UNIVERSITY OF PORTSMOUTH

INTUITION BASED DECISION MAKING METHODOLOGY FOR RANKING FUZZY NUMBERS USING CENTROID POINT AND SPREAD

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ABSTRACT

The concept of ranking fuzzy numbers has received significant attention from the research community due to its successful applications for decision making. It complements the decision maker exercise their subjective judgments under situations that are vague, imprecise, ambiguous and uncertain in nature. The literature on ranking fuzzy numbers show that numerous ranking methods for fuzzy numbers are established where all of them aim to correctly rank all sets of fuzzy numbers that mimic real decision situations such that the ranking results are consistent with human intuition. Nevertheless, fuzzy numbers are not easy to rank as they are represented by possibility distribution, which indicates that they possibly overlap with each other, having different shapes and being distinctive in nature. Most established ranking methods are capable to rank fuzzy numbers with correct ranking order such that the results are consistent with human intuition but there are certain circumstances where the ranking methods are particularly limited in ranking non – normal fuzzy numbers, non – overlapping fuzzy numbers and fuzzy numbers of different spreads.

As overcoming these limitations is important, this study develops an intuition based decision methodology for ranking fuzzy numbers using centroid point and spread approaches. The methodology consists of ranking method for type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers where all of them are theoretically and empirically validated. Theoretical validation highlights the capability of the ranking methodology to satisfy all established theoretical properties of ranking fuzzy quantities. On contrary, the empirical validation examines consistency and efficiency of the ranking methodology on ranking fuzzy numbers correctly such that the results are consistent with human intuition and can rank more than two fuzzy numbers simultaneously. Results obtained in this study justify that the ranking methodology not only fulfills all established theoretical properties but also ranks consistently and efficiently the fuzzy numbers. The ranking methodology is implemented to three related established case studies found in the literature of fuzzy sets where the methodology produces consistent and efficient results on all case studies examined. Therefore, based on evidence illustrated in this study, the ranking methodology serves as a generic decision making procedure, especially when fuzzy numbers are involved in the decision process.

TABLE OF CONTENTS

AUTHOR'S DECLARATION

Whilst registered as a candidate for the above degree, I have not been registered for any other research award. The results and conclusions embodied in this thesis are the work of the named candidate and have not submitted for any other academic award.

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LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

Symbols

A_i	Fuzzy Number A_i / Generalised Fuzzy Number A_i
$\tilde{A}_i$	Standardised Generalised Fuzzy Number A_i
$\mu_{\tilde{A}_i}(x)$	Membership function of fuzzy number A_i
$A_i \cup A_j$	Fuzzy union of A_i and A_j
$A_i \cap A_j$	Fuzzy intersection of A_i and A_j
$x^*_{A_i}$	Horizontal – x centroid / Horizontal – x value
$y^*_{A_i}$	Vertical – y centroid / Vertical – y value
$\left(x^*_{A_i}, y^*_{A_i}\right)$	Centroid Point for A_i
i_{A_i}	Distance of the horizontal – x axis
ii_{A_i}	Distance of the vertical – y axis
$s\left(\tilde{A}_i\right)$	Spread of fuzzy number A_i
w_{A_i}	Height of fuzzy number A_i
$\times$	Scalar Multiplication
$\succ$	Ranking order for ‘greater ranking’
$\prec$	Ranking order for ‘lower than’
$\approx$	Ranking order for ‘equal ranking’
$g(x)$	Weightage of x

LIST OF ABBREVIATIONS

Abbreviations

CPS	Ranking Methodology Using Centroid Point and Spread
CPS_I	Ranking Method Using Centroid Point and Spread For Type – I Fuzzy Numbers
CPS_{II}	Ranking Method Using Centroid Point and Spread For Type – II Fuzzy Numbers
CPS_Z	Ranking Method Using Centroid Point and Spread For Z – Numbers

PUBLICATIONS

[1] Bakar, A. S. A. & Gegov, A. (2014). Ranking of Fuzzy Numbers Based Centroid Point and Spread, *Journal of Intelligent and Fuzzy Systems 27*, pp. 1179 – 1186.

[2] Bakar, A. S. A. & Gegov, A. (2015). Multi – Layer Decision Methodology for Ranking Z - Numbers, *International Journal of Computational Intelligence Systems 8 (2)*, pp. 395 – 406.

[3] Bakar, A. S. A. & Gegov, A. (2015). Intuition Based Decision Methodology For Ranking Type – II Fuzzy Numbers, *16th World Congress of the International Fuzzy Systems Association* – 9th *European Society for Fuzzy Logic and Technology*, Gijon, Spain, pp. 593 – 600.

[4] Gegov, A. & Bakar, A. S. A. (2015). Validation of Methods for Ranking Fuzzy Numbers in Decision Making, *Journal of Intelligent and Fuzzy Systems*, accepted.

CHAPTER ONE

INTRODUCTION

1.1 OVERVIEW

Modern science is introduced in decision making environment as handling and solving current decision making problems are crucial and necessary. It suggests development or utilisation of computer or mathematical models to appropriately solve various decision making problems. In the literature of decision making, utilisation of established mathematical model to solve a decision making problem is clearly indicated as a much easier way than developing a mathematical model because the former involves only the application of a suitable established mathematical model while the latter requires a novel mathematical model development to handle the problem. Although development of a novel mathematical model is not easy, it suggests better quality in terms of describing and observing the situation than utilising the established model.

As far as the current decision making environments are concerned, involvement of human perception in the mathematical based decision model is pointed out as one of the seriously considered factors in many research areas such as economic, engineering, artificial intelligent and socio-economic. This is because of human always involves in every investigation of the decision making conducted. Human perception is defined in a generic way as human expressions towards a situation perceived using their subjective judgments and preferences. Therefore, in developing an effective mathematical model for decision making, the model is first expected to have the capability to represent linguistic terms appropriately because human perception is often associated with natural language. Secondly, the model is anticipated to produce correct decision results such that the results obtained are consistent with human intuition. Nonetheless, both expectations are not easy to achieve as solving a human based decision making problem which is represented by linguistic terms using mathematical knowledge is impractical. This is due to the fact that one cannot solve linguistic terms as part of natural language using numbers.

As a linguistic term is not easy to be interpreted using mathematical knowledge, a mathematical theory named fuzzy set theory is introduced as the medium of representation for human perception. Fuzzy set theory is a mathematical field that is capable to effectively deal with situations that are vague, imprecise and ambiguous in nature, like human decision making. It provides proper representation for the mathematical model in representing human perception appropriately. Since, application of fuzzy set theory in human decision making is relevant and suitable, this study aims at developing a fuzzy based mathematical decision model that is capable to well represent the linguistic terms and produces correct decision results such that the results obtained are consistent with human intuition. The model is also expected to serve as a generic decision model for human based decision making problems.

1.2 THESIS ORGANISATION

This section illustrates the overview in terms of organisation of the thesis. There are altogether nine chapters presented in the thesis including this chapter where the remaining eight chapters are described as follows.

Chapter 2 discusses the literature review of the study whereby problem statements, objectives and significance of the study are pointed out. Chapter 3 outlines theoretical preliminaries of the thesis such that definitions and formulations used in this study are given. In Chapter 4, research methodology of this study is thoroughly discussed where information provided in this chapter underpins development of the methodology in Chapter 5, 6 and 7. Thus, all discussions in Chapter 4, 5, 6 and 7 cover on the methodology section of the thesis. Chapter 8 focuses the implementation of the proposed work in solving established case studies while contributions are given in Chapter 9 together with concluding remarks and recommendations for future work.

1.3 SUMMARY

In this chapter, introductory section of thesis is provided. The thesis first mentions the overview on this study and this is later followed by the thesis organisation. In Chapter 2, the thesis discusses the literature review of this study.

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

This chapter illustrates details on the literature review of the thesis. It discusses established works found in the literature which are related to this study. The chapter starts its discussion with the description of basic notions of fuzzy sets which justify the applicability of fuzzy sets in human decision making. Then, chronological development of fuzzy sets tools is highlighted where overview on type – I fuzzy numbers and its extensions namely type – II fuzzy numbers and Z – numbers are covered. The main focus of this study is next addressed such that comprehensive reviews on ranking fuzzy numbers are provided. Two main areas of ranking fuzzy numbers namely ranking method based on centroid point and ranking method based on spread are thoroughly discussed in this chapter. Later on, more descriptions with regard to this study are underlined such as research problems, research questions and research objectives of this study. At the end of this chapter, research contribution is presented. Therefore, details on those aforementioned points are extensively discussed in sections and subsections of this chapter.

2.2 NOTIONS UNDERLYING FUZZY SETS

This section discusses the suitability and reliability of fuzzy sets when dealing with human decision making. In human decision making processes, natural language is often used as the medium of indication towards a situation perceived. This is because subjective perceptions expressed by humans are only appropriate when they are described using linguistic terms as part of natural language (Yeh et al., 2010). In research works done by Kwang & Lee (1999), Chen & Lu (2001), Lazzerini & Mkrtchyan (2009) and Chen & Chen (2009), fuzzy sets are pointed out as a suitable tool to deal with natural language. This is due to the fact that fuzzy sets theory underpins three basic notions namely graduality, epistemic uncertainty and bipolarity factors which are capable to represent the natural language well (Dubois & Prade, 2012). Therefore, without loss of generality on Dubois & Prade (2012) investigation, descriptions of all the three notions of fuzzy sets are as follows.

2.2.1 Graduality

According to Zadeh (1965), the concept of natural language is often regarded as a matter of degree, including the truth. This is because natural language used by humans on describing a subject is distinguished by different degrees of beliefs. For example in the case of height of a man, if height of a man is considered as 'tall' with 1.65 meters, then 1.75 meters is not regarded as 'tall' but is classified as 'very tall', where 'very tall' is another natural language used to described the height of a man. Utilisation of both 'tall' and 'very tall' in this case, implies that there is a transitional process occurs in terms of degree of belief used when information about the subject perceived is changed. This is expressed when degree of belief 'tall' decreases and degree of belief 'very tall' increases as values of height approaches 1.75 meters. The continuous but alternate pattern transition between these degrees of belief implies that natural languages conveyed by humans are gradual and not abrupt (Zadeh, 1965; Dubois & Prade, 2012).

2.2.2 Epistemic Uncertainty

Epistemic uncertainty of fuzzy sets is viewed as representation of incomplete information about a situation (Dubois, 2008). This underpins the effort on gaining better knowledge of decision processes because natural language used in human decision making are sometimes incomplete (Lazzerini & Mktrchyan, 2009). Among examples of the decision making situations involve in this case are forecasting and group decision making (Chen & Chen, 2007). In representing the natural language, epistemic uncertainty complements the capability of membership functions of fuzzy sets so that the ill – known situations are represented appropriately (Dubois, 2008).

2.2.3 Bipolarity

Bipolarity or double – sided nature refers to a process where human tend to follow their positive and negative attributes in decision making. This is expressed the fact that even if enough information about a decision is collected, human sometimes relies on their corresponding positive, negative or neutral effects on a situation. For example, options under consideration are separated based on good or bad alternatives and a decision is made in accordance to the strongest attribute produced by one of the alternatives. According to Cacioppo et al. (1997), results in cognitive psychology highlight the importance of bipolar reasoning in human cognitive activities. This is due to the fact that in multi-agent decision analysis, doubled – sided judgment are always applied to solve human based decision making problems (Zhang, 1994). Moreover, bipolarity perspective complements the capability of membership functions in representing both causal relations of positive and negative attributes of a situation appropriately (Zhang et al., 1989; Uehara & Fujise, 1993).

Even though, it is notable that human based decision making are usually subjective, vague and linguistically defined (natural language), basic notions of fuzzy sets namely the graduality, epistemic uncertainty and bipolarity prove that fuzzy sets are capable to represent human based decision making appropriately.

2.3 DEVELOPMENT OF FUZZY SETS

This section discusses the chronological development of fuzzy sets, specifically on tools used in decision making process. In section 2.2, fuzzy sets are pointed out as a suitable knowledge for human decision making where this is justified when basic notions of fuzzy sets capable to represent the natural language appropriately. Even though, fuzzy sets represent the natural language well, it is not easy to distinguish two or more natural languages used in a decision making problem as they are all defined qualitatively. Due to this, Zadeh (1965) suggests a quantitative definition for fuzzy sets which is well – suited for natural language known as fuzzy numbers.

In the literature of fuzzy sets, there are three kinds of fuzzy numbers found namely type – I fuzzy number, type – II fuzzy number and Z – number. These fuzzy numbers are considered in this study because they are all introduced by Zadeh. Among those three, a type – I fuzzy number is the most utilised fuzzy number in the literature of fuzzy sets followed by a type – II fuzzy number and then a Z – number. This happens because the chronological development of these fuzzy numbers, type – I is developed in 1965, type – II (1975) and Z – number (2011), which affect their utilisation frequency in the literature of fuzzy sets. Even though there are three types of fuzzy numbers considered in this study, they are not utilised simultaneously in representing the natural language. This is because they are all different in theoretical nature, thus indicate that only one type of fuzzy numbers is used at one time. Therefore, with respect to all fuzzy numbers considered in this study and literature of fuzzy sets, details on type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers are as follows.

2.3.1 Type – I Fuzzy Numbers

Type – I fuzzy number or the classical fuzzy number is the first fuzzy numbers introduced in the literature of fuzzy sets. In some established research studies done by Chen & Lu (2001), Wang et al. (2006), Thorani et al. (2012) and Yu et al. (2013), the term fuzzy number is used in their discussions as this is the original fuzzy number established in the literature of fuzzy sets. The term fuzzy number is changed into type – I fuzzy number only when type – II fuzzy numbers are introduced in the literature of fuzzy sets. This is because both type – I fuzzy numbers and type – II fuzzy numbers are themselves fuzzy numbers but they are differed in nature. According to Chen & Chen (2009), type – I fuzzy numbers consist of both membership degree and the spread features which are later discussed in detailed in Section 3.2 and subsection 3.5.3 respectively, correspond to confidence level and opinion of decision makers respectively. Due to this, type – I fuzzy numbers are applied in many decision making problems such as in evaluating Taiwan’s urban public transport system performance (Yeh et al., 2000), evaluation of engineering consultants’ performances (Chow & Ng, 2007), fuzzy risk analysis (Chen & Chen, 2009), selection of beneficial project investment (Jiao et al., 2009) and solving air fighter selection problem (Vencheh & Mokhtarian, 2011).

2.3.2 Type – II Fuzzy Numbers

Type – II fuzzy number is introduced in literature of fuzzy sets by Zadeh (1975) as an extension of type – I fuzzy numbers to model perceptions. This is because the uncertainty representation of type – I fuzzy number on natural language is insufficient to model perception (Dereli et al., 2011). Furthermore, imprecision level about a situation increases when number is translated into word (natural language) and finally to perceptions (John & Coupland, 2009). This implies that the representation adequacy of type – I fuzzy numbers on uncertainty is arguable. According to Wallsten & Budescu (1995), there are two types of uncertainties that are related with natural language namely intra – personal uncertainty and inter – personal uncertainty where both uncertainties are viewed as a group of type – I fuzzy numbers. Among research studies utilised type – II fuzzy numbers in their decision making applications are Figueroa et al. (2005) in mobile object based control tracking, Zeng & Liu (2006) in speech database classification and recognition, Seremi & Montazer (2008) in selection of website structures, Own (2009) in pattern recognition involving medical diagnosis reasoning problem, Bajestani & Zare (2009) in prediction of stock market index in Taiwan and Akay et al. (2011) in selection of appropriate adhesive tape dispenser. Although, type – II fuzzy numbers are introduced to enhance type – I fuzzy numbers in modelling perceptions, they are not often used for decision making applications as type – II fuzzy numbers are more complex than type – I fuzzy numbers in nature.

2.3.3 Z – Numbers

As compared to type – I fuzzy number and type – II fuzzy number, Z – number is the newest presented fuzzy numbers in the literature of fuzzy sets. Z – number is introduced by Zadeh (2011) as an extension of type – I fuzzy numbers but is completely differed from type – II fuzzy number. Even though both Z – number and type – II fuzzy number are extensions of type – I fuzzy numbers, the former is capable in measuring the reliability of the decision made as compared to the latter. Since, fuzzy numbers are the medium of quantitative representation for natural language, Z – number enhances the capability of both type – I and type – II fuzzy numbers by taking into account the reliability of the numbers used (Zadeh, 2011). According to Kang et al. (2012a), Z – number is represented by two embedded type – I fuzzy numbers where one of them plays the role that is similar as in subsection 2.3.1, while the other defines the reliability of the first one. Research on utilising Z – numbers in decision making applications is inadequate as compared to other fuzzy numbers, as it is a new fuzzy concept developed in the theory of fuzzy sets. As far as this study is concerned, only two decision making applications are found in literature of fuzzy sets namely the vehicle selection under uncertain environment (Kang et al., 2012b) and ranking of financial institutes in India based on their financing technical aspect (Azadeh et al., 2013).

Despite all aforementioned capabilities of fuzzy sets, in particular fuzzy numbers, when dealing with subjective human judgment and representing natural language quantitatively, it is not easy to presume one fuzzy number is greater or smaller than other fuzzy numbers under consideration. This is due to the fact that fuzzy numbers are represented by possibility distributions which indicate that they may overlap among them (Zimmerman, 2000; Kumar et al., 2010). This implies that each natural language represented by fuzzy number is hard to differentiate or distinguish, thus evaluating the natural language used in decision making is a difficult task. Therefore, one fundamental concept known as ranking fuzzy numbers (Jain, 1976) is introduced in the literature of fuzzy sets to solve this issue.

2.4 RANKING OF FUZZY NUMBERS

This section illustrates a thorough review on ranking fuzzy numbers which stands as the basis in handling fuzzy numbers appropriately. It is worth mentioning that descriptions made in this section consider only discussions on ranking of type – I fuzzy numbers as discussion on ranking of type – II fuzzy numbers and Z – numbers are inadequate in the literature of ranking fuzzy numbers. However, this aspect can be disregarded given that both type – II fuzzy numbers and Z – numbers are defined as the extensions of type – I fuzzy numbers as discussed in Section 2.3.2 and 2.3.3. This indicates that details associated with ranking of type – I fuzzy numbers are applicable for ranking of type – II fuzzy numbers and Z – numbers. Thus, all discussions made on ranking fuzzy numbers, especially ranking of type – I fuzzy numbers, are also relevant for ranking of type – II fuzzy numbers and Z – numbers. Hence, the phrase ranking fuzzy numbers is used in this case as a generic phrase for ranking of type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers. It is also worth noting here that several crucial terms such as embedded fuzzy numbers, spread of fuzzy numbers, singleton fuzzy numbers, trapezoidal fuzzy numbers, triangular fuzzy numbers, overlapping fuzzy numbers, non – overlapping fuzzy numbers, normal fuzzy numbers, non – normal fuzzy numbers, height of fuzzy numbers and $\alpha -$ cuts are extensively used in this chapter but information with regard to them are given in detailed in Chapter 3 and Chapter 4.Therefore, with no loss of generality, the literature on established existing works of ranking fuzzy numbers are as follows.

Ranking fuzzy numbers is introduced in fuzzy sets as a concept that determines which fuzzy number is greater when two or more fuzzy numbers are compared. A definition by Collan (2009) refers ranking fuzzy numbers as a process of comparing and organising fuzzy numbers in a specific ordering. This definition indicates that each fuzzy number under consideration is assigned a value whereby this value is used as comparing measure with other fuzzy numbers. Values obtained from each fuzzy number under consideration are then compared accordingly. As far as investigations on ranking fuzzy numbers are concerned, there are ranking methods that rank fuzzy numbers simultaneously (Chen & Chen, 2009) and some utilise pairwise ranking (Zhang & Yu, 2010) to rank fuzzy numbers. In ranking fuzzy numbers, simultaneous ranking refers to the capability of ranking method to simultaneously rank any quantity of fuzzy numbers at one time while pairwise ranking is the capability of ranking method to rank only two fuzzy numbers at one time. In this case, the capability of ranking methods to rank more than two fuzzy numbers determines the efficiency level of the ranking method. Baas & Kwakernaak (1977), Jain (1978) and Dubois & Prade (1978) are the first research groups that explore this area whereby notions underlying ranking of fuzzy numbers are discussed. Then, numerous efforts on finding appropriate ranking fuzzy numbers methods are demonstrated. Even though, fuzzy numbers are represented by possibility distributions and are not easily compared (Lee et al., 1999), there are numerous ranking methods are presented such as ranking methods based on area such as ranking methods by Wang et al. (2005), Kumar et al. (2010), Chen & Sanguatsan (2011), and Thorani et al. (2013), ranking using centroid approach (Cheng, 1998; Chu & Tsao, 2002; Wang & Yang (2006), Chen & Chen, 2009; Wang & Lee, 2009; Bakar et al., 2010) and ranking methods based on distance (Yao & Wu, 2000; Asady & Zendehnam, 2007; Asady, 2009, Asady & Abbasbandy, 2009; Rao & Shankar, 2013; Wang et al., 2013). Although, all aforementioned methods are of different perspectives, they aim to rank all types of fuzzy numbers in a correct ranking order such that ranking results obtained are consistent with human intuition.

A comprehensive survey on ranking fuzzy numbers method is conducted by Wang & Kerre (2001) where categorisation of ranking fuzzy numbers methods is presented. According to Wang & Kerre (2001), there are three categories of ranking fuzzy numbers methods in the literature of fuzzy sets namely preference relation, fuzzy mean and spread and fuzzy scoring. Under preference relation, ranking methods presented are those that usually map fuzzy numbers to respective real numbers where natural ordering exist (Deng, 2009). Among them are preference weighting function expectations based ranking method (Liu & Han, 2005), utilisation of distance minimisation to ranking fuzzy numbers (Asady, 2011), ranking fuzzy numbers based on maximum and minimum sets (Chou et al., 2011), ranking fuzzy numbers using left and right transfer coefficient (Yu et a., 2013) and ranking based on integral value (Yu & Dat, 2014).

In fuzzy mean and spread, ranking methods considered usually determine their ranking values by computing values of mean and spread for each fuzzy numbers. Then, using both values, a fuzzy number with greater mean value but lower spread value is ranked higher compared to other fuzzy numbers under consideration (Lee & Kwang, 1999). Among methods considered under this category are ranking fuzzy numbers based on α – cut, beliefs features and ratio between signal and noise (Chen & Wang, 2009), ranking based on deviation degree (Wang et al., 2009; Hajjari & Abbasbandy, 2011), ranking fuzzy numbers based on epsilon deviation (Yu et al., 2013).

Under fuzzy scoring, ranking methods considered generally utilise proportional optimal, left or right scores, centroid index and area measurement techniques to ranking fuzzy numbers. For ranking fuzzy numbers purposes, fuzzy numbers with the highest ranking value using one of the aforementioned techniques is ranked higher than the rest of fuzzy numbers under consideration. Among ranking fuzzy numbers methods that are considered under this category are ranking method using lexicographic screening procedure (Wang et al., 2005), ranking method based radius of gyration (Wang & Lee, 2009), ranking fuzzy numbers of different heights and spreads (Chen & Chen, 2009; 2012), ranking method using deviation degree (Asady, 2010) and centroid – based technique (Xu & Wei, 2010), ranking using area on the left and right of fuzzy numbers (Nejad & Mashinci, 2011), ranking method based on deviation degree (Phuc et al., 2012), ranking based on distance from largest value of a fuzzy numbers to original point (Shureshjani & Darehmiraki, 2013), ranking fuzzy numbers based on ideal solution (Deng, 2014) and ranking using altitudinal expected score and accuracy function (Wu & Chiclana, 2014). The following Table 2.1 illustrates list of ranking methods with their respective categories.

Table 2.1: Categorisation of Ranking Fuzzy Numbers

Ranking Method	Category		
	Preference Relation	Fuzzy Mean and Spread	Fuzzy Scoring
Fortemps & Roubens (1996)			√
Cross & Setnes (1998)	√		
Kwang & Lee (1999)		√	
Lee (2000)	√		
Kwang & Lee (2001)		√	

Chen & Lu (2001)		√	
Facchinetti (2002)			√
Chen & Lu (2002)		√	
Wang *et al.* (2005)			√
Nojavan & Ghazanfari (2006)	√		
Asady & Zendehnam (2007)	√		
Wang & Lee (2008)			
Ramli & Mohamad (2009)	√		
Chen & Chen (2009)		√	√
Chen *et al.* (2010)			√
Vencheh & Mokhtarian (2011)			√
Nejad & Mashinci (2011)			√
Phuc et al. (2012)			√
Shureshjani & Darehmiraki (2013)			√
Wu & Chiclana, 2014			√

'√' denotes as ranking method falls in this category

Although, there are three main categories in term of methods in ranking fuzzy numbers as shown in Table 2.1, many studies in the literature of ranking fuzzy numbers combine more than one category in ranking fuzzy numbers. This is shown when Chen & Chen (2009), Nejad & Mashinchi (2011) and Yu et al. (2013) contribute their research works using this direction. Chen & Chen (2009) ranking method merges fuzzy scoring and fuzzy mean and spread categories where the method utilises defuzzified value, height and spread to ranking fuzzy numbers. Nejad & Mashinchi (2011) ranking method combines fuzzy mean and spread category and preference relation category as this method ranks fuzzy numbers using transfer coefficient and deviation degree. Yu et al. (2013) ranking method on the other hand utilises fuzzy scoring and fuzzy mean and spread categories as the method focusing on combinations of centroid and epsilon deviation degree.

It is worth mentioning here that even if there are numerous methods for ranking fuzzy numbers are discussed in the literature of fuzzy sets, all of them posses their own advantages and disadvantages. In this study, the centroid point and spread are chosen as methods for ranking fuzzy numbers as both are capable to ranking fuzzy numbers correctly such that the ranking results are consistent with human intuition. Centroid, a defuzzification technique that transforms a fuzzy number into a crisp value, interprets a decision in an easy way as compared to other approaches because it provides only one value to represent a fuzzy number. Apart from that, centroid point enables ranking methods to ranking fuzzy numbers simultaneously. Spread on the other hand captures decision makers' opinions well by viewing optimistic, pessimistic and neutral decision makers' viewpoints using different spreads. These justifications imply that both centroid point and spread methods are worth considering and discussed in this study as both are in line with human intuition.

2.4.1 Ranking Using Centroid Point Approach

Literature of ranking fuzzy numbers indicate that a centroid point is made up by horizontal – x component and vertical – y component where both are utilised to determine the ranking value for each fuzzy number under consideration (Wang et al., 2005; Shieh, 2007). Values for the horizontal – x component and vertical – y component are calculated based values cover along the x – axis and y – axis respectively. Both values are then combined as the centroid point of a fuzzy number. However, in some exceptional cases, only the horizontal – x component is used to ranking fuzzy numbers.

Research on utilising centroid point in ranking fuzzy numbers is first initiated by Yager (1981) where only the horizontal – x component is considered in the ranking formulation. In the investigation, $g(x)$ is introduced as the weight function in measuring the important of x values where $g(x)$ complements the calculation for the horizontal – x component. The value obtained from the process represents the ranking value for each fuzzy number under consideration and is used to determine the ordering of fuzzy numbers. According to Yager (1981) ranking method, a fuzzy number with the greatest horizontal – x component value among other fuzzy numbers under consideration is classified as the highest ranked fuzzy number. Although, appropriate ranking results are obtained when this method is utilised, the method neglects the normality (heights of fuzzy number) and convexity components of fuzzy numbers in the ranking formulation where both components are crucial when cases involving non – normal fuzzy numbers are considered (Ramli & Mohamad, 2009).

Effort by Yager (1981) in ranking fuzzy numbers is then continued by Murakami et al. (1983) where a vertical – y component is introduced for the first time in the literature of ranking fuzzy numbers. This component is calculated by multiplying the value of the horizontal – x component with function of fuzzy number and is later paired up with the horizontal – x component to ranking fuzzy numbers. It has to be noted here that the horizontal – x component is the same as in Yager (1981). According to Murakami et al. (1983), fuzzy numbers with greater value of horizontal – x component and (or) vertical – y component is ranked higher than other fuzzy numbers under consideration. However, this ranking method gives unreasonable ranking results for all cases of fuzzy numbers considered where the values obtained for the vertical – y component are the same for all fuzzy numbers under consideration (Bortolan & Degani, 1985).

A different perspective from Murakami et al. (1983) point of view in ranking fuzzy numbers is proposed by Cheng (1998). If Murakami et al. (1983) ranking method considers at least one component, either horizontal – x component or (and) vertical – y component, then Cheng (1998) ranking method utilises both components in ranking fuzzy numbers. Cheng (1998) ranking method enhances Murakami et al. (1983) ranking method by introducing a new formulation for the vertical – y component as Murakami et al. (1983) vertical – y component is unable to differentiate each fuzzy number under consideration effectively. Cheng (1998) defines the vertical – y component as the inverse function of the horizontal – x component where the horizontal – x component is equivalent as in Yager (1981) and Murakami et al. (1983).

Even if Cheng (1998) ranking method enhances Murakami et al. (1983) ranking method, the former produces incorrect ranking result such that the ranking result is inconsistent with human intuition on non – overlapping cases fuzzy numbers of different spreads but same height (Chu & Tsao, 2002). Therefore, Chu & Tsao (2002) present a novel method for ranking fuzzy numbers where it is based on the area between the centroid point and the point of origin. In the investigation, computational works for both the horizontal – x component and vertical – y component utilised in this method are the same as Murakami et al. (1983) and Cheng (1998) ranking methods where values for both components are in this case multiplied with each other in obtaining the ranking values for all fuzzy numbers under consideration.

