a\Lambda + \varepsilon_{1ij}\Delta J_a\Delta^T, \\
\tilde{\bar{\Xi}}_{ij}^{2} \;=\; & \begin{bmatrix} \hat{\Theta}_{11}^{ij} + \mathrm{sym}\left(\hat{\Pi}_2\right) & \sqrt{\bar{d}}\hat{N} & \sqrt{\bar{d}}\Upsilon R \\ * & -R & 0 \\ * & * & -R \end{bmatrix} \\
\leq\; & \begin{bmatrix} \Theta_{11}^{ij} & \sqrt{\bar{d}}\hat{N} & \tilde{\Phi}_{14}^{ij} \\ * & -R & 0 \\ * & * & -R \end{bmatrix} + \varepsilon_{2ij}^{-1}\Lambda^T J_a\Lambda + \varepsilon_{2ij}\Delta J_a\Delta^T,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\Phi}_{14}^{ij} &= \begin{bmatrix} \sqrt{\bar{d}}RA_i & \sqrt{\bar{d}}RB_iM_{a0}K_{aj} & 0 \end{bmatrix}^T, \\
\Lambda &= \begin{bmatrix} B_i^T P & 0 & 0 & \sqrt{\bar{d}}B_i^T R & 0 \end{bmatrix}, \\
\Delta^T &= \begin{bmatrix} 0 & M_{a0}K_{aj} & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

From (5.8)–(5.9) and according to Schur complement, $\tilde{\Xi}_{ij}^1 < 0$ and $\tilde{\Xi}_{ij}^2 < 0$ are obtained, it is to say that

$$
\hat{\Xi}_{ij} + \bar{d}\hat{M}R^{-1}\hat{M}^T < 0, \quad \hat{\Xi}_{ij} + \bar{d}\hat{N}R^{-1}\hat{N}^T < 0.
$$

It leads to $\dot{V}(t) < 0$, then the system in (5.6) is asymptotically stable for the delay $d(t)$ satisfying (5.2). Next, the H_∞ performance of the system in (5.6) is established under zero initial conditions. Firstly, the Lyapunov functional is defined as shown in (5.11). It is not difficult to achieve:

$$
\begin{aligned}
&\dot{V}(t) + z_1^T(t)z_1(t) - \gamma^2 w^T(t)w(t) \\
\leq\ & \sum_{i=1}^4 \sum_{j=1}^4 h_i h_j^d \bar{\xi}^T(t) \left[\check{\Xi}_{ij} + d(t)MR^{-1}M^T + (\bar{d} - d(t))NR^{-1}N^T \right] \bar{\xi}(t) \\
=\ & \sum_{i=1}^4 \sum_{j=1}^4 h_i h_j^d \bar{\xi}^T(t) \left[\frac{d(t)}{\bar{d}} \left(\check{\Xi}_{ij} + \bar{d}MR^{-1}M^T \right) \right. \\
& \left. + \frac{\bar{d} - d(t)}{\bar{d}} \left(\check{\Xi}_{ij} + \bar{d}NR^{-1}N^T \right) \right] \bar{\xi}(t),
\end{aligned}
$$

where

$$
\check{\Xi}_{ij} = \check{\Phi}_{11}^{ij} + \check{\Phi}_{13}^{ij}\check{\Phi}_{13}^{ijT} + \check{\Phi}_{14}^{ij}\check{\Phi}_{14}^{ijT}, \quad \bar{\xi}^T(t) = \begin{bmatrix} \xi^T(t) & w^T(t) \end{bmatrix},
$$

and $\check{\Phi}_{11}^{ij}$, $\check{\Phi}_{13}^{ij}$ and $\check{\Phi}_{14}^{ij}$ are the matrices Φ_{11}^{ij}, Φ_{13}^{ij} and Φ_{14}^{ij} in which the terms $PB_iM_{a0}K_{aj}$, $K_{aj}^T M_{a0}D_{1i}^T$ and $\sqrt{\bar{d}}K_{aj}^T M_{a0}B_i^T R$ are replaced by the terms $PB_i m_a K_{aj}$, $K_{aj}^T m_a D_{1i}^T$ and $K_{aj}^T m_a B_i^T$ respectively. According to Schur complement and the above method, we develop

$$
\dot{V}(t) + z_1^T(t)z_1(t) - \gamma^2 w^T(t)w(t) < 0, \tag{5.13}
$$

for all nonzero $w \in L_2[0, \infty)$. Under zero initial conditions, we have $V(0) = 0$ and $V(\infty) \geq 0$. Integrating both sides of (5.13) yields $\|z_1\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0, \infty)$, and the H_∞ performance is established.

In what follows, we will show that the hard constraints in (5.7) are guaranteed. Inequality (5.13) guarantees $\dot{V}(t) - \gamma^2 w^T(t)w(t) < 0$. Integrating both sides of the above inequality from zero to any $t > 0$, we obtain

$$V(t) - V(0) < \gamma^2 \int_0^t w^T(s)w(s)\mathrm{d}s < \gamma^2 \|w\|_2^2. \tag{5.14}$$

From the definition of the Lyapunov functional in (5.11), we obtain that $x^T(t)Px(t) < \rho$ with $\rho = \gamma^2 w_{\max} + V(0)$. Similar to (Gao *et al.*, 2010a), the following inequality hold

$$
\begin{aligned}
&\max_{t>0} |\{z_2(t)\}_q|^2 \\
\leq\ & \max_{t>0} \left\| \sum_{i=1}^4 h_i x^T(t)\{C_{2i}\}_q^T\{C_{2i}\}_q x(t) \right\|_2 \\
=\ & \max_{t>0} \left\| \sum_{i=1}^4 h_i x^T(t)P^{\frac{1}{2}}P^{-\frac{1}{2}}\{C_{2i}\}_q^T\{C_{2i}\}_q P^{-\frac{1}{2}}P^{\frac{1}{2}}x(t) \right\|_2 \\
<\ & \rho \cdot \theta_{\max}\left(\sum_{i=1}^4 h_i P^{-\frac{1}{2}}\{C_{2i}\}_q^T\{C_{2i}\}_q P^{-\frac{1}{2}} \right), \quad q = 1, 2,
\end{aligned}
$$

where $\theta_{\max}(\cdot)$ represents maximal eigenvalue. From the above inequality, it leads to that the constraints in (5.7) are guaranteed, if

$$\rho \cdot \sum_{i=1}^4 h_i P^{-\frac{1}{2}}\{C_{2i}\}_q^T\{C_{2i}\}_q P^{-\frac{1}{2}} < I, \tag{5.15}$$

which means

$$\sum_{i=1}^4 h_i \left(\rho \cdot P^{-\frac{1}{2}}\{C_{2i}\}_q^T\{C_{2i}\}_q P^{-\frac{1}{2}} - I \right) < 0,$$

which is guaranteed by the feasibility of (5.10). The proof is completed. ∎

Remark 5.1 In this chapter, the free-weight matrices method (He *et al.*, 2004) has been utilized to propose the delay-dependent H_∞ performance analysis condition for the time-varying actuator delay $d(t)$. How to develop the less conservative condition is still a challenging research topic. The interval time-varying delay and present less conservative results have been targeted in our future work.

In what follows, the reliable fuzzy H_∞ controller existence condition is presented for the active suspension system in (5.6), based on reliable fuzzy H_∞ performance analysis criterion in Theorem 5.1.

Theorem 5.2 Consider the closed-loop system in (5.6). For given scalars $\bar{d} > 0$ and μ, if there exist matrices $\bar{P} > 0$, $\bar{Q} > 0$, $\bar{S} > 0$, $\bar{R} > 0$, Y_{aj}, $\bar{N}_j$, and $\bar{M}_j$ with appropriate dimensions and positive scalars $\bar{\varepsilon}_{1ij} > 0$ and $\bar{\varepsilon}_{2ij} > 0$ ($i, j = 1, 2, 3, 4$) such that the following LMIs hold for $q = 1, 2$:

$$\begin{bmatrix} \bar{\Phi}_{11}^{ij} & \sqrt{\bar{d}}\bar{M} & \bar{\Phi}_{13}^{ij} & \bar{\Phi}_{14}^{ij} & \bar{\Phi}_{15}^{1ij} & \bar{\Phi}_{16}^{ij} \\ 0 & \bar{R} - 2\bar{P} & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & D_{1i} & 0 \\ 0 & 0 & 0 & -R & \bar{\varepsilon}_{1ij}\sqrt{\bar{d}}B_i & 0 \\ 0 & 0 & 0 & 0 & -\bar{\varepsilon}_{1ij}J_a^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\bar{\varepsilon}_{1ij}J_a^{-1} \end{bmatrix} < 0, \quad (5.16)$$

$$\begin{bmatrix} \bar{\Phi}_{11}^{ij} & \sqrt{\bar{d}}\bar{N} & \bar{\Phi}_{13}^{ij} & \bar{\Phi}_{14}^{ij} & \bar{\Phi}_{15}^{2ij} & \bar{\Phi}_{16}^{ij} \\ 0 & \bar{R} - 2\bar{P} & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & D_{1i} & 0 \\ 0 & 0 & 0 & -R & \bar{\varepsilon}_{2ij}\sqrt{\bar{d}}B_i & 0 \\ 0 & 0 & 0 & 0 & -\bar{\varepsilon}_{2ij}J_a^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\bar{\varepsilon}_{2ij}J_a^{-1} \end{bmatrix} < 0, \quad (5.17)$$

$$\begin{bmatrix} -\bar{P} & \sqrt{\rho}\bar{P}\{C_{2i}\}_q^T \\ * & -I \end{bmatrix} < 0, \quad (5.18)$$

where

$$\bar{\Phi}_{11}^{ij} = \bar{\Xi}_{11}^{ij} + \text{sym}\left(\bar{\Xi}_2\right), \quad \bar{\Xi}_{11}^{ij} = \begin{bmatrix} \bar{\Theta}_{11}^{ij} & \bar{\Theta}_{12}^{ij} \\ * & -\gamma^2 I \end{bmatrix},$$

$$\bar{\Theta}_{11}^{ij} = \begin{bmatrix} \text{sym}\left(A_i\bar{P}\right) + \bar{Q} + \bar{S} & B_iY_{aj} & 0 \\ * & -(1-\mu)\bar{S} & 0 \\ * & * & -\bar{Q} \end{bmatrix},$$

$$\bar{\Theta}_{12}^{ij} = \begin{bmatrix} B_{1i} \\ 0 \\ 0 \end{bmatrix}, \bar{\Xi}_2 = \begin{bmatrix} \bar{M} & \bar{N} - \bar{M} & -\bar{N} & 0 \end{bmatrix},$$

$$\bar{\Phi}_{13}^{ij} = \begin{bmatrix} C_{1i}\bar{P} & D_{1i}Y_{aj} & 0 & 0 \end{bmatrix}^T, \quad \bar{\Phi}_{16}^{ij} = \begin{bmatrix} 0 & Y_{aj} & 0 & 0 \end{bmatrix}^T,$$

$$\bar{\Phi}_{14}^{ij} = \begin{bmatrix} \sqrt{\bar{d}}A_i & \sqrt{\bar{d}}B_iY_{aj} & 0 & \sqrt{\bar{d}}B_{1i} \end{bmatrix}^T,$$

$$\bar{\Phi}_{15}^{1ij} = \begin{bmatrix} \bar{\varepsilon}_{1ij}B_i^T & 0 & 0 & 0 \end{bmatrix}^T, \quad \bar{\Phi}_{15}^{2ij} = \begin{bmatrix} \bar{\varepsilon}_{2ij}B_i^T & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\bar{M} = \begin{bmatrix} \bar{M}_1^T & \bar{M}_2^T & \bar{M}_3^T & \bar{M}_4^T \end{bmatrix}^T, \quad \bar{N} = \begin{bmatrix} \bar{N}_1^T & \bar{N}_2^T & \bar{N}_3^T & \bar{N}_4^T \end{bmatrix}^T.$$

Then a reliable controller in the form of (5.5) exists, such that

(1) the closed-loop system is asymptotically stable;

(2) the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to output constraints (5.7) with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma^2$.

Moreover, if inequalities (5.16)–(5.18) have a feasible solution, then the control gain K_{aj} in (5.5) is given by $K_{aj} = M_{a0}^{-1} Y_{aj} \bar{P}^{-1}$.

Proof. From $\left(\bar{R} - \bar{P}\right) \bar{R}^{-1} \left(\bar{R} - \bar{P}\right) \geq 0$, we have $-\bar{P}\bar{R}^{-1}\bar{P} \leq \bar{R} - 2\bar{P}$. After replacing $\bar{R} - 2\bar{P}$ in (5.16)–(5.17) with $-\bar{P}\bar{R}^{-1}\bar{P}$ and performing corresponding congruence transformation by

$$\operatorname{diag}\left\{\bar{P}^{-1}, \bar{P}^{-1}, \bar{P}^{-1}, I, \bar{P}^{-1}, I, \bar{R}^{-1}, \bar{\varepsilon}_{1ij}^{-1}I, \bar{\varepsilon}_{1ij}^{-1}I\right\},$$

and by

$$\operatorname{diag}\left\{\bar{P}^{-1}, \bar{P}^{-1}, \bar{P}^{-1}, I, \bar{P}^{-1}, I, \bar{R}^{-1}, \bar{\varepsilon}_{2ij}^{-1}I, \bar{\varepsilon}_{2ij}^{-1}I\right\},$$

together with the change of matrix variables defined by

$$
\begin{aligned}
P &= \bar{P}^{-1}, \quad R = \bar{R}^{-1}, \quad Q = \bar{P}^{-1}\bar{Q}\bar{P}^{-1}, \\
K_j &= M_{a0}^{-1} Y_j \bar{P}^{-1}, \qquad S = \bar{P}^{-1}\bar{S}\bar{P}^{-1}, \quad \varepsilon_{1ij} = \bar{\varepsilon}_{1ij}^{-1}, \\
\varepsilon_2 &= \bar{\varepsilon}_{2ij}^{-1}, \quad M = \operatorname{diag}\left\{\bar{P}^{-1}, \bar{P}^{-1}, \bar{P}^{-1}, I\right\} \bar{M}\bar{P}^{-1}, \\
N &= \operatorname{diag}\left\{\bar{P}^{-1}, \bar{P}^{-1}, \bar{P}^{-1}, I\right\} \bar{N}\bar{P}^{-1}.
\end{aligned}
$$

It is concluded that the conditions in (5.8) and (5.9) hold. On the other hand, (5.18) is equivalent to (5.10) by performing a simple congruence transformation with $\operatorname{diag}\left\{\bar{P}^{-1}, I\right\}$. Therefore, all the conditions in Theorem 1 are satisfied. The proof is completed. ∎

Remark 5.2 In the study, the conservativeness will be reduced if the matrices Q, S, R, M and N are replaced by $\sum_{i=1}^4 h_i Q_i$, $\sum_{i=1}^4 h_i S_i$, $\sum_{i=1}^4 h_i R_i$, $\sum_{i=1}^4 h_i M_i = \sum_{i=1}^4 h_i \begin{bmatrix} M_{1i}^T & M_{2i}^T & M_{3i}^T & M_{4i}^T \end{bmatrix}^T$ and $\sum_{i=1}^4 h_i N_i = \sum_{i=1}^4 h_i \begin{bmatrix} N_{1i}^T & N_{2i}^T & N_{3i}^T & N_{4i}^T \end{bmatrix}^T$. However, computation complexion of the existence condition in Theorem 2 of reliable fuzzy H_∞ controller design will be increased intensively. Thus, the above proof is employed to handle the tradeoff in this study.