A new direction of computing the centroid point is then presented by Chen & Chen (2003) where both formulations of horizontal – x component and vertical – y component are calculated using the medium curve approach. Medium curve is an approach of finding the median where the median is calculated based on the values between infimum and supremum of α – cuts of a fuzzy number. The median is used in this case to obtain a straight line that determines the values for both horizontal – x component and vertical – y component. According to Chen & Chen (2003), advantage of using this approach in ranking fuzzy numbers is the approach capable to appropriately deal with both symmetric and asymmetric fuzzy numbers. Nonetheless, Chen & Chen (2003) ranking method is limited to overlapping fuzzy numbers cases while no work on non – overlapping fuzzy numbers cases is investigated.

A novel formulation of the centroid point for ranking fuzzy numbers purposes is presented by Wang et al. (2006) in the literature of ranking fuzzy numbers where both horizontal – x component and vertical – y component are introduced based on analytical geometric point of views (Ramli & Mohamad, 2009). In Wang et al. (2006) research work, Cheng (1998) and Chu & Tsao (2002) ranking methods are pointed out as methods that are not suitable for ranking fuzzy numbers. This is because Cheng’s (1998) ranking method neglects negative fuzzy numbers case as it only deals with positive fuzzy numbers case while Chu & Tsao (2002) ranking method treats mirror image cases of fuzzy numbers with equal ranking (Wang et al., 2006). Wang et al. (2006) also proves that formulations in term of horizontal – x component and vertical – y component by both Cheng (1998) and Chu & Tsao (2002) dissatisfy their two properties of correct centroid formulations. The properties are

Property 1: If A_i and A_j are fuzzy numbers with their membership functions $\mu_{A_i}(x)$ and $\mu_{A_j}(x)$ respectively have the relation of $\mu_{A_j}(y)=\mu_{A_i}(x)$, where $y=x+\delta$, then $x^*\left(A_j\right)=x^*(A_i)+\delta$, $y^*\left(A_j\right)= y^*(A_i)$.

Property 2: I If A_i and A_j are fuzzy numbers with their membership functions $\mu_{A_i}(x)$ and $\mu_{A_j}(x)$ respectively have the relation of $\mu_{A_j}(y)=\mu_{A_i}(y)$, for all $y \in \mathcal{R}$, then $x^*\left(A_j\right)=x^*(A_i)$.

In Wang et al. (2006) centroid point formulation, horizontal – x component is calculated by associating the height of fuzzy numbers, w. For vertical – y component, it is computed by finding inverse function of membership function of fuzzy numbers. Even if the centroid point method proposed by Wang et al. (2006) is justified as correct based on the two aforementioned properties, there is no evidence that indicates that the method is suitable for ranking fuzzy numbers.

According to Shieh (2007), Wang et al.'s (2006) centroid point method is inappropriate for ranking fuzzy numbers as it dissatisfies the condition on computing the value of vertical – y component. In order to compute the vertical – y component value, the membership function of fuzzy numbers must always be the same even if x – axis and y – axis are changed in position (Shieh, 2007). Due to this, Shieh (2007) introduces a new vertical – y component for fuzzy numbers the where the component is computed using distance of an α - cut of a fuzzy number. It is worth mentioning here that the horizontal – x component by Shieh (2007) is the same as Wang et al. (2006). It is proven by Bakar et al. (2012) that the Shieh (2007) centroid point method satisfies properties of correct centroid point formulation by Wang et al. (2006).

Another ranking method is introduced in the literature of ranking fuzzy numbers where Chen & Chen (2007) incorporate the centroid point in the standard deviation formulation to replace the mean. In Chen & Chen (2007), fuzzy numbers with greater standard deviation are ranked lower than other fuzzy numbers under consideration. This method ranks all cases of trapezoidal fuzzy numbers appropriately but no discussion is made on other types of fuzzy numbers.

Further investigation on finding appropriate ranking fuzzy numbers method is conducted by Wang & Lee (2008) where Chu & Tsao's (2002) ranking method on area between centroid and original point is enhanced. Using the same viewpoint as Murakami et al. (1983), Wang & Lee (2008) also considers the horizontal – x component as a more important component than vertical – y component in ranking fuzzy numbers. This is because multiplication process between the horizontal – x component and vertical – y component by Chu & Tsao (2002) reduces the importance of the horizontal – x component when ranking fuzzy numbers.

A wide – range study on the development of ranking of fuzzy numbers based on the centroid point method is thoroughly prepared by Ramli & Mohamad (2009) where the study investigates the advantages and weaknesses of all centroid point based methods in the literature of ranking fuzzy numbers. In Ramli & Mohamad (2009), ranking methods by Yager (1981), Murakami et al. (1983), Chen & Chen (2003), Wang et al. (2006), Shieh (2007) and Wang & Lee (2008) are explicitly discussed. Nonetheless, no ranking method is introduced by Ramli & Mohamad (2009). In a research work done by Chen & Chen (2009), twelve benchmarking examples of fuzzy numbers that mimic real world situations are introduced. Using these benchmarking examples, many drawbacks by previous established work are discovered. Among them are ranking methods by Yager (1981) and Murakami et al. (1983) where both ranking methods treat embedded or fully overlapped fuzzy numbers of different spreads as equal ranking and are unable to calculate ranking value for singleton fuzzy numbers. Limitations of Cheng (1998), Chu & Tsao (2002), Chen & Chen (2007) ranking methods are also mentioned in Chen & Chen (2009). Another improvement of Chu & Tsao (2002) is introduced by Xu & Wei (2010) where the ranking method ranks symmetrical fuzzy numbers with the same centroid point appropriately and solves Cheng (1998) problem on ranking fuzzy numbers with their images well.

Later on, Dat et al. (2012) apply Shieh (2007) centroid point formulation to rank fuzzy numbers. In the study by Dat et al. (2012), all cases of fuzzy numbers are correctly ranked such that the ranking results are consistent with human intuition. However, in a research work by Bakar & Gegov (2014), drawback of Dat et al. (2012)

work on ranking embedded fuzzy numbers is discovered. In order to rank two embedded symmetrical fuzzy numbers of same shape but different spread, Dat et al. (2012) ranking method gives both fuzzy numbers with equal ranking values. The result is considered to be misleading as fuzzy numbers examined are not the same.

With respects to all ranking methods using centroid points mentioned above, it is noticeable that every ranking method performs its own advantages and weaknesses. It is also found that along with discussions made in this subsection, no single method which utilises centroid point is capable to rank all cases of fuzzy numbers appropriately. Hence, this study suggests that centroid point needs at least a complementary approach to ranking fuzzy numbers correctly such that the ranking results are consistent with human intuition. As far as research in ranking fuzzy numbers are concerned, a fuzzy number is ranked higher than other fuzzy numbers under consideration when it has the larger mean and lower spread values (Lee et al., 1999; Chu & Tsao, 2002; Chen & Chen, 2009). This indicates that the spread is suitable in complementing the centroid point in ranking fuzzy numbers. Therefore, in the following subsection, discussions on the utilisation of the spread method in ranking fuzzy numbers are reviewed.

2.4.2 Ranking Using Spread Approach

Spread in the literature of ranking fuzzy numbers is first proposed by Chen & Lu (2001) whereby it is defined based on total dominance of a fuzzy number. In Chen & Lu (2001) ranking method, an area dominance based approach is utilised where spreads of fuzzy numbers are calculated in determining the total dominance of fuzzy numbers. Total dominance of fuzzy numbers in this case reflects as the ranking value for each fuzzy number under consideration. According to Chen & Lu (2001), computation of the total dominance of fuzzy numbers is evaluated in accordance to decision maker's index of optimism which are classified into three namely pessimistic, optimistic and neutral. Using this method, large index of optimism implies that right area dominance is more important than area dominance on the left and vice versa. It is worth mentioning here that Chen & Lu's (2001) ranking method is capable to appropriately rank embedded fuzzy numbers and non – overlapping fuzzy numbers cases but gives incorrect ranking results such that the results are inconsistent with human intuition for non – normal fuzzy numbers.

Chen & Lu (2002) gives another version of spread based – ranking method where the spread in this case is defined using both indices of quantity and quality aspects of a fuzzy number. Quantity index refers to dominance value of a fuzzy number which is expressed using α – cuts while quality index is signified by the ratio of signal and noise which is represented by midpoint and spreads of each α – cuts respectively (Chen & Lu, 202). According to Chen & Lu (2002), if a fuzzy number is described with stronger signal but weaker noise than other fuzzy numbers under consideration, then the fuzzy number is ranked higher than the others. Utilisation of various α – cuts in addressing the quality aspect of fuzzy numbers by Chen & Lu (2002), complements the ordering of fuzzy numbers where each fuzzy number is ranked by aggregating both quantity and quality aspects of a fuzzy number. It has to be noted that the ranking method by Chen & Lu (2002) is capable to rank many types of fuzzy numbers appropriately but discussion on non – normal fuzzy numbers is again neglected.

A different viewpoint in terms of formulation for the spread is introduced by Chen & Chen (2007) in ranking fuzzy numbers. According to Chen & Chen (2007), spread is defined as a standard deviation between the mean and points along the x – axis of a fuzzy number. Another research work by Chen & Chen (2009), same spread formulation as Chen & Chen (2007) method is used for ranking fuzzy numbers. Chen & Chen (2009) apply their ranking method on risk analysis problem but the ranking method produces incorrect ranking order such that the ranking result is inconsistent with human intuition on embedded fuzzy numbers of different spread. This is because Chen & Chen (2009) ranking method considers the spread as a component that is not as important as the centroid point and the height when ranking fuzzy numbers. A different direction on utilising the spread in ranking fuzzy numbers is prepared by Yu et al. (2013) where the ranking method also treats the spread as unimportant factor in ranking fuzzy numbers compared to centroid point. However, in Yu et al. (2013) investigation, the spread method is utilised when the centroid point gives incorrect ranking results such that the ranking results are inconsistent with intuition on cases of fuzzy numbers observed.

Although, literature on the spread of fuzzy numbers in ranking method is not as extensive as the centroid point, the spread is crucial whene cases of fuzzy numbers are unsolved by the centroid point (Yu et al., 2013). Table 2.2 outlines summary of ranking methods that utilise centroid point and spread components.

Table 2.2: Summary of Components Used In Ranking Fuzzy Numbers Methods

Ranking Method	Component		
	Horizontal – *x*	Vertical - *y*	Spread
Yager (1981)	√		
Murakami et. al (1983)	√	√	
Cross & Setnes (1998)			
Cheng (1998)			√
Chen & Lu (2001)			√
Chen & Lu (2002)			
Chu & Tsao (2002)		√	
Chen & Chen (2003)		√	
Deng & Liu (2005)		√	
Wang et. al (2006)		√	
Shieh (2007)		√	
Chen & Chen (2007)		√	√
Wang & Lee (2008)		√	
Ramli & Mohamad (2009)			
Chen & Chen (2009)	√		√
Bakar et al. (2010)	√	√	
Xu &Wei (2010)		√	
Bakar et al.(2012)	√	√	√
Dat et al. (2012)	√	√	
Yu et al. (2013)	√		√
Zhang et al. (2014)	√		√
Bakar & Gegov (2014)	√	√	√

Note: '√'indicates component is used by given ranking method.

2.5 RESEARCH PROBLEMS

This section discusses research problems of this study. It covers gaps and limitations faced by established ranking methods when ranking fuzzy numbers. The following details signify gaps and limitations of the established methods in the literature of ranking fuzzy numbers.

The first main gap in the literature of ranking fuzzy numbers is the incapability of ranking methods on ranking some cases of fuzzy numbers appropriately. Cheng's (1998) ranking method is incapable to rank singleton fuzzy numbers as the method only takes into account fuzzy numbers with area such as triangular and trapezoidal fuzzy numbers. Hence, there is no ranking result obtained for singleton fuzzy numbers when Cheng (1998) ranking method is used. Another drawback by Cheng (1998) ranking method is the method distinguishes embedded fuzzy numbers of different spreads with incorrect result such that the ranking result is inconsistent with human intuition because this method only considers fuzzy numbers with same spread. Apart from Cheng (1998), Chen & Lu (2001) ranking method is found out to have limitation on appropriately ranking fuzzy numbers of non – normal as the method considers only fuzzy numbers which are normal. A different weakness is found in Chu & Tsao (2002) ranking method where this method is unable to treat singleton fuzzy numbers well and provides incorrect ranking order such that the ranking result is inconsistent with human intuition for most cases of embedded fuzzy numbers. A crucial decision making problem is not covered by Cheng (1998), Chu & Tsao (2002) and Wang et al. (2006) where all of them neglect negative fuzzy numbers in their analyses. Chen & Chen (2009), Bakar et al. (2010) and Dat et al. (2012) give incorrect ranking order such that the ranking result is inconsistent with human intuition on embedded fuzzy numbers of different shapes and spreads.

The second main gap in the literature of ranking fuzzy numbers is there are some established ranking methods that are not applicable to solve decision making problems. With regards to discussions made on the first gap in terms of ranking fuzzy numbers, it is worth considering that some of the aforementioned ranking methods are capable to deal with real decision making problems appropriately while some provide inappropriate results. This is due to the fact that every ranking method has their own strength and weaknesses when dealing with fuzzy numbers. Thus, capabilities of each ranking method introduced in solving real decision making problems are vary from one to another.

Though the literature of ranking fuzzy numbers indicates that methods for ranking fuzzy numbers is extensive, gaps and limitations faced by established research works are still unsolved. Therefore, this study is carried out to solve these limitations appropriately.

2.6 RESEARCH QUESTIONS

This section lists relevant research questions based on research problems mentioned in Section 2.5 shown as follows.

a) Is there any established ranking method that integrates centroid point and spread in their formulation which is capable to correctly rank all types of fuzzy numbers such that the ranking results are consistent with human intuition?

b) Is there any established ranking method in literature of ranking fuzzy numbers which is capable to produce correct ranking results such that the ranking results are consistent with human intuition for every type of fuzzy numbers considered in literature, and efficiently rank more than two fuzzy numbers at one time or simultaneously?

c) Is there any established ranking method in literature of ranking fuzzy numbers which is capable to consistently and efficiently solving real decision making problem correctly such that the results are consistent with human intuition?

2.7 RESEARCH OBJECTIVES

This study embarks on the following objectives which are in accordance with Section 2.6.

a) To develop a methodology for ranking type – I fuzzy numbers based on centroid point and spread.

b) To extend the methodology for ranking type – I fuzzy numbers based on centroid point and spread on ranking type – II fuzzy number and Z – numbers.

c) To validate the consistency and efficiency of the methodology for ranking type – I fuzzy numbers based on centroid point and spread, its extension on ranking type – II fuzzy number and Z – numbers theoretically and empirically.

d) To develop theoretical properties and benchmark test sets for Z – numbers.

e) To apply the methodology for ranking type – I fuzzy numbers based on centroid point and spread, its extension on ranking type – II fuzzy number and Z – numbers to established decision – making case studies in the literature of fuzzy sets.

2.8 RESEARCH CONTRIBUTIONS

This section points out the main contribution of this study, especially in ranking fuzzy numbers. There are three main contributions displayed by this study where all of them are based on Section 2.7 and are described as follows.

The first main contribution of this study is that the development of methodology for ranking fuzzy numbers based on centroid point and spread is proposed to solve gaps and limitations by established works as mentioned in section 2.5. Development work on the ranking method is validated using established benchmarking examples of fuzzy numbers, namely overlapping and non – overlapping fuzzy numbers, embedded and trivial cases of fuzzy numbers. This ensures that the ranking method proposed ranks fuzzy numbers correctly such that the ranking results are consistent with human intuition.

The second contribution of this study is that the suggested ranking method in the first highlight is extended to a methodology for ranking fuzzy numbers. This extension points out in this study to illustrate the significant capability of the suggested work to ranking other types of fuzzy numbers. The methodology is examined in terms of its consistency and efficiency to ranking fuzzy numbers using both theoretical and empirical validations.

The third contribution of this study is that the methodology suggested in the second significant is applied to solve real decision making case studies in the literature of fuzzy sets. These implementations are necessary as in fuzzy decision making environment, fuzzy numbers are utilised as data representation. Thus, this indicates that the proposed ranking method is introduced not only to rank fuzzy numbers but able to solve decision making problems.

2.9 SUMMARY

In this chapter, a literature review with regards to this study is presented. Notions underlying fuzzy sets are first discussed in this chapter and this is followed by developments of fuzzy sets. Literature on ranking fuzzy numbers is then reviewed whereby thorough reviews on centroid point based ranking method and ranking method based on spread are explicitly illustrated. Later on, the research problem, research objectives and research highlights are presented such that all of them are gaps, targets and contributions by this study respectively. In Chapter 3, the thesis discusses the theoretical preliminaries of this study.

CHAPTER THREE

THEORETICAL PRELIMINARIES

3.1 INTRODUCTION

This chapter illustrates theoretical preliminaries of the thesis. It discusses fuzzy concepts and terminology used throughout the thesis where some of the concepts are defined using definitions by the experts while the remaining concepts are provided with theoretical proves. Details on those aforementioned points are intensively discussed in sections and subsections provided in this chapter.

3.2 BASIC UNDERSTANDING OF FUZZY SET

Many research articles in the literature of decision making indicate that the classical set theory serves as a useful tool in solving decision making problems. It defines the membership degree of elements in a set using binary representation of 0 and 1 to indicate whether an element is not a member and a member of a set respectively. If weather condition for today is considered as an example, then today weather is either 'hot' or 'not hot' when the classical sets are used. However, consideration only to two binary terms by classical sets is inadequate as human perceptions are vary among people, as different people employ different types of perceptions which are vague and fuzzy (Cheng, 1998).

Due to the limitation of the classical sets, fuzzy sets theory is introduced in decision making environment as dealing with situations that are fuzzy in nature is important. In contrast with classical sets, fuzzy sets theory allows gradual assessments of an element's degree of belongingness in the interval of 0 and 1 where these values indicate variation in terms of human perceptions about a situation perceived. Using definition by Cheng (1998), definition of fuzzy sets is given as follows.

Definition 3.1 (Cheng, 1998) A *fuzzy set* A_i in a universe of discourse U is characterized by a membership function $\mu_{A_i}(x)$ which maps each element x in U such that x is real number in the interval [0, 1].

Membership function for A_i, $\mu_{A_i}(x)$ is given as

$$\mu_{A_i}(x): X \rightarrow [0,1] \tag{3.1}$$

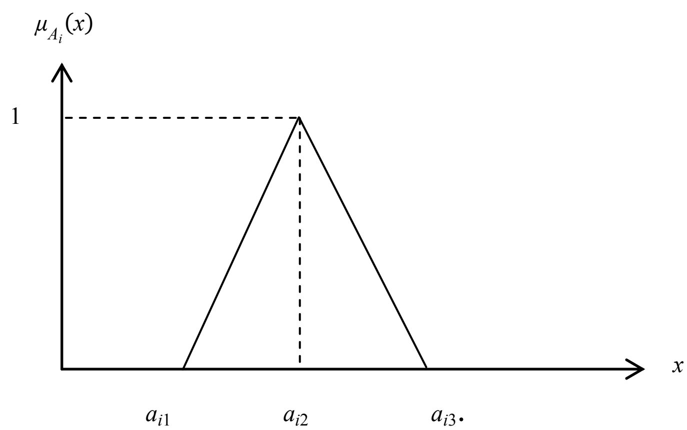


Fig 3.1: Membership function of a fuzzy set

Equation (3.1) and Figure (3.1) indicate that value of membership degree of fuzzy set is defined within interval [0, 1]. For instance, if $\mu_{hot}(x)$ is defined as membership function of 'hot' as weather condition for today and the membership value is approaching 0, then x is closer to 'not hot' or 'very hot'. In contrary, x is closer to 'hot' when the membership value is approaching 1. The following Table 3.1 illustrates differences between classical set theory and fuzzy set theory.

Table 3.1: Differences between classical sets and fuzzy sets theories

Theory	Representation	Membership degree
Classical	Binary	0 and 1
Fuzzy	Gradual	[0, 1]

3.2.1 Basic Fuzzy Sets Operations

There are three basic operations of fuzzy sets defined in the literature of fuzzy sets namely fuzzy union, fuzzy intersection and fuzzy complement. All of these operations are defined in Klir (1997) by the following definitions.

Let A_i and A_j be two fuzzy subsets of the universal interval U with membership functions for A_i and A_j are denoted by $\mu_{A_i}(x)$ and $\mu_{A_j}(x)$ respectively. Definitions of fuzzy union, fuzzy intersection and fuzzy complement based on Klir (1997) are given as

a) Fuzzy union of A_i and A_j is denoted by $A_i \cup A_j$ such that the membership function is defined as

$$\mu_{A_i \cup A_j}(x) = \max\left[\mu_{A_i}(x), \mu_{A_j}(x)\right], \text{ for all } x \in U$$

b) Fuzzy intersection of A_i and A_j is denoted by $A_i \cap A_j$ such that the membership function is defined as

$$\mu_{A_i \cap A_j}(x) = \min\left[\mu_{A_i}(x), \mu_{A_j}(x)\right], \text{ for all } x \in U$$

c) Fuzzy complement of A_i is denoted by $\mu_{\overline{A}_i}(x)$ such that the membership function is defined as

$$\mu_{\overline{A}_i}(x) = 1 - \mu_{A_i}(x), \text{ for all } x \in U$$

3.3 FUZZY NUMBERS

As discussed in section 2.3, three types of fuzzy numbers are pointed out in the literature of fuzzy sets namely type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers where all of them are defined chronologically as follows.

3.3.1 Type – I Fuzzy Numbers

In subsection 2.3.1, type – I fuzzy number is chronologically developed as the first fuzzy numbers are established in literature of fuzzy sets (Zadeh, 1965). As fuzzy numbers are actually type – I fuzzy numbers, definition of fuzzy number given by Dubois & Prade (1983) which reflects as the definition of type – I fuzzy number, is as follows.

Definition 3.2: (Dubois & Prade, 1978) A *type – I fuzzy number* A_i is a fuzzy subset of the real line $\mathcal{R}$ that is both convex and normal and satisfies the following properties:

i. μ_{A_i} is a continuous mapping from $\mathcal{R}$ to the closed interval $[0,w]$, $0 \leq w \leq 1$,

ii. $\mu_{A_i}(x)=0$, for all $x \in [-\infty, a]$,

iii. μ_{A_i} is strictly increasing on $[a,b]$,

iv. $\mu_{A_i}(x)=w$, for all $x \in [b,c]$ where w is a constant and $0 \leq w \leq 1$,

v. μ_{A_i} is strictly decreasing on $[c,d]$,

vi. $\mu_{A_i}(x)=0$, for all $x \in [d, \infty]$,

where $a \leq b \leq c \leq d$; a, b, c and d are components of a type – I fuzzy number and real while w represents the height of a type – I fuzzy number.

3.3.2 Type – II Fuzzy Numbers

Type – II fuzzy numbers are developed in the literature of fuzzy sets as the extension of type – I fuzzy numbers as the capability of type – I fuzzy numbers to represent human perception is inadequate (Walsten & Budescu, 1995). As type – II fuzzy sets are used in this stud, thus definition of type – II fuzzy sets by Mendel et al. (2006) is as follows.

Definition 3.3: (Mendel et al., 2006) A *type – II fuzzy set* A_i in a universe of discourse U is characterized by a type – II membership function $\mu_{A_i}(x)$ which maps each element x in U a real number in the interval [0, 1].

The membership function for A_i, $\mu_{A_i}(x)$ is given as

$$A_i = \{((x,u), \mu_{A_i}(x,u)) | \forall x \in U, \forall u \in J_x \subseteq [0,1], 0 \leq \mu_{A_i}(x,u) \leq 1\} \tag{3.2}$$

where J_x represents an interval in [0, 1].

According to Mendel et al. (2006), another representation of type – II fuzzy set is given in the following equation depicted as

$$A_i = \int_{x \in U} \int_{u \in J_x} \mu_A(x,u)/(x,u) \tag{3.3}$$

where $J_x \subseteq$[0, 1] and $\int\int$ represents the union over all allowable x and u.

It has to be noted that from equation (3.3), if $\mu_A(x,u) = 1$, then A_i is known as an interval type – IIfuzzy set. It is worth mentioning that interval type – II fuzzy set is a special case of type – II fuzzy set (Mendel et al., 2006) where it can be represented by the following equation

$$A_i = \int_{x \in U} \int_{u \in J_x} 1/(x,u) \tag{3.4}$$

where $J_x \subseteq$[0, 1].

Interval type – II fuzzy set is utilised in this study as this is the frequently used type – II fuzzy set in the literature. According to Zadeh (1975), representation of interval type – II fuzzy set using number is called as interval type – II fuzzy numbers. The following Figure 3.2 illustrates interval type – II fuzzy number.

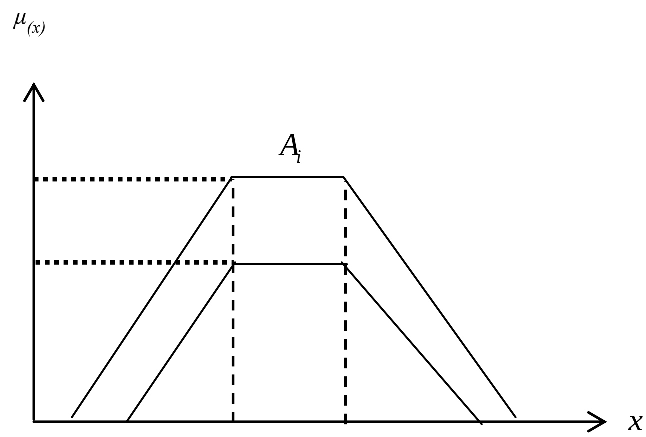


Fig 3.2: Interval type – II fuzzy number

It is noticeable that type – II fuzzy number in Figure 3.2 is more complex than than type – I in terms of representation where this indicates that type – II fuzzy number needs a more complicated computational technique than type – I fuzzy number. According to Greenfield & Chiclana (2013), there are numerous defuzzification strategies developed in the literature of fuzzy sets which plan on converting type – II fuzzy number into type – I fuzzy number. This strategy is intentionally introduced to reduce the complexity of type – II fuzzy numbers without losing information on the computational results. Among them that consider this strategy are Karnik & Mendel (2001), Nie & Tan (2008), Wu & Mendel (2009) and Greenfield & Chiclana (2009; 2013). Nevertheless, based on a thorough comparative analysis made by Greenfield and Chiclana (2013) on all the aforementioned methods, Nie & Tan (2008) reduction method outperforms other approaches on reducing type – II fuzzy number into type – I fuzzy number. Therefore, without loss of generality of Nie & Tan (2008), the reduction method is as follows.

$$\mu_T(x_A) = \frac{1}{2}\left(\mu_L(x_A) + \mu_U(x_A)\right) \tag{3.5}$$

where T is the resultant type – I fuzzy numbers.

3.3.3 Z – Number

According to Zadeh (2011), Z – numbers are the newest type of fuzzy numbers introduced in the literature of fuzzy sets. Definition of Z – numbers given by Kang et al. (2012) is as follows.

Definition 3.4: (Kang et al., 2012) A *Z – number* is an ordered pair of fuzzy number denoted as $Z = (\tilde{A}, \tilde{B})$. The first component, $\tilde{A}$ is known as the restriction component where it is a real – valued uncertain on X whereas the second component $\tilde{B}$, is a measure of reliability for $\tilde{A}$. The following Figure 3.3 illustrates Z – number based on Kang et al. (2012) definition.

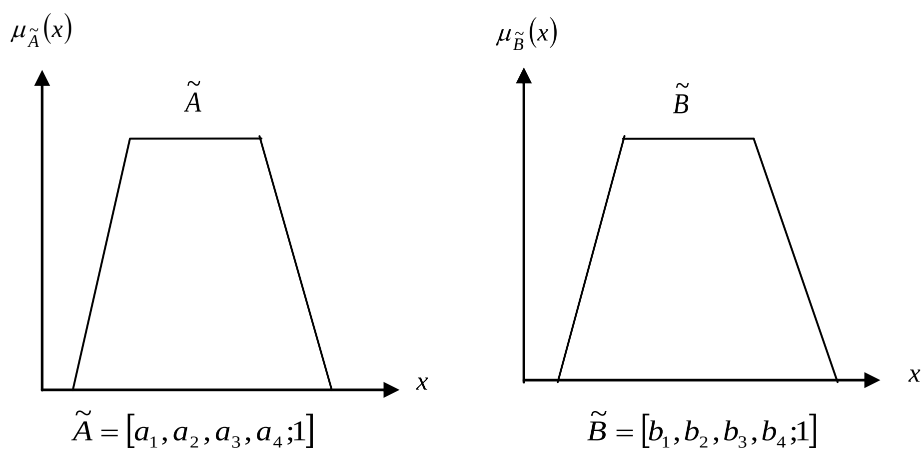


Fig 3.3: A Z – number, $Z = (\tilde{A}, \tilde{B})$

As mentioned in Chapter 2, Z – numbers are better in terms of their representation as compared to type – I fuzzy number and type – II fuzzy number fuzzy numbers. This is due to the fact that Z – numbers (level 3) are classified as the highest level in terms of generalised numbers than type – I fuzzy number and type – II fuzzy number which level 2 (Zadeh, 2011). Therefore, Zadeh (2011) suggests any computational work involving Z – numbers needs first reduce the Z – numbers into certain level without losing the informativeness of the computational results. This suggestion is taken into account by Kang et al. (2012a) where a method of converting *Z* – numbers into fuzzy numbers based on Fuzzy Expectation of a fuzzy set is proposed. With no loss of generality of Kang et al. (2012a) work, the conversion of *Z* – numbers into fuzzy numbers is as follows.