5.4   Fuzzy Controller Design

In the section, fuzzy H_∞ controller design is presented for active suspension systems with actuator delay based on T-S fuzzy model method. If there is no actuator fault in the active suspension system, then we obtain,

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B(t)u(t - d(t)), \\
z_1(t) &= C_1(t)x(t) + D_1(t)u(t - d(t)), \\
z_2(t) &= C_2(t)x(t),
\end{aligned}
\tag{5.19}
$$

Based on the above presented fuzzy modeling, the overall fuzzy model is inferred as follows:

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{4} h_i(\xi(t)) \left[A_i x(t) + B_i u(t - d(t)) + B_{1i} w(t) \right], \\
z_1(t) &= \sum_{i=1}^{4} h_i(\xi(t)) \left[C_{1i} x(t) + D_i u(t - d(t)) \right], \\
z_2(t) &= \sum_{i=1}^{4} h_i(\xi(t)) C_{2i} x(t).
\end{aligned}
\tag{5.20}
$$

In addition, the overall fuzzy control law is represented by

$$
u(t) = \sum_{j=1}^{4} h_j(\xi(t)) K_{sj} x(t)
\tag{5.21}
$$

For the case of the standard controller (5.21), the closed-loop system is given by

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{4} \sum_{j=1}^{4} h_i h_j^d \left[A_i x(t) + B_i K_{sj} x(t - d(t)) + B_{1i} w(t) \right], \\
z_1(t) &= \sum_{i=1}^{4} \sum_{j=1}^{4} h_i h_j^d \left[C_{1i} x(t) + D_{1i} K_{sj} x(t - d(t)) \right], \\
z_2(t) &= \sum_{i=1}^{4} h_i C_{2i} x(t).
\end{aligned}
\tag{5.22}
$$

By employing the similar method proposed in the previous section, the following corollary is obtained for the fuzzy H_∞ performance analysis at the context of the system in (5.22) with actuator delay.

Corollary 5.1 Consider the closed-loop system in (5.22). Given scalars $\bar{d} > 0$, μ and matrices K_{sj}, if there exist matrices $P > 0$, $Q > 0$, $S > 0$, $R > 0$, N_j, and M_j ($j = 1, 2, 3, 4$) with appropriate dimension such that the following LMIs hold for $q = 1, 2$:

$$\begin{bmatrix} \acute{\Phi}_{11}^{ij} & \sqrt{\bar{d}}M & \acute{\Phi}_{13}^{ij} & \acute{\Phi}_{14}^{ij} \\ 0 & -R & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -R \end{bmatrix} < 0, \tag{5.23}$$

$$\begin{bmatrix} \acute{\Phi}_{11}^{ij} & \sqrt{\bar{d}}N & \acute{\Phi}_{13}^{ij} & \acute{\Phi}_{14}^{ij} \\ 0 & -R & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -R \end{bmatrix} < 0, \tag{5.24}$$

$$\begin{bmatrix} -P & \sqrt{\rho}\left\{C_{2i}\right\}_q^T \\ * & -I \end{bmatrix} < 0, \tag{5.25}$$

where

$$\acute{\Phi}_{11}^{ij} = \acute{\Xi}_{11}^{ij} + \text{sym}\left(\Xi_2\right), \acute{\Xi}_{11}^{ij} = \begin{bmatrix} \acute{\Theta}_{11}^{ij} & \Theta_{12}^{ij} \\ * & -\gamma^2 I \end{bmatrix},$$

$$\acute{\Phi}_{13}^{ij} = \begin{bmatrix} C_{1i} & D_{1i}K_{sj} & 0 & 0 \end{bmatrix}^T,$$

$$\acute{\Theta}_{11}^{ij} = \begin{bmatrix} \text{sym}\left(PA_i\right) + Q + S & PB_iK_{sj} & 0 \\ * & -\left(1-\mu\right)S & 0 \\ * & * & -Q \end{bmatrix},$$

$$\acute{\Phi}_{14}^{ij} = \begin{bmatrix} \sqrt{\bar{d}}RA_i & \sqrt{\bar{d}}RB_iK_{sj} & 0 & \sqrt{\bar{d}}RB_{1i} \end{bmatrix}^T,$$

Take into account the matrices Ξ_2, Θ_{12}^{ij}, M and N in Theorem 5.1, we obtain,

(1) the closed-loop system is asymptotically stable;

(2) the performance $\|T_{z_1w}\|_\infty < \gamma$ is minimized subject to output constraints (5.7).

Similarly, the fuzzy H_∞ controller design condition as below is derived from Theorem 5.2.

Corollary 5.2 Consider the closed-loop system in (5.22). Given scalars $\bar{d} > 0$ and μ, the closed-loop system (5.6) is asymptotically stable with an H_∞ disturbance attenuation level γ, if there exist matrices $\bar{P} > 0$, $\bar{Q} > 0$, $\bar{S} > 0$, $\bar{R} > 0$, Y_{sj}, $\bar{N}_j$, and $\bar{M}_j$ ($j = 1, 2, 3, 4$) with appropriate dimensions such that the following

LMIs hold for $q = 1, 2$:

$$\begin{bmatrix} \grave{\Phi}_{11}^{ij} & \bar{\Phi}_{12}^{1ij} & \grave{\Phi}_{13}^{ij} & \grave{\Phi}_{14}^{ij} \\ 0 & \bar{R} - 2\bar{P} & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -R \end{bmatrix} < 0, \quad (5.26)$$

$$\begin{bmatrix} \grave{\Phi}_{11}^{ij} & \bar{\Phi}_{12}^{2ij} & \grave{\Phi}_{13}^{ij} & \grave{\Phi}_{14}^{ij} \\ 0 & \bar{R} - 2\bar{P} & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -R \end{bmatrix} < 0, \quad (5.27)$$

$$\begin{bmatrix} -\bar{P} & \sqrt{\rho}\bar{P}\{C_{2i}\}_q^T \\ * & -I \end{bmatrix} < 0, \quad (5.28)$$

where

$$\begin{aligned} \grave{\Phi}_{11}^{ij} &= \grave{\Xi}_{11}^{ij} + \text{sym}\left(\bar{\Xi}_2\right), \quad \grave{\Xi}_{11}^{ij} = \begin{bmatrix} \grave{\Xi}_{11}^{ij} & \bar{\Theta}_{12}^{ij} \\ * & -\gamma^2 I \end{bmatrix}, \\ \grave{\Phi}_{13}^{ij} &= \begin{bmatrix} C_{1i}\bar{P} & D_{1i}Y_{sj} & 0 & 0 \end{bmatrix}^T, \\ \Theta_{11}^{ij} &= \begin{bmatrix} \text{sym}\left(A_i\bar{P}\right) + \bar{Q} + \bar{S} & B_i Y_{sj} & 0 \\ * & -\left(1 - \mu\right)\bar{S} & 0 \\ * & * & -\bar{Q} \end{bmatrix}, \\ \grave{\Phi}_{14}^{ij} &= \begin{bmatrix} \sqrt{\bar{d}}A_i & \sqrt{\bar{d}}B_i Y_{sj} & 0 & \sqrt{\bar{d}}B_{1i} \end{bmatrix}^T, \end{aligned}$$

$\bar{\Xi}_2$, $\bar{\Theta}_{12}^{ij}$, $\bar{M}$ and $\bar{N}$ are defined in Theorem 5.2. Then a standard controller in the form of (5.21) exists, such that

(1) the closed-loop system is asymptotically stable;

(2) the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized subject to output constraint (5.7).

Moreover, if inequalities (5.26)–(5.28) have a feasible solution, then the control gain K_{sj} in (5.21) is given by $K_{sj} = Y_{sj}\bar{P}^{-1}$.

Remark 5.3 When the derivative of $d(t)$ is unknown, and the delay $d(t)$ satisfies $0 < d(t) \leq \bar{d}$, by setting $S = 0$ in (5.12) and the LMIs-based conditions in Theorems 5.1-5.2 and Corollary 5.1-5.2, the reliable fuzzy H_∞ controller and fuzzy H_∞ controller can be obtained for the systems in (5.6) and (5.22) under the condition that the actuator delay $d(t)$ satisfies $0 < d(t) \leq \bar{d}$ respectively.

5.5 Case Study

The sprung mass $m_s(t)$ is assumed to set as the range [873kg, 1073kg] and the unsprung mass $m_u(t)$ to [104kg, 124kg]. In this study, the maximum allowable suspension stroke is set as $z_{\max} = 0.1$ m with $\rho = 1$. For the actuator delay $d(t) = 5 + 5\sin(\frac{1}{50})$ ms satisfying $\bar{d} = 10$ ms and $\mu = 0.1$, we consider fuzzy H_∞ controller design for the uncertain active suspension systems in (5.22). By using the convex optimization method, it is found that the minimum guaranteed closed-loop H_∞ performance index γ_{min} is 5.3011 and the fuzzy controller gain matrices

$$K_{si} = 10^4 \times \left[\begin{array}{cccc} -3.3260 & 5.6998 & -2.5167 & 0.2824 \end{array} \right], \tag{5.29}$$

where $i = 1, 2, 3, 4$.

It is expected that the desired fuzzy H_∞ controller in (5.21) with the parameters in (5.29) can be designed such that: 1) the sprung mass acceleration $z_1(t)$ is as small as possible; 2) the suspension deflection is below the maximum allowable suspension stroke $z_{\max} = 0.1$ m, which means that $x_1(t)/z_{\max}$ below 1; 3) the relation dynamic tire load $k_t x_2(t)/(m_s(t) + m_u(t))g < 1$. We first consider the following road disturbance (Du & Zhang, 2009) as

$$z_r(t) = 0.0254 \sin 2\pi t + 0.005 \sin 10.5\pi t + 0.001 \sin 21.5\pi t (\text{m}). \tag{5.30}$$

According to (Du & Zhang, 2009), the road disturbance has a similar frequency as the car body resonance frequency (1Hz) under the condition that high-frequency disturbance is added to simulate the rough road surface. In order to carry out the simulation for the fuzzy H_∞ controller as in (5.22), the variational sprung mass $m_s(t)$ and the variational unsprung mass $m_u(t)$ are set as: $m_s(t) = 973 + 100 \sin(t)$ kg and $m_u(t) = 114 + 10 \cos(t)$ kg, for deriving the fuzzy membership functional $h_i(\xi(t))$ ($i = 1, 2, 3, 4$). By using the fuzzy H_∞ controller in (5.21) with the parameters in (5.29), we derive the corresponding closed-loop fuzzy system. Fig. 5.2 depicts the responses of body vertical accelerations and the actuator force for the open- (e.g., passive) and closed-loop (e.g., active) systems. Fig. 5.3 demonstrates the responses of suspension stroke and tire deflection constraint for both the passive and active systems. It is observed from Fig. 5.2

that the proposed fuzzy H_∞ control strategy reduces the sprung mass acceleration significantly in comparison with the passive suspension under the same road disturbance. The designed fuzzy H_∞ controller achieve less value of the maximum body acceleration for the active suspension system than the passive system, and passenger acceleration in the active suspension system is reduced significantly, which guarantees better ride comfort. In addition, it can be seen from Fig. 5.3 that , the suspension deflection constraint $x_1(t)/z_{\max} < 1$ and the relation dynamic tire load constraint $k_t x_2(t)/(m_s(t) + m_u(t))g < 1$ are guaranteed, which implies that the road holding capability is ensured by the desired fuzzy controller. These two figures confirm that the designed standard state-feedback fuzzy H_∞ controller can achieve better ride comfort and road handling, and guarantee constraint suspension deflection for the active suspension system.

(a)   (b)
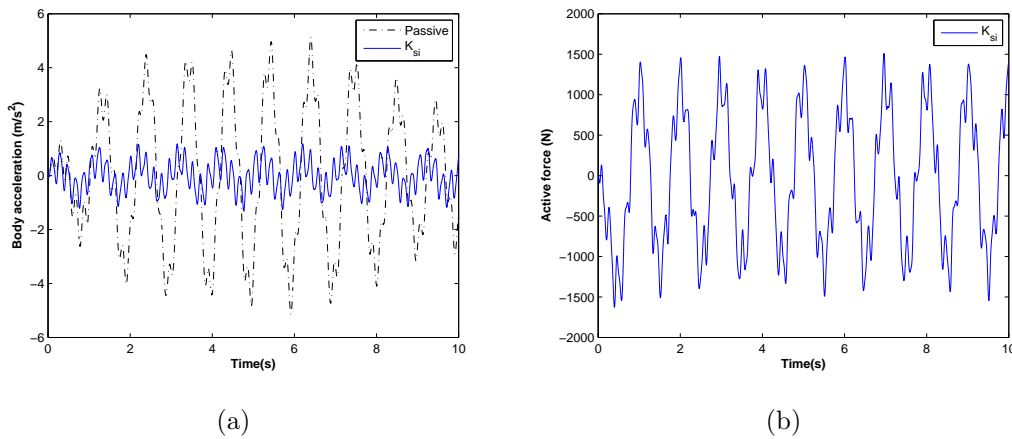
Figure 5.2: (a) Responses of body vertical accelerations, (b) Response of active force.

To further evaluate the effectiveness of the proposed fuzzy H_∞ controller design strategy with actuator delays, the road disturbance as below is taken into account. In the context of active suspension performance, the road disturbance can be generally assumed as discrete events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pothole on an smooth road surface. As (Du *et al.*, 2008), the road surface is represented by,

$$z_r(t) = \begin{cases} \frac{A}{2}(1 - \cos(\frac{2\pi V}{L}t)), & \text{if } 0 \le t \le \frac{L}{V}, \\ 0, & \text{if } t > \frac{L}{V}, \end{cases} \tag{5.31}$$

(a)   (b)
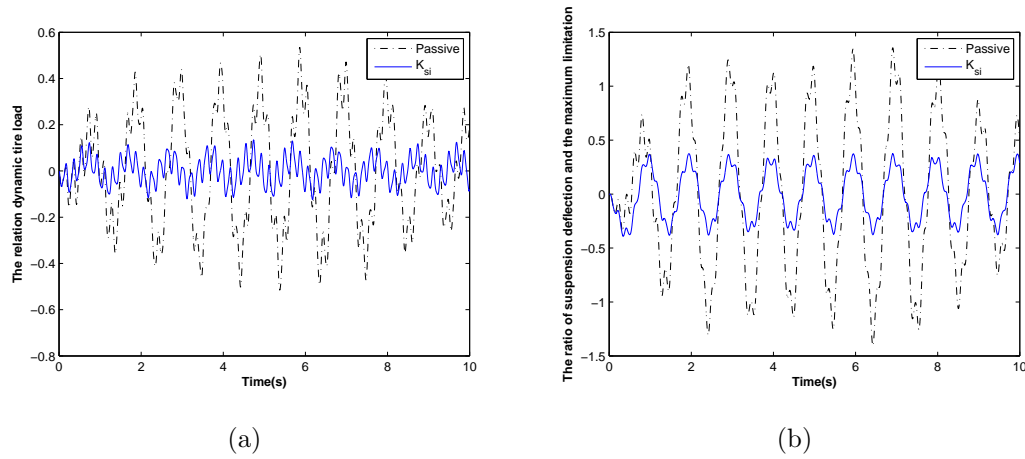
Figure 5.3: (a) Responses of suspension deflection constraint, (b) Responses of tire stroke constraint.

where A and L are the height and the length of the bump. Assume $A = 50$ mm, $L = 6$ m and the vehicle forward velocity as $V = 35$ (km/h). Fig. 5.4 illustrates the responses to body vertical accelerations and the actuator force; Fig. 5.5 presents the responses to suspension stroke and tire deflection constraint for the passive and active systems under the introduced road disturbance, respectively. The simulation results convincingly demonstrate that the fuzzy H_∞ controller offers better suspension performance than the open-loop suspension system.