Step 1: Convert the reliability component, $\tilde{B}$ into a crisp number, α using the following equation

$$\alpha = \frac{\int_{-\infty}^{\infty} x\mu_{\tilde{B}}(x)dx}{\int_{-\infty}^{\infty} \mu_{\tilde{B}}(x)dx} \tag{3.6}$$

Note that, α represents the weight of the reliability component of a Z – number.

Step 2: Add the weight of the reliability component $\tilde{B}$ to the restriction component $\tilde{A}$. The *Z* – number is now defined as weighted restriction of Z – number and can be denoted as

$$\tilde{Z}^{\alpha} = \left\{ \left\langle x, \mu_{\tilde{A}^{\alpha}}(x) \right\rangle \middle| \mu_{\tilde{A}^{\alpha}}(x) = \alpha\mu_{\tilde{A}}(x), x \in [0,1] \right\}. \tag{3.7}$$

Step 3: Convert the weighted restriction of Z – number into a fuzzy number which can be represented as

$$\tilde{Z}' = \left\{ \left\langle x, \mu_{\tilde{Z}'}(x) \right\rangle \middle| \mu_{\tilde{Z}'}(x) = \mu_{\tilde{A}}\left(\frac{x}{\sqrt{\alpha}}\right), x \in [0,1] \right\}. \tag{3.8}$$

In Kang et al. (2012), it is shown that the process of converting *Z* – numbers into fuzzy numbers was sensible and logical because the result obtained by the study indicates that a *Z* – number is reduced into a lower level of generality which is a fuzzy number, but the computational informativeness is unaffected. Moreover, the conversion of a Z – number into a fuzzy number is reasonable due to the fact that both $\tilde{Z}^{\alpha}$ and $\tilde{Z}'$ are basically the same when the Fuzzy Expectation Theorem is applied.

3.4 FORMS OF FUZZY NUMBERS

This section covers discussions in terms of several forms of fuzzy numbers which are found in the literature of fuzzy sets. It has to be noted that all descriptions provided in this section focus only on type – I fuzzy numbers. As for type – II and Z – numbers, their discussions are similar to in type – I fuzzy numbers as both type – II numbers and Z – numbers are extension of type – I fuzzy numbers. Therefore, any descriptions of type – I fuzzy numbers provide in the following subsections are applicable to type – II fuzzy numbers and Z – numbers as well. Therefore, a generic term fuzzy numbers is used in this case to indicate that it covers type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers.

3.4.1 Linear Fuzzy Numbers

According to Chen & Chen (2003), fuzzy numbers are divided into two types namely linear and non – linear. Nevertheless, linear fuzzy numbers are often used in many decision making situations as non – linear fuzzy numbers are too complex to handle and they are normally transformed into linear type for convenience (Chen & Linkens, 2004). In literature of fuzzy sets, there are two linear types fuzzy numbers which are often utilised namely triangular and trapezoidal fuzzy numbers. Nonetheless, there is another fuzzy number that is rather extensively used in the literature of decision making which is a singleton fuzzy number. It is worth mentioning here that all of these mentioned fuzzy numbers are used throughout the thesis. Thus, the following definition 3.5 and Figure 3.4 are definition and illustrations of triangular fuzzy number respectively while definition (3.6) and Figure (3.5) are definition and illustration for trapezoidal fuzzy number respectively.

Definition 3.5: (Laarhoven & Pedrycz, 1983) A *triangular fuzzy number* A_i is represented by the following membership function. Figure 3.1 illustrates triangular fuzzy numbers.

$$\mu_{A_i}(x) = (a_{i1}, a_{i2}, a_{i3}) = \begin{cases} \dfrac{x - a_{i1}}{a_{i2} - a_{i1}} & if \quad a_{i1} \le x \le a_{i2} \\ \dfrac{a_{i3} - x}{a_{i3} - a_{i2}} & if \quad a_{i2} \le x \le a_{i3} \\ 0 & otherwise \end{cases}$$

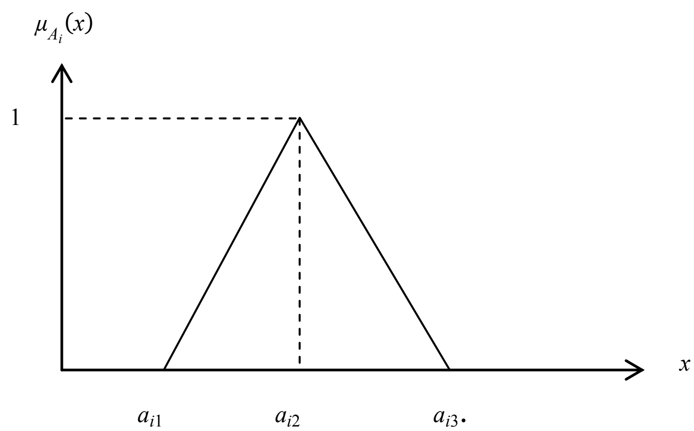


Fig 3.4: A Triangular Fuzzy Number

Definition 3.6: A *trapezoidal fuzzy number* A_i is represented by the following membership function given by

$$\mu_{A_i}(x) = (a_{i1}, a_{i2}, a_{i3}, a_{i4}) = \begin{cases} \dfrac{x - a_{i1}}{a_{i2} - a_{i1}} & if \quad a_{i1} \le x \le a_{i2} \\ 1 & if \quad a_{i2} \le x \le a_{i3} \\ \dfrac{a_{i4} - x}{a_{i4} - a_{i3}} & if \quad a_{i3} \le x \le a_{i4} \\ 0 & if \quad otherwise \end{cases}$$

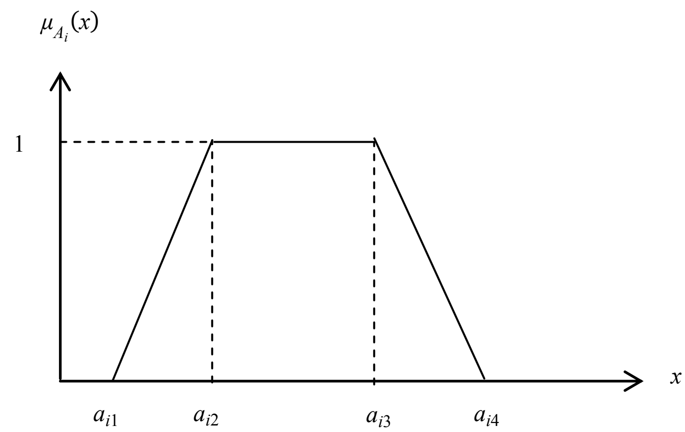


Fig 3.5: A Trapezoidal Fuzzy Number

It has to be noted here that for trapezoidal fuzzy numbers, if $a_{i2} = a_{i3}$, then a fuzzy number is in the form of a triangular fuzzy number (Cheng, 1998). While, if $a_{i1} = a_{i2} = a_{i3} = a_{i4}$ or $a_{i1} = a_{i2} = a_{i3}$ for both trapezoidal and triangular fuzzy numbers, respectively, then both are in the form of singleton fuzzy number (Chen & Chen, 2009). The following Figure 3.6 illustrates singleton fuzzy numbers.

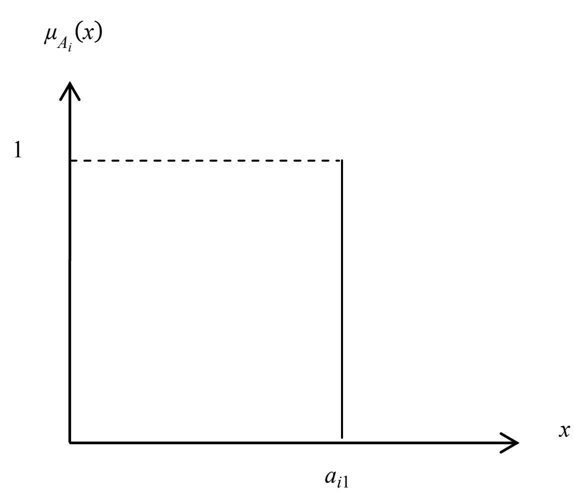


Fig 3.6: A Singleton Fuzzy Number

3.4.2 Generalised Fuzzy Numbers

This subsection provides discussions on another form of fuzzy numbers which is generalised fuzzy numbers. According to Chen & Chen (2003), a fuzzy number is better represented by generalised fuzzy numbers. This is because generalised fuzzy numbers provide a consistent representation for any fuzzy number even if any shape of fuzzy numbers is utilised. It has to be noted here that starting from this point until the last part of this chapter, only trapezoidal fuzzy numbers are utilised as medium of representation. This is due to the fact that both triangular and singleton fuzzy numbers are special cases of trapezoidal fuzzy numbers (Cheng, 1998 and Chen & Chen, 2003). Therefore, without loss of generality, definition of generalised trapezoidal fuzzy numbers is as follows.

Definition 3.7: (Chen & Chen, 2003) *Generalised Trapezoidal Fuzzy Number* A_i is a fuzzy number $A_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}\,;\, w_{A_i})$ where $0 \le a_{i1} \le a_{i2} \le a_{i3} \le a_{i4} \le 1$ with height, $w_{A_i} \in [0,1]$.

As consideration only on positive values by generalised fuzzy numbers limits the capability of fuzzy numbers on decision making, Chen & Chen (2007) extend generalised fuzzy numbers to standardised generalised fuzzy numbers so that both positive and negative values are considered in the analysis. Based on Chen & Chen (2007), definition of standardised generalised fuzzy numbers is given as follows.

Definition 3.8: (Chen & Chen, 2007) If fuzzy number A_i has the property such that $-1 \le a_{i1} \le a_{i2} \le a_{i3} \le a_{i4} \le 1$, then $\tilde{A}_i$ is called a *standardised generalised trapezoidal fuzzy number* and is denoted as

$$\tilde{A}_i = \left(\tilde{a}_{i1}, \tilde{a}_{i2}, \tilde{a}_{i3}, \tilde{a}_{i4}; w_{A_i}\right)$$

Any non – generalised fuzzy number is transformed into standardised generalised fuzzy numbers using a normalisation process depicted in equation (3.9).

$$\tilde{A}_i = \left(\frac{a_{i1}}{|k|}, \frac{a_{i2}}{|k|}, \frac{a_{i3}}{|k|}, \frac{a_{14}}{|k|}; w_{A_i} \right)$$

$$= \left(\tilde{a}_{i1}, \tilde{a}_{i2}, \tilde{a}_{i3}, \tilde{a}_{i4}; w_{A_i} \right) \tag{3.9}$$

where $|k| = \max(a_{i1}, a_{i2}, a_{i3}, a_{i4})$.

It is worth mentioning here that in the normalisation process, only components of fuzzy numbers are changed where a_{i1}, a_{i2}, a_{i3}, a_{i4} change to $\tilde{a}_{i1}, \tilde{a}_{i2}, \tilde{a}_{i3}, \tilde{a}_{i4}$, but this does not apply to the height of fuzzy number (Chen & Chen, 1986).

3.5 COMPONENTS OF FUZZY NUMBERS

This section illustrates components of fuzzy numbers utilised in this study. It is worth mentioning that many components of fuzzy numbers are discussed in the literature of fuzzy sets but only components that are related to this study are considered in this section. Details with regards to components of fuzzy numbers considered in this study are described extensively as follows.

3.5.1 Centroid Point

Section 2.4 highlights some important points of centroid points in ranking fuzzy numbers where it consists of two values namely horizontal – x value and vertical – y value. Wang (2009) defined a centroid point, as in Figure 3.8, as a point which is situated at the middle of a fuzzy number which reflects as a representation of a fuzzy number using crisp value. The conversion of fuzzy numbers into one crisp value for each horizontal – x value and vertical – y value are known as defuzzification. In the literature of fuzzy sets, some research works used only the horizontal – x value while some considered both horizontal – x value and vertical – y value. Nonetheless, in this study, both values are considered and are used throughout the thesis as considering only horizontal – x value is inadequate in representing a fuzzy number (Murakami et al., 1983; Cheng, 1998, Chen & Chen, 2009; Dat et al., 2012). In order to obtain these values, formulas given by Shieh (2007) are utilised in this study. The following are centroid point formulation by Shieh (2007) which define horizontal – x value as

$$x^*_{A_i} = \frac{\int_{-\infty}^{\infty} x f(x) dx}{\int_{-\infty}^{\infty} f(x)\ dx} = \frac{1}{3}\left[a_{i1} + a_{i2} + a_{i3} + a_{i4} - \frac{(a_{i4}a_{i3} - a_{i1}a_{i2})}{(a_{i4}+a_{i3}) - (a_{i1}+a_{i2})}\right] \tag{3.10}$$

and vertical – y value as

$$y^*_{A_i} = \frac{\int_0^{w_{A_i}} \alpha |A_i{}^{\alpha}| d\alpha}{\int_0^{w_{A_i}} |A_i{}^{\alpha}|\ d\alpha} = \frac{w_{A_i}}{3}\left[1 + \frac{a_{i3} - a_{i2}}{(a_{i4}+a_{i3}) - (a_{i1}+a_{i2})}\right] \tag{3.11}$$

where $|A_i{}^{\alpha}|$ is the length of the α-cut of A_i and $\left(x^*_{A_i}, y^*_{A_i}\right)$ is the centroid point for fuzzy numbers A_i.

It is worth mentioning here that for standardised generalised fuzzy numbers, the centroid point for the fuzzy number $\widetilde{A}_i$ is denoted as $(x^*_{\widetilde{A}_i}, y^*_{\widetilde{A}_i})$ with $x^*_{\widetilde{A}_i} \in [-1,1]$ and $y^*_{\widetilde{A}_i} \in [0,1]$. Based on Wang *et al.* (2006), properties of the correct centroid formula are used to validating the centroid formula by Shieh (2007) which is shown as follows.

Property 1:

If A_i and A_j are standardised generalised fuzzy numbers with their membership functions $\mu_{A_i}(x)$ and $\mu_{A_j}(x)$ respectively are $\mu_{A_j}(y) = \mu_{A_i}(x)$, where $y = x + \delta$, then $x^*(A_j) = x^*(A_i) + \delta$, $y^*(A_j) = y^*(A_i)$.

Proof:

When $\mu_{A_j}(y) = \mu_{A_i}(x)$, $y = x + \delta$, we have $\mu_{A_j}(x+\delta) = \mu_{A_i}(x)$. We obtain from equation (3.10)

$$y^*(A_j) = \frac{w}{3}\left[1 + \frac{(a_3+\delta) - (a_2+\delta)}{\left((a_4+\delta)+(a_3+\delta)\right) - \left((a_1+\delta)+(a_2+\delta)\right)}\right]$$

$$=\frac{w}{3}\left[1+\frac{(a_3-a_2)+(\delta-\delta)}{\left((a_4+a_3)+(2\delta)\right)-\left((a_1+a_2)+(2\delta)\right)}\right]$$

$$=\frac{w}{3}\left[1+\frac{(a_3-a_2)}{(a_4+a_3)-(a_1+a_2)}\right]$$

$$= y^*(A_i).$$

Hence,

$$y^*\left(A_j\right)= y^*(A_i). \tag{3.12}$$

We also have,

$$x^*\left(A_j\right)=\frac{1}{3}\left[(a_1+\delta)+(a_2+\delta)+(a_3+\delta)+(a_4+\delta)-\frac{\left(a_4+\delta)(a_3+\delta\right)-(a_1+\delta)(a_2+\delta)}{(a_4+\delta)+(a_3+\delta)-\left(a_1+\delta)+(a_2+\delta\right)}\right]$$

$$=\frac{1}{3}\left[(a_1+a_2+a_3+a_4)+(4\delta)-\frac{(a_4a_3+a_3\delta+a_4\delta+\delta^2)-\left(a_1a_2+a_2\delta+a_1\delta+\delta^2\right)}{(a_4+a_3)-(a_1+a_2)}\right]$$

$$=\frac{1}{3}\left[\frac{((a_4+a_3)-(a_1+a_2))((a_1+a_2+a_3+a_4+(4\delta))-(a_4a_3)-(a_1a_2)+\delta((a_4+a_3)-(a_2+a_1))}{(a_4+a_3)-(a_1+a_2)}\right]$$

$$=\frac{1}{3}\left[\frac{\left(a_1+a_2+a_3+a_4\right)\left((a_4+a_3)-(a_1+a_2)\right)-(a_4a_3-a_1a_2)}{(a_4+a_3)-(a_1+a_2)}\right]+\frac{1}{3}\left[\frac{(4\delta-\delta)(a_4+a_3)-a_2+a_1}{(a_4+a_3)-(a_1+a_2)}\right]$$

$$=\frac{1}{3}\left[\frac{\left((a_1+a_2+a_3+a_4)\left((a_4+a_3)-(a_1+a_2)\right)\right)-(a_4a_3-a_1a_2)}{(a_4+a_3)-(a_1+a_2)}\right]+\frac{1}{3}\left[\frac{(3\delta)(a_4+a_3)-(a_2+a_1)}{(a_4+a_3)-(a_1+a_2)}\right]$$

$$=\frac{1}{3}\left[a_1+a_2+a_3+a_4-\frac{a_4a_3-a_1a_2}{a_4+a_3-a_1+a_2}\right]+\frac{1}{3}[(3\delta)]$$

$$=x^*(A_i)+\delta$$

Thus,

$$x^*(A_j) = x^*(A_i) + \delta \tag{3.13}$$

Property 2:

If A_i and A_j are standardised generalised fuzzy numbers with their membership functions $\mu_{A_i}(x)$ and $\mu_{A_j}(x)$ respectively have the relation $\mu_{A_j}(y) = w\,\mu_{A_i}(y)$ for all $y \in \mathcal{R}$, then $x^*(A_j) = x^*(A_i)$.

Proof:

From equation (3.11), we obtain

$$y^*(A_j) = w\left(\frac{w}{3}\left[1 + \frac{a_3 - a_2}{((a_4 + a_3)) - ((a_1 + a_2))}\right]\right)$$

$$= w\,y^*(A_i).$$

Hence,

$$y^*(A_j) = w\,y^*(A_i) \tag{3.14}$$

Then, from equation (3.10), we have

$$x^*(A_j) = \left(\frac{1}{3}\left[a_1 + a_2 + a_3 + a_4 - \frac{a_4a_3 - a_1a_2}{(a_4 + a_3) - (a_1 + a_2)}\right]\right)$$

$$= x^*(A_i).$$

Therefore, we have

$$x^*(A_j) = x^*(A_i). \tag{3.15}$$

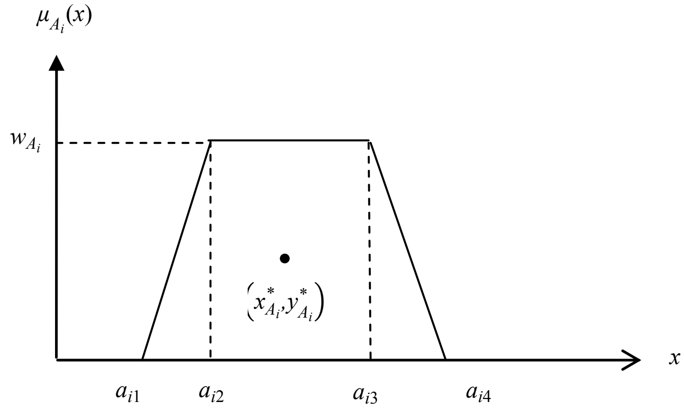


Fig 3.8: The Centroid Point, $\left(x^*_{A_i}, y^*_{A_i}\right)$ of A Trapezoidal Fuzzy Number

It is worth emphasising here that equation (3.12) until equation (3.15) indicate that centroid point formulation by Shieh (2007) is relevant and suitable for this study as the formulation fulfils both properties given by Wang et al. (2006).

3.5.2 Height

This subsection discusses the description of another basic component of fuzzy numbers which is height. Height of fuzzy numbers plays a very significant role in fuzzy decision making problems especially when confidence levels of decision makers vary. According to Chen & Chen (2003), if the height of a fuzzy number is high, then confidence level a decision maker is high. Based on Collan (2009), height of fuzzy numbers is defined as follows.

Definition 3.9: (Collan, 2009) *Height of fuzzy number* A_i is the largest value within a given set of $\mu_{A_i}(x)$ over X. The height of a fuzzy number is denoted as

$w_{A_i} = \sup \mu_{A_i}(x)$, where $w_{A_i} \in [0,1]$.

If the height of a fuzzy number A_i, is equal to 1, $w_{A_i} = 1$, then A_i is known as a *normal fuzzy number*. Otherwise it is called as a *non-normal fuzzy number*. In Figure 3.9, two fuzzy numbers with different heights are illustrated.

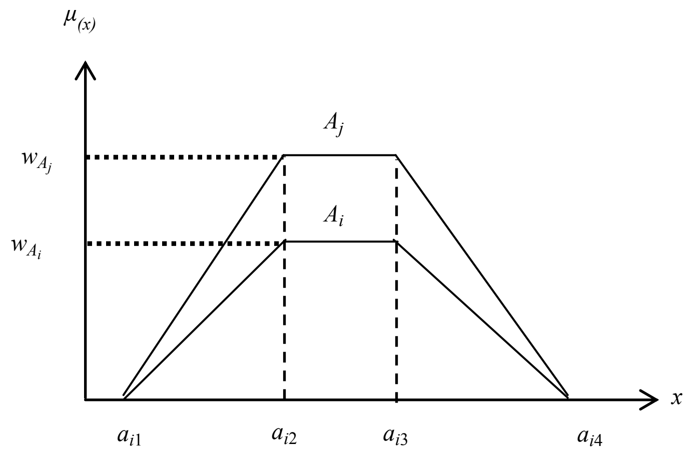


Fig 3.9: Two Fuzzy Numbers of Different Heights, w_{A_i} and w_{A_j}.

3.5.3 Spread

Spread is another component of fuzzy numbers which is important in fuzzy decision making. Main importance of spread in the decision making process is its capability to interpreting decision makers' viewpoints very well. According to Kwang & Lee (2000), different decision maker viewpoints are reflected with different spreads. This is due to the fact that the viewpoint of a decision maker is categorised into three namely pessimistic, normal and optimistic (Kwang & Lee, 1999). The following Figure 3.10 illustrates the maximum spread area of a fuzzy number (Chen & Chen, 2009) while definition of spread given by Lee & Li (1998) is as follows.

Definition 3.10: (Lee & Li, 1988) *Spread* is defined as the measure of variability length of the support of fuzzy numbers. In this case, it refers to the variability between points of fuzzy number with its centroid of horizontal – *x* value.

Definition 3.11: (Dutta et al., 2011) *Support* of fuzzy number *A* defined in *X* is the crisp set defined as

$$Supp(A) = \{x \in X : \mu_A(A) > 0\}$$

Using the method given by Chen & Lu (2001), spread of fuzzy number is expressed and calculated as

$$s_A = dist\ (a_4 - a_1) = |a_4 - x_A| + |x_A - a_1|$$

$$= |a_4 - a_1| \tag{3.16}$$

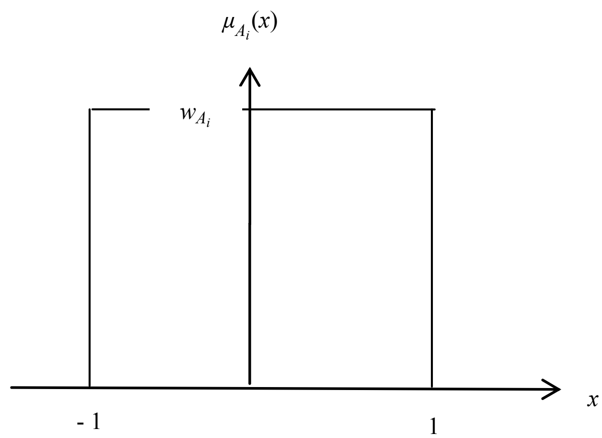


Fig 3.10: Maximum Spread Area of Fuzzy Number (Chen & Chen, 2009).

Lee & Li (1988) states that the spread is used in many ranking fuzzy numbers methodologies. This is also shown in Bakar & Gegov (2014) when the spread complements the capability of centroid point in ranking all cases of fuzzy numbers. According to Bakar & Gegov (2014), the role played by the spread is twofold namely complementing centroid point in ranking fuzzy numbers and supporting decision makers in the decision making process. This is illustrated when spread provides great effect in ranking fuzzy numbers especially when the centroid point is incapable to rank the fuzzy numbers of different spreads and embedded fuzzy numbers of different shapes. In addition, spread complements different types of decision makers namely pessimistic, neutral and optimistic in decision making process.

As mentioned earlier in this subsection, there are three types of decision makers which are pessimistic, neutral and optimistic. This has also been shown in the literature of decision making (Kwang & Lee, 1999 and Ramli & Mohamad, 2009). They basically view the same situation but define the situation using different interpretations. These variations in terms of decision makers' interpretations allow the utilisation of different spread when fuzzy numbers are used.

3.6 SUMMARY

In this chapter, the theoretical preliminaries of this thesis are presented. It covers definitions, terminology and fuzzy concepts utilised throughout the thesis. In Chapter 4, the thesis discusses the research methodology.

CHAPTER FOUR

RESEARCH METHODOLOGY

4.1 INTRODUCTION

This chapter illustrates details on the research methodology of the thesis. Main subject of this chapter focuses on development of the proposed novel methodology for ranking fuzzy numbers based on centroid point and spread. In developing the ranking methodology, a novel way of calculating the spread is proposed where this method is incorporated with an established centroid point method as a novel ranking fuzzy numbers approach. Since this is the first time the spread method is developed, the capability of the spread method in complementing the centroid point method for ranking fuzzy numbers is validated using relevant theoretical properties which are introduced in this study. As for the novel ranking methodology developed, it is validated based on theoretical and empirical validations which determine reliability, consistency and efficiency of the new ranking method. Reliability, a theoretical based – validation, validates the novel ranking methodology using several established ranking properties. The other two criteria namely consistency and efficiency, which are two distinct empirically based – validations, evaluate the capability of the novel ranking methodology to correctly rank fuzzy numbers such that the ranking results are consistent with human intuition and ranking more than two fuzzy numbers at one time respectively. Both theoretical and empirical validations mentioned are thoroughly defined in this chapter but their implementations are illustrated in the following three chapters of the thesis. This indicates that this chapter underpins the next three chapters of the thesis. Details on those aforementioned points are extensively discussed in sections and subsections of this chapter.

4.2 CENTROID POINT BASED – SPREAD METHOD

In this section, a novel formulation on calculating the spread of fuzzy numbers is developed. The novel spread method is a distance – based approach where it employs distance from the centroid point of a fuzzy number in obtaining the spread value. This spread method is an extension of Chen & Lu (2001) spread method where it considers both distances on horizontal x – axis and vertical – y axis to find the spread value of a fuzzy number. Chen & Lu (2001) spread method utilised only distance on horizontal x – axis to find the spread value, then involvement of both distances of the horizontal x – axis and vertical – y axis by the novel spread formulation is illustrated as follows.

Let $\tilde{A}_1 = \left(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4; w_{\tilde{A}_1}\right)$ be a standardised generalised trapezoidal fuzzy number and $\left(x_{\tilde{A}_1}, y_{\tilde{A}_1}\right)$ be the centroid point for $\tilde{A}_1$ such that $x_{\tilde{A}_1}$ and $y_{\tilde{A}_1}$ are the horizontal x – axis and vertical y – axis of the standardised generalised fuzzy number $\tilde{A}_1$, respectively. It has to be noted here that $x_{\tilde{A}_1}$ and $y_{\tilde{A}_1}$ are obtained using equations (3.10) and (3.11) respectively.

Step 1: Compute the distance along the horizontal x – axis of the standardised generalised fuzzy number $\tilde{A}_1$ using the following distance formula.

$$\begin{aligned} i_{\tilde{A}_1} = dist\left(\tilde{a}_4 - \tilde{a}_1\right) &= \left|\tilde{a}_4 - x_{\tilde{A}_1}\right| + \left|x_{\tilde{A}_1} - \tilde{a}_1\right| \\ &= \left|\tilde{a}_4 - \tilde{a}_1\right| \end{aligned} \tag{4.1}$$

where $i_{\tilde{A}_1}$ is the distance along horizontal x – axis of standardised generalised fuzzy number $\tilde{A}_1$.

Step 2: Find the distance on the vertical y – axis of standardised generalised fuzzy number $\tilde{A}_1$ which is given as

$$ii_{\tilde{A}_1} = y_{\tilde{A}_1} \quad (4.2)$$

In this step, distance on vertical y – axis, $ii_{\tilde{A}_1}$ is the same as the value of vertical y – axis. The purpose of introducing this step in the spread formulation is to address fuzzy numbers of different heights and cater limitation of Chen & Lu (2001) spread method. This is because spread value of a fuzzy number is not the same as other fuzzy numbers under consideration given if all of them are of different heights.

When both distances of horizontal x – axis and the vertical y – axis of a standardised generalised fuzzy number $\tilde{A}_1$ are obtained, spread value of the fuzzy number is then computed.

Step 3: Obtain spread value of standardised generalised fuzzy number $\tilde{A}_1$ using the following formula given as

$$s(\tilde{A}_1) = i_{\tilde{A}_1} \text{ x } ii_{\tilde{A}_1} \quad (4.3)$$

where $i_{\tilde{A}_1}$ and $ii_{\tilde{A}_1}$ are $dist(\tilde{a}_4 - \tilde{a}_1)$ and $y_{\tilde{A}_1}$ respectively.

$s(\tilde{A}_1)$, $i_{\tilde{A}_1}$, $ii_{\tilde{A}_1}$, $dist(\tilde{a}_4 - \tilde{a}_1) \in [0, 1]$ and equation (4.3) is a scalar multiplication of $i_{\tilde{A}_1}$ and $ii_{\tilde{A}_1}$

The following Figure 4.1 illustrates the components of spread namely the distance along the horizontal x – axis, $i_{\tilde{A}_1}$, distance on the vertical y – axis, $ii_{\tilde{A}_1}$, and the centroid point, ($x_{\tilde{A}_1}$, $y_{\tilde{A}_1}$) of fuzzy number $\tilde{A}_1$.

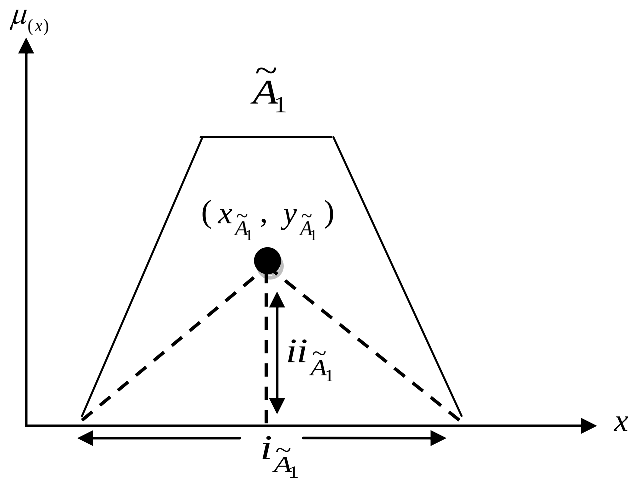


Fig 4.1: Component of spread, $i_{\tilde{A}_1}$ and $ii_{\tilde{A}_1}$ and the centroid point, $(x_{\tilde{A}_1}, y_{\tilde{A}_1})$ of fuzzy number $\tilde{A}_1$.

4.2.1 Illustrative Example

This subsection illustrates a numeric – based example adopted from Chen & Chen (2009) which is used to demonstrate the utilisation of the spread method developed in Section 4.2. Complete illustration of utilising the centroid point based spread method on this example is as follows.

Let $\tilde{A}_1$ = (0.1, 0.3, 0.3, 0.5; 1.0) be a standardised generalised fuzzy number for which to be calculated its spread and (0.3, 0.3333) as the centroid point for $\tilde{A}_1$ which is obtained using equations (3.10) and (3.11).

Step 1: Compute the distance along the horizontal x – axis of standardised generalised fuzzy number $\tilde{A}_1$ given as

$$
\begin{aligned}
i_{\tilde{A}_1} = dist(\tilde{a}_4 - \tilde{a}_1) &= |0.5 - 0.3| + |0.3 - 0.1| \\
&= |0.5 - 0.1| \\
&= 0.4
\end{aligned}
$$

Step 2: Find the distance on the vertical y – axis of standardised generalised fuzzy number $\widetilde{A}_1$.

Since, centroid of vertical y – value is the distance on the vertical y – axis, hence

$$ii_{\widetilde{A}_1} = y_{\widetilde{A}_1} = 0.3333$$

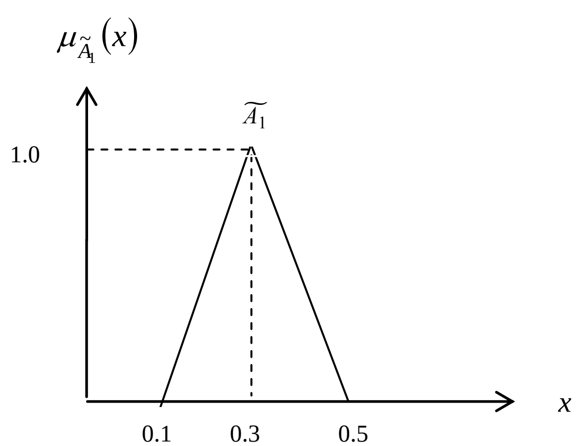
Step 3: Obtain the spread value of standardised generalised fuzzy number $\widetilde{A}_1$ using the following formula given as

$$s(\widetilde{A}_1) = 0.4 \times 0.3333$$

$$= 0.1333$$

Thus, spread value of fuzzy number $\widetilde{A}_1$ is 0.1333.

$s(\widetilde{A}_1)$, $i_{\widetilde{A}_1}$, $ii_{\widetilde{A}_1}$, $dist(\widetilde{a}_4 - \widetilde{a}_1) \in [0 ,1]$.



$\widetilde{A}_1$ = (0.1, 0.3, 0.3, 0,5; 1.0)

Fig 4.2: Fuzzy number $\widetilde{A}_1$.

4.2.2 Theoretical Validation

This subsection validates theoretically the proposed centroid point based spread method using several theoretical properties which are introduced in this study. These relevant properties justify the capability of the centroid point based spread method in complementing the centroid point in ranking fuzzy numbers. It is worth mentioning here that this validation focuses mainly on the embedded cases of fuzzy numbers where centroid point is incapable to rank them appropriately (Bakar & Gegov, 2014). Therefore, capability of centroid point based spread method in ranking fuzzy numbers especially on embedded case of fuzzy numbers is validated using the following theoretical properties.

Let $\tilde{A}_1$ and $\tilde{A}_2$ be trapezoidal and triangular standardised generalised fuzzy numbers respectively.

***Property A1*:** If $\tilde{A}_1$ and $\tilde{A}_2$ are embedded and having different centroid points but similar support, then $s(\tilde{A}_1) > s(\tilde{A}_2)$.

***Proof*:**

Since $\tilde{A}_1$ and $\tilde{A}_2$ are embedded and having similar support, hence it has to be noted that

$x_{\tilde{A}_1} = x_{\tilde{A}_2}$ and $y_{\tilde{A}_1} > y_{\tilde{A}_2}$ .Then, from equation (4.3), the following are obtained such that

$i_{\tilde{A}_1} = i_{\tilde{A}_2}$ and $ii_{\tilde{A}_1} > ii_{\tilde{A}_2}$. Therefore, $s(\tilde{A}_1) > s(\tilde{A}_2)$.

***Property A2*:** If $\tilde{A}_1$ is a singleton fuzzy numbers, then $s(\tilde{A}_1) = 0$.

***Proof*:**

For any crisp (real) numbers, it has to be noted that $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4$ implies that $i_{\tilde{A}_1} = 0$ and $ii_{\tilde{A}_1} = w/3$. Therefore, $s(\tilde{A}_1) = 0$.

***Property 3*:** If $\widetilde{A}_1$ is an asymmetrical triangular fuzzy numbers then $s(\widetilde{A}_1) = i_{\widetilde{A}_1}$ x $ii_{\widetilde{A}_1}$.

***Proof*:**

For any asymmetrical triangular fuzzy numbers, it is obvious that $\widetilde{a}_2 = \widetilde{a}_3 \neq x_{\widetilde{A}_1}$. By definition, the following are obtained where

$$dist(\widetilde{a}_4 - \widetilde{a}_3) + dist(\widetilde{a}_3 - \widetilde{a}_1) = dist(\widetilde{a}_4 - \widetilde{a}_2) + dist(\widetilde{a}_2 - \widetilde{a}_1) = dist(\widetilde{a}_4 - \widetilde{a}_1) = i_{\widetilde{A}_1} .$$

Therefore, $s(\widetilde{A}_1) = i_{\widetilde{A}_1}$ x $ii_{\widetilde{A}_1}$.

The above theoretical validation clearly signifies that the proposed centroid point based spread method is capable to complement centroid point in ranking fuzzy numbers. Although, the main focus of this validation is on the embedded case of fuzzy numbers, other cases of fuzzy numbers such as overlapping and non – overlapping cases of fuzzy numbers are well considered in this validation. This is because embedded case of fuzzy numbers is the only case which the centroid point method is incapable to deal with. For other cases, centroid point differentiate them appropriately. It is worth mentioning here that details with regard to embedded, overlapping and non – overlapping cases of fuzzy numbers are given later in Section 4.5. In the next section, the centroid point based spread method is incorporated with the centroid point approach to develop a novel ranking fuzzy numbers methodology.

4.3 HYBRID APPROACH FOR RANKING FUZZY NUMBERS

In this section, a novel methodology for ranking fuzzy numbers is proposed. The methodology is developed using the established centroid point method by Shieh (2007) and the novel spread approach presented in Section 4.2 where it is applied to ranking fuzzy numbers. As mentioned in Section 2.2, fuzzy numbers are a generic term for type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers, thus indicating that the novel ranking methodology is a ranking methodology for type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers. Therefore, illustrations of the ranking methodology are as follows.

4.3.1 Ranking Methodology for Type – I Fuzzy Numbers

This subsection illustrates the methodology for ranking type – I fuzzy numbers based on centroid point and spread, CPS_I which is given as follows.

Let $\widetilde{A}_1 = \left(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3, \widetilde{a}_4; w_{\widetilde{A}_1}\right)$ be a standardised generalised trapezoidal type – I fuzzy number and ranked.

Step 1: Calculate centroid point $\left(x^*_{\widetilde{A}}, y^*_{\widetilde{A}}\right)$ of standardised generalised type – I fuzzy number $\widetilde{A}_1$ using Shieh (2007). The horizontal – x centroid of type – I fuzzy number $\widetilde{A}_1$, $x^*_{\widetilde{A}_1}$ is calculated as

$$x^*_{\widetilde{A}_1} = \frac{\int_{-\infty}^{\infty} x f(x) dx}{\int_{-\infty}^{\infty} f(x) dx} \tag{4.4}$$

and the vertical – y centroid of the type – I fuzzy number $\widetilde{A}_1$, $y^*_{\widetilde{A}_1}$ is given as

$$y^*_{\widetilde{A}_1} = \frac{\int_0^{w_{\widetilde{A}_i}} \alpha \left|\widetilde{A}_i^{\alpha}\right| d\alpha}{\int_0^{w_{\widetilde{A}_i}} \left|\widetilde{A}_i^{\alpha}\right| d\alpha} \tag{4.5}$$

where

$\left|\widetilde{A}_i^{\alpha}\right|$ is length of α – cuts of type – I fuzzy number $\widetilde{A}_1$, $x^*_{\widetilde{A}_1} \in [-1 , 1]$ and $y^*_{\widetilde{A}_1} \in [0 , w_A]$.

Note that, the centroid point by Shieh (2007) used in this step is applied to standardised generalised type – I fuzzy numbers.

Step 2: Obtain spread value of standardised generalised type – I fuzzy number $\widetilde{A}_1$ using the following formula given as

$$s(\widetilde{A}_1) = i_{\widetilde{A}_1} \text{ x } ii_{\widetilde{A}_1} \tag{4.6}$$

where $i_{\widetilde{A}_1}$ and $ii_{\widetilde{A}_1}$ are $dist(\widetilde{a}_4 - \widetilde{a}_1)$ and $y^*_{\widetilde{A}_1}$ respectively.

$s(\widetilde{A}_1)$, $i_{\widetilde{A}_1}$, $ii_{\widetilde{A}_1}$, $dist(\widetilde{a}_4 - \widetilde{a}_1) \in [0, 1]$.

Note that, the spread formulation in equation (4.6) is the same as equation (4.3). This indicates that the centroid point based spread method developed in Section 4.2 is utilised in this step.

Step 3: Compute ranking value for $\widetilde{A}_1$ using *CPS* ranking method which is defined as

$$CPS_I(\widetilde{A}_1) = x^*_{\widetilde{A}_1} \times y^*_{\widetilde{A}_1} \times (1 - s(\widetilde{A}_1)) \tag{4.7}$$

where

$x^*_{\widetilde{A}_1}$ is horizontal – *x* centroid for standardised generalised type – I fuzzy number $\widetilde{A}_1$

$y^*_{\widetilde{A}_1}$ is horizontal – *y* centroid for standardised generalised type – I fuzzy number $\widetilde{A}_1$

$s(\widetilde{A}_1)$ is spread for standardised generalised type – I fuzzy number $\widetilde{A}_1$.

and $CPS_I(\widetilde{A}_1) \in [-1, 1]$.

If $CPS_I(\widetilde{A}_1) > CPS_I(\widetilde{A}_2)$, then $\widetilde{A}_1 \succ \widetilde{A}_2$. (i.e. $\widetilde{A}_1$ is ranked higher than $\widetilde{A}_2$).

If $CPS_I(\widetilde{A}_1) < CPS_I(\widetilde{A}_2)$, then $\widetilde{A}_1 \prec \widetilde{A}_2$. (i.e. $\widetilde{A}_1$ is ranked lower than $\widetilde{A}_2$).

If $CPS_I(\widetilde{A}_1) = CPS_I(\widetilde{A}_2)$, then $\widetilde{A}_1 \approx \widetilde{A}_2$. (i.e. the ranking for $\widetilde{A}_1$ and $\widetilde{A}_2$ is equal).

Notice that, $\left(1 - s\left(\tilde{A}_1\right)\right)$ is introduced in the ranking formulation to ensure that any type – I fuzzy number with greater spread value, $s(\tilde{A}_1)$, than other type – I fuzzy number under consideration is treated as the smallest type – I fuzzy number among them.

4.3.2 Ranking Methodology for Interval Type – II Fuzzy Numbers

This subsection signifies the methodology for ranking type – II fuzzy numbers based on centroid point and spread, CPS_{II}. As there are two distinct ways of ranking type – II fuzzy numbers considered in the literature of fuzzy sets namely the direct and indirect, the CPS_{II} ranking method developed in this study also takes into account both ways to demonstrate its capability to ranking type – II fuzzy numbers. It is worth mentioning here that the interval type – II fuzzy numbers are utilised in this study as they are the generalisation of type – II fuzzy numbers (Mitchel, 2006) and are viewed as the special case and require less computational works compared to type – II fuzzy numbers (Hu et al., 2013). Therefore, without loss of generality, definition of interval type – II fuzzy number is given as follows.

Let $\hat{A} = \left[\left(\hat{a}_1^U, \hat{a}_2^U, \hat{a}_3^U, \hat{a}_4^U; 1; 1\right)\left(\hat{a}_1^L, \hat{a}_2^L, \hat{a}_3^L, \hat{a}_4^L; w_{\hat{A}_{i1}}; w_{\hat{A}_{j1}}\right)\right]$ be an interval type – II fuzzy number whereby components $\hat{a}_i^U$ and $\hat{a}_i^L$ such that i = 1, 2, 3, 4, are the upper membership function, UMF and lower membership function, LMF respectively (Wu & Mendel, 2009). Notice that, $\hat{A}$ is transformed into standardised generalised interval type – II fuzzy numbers using the following normalisation steps which are proposed in this study.

If an interval type – II fuzzy number $\hat{A}$ has the property such that $-1 < a_1'^U < a_2'^U < a_3'^U < a_4'^U < 1$ and $-1 < a_1'^L < a_2'^L < a_3'^L < a_4'^L < 1$ then A' is called as a standardised generalised interval type – II trapezoidal fuzzy number and is denoted as

$$A' = \left[\left(a_1'^U, a_2'^U, a_3'^U, a_4'^U; 1; 1\right)\left(a_1'^L, a_2'^L, a_3'^L, a_4'^L; w_{A'_{i1}}; w_{A'_{j1}}\right)\right] \tag{4.8}$$

Any interval type – II fuzzy numbers may be transformed into a standardised generalised interval type – II fuzzy numbers by normalisation process as described in (3.4.2).

$$A' = \left[\left(\frac{\hat{a}_1^U}{|k|}, \frac{\hat{a}_2^U}{|k|}, \frac{\hat{a}_3^U}{|k|}, \frac{\hat{a}_4^U}{|k|};1;1\right)\left(\frac{\hat{a}_1^L}{|m|}, \frac{\hat{a}_2^L}{|m|}, \frac{\hat{a}_3^L}{|m|}, \frac{\hat{a}_4^L}{|m|}; w_{A'}^L; w_{A'}^L\right)\right]$$

$$= \left[\left(a_1'^U, a_2'^U, a_3'^U, a_4'^U;1;1\right)\left(a_1'^L, a_2'^L, a_3'^L, a_4'^L; w_{A'}^L; w_{A'}^L\right)\right] \tag{4.9}$$

where $|k| = \max\left(\hat{a}_1^U, \hat{a}_2^U, \hat{a}_3^U, \hat{a}_4^U\right), |m| = \left(\hat{a}_1^L, \hat{a}_2^L, \hat{a}_3^L, \hat{a}_4^L\right)$.

In the normalisation process, only the components of interval type – II fuzzy numbers where $\hat{a}_1^U, \hat{a}_2^U, \hat{a}_3^U, \hat{a}_4^U$ and $\hat{a}_1^L, \hat{a}_2^L, \hat{a}_3^L, \hat{a}_4^L$ are change to $a_1'^U, a_2'^U, a_3'^U, a_4'^U$ and $a_1'^L, a_2'^L, a_3'^L, a_4'^L$ respectively while the heights of interval type – II fuzzy numbers remain the same.

As there are two ways of ranking type – II fuzzy numbers found in the literature namely the direct and indirect ways, the capability of the CPS_{II} ranking method in ranking type – II fuzzy numbers using both ways are demonstrated as the following. Note that, the type – II fuzzy numbers utilised in this case are in the form of standardised generalised interval type – II fuzzy numbers.

Let $A' = \left[\left(a_1'^U, a_2'^U, a_3'^U, a_4'^U;1;1\right)\left(a_1'^L, a_2'^L, a_3'^L, a_4'^L; w_{A'}^L; w_{A'}^L\right)\right]$ be a standardised generalised interval type – II fuzzy number.

Step 1: Compute the centroid point for A' by finding the horizontal – x centroid using the following equation

$$x_{A'}^* = \frac{\int_{-\infty}^{\infty} xf(x)dx}{\int_{-\infty}^{\infty} f(x)dx} \tag{4.10}$$

and the vertical – y centroid value of A' as

$$y^*_{A'} = \frac{\int_0^{w_{\tilde{A}_i}} \alpha |A'| d\alpha}{\int_0^{w_{\tilde{A}_i}} |A'| d\alpha} \tag{4.11}$$

where

$|A'|$ is length of α - cuts of A'. $x^*_{A'} \in [-1\ ,\ 1]$ and $y^*_{A'} \in [0\ , w_A]$.

In this step, two centroid points are obtained for A' whereby the centroid points are $\left(x^{*\ U}_{A'}, y^{*\ U}_{A'}\right)$ and $\left(x^{*\ L}_{A'}, y^{*\ L}_{A'}\right)$ for each $\hat{a}_i^U$ and $\hat{a}_i^L$ respectively.

Step 2: Calculate the spread values for A' such that the distance along the x – axis from the horizontal – x is

$$i_{A'} = dist\left[\left(a'^U_4 - a'^U_1\right), \left(a'^U_4 - a'^U_1\right)\right] = \left[\left|a'^U_4 - x^{*\ U}_{A'}\right| + \left|x^{*\ U}_{A'} - a'^U_1\right|, \left|a'^L_4 - x^{*\ L}_{A'}\right| + \left|x^{*\ L}_{A'} - a'^L_1\right|\right]$$
$$= \left|a'^U_4 - a'^U_1\right|, \left|a'^L_4 - a'^L_1\right| \tag{4.12}$$

While the distance along the vertical y – axis from the vertical y – value is depicted as

$$ii_{A'} = y^{*\ U}_{A'}, y^{*\ L}_{A'} \tag{4.13}$$

Therefore, the spread of A', $s(A')$ is defined as

$$s(A') = i_{A'} \times ii_{A'}$$
$$= \left(\left|a'^U_4 - a'^U_1\right| \times y^{*\ U}_{A'}\right), \left(\left|a'^L_4 - a'^L_1\right| \times y^{*\ U}_{A'}\right)$$

where $i_{A'}$ and $ii_{A'}$ are $dist\left[\left(a'^U_4 - a'^U_1\right), \left(a'^U_4 - a'^U_1\right)\right]$ and $y^*_{A'}$ respectively.

$s(A')$, $i_{A'}$, $ii_{A'}$, $dist\left[\left(a'^U_4 - a'^U_1\right), \left(a'^U_4 - a'^U_1\right)\right] \in [0\ ,1]$.

This step also produces two values like in Step 1 but in this case, both values are the spread for $\hat{a}_i^U$ and $\hat{a}_i^L$ which are separated by ','.

Step 3: Determine the ranking value for A' using the following equation

$$CPS_{II}(A') = \bar{x}^*_{A'} \times \bar{y}^*_{A'} \times (1 - \bar{s}(A')) \tag{4.15}$$

where

$\bar{x}^*_{A'}$ is the average of the horizontal – x centroid for A'

$\bar{y}^*_{A'}$ is average of the vertical – y centroid for A'

$\bar{s}(A')$ is the average of the spread for A'.

$CPS_{II}(A') \in [-1, 1]$.

If $CPS_{II}(A') > CPS_{II}(B')$, then $A' \succ B'$. (i.e. A' is greater than B').
If $CPS_{II}(A') < CPS_{II}(B')$, then $A' \prec B'$. (i.e. A' is lesser than B').
If $CPS_{II}(A') = CPS_{II}(B')$, then $A' \approx B'$. (i.e. A' and B' are equal ranked).

Notice that, $(1 - \bar{s}(A'))$ is introduced in the ranking formulation to ensure that any type – II fuzzy number with greater spread value, $\bar{s}(A')$ than other type – II fuzzy number under consideration is treated as the smallest type – II fuzzy number among them. Computations on finding the average in Step 3 are introduced in this methodology to ensure that *CPSII* ranking method is applicable to ranking interval type – II fuzzy numbers. It is also worth adding that computation of average introduced in this methodology is a generalisation of Wu & Mendel (2009) work on ranking type – II fuzzy number using approximation to the end points of type – reduced interval (Greenfield & Chiclana, 2013).

Indirect Approach

This study defines the indirect way to ranking interval type – II fuzzy numbers as the involvement of additional process before the ranking procedure is carried out. In this case, interval type – II fuzzy numbers under consideration are reduced into other suitable form, which is type – I fuzzy numbers, before they are ranked accordingly. Consideration of the reduction process in this study is in line with reduction – based methods developed by Mendel (2001), Mendel & John (2002), Nie & Tan (2008), Greefield et al. (2009) and Greenfield & Chiclana (2012). Although, interval type – II fuzzy numbers are directly ranked by the CPS_{II} ranking method in the previous subsection, the indirect way for ranking interval type – II fuzzy numbers is also provided in this study as this is another direction found in the literature of fuzzy sets. As the indirect approach requires reduction of the interval type – I fuzzy numbers into type – I fuzzy numbers, this study first extends the definition of interval type – II fuzzy numbers in Definition (3.12) into standardised generalised interval type – II fuzzy numbers shown as follows.

Let $A' = \left[\left(a_1'^U, a_2'^U, a_3'^U, a_4'^U; 1; 1\right)\left(a_1'^L, a_2'^L, a_3'^L, a_4'^L; w_{A'}^L; w_{A'}^L\right)\right]$ be a standardised generalised interval type – II fuzzy number. $A' = \left[\left(a_1'^U, a_2'^U, a_3'^U, a_4'^U; 1; 1\right)\left(a_1'^L, a_2'^L, a_3'^L, a_4'^L; w_{A'}^L; w_{A'}^L\right)\right]$ is reduced into standardised generalised type – I fuzzy numbers using Nie – Tan (2008) reduction method shown as follows.

$$A' = \left[\left(\frac{a_1'^U + a_1'^L}{2}, \frac{a_2'^U + a_2'^L}{2}, \frac{a_3'^U + a_3'^L}{2}, \frac{a_4'^U + a'^L}{2}_4; \frac{1 + w_{A'}^L}{2}; \frac{1 + w_{A'}^L}{2}\right)\right]$$

After the reduction process, it is noticeable that A' is currently in the form of standardised generalised type – I fuzzy number, $\tilde{A}$ such that it is the same as $\tilde{A}$ defined in 4.3.1. Therefore, with no loss of generality, the procedure to indirectly rank interval type – II fuzzy numbers is as follows.

Step 1: Calculate centroid point $\left(x^*_{\tilde{A}}, y^*_{\tilde{A}}\right)$ of standardised generalised type – I fuzzy number $\tilde{A}_1$ using Shieh (2007). The horizontal – x value of type – I fuzzy number $\tilde{A}_1$, $x^*_{\tilde{A}_1}$ is calculated as

$$x^*_{\tilde{A}_1} = \frac{\int_{-\infty}^{\infty} x f(x) dx}{\int_{-\infty}^{\infty} f(x) dx} \tag{4.4}$$

and the vertical – y value of the type – I fuzzy number $\tilde{A}_1$, $y^*_{\tilde{A}_1}$ is given as

$$y^*_{\tilde{A}_1} = \frac{\int_0^{w_{\tilde{A}_i}} \alpha \left|\tilde{A}_i^\alpha\right| d\alpha}{\int_0^{w_{\tilde{A}_i}} \left|\tilde{A}_i^\alpha\right| d\alpha} \tag{4.5}$$

where

$\left|\tilde{A}_i^\alpha\right|$ is length of α – cuts of type – I fuzzy number $\tilde{A}_1$, $x^*_{\tilde{A}_1} \in [-1, 1]$ and $y^*_{\tilde{A}_1} \in [0, w_A]$.

Note that, the centroid point by Shieh (2007) used in this step is applied to standardised generalised type – I fuzzy numbers.

Step 2: Obtain spread value of standardised generalised type – I fuzzy number $\tilde{A}_1$ using the following formula given as

$$s(\tilde{A}_1) = i_{\tilde{A}_1} \text{ x } ii_{\tilde{A}_1} \tag{4.6}$$

where $i_{\tilde{A}_1}$ and $ii_{\tilde{A}_1}$ are $dist\left(\tilde{a}_4 - \tilde{a}_1\right)$ and $y^*_{\tilde{A}_1}$ respectively.

$s(\tilde{A}_1)$, $i_{\tilde{A}_1}$, $ii_{\tilde{A}_1}$, $dist\left(\tilde{a}_4 - \tilde{a}_1\right) \in [0, 1]$.

Note that, the spread formulation in equation (4.6) is the same as equation (4.3). This indicates that the centroid point based spread method developed in Section 4.2 is utilised in this step.

Step 3: Compute ranking value for $\tilde{A}_1$ using CPS ranking method which is defined as

$$CPS_I\left(\tilde{A}_1\right) = x^*_{\tilde{A}_1} \times y^*_{\tilde{A}_1} \times \left(1 - s\left(\tilde{A}_1\right)\right) \tag{4.7}$$

where

$x^*_{\tilde{A}_1}$ is horizontal – x centroid for standardised generalised type – I fuzzy number $\tilde{A}_1$

$y^*_{\tilde{A}_1}$ is horizontal – y centroid for standardised generalised type – I fuzzy number $\tilde{A}_1$

$s(\tilde{A}_1)$ is spread for standardised generalised type – I fuzzy number $\tilde{A}_1$.

and $CPS_I\left(\tilde{A}_1\right) \in [-1, 1]$.

If $CPS_I\left(\tilde{A}_1\right) > CPS_I\left(\tilde{A}_2\right)$, then $\tilde{A}_1 \succ \tilde{A}_2$. (i.e. $\tilde{A}_1$ is ranked higher than $\tilde{A}_2$).
If $CPS_I\left(\tilde{A}_1\right) < CPS_I\left(\tilde{A}_2\right)$, then $\tilde{A}_1 \prec \tilde{A}_2$. (i.e. $\tilde{A}_1$ is ranked lower than $\tilde{A}_2$).
If $CPS_I\left(\tilde{A}_1\right) = CPS_I\left(\tilde{A}_2\right)$, then $\tilde{A}_1 \approx \tilde{A}_2$. (i.e. the ranking for $\tilde{A}_1$ and $\tilde{A}_2$ is equal).

4.3.3 Ranking of Z – Fuzzy Numbers

This section discusses the methodology for ranking Z – fuzzy numbers based on centroid point and spread, *CPS*$_Z$. As there is inadequate information on dealing with Z – fuzzy numbers, this study develops a method for ranking Z – fuzzy numbers using the following descriptions. Thus, with no loss of generality, the following description of Z – numbers is given.

Let $Z_A = \{(A_i = \tilde{a}_{i1}, \tilde{a}_{i2}, \tilde{a}_{i3}, \tilde{a}_{i4}; w_{A_i}), (A_j = \tilde{a}_{j1}, \tilde{a}_{j2}, \tilde{a}_{j3}, \tilde{a}_{j4}; w_{A_j})\}$ be a Z – number where components A_i and A_j such that $i, j = 1,2,...,n$ are restriction and reliability components for A respectively. A multi – layer decision making methodology for ranking Z – numbers is illustrated where it consists of two layers which are listed as follows.

1. Layer One: Z – numbers conversion method (B. Kang et al., 2012a).
2. Layer Two: *CPS* ranking method

Full description for both layers is described as follows:

Layer One

Step A1: Convert the reliability component, $\bar{B}$ into a crisp number, α (weight of the reliability component) using (3.6)

Step A2: Add α to restriction component, $\bar{A}$ to form a weighted restriction of Z – number as in (3.7).