(a)   (b)

Figure 5.4: (a) Responses of body vertical accelerations, (b) Response of active force.
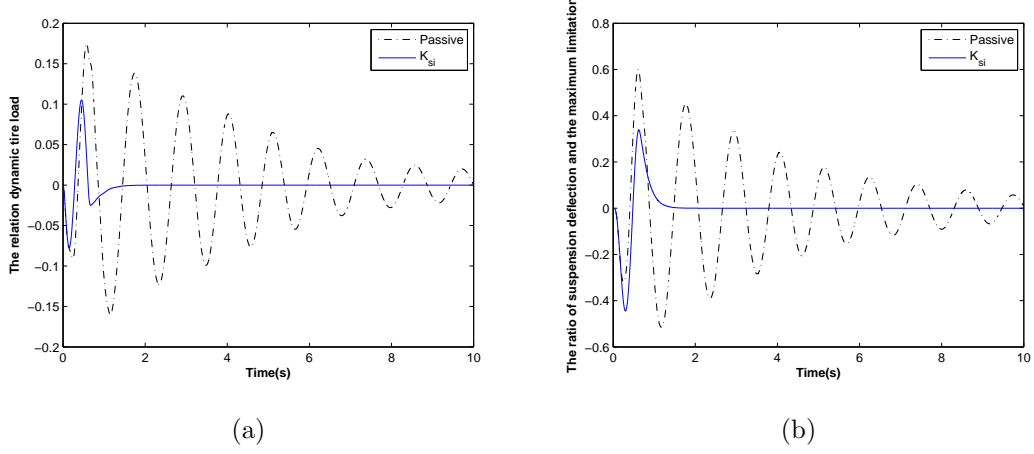
Figure 5.5: (a) Responses of suspension deflection constraint, (b) Responses of tire stroke constraint.

The effectiveness and advantages of the proposed reliable fuzzy H_∞ controller design for active suspension systems with actuator delay and fault will be demonstrated in what follows. The parameters notation in the fuzzy H_∞ controller design in the above section is applied here as well. It is assumed that there exists the following actuator fault, namely, $\check{m}_a = 0.1$, $\hat{m}_a = 0.5$, which implies $M_{a0} = 0.3$ and $J_a = 0.2$. Based on the convex optimization method, we can obtain the minimum guaranteed closed-loop H_∞ performance index γ_{min} is 28.6991 and the reliable fuzzy controller gain matrices

$$
\begin{aligned}
K_{a1} &= 10^4 \times \begin{bmatrix} 4.1910 & -0.9700 & -2.5381 & 0.5713 \end{bmatrix}, \\
K_{a2} &= 10^4 \times \begin{bmatrix} 4.1916 & -0.9829 & -2.5381 & 0.5711 \end{bmatrix}, \\
K_{a3} &= 10^4 \times \begin{bmatrix} 4.1964 & -0.9751 & -2.5382 & 0.5706 \end{bmatrix}, \\
K_{a4} &= 10^4 \times \begin{bmatrix} 4.2149 & -0.9439 & -2.5388 & 0.5701 \end{bmatrix}.
\end{aligned}
\tag{5.32}
$$

For two different cases of road disturbances, namely, the first case road disturbance is shown in (5.30) and the second case road disturbance is given in (5.31). In Figs. 5.6–5.9, the responses to the open and closed-loop systems with the actuator delay and fault via the standard fuzzy H_∞ controller K_{si} and reliable controller K_{ai} ($i = 1, 2, 3, 4$) are based on the two different types of road disturbances. These figures show that the less value of the maximum body acceleration is achieved for

the active suspension system, the suspension deflection constraint $x_1(t)/z_{\max} < 1$ is guaranteed and the relation dynamic tire load $k_t x_2(t)/(m_s(t) + m_u(t))g$ is below 1 in comparison with the passive suspension system, by utilizing the standard fuzzy H_∞ controller K_{si} and reliable controller K_{ai} ($i = 1, 2, 3, 4$) for three different types of road disturbances respectively. However, it can be observed from Figs. 5.6 and 5.8 that the reliable fuzzy H_∞ controller achieves less value of the maximum body acceleration than the standard H_∞ controller for the active suspension system with actuator delay and fault. From Fig. 5.7 and 5.9, it can be seen that K_{ai} ($i = 1, 2, 3, 4$) is capable to provide a much more steady control force in fault condition than conventional controller K_{si} ($i = 1, 2, 3, 4$).

(a)

(b)
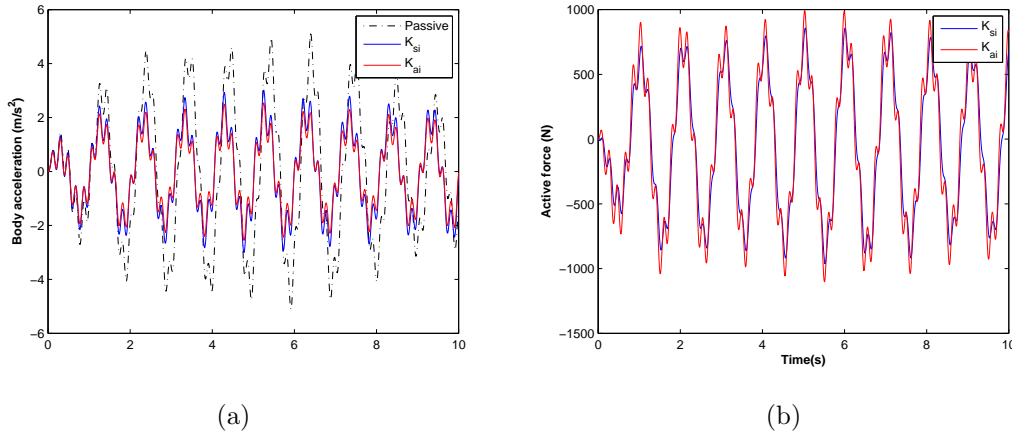
Figure 5.6: (a) Responses of body vertical accelerations, (b) Response of active force.

To further evaluate the suspension system performance under different fuzzy controllers K_{si} and K_{ai} ($i = 1, 2, 3, 4$), the root mean square (RMS) values of the body acceleration are exploited to demonstrate its advantages. The road disturbances can also be generally assumed as random vibrations, which are consistent and typically specified as random process with a given ground displacement power spectral density (PSD) of

$$G_q(n) = G_q(n_0) \left(\frac{n}{n_0} \right)^{-c}, \tag{5.33}$$

where n_0 denotes the spatial frequency and n_0 is the reference spatial frequency of $n_0 = 0.1$ (1/m); $G_q(n_0)$ is used to stand for the road roughness coefficient;
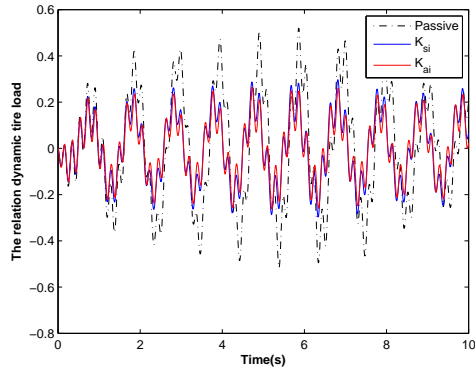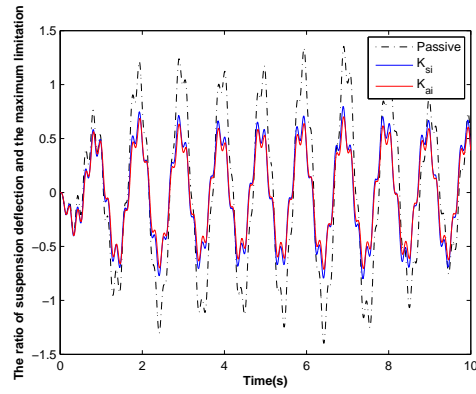
Figure 5.7: (a) Responses of suspension deflection constraint, (b) Responses of tire stroke constraint.
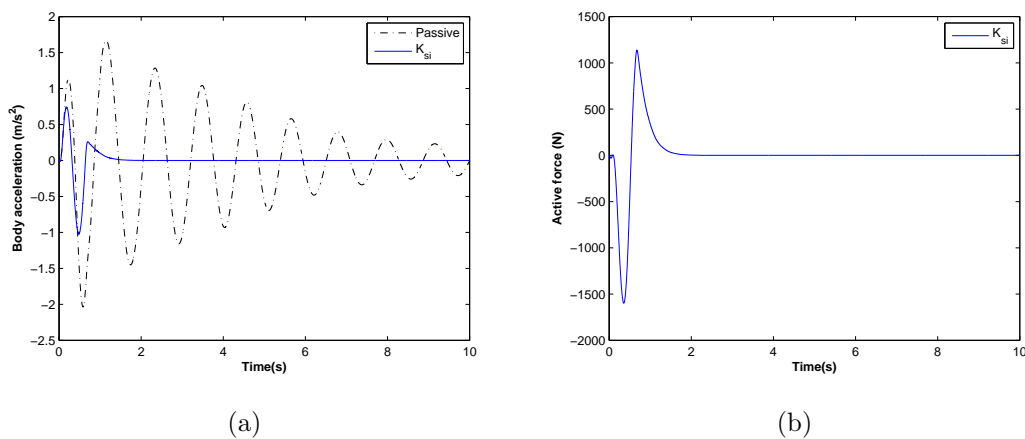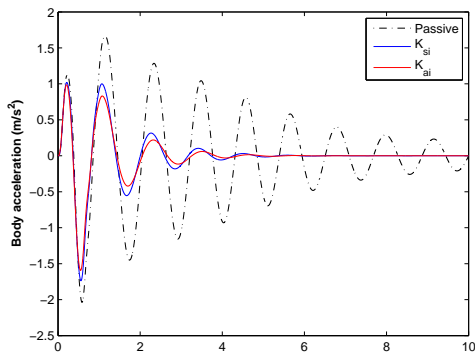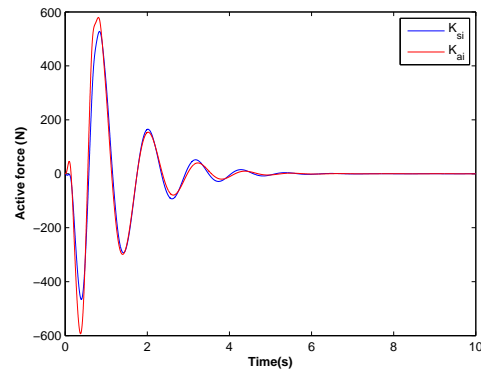

Figure 5.8: (a) Responses of body vertical accelerations, (b) Response of active force.
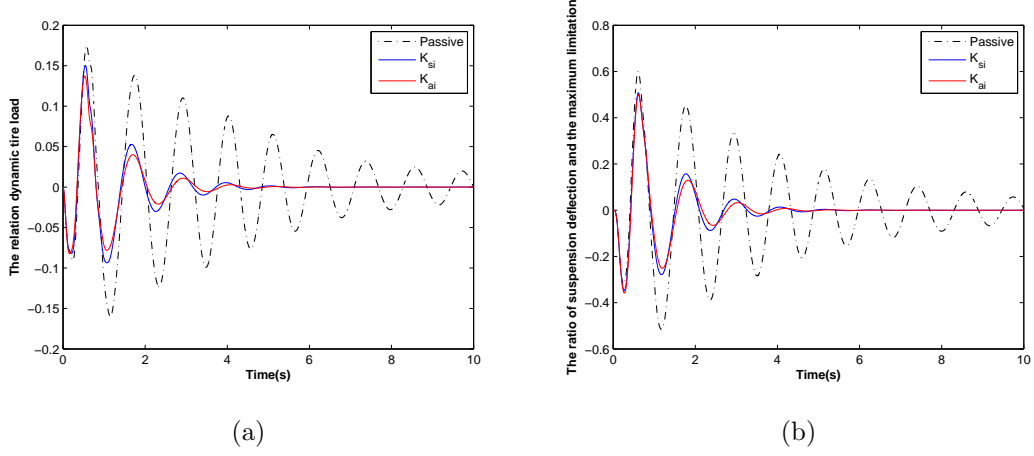
Figure 5.9: (a) Responses of suspension deflection constraint, (b) Responses of tire stroke constraint.

$c = 2$ is the road roughness constant. Related to the time frequency f, we have $f = nV$ with V for the vehicle forward velocity. Based on the equation (5.33), we can obtain the PSD ground displacement:

$$G_q\left(f\right) = G_q\left(n_0\right) n_0^{-2} \frac{V}{f^2}. \tag{5.34}$$

Accordingly, PSD ground velocity is given by

$$G_{\dot{q}}\left(f\right) = \left(2\pi f\right)^2 G_q\left(f\right) = 4\pi G_q\left(n_0\right) n_0^2 V, \tag{5.35}$$

which is only related to the vehicle forward velocity. When the vehicle forward velocity is fixed, the ground velocity can be viewed as a white-noise signal. We choose four difference road roughness $G_q\left(n_0\right) = 16 \times 10^{-6}$ m^3, 64×10^{-6} m^3, 256×10^{-6} m^3 and 1024×10^{-6} m^3, which are corresponded to B Grade (Good), C Grade (Average), D Grade (Poor) and E Grade (Very Poor) for the vehicle forward velocity $V = 35$ (km/h), respectively.

RMS are strictly related to the ride comfort, which are often used to quantify the amount of acceleration transmitted to the vehicle body. The RMS value of variable $x(t)$ is calculated as $\text{RMS}_x = \sqrt{(1/T) \int_0^T x^T(t)x(t)\mathrm{d}t}$. In our study, we choose $T = 100$ s to calculate the RMS values of the body acceleration, suspension stroke and relative dynamics tire load for different road roughness

coefficient $G_q(n_0)$, which are listed in Tables 5.2–5.4 by using the fuzzy controller K_{si} and reliable fuzzy controller K_{ai}, respectively. It can be observed that these tables indicate that the improvement in ride comfort and the satisfaction of hard constraints can be achieved for the different load conditions by using reliable fuzzy controller K_{ai} compared with the fuzzy controller K_{si} for the uncertain suspension systems with actuator delay and fault.