Step A3: Convert the weighted restriction of Z – number into standardised generalised type – I fuzzy numbers as in (3.8).

It has to be noted that Step A1 until Step A3 of *Layer One* are the same as in Section 3.3.3. However, Step A3 of *Layer One* extends the fuzzy numbers used by Kang et al. (2012) to standardised generalised type – I fuzzy numbers as defined in Section 3.4.2.

Layer Two

Let $Z_A = [z_{a1}, z_{a2}, z_{a3}, z_{a4}]$ be standardised generalised type – I fuzzy number obtained from *Layer One* known Z – fuzzy number A and description for *Layer Two* is as the following. Notice that, a Z – fuzzy number, Z_A, which is referred to as the standardised generalised type – I fuzzy number after conversion from Z – number in Layer One is equivalent to type – I fuzzy number, $\tilde{A}$ defined in Chapter 4. Therefore, with no loss of generality, the procedure to rank Z – fuzzy number is the same as ranking procedure in Section 4.3. Thus, the complete procedure for ranking Z – fuzzy numbers using the CPSZ ranking method is not given in this chapter as repeating the same procedure in the thesis is redundant.

4.4 THEORETICAL VALIDATION OF RANKING METHODOLOGY

According to Brunelli & Mezei (2013), theoretical validation of ranking fuzzy numbers is an axiomatic based – research as it concerns a broad scope of ranking fuzzy numbers where ranking methods are validated based on properties for ranking fuzzy quantities. Fuzzy quantities defined by Wang & Kerre (2001) are in principle more generic than fuzzy numbers, but they are not often used in the literature of fuzzy sets. Since fuzzy numbers are subsets of fuzzy quantities, hence any properties that are related to the latter are also applicable to the former. In the literature of fuzzy sets, reasonable properties for ranking fuzzy quantities are presented by Wang & Kerre (2001; 2002) where these properties are purposely developed for type – 1 and type – 2 fuzzy numbers. Wu & Mendel (2009), Kumar et al. (2010), Asady (2010) and Yu et al. (2013) are among the recently established ranking methods that utilise these properties in validating their methods. Therefore, based on Wang & Kerre (2001, 2002), reasonable properties for ranking fuzzy quantities which are fuzzy numbers are as follows.

Let $\tilde{A}_1$ and $\tilde{A}_2$ be two standardised generalised fuzzy numbers where $\tilde{A}_1$ and $\tilde{A}_2$ are of any type of fuzzy numbers.

***Property 1*:** If $\tilde{A}_1 \succcurlyeq \tilde{A}_2$ and $\tilde{A}_2 \succcurlyeq \tilde{A}_1$, then $\tilde{A}_1 \approx \tilde{A}_2$

***Property 2*:** If $\tilde{A}_1 \succcurlyeq \tilde{A}_2$ and $\tilde{A}_2 \succcurlyeq \tilde{A}_3$, then $\tilde{A}_1 \succcurlyeq \tilde{A}_3$

***Property 3*:** If $\tilde{A}_1 \cap \tilde{A}_2 = \emptyset$ and $\tilde{A}_1$ is on the right side of $\tilde{A}_2$, then $\tilde{A}_1 \succcurlyeq \tilde{A}_2$

***Property 4*:** The order of $\tilde{A}_1$ and $\tilde{A}_2$ is not affected by the other fuzzy numbers under comparison.

If a ranking method fulfils all the aforementioned ranking properties suggested by Wang & Kerre (2001; 2002), then the method is considered to be an effective ranking method theoretically. Table 4.1 illustrates the applicability of the properties of ranking fuzzy quantities towards fuzzy numbers.

Table 4.1: Applicability of the properties of ranking fuzzy quantities towards fuzzy numbers.

Fuzzy Numbers	Properties Applicability
Type – I	Yes
Type – II	Yes
Type – II after reduction into Type – I	Yes
Z – numbers	No
Z – numbers after reduction into Type – I	Yes

Although the aforementioned properties are not developed for Z – numbers in the first place, as Z – numbers are new in the literature of fuzzy sets (Zadeh, 2011), they are all applicable whenever Z – numbers are reduced into type – I fuzzy numbers (Kang et al., 2012).

4.5 EMPIRICAL VALIDATION OF RANKING METHODOLOGY

In this section, the empirical validation of a ranking fuzzy numbers method is extensively discussed. Discussions of this validation are made in accordance to case studies found in the literature for fuzzy sets. Among the case studies found are risk analysis under uncertainty (Chen & Chen, 2009), fuzzy programming in textile industry (Elamvazuthi et al., 2009), a fuzzy approach in torque – sensorless control of DC motor (Liem et al., 2015), inspection planning in manufacturing problem (Mousavi et al., 2015) and uncertain stochastic nonlinear systems with input saturation (Sui et al., 2015). Based on these case studies, Cheng (1998), Wang et al. (2005), Asady (2009), Chen & Chen (2007, 2009), Dat et al. (2012), Yu et al. (2013) and Bakar & Gegov (2014; 2015) suggest several numerical examples that generically represent all of the aforementioned case studies. All numerical examples presented in the literature are explained and illustrated as follows.

Trivial Case

Trivial case category covers cases of fuzzy numbers which are simple and easy to differentiate. This is because ranking orders of all cases under this category are determined by observing the nature of fuzzy numbers under consideration. Thus, this category is carried out to assess the capability of ranking methods including CPS ranking method to appropriately rank simple cases of fuzzy numbers first before more complex fuzzy numbers cases are considered. In this study, two trivial cases of fuzzy numbers are considered.

Embedded Case

Embedded case category involves sets of fuzzy numbers which are fully overlapped with one another. Regardless whether the fuzzy numbers are of different heights or spreads, as long as they are fully overlapping with each other, they are considered to represent a embedded case. Under this category, three different kinds of embedded cases of fuzzy numbers are investigated.

Overlapping Case

Overlapping case category is among the most important cases in ranking fuzzy numbers area of research. If embedded fuzzy numbers cases are fuzzy numbers which are fully overlapped with each other, this category considers fuzzy numbers that are partially overlapping from one to another. For this category, two distinct cases of overlapping fuzzy numbers are examined.

Non – overlapping Case

Non – overlapping case category involves cases of fuzzy numbers that are separated from each other. This category is considered as the opposite of the overlapping case category where two distinct non – overlapping cases of fuzzy numbers are considered in this study.

4.5.1 Evaluation of Consistency

Consistency is defined in the literature for ranking fuzzy numbers as the capability of a ranking method to produce correct ranking order such that the ranking result is consistent with human intuition. This evaluation is a common validation done by many established ranking methods like Cheng (1998), Chen & Lu (2001), Wang et al. (2006), Chen & Chen (2009), Dat et al. (2012), Bakar & Gegov (2014) where ordering results of a ranking method is compared based on several sets of fuzzy numbers with other ranking methods under consideration for their consistency evaluation. If a method ranks fuzzy numbers correctly such that the ranking results are consistent with human intuition, then the ranking result is justified as consistent, otherwise the ranking result is inconsistent.

Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three fuzzy numbers to be ranked and Table 4.2 indicates the possible ranking order for $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ with respective level of consistency.

Table 4.2: Evaluation of Consistency

Human intuition = $\tilde{A} \succ \tilde{B} \succ \tilde{C}$	
Ranking order	Consistency
$\tilde{A} \succ \tilde{B} \succ \tilde{C}$	100
$\tilde{A} \succ \tilde{C} \succ \tilde{B}$	50
$\tilde{B} \succ \tilde{A} \succ \tilde{C}$	50
$\tilde{B} \succ \tilde{C} \succ \tilde{A}$	50
$\tilde{C} \succ \tilde{B} \succ \tilde{A}$	0
$\tilde{C} \succ \tilde{A} \succ \tilde{B}$	50
$\tilde{A} \approx \tilde{B} \succ \tilde{C}$	50
$\tilde{A} \succ \tilde{B} \approx \tilde{C}$	50
$\tilde{B} \succ \tilde{C} \approx \tilde{A}$	50
$\tilde{B} \approx \tilde{C} \succ \tilde{A}$	0
$\tilde{C} \approx \tilde{B} \succ \tilde{A}$	0
$\tilde{C} \succ \tilde{B} \approx \tilde{A}$	0
$\tilde{A} \approx \tilde{B} \approx \tilde{C}$	0

Table 4.2 clearly indicates that whenever three fuzzy numbers are used to represent cases of fuzzy numbers, two ranking operators are used to indicate the level of ordering consistency of a ranking fuzzy numbers method. In this case, the consistency evaluation provided by this study are categorised into three which are explained as follows.

1) For any ranking methods that rank any cases of fuzzy numbers using two correct ranking operators, the ranking results obtained by these methods are classified as correct such that the ranking results are 100% consistent with human intuition.

2) For any ranking methods that rank any cases of fuzzy numbers using one out of two correct ranking operators, the ranking results obtained by these methods are classified as partially correct such that the ranking results are 50% consistent with human intuition.

3) For any ranking methods that rank any cases of fuzzy numbers using two incorrect ranking operators, the ranking results obtained by these methods are classified as incorrect such that the ranking results are 0% consistent with human intuition.

The consistency evaluations on the ranking order of fuzzy numbers provided in this study indicate that the levels of consistency for any ranking fuzzy numbers methods are varied from one to another. Since explanations in term of consistency evaluation provided in this study are applicable for cases with three fuzzy numbers, they are relevant for any ranking method in the literature of ranking fuzzy numbers which also take into account three fuzzy numbers in their analysis. Therefore, this study presents a generic consistency validation for ranking fuzzy numbers in the literature of fuzzy sets.

4.5.2 Evaluation of Efficiency

This subsection describes the efficiency evaluation of ranking fuzzy numbers methods including the *CPS* ranking methodology when ranking fuzzy numbers. According to Allahviranloo et al. (2013), Fries (2014) and Jahantigh & Hajighasemi (2014), efficiency of a ranking method is often determined in accordance to its computational complexity when ranking fuzzy numbers. In the literature of fuzzy sets, two kinds of ranking method are found namely simultaneous ranking and pairwise ranking. Simultaneous ranking refers to the capability of a method to ranking any quantity of fuzzy numbers simultaneously like Chen & Chen (2009) and Bakar & Gegov (2014; 2015) while pairwise ranking is the capability of a method to ranking only two fuzzy numbers at one time such as Bakar et al. (2010; 2012) and Dat et al. (2012). Although there are different capabilities in terms of ranking fuzzy numbers, these are not empirically proven in the literature of fuzzy sets. Thus, this study provides empirical justification in terms of validating the efficiency level of ranking methods by taking into consideration the capability of ranking methods to rank more than two fuzzy numbers simultaneously. The complete explanation of the efficiency evaluation developed in this study is as follow.

As far as the literature on ranking fuzzy numbers methods is concerned, both kinds of capability of ranking fuzzy numbers methods follow the same basic algorithms when ranking fuzzy numbers. In accordance to aforementioned ranking methods, basic algorithms for ranking fuzzy numbers are signified as Basic Algorithm and are shown as follows.

Basic Algorithm for Ranking Fuzzy Numbers

1) Assign a value to each fuzzy number under consideration whereby this value is called an assignment.
2) Make a comparison based on the assignment obtained in 1). This step is also known as sorting stage.

Regardless to whether a ranking method utilises simultaneous or pairwise rankings, each ranking way underpins the same number of assignments and comparisons when fuzzy numbers under consideration. If fuzzy numbers examined are more than two, then number of assignments and comparisons are varied as simultaneous ranking ranks simultaneously all fuzzy numbers under consideration while pairwise ranking requires more steps to ranking the fuzzy numbers even if the quantity of fuzzy numbers are the same when simultaneous ranking is used. Therefore, depending on the quantity of fuzzy numbers considered, differences between simultaneous ranking and pairwise ranking in terms of number of assignments and comparison are summarised in Table 4.3.

Table 4.3: Differences between simultaneous ranking and pairwise ranking in terms of number of assignments and comparisons.

No of Fuzzy Numbers	Simultaneous Ranking		Pairwise Ranking	
	No. of Assignments	No. of Comparisons	No. of Assignments	No. of Comparisons
2	2	1	2	1
3	3	1	6	3
4	4	1	12	6
5	5	1	20	10
6	6	1	30	15
7	7	1	42	21
8	8	1	56	28

Even though, Table 4.3 clearly indicates that number of assignments and comparisons for simultaneous and pairwise rankings methods are different even if the number of fuzzy numbers under consideration is the same, both ranking ways sometimes require additional operations to ranking fuzzy numbers appropriately. This is because in certain situations, a ranking method is incapable to rank fuzzy numbers appropriately only if one approach is used. Therefore, incorporation of other approaches as such the additional operation along with the established ranking method complements the ranking method in ranking fuzzy numbers appropriately. Among ranking methods found in the literature of fuzzy sets that rely on additional operations to ranking fuzzy numbers appropriately are Cheng (1998), Kumar & Kaur (2012), Yu et al. (2013) and Zhang et al. (2014). Thus, incorporation of additional operation by some ranking methods creates further extension of the basic algorithm mentioned earlier, where this study lists the extension algorithms as follows.

Extension of Basic Algorithm used for Ranking Fuzzy Numbers

1) Assign a value to each fuzzy number under consideration whereby this value is called an initial assignment.
2) Make a comparison based on assignment obtained in 1) where this is defined as initial comparison.
3) Assign a value to each fuzzy number under consideration for second time whereby this value is called the secondary assignment.
4) Make a comparison based on secondary assignment obtained in 3) which is defined as secondary comparison.

It has to be noted that steps 1 and 2 of basic algorithm are changed to initial assignment and initial comparison in this algorithm respectively as both steps are repeated in steps 3 and 4 respectively. The terms initial assignment and initial comparison are introduced in this case as to avoid confusion between the steps used and to indicate that the ranking methods require additional operations in the methodology. Therefore, regardless if a ranking method uses simultaneous ranking or pairwise ranking, if the method incorporates an additional approach to ranking fuzzy numbers, then an extension of the basic algorithm is used where secondary assignment and secondary comparison are obtained in its result. The following Table 4.4 illustrates comparisons in terms of the algorithm used between simultaneous ranking, simultaneous ranking with additional operation, pairwise ranking and pairwise ranking with additional operation.

Table 4.4: Algorithm comparison between simultaneous ranking, simultaneous ranking with additional operation, pairwise ranking and pairwise ranking with additional operation.

Algorithm	Simultaneous Ranking		Pairwise Ranking	
	Without Additional Operation	With Additional Operation	Without Additional Operation	With Additional Operation
Initial Assignment	Yes	Yes	Yes	Yes
Initial Comparison	Yes	Yes	Yes	Yes
Secondary Assignment	No	Yes	No	Yes
Secondary Comparison	No	Yes	No	Yes

Based on all discussions made above, a complete evaluation of efficiency for ranking fuzzy numbers methods is suggested. There are four classes of efficiency evaluations are introduced in this study namely very efficient, slightly efficient, slightly inefficient and very inefficient. All of these classes are determined through examining the capability of a method in ranking more than two fuzzy numbers simultaneously. Based on Table 4.3 and Table 4.4, the following Table 4.5 and Figure 4.3 are developed.

Table 4.4: Evaluation of Efficiency.

No. of Fuzzy numbers	Efficiency			
	Simultaneous Ranking		Pairwise Ranking	
	Without Additional Computation	With Additional Computation	Without Additional Computation	With Additional Computation
2	2_{IA}	2_{IA}	2_{IA}	2_{IA}
3	3_{IA}	$3_{IA} + 3_{IC} + 3_{SA} + 3_{SC}$	$6_{IA} + 3_{IC}$	$6_{IA} + 3_{IC} + 6_{SA} + 3_{SC}$
4	4_{IA}	$4_{IA} + 4_{IC} + 4_{SA} + 4_{SC}$	$12_{IA} + 6_{IC}$	$12_{IA} + 6_{IC} + 12_{SA} + 6_{SC}$
5	5_{IA}	$5_{IA} + 5_{IC} + 5_{SA} + 5_{SC}$	$20_{IA} + 10_{IC}$	$20_{IA} + 10_{IC} + 20_{SA} + 10_{SC}$
6	6_{IA}	$6_{IA} + 6_{IC} + 6_{SA} + 6_{SC}$	$30_{IA} + 15_{IC}$	$30_{IA} + 15_{IC} + 30_{SA} + 15_{SC}$
7	7_{IA}	$7_{IA} + 7_{IC} + 7_{SA} + 7_{SC}$	$42_{IA} + 21_{IC}$	$42_{IA} + 21_{IC} + 42_{SA} + 21_{SC}$
8	8_{IA}	$8_{IA} + 8_{IC} + 8_{SA} + 8_{SC}$	$56_{IA} + 28_{IC}$	$56_{IA} + 28_{IC} + 56_{SA} + 28_{SC}$
N	$f(N) = N$	$f(N) = 4N$	$f(N) = \frac{3}{2}N^2 - \frac{3}{2}N$	$f(N) = 3N^2 - 3N$
Efficiency Classification	Very Efficient	Slightly Efficient	Slightly Inefficient	Very Inefficient

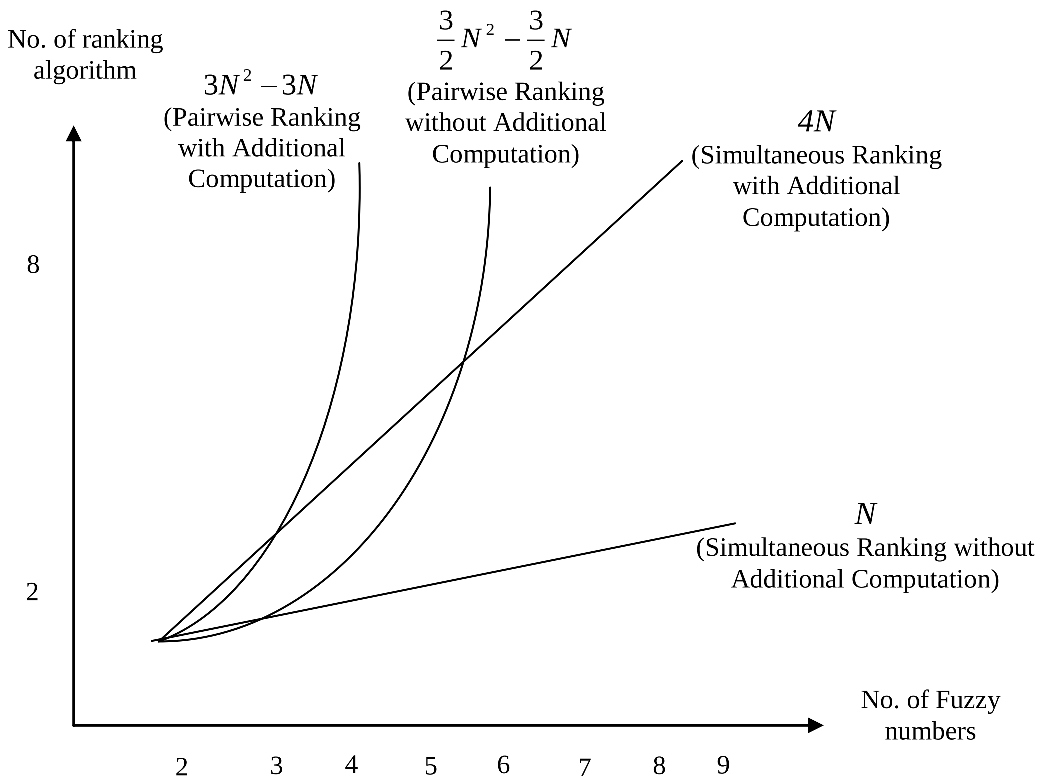


Fig 4.3: Evaluation of efficiency.

It is clearly indicate in Table 4.5 and Figure 4.3, a simultaneous – based ranking method like the *CPS* ranking method is more efficient than methods with pairwise ranking because it is represented by a linear function while the latter are signified by quadratic functions. Apart from that, the *CPS* ranking methodology and other simultaneous ranking methods are four times (4 times) more efficient than a simultaneous ranking method that requires additional operation in the formulation. This is shown when functions obtained for the *CPS* ranking methodology (simultaneous ranking without additional operation) and simultaneous ranking with additional operation are $f(N) = N$ and $f(N) = 4N$ respectively. For pairwise ranking, methods that require additional operation to ranking fuzzy numbers are twice less efficient than one without additional operation where the functions are $f(N) = 3N^2 - 3N$ and $f(N) = \frac{3}{2}N^2 - \frac{3}{2}N$ for a method with additional operation and method without additional operation respectively. Therefore, based on these discussions, *CPS* ranking methodology or any simultaneous ranking methods which requires no additional operation is classified as very efficient, simultaneous ranking with additional operation as slightly efficient, pairwise ranking without additional operation as slightly inefficient and pairwise ranking with additional operation as very inefficient. Therefore, similarly as subsection 4.4.1, descriptions mentioned in this subsection are also utilised on the following three chapters of the thesis for validation purposes.

4.3 SUMMARY

In this chapter, the research methodology of the thesis is thoroughly discussed. A novel methodology for ranking fuzzy numbers is developed in this chapter which consists of centroid point and spread method, *CP*S. The spread method which is proposed based on distance from the centroid point, fulfils all relevant theoretical properties on differentiating fuzzy numbers introduced in this study. Then, the spread method is incorporated with an established centroid point method as a novel methodology for ranking fuzzy numbers where the ranking method satisfies all the ordering properties under consideration. Together with those discussions, two types of evaluation, namely the consistency and efficiency, are introduced in this chapter as the empirical validation for ranking fuzzy numbers methods. Descriptions on both types of evaluation in this chapter underpin discussions on the empirical validation for the next three chapters of the thesis. This indicates that Chapter 4 underpins Chapter 5, Chapter 6 and Chapter 7 of the thesis. In Chapter 5, the thesis discusses the capability of *CPS* ranking methodology in ranking type – I fuzzy numbers.

CHAPTER FIVE

RANKING OF TYPE – I FUZZY NUMBERS

5.1 INTRODUCTION

This chapter discusses details on validation of the proposed new methodology for ranking type – I fuzzy numbers based on centroid point and spread, *CPS_I*. Theoretical and empirical validation defined in Section 4.4 and 4.5 respectively are demonstrated in this chapter. These validations which are associated with properties of ranking fuzzy quantities as well as consistency and efficiency evaluation of ranking operations are described in detail here. Therefore, without loss of generality of Section 4.4 and 4.5, details on those aforementioned both validations are extensively discussed in sections and subsections of this chapter.

5.1 THEORETICAL VALIDATION

This subsection validates theoretically the CPS_I ranking method using theoretical properties adopted from Wang & Kerre (2001, 2002). These properties justify the capability of the CPS_I ranking method to ranking fuzzy numbers appropriately. It is worth mentioning that proofs provided for all of the theoretical properties considered are applicable to CPS_I ranking method. With no loss of generality, theoretical ordering properties by Wang & Kerre (2001, 2002) which are prepared for CPS_I ranking method are presented as follows.

Let $\tilde{A}_1$ and $\tilde{A}_2$ be two standardised generalised fuzzy numbers where $\tilde{A}_1$ and $\tilde{A}_2$ are of any types of fuzzy numbers.

***Property 1*:** If $\tilde{A}_1 \succcurlyeq \tilde{A}_2$ and $\tilde{A}_2 \succcurlyeq \tilde{A}_1$, then $\tilde{A}_1 \approx \tilde{A}_2$

***Proof*:**

Since, $\tilde{A}_1 \succcurlyeq \tilde{A}_2$ implies that $CPS_I\left(\tilde{A}_1\right) \geq CPS_I\left(\tilde{A}_2\right)$, and $\tilde{A}_2 \succcurlyeq \tilde{A}_1$ implies that $CPS_I\left(\tilde{A}_2\right) \geq CPS_I\left(\tilde{A}_1\right)$, hence indicates that, $CPS_I\left(\tilde{A}_1\right) = CPS_I\left(\tilde{A}_2\right)$, which is $\tilde{A}_1 \approx \tilde{A}_2$

***Property 2*:** If $\tilde{A}_1 \succcurlyeq \tilde{A}_2$ and $\tilde{A}_2 \succcurlyeq \tilde{A}_3$, then $\tilde{A}_1 \succcurlyeq \tilde{A}_3$

***Proof*:**

For CPS ranking method, $\tilde{A}_1 \succcurlyeq \tilde{A}_2$ implies that $CPS_I\left(\tilde{A}_1\right) \geq CPS_I\left(\tilde{A}_2\right)$, and $\tilde{A}_2 \succcurlyeq \tilde{A}_3$, implies that $CPS_I\left(\tilde{A}_2\right) \geq CPS_I\left(\tilde{A}_3\right)$. This indicates that $CPS_I\left(\tilde{A}_1\right) \geq CPS_I\left(\tilde{A}_3\right)$, which is $\tilde{A}_1 \succcurlyeq \tilde{A}_3$.

***Property 3*:** If $\tilde{A}_1 \cap \tilde{A}_2 = \emptyset$ and $\tilde{A}_1$ is on the right side of $\tilde{A}_2$, then $\tilde{A}_1 \succcurlyeq \tilde{A}_2$

***Proof*:**

Since, $\tilde{A}_1 \cap \tilde{A}_2 = \emptyset$ and $\tilde{A}_1$ is on the right side of $\tilde{A}_2$, hence, implies that $CPS_I\left(\tilde{A}_1\right) \geq CPS_I\left(\tilde{A}_2\right)$, thus, $\tilde{A}_1 \succcurlyeq \tilde{A}_2$.

***Property 4*:** The order of $\tilde{A}_1$ and $\tilde{A}_2$ is not affected by the other fuzzy numbers under comparison.

***Proof*:**

Since, the ordering of $\tilde{A}_1$ and $\tilde{A}_2$ is completely determined by $CPS_I\left(\tilde{A}_1\right)$ and $CPS_I\left(\tilde{A}_2\right)$ respectively, hence indicates that the ordering of $\tilde{A}_1$ and $\tilde{A}_2$ is not affected by the other fuzzy numbers under comparison.

The above theoretical validation clearly indicates that the CPS_I ranking method is capable to ranking fuzzy numbers appropriately. This is signified through proof based – properties fulfilment by the CPS_I ranking method on all theoretical validations considered in this subsection. In the next section, a generic empirical validation for any ranking fuzzy numbers methods is thoroughly discussed.

5.2 EMPIRICAL VALIDATION

This section discusses empirical validation of the CPS_I ranking method on ranking type – I fuzzy numbers. The empirical validation is a comparative – based ranking order analysis between the CPS_I ranking method and established ranking methods under consideration on their consistency and efficiency in ranking type – I fuzzy numbers. All established ranking methods considered in this validation are methods for ranking type – I fuzzy numbers found in literature of fuzzy sets. These methods are chosen according to their high referencing frequency by many established ranking methods. Therefore, without loss of generality in terms of information in Section 4.5, the consistency and efficiency evaluations of the CPS_I ranking method are given as follows.

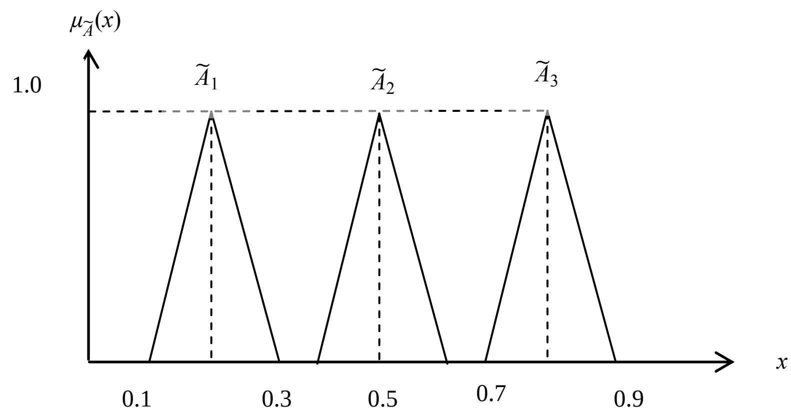
5.3.1 Evaluation of Consistency

This subsection provides details on consistency evaluation of the CPS_I ranking method on ranking type – I fuzzy numbers. Nine sets of type – I fuzzy numbers adopted from Chen & Chen (2009) with modifications are utilised as benchmarking examples in this case where all of them are often used in validating many ranking methods such as Kumar et al. (2010), Bakar et al. (2010), Chen & Sanguatsan (2011), Dat et al. (2012) and Zhang et al. (2014). Therefore, with no loss of generality, all of the nine benchmarking examples which fall under the four categories mentioned in Section 4.5 are illustrated as follows.

Trivial Case

Trivial Case 1

Trivial case 1 involves three triangular type – I fuzzy numbers of similar shapes and not overlapped which is illustrated in Figure 5.1.



$\tilde{A}_1 = (0.1, 0.2, 0.2, 0.3; 1.0)$ $\tilde{A}_2 = (0.4, 0.5, 0.5, 0.6; 1.0)$ $\tilde{A}_3 = (0.7, 0.8, 0.8, 0.9; 1.0)$

Fig 5.1: Trivial Case 1

Using the CPS_I ranking method, the ranking order for $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ which in this case is determined as follows.

Step 1: Calculate the centroid point (x^*, y^*) for $\tilde{A}_1$ such that the value of $x^*_{\tilde{A}_1}$ is computed using equation (4.4) as

$$x^*_{\tilde{A}_1} = \frac{1}{3}\left[0.1 + 0.2 + 0.2 + 0.3 - \frac{(0.06 - 0.02)}{(0.5 - 0.3)}\right]$$

$$= 0.2$$

whereas, the value of $y^*_{\tilde{A}_1}$ is obtained using equation (4.5) as

$$y^*_{\tilde{A}_1} = \frac{1}{3}\left[1 + \frac{0}{(0.5 - 0.3)}\right]$$
$$= 0.3333$$

Hence, the centroid point for $\tilde{A}_1$ is (0.2, 0.3333).

Using the same procedure as in Step 1, the centroid point values for $\tilde{A}_2$ and $\tilde{A}_3$ are as follows:

$$(x^*_{\tilde{A}_2}, y^*_{\tilde{A}_2}) = (0.5, 0.3333)$$

$$(y^*_{\tilde{A}_3}, y^*_{\tilde{A}_3}) = (0.8, 0.3333)$$

Step 2: Compute the spread values of $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}_3$ where the spread of $\tilde{A}_1$ is

$$s(\tilde{A}_1) = 0.2 \times 0.3333$$

$$= 0.0667$$

and the spread values for $\tilde{A}_2$ and $\tilde{A}_3$ are

$$s(\tilde{A}_2) = 0.0667$$

$$s(\tilde{A}_3) = 0.0667$$

Step 3: Obtain the ranking values of $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}_3$ such that the ranking value for $\tilde{A}_1$ is

$$CPS_I\left(\tilde{A}_1\right) = 0.2 \times 0.3333 \times (1 - 0.0667)$$

$$= 0.0662$$

and ranking values for $\tilde{A}_2$ and $\tilde{A}_3$ are

$$CPS_I\left(\tilde{A}_2\right) = 0.1555$$

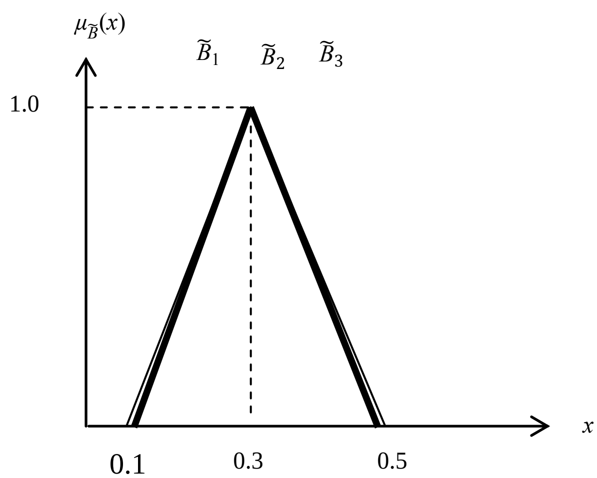
$$CPS_I\left(\tilde{A}_3\right) = 0.2489$$

Since $CPS_I\left(\tilde{A}_3\right) > CPS_I\left(\tilde{A}_2\right) > CPS_I\left(\tilde{A}_1\right)$, hence the ranking order result for type – I fuzzy numbers $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}_3$ in this case is $\tilde{A}_3 \succ \tilde{A}_2 \succ \tilde{A}_1$.

It is worth mentioning here that the entire steps utilised by the CPS_I ranking method in ranking type – I fuzzy numbers are only demonstrated in Trivial Case 1. This is because these steps are also applied to the remaining eight cases of benchmarking examples considered in this study, thus repeating the entire steps are redundant. Therefore, only definition, illustration, the ranking results and discussions on each case considered are provided.

Trivial Case 2

Trivial case 2 involves three identical triangular type – I fuzzy numbers which are embedded with each other. The following Figure 5.2 illustrates type – I fuzzy numbers of trivial case 2.



$$\tilde{B}_1 = (0.1, 0.3, 0.3, 0.5; 1.0) \quad \tilde{B}_2 = (0.1, 0.3, 0.3, 0.5; 1.0) \quad B_3 = (0.1, 0.3, 0.3, 0.5; 1.0)$$

Fig 5.2: Trivial Case 2

Results and Validation

Comparisons of ranking order for trivial case 1 and 2 between the CPS_I ranking method and established ranking methods considered in this study are shown in Table 5.1 and 5.2 respectively.

Table 5.1: Ranking Results for Trivial Case 1

Methods	Fuzzy Numbers $\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	Ranking Results	Level of Consistency
Cheng (1998)	0.583	0.583	0.583	$\tilde{A}_1 \approx \tilde{A}_2 \approx \tilde{A}_3$	0
Kumar et al. (2010)	0.300	0.300	0.300	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100
Dat et al. (2012)	0.000 / 0.300	0.300 / 0.600	0.600 / 0.000	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100
Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100
Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100
Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100
Zhang et al. (2014) for $\alpha = 0$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100
Zhang et al. (2014) for $\alpha = 0.5$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100
Zhang et al. (2014) for $\alpha = 1$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100
CPS_I	0.089	0.107	0.119	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	100

Table 5.2: Ranking Results for Trivial Case 2

Methods	Fuzzy Numbers $\tilde{B}_1$	$\tilde{B}_2$	$\tilde{B}_3$	Ranking Results	Level of Consistency
Cheng (1998)	0.583	0.583	0.583	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
Kumar et al. (2010)	0.300	0.300	0.300	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
Dat et al. (2012)	0.333 / 0.333	0.333 / 0.333	0.333 / 0.333	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
Zhang et al. (2014) for $\alpha = 0$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
Zhang et al. (2014) for $\alpha = 0.5$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
Zhang et al. (2014) for $\alpha = 1$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100
CPS_I	0.089	0.089	0.089	$\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$	100

It is worth noting here that ranking values obtained by Dat et al (2012) and Zhang et al. (2014) are separated by separator (/) in both Table 5.1 and Table 5.2. This is to point out that both methods adopted pairwise ranking approach to ranking type – I fuzzy numbers. Also indicated in both tables is Yu et al. (2013) ranking method where this method provides equal ranking values for all type – I fuzzy numbers under consideration but gives different ranking orders for different α. This happens because Yu et al. (2013) ranking method considers different type of decision makers' opinions which is reflected by α when ranking type – I fuzzy numbers, thus different ranking orders are computed for different values of α even if the ranking values obtained are the same at the first place. Notice that, these conditions of Dat et al. (2012) and Zhang et al. (2014) ranking methods apply to all cases of benchmarking examples considered in this chapter while only some cases apply to Yu et al. (2013) ranking method.

Discussions

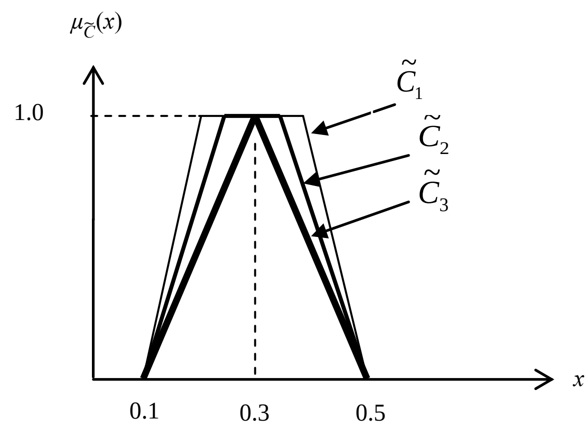
For trivial case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$. This is because $\tilde{A}_3$ is located at the farthest right compared to $\tilde{A}_2$, while $\tilde{A}_2$ is on the right of $\tilde{A}_1$. In Table 5.1, all established ranking methods considered in this study including the CPS_I ranking method except Cheng (1998), produce correct ranking order for this case such that the ranking result is 100% consistent with human intuition. Cheng (1998) ranking method in this case, produces equal ranking which is 0% consistent with human intuition. This indicates that the CPS_I ranking method is capable to deal with type – I fuzzy numbers of different locations.

For trivial case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $\tilde{B}_1 \approx \tilde{B}_2 \approx \tilde{B}_3$. This is due to the fact that all type – I fuzzy numbers under consideration are the same in term of their shapes, spreads, heights and centroids. Shown in Table 5.2, all ranking results obtained by all established ranking methods considered in this study and the $CPSI$ ranking method are the correct ranking order such that the results are 100% consistent with human intuition. This points out that the $CPSI$ ranking method is capable to give same ranking value for each type – I fuzzy numbers even if same type – I fuzzy numbers are compared.

Embedded Case

Embedded Case 1

Embedded case 1 involves three embedded type – I fuzzy numbers where two of them are in trapezoidal type – I fuzzy numbers while the other is a triangular type – I fuzzy number. All of these type – I fuzzy numbers are of same height but differed in centroid point and spread as shown in Figure 5.3.



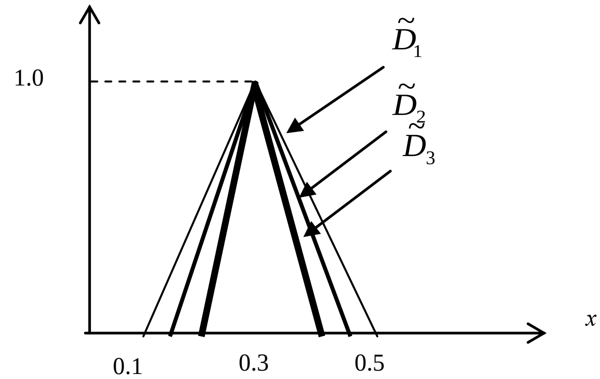
$\tilde{C}_1 = (0.1, 0.2, 0.4, 0.5; 1.0)$ $\tilde{C}_2 = (0.1, 0.25, 0.35, 0.5; 1.0)$ $\tilde{C}_3 = (0.1, 0.3, 0.3, 0.5; 1.0)$

Fig 5.3: Embedded Case 1

Embedded Case 2

Embedded case 2 involves three triangular type – I fuzzy numbers where they are embedded with each other, same height and same centroid point but different in term of their spread. Figure 5.4 best is the illustration for this case.

$\mu_{\tilde{D}}(x)$

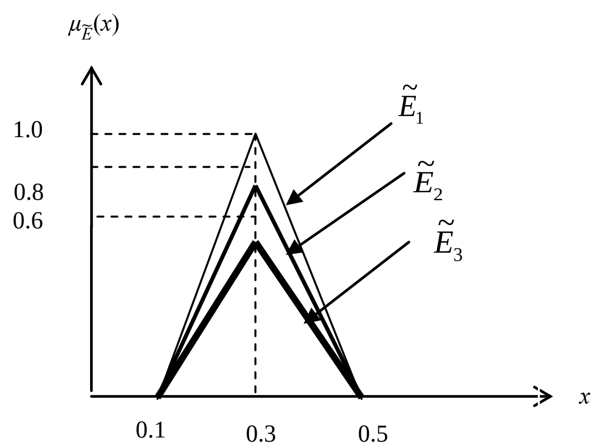


$\widetilde{D}_1 = (0.1, 0.3, 0.3, 0.5; 1.0)$ $\widetilde{D}_2 = (0.15, 0.3, 0.3, 0.45; 1.0)$ $\widetilde{D}_3 = (0.2, 0.3, 0.3, 0.4; 1.0)$

Fig 5.4: Embedded Case 2

Embedded Case 3

Embedded case 3 shown in Figure 5.5 involves three triangular type – I fuzzy numbers that are embedded with each other and having the same horizontal – *x* centroid but different in spread and vertical – *y* centroid.



$\widetilde{E}_1 = (0.1, 0.3, 0.3, 0.5; 1.0)$ $\widetilde{E}_2 = (0.1, 0.3, 0.3, 0.5; 0.8)$ $\widetilde{E}_3 = (0.1, 0.3, 0.3, 0.5; 0.6)$

Fig 5.5: Embedded Case 3

Results and Validation

Comparisons of ranking order for embedded case 1, 2 and 3 between the CPS_I ranking method and established ranking methods considered in this study are illustrated in Table 5.3, 5.4 and 5.5 respectively.

Table 5.3: Ranking Results for Embedded Case 1

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$		
Cheng (1998)	0.583	0.583	0.583	$\tilde{C}_1 \approx \tilde{C}_2 \approx \tilde{C}_3$	0
Kumar et al. (2010)	0.300	0.300	0.300	$\tilde{C}_1 \approx \tilde{C}_2 \approx \tilde{C}_3$	0
Dat et al. (2012)	0.333 / 0.333	0.333 / 0.333	0.222 / 0.333	$\tilde{C}_1 \approx \tilde{C}_2 \succ \tilde{C}_3$	50
Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$\tilde{C}_1 \prec \tilde{C}_2 \prec \tilde{C}_3$	0
Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$\tilde{C}_1 \approx \tilde{C}_2 \approx \tilde{C}_3$	0
Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$\tilde{C}_1 \succ \tilde{C}_2 \succ \tilde{C}_3$	100
Zhang et al. (2014) for $\alpha = 0$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{C}_1 \prec \tilde{C}_2 \prec \tilde{C}_3$	0
Zhang et al. (2014) for $\alpha = 0.5$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{C}_1 \approx \tilde{C}_2 \approx \tilde{C}_3$	0
Zhang et al. (2014) for $\alpha = 1$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{C}_1 \succ \tilde{C}_2 \succ \tilde{C}_3$	100
CPS_I	0.119	0.107	0.089	$\tilde{C}_1 \succ \tilde{C}_2 \succ \tilde{C}_3$	100

Table 5.4: Ranking Results for Embedded Case 2

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	$\tilde{D}_1$	$\tilde{D}_2$	$\tilde{D}_3$		
Cheng (1998)	0.583	0.583	0.583	$\tilde{D}_1 \approx \tilde{D}_2 \approx \tilde{D}_3$	0
Kumar et al. (2010)	0.300	0.300	0.300	$\tilde{D}_1 \approx \tilde{D}_2 \approx \tilde{D}_3$	0
Dat et al. (2012)	0.333 / 0.333	0.333 / 0.333	0.333 / 0.333	$\tilde{D}_1 \approx \tilde{D}_2 \approx \tilde{D}_3$	0
Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$\tilde{D}_1 \succ \tilde{D}_2 \succ \tilde{D}_3$	0
Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$\tilde{D}_1 \approx \tilde{D}_2 \approx \tilde{D}_3$	0
Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$\tilde{D}_1 \prec \tilde{D}_2 \prec \tilde{D}_3$	100
Zhang et al. (2013) for $\alpha = 0$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{D}_1 \succ \tilde{D}_2 \succ \tilde{D}_3$	0
Zhang et al. (2013) for $\alpha = 0.5$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{D}_1 \approx \tilde{D}_2 \approx \tilde{D}_3$	0
Zhang et al. (2013) for $\alpha = 1$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$\tilde{D}_1 \prec \tilde{D}_2 \prec \tilde{D}_3$	100
CPS_I	0.089	0.107	0.119	$\tilde{D}_1 \prec \tilde{D}_2 \prec \tilde{D}_3$	100

Table 5.5: Ranking Results for Embedded Case 3

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	$\tilde{E}_1$	$\tilde{E}_2$	$\tilde{E}_3$		
Cheng (1998)	0.583	0.461	0.346	$\tilde{E}_1 \succ \tilde{E}_2 \succ \tilde{E}_3$	100
Kumar et al. (2010)	0.240	0.240	0.240	$\tilde{E}_1 \approx \tilde{E}_2 \approx \tilde{E}_3$	0
Dat et al. (2012)	0.266 / 0.244	0.244 / 0.133	0.133 / 0.266	$\tilde{E}_1 \succ \tilde{E}_2 \succ \tilde{E}_3$	100
Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$\tilde{E}_1 \prec \tilde{E}_2 \prec \tilde{E}_3$	0
Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$\tilde{E}_1 \approx \tilde{E}_2 \approx \tilde{E}_3$	0
Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$\tilde{E}_1 \succ \tilde{E}_2 \succ \tilde{E}_3$	100
Zhang et al. (2013) for $\alpha = 0$	x	x	x	-	N/A
Zhang et al. (2013) for $\alpha = 0.5$	x	x	x	-	N/A
Zhang et al. (2013) for $\alpha = 1$	x	x	x	-	N/A
CPS_I	0.119	0.107	0.089	$\tilde{E}_1 \succ \tilde{E}_2 \succ \tilde{E}_3$	100

Note: 'x' denotes method as unable to calculate the ranking value.
'-' denotes no ranking order is obtained.

Discussions

For embedded case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $\tilde{C}_1 \succ \tilde{C}_2 \succ \tilde{C}_3$. This is because the vertical – y centroid of type – I fuzzy number $\tilde{C}_1$ is the largest among the three, followed by $\tilde{C}_2$ and then $\tilde{C}_3$. In Table 5.3, Cheng (1998) and Kumar et al. (2010) ranking methods produce incorrect ranking order such that the ranking result is 0% consistent with human intuition for this case where both methods give equal ranking, $\tilde{C}_1 \approx \tilde{C}_2 \approx \tilde{C}_3$ as they treat all type – I fuzzy numbers under consideration as having the same area. A partially incorrect ranking order such that the ranking result is 50% consistent with human intuitions is obtained by Dat et al. (2012) where this method is incapable to differentiate $\tilde{C}_1$ and $\tilde{C}_2$ effectively. Different ranking orders are produced by Yu et al. (2013) and Zhang et al. (2014) as both ranking methods depend on decision maker's opinion to raking fuzzy numbers. The CPS_I ranking method on the other hand, ranks this case with correct ranking order such that the ranking result is 100% consistent with human intuition which emphasises that this method is capable to deal with embedded type – I fuzzy numbers of different shapes.

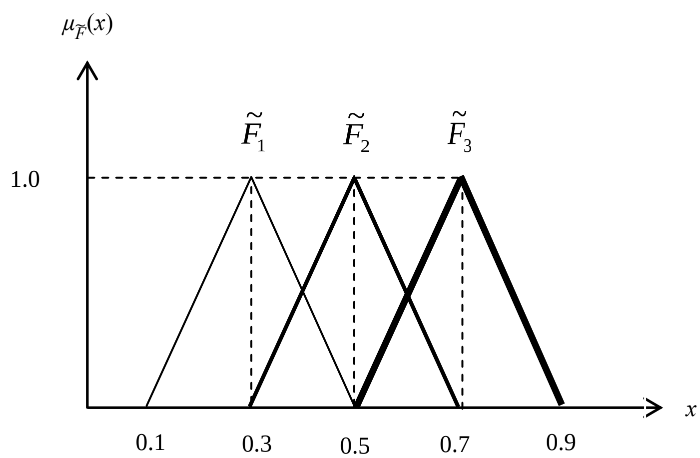
For embedded case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $\widetilde{D}_1 \prec \widetilde{D}_2 \prec \widetilde{D}_3$. This is due to the fact that the spread value for $\widetilde{D}_3$ is the smallest among the three, followed by $\widetilde{D}_2$ and then $\widetilde{D}_1$. Clearly indicate in Table 5.4 is the incorrect ranking results by Cheng (1998), Kumar et al. (2010) and Dat et al. (2012) such that the results are 0% consistent with human intuition. All of them give equal ranking for this case, $\widetilde{D}_1 \approx \widetilde{D}_2 \approx \widetilde{D}_3$, because Cheng (1998) and Kumar et al. (2010) ranking methods treat all type – I fuzzy numbers under consideration as the same area whereas Dat et al. (2012) ranking method produces same distance for all type – I fuzzy numbers in this case. Yu et al. (2013) and Zhang et al. (2014) ranking methods produce many ranking results for this case since both take into account decision makers' opinion when ranking fuzzy numbers. Only the *CPSI* ranking method obtains the correct ranking order such that the ranking result is 100% consistent with human intuition for this case which signaling that this method is capable to differentiate type – I fuzzy numbers with different spread appropriately.

For embedded case 3, the correct ranking order such that the ranking result is 100% consistent with human intuition is $\widetilde{E}_1 \succ \widetilde{E}_2 \succ \widetilde{E}_3$. $\widetilde{E}_1$ is considered as the greatest type – I fuzzy numbers among the three because the height of $\widetilde{E}_1$ is the largest, followed by $\widetilde{E}_2$ and then $\widetilde{E}_3$. In Table 5.5, ranking method by Kumar et al. (2010) treats this case with equal ranking, $\widetilde{E}_1 \approx \widetilde{E}_2 \approx \widetilde{E}_3$ as this method considers all type – I fuzzy numbers under consideration as the same area. Yu et al. (2013) ranking method produces different ranking order for different decision makers' opinions while Zhang et al. (2014) ranking method is incapable to come out with any ranking order as the method is not applicable to non – normal fuzzy numbers. Nonetheless, correct ranking orders such that the ranking result is 100% consistent with human intuition are obtained by Cheng (1998), Dat et al. (2012) and the CPS_I ranking method. This result implies that the CPS_I ranking method is capable to deal with type – I fuzzy numbers of different heights effectively.

Overlapping Case

Overlapping Case 1

Overlapping case 1 illustrates in Figure 5.6 involves three overlapping identical triangular type – I fuzzy numbers which are same in spread and height. Nevertheless, they are differed in terms of their positions.

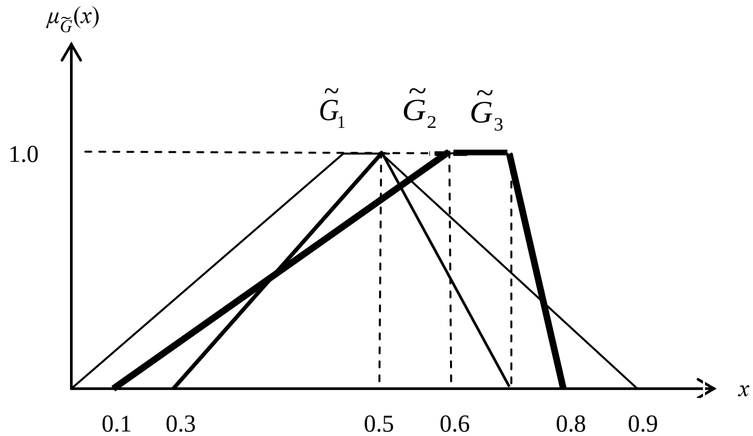


$$\tilde{F}_1 = (0.1, 0.3, 0.3, 0.5; 1.0) \ \tilde{F}_2 = (0.3, 0.5, 0.5, 0.7; 1.0) \ \tilde{F}_3 = (0.5, 0.7, 0.7, 0.9; 1.0)$$

Fig 5.6: Overlapping Case 1

Overlapping Case 2

Overlapping case 2 involves three overlapping type – I fuzzy numbers comprise two trapezoidal type – I fuzzy numbers and a triangular type – I fuzzy numbers as illustrate in Figure 5.7. All of them are same of height but different of centroid point and spread.



$$\tilde{G}_1 = (0.0,0.4,0.5,0.7;1.0)\ \tilde{G}_2 = (0.2,0.5,0.5,0.9;1.0)\ \tilde{G}_3 = (0.1,0.6,0.7,0.8;1.0)$$

Fig 5.7: Overlapping Case 3

Results and Validation

Comparisons of ranking order for overlapping case 1 and 2 between the CPS_I ranking method and established ranking methods considered in this study are illustrated in Table 5.6 and 5.7 respectively.

Table 5.6: Ranking Results for Overlapping Case 1

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	$\tilde{F}_1$	$\tilde{F}_2$	$\tilde{F}_3$		
Cheng (1998)	0.583	0.707	0.831	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
Kumar et al. (2010)	0.3	0.5	0.8	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
Dat et al. (2012)	0 .000 / 0.040	0.040 / 0.400	0. 000 / 0.400	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
Yu et al. (2013) for α = 0	0.300	0.500	0.700	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
Yu et al. (2013) for α = 0.5	0.300	0.500	0.700	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
Yu et al. (2013) for α = 1	0.300	0.500	0.700	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
Zhang et al. (2013) for α = 0	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
Zhang et al. (2013) for α = 0.5	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
Zhang et al. (2013) for α = 1	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100
CPS_I	0.089	0.107	0.119	$\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$	100

Table 5.7: Ranking Results for Overlapping Case 2

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	$\tilde{G}_1$	$\tilde{G}_2$	$\tilde{G}_1$		
Cheng (1998)	0.680	0.726	0.746	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
Kumar et al. (2010)	0.300	0.500	0.700	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
Dat et al. (2012)	0.000 / 0.040	0.040 / 0.400	0. 400 / 0.000	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
Yu et al. (2013) for $\alpha = 0$	0.300	0.500	0.700	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
Yu et al. (2013) for $\alpha = 0.5$	0.300	0.500	0.700	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
Yu et al. (2013) for $\alpha = 1$	0.500	0.7200	0.969	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
Zhang et al. (2013) for $\alpha = 0$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
Zhang et al. (2013) for $\alpha = 0.5$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
Zhang et al. (2013) for $\alpha = 1$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100
CPS_I	0.089	0.107	0.119	$\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$	100

Discussions

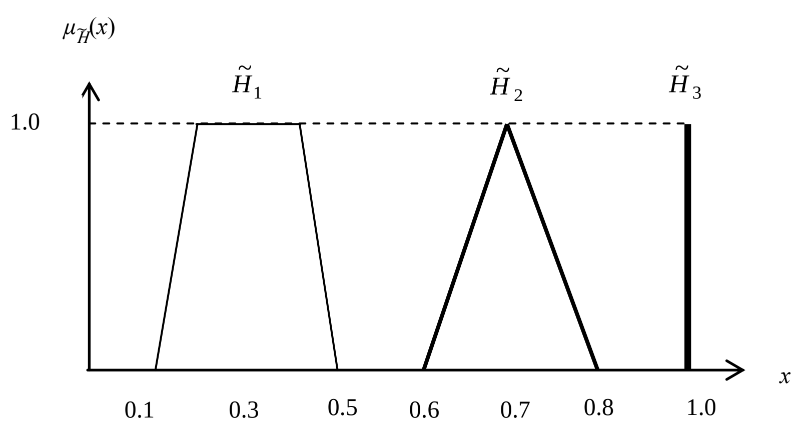
For overlapping case 1, the correct ranking order such that the ranking results is 100% consistent with human intuition is $\tilde{F}_1 \prec \tilde{F}_2 \prec \tilde{F}_3$. This is because $\tilde{F}_3$ is situated on the farthest right among the three, followed by $\tilde{F}_2$ and then $\tilde{F}_1$. Table 5.6 indicates that all ranking methods considered in this study including the *CPSI* ranking method produce correct ranking order such that the ranking result is 100% consistent with human intuition. All ranking methods obtain correct ranking result because this case is easy to distinguish. The result of the *CPSI* ranking method obtained in this case indicates that this method is capable to appropriately differentiate partial overlapping type – I fuzzy numbers.

For overlapping case 2, the correct ranking order such that the ranking results is 100% consistent with human intuition is $\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$. This is due to the fact that when combining both values of centroi point and spread of each type – I fuzzy number under consideration, $\tilde{G}_3$ is the greatest followed b y $\tilde{G}_2$ and then $\tilde{G}_1$. Table 5.7 shows all ranking methods under consideration including the CPS_I ranking method produce the same correct ranking order such that the ranking result is 100% consistent with human intuition because this case is trivial. This indicates that the CPS_I ranking method is capable to appropriately deal with overlapping case of type – I fuzzy numbers like other established ranking methods.

Non - Overlapping Case

Non - Overlapping Case 1

Non **-** overlapping Case 1 involves different types of type – I fuzzy numbers namely trapezoidal, triangular and singleton that are not overlapped as shown in Figure 5.8. In this case, all of the type – I fuzzy numbers considered are differed in terms of the centroid point and spread but are the same of height.

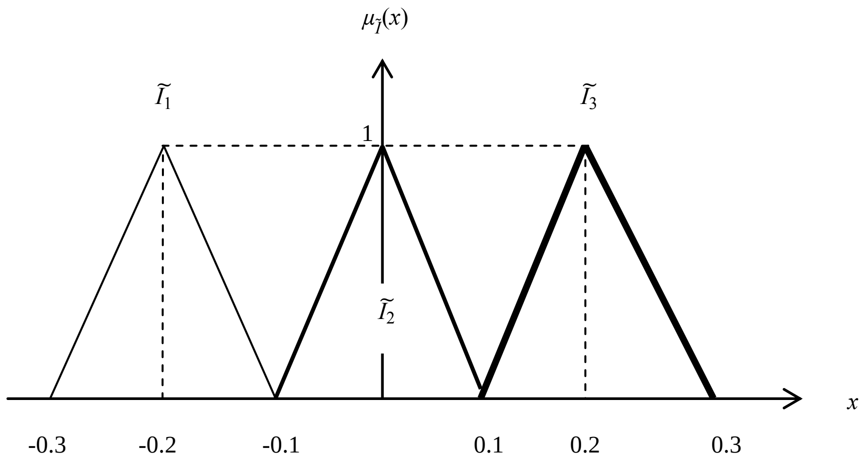


$\tilde{H}_1 = (0.1,0.3,0.3,0.5;1.0)$ $\tilde{H}_2 = (0.6,0.7,0.7,0.8;1.0)$ $\tilde{H}_3 = (1.0,1.0,1.0,1.0;1.0)$

Fig 5.8: Non - Overlapping Case 1

Non - Overlapping Case 2

Non – overlapping case 2 involves three identical triangular type – I fuzzy numbers of same spread and height. The only distinction between them is their position. One of them is situated on the negative side, one is on positive side and the other is in the middle of positive and negative values. This case is classified as the mirror image situation or reflection case of type – I fuzzy numbers (Asady, 2009) which is illustrated in Figure 5.9.