Table 5.2: RMS body acceleration

Grade	Passive systems	Fuzzy Controller	Reliable Fuzzy Controller
B	0.0081	0.0046	0.0041
C	0.0152	0.0092	0.0083
D	0.0284	0.0183	0.0166
E	0.0644	0.0387	0.0351

Table 5.3: RMS suspension stroke

Grade	Passive systems	Fuzzy Controller	Reliable Fuzzy Controller
B	1.7635×10^{-4}	9.7651×10^{-5}	9.5584×10^{-5}
C	3.3536×10^{-4}	1.9626×10^{-4}	1.9057×10^{-4}
D	6.2909×10^{-4}	3.9088×10^{-4}	3.8283×10^{-4}
E	0.0014	8.2616×10^{-4}	8.0992×10^{-4}

Table 5.4: RMS relative dynamics tire load

Grade	Passive systems	Fuzzy Controller	Reliable Fuzzy Controller
B	8.3596×10^{-4}	5.2554×10^{-4}	4.9612×10^{-4}
C	0.0016	0.0010	9.9561×10^{-4}
D	0.0030	0.0021	0.0020
E	0.0067	0.0044	0.0042

5.6   Summary

This chapter has investigated the problem of reliable fuzzy H_∞ control for active suspension systems with actuator delay and fault. The sprung and unsprung mass

variations, the actuator delay and fault, and the suspension performance have all been taken into account to construct the T-S fuzzy system for the control design objective. Based on the PDC scheme and stability theory, the reliable fuzzy H_∞ performance analysis condition has been derived for the proposed T-S fuzzy system presenting the active suspension system with uncertainty. Then, the reliable fuzzy H_∞ controller has been designed such that the resulting closed-loop T-S fuzzy system is asymptotically stable with H_∞ performance, and simultaneously satisfies the constraint suspension performance. A quarter-vehicle suspension model has been used to validate the effectiveness of the proposed design method. Simulation results have clearly demonstrated that the designed reliable fuzzy controller has the capability of guaranteeing a better suspension performance under sprung and unsprung mass variations, actuator delay and faults.

In this chapter, the standard actuator fault model has been used for this research. It is very difficult to synthesis the controller for the Markovian jumping fuzzy control systems. Therefore, this chapter does not exploit the more general actuator fault model proposed in Chapter 4 to the fuzzy control design problem. In future work, new methods will be developed to solve the controller for the Markovian jumping fuzzy control systems.

Chapter 6

Adaptive Sliding Mode Control for Nonlinear Vehicle Active Suspension Systems

6.1 Introduction

For active suspension systems, the vehicle sprung and unsprung masses vary with the loading conditions, such as the payload and number of passengers. If we do not take into account the variations of the vehicle sprung and unsprung masses in the control design process, the performance of the vehicle suspension systems will be affected. More recently, the authors in (Du *et al.*, 2008; Gao *et al.*, 2006, 2010a) used the polytopic parameter uncertainties to model the varying vehicle sprung or unsprung masses. Among (Du *et al.*, 2008; Gao *et al.*, 2006), the parameter-dependent controllers have been proposed for the quarter-car suspension systems with sprung mass variation. The parameter-independent sampled-data H_∞ control strategy has been provided in (Gao *et al.*, 2010a) for quarter-car suspension systems with both sprung and unsprung mass variations. However, the active suspension system models in (Du *et al.*, 2008; Gao *et al.*, 2006, 2010a) are linear and the nonlinear term caused by the actuator dynamic which has not been considered in these references (Chen & Guo, 2005; Du & Zhang, 2007; Du *et al.*, 2008; Gao *et al.*, 2006, 2010a). Thus, it is an urgent

task to design the active controller for the nonlinear uncertain active suspension systems.

For the uncertain active suspension systems, we can apply the Takagi-Sugeno (T-S) fuzzy approach to handle the uncertainty since T-S fuzzy model is very effective in representing complex nonlinear systems (Sugeno, 1985; Tanaka & Wang, 2001). Since fuzzy sets were proposed by Zadeh (Zadeh, 1965), fuzzy logic control has developed into a conspicuous and successful branch of automation and control theory. The uncertain or nonlinear systems can be described as a weighted sum of some simple linear subsystems by using the T-S fuzzy approach, and thus are easily to be analyzed. Recently, many results on stability analysis and controller synthesis problems for T-S fuzzy systems via various techniques have been obtained during the past decades (Chen *et al.*, 2008; Dong *et al.*, 2009; Dong & Yang, 2008; Feng, 2006; Lam & Narimani, 2009; Lin *et al.*, 2007; Nguang & Shi, 2003; Wu & Li, 2007; Zhang & Xu, 2009; Zhou *et al.*, 2005). In particular, for the active controller design problems, the authors in (Du & Zhang, 2009) presented T-S model-based fuzzy control design approach for electrohydraulic active suspension systems with input constraint.

On the other hand, it is well-known that the sliding mode control method is an effective robust control approach for the nonlinear systems. Moreover, the sliding mode control has received relatively much attention since it has various attractive features such as fast response, good transient performance, order-reduction and so on (Edwards & Spurgeon, 1998; Feng *et al.*, 2009; Ho & Niu, 2007; Niu *et al.*, 2005, 2007; Utkin, 1993; Wang *et al.*, 2009a; Yu & Kaynak, 2009). Recently, the sliding mode controller design problems have been extensively investigated for nonlinear suspension systems in (Al-Holou *et al.*, 2002; Chen & Huang, 2008; Kim & Ro, 1998; Sam *et al.*, 2004; Yagiz & Yuksek, 2001; Yoshimura *et al.*, 2001). In addition, the authors in (Huang & Chen, 2006; Huang & Lin, 2003b; Yagiz *et al.*, 2008) considered the fuzzy sliding mode control design problems for the suspension systems. When carrying out the sliding mode controller design for the suspension systems, however, it can be found that the suspension performance including ride comfort, road holding and suspension deflection, have not been fully taken into account, which may affect the suspension performance. Furthermore,

the uncertainty for the sprung and unsprung masses have not been considered in the above sliding mode controller design process.

This chapter deals with the adaptive sliding mode control problem for the nonlinear active suspension systems by means of T-S fuzzy approach. The varying sprung and unsprung masses, the unknown actuator nonlinearity and the suspension performances are taken into account simultaneously, and the corresponding mathematical model is established. By using sector nonlinearity approach, the T-S fuzzy model of the suspension system is developed to achieve the objective of the sliding mode controller design. An adaptive sliding mode controller is designed to guarantee the reachability of the specified switching surface. Simulation results are provided to demonstrate the effectiveness of the proposed method.

The reminder of this chapter is organized as follows. Section 6.2 formulates the problem to be addressed and Section 6.3 presents the adaptive fuzzy sliding mode controller design results. We provide the simulation results in Section 6.4 and conclude the chapter in Section 6.5.

6.2 Problem Formulation

It can be found that the suspension system in (2.16) is a model containing the sprung mass m_s and the front and rear wheels unsprung masses m_{uf} and m_{ur} vary in given ranges. In addition, when building the modeling of the suspension systems, the actuator uncertainty should be taken into account, which can be expressed as:

$$
\begin{aligned}
\dot{x}(t) &= A(t)\,x(t) + B_1(t)\,w(t) + B(t)\left[u(t) + g(x(t))\right], \\
z_1(t) &= C_1(t)\,x(t) + \left[D_1(t) + g(x(t))\right]u(t), \\
z_2(t) &= C_2(t)\,x(t).
\end{aligned}
\tag{6.1}
$$

The unknown nonlinear function $g(x(t))$ represents the parameter uncertainty for the control input and satisfies the following form:

$$
\|g(x(t))\| \le \delta \|x(t)\|
\tag{6.2}
$$

with $\delta > 0$ a known constant.

The sprung mass m_s, the front and rear wheels unsprung masses m_{uf} and m_{ur} are uncertainties, which vary in a given range, i.e. $m_s \in [m_{s\min}, m_{s\max}]$, $m_{uf} \in [m_{uf\min}, m_{uf\max}]$ and $m_{ur} \in [m_{ur\min}, m_{ur\max}]$. This means the uncertain mass m_s is bounded by its minimum value $m_{s\min}$ and its maximum value $m_{s\max}$. In addition, the uncertain mass m_{uf} is bounded by its minimum value $m_{uf\min}$ and its maximum value $m_{uf\max}$, m_{ur} is bounded by its minimum value $m_{ur\min}$ and its maximum value $m_{ur\max}$. Next, we can obtain the values of $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ from $m_s \in [m_{s\min}, m_{s\max}]$, $m_{uf} \in [m_{uf\min}, m_{uf\max}]$ and $m_{ur} \in [m_{ur\min}, m_{ur\max}]$. Then we have

$$
\begin{aligned}
\max \frac{1}{m_s} &= \frac{1}{m_{s\min}} =: \hat{m}_s, \quad \min \frac{1}{m_s} = \frac{1}{m_{s\max}} =: \check{m}_s, \\
\max \frac{1}{m_{uf}} &= \frac{1}{m_{uf\min}} =: \hat{m}_{uf}, \quad \min \frac{1}{m_{uf}} = \frac{1}{m_{uf\max}} =: \check{m}_{uf}, \\
\max \frac{1}{m_{ur}} &= \frac{1}{m_{ur\min}} =: \hat{m}_{ur}, \quad \min \frac{1}{m_{ur}} = \frac{1}{m_{ur\max}} =: \check{m}_{ur},
\end{aligned}
$$

We can represent $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ by

$$
\begin{aligned}
\frac{1}{m_s} &= M_1(\xi_1(t))\hat{m}_s + M_2(\xi_1(t))\check{m}_s, \\
\frac{1}{m_{uf}} &= N_1(\xi_2(t))\hat{m}_{uf} + N_2(\xi_2(t))\check{m}_{uf}, \\
\frac{1}{m_{ur}} &= O_1(\xi_3(t))\hat{m}_{ur} + O_2(\xi_3(t))\check{m}_{ur},
\end{aligned}
$$

where $\xi_1(t) = \frac{1}{m_s}$, $\xi_2(t) = \frac{1}{m_{uf}}$ and $\xi_3(t) = \frac{1}{m_{ur}}$,

$$
\begin{aligned}
M_1(\xi_1(t)) + M_2(\xi_1(t)) &= 1, \\
N_1(\xi_2(t)) + N_2(\xi_2(t)) &= 1, \\
O_1(\xi_3(t)) + O_2(\xi_3(t)) &= 1.
\end{aligned}
$$

The membership functions can be calculated as

$$
\begin{aligned}
M_1(\xi_1(t)) &= \frac{\frac{1}{m_s} - \check{m}_s}{\hat{m}_s - \check{m}_s}, \quad M_2(\xi_1(t)) = \frac{\hat{m}_s - \frac{1}{m_s}}{\hat{m}_s - \check{m}_s}, \\
N_1(\xi_2(t)) &= \frac{\frac{1}{m_{uf}} - \check{m}_{uf}}{\hat{m}_{uf} - \check{m}_{uf}}, \quad N_2(\xi_2(t)) = \frac{\hat{m}_{uf} - \frac{1}{m_{uf}}}{\hat{m}_{uf} - \check{m}_{uf}}, \\
O_1(\xi_3(t)) &= \frac{\frac{1}{m_{ur}} - \check{m}_{ur}}{\hat{m}_{ur} - \check{m}_{ur}}, \quad O_2(\xi_3(t)) = \frac{\hat{m}_{ur} - \frac{1}{m_{ur}}}{\hat{m}_{ur} - \check{m}_{ur}}.
\end{aligned}
$$

Table 6.1: List of Fuzzy Rules

Rule No.	Premise variables		
	$\xi_1(t)$	$\xi_2(t)$	$\xi_3(t)$
1	**Heavy**	**Heavy**	**Heavy**
2	**Heavy**	**Light**	**Heavy**
3	**Light**	**Heavy**	**Heavy**
4	**Light**	**Light**	**Heavy**
5	**Light**	**Light**	**Light**
6	**Light**	**Heavy**	**Light**
7	**Heavy**	**Light**	**Light**
8	**Heavy**	**Heavy**	**Light**

Then, the uncertain systems in (6.1) is represented by the following fuzzy model. Table 6.1 shows the fuzzy rules of this fuzzy systems.

Model Rule 1: **IF** $\xi_1(t)$ is **Heavy**, $\xi_2(t)$ is **Heavy**, and $\xi_3(t)$ is **Heavy**, **THEN**

$$
\begin{aligned}
\dot{x}(t) &= A_1 x(t) + B_1 [u(t) + g(x(t))] + B_{11} w(t), \\
z_1(t) &= C_{11} x(t) + D_{11}[u(t) + g(x(t))], \\
z_2(t) &= C_{21} x(t),
\end{aligned}
$$

matrices A_1, B_1, B_{11}, C_{11}, D_{11} and C_{21} are obtained by replacing $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ with matrices $A(t)$, $B(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$ and $C_2(t)$ with $\hat{m}_s$, $\hat{m}_{uf}$ and $\hat{m}_{ur}$ respectively.

Model Rule 2: **IF** $\xi_1(t)$ is **Heavy**, $\xi_2(t)$ is **Light**, and $\xi_3(t)$ is **Heavy**, **THEN**

$$
\begin{aligned}
\dot{x}(t) &= A_2 x(t) + B_2[u(t) + g(x(t))] + B_{12} w(t), \\
z_1(t) &= C_{12} x(t) + D_{12}[u(t) + g(x(t))], \\
z_2(t) &= C_{22} x(t),
\end{aligned}
$$

matrices A_2, B_2, B_{12}, C_{12}, D_{12} and C_{22} are obtained by replacing $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ with matrices $A(t)$, $B(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$ and $C_2(t)$ with $\hat{m}_s$, $\check{m}_{uf}$ and $\hat{m}_{ur}$ respectively.

Model Rule 3: **IF** $\xi_1(t)$ is **Light**, $\xi_2(t)$ is **Heavy**, and $\xi_3(t)$ is **Heavy**,
THEN

$$
\begin{aligned}
\dot{x}(t) &= A_3 x(t) + B_3[u(t) + g(x(t))] + B_{13}w(t), \\
z_1(t) &= C_{13}x(t) + D_{13}[u(t) + g(x(t))], \\
z_2(t) &= C_{23}x(t),
\end{aligned}
$$

matrices A_3, B_3, B_{13}, C_{13}, D_{13} and C_{23} are obtained by replacing $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ with matrices $A(t)$, $B(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$ and $C_2(t)$ with $\breve{m}_s$, $\hat{m}_{uf}$ and $\hat{m}_{ur}$ respectively.

Model Rule 4: **IF** $\xi_1(t)$ is **Light**, $\xi_2(t)$ is **Light**, and $\xi_3(t)$ is **Heavy**,
THEN

$$
\begin{aligned}
\dot{x}(t) &= A_4 x(t) + B_4[u(t) + g(x(t))] + B_{14}w(t), \\
z_1(t) &= C_{14}x(t) + D_{14}[u(t) + g(x(t))], \\
z_2(t) &= C_{24}x(t),
\end{aligned}
$$

matrices A_4, B_4, B_{14}, C_{14}, D_{14} and C_{24} are obtained by replacing $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ with matrices $A(t)$, $B(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$ and $C_2(t)$ wit $\breve{m}_s$, $\breve{m}_{uf}$ and $\hat{m}_{ur}$ respectively.