$\tilde{I}_1 = (-0.3,-0.2,-0.2,-0.1;1,0)$ $\tilde{I}_2 = (-0.1,0.0,0.0,0.1;1,0)$ $\tilde{I}_3 = (0.1,0.2,0.2,0.3;1,0)$

Fig 5.9: Non – Overlapping Case 2

Results and Validation

Comparisons of ranking order for non – overlapping Case 1 and 2 between the CPS_I ranking method and other established ranking methods considered in this study are illustrated in Table 5.8 and 5.9 respectively.

Table 5.8: Ranking Results for Non – Overlapping Case 1

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	$\tilde{H}_1$	$\tilde{H}_2$	$\tilde{H}_1$		
Cheng (1998)	0.424	0.583	x	-	N/A
Kumar et al. (2010)	0.300	0.300	x	-	N/A
Dat et al. (2012)	0.000 / 0.300	0.300 / 0.600	0. 600 / 0.000	$\tilde{H}_1 \prec \tilde{H}_2 \prec \tilde{H}_3$	100
Yu et al. (2013) for α = 0	0.700	0.300	x	-	N/A
Yu et al. (2013) for α = 0.5	1.000	1.000	x	-	N/A
Yu et al. (2013) for α = 1	0.300	0.7200	x	-	N/A
Zhang et al. (2013) for α = 0	1.000	1.000	x	-	N/A
Zhang et al. (2013) for α = 0.5	1.000	1.000	x	-	N/A
Zhang et al. (2013) for α = 1	1.000	1.000	x	-	N/A
CPS_I	0.089	0.107	0.119	$\tilde{H}_1 \prec \tilde{H}_2 \prec \tilde{H}_3$	100

Table 5.9: Ranking Results for Non – Overlapping Case 2

Methods	Fuzzy Numbers $\tilde{I}_1$	$\tilde{I}_2$	$\tilde{I}_3$	Ranking Results	Level of Consistency (%)
Cheng (1998)	0.583	0.583	0.583	$\tilde{I}_1 \approx \tilde{I}_2 \approx \tilde{I}_3$	0
Kumar et al. (2010)	0.300	0.300	0.300	$\tilde{I}_1 \approx \tilde{I}_2 \approx \tilde{I}_3$	0
Dat et al. (2012)	0.000 / 0.300	0.300 / 0.600	0. 600 / 0.000	$\tilde{I}_1 \prec \tilde{I}_2 \prec \tilde{I}_3$	100
Yu et al. (2013) for $\alpha = 0$	751	0.000	0.001	$\tilde{I}_1 \succ \tilde{I}_3 \succ \tilde{I}_2$	0
Yu et al. (2013) for $\alpha = 0.5$	1.000	1.000	1.000	$\tilde{I}_1 \approx \tilde{I}_2 \approx \tilde{I}_3$	0
Yu et al. (2013) for $\alpha = 1$	0.001	0.000	751	$\tilde{I}_2 \prec \tilde{I}_1 \prec \tilde{I}_3$	100
Zhang et al. (2013) for $\alpha = 0$	1.000	1.000	1.000	$\tilde{I}_1 \succ \tilde{I}_2 \succ \tilde{I}_3$	0
Zhang et al. (2013) for $\alpha = 0.5$	1.000	1.000	1.000	$\tilde{I}_1 \approx \tilde{I}_2 \approx \tilde{I}_3$	0
Zhang et al. (2013) for $\alpha = 1$	1.000	1.000	1.000	$\tilde{I}_1 \prec \tilde{I}_2 \prec \tilde{I}_3$	100
CPS_I	0.089	0.107	0.119	$\tilde{I}_1 \prec \tilde{I}_2 \prec \tilde{I}_3$	100

Discussions

For non – overlapping case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $\tilde{H}_1 \prec \tilde{H}_2 \prec \tilde{H}_3$. This is because $\tilde{H}_3$ is situated on the farthest right among the three and followed by $\tilde{H}_2$ and then $\tilde{H}_1$. Table 5.8 clearly signifies that only Dat et al. (2012) and the CPS_I ranking methods are capable to rank this case correctly such that the ranking result is 100% consistent with human intuition. For other ranking methods considered in this study, all of them are incapable to rank singleton type – I fuzzy numbers appropriately, thus all of them are not applicable for ranking fuzzy numbers. This shows that the CPS_I ranking method is capable to appropriately deal with non – overlapping type – I fuzzy numbers and singleton type – I fuzzy numbers.

For non – overlapping case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $\tilde{I}_1 \prec \tilde{I}_2 \prec \tilde{I}_3$. This is due to the fact that $\tilde{I}_3$ is located on the farthest right which is on the positive side, followed by $\tilde{I}_2$ and then $\tilde{I}_1$. In Table 5.9, Cheng (1998) and Kumar et al. (2010) ranking methods produce equal ranking, $\tilde{I}_1 \approx \tilde{I}_2 \approx \tilde{I}_3$ for this case which is incorrect such that the ranking result is 0% consistent with human intuition. Yu et al. (2013) and Zhang et al. (2014) ranking methods also come out with many ranking orders for this case as they depend on decision makers' opinions when ranking fuzzy numbers. Only Dat et al. (2012) and the CPS_I ranking methods capable to give correct ranking order for this case such that the ranking result is 100% consistent with human intuition. This directly emphasise that the CPS_I ranking method is capable to effectively deal with negative and positive type – I fuzzy numbers simultaneously.

Summary of Consistency Evaluation

This subsection covers the summary on the consistency evaluations for all ranking methods considered in section 5.2.1 including the CPS_I ranking method. The summary provides clear observation in terms of number of consistent ranking result produced by all ranking methods considered in this study and their performance percentage. Using Section 4.4 as guideline and information obtained from Table 5.1 until Table 5.9, the following Table 5.10 summaries the consistency evaluation of all ranking methods considered in this study including the CPS_I ranking method on ranking type – I fuzzy numbers.

Table 5.10: Summary of Consistency Evaluation

Methods	Consistency Evaluation	
	Proportion of Result with 100% Level of Consistency	Percentage of Result with 100% Level of Consistency
Cheng (1998)	4/9	44.44%
Kumar et al. (2010)	3/9	33.33%
Dat et al. (2012)	7/9	77.75%
Yu et al. (2013) for $\alpha = 0$	4/9	44.44%
Yu et al. (2013) for $\alpha = 0.5$	4/9	44.44%
Yu et al. (2013) for $\alpha = 1$	4/9	44.44%
Zhang et al. (2014) for $\alpha = 0$	4/9	55.55%
Zhang et al. (2014) for $\alpha = 0.5$	4/9	55.55%
Zhang et al. (2014) for $\alpha = 1$	4/9	55.55%
CPS_I	9/9	100%

Results in Table 5.8 show that Kumar et al. (2010) ranking method obtains the least number of consistent ranking results where the method ranks three out of nine (33.33%) cases of benchmark examples provided in this study. Cheng (1998) and Yu et al. (2013) with $\alpha = 0$ and 0.5 share the same number of consistent ranking results with four out of nine cases which is equivalence to 44.44%. Zhang et al. (2013) with $\alpha = 0$ and 0.5 ranking methods successfully rank five out of nine (55.55%) benchmark examples. Dat et al. (2012) and Zhang et al. (2014) with $\alpha = 1$ ranking methods achieve seven out nine cases while Yu et al. (2013) ranking method ranks eight out of nine cases of benchmarking examples prepared in this study. Among all ranking methods considered in this evaluation, only the CPS_I ranking method perfectly ranks all nine (100%) cases of benchmarking examples with correct ranking order such that all results obtained are 100% consistent with human intuition. Therefore, this evaluation clearly indicates that the CPS_I ranking method is considered as a ranking method that correctly ranks all type – I fuzzy numbers such that the ranking results are 100% consistent with human intuition.

5.2.2 Evaluation of Efficiency

This subsection discusses the efficiency evaluations of all the ranking methods considered in this study including the CPS_I ranking method. It is intentionally prepared as a separate subsection from the summary of the consistency evaluation because all ranking methods considered in this study and the CPS_I ranking method, perform similar efficiency capability when ranking three type – I fuzzy numbers. This is because the efficiency result of a ranking method is the same for all benchmarking examples provided in this study even if the consistency evaluations are different. Therefore, without loss of generality of Section 4.5, the efficiency evaluations of all ranking methods considered in this study including the CPS_I ranking method are summarised in Table 5.11.

Table 5.11: Summary of Efficiency Evaluation

Methods	Efficiency Evaluation
Cheng (1998)	Slightly Efficient
Kumar et al. (2010)	Slightly Efficient
Dat et al. (2012)	Slightly Inefficient
Yu et al. (2013) for $\alpha = 0$	Slightly Efficient
Yu et al. (2013) for $\alpha = 0.5$	Slightly Efficient
Yu et al. (2013) for $\alpha = 1$	Slightly Efficient
Zhang et al. (2013) for $\alpha = 0$	Very Inefficient
Zhang et al. (2013) for $\alpha = 0.5$	Very Inefficient
Zhang et al. (2013) for $\alpha = 1$	Very Inefficient
CPS_I	Very Efficient

In Table 5.11, Zhang et al. (2014) ranking method with α = 0, 0.5 and 1, is classified as a very inefficient ranking method as this method is a pairwise ranking method and needs additional operation to ranking type – I fuzzy number appropriately. Dat et al. (2012) ranking method is evaluated as a slightly inefficient ranking method because it is a pairwise ranking method but does not need additional operation when ranking type – I fuzzy numbers appropriately. Cheng (1998) and Yu et al. (2012) ranking methods are considered as slightly efficient ranking methods in this evaluation as both simultaneously rank the type – I fuzzy numbers but incorporate additional operation in obtaining the final ranking order. In this evaluation, the CPS_I ranking method is regarded as a very efficient ranking method as this method ranks fuzzy numbers correctly such that the ranking result is consistent with human intuition using simultaneous ranking without incorporating any additional operation. Therefore, this evaluation signifies that the CPS_I ranking method is capable to rank three type – I fuzzy numbers simultaneously without incorporating additional operation when ranking type – I fuzzy numbers.

5.3 SUMMARY

In this chapter, the capability of the CPS_I ranking method to ranking type – I fuzzy numbers is provided. Two main empirical validations namely the consistency andefficiency of the CPS_I ranking method are also highlighted in this chapter. In the validation, the capability of the CPS_I ranking method to correctly ranks all cases of type – I fuzzy numbers such that the ranking results are consistent with human intuition is addressed. The efficiency of the CPS_I ranking method on ranking three type – I fuzzy numbers simultaneously is also demonstrated in this chapter where the method is capable to ranking three type – I fuzzy numbers simultaneously without incorporating additional operation. In this respect, the CPS_I ranking method is considered as a ranking method that is capable on ranking type – I fuzzy numbers consistently and efficiently. In Chapter 6, the thesis extends the applicability of the *CPS* ranking methodology in ranking type – II fuzzy numbers.

CHAPTER SIX

RANKING OF TYPE – II FUZZY NUMBERS

6.1 INTRODUCTION

This chapter discusses details on validation of the novel methodology for ranking type – II fuzzy numbers based on centroid point and spread, CPS_{II}. Theoretical and empirical validation defined in Section 4.4 and 4.5 respectively are demonstrated in this chapter. These validation which are associated with properties of ranking fuzzy quantities as well as consistency and efficiency evaluation of ranking operations are described in detail here. Therefore, without loss of generality of Section 4.4 and 4.5, details on those aforementioned both validation are extensively discussed in sections and subsections of this chapter.

6.2 THEORETICAL VALIDATION

This subsection validates theoretically the novel CPS_{II} ranking method using theoretical properties adopted from Wang & Kerre (2001, 2002). These properties justify the capability of the CPS_{II} ranking method to ranking interval type – II fuzzy numbers appropriately by proofs provided which are applicable to CPS_{II} ranking method. It has to be noted that only theoretical validation for direct approach of ranking interval type – II fuzzy numbers is demonstrated here. This is because theoretical validation for the indirect approach is the same as in theoretical validation for type – I fuzzy numbers. Therefore, with no loss of generality, theoretical ordering properties by Wang & Kerre (2001, 2002) which are prepared for CPS_{II} ranking method are presented as follows.

Let A_1' and A_2' be two standardised generalised type – II fuzzy numbers.

***Property 1*:** If $A_1' \succcurlyeq A_2'$ and $A_2' \succcurlyeq A_1'$, then $A_1' \approx A_2'$

***Proof*:**

Since, $A_1' \succcurlyeq A_2'$ implies that $CPS_{II}(A_1') \geq CPS_{II}(A_2')$, and $A_2' \succcurlyeq A_1'$ implies that $CPS_{II}(A_2') \geq CPS_{II}(A_1')$ hence indicates that, $CPS_{II}(A_1') = CPS_{II}(A_2')$, which is $A_1' \approx A_2'$

***Property 2*:** If $A_1' \succcurlyeq A_2'$ and $A_2' \succcurlyeq A_3'$, then $A_1' \succcurlyeq A_3'$

***Proof*:**

For CPS_{II} ranking method, $A_1' \succcurlyeq A_2'$ implies that $CPS_{II}(A_1') \geq CPS_{II}(A_2')$, and $A_2' \succcurlyeq A_3'$, implies that $CPS_{II}(A_2') \geq CPS_{II}(A_3')$. This indicates that $CPS_{II}(A_1') \geq CPS_{II}(A_3')$, which is $A_1' \succcurlyeq A_3'$.

***Property 3*:** If $A_1' \cap A_2' = \emptyset$ and A_1' is on the right side of A_2', then $A_1' \succcurlyeq A_2'$

***Proof*:**

Since, $A_1' \cap A_2' = \emptyset$ and A_1' is on the right side of A_2', hence, implies that $CPS_{II}(A_1') \geq CPS_{II}(A_2')$, thus, $A_1' \succcurlyeq A_2'$.

***Property 4*:** Ordering of A_1' and A_2' is not affected by the other type – II fuzzy numbers under comparison.

***Proof*:**

Since, the order of A_1' and A_2', is completely determined by $CPS_{II}(A_1')$ and $CPS_{II}(A_2')$ respectively, which indicates that it has nothing to do by the other type – II fuzzy numbers under comparison, thus, the ordering of A_1' and A_2' is not affected by the other type – II fuzzy numbers under comparison.

The above theoretical validation clearly indicates that the *CPS*$_{II}$ ranking method is capable to ranking fuzzy numbers appropriately. This is signified through proof based – properties fulfilment by the *CPS*$_{II}$ ranking method on all theoretical validations considered in this subsection. In next section, empirical validation for the CPS_{II} ranking method and established ranking methods considered in this study is thoroughly discussed.

6.3 EMPIRICAL VALIDATION

This section discusses empirical validation of the CPS_{II} ranking method and established ranking methods considered in this study on ranking interval type – II fuzzy numbers. The empirical validation provided is a comparative – based ranking order analysis between the CPS_{II} ranking method and established ranking methods under consideration on their consistency and efficiency to ranking interval type – II fuzzy numbers. Most of the established ranking methods considered in this validation are methods for ranking type – I fuzzy numbers while the remaining methods are for ranking type – II fuzzy numbers. These methods are chosen according to their high referencing frequency by many established ranking methods found in literature of fuzzy sets. For those ranking methods that are developed for ranking type – I fuzzy numbers, they are denoted with 'II' (for example: II – Cheng (1998)) in this study to indicate that they are applied to ranking interval type – II fuzzy numbers for the first time. Therefore, based on information in Section 4.5, the consistency and efficiency evaluation of the CPS_{II} ranking method and established ranking methods considered in this study are as follows.

6.3.1 Evaluation of Consistency

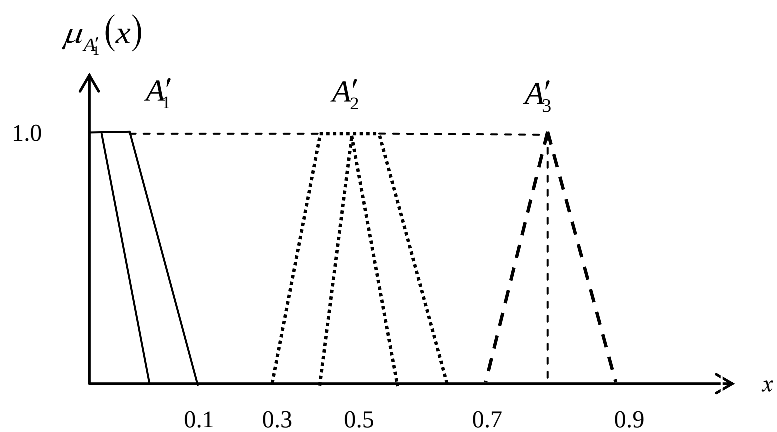
In this subsection, 9 benchmarking sets of interval type – II fuzzy numbers with modification adopted from Wu & Mendel (2009) are used. Modifications are made in this subsection as this study covers more generic and complex cases which are more important in decision making than previous work by Wu & Mendel (2009). Among generic and complex cases of interval type – II fuzzy numbers that are neglected in Wu & Mendel (2009) but considered in this study are non – overlapping, negative data value and crisp value cases. Furthermore, three interval type – II fuzzy numbers which are suitable for each case considered in this study are chosen from the 32 interval type – II fuzzy numbers by Wu & Mendel (2009). The utilisation of selected three type – II fuzzy numbers in each case not only provides better view on cases similar as in real world problems but also give same effect on ranking results as of the 32 interval type – II fuzzy numbers in Wu & Mendel (2009). Thus, the following are details on consistency evaluation based on 9 benchmark examples of all ranking methods considered in this study including both direct and indirect ways using the CPS_{II} ranking method.

Using direct based – CPS_{II} ranking method, the ranking order for A_1', A_2' and A_3' in this case is determined as follows.

Trivial Case

Trivial Case 1

Trivial case 1 involves three interval type – II fuzzy numbers that are not overlapped as shown in Figure 6.1.



$$A_1' = (0.000,0.000,0.014,0.197;1.000),(0.000,0.000,0.005,0.066;1.000)$$

$$A_2' = (0.359,0.475,0.550,0.691;1.000),(0.486,0.503,0.503,0.514;1.000)$$

$$A_3' = (0.598,0.775,0.860,0.952;1.000),(0.803,0.836,0.836,0.917;1.000)$$

Fig 6.1: Trivial Case 1

Step 1: Compute the centroid point for A_1' by finding the horizontal – x centroid of A_1' using equation (6.1) as

$$x_{A_1'} = \left(\frac{1}{3}\left[0+0+0.014+0.1971-\frac{(0.002-0)}{(0.211-0)}\right], \frac{1}{3}\left[0+0+0.005+0.066-\frac{(0.0003-0)}{(0.071-0)}\right]\right)$$

$$= (0.0660, 0.0221)$$

Whereas, using equation (6.2), the value of $y_{A_1'}$ is

$$x_{A_1'} = \left(\frac{1}{3}\left[1+\frac{0.014}{(0.211-0)}\right], \frac{1}{3}\left[1+\frac{0.005}{(0.071-0)}\right]\right)$$

Hence, the centroid point for A_1' is (0.0660, 0.355) and (0.0221, 0.3568).

Utilising the same procedure as shown above, the centroid points of A_2' and A_3' calculated accordingly and the results are as follows.

$$\left(x_{A_2'}, y_{A_2'}\right) = (0.5201, 0.3948), (0.5201, 0.3948)$$

$$\left(x_{A_3'}, y_{A_3'}\right) = (0.8520, 0.3333), (0.5201, 0.3948)$$

Step 2: Calculate the spread values for A_1' such that

$$s(A_1') = (0.1971 \times 0.3555), (0.0660 \times 0.3568)$$

$$= (0.0700), (0.0235)$$

While for A_2' and A_3', their spread values are

$$s(A_2') = (0.1311), (0.0093)$$

$$s(A_3') = (0.1408), (0.0380)$$

Step 3: Determine the ranking value for A_1' using the following equation

$$CPS_{II} = \left(\frac{0.0660 + 0.0221}{2}\right) \times \left(\frac{0.3555 + 0.3568}{2}\right) \times \left(\frac{(1 - 0.0700) + (1 - 0.0235)}{2}\right)$$

$$= 0.0150$$

and ranking values for A_2' and A_3' are

$$CPS_{II}(A_2') = 0.1728$$

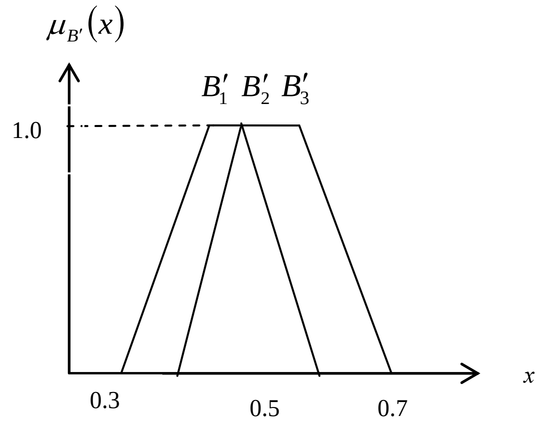
$$CPS_{II}(A_3') = 0.2736$$

Since $CPS_{II}(A_3') > CPS_{II}(A_2') > CPS_{II}(A_1')$, hence the ranking order result for interval type – II fuzzy numbers A_1', A_2' and A_3' is $A_3' \succ A_2' \succ A_1'$.

It is worth mentioning here that the entire steps utilised by the CPS_{II} ranking method in ranking interval type – II fuzzy numbers are only demonstrated in Trivial Case 1. This is because these steps are also applied to the remaining eight cases of benchmarking examples considered in this study, thus repeating the entire steps in the thesis are redundant. Therefore, only definition, illustration, the ranking results and discussions on each case considered are provided.

Trivial Case 2

Trivial case 2 involves three identical interval type – II fuzzy numbers which are embedded with each other. The following Figure 6.2 illustrates interval type – II fuzzy numbers of trivial case 2.



$$B_1' = (0.359,0.475,0.550,0.691;1.000),(0.486,0.503,0.503,0.514;1.000)$$
$$B_2' = (0.359,0.475,0.550,0.691;1.000),(0.486,0.503,0.503,0.514;1.000)$$
$$B_3' = (0.359,0.475,0.550,0.691;1.000),(0.486,0.503,0.503,0.514;1.000)$$

Fig 6.2: Trivial Case 2

Results and Validation

Comparisons of ranking order for trivial case 1 and 2 between CPS_{II} ranking method and established ranking methods considered in this study are illustrated in Table 6.1 and 6.2 respectively.

Table 6.1: Ranking Results for Trivial Case 1

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	A_1'	A_2'	A_3'		
Mitchell (2006)	0.583	0.583	0.583	$A_1' \approx A_2' \approx A_3'$	100
Wu & Mendel (2009)	0.047	0.519	0.812	$A_1' \prec A_2' \prec A_3'$	100
II – Cheng (1998)	0.583	0.583	0.583	$A_1' \approx A_2' \approx A_3'$	0
II – Kumar et al. (2010)	0.300	0.300	0.300	$A_1' \approx A_2' \approx A_3'$	0
II – Dat et al. (2012)	0.000 / 0.222	0.222 / 0.333	0.333 / 0.000	$A_1' \prec A_2' \prec A_3'$	100
II – Yu et al. (2013) for α = 0	1.00	1.00	1.00	$A_1' \prec A_2' \prec A_3'$	100
II – Yu et al. (2013) for α = 0.5	1.00	1.00	1.00	$A_1' \prec A_2' \prec A_3'$	100
II – Yu et al. (2013) for α = 1	1.00	1.00	1.00	$A_1' \prec A_2' \prec A_3'$	100
II – Zhang et al. (2013) for α = 0	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$A_1' \prec A_2' \prec A_3'$	100
II – Zhang et al. (2013) for α = 0.5	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$A_1' \prec A_2' \prec A_3'$	100
II – Zhang et al. (2013) for α = 1	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$A_1' \prec A_2' \prec A_3'$	100
CPS_{II} – direct	0.089	0.107	0.119	$A_1' \prec A_2' \prec A_3'$	100

CPS_{II} – indirect	0.089	0.107	0.119	$A_1' \prec A_2' \prec A_3'$	100

Table 6.2: Ranking Results for Trivial Case 2

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	B_1'	B_2'	B_3'		
Mitchell (2006)	0.583	0.583	0.583	$B_1' \approx B_2' \approx B_3'$	100
Wu & Mendel (2009)	0.519	0.519	0.519	$B_1' \approx B_2' \approx B_3'$	100
II – Cheng (1998)	0.583	0.583	0.583	$B_1' \approx B_2' \approx B_3'$	100
II – Kumar et al. (2010)	0.300	0.300	0.300	$B_1' \approx B_2' \approx B_3'$	100
II – Dat et al. (2012)	0.333 / 0.333	0.333 / 0.333	0.333 / 0.333	$B_1' \approx B_2' \approx B_3'$	100
II – Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$B_1' \approx B_2' \approx B_3'$	100
II – Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$B_1' \approx B_2' \approx B_3'$	100
II – Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$B_1' \approx B_2' \approx B_3'$	100
II – Zhang et al. (2013) for $\alpha = 0$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$B_1' \approx B_2' \approx B_3'$	100
II – Zhang et al. (2013) for $\alpha = 0.5$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$B_1' \approx B_2' \approx B_3'$	100
II – Zhang et al. (2013) for $\alpha = 1$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$B_1' \approx B_2' \approx B_3'$	100
CPS_{II} – direct	0.1728	0.1728	0.1728	$B_1' \approx B_2' \approx B_3'$	100
CPS_{II} – indirect	0.1728	0.1728	0.1728	$B_1' \approx B_2' \approx B_3'$	100

It is worth notifying here that ranking values obtained by Dat et al (2012) and Zhang et al. (2014) ranking methods are separated by separator (/) in both Table 6.1 and Table 6.2. This is to point out that both methods adopted pairwise ranking approach to ranking interval type – II fuzzy numbers. Also indicated in both tables is Yu et al. (2013) ranking method where this method provides equal ranking values for all interval type – II fuzzy numbers under consideration but gives different ranking orders for different α . This happens because Yu et al. (2013) ranking method considers different type of decision makers' opinions which is reflected by α when ranking interval type – II fuzzy numbers, thus different ranking orders are computed for different values even if the ranking values obtained are the same at the first place. Notice that, these conditions of Dat et al (2012) and Zhang et al. (2014) ranking methods apply to all cases of benchmarking examples considered in this chapter while only some cases apply to Yu et al. (2013) ranking method.

Discussions

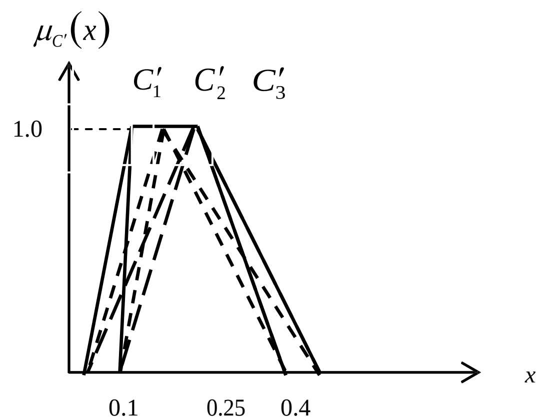
For trivial case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $A_3' \succ A_2' \succ A_1'$. This is because A_3' is located at the farthest right compared to A_2' and A_1', while A_2' is on the right of A_1'. In Table 6.1, only II – Cheng (1998) and II – Kumar et al. (2010) ranking methods produce incorrect ranking result such that the ranking results are 0% consistent with human intuition. While, other established ranking methods considered in this study including both direct and indirect ways of the CPS_{II} ranking method produce correct ranking order for this case such that the ranking result is 100% consistent with human intuition. This indicates that the CPS_{II} ranking method is capable to directly and indirectly deal with the interval type – II fuzzy numbers of different locations.

For trivial case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $B_1' \approx B_2' \approx B_3'$. This is due to the fact that all type – II fuzzy numbers under consideration are the same such that they are the same in term of their shapes, spreads, heights and centroids. Shown in Table 6.2, all ranking results obtained by all established ranking methods considered in this study and both direct and indirect ways of the CPS_{II} ranking method are the correct ranking order such that the results are 100% consistent with human intuition. This points out that the CPS_{II} ranking method is capable to give same ranking value for each interval type – II fuzzy numbers even if same type – II fuzzy numbers are compared regardless direct or indirect way is used.

Embedded Cases

Embedded Case 1

Embedded case 1 involves three embedded interval type – II fuzzy numbers which is illustrated in Figure 6.3.

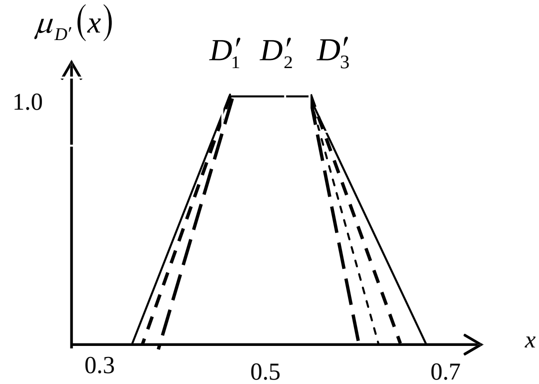


$$C_1' = (0.038,0.150,0.250,0.462;1.000),(0.109,0.150,0.250,0.421;1.000)$$
$$C_2' = (0.038,0.200,0.200,0.462;1.000),(0.109,0.200,0.200,0.421;1.000)$$
$$C_3' = (0.038,0.250,0.250,0.462;1.000),(0.109,0.250,0.250,0.421;1.000)$$

Fig 6.3: Embedded Case 1

Embedded Case 2

Embedded Case 2 involves three type – II fuzzy numbers where all of them are embedded, normal and having same centroid point for both upper and lower membership functions. Figure 6.4 best is the illustration for this case.