Model Rule 5: **IF** $\xi_1(t)$ is **Light**, $\xi_2(t)$ is **Light**, and $\xi_3(t)$ is **Light**,
THEN

$$
\begin{aligned}
\dot{x}(t) &= A_5 x(t) + B_5[u(t) + g(x(t))] + B_{15}w(t), \\
z_1(t) &= C_{15}x(t) + D_{15}[u(t) + g(x(t))], \\
z_2(t) &= C_{25}x(t),
\end{aligned}
$$

matrices A_5, B_5, B_{15}, C_{15}, D_{15} and C_{25} are obtained by replacing $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ with matrices $A(t)$, $B(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$ and $C_2(t)$ wit $\breve{m}_s$, $\breve{m}_{uf}$ and $\breve{m}_{ur}$ respectively.

Model Rule 6: **IF** $\xi_1(t)$ is **Light**, $\xi_2(t)$ is **Heavy**, and $\xi_3(t)$ is **Light**,
THEN

$$
\begin{aligned}
\dot{x}(t) &= A_6 x(t) + B_6[u(t) + g(x(t))] + B_{16}w(t), \\
z_1(t) &= C_{16}x(t) + D_{16}[u(t) + g(x(t))], \\
z_2(t) &= C_{26}x(t),
\end{aligned}
$$

matrices A_6, B_6, B_{16}, C_{16}, D_{16} and C_{26} are obtained by replacing $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ with matrices $A(t)$, $B(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$ and $C_2(t)$ with $\breve{m}$, $\hat{m}_{uf}$ and $\breve{m}_{ur}$ respectively.

Model Rule 7: **IF** $\xi_1(t)$ is **Heavy**, $\xi_2(t)$ is **Light**, and $\xi_3(t)$ is **Light**, **THEN**

$$
\begin{aligned}
\dot{x}(t) &= A_7 x(t) + B_7[u(t) + g(x(t))] + B_{17}w(t), \\
z_1(t) &= C_{17}x(t) + D_{17}[u(t) + g(x(t))], \\
z_2(t) &= C_{27}x(t),
\end{aligned}
$$

matrices A_7, B_7, B_{17}, C_{17}, D_{17} and C_{27} are obtained by replacing $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ with matrices $A(t)$, $B(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$ and $C_2(t)$ with $\hat{m}_s$, $\breve{m}_{uf}$ and $\breve{m}_{ur}$ respectively.

Model Rule 8: **IF** $\xi_1(t)$ is **Heavy,** $\xi_2(t)$ is **Heavy**, and $\xi_3(t)$ is **Light**, **THEN**

$$
\begin{aligned}
\dot{x}(t) &= A_8 x(t) + B_8[u(t) + g(x(t))] + B_{18}w(t), \\
z_1(t) &= C_{18}x(t) + D_{18}[u(t) + g(x(t))], \\
z_2(t) &= C_{28}x(t),
\end{aligned}
$$

matrices A_8, B_8, B_{18}, C_{18}, D_{18} and C_{28} are obtained by replacing $\frac{1}{m_s}$, $\frac{1}{m_{uf}}$ and $\frac{1}{m_{ur}}$ with matrices $A(t)$, $B(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$ and $C_2(t)$ with $\hat{m}_s$, $\hat{m}_{uf}$ and $\breve{m}_{ur}$ respectively. By fuzzy blending, the overall fuzzy model is inferred as follows:

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{8} h_i(\xi(t)) \{A_i x(t) + B_i[u(t) + g(x(t)) \\
&\quad + B_{1i}w(t)\}, \\
z_1(t) &= \sum_{i=1}^{8} h_i(\xi(t)) \{C_{1i}x(t) + D_{1i}[u(t) + g(x(t))]\}, \\
z_2(t) &= \sum_{i=1}^{8} h_i(\xi(t)) C_{2i}x(t), \tag{6.3}
\end{aligned}
$$

where

$$
\begin{aligned}
h_1\left(\xi\left(t\right)\right) &= M_1\left(\xi_1\left(t\right)\right) \times N_1\left(\xi_2\left(t\right)\right) \times O_1\left(\xi_2\left(t\right)\right), \\
h_2\left(\xi\left(t\right)\right) &= M_1\left(\xi_1\left(t\right)\right) \times N_2\left(\xi_2\left(t\right)\right) \times O_1\left(\xi_2\left(t\right)\right), \\
h_3\left(\xi\left(t\right)\right) &= M_2\left(\xi_1\left(t\right)\right) \times N_1\left(\xi_2\left(t\right)\right) \times O_1\left(\xi_2\left(t\right)\right), \\
h_4\left(\xi\left(t\right)\right) &= M_2\left(\xi_1\left(t\right)\right) \times N_2\left(\xi_2\left(t\right)\right) \times O_1\left(\xi_2\left(t\right)\right), \\
h_5\left(\xi\left(t\right)\right) &= M_2\left(\xi_1\left(t\right)\right) \times N_2\left(\xi_2\left(t\right)\right) \times O_2\left(\xi_2\left(t\right)\right), \\
h_6\left(\xi\left(t\right)\right) &= M_2\left(\xi_1\left(t\right)\right) \times N_1\left(\xi_2\left(t\right)\right) \times O_2\left(\xi_2\left(t\right)\right), \\
h_7\left(\xi\left(t\right)\right) &= M_1\left(\xi_1\left(t\right)\right) \times N_2\left(\xi_2\left(t\right)\right) \times O_2\left(\xi_2\left(t\right)\right), \\
h_8\left(\xi\left(t\right)\right) &= M_1\left(\xi_1\left(t\right)\right) \times N_1\left(\xi_2\left(t\right)\right) \times O_2\left(\xi_2\left(t\right)\right).
\end{aligned}
$$

It is obvious that the fuzzy weighting function $h_i(\xi\left(t\right))$ satisfies

$$
h_i\left(\xi\left(t\right)\right) \geq 0, \quad \sum_{i=1}^{8} h_i\left(\xi\left(t\right)\right) = 1.
$$

Remark 6.1 Since the sprung mass m_s, the front and rear wheels unsprung masses m_{uf} and m_{ur} are uncertainties, which vary in given ranges, i.e. $m_s \in [m_{s\min}, m_{s\max}]$, $m_{uf} \in [m_{uf\min}, m_{uf\max}]$ and $m_{ur} \in [m_{ur\min}, m_{ur\max}]$. In this study, the masses m_s, m_{uf} and m_{ur} are selected constants in the given ranges. Thus, we know that the corresponding fuzzy weighting function h_i is a constant.
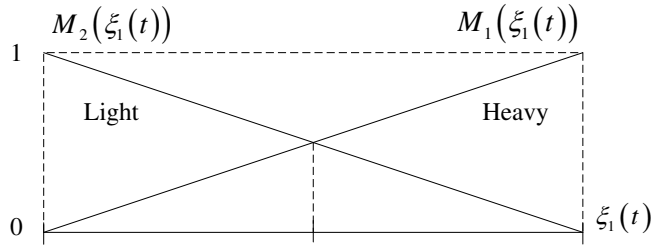
Figure 6.1: Membership functions $M_1\left(\xi_1\left(t\right)\right)$ and $M_2\left(\xi_1\left(t\right)\right)$
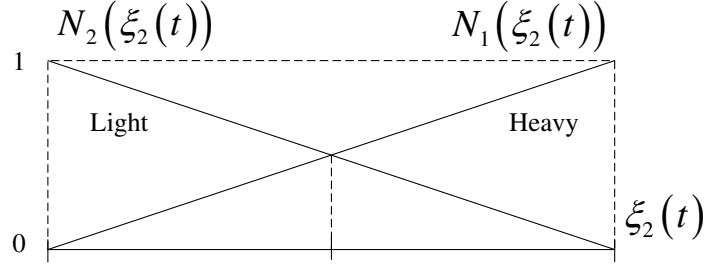
Figure 6.2: Membership functions $N_1\left(\xi_2\left(t\right)\right)$ and $N_2\left(\xi_2\left(t\right)\right)$
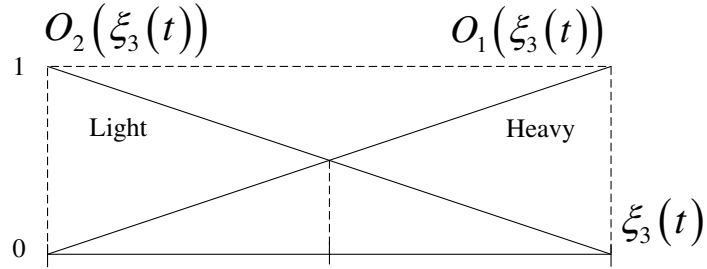
Figure 6.3: Membership functions $O_1\left(\xi_3\left(t\right)\right)$ and $O_2\left(\xi_3\left(t\right)\right)$

6.3   Design of Adaptive Sliding Mode Controller

The control design objective for the half-vehicle active suspension system based on T-S fuzzy model in (6.3) is to synthesize an SMC law such that the state trajectories of (6.3) are globally driven onto (with probability 1) the specified sliding surface. Moreover, the designing sliding motion should be asymptotically stable.

6.3.1   Switching Surface

At the first step of design procedure, in this work, we construct integral-type sliding surface function as follows:

$$s\left(t\right) = Gx\left(t\right) - \int_0^t G\left(\bar{A}_i + \bar{B}_i\bar{K}_j\right)x\left(z\right)\mathrm{d}z, \tag{6.4}$$

where

$$\bar{A}_i + \bar{B}_i\bar{K}_j = \sum_{i=1}^8\sum_{j=1}^8 h_i\left(\xi\left(t\right)\right)h_j\left(\xi\left(t\right)\right)\left(A_i + B_iK_j\right),$$

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$G \in \mathbf{R}^{2 \times 8}$ is a constant matrix to be designed satisfying that GB_i is nonsingular and $GB_{1i} = 0$ $(i = 1, 2, \ldots, 8)$. $K_j \in \mathbf{R}^{2 \times 8}$ $(j = 1, 2, \ldots, 8)$ is the state feedback gain matrix to be designed.

Remark 6.2 Here, due to the structure of B_i and B_{1i}, we can easily obtain the constant matrix G to satisfy GB_i is nonsingular and $GB_{1i} = 0$ $(i = 1, 2, \ldots, 8)$. The detailed method is provided in simulation part.

According to the necessary condition for the reaching of the sliding surface: $s(t) = 0$ and $\dot{s}(t) = 0$. We have the following equivalent control law:

$$u(t) = \sum_{j=1}^{8} h_j(\xi(t)) K_j x(t) - g(x(t)). \tag{6.5}$$

Then, substituting (6.5) into (6.3) yields

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{8} \sum_{j=1}^{8} h_i(\xi(t)) h_j(\xi(t)) [(A_i + B_i K_j) x(t) \\
&\quad + B_i w(t)], \\
z_1(t) &= \sum_{i=1}^{8} \sum_{j=1}^{8} h_i(\xi(t)) h_j(\xi(t)) (C_{1i} + D_{1i} K_j) x(t), \\
z_2(t) &= \sum_{i=1}^{8} h_i(\xi(t)) C_{2i} x(t).
\end{aligned}
\tag{6.6}
$$

We know that the above expression (6.6) is the sliding-mode dynamics of the fuzzy system (6.3) in the specifies switching surface $s(t) = 0$.

Without loss of generality, it is assumed, $w \in L_2[0, \infty)$, and $\|w\|_2^2 \le w_{\max} < \infty$. The main aim of this chapter is to design the adaptive sliding mode controller $u(t)$ in (6.5) such that the sliding mode is asymptotically stable and under zero initial condition, the closed-loop system guarantees that $\|z_1\|_2 < \gamma \|w\|_2$, and the following control output constraints are guaranteed:

$$|\{z_2(t)\}_q| \le 1, \quad q = 1, 2, 3, 4, \quad t > 0. \tag{6.7}$$

6.3.2 Stability of Sliding Motion

In this subsection, we will analyze the asymptotic stability and H_∞ performance constraint for the sliding mode dynamic. Based on the linear matrix inequality (LMI) method, we have the following theorem.

Theorem 6.1 For system in (6.6), $i, j = 1, 2, \ldots, 8$ and $q = 1, 2, 3, 4$, if there exist matrices $\bar{P} > 0$ and $\bar{K}_j$ such as the following LMIs hold:

$$\Psi_{ii} < 0, \tag{6.8}$$

$$\Psi_{ij} + \Psi_{ji} < 0, \quad i < j, \tag{6.9}$$

$$\begin{bmatrix} -\bar{P} & \sqrt{\rho}\bar{P}\{C_{2i}\}_q^T \\ * & -I \end{bmatrix} < 0, \tag{6.10}$$

where

$$\Psi_{ij} = \begin{bmatrix} \text{sym}(A_i\bar{P} + B_i\bar{K}_j) & B_{1i} & \bar{P}C_{1i}^T + \bar{K}_j^T D_{1i}^T \\ * & -\gamma^2 & 0 \\ * & * & -I \end{bmatrix},$$

then the sliding motion (6.6) is asymptotically stable; the performance $\|T_{z_1 w}\|_\infty < \gamma$ is minimized with the disturbance energy under the bound $w_{\max} = (\rho - V(0))/\gamma$; and the following control output constraints are guaranteed. Then matrix can be obtained $K_j = \bar{K}_j\bar{P}^{-1}$

Proof. Considering the Lyapunov-Krasovskii functional as follows:

$$V_3(t) = x^T(t)Px(t).$$

We will first establish the H_∞ performance of the system in (6.6) under zero initial conditions,

$$\dot{V}_3(t) + z_1^T(t)z_1(t) - \gamma^2 w^T(t)w(t)$$

$$\leq \sum_{i=1}^{8}\sum_{j=1}^{8} h_i h_j \left[x^T(t)\left(P(A_i + B_i K_j)\right.\right.$$

$$\left.\left. + (A_i + B_i K_j)^T P\right)x(t) + 2x^T(t)PB_{1i}w(t)\right]$$

$$+ \sum_{i=1}^{8}\sum_{j=1}^{8} h_i h_j x(t)(C_i + D_{1i}K_j)^T$$

$$\times (C_i + D_{1i}K_j)x(t) - \gamma^2 w^T(t)w(t)$$

$$= \sum_{i=1}^{8}\sum_{j=1}^{8} h_i h_j \begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix} \hat{\Pi}_{ij} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$

$$= \begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix} \left(\sum_{i=1}^{8} h_i^2 \hat{\Pi}_{ii} + \sum_{i=1}^{7}\sum_{j=i+1}^{8} h_i h_j \left(\hat{\Pi}_{ij} + \hat{\Pi}_{ji}\right) \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}.$$

$$\tag{6.11}$$

where

$$\hat{\Pi}_{ij} = \begin{bmatrix} P\left(A_i + B_i K_j\right) + \left(A_i + B_i K_j\right)^T P \\ + \left(C_i + D_{1i} K_j\right)^T \left(C_i + D_{1i} K_j\right) & PB_{1i} \\ * & -\gamma^2 I \end{bmatrix}.$$

For inequalities (6.8) and (6.9), by performing congruence transformations with $\mathrm{diag}\,\{P, I, I, I\}$ ($K_j = \bar{K}_j \bar{P}^{-1}$ and $P = \bar{P}^{-1}$) and using Shur complement, it is derived that

$$\dot{V}_3(t) + z_1^T(t) z_1(t) - \gamma^2 w^T(t) w(t) < 0, \tag{6.12}$$

for all nonzero $w \in L_2[0, \infty)$. In addition, when $w(t) = 0$, the derivative of $V_3(t)$ along the solution of the system in (6.6) is expressed as

$$\dot{V}_3(t) = \sum_{i=1}^{8} \sum_{j=1}^{8} h_i h_j \left[x^T(t) \left(P\left(A_i + B_i K_j\right) + \left(A_i + B_i K_j\right)^T P \right) x(t) \right],$$

which means the system in (6.6) is asymptotically stable from Theorem 6.3.2.

Under zero initial conditions, we have $V_3(0) = 0$ and $V(\infty) \geq 0$. Integrating both sides of (6.12) yields $\|z_1\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0, \infty)$, and the H_∞ performance is established. In the following, we will show that the hard constraints in (6.7) can be guaranteed. Inequality (6.12) guarantees $\dot{V}_3(t) - \gamma^2 w^T(t) w(t) < 0$. Integrating both sides of the above inequality from zero to any $t > 0$, we obtain

$$V_3(t) - V_3(0) < \gamma^2 \int_0^t w^T(s) w(s) \mathrm{d}s < \gamma^2 \|w\|_2^2. \tag{6.13}$$

From the definition of the Lyapunov functional $V_3(t)$, we know that $x^T(t) P x(t) < \rho$ with $\rho = \gamma^2 w_{\max} + V_3(0)$. Similar to Gao $et\ al.$ (2010a), the following inequality holds

$$\max_{t>0} \left| \{z_2(t)\}_q \right|^2$$
$$\leq \max_{t>0} \left\| \sum_{i=1}^{8} h_i x^T(t) \{C_{2i}\}_q^T \{C_{2i}\}_q x(t) \right\|_2$$
$$< \rho \cdot \theta_{\max} \left(\sum_{i=1}^{8} h_i P^{-\frac{1}{2}} \{C_{2i}\}_q^T \{C_{2i}\}_q P^{-\frac{1}{2}} \right),$$

where $q = 1, 2, 3, 4$, $\theta_{\max}(\cdot)$ represents maximal eigenvalue. From the above inequality, we know that the constraints in (6.7) are guaranteed, if

$$\rho \cdot \sum_{i=1}^{8} h_i P^{-\frac{1}{2}} \{C_{2i}\}_q^T \{C_{2i}\}_q P^{-\frac{1}{2}} < I, \tag{6.14}$$

which means

$$\sum_{i=1}^{8} h_i \left(\rho \cdot P^{-\frac{1}{2}} \{C_{2i}\}_q^T \{C_{2i}\}_q P^{-\frac{1}{2}} - I \right) < 0,$$

which can be guaranteed by the feasibility of the following inequality,

$$\begin{bmatrix} -P & \sqrt{\rho}\,\{C_{2i}\}_q^T \\ * & -I \end{bmatrix} < 0, \quad q = 1, 2, 3, 4. \tag{6.15}$$

(6.10) is equivalent to (6.15) by performing a simple congruence transformation with diag $\left\{ \bar{P}^{-1}, I \right\}$. The proof is completed. ∎

6.3.3 Reachability Analysis

In this subsection, an adaptive sliding mode controller will be designed such that the trajectory of the closed-loop system will be driven onto the sliding surface in finite time, and thus the reachability is guaranteed.

Theorem 6.2 Consider the system in (6.3) with assumption in (6.2). Under the following sliding mode controller

$$u(t) = \sum_{j=1}^{8} h_j K_j x(t) - \rho(t) \operatorname{sgn}(x(t)) \tag{6.16}$$

where

$$\rho(t) = \lambda + \delta \|x(t)\|$$

with $\lambda > 0$ is a known small constant, the state trajectories of the system in (6.3) will be driven onto the switching surface $s(t) = 0$ in finite time with probability 1.

Proof. Choosing the following Lyapunov function candidate as

$$V_1(t) = \frac{1}{2} s^T(t) \left(\sum_{i=1}^{8} h_i GB_i \right)^{-1} s(t).$$

From (6.4) and (6.16), we can see that

$$
\begin{aligned}
\dot{s}\left(t\right) &= G\sum_{i=1}^{8}h_i\{A_i x\left(t\right)+B_i[u\left(t\right)+g\left(x\left(t\right)\right)]\} \\
&\quad -G\sum_{i=1}^{8}\sum_{j=1}^{8}h_i h_j\left[\left(A_i+B_i K_j\right)x\left(t\right)\right] \\
&= G\sum_{i=1}^{8}h_i B_i\left(-\rho\left(t\right)\operatorname{sgn}\left(x\left(t\right)\right)+g\left(x\left(t\right)\right)\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\dot{V}_1\left(t\right) &= s^T\left(t\right)\left(\sum_{i=1}^{8}h_i GB_i\right)^{-1}\dot{s}\left(t\right) \\
&\leq -\rho\left(t\right)\left\|s\left(t\right)\right\|_1+\delta\left\|s\left(t\right)\right\|\left\|x\left(t\right)\right\| \\
&\leq -\rho\left(t\right)\left\|s\left(t\right)\right\|+\delta\left\|s\left(t\right)\right\|\left\|x\left(t\right)\right\| \\
&= -\lambda\left\|s\left(t\right)\right\|<0 \quad\text{for}\quad \left\|s\left(t\right)\right\|\neq 0.
\end{aligned}
$$

This implies that the trajectories of the system (6.3) will be globally driven onto the specified switching surface $s(t)=0$ with probability 1 in finite time. The proof is completed. ∎

It is shown that the bound of $g\left(x\left(t\right)\right)$ is required to synthesize the sliding mode control law (6.16). In practice, it is difficult to obtain the exact knowledge of the bound δ in practical application. In the following Theorem, an adaptive sliding mode control law is further presented for the case when the bound is unavailable. First, let $\hat{\delta}\left(t\right)$ represent the estimation of the unknown real constant δ, then the corresponding estimation error can be given as $\tilde{\delta}\left(t\right)=\hat{\delta}\left(t\right)-\delta$.

Theorem 6.3 Consider the system (6.3) and suppose that the exact value of the bound δ is unknown. If the adaptive sliding mode control law is given by

$$
u\left(t\right)=\sum_{j=1}^{8}h_j K_j x\left(t\right)-\hat{\rho}\left(t\right)\operatorname{sgn}\left(x\left(t\right)\right), \tag{6.17}
$$

where

$$
\hat{\rho}\left(t\right)=\lambda+\hat{\delta}\left(t\right)\left\|x\left(t\right)\right\|
$$

and parametric updating law as

$$\dot{\hat{\delta}}(t) = \eta \left\| s(t) \right\| \left\| x(t) \right\|$$

with $\lambda > 0$ and $\eta > 0$ are known small scalers, then the state trajectories of the system (6.3) will be driven onto the switching surface $s(t) = 0$ with probability 1 in finite time.

Proof. Choosing the following Lyapunov function candidate as

$$V_2(t) = \frac{1}{2} s^T(t) \left(\sum_{i=1}^{8} h_i G B_i \right)^{-1} s(t) + \frac{1}{2\eta} \tilde{\delta}^2(t),$$

then we have

$$
\begin{aligned}
\dot{V}_2(t) &= s^T(t) \left(\sum_{i=1}^{8} h_i G B_i \right)^{-1} \dot{s}(t) + \frac{1}{\eta} \tilde{\delta}(t) \dot{\tilde{\delta}}(t) \\
&= s^T(t) \left(\sum_{i=1}^{8} h_i G B_i \right)^{-1} \left(\sum_{i=1}^{8} h_i G B_i \right) \\
&\quad \times \left(-\hat{\rho}(t) \operatorname{sgn}(x(t)) + g(x(t)) \right) + \frac{1}{\eta} \tilde{\delta}(t) \dot{\tilde{\delta}}(t) \\
&\leq -\hat{\rho}(t) \left\| s(t) \right\| + \delta \left\| s(t) \right\| \left\| x(t) \right\| \\
&\quad + \tilde{\delta}(t) \left\| s(t) \right\| \left\| x(t) \right\| \\
&= -\lambda \left\| s(t) \right\| < 0 \quad \text{for} \quad \left\| s(t) \right\| \neq 0.
\end{aligned}
$$

This implies that the trajectories of the system (6.3) will be globally driven onto the specified switching surface $s(t) = 0$ with probability 1 despite the actuator uncertainty. The proof is completed. ∎

6.4 Case Study

The sprung mass m_s, the front and rear unsprung masses m_{uf} and m_{ur} are assumed that m_s belongs to the range [621 kg 759 kg], m_{uf} belongs to the range [39.6 kg 40.4 kg] and m_{ur} belongs to the range [44.55 kg 45.45 kg] respectively. The problem at hand is to design an adaptive sliding mode controller such that the sliding motion in the specified switching surface is asymptotically stable and

satisfies the H_∞ performance under the suspension constrained performance in (6.7), and the state trajectories can be driven onto the switching surface. Choosing

$$
G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \tag{6.18}
$$

which yields that GB_i is nonsingular and $GB_{1i} = 0$. Here, we choose $\rho = 1$ as discussed in (Chen & Guo, 2005). The maximum allowable front and rear suspension strokes are assumed as $z_{f\max} = 0.1$ m and $z_{r\max} = 0.1$ m respectively. In addition, the nonlinear term $g(x(t))$ is assumed as $g(x(t)) = [0.5x_1(t) \quad 0.5x_2(t)]^T$. By using the convex optimization to Corollary 1, the minimum guaranteed closed-loop H_∞ performance index can be computed as $\gamma_{\min} = 5.8483$ and admissible control gain matrices are given at the next page.

To check the effectiveness of the design controller, we hope that the desired controller to satisfy: 1) the first control output $z_1(t)$ including the heave acceleration $\ddot{z}_c(t)$ and the pitch acceleration $\ddot{\varphi}(t)$ is as small as possible; 2) the suspension deflection is below the maximum allowable suspension strokes $z_{f\max} = 0.1$ m and $z_{r\max} = 0.1$ m, which means that $z_2(t)_1 < 1$ and $z_2(t)_1 < 1$; 3) the controlled output satisfies $z_2(t)_3 < 1$ and $z_2(t)_4 < 1$. In order to evaluate the suspension characteristics with respect to ride comfort, vehicle handling, and working space of the suspension, the variability of the road profiles is taken into account. In the context of active suspension performance, road disturbances can be generally assumed as shocks. Shocks are discrete events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pothole on an smooth road surface. In this work, this case of road profile is considered to reveal the transient response characteristic, which is given by

$$
z_{rf}(t) = \begin{cases} \frac{A}{2}(1 - \cos(\frac{2\pi V}{L}t)), & \text{if } 0 \le t \le \frac{L}{V}, \\ 0, & \text{if } t > \frac{L}{V}, \end{cases} \tag{6.19}
$$

where A and L are the height and the length of the bump. Assume $A = 0.1$ m, $L = 2.5$ m and the vehicle forward velocity as $V = 20(\text{km/h})$. In this section, we assume that the road condition $z_{rr}(t)$ for the rear wheel is the same as the front wheel but with a time delay of $(l_1 + l_2)/V$.

$$K_1 = 10^6 \times \begin{bmatrix} -0.1423 & -0.0077 & 0.0220 & -0.4431 & -0.0281 & -0.0009 & -0.0029 & -0.0086 \\ 0.3814 & -0.0399 & -0.2177 & 2.0060 & 0.0641 & -0.0229 & 0.0097 & 0.0303 \end{bmatrix},$$

$$K_2 = 10^6 \times \begin{bmatrix} -0.1447 & -0.0073 & 0.0234 & -0.4551 & -0.0284 & -0.0007 & -0.0029 & -0.0088 \\ 0.3830 & -0.0402 & -0.2185 & 2.0140 & 0.0643 & -0.0230 & 0.0097 & 0.0304 \end{bmatrix},$$

$$K_3 = 10^6 \times \begin{bmatrix} -0.1456 & -0.0076 & 0.0239 & -0.4601 & -0.0286 & -0.0007 & -0.0030 & -0.0089 \\ 0.3791 & -0.0397 & -0.2163 & 1.9932 & 0.0637 & -0.0228 & 0.0096 & 0.0301 \end{bmatrix},$$

$$K_4 = 10^6 \times \begin{bmatrix} -0.1481 & -0.0071 & 0.0254 & -0.4731 & -0.0290 & -0.0006 & -0.0030 & -0.0091 \\ 0.3802 & -0.0399 & -0.2167 & 1.9986 & 0.0638 & -0.0228 & 0.0097 & 0.0301 \end{bmatrix},$$

$$K_5 = 10^6 \times \begin{bmatrix} -0.1178 & -0.0153 & 0.0010 & -0.2701 & -0.0242 & -0.0036 & -0.0020 & -0.0061 \\ 0.4121 & -0.0480 & -0.2415 & 2.2080 & 0.0690 & -0.0258 & 0.0107 & 0.0332 \end{bmatrix},$$

$$K_6 = 10^6 \times \begin{bmatrix} -0.1159 & -0.0156 & -0.0002 & -0.2609 & -0.0239 & -0.0037 & -0.0020 & -0.0060 \\ 0.4105 & -0.0477 & -0.2408 & 2.1996 & 0.0687 & -0.0257 & 0.0107 & 0.0331 \end{bmatrix},$$

$$K_7 = 10^6 \times \begin{bmatrix} -0.1146 & -0.0154 & -0.0009 & -0.2533 & -0.0236 & -0.0037 & -0.0019 & -0.0059 \\ 0.4116 & -0.0477 & -0.2413 & 2.2043 & 0.0689 & -0.0257 & 0.0107 & 0.0331 \end{bmatrix},$$

$$K_8 = 10^6 \times \begin{bmatrix} -0.1125 & -0.0157 & -0.0022 & -0.2432 & -0.0233 & -0.0038 & -0.0019 & -0.0057 \\ 0.4102 & -0.0474 & -0.2406 & 2.1965 & 0.0687 & -0.0257 & 0.0106 & 0.0330 \end{bmatrix}.$$

The switching function in (6.4) is designed as

$$s\left(t\right) = Gx\left(t\right) - \int_0^t G \sum_{i=1}^8 \sum_{j=1}^8 h_i\left(\xi\left(t\right)\right) h_j\left(\xi\left(t\right)\right) \times \left(A_i + B_i K_j\right) x\left(z\right) \mathrm{d}z,$$

where the matrix G is given in (6.18) and the controller gain matrices K_j ($j = 1, 2, \ldots, 8$) have been calculated. And the desired sliding mode control law in Theorem 2 can be obtained as

$$u\left(t\right) = \sum_{j=1}^8 h_j\left(\xi\left(t\right)\right) K_j x\left(t\right) - \hat{\rho}\left(t\right) \mathrm{sgn}\left(x\left(t\right)\right), \tag{6.20}$$

where

$$\hat{\rho}\left(t\right) = 0.5 + \hat{\delta}\left\|x\left(t\right)\right\|$$

and the parametric updating law is

$$\dot{\hat{\delta}}\left(t\right) = 0.1\left\|s\left(t\right)\right\|\left\|x\left(t\right)\right\|.$$

In the built T-S model, we assume that $m_s = 700$ kg, $m_{uf} = 40.4$ and $m_{uf} = 45$, which are all within their given ranges respectively. Then, we can have the membership functions $h_i(\xi(t))$ ($i = 1, 2, \ldots, 8$) Fig. 6.4–6.6 plot responses of the heave and pitch accelerations, the front and rear suspension deflection constrains, the relation of dynamic front and rear tire deflection constrains of the open- and closed-loop systems under above sliding mode control law. In this chapter, we mainly pay attention to the suspension performances such as ride comfort, vehicle handling, and working space of the suspension. We can see from Fig. 6.4 that an improved ride comfort has been achieved compared with the passive systems. Furthermore, it can be seen Fig. 6.5 shows that the suspension strokes constraints are guaranteed. It can be seen from Fig. 6.6 that the dynamic front and rear tire stroke constrains have also been convinced. Therefore, it can be observed from Fig. 6.4–6.6 show that an improvement in ride comfort has been made through the designed the state feedback controller. Fig. 6.7 depicts the trajectories of the front and rear actuator forces. Furthermore, It can observed from Fig. 6.8 and 6.9 that the sliding mode is attained in finite time. For the different membership functions $h_i(\xi(t))$ ($i = 1, 2, \ldots, 8$) with $m_s = 650$ kg, $m_{uf} = 40$ kg and $m_{uf} = 45.2$ kg, the simulation results are provided in Fig. 6.10–6.15 to further illustrate the effectiveness of the proposed controller design method in this chapter.
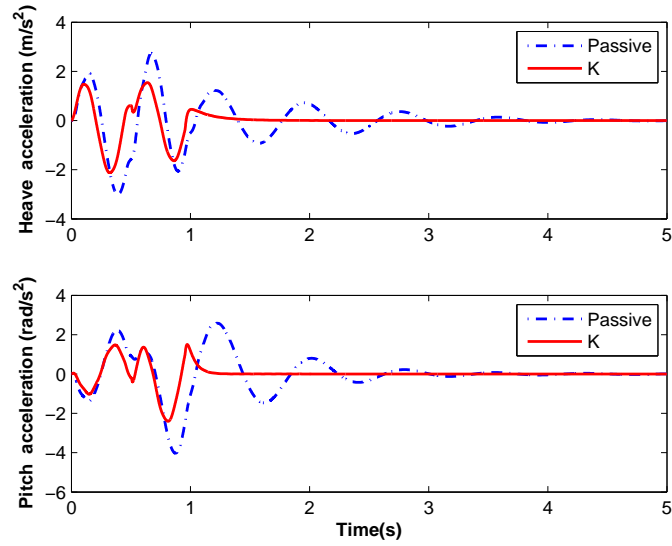
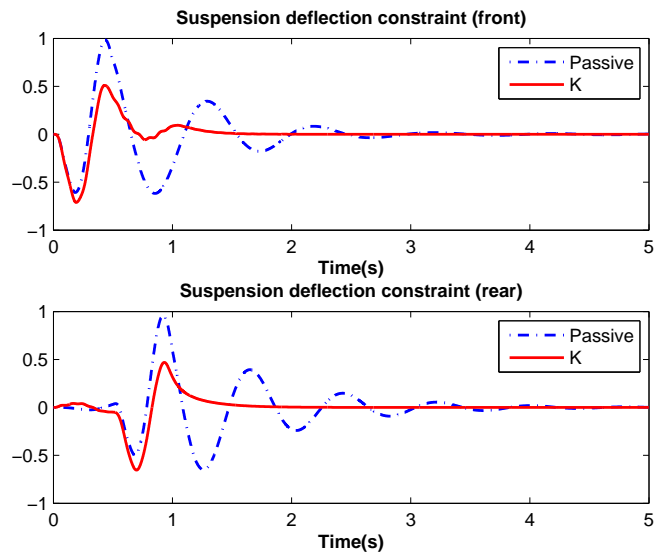Figure 6.4: Responses of the heave accelerations and the pitch acceleration

Figure 6.5: Responses of the front and rear suspension deflection constraints

Figure 6.6: Responses of the dynamic front and rear tire stroke constraints

Figure 6.7: Responses of the dynamic front and rear actuator force

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Figure 6.8: Trajectories of sliding variable $s(t)$

Figure 6.9: Trajectory of adaptive parameter

Figure 6.10: Responses of the heave accelerations and the pitch acceleration

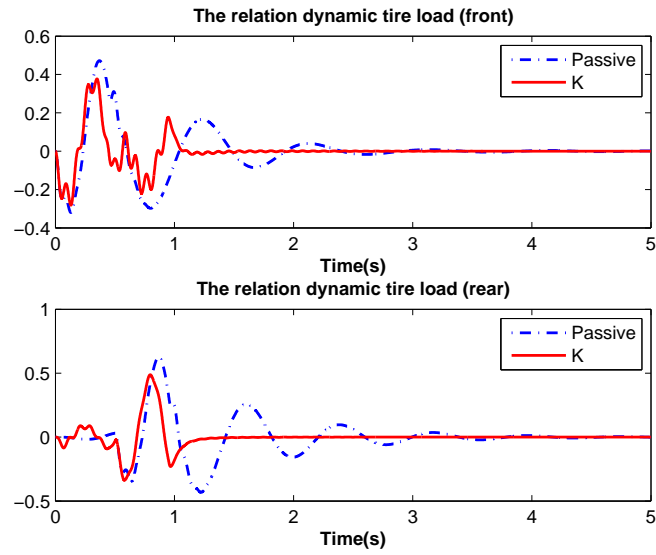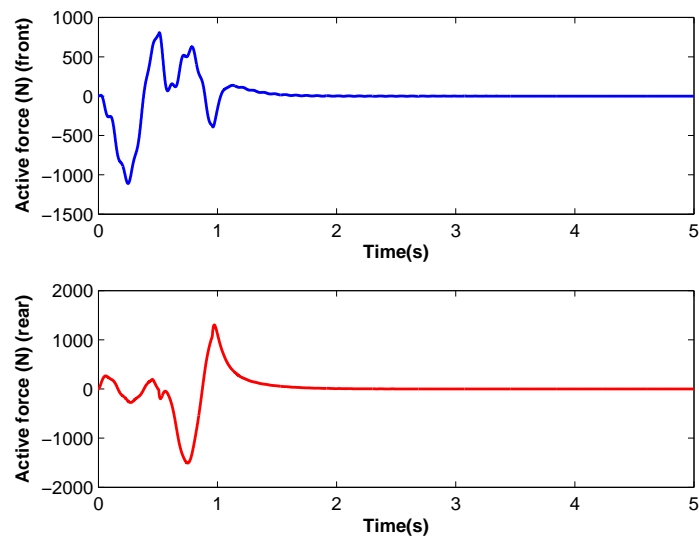Figure 6.11: Responses of the front and rear suspension deflection constraints
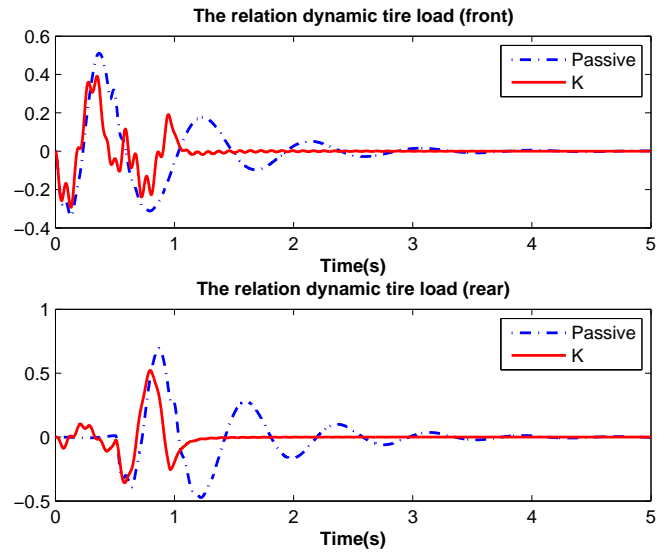
Figure 6.12: Responses of the dynamic front and rear tire stroke constraints
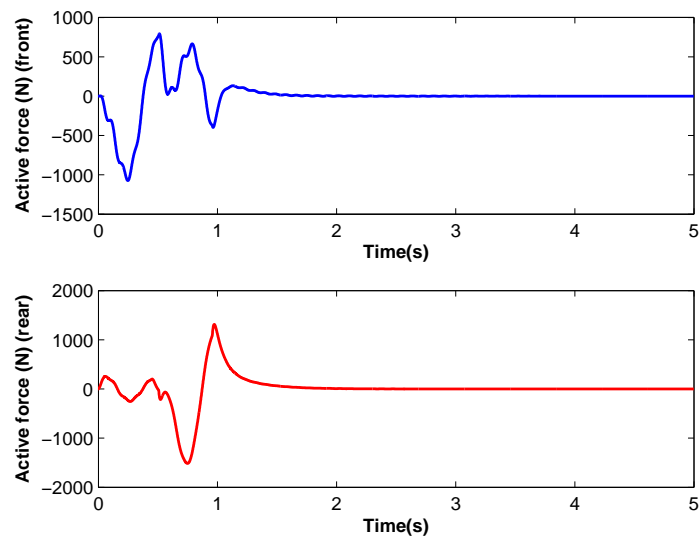
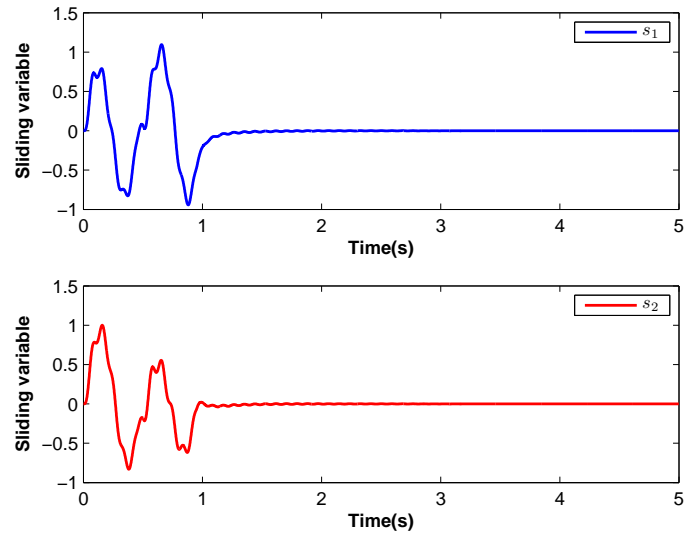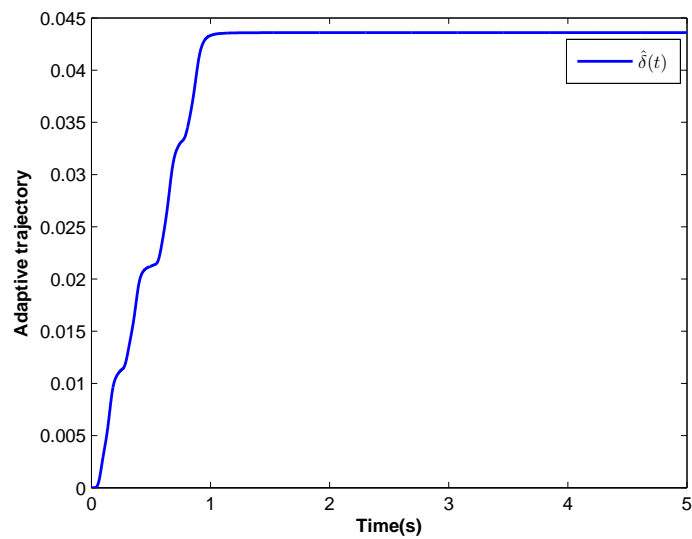Figure 6.13: Responses of the dynamic front and rear actuator force

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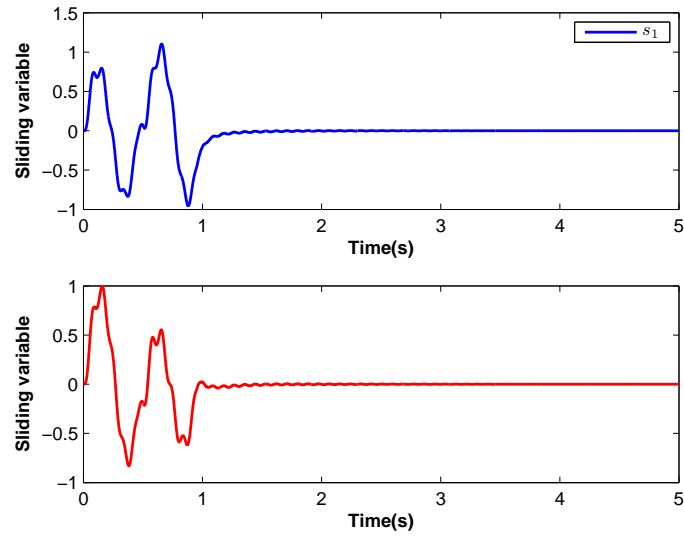
Figure 6.14: Trajectories of sliding variable $s(t)$

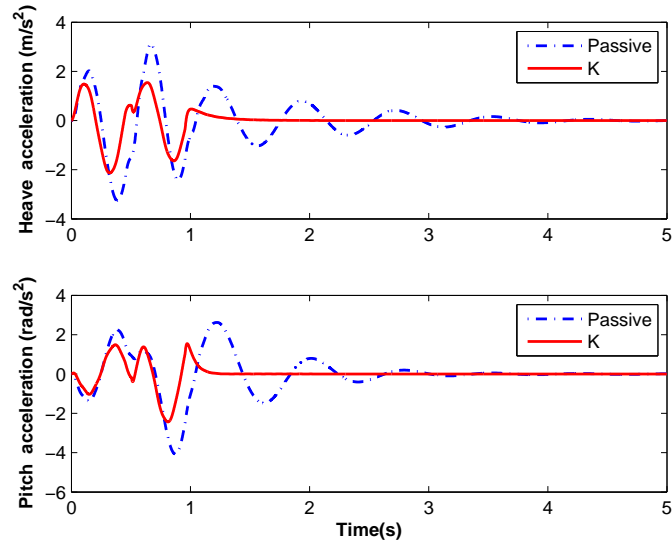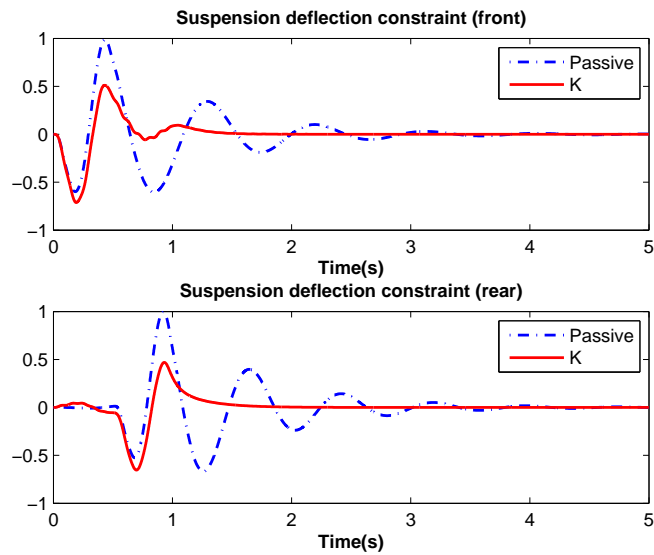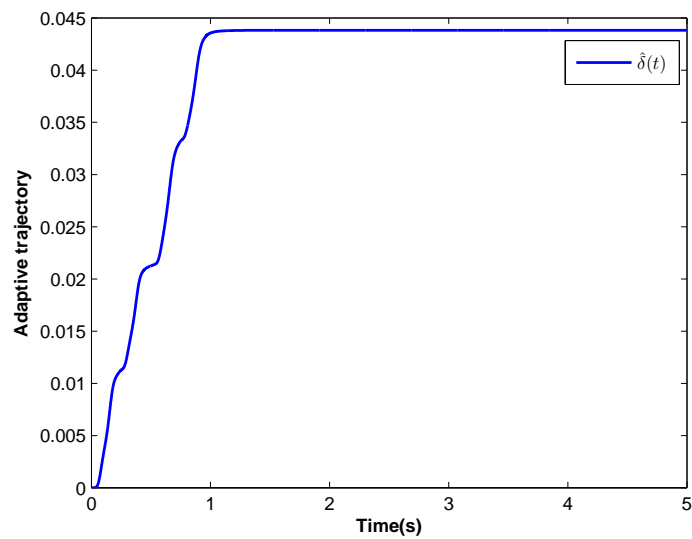Figure 6.15: Trajectory of adaptive parameter

6.5 Summary

This chapter has studied the problem of adaptive sliding mode control design for nonlinear active suspension systems by means of T-S fuzzy approach. The corresponding dynamic system has been built after considering the variations of the sprung mass, the front and rear unsprung masses, the nonlinear actuator dynamic and the suspension performances. The adaptive sliding mode controller has been designed to guarantee the reachability of the specified switching surface. Then, we have developed the sufficient conditions to guarantee the asymptotical stability of the dynamics in the specified switching surface with H_∞ norm performance under the constraints of the suspension performances. The convex optimization method has been used to present these conditions, which has been solved by means of the standard software. Simulation results have been provided to illustrate the effectiveness of the proposed method.

Chapter 7

Conclusions and Future Work

7.1 Overview

This thesis startes by considering four key contributions as the main objectives of this research project. The first contribution is focused on robust H_∞ control for active suspension systems with actuator delay under the assumption that the state signals are fully or partially measurable. Then, the novel state-feedback and output-feedback controllers are designed to guarantee the stability and improve the suspension performances of the closed-loop system. The second achievement is to model a new type actuator fault in vehicle active suspension system and design a novel fault-tolerant controller to minimize the vertical vibrations of vehicle body to improve the ride comfort and satisfy the road good holding and suspension deflection suspension performances. The third contribution is to propose the fuzzy control algorithm for uncertain active suspension systems where the uncertainties are coming from road inputs and suspension parameters. T-S fuzzy model control method has been utilized to improve the suspension performances. The final contribution is to investigate the adaptive sliding mode control design problem for nonlinear vehicle active suspension systems with uncertainty under the frame of multi-objective control. The suspension performances are considered and the T-S fuzzy model control approach is utilized to represent the nonlinear uncertain suspension system by T-S fuzzy system. The sliding mode controller is designed to guarantee the stability of the system and improve the suspension performances.

7.2 Contributions

The thesis is mainly focused on the control design for vehicle active suspension systems with disturbance and uncertainty. More specifically, four aspects have been considered in details.

7.2.1 Robust H_∞ Controller Design for Active Suspension Systems with Actuator Time-varying Delay

In Chapter 3, a novel half-vehicle active suspension system with polytopic uncertainties and actuator time-varying delay has been first modelled. Under the assumption that the state signals are fully known, the new robust H_∞ controller has been designed for the uncertain suspension system to minimize the vertical and longitudinal vibrations of vehicle body to improve the ride comfort, road handling and suspension deflection performances in Chapter 3.2. The main technique used in this Chapter 3.2 was to construct a novel Lyapunov functional and develop some novel delay-dependent stability analysis methods. In Chapter 3.3, for the partial measurable state signals, the new type dynamic output-feedback controller was constructed first for the active suspension systems with actuator time-varying delays. Based on the Lyapunov stability theory, a output-feedback H_∞ controller has been designed to guarantee the closed-loop systems stability and improve the suspension performance in Chapter 3.3.1. We can observe from the simulation results that the improvement in suspension performance can be achieved for the different road conditions by using the output-feedback controller by considering actuator delay compared with the output-feedback controller without considering actuator delay.

7.2.2 Fault-Tolerant H_∞ Controller Design for Active Suspension Systems with Actuator Faults

In Chapter 4, the fault-tolerant H_∞ control problem has been studied for active suspension systems with actuator faults. In Chapter 4.2, we formulated the active

suspension systems with actuator faults and proposed a novel actuator failure process based on continuous-time homogeneous Markov jump modes. In Chapter 4.3, a novel fault-tolerant H_∞ controller has been designed such that the resulting control system is tolerant in the sense that it guarantees asymptotic stability and H_∞ performance, and simultaneously satisfies the constrained performance when possible actuator failures exist. In Chapter 4.4, the efficiency of the developed method has been demonstrated with a quarter-vehicle active suspension model.

7.2.3 Fuzzy Controller Design for Active Suspension Systems with Uncertainty

In Chapter 5, the fuzzy reliable H_∞ control problem has been considered for uncertain active suspension systems with actuator delay and fault based on T-S fuzzy model approach. In Chapter 5.2, the T-S fuzzy nonlinear sector method has been utilized to represent the uncertain active suspension systems with sprung and unsprung mass variations, and suspension performances. In Chapter 5.3, novel LMI-based reliable fuzzy H_∞ controller existence conditions have been derived for the T-S fuzzy systems with actuator faults and time-varying delay. In Chapter 5.4, fuzzy controller has been designed to improve suspension performances. Simulation results have been provided to illustrate the effectiveness of the proposed approaches in Chapter 5.5.

7.2.4 Adaptive Sliding Mode Controller Design for Nonlinear Active Suspension Systems

In Chapter 6, the problem of adaptive sliding mode control has been studied for the active suspension systems with uncertainty and nonlinearity using multi-objective control. In Chapter 6.2, the corresponding dynamic system has been built by considering the variations of the sprung mass, the front and rear unsprung masses, the nonlinear actuator dynamic and the suspension performances. This control design process is different from the existing sliding mode control methods

as the suspension performances have been considered and the T-S fuzzy model approach has been utilized to represent the nonlinear uncertain suspension system. In Chapter 6.3, the sliding mode controller has been designed to guarantee the asymptotical stability of the dynamics in the specified switching surface with H_∞ norm performance under the constraints of the suspension performances. The convex optimization method has been used to present these conditions, which has been solved by means of the standard software. In Chapter 6.4, simulation results for a half-vehicle model have been provided to demonstrate the effectiveness of the presented method.

7.3   Future Work

Related topics for future research are listed below.

7.3.1   Relaxation on Stability Analysis and Controller Synthesis Conditions

In this thesis, the quadratic Lyapunov stability theory has been used to investigate the fuzzy control problem for the active suspension systems with uncertainty. In future work, piecewise and parameter dependent Lyapunov function methods will be exploited to further improve the suspension performances compared with the quadratic Lypaunov method. In addition, in order to propose more general fuzzy control results, future work will be done without requiring that both the T-S fuzzy model and the fuzzy controller share the same number of rules and/or the same set of premise membership functions. Thus, it offers a greater design flexibility for the fuzzy controller and is possible to lower the controller complexity by employing a smaller number of rules and simple membership functions. Based on these methods, the controller design criteria will be presented in terms of LMIs, which can be checked efficiently by using the standard software (Matlab LMI Control Toolbox). By utilizing Matlab simulick and M functions, simulation results can be done to illustrate the effectiveness of the proposed fuzzy control method.

7.3.2 Adaptive Direct Fuzzy Control

Consider the presence of non-linearities such as a hardening spring, a quadratic damping force and the tyre lift-off phenomenon in a real suspension system and establish a proper nonlinear half (full)-vehicle suspension system. Fuzzy logic systems will be used to approximate these nonlinear systems. We will develop novel direct fuzzy backstepping control methods to handle the control design problems for the systems.

7.3.3 LPV Gain-scheduling Control

We will present a novel LPV gain-scheduling controller design approach for nonlinear active suspension system that takes nonlinear hydraulic actuator and the nonlinearity characteristic of the spring force, the damping force and the mass variations into account. For the nonlinear LPV model of the system, the gain-scheduling technique is based on the suspension deflection and the mass variations of the vehicle and parameter variations of the spring and damping elements. Under the suspension performance constrains of suspension deflection and road holding, the improved ride comfort can be achieved under the multi-objective control frame. The state of the art is that the LPV control method does not require full state feedback and it does not require severe structural assumptions on the plant model for the novel nonlinear full-vehicle active suspension system with hydraulic actuator.

7.3.4 Multi-objective Finite Frequency Control

The multi-objective control problem of vehicle active suspension systems with frequency band constraints will be investigated. In previous work, the control design model is based on a quarter-vehicle suspension model. The infinite frequency control is difficult when we consider the complex full-vehicle suspension system. Under the frame of multi-objective control, the following infinite frequency control design problems will be investigated.

Based on the premise that all the state variables are online measurable, the multi-objective state-feedback control problem will be considered for the full-vehicle active suspension systems with frequency band constraints based on the generalized Kalman-Yakubovich-Popov lemma. The frequency domain inequalities are transformed into linear matrix inequalities, and our attention is focused on developing methods to design a state feedback control law based on matrix inequalities such that the resulting closed-loop system is asymptotically stable with a prescribed level of disturbance attenuation in certain frequency domain. Then, the finite frequency method is further developed to deal with the problem of the full-vehicle suspension control systems with hydraulic actuator dynamic, actuator input delay and actuator saturation. As is well known, in vehicle active suspension systems, real hydraulic actuator dynamic, actuator input delay and actuator saturation are important issues that need careful treatment to avoid poor performance of the closed-loop system.

The online measurable state variables sometimes introduce higher cost and additional complexity by measuring all the state components. In the cases where not all the state variables can be measured on-line, output feedback control is an alternative, which can conduct effective control according to part of the measured state components. In other words, output feedback strategy requires less sensors, compared with the state feedback counterparts. Considering a practical situation of active suspension systems, a dynamic output feedback controller will be designed to match the finite frequency characteristics for the full-vehicle suspension control system. Furthermore, some infinite frequency control strategies will be proposed to handle the full-vehicle suspension systems with hydraulic actuator dynamic, actuator input delay and actuator saturation via output feedback control approach.

7.3.5 Generalization

The proposed control objectives and control strategies should be generalized to a number of different situations. In this thesis, the generalization for this framework was only tested upon the quarter-vehicle and half-vehicle suspension systems. The

effectiveness of the proposed method in this thesis should be verified in the coupled states of four quarter-vehicle suspension systems and the full-vehicle suspension systems. Moreover, a hybrid model including four wheel vehicle integrated control systems (e.g., braking and traction control systems) will be investigated under the proposed control approaches in the future.

7.3.6 Application

This thesis develops the theoretical research to service the project (design and control of active suspension systems for in-wheel motor electric vehicles, funded by Protean Electric Ltd.). It should be pointed out that the proposed control methods on the active suspension systems can be implemented in the in-wheel motor electric vehicle active suspension systems. However, the detailed in-wheel motor electric vehicle active suspension model and the corresponding parameters should be known before using the proposed method to the real vehicle active suspension system. We will collaborate with the Protean Electric Ltd to establish an in-wheel motor electric active suspension dynamic model, in which the hydraulic actuator should be taken into account. The effect of electric current and voltage for the active suspension system should be considered. The stability analysis problem for this control system should be investigated and then the control design approaches proposed in this thesis will be applied to this system. In particular, we will collaborate with the company and evaluate the proposed controllers on the in-wheel motor electric vehicle active suspension system.

7.4 Summary

In this dissertation, novel robust control design approaches were proposed for vehicle active suspension systems with uncertainty. Firstly, novel state-feedback and output-feedback controller was designed to guarantee the stability and improve the suspension performances of vehicle suspension systems with actuator time-varying delays. Secondly, a new type actuator fault model was built in vehicle active suspension system and a novel tolerant-fault controller was designed to

minimize the vertical vibrations of vehicle body to improve the ride comfort and satisfy the road good holding and suspension deflection suspension performances. Thirdly, novel fuzzy control algorithm was proposed for the uncertain active suspension systems to improve the suspension performances. Finally, an adaptive sliding mode controller was designed for vehicle active suspension systems with uncertainty and nonlinearity, and the sufficient controller existence condition was derived.

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Appendix A

Publications

1. Li, H., Liu, H., Gao, H. and Shi, P. (2012) Reliable Fuzzy Control for Active Suspension Systems with Actuator Delay and Fault, *IEEE Transactions on Fuzzy Systems*, 20 (2) 342–357.

2. Li, H., Yu, J., Liu, H. and Hilton, C. (2012) Adaptive Sliding Mode Control for Nonlinear Active Suspension Systems Via T-S Fuzzy Model. *IEEE Transactions on Industrial Electronic*, accepted for publication.

3. Li, H., Gao, H. and Liu, H. (2011) Robust Quantized Control for Active Suspension Systems. *IET Control Theory and Applications*, 5 (17), 1955–1969.

4. Li, H., Liu, H., Hand, S. and Hilton, C. (2011) Non-Fragile H_∞ Control for Half-Vehicle Active Suspension Systems with Actuator Uncertainties. *Journal of Vibration and Control*, DOI: 10.1177/1077546311434972.

5. Li, H., Liu, H., Gao, H. and Liu M. (2012) Fault-Tolerant H_∞ Control for Active Suspension Vehicle Systems with Actuator Faults. *Proceedings of the Institution of Mechanical Engineers, Part I, Journal of Systems and Control Engineering*, 3 (226), 348–363.

6. Li, H., Liu, H., Hand, S. and Hilton, C. (2011) Multi-objective H_∞ Control for Vehicle Active Suspension Systems with Random Actuator Delay. *International Journal of Systems Science*, DOI:10.1080/00207721.2011.566640.

7. Li, H., Liu, H., Hand, S. and Hilton, C. (2011) Robust H_∞ Control for a Half-Car Active Suspension System with Input Time-Varying Delay. *International Journal of Systems Science*, DOI:10.1080/00207721.2011.617895.

8. Li, H., Zhou, Q., Chen, B. and Liu, H. (2011) Parameter-Dependent Robust Stability for Uncertain Markovian Jump Systems with Time Delay. *Journal of the Franklin Institute*, 348 (4), 738–748.

9. Li, H., Liu, H., Hand, S. and Hilton, C. (2011) A Study on Half-Vehicle Active Suspension Control Using Sampled-Data Control, *2011 23rd Chinese Control and Decision Conference, Mianyang, China*, 2635–2640.

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11. Li, H., Liu, H. and Liu M. (2011) Active Control for Suspension Vehicle Systems with Nonlinear Actuator Uncertainty Using Sliding-mode Control, *International Workshop on Advanced Computational Intelligence and Intelligent Informatics, Suzhou, China,* accepted for publication.