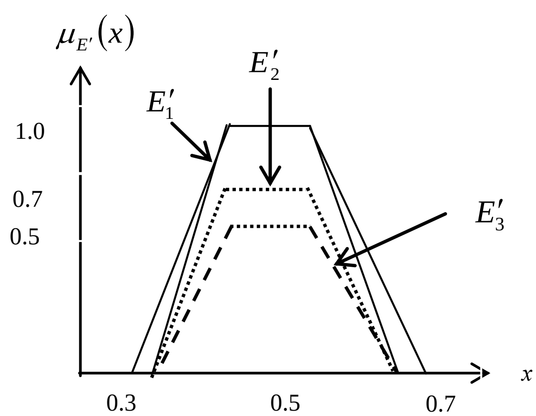


$$D_1' = (0.359, 0.475, 0.550, 0.691; 1.000), (0.400, 0.475, 0.550, 0.660; 1.000)$$
$$D_2' = (0.359, 0.475, 0.550, 0.691; 1.000), (0.430, 0.475, 0.550, 0.640; 1.000)$$
$$D_3' = (0.359, 0.475, 0.550, 0.691; 1.000), (0.450, 0.475, 0.550, 0.600; 1.000)$$

Fig 6.4: Embedded Case 2

Embedded Case 3

Embedded case 3 shown in Figure 6.5 involves three trapezoidal interval type – II fuzzy numbers that are embedded with each other.



$$E_1' = (0.359, 0.475, 0.550, 0.691; 1.000), (0.430, 0.475, 0.550, 0.640; 1.000)$$
$$E_2' = (0.359, 0.475, 0.550, 0.691; 1.000), (0.430, 0.475, 0.550, 0.640; 0.740)$$
$$E_3' = (0.359, 0.475, 0.550, 0.691; 1.000), (0.430, 0.475, 0.550, 0.640; 0.530)$$

Fig 6.5: Embedded Case 3

Results and Validation

Comparisons of ranking order for embedded case 1, 2 and 3 between the CPS_{II} ranking method and established ranking methods considered in this study are illustrated in Table 6.3, 6.4 and 6.5 respectively.

Table 6.3: Ranking Results for Embedded Case 1

Methods	Fuzzy Numbers C_1'	Fuzzy Numbers C_2'	Fuzzy Numbers C_3'	Ranking Results	Level of Consistency (%)
Mitchell (2006)	0.583	0.583	0.583	$C_1' \approx C_2' \approx C_3'$	0
Wu & Mendel (2009)	0.583	0.583	0.583	$C_1' \approx C_2' \approx C_3'$	0
II – Cheng (1998)	0.583	0.583	0.583	$C_1' \approx C_2' \approx C_3'$	0
II – Kumar et al. (2010)	0.300	0.300	0.300	$C_1' \approx C_2' \succ C_3'$	50
II – Dat et al. (2012)	0.333 / 0.333	0.333 / 0.333	0.222 / 0.333	$C_1' \approx C_2' \succ C_3'$	100
II – Yu et al. (2013) for α = 0	1.00	1.00	1.00	$C_1' \prec C_2' \prec C_3'$	0
II – Yu et al. (2013) for α = 0.5	1.00	1.00	1.00	$C_1' \approx C_2' \approx C_3'$	0
II – Yu et al. (2013) for α = 1	1.00	1.00	1.00	$C_1' \succ C_2' \succ C_3'$	100
II – Zhang et al. (2013) for α = 0	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$C_1' \approx C_2' \approx C_3'$	0
II – Zhang et al. (2013) for α = 0.5	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$C_1' \succ C_2' \succ C_3'$	100
II – Zhang et al. (2013) for α = 1	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$C_1' \succ C_2' \succ C_3'$	100
CPS_{II} – direct	0.119	0.107	0.089	$C_1' \succ C_2' \succ C_3'$	100
CPS_{II} – indirect	0.119	0.107	0.089	$C_1' \succ C_2' \succ C_3'$	100

Table 6.4: Ranking Results for Embedded Case 2

Methods	Fuzzy Numbers D_1'	Fuzzy Numbers D_2'	Fuzzy Numbers D_3'	Ranking Results	Level of Consistency (%)
Mitchell (2006)	0.583	0.583	0.583	$D_1' \approx D_2' \approx D_3'$	0
Wu & Mendel (2009)	0.519	0.519	0.519	$D_1' \approx D_2' \approx D_3'$	0
II – Cheng (1998)	0.300	0.300	0.300	$D_1' \approx D_2' \approx D_3'$	0
II – Kumar et al. (2010)	0.300	0.300	0.300	$D_1' \approx D_2' \approx D_3'$	0
II – Dat et al. (2012)	0.222 / 0.333	0.333 / 0.555	0.555 / 0.222	$D_1' \prec D_2' \prec D_3'$	100
II – Yu et al. (2013) for α = 0	1.00	1.00	1.00	$D_1' \succ D_2' \succ D_3'$	0
II – Yu et al. (2013) for α = 0.5	1.00	1.00	1.00	$D_1' \approx D_2' \approx D_3'$	0
II – Yu et al. (2013) for α = 1	1.00	1.00	1.00	$D_1' \prec D_2' \prec D_3'$	100
II – Zhang et al. (2013) for α = 0	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$D_1' \succ D_2' \succ D_3'$	0
II – Zhang et al. (2013) for α = 0.5	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$D_1' \approx D_2' \approx D_3'$	0
II – Zhang et al. (2013) for α = 1	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$D_1' \prec D_2' \prec D_3'$	100

CPS_{II} – direct	0.161	0.167	0.173	$D_1' \prec D_2' \prec D_3'$	100
CPS_{II} – indirect	0.161	0.167	0.173	$D_1' \prec D_2' \prec D_3'$	100

Table 6.5: Ranking Results for Embedded Case 3

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	E_1'	E_2'	E_3'		
Mitchell (2006)	0.583	0.461	0.346	$E_1' \succ E_2' \succ E_3'$	100
Wu & Mendel (2009)	0.175	0.175	0.175	$E_1' \approx E_2' \approx E_3'$	0
II – Cheng (1998)	0.240	0.240	0.240	$E_1' \approx E_2' \approx E_3'$	0
II – Kumar et al. (2010)	0.300	0.300	0.300	$E_1' \approx E_2' \approx E_3'$	0
II – Dat et al. (2012)	0.244 / 0.196	0.196 / 0.067	0.067 / 0.244	$E_1' \succ E_2' \succ E_3'$	100
II – Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$E_1' \prec E_2' \prec E_3'$	0
II – Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$E_1' \approx E_2' \approx E_3'$	0
II – Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$E_1' \succ E_2' \succ E_3'$	100
II – Zhang et al. (2013) for $\alpha = 0$	x	x	X	-	N/A
II – Zhang et al. (2013) for $\alpha = 0.5$	x	x	X	-	N/A
II – Zhang et al. (2013) for $\alpha = 1$	x	x	X	-	N/A
CPS_{II} – direct	0.051	0.045	0.040	$E_1' \succ E_2' \succ E_3'$	100
CPS_{II} – indirect	0.051	0.045	0.040	$E_1' \succ E_2' \succ E_3'$	100

Note: 'x' denotes method as unable to calculate the ranking value.
'-' denotes no ranking order is obtained.

Discussions

For embedded case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $C_1' \succ C_2' \succ C_3'$. This is because the vertical – *y* centroid of interval type – II fuzzy numbers C_1' is the largest among the three, followed by C_2' and then C_3'. In Table 6.3, Mitchel (2006), Wu & Mendel (2009), II – Cheng (1998) and II – Kumar et al. (2010) ranking methods produces incorrect ranking order such that the ranking result is 0% consistent with human intuition for this case where both methods give equal ranking, $C_1' \approx C_2' \approx C_3'$ as they treat all interval type – II fuzzy numbers under consideration as having the same area. A partially correct ranking order such that the ranking result is 50% consistent with human intuitions is obtained by II – Dat et al. (2012) where this method is incapable to differentiate C_1' and C_2' effectively.

Different ranking orders are produced by II – Yu et al. (2013) and II – Zhang et al. (2014) as both ranking methods depend on decision maker's opinion to ranking interval type – II fuzzy numbers. The CPS_{II} ranking methods for both direct and indirect ways on the other hand, rank this case with correct ranking order such that the ranking result is 100% consistent with human intuition which emphasises that this method is capable to deal with embedded interval type – II fuzzy numbers of different shapes.

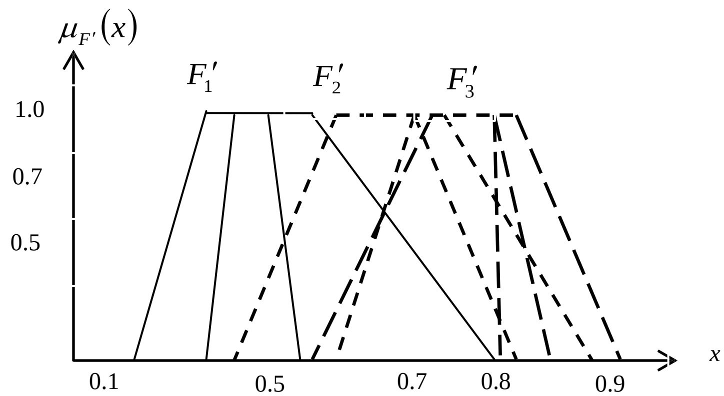
For embedded case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $D_1' \prec D_2' \prec D_3'$. This is due to the fact that the spread value for D_3' is considered as the smallest among the three, followed by D_2' and then D_1'. Clearly indicated in Table 6.4, Mitchel (2006), Wu & Mendel (2009), II – Cheng (1998), II – Kumar et al. (2010) and II – Dat et al. (2012) give equal ranking for this case, $D_1' \approx D_2' \approx D_3'$ because II – Cheng (1998) and II – Kumar et al. (2010) ranking methods treat all interval type – II fuzzy numbers under consideration as the same area whereas II – Dat et al. (2012) ranking method produces same distance for all interval type – II fuzzy numbers in this case. II – Yu et al. (2013) and II – Zhang et al. (2014) ranking methods produce many ranking results for this case since both take into account decision makers' opinion when ranking fuzzy numbers. Only the CPS_{II} ranking methods for both direct and indirect ways obtain the correct ranking order such that the ranking result is 100% consistent with human intuition for this case which signalling that these methods capable to differentiate interval type – II fuzzy numbers with different spread appropriately.

For embedded case 3, the correct ranking order such that the ranking result is 100% consistent with human intuition $E_1' \succ E_2' \succ E_3'$. E_1' is considered as the greatest interval type – II fuzzy numbers among the three because height of E_1' is the largest, followed by E_2' and then E_3'. In Table 6.5, ranking methods by Wu & Mendel (2009) and II – Kumar et al. (2010) treat this case with equal ranking, $E_1' \approx E_2' \approx E_3'$ as this method considers all interval type II fuzzy numbers under consideration as the same area. II – Yu et al. (2013) ranking method produces different ranking order for different decision makers' opinions while II – Zhang et al. (2014) ranking method is incapable to come out with any ranking order as the method is not applicable to non – normal interval type – II fuzzy numbers. Nonetheless, correct ranking orders such that the ranking result is 100% consistent with human intuition are obtained by II – Cheng (1998), II – Dat et al. (2012) and the CPS_{II} ranking methods for both direct and indirect ways. This result implies that the CPS_{II} ranking methods capable to deal with interval type – II fuzzy numbers of different heights effectively.

Overlapping Cases

Overlapping Case 1

Overlapping case 1 illustrates in Figure 6.6 involves three overlapping interval type – II fuzzy numbers which are of same height. Nonetheless, they are differed in terms of their positions.

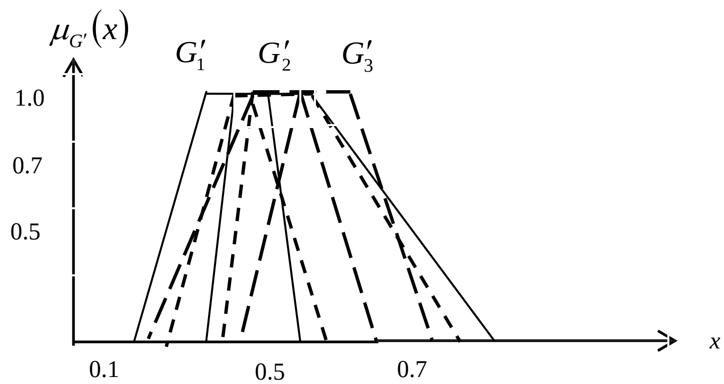


$$F_1' = (0.117,0.350,0.550,0.780;1.000),(0.409,0.465,0.465,0.541;1.000)$$
$$F_2' = (0.438,0.650,0.800,0.941;1.000),(0.679,0.738,0.738,0.821;1.000)$$
$$F_3' = (0.598,0.775,0.860,0.952;1.000),(0.803,0.836,0.836,0.917;1.000)$$

Fig 6.6: Overlapping Case 1

Overlapping Case 2

Overlapping Case 2 involves three overlapping interval type – II fuzzy numbers which have different spread and centroid point. All of them are normal and asymmetric. Figure 6.7 illustrates overlapping case 1 of interval type – II fuzzy numbers.



$$G_1' = (0.117, 0.350, 0.550, 0.780; 1.000), (0.409, 0.465, 0.465, 0.541; 1.000)$$

$$G_2' = (0.259, 0.400, 0.550, 0.762; 1.000), (0.429, 0.475, 0.475, 0.521; 1.000)$$

$$G_3' = (0.217, 0.425, 0.600, 0.741; 1.000), (0.479, 0.529, 0.529, 0.602; 1.000)$$

Fig 6.7: Overlapping Case 2

Results and Validation

Comparisons in terms of ranking order results for Overlapping Case 1 and 2 between the CPS_{II} ranking method and established ranking methods considered in this study are illustrated in Table 6.6 and 6.7 respectively.

Table 6.6: Ranking Results for Overlapping Case 1

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	F_1'	F_2'	F_3'		
Mitchell (2006)	0.583	0.583	0.583	$F_1' \approx F_2' \approx F_3'$	0
Wu & Mendel (2009)	0.456	0.716	0.812	$F_1' \prec F_2' \prec F_3'$	100
II – Cheng (1998)	0.088	0.088	0.088	$F_1' \approx F_2' \approx F_3'$	0
II – Kumar et al. (2010)	0.300	0.300	0.300	$F_1' \approx F_2' \approx F_3'$	0
II – Dat et al. (2012)	0.222 / 0.333	0.333 / 0.555	0.555 / 0.222	$F_1' \prec F_2' \prec F_3'$	100
II – Yu et al. (2013) for $\alpha = 0$	0.300	0.500	0.700	$F_1' \prec F_2' \prec F_3'$	100
II – Yu et al. (2013) for $\alpha = 0.5$	0.300	0.500	0.700	$F_1' \prec F_2' \prec F_3'$	100
II – Yu et al. (2013) for $\alpha = 1$	0.300	0.500	0.700	$F_1' \prec F_2' \prec F_3'$	100
II – Zhang et al. (2013) for $\alpha = 0$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$F_1' \prec F_2' \prec F_3'$	100
II – Zhang et al. (2013) for $\alpha = 0.5$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$F_1' \prec F_2' \prec F_3'$	100
II – Zhang et al. (2013) for $\alpha = 1$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$F_1' \prec F_2' \prec F_3'$	100
CPS_{II} – direct	0.144	0.235	0.274	$F_1' \prec F_2' \prec F_3'$	100
CPS_{II} – indirect	0.144	0.235	0.274	$F_1' \prec F_2' \prec F_3'$	100

Table 6.7: Ranking Results for Overlapping Case 2

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	G_1'	G_2'	G_3'		
Mitchell (2006)	0.680	0.726	0.746	$G_1' \prec G_2' \prec G_3'$	100
Wu & Mendel (2009)	0.456	0.495	0.513	$G_1' \prec G_2' \prec G_3'$	100
II – Cheng (1998)	0.240	0.240	0.240	$G_1' \prec G_2' \prec G_3'$	100
II – Kumar et al. (2010)	0.300	0.300	0.300	$G_1' \prec G_2' \prec G_3'$	100
II – Dat et al. (2012)	0.040 / 0.140	0.140 / 0.266	0.266 / 0.040	$G_1' \prec G_2' \prec G_3'$	100
II – Yu et al. (2013) for $\alpha = 0$	0.300	0.500	0.700	$G_1' \prec G_2' \prec G_3'$	100
II – Yu et al. (2013) for $\alpha = 0.5$	0.300	0.500	0.700	$G_1' \prec G_2' \prec G_3'$	100
II – Yu et al. (2013) for $\alpha = 1$	0.300	0.500	0.700	$G_1' \prec G_2' \prec G_3'$	100
II – Zhang et al. (2013) for $\alpha = 0$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$G_1' \prec G_2' \prec G_3'$	100
II – Zhang et al. (2013) for $\alpha = 0.5$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$G_1' \prec G_2' \prec G_3'$	100
II – Zhang et al. (2013) for $\alpha = 1$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$G_1' \prec G_2' \prec G_3'$	100
CPS_{II} – direct	0.144	0.159	0.168	$G_1' \prec G_2' \prec G_3'$	100
CPS_{II} – indirect	0.144	0.159	0.168	$G_1' \prec G_2' \prec G_3'$	100

Discussions

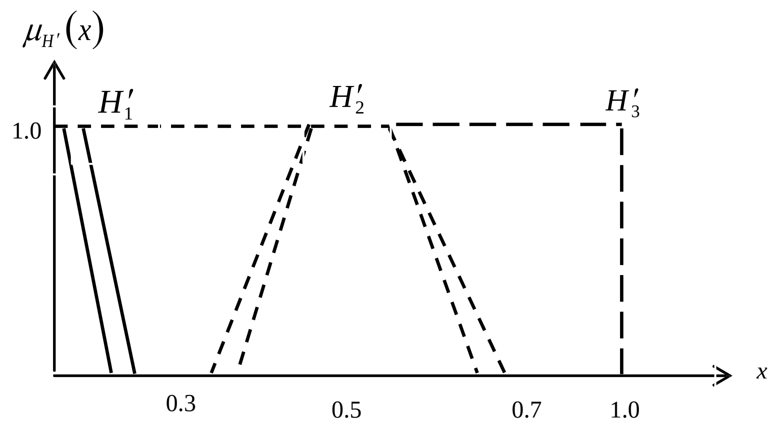
For overlapping case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $F_1' \prec F_2' \prec F_3'$. This is because F_3' is situated on the farthest right among the three, followed by F_2' and then F_1'. Table 6.6 indicates that only Mitchel (2006), II – Cheng (1998) and II – Kumar et al. (2010) ranking methods produce incorrect ranking order for this such that the result is 0% consistent with human intuition where they give equal ranking, $F_1' \approx F_2' \approx F_3'$ for this case. For other ranking methods considered in this study including both direct and indirect ways of the CPS_{II} ranking method, all of them produce correct ranking order such that the ranking results are consistent with human intuition. The result of the CPS_{II} ranking method obtains in this case indicates that this method is capable to appropriately differentiate partial overlapping interval type – II fuzzy numbers.

For overlapping case 2, the correct ranking order such that the ranking results is 100% consistent with human intuition is $G_1' \prec G_2' \prec G_3'$. This is due to the fact that when combining both values of centroid point and spread of each type – I fuzzy number under consideration, G_3' is the greatest followed by G_2' and G_1'. Table 6.7 shows all ranking methods under consideration including both direct and indirect ways of the CPS_{II} ranking method produce the same correct ranking order such that the ranking result is 100% consistent with human intuition. This signifies that the CPS_{II} ranking method is capable to appropriately deal with overlapping case of interval type – II fuzzy numbers like other established ranking methods.

Non – Overlapping Cases

Non - Overlapping Case 1

Non - overlapping Case 1 involves different types of interval type – II fuzzy numbers namely trapezoidal, triangular and singleton that are not overlapped as shown in Figure 6.8. In this case, all of the interval type – II fuzzy numbers considered are differed in terms of the centroid point and spread but are the same of height.

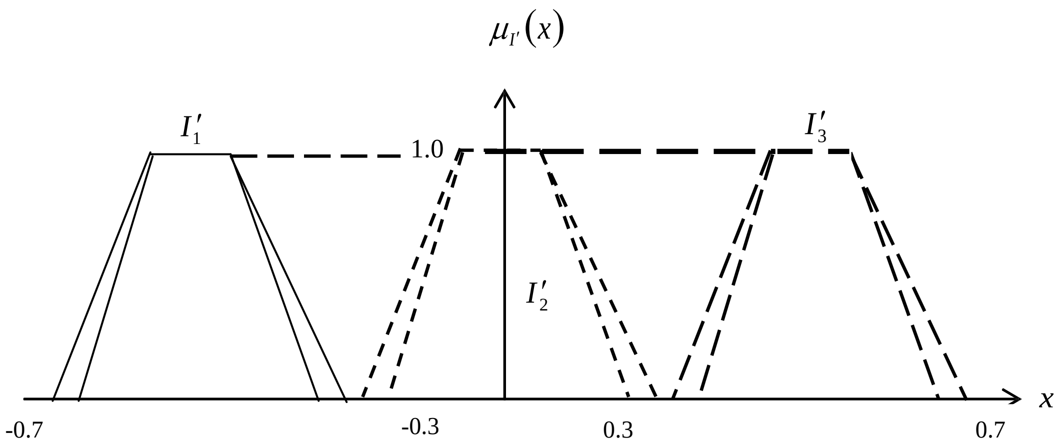


$$H_1' = (0.000, 0.000, 0.014, 0.197; 1.000), (0.000, 0.000, 0.005, 0.150; 1.000)$$
$$H_2' = (0.359, 0.475, 0.550, 0.691; 1.000), (0.486, 0.503, 0.503, 0.514; 1.000)$$
$$H_3' = (1.000, 1.000, 1.000, 1.000; 1.000), (1.000, 1.000, 1.000, 1.000; 1.000)$$

Fig 6.8: Non – overlapping Case 1

Non - Overlapping Case 2

Non – overlapping case 2 involves three identical interval type – II fuzzy numbers of same spread and height. The only distinction between them is their position. One of them is situated on the negative side, one is on positive side and the other is in the middle of positive and negative values. This case is classified as the mirror image situation or reflection case of interval type – II fuzzy numbers which is illustrated in Figure 6.9.



$$I_1' = (-0.691,-0.550,-0.475,-0.359;1,000),(-0.514,-0.503,-0.503,-0.486;1,000)$$
$$I_2' = (-0.313,-0.2800,0.250,0.313;1,0000),(-0.186,-0.150,0.230,0.186;1,0000)$$
$$\tilde{I}_3' = (0.359,0.475,0.550,0.691;1,000),(0.486,0.503,0.503,0.5141;1,000)$$

Fig 6.9: Non – overlapping Case 2

Results and Validation

Comparisons of ranking order for non – overlapping Case 1 and 2 between the CPS_{II} ranking method and established existing methods considered in this study are illustrated in Table 6.8 and 6.9 respectively.

Table 6.8: Ranking Results for Non – Overlapping Case 1

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	H_1'	H_2'	H_3'		
Mitchell (2006)	0.424	0.583	x	-	N/A
Wu & Mendel (2009)	0.047	0.519	x	-	N/A
II – Cheng (1998)	0.583	0.583	x	-	N/A
II – Kumar et al. (2010)	0.300	0.300	x	-	N/A
II – Dat et al. (2012)	0.222 / 0.333	0.333 / 0.555	0.555 / 0.222	$H_1' \prec H_2' \prec H_3'$	100
II – Yu et al. (2013) for α = 0	0.700	0.300	x	-	N/A
II – Yu et al. (2013) for α = 0.5	1.000	1.000	x	-	N/A
II – Yu et al. (2013) for α = 1	0.300	0.700	x	-	N/A
II – Zhang et al. (2013) for α = 0	1.000	1.000	x	-	N/A
II – Zhang et al. (2013) for α = 0.5	1.000	1.000	x	-	N/A
II – Zhang et al. (2013) for α = 1	1.000	1.000	x	-	N/A

CPS_{II} – direct	0.235	0.144	0.1274	$H_1' \prec H_2' \prec H_3'$	100
CPS_{II} – indirect	0.235	0.144	0.1274	$H_1' \prec H_2' \prec H_3'$	100

Note: 'x' denotes method as unable to calculate the ranking value.

'-' denotes no ranking order is obtained.

Table 6.9: Ranking Results for Non – Overlapping Case 2

Methods	Fuzzy Numbers			Ranking Results	Level of Consistency (%)
	I_1'	I_2'	I_3'		
Mitchell (2006)	x	x	0.583	-	N/A
Wu & Mendel (2009)	x	x	0.519	-	N/A
II – Cheng (1998)	0.240	0.240	0.240	$I_1' \approx I_2' \approx I_3'$	0
II – Kumar et al. (2010)	0.300	0.300	0.300	$I_1' \approx I_2' \approx I_3'$	0
II – Dat et al. (2012)	-0.400 / 0.000	0.000 / 0.400	0.400 / -0.400	$I_1' \prec I_2' \prec I_3'$	100
II – Yu et al. (2013) for α = 0	751	0.000	0.001	$I_1' \succ I_3' \succ I_2'$	0
II – Yu et al. (2013) for α = 0.5	1.000	1.000	1.000	$I_1' \approx I_2' \approx I_3'$	0
II – Yu et al. (2013) for α = 1	0.001	0.000	751	$I_2' \prec I_1' \prec I_3'$	100
II – Zhang et al. (2013) for α = 0	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$I_1' \prec I_2' \prec I_3'$	100
II – Zhang et al. (2013) for α = 0.5	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$I_1' \prec I_2' \prec I_3'$	100
II – Zhang et al. (2013) for α = 1	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$I_1' \prec I_2' \prec I_3'$	100
CPS_{II} – direct	-0.173	0.000	0.173	$I_1' \prec I_2' \prec I_3'$	100
CPS_{II} – indirect	-0.173	0.000	0.173	$I_1' \prec I_2' \prec I_3'$	100

Note: 'x' denotes method as unable to calculate the ranking value.

'-' denotes no ranking order is obtained.

Discussion

For non – overlapping case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $H_1' \prec H_2' \prec H_3'$. This is because H_3' is situated on the farthest right among the three and followed by H_2' and H_1'. Table 6.8 clearly signifies that only II – Dat et al. (2012) and both direct and indirect ways of the CPS_{II} ranking methods are capable to rank this case correctly such that the ranking result is 100% consistent with human intuition. For other ranking methods considered in this study, all of them are incapable to rank singleton interval type – II fuzzy numbers appropriately, thus all of them are not applicable for ranking interval type – II fuzzy numbers. This shows that the CPS_{II} ranking method is capable to appropriately deal with non – overlapping interval type – II fuzzy numbers and singleton interval type – II fuzzy numbers.

For non – overlapping case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $I_1' \prec I_2' \prec I_3'$. This is due to the fact I_3' is located on the farthest right which is on the positive side, followed by I_2' and then I_1' . In Table 6.9, II – Cheng (1998) and II – Kumar et al. (2010) ranking methods obtain equal ranking, $I_1 \approx 'I_2 \approx 'I_3'$ for this case which is incorrect such that the ranking result is 0% consistent with human intuition. II – Yu et al. (2013) ranking method comes out with many ranking orders for this case as they depend on decision makers' opinions when ranking fuzzy numbers. Only II – Dat et al. (2012), Zhang et al. (2014) and both ways of the CPS_{II} ranking methods to give correct ranking order for this case such that the ranking result is 100% consistent with human intuition. This directly emphasise that the CPS_{II} ranking method is capable to effectively deal with negative and positive interval type – II fuzzy numbers simultaneously.

Summary of Consistency Evaluation

This subsection covers the summary on the consistency evaluations for all ranking methods considered in section 6.2.1 including the CPS_{II} ranking method. The summary provides clear observation in terms of number of consistent ranking result produced by all ranking methods considered in this study and their performance percentage. Using Section 4.5 as guideline and information obtained from Table 6.1 until Table 6.9, the following Table 6.10 summaries the consistency evaluation of all ranking methods considered in this study including the CPS_{II} ranking method on ranking interval type – II fuzzy numbers.

Table 6.10: Summary of Consistency Evaluation

Methods	Consistency Evaluation	
	Proportion of Result with 100% Level of Consistency	Percentage of Result with 100% Level of Consistency
Mitchell (2006)	4/9	44.44%
Wu & Mendel (2009)	4/9	44.44%
II – Cheng (1998)	4/9	44.44%
II – Kumar et al. (2010)	3/9	33.33%
II – Dat et al. (2012)	7/9	77.75%
II – Yu et al. (2013) for $\alpha = 0$	4/9	44.44%
II – Yu et al. (2013) for $\alpha = 0.5$	4/9	44.44%
II – Yu et al. (2013) for $\alpha = 1$	4/9	44.44%
II – Zhang et al. (2014) for $\alpha = 0$	4/9	55.55%
II – Zhang et al. (2014) for $\alpha = 0.5$	4/9	55.55%
II – Zhang et al. (2014) for $\alpha = 1$	4/9	55.55%
CPS_{II} – direct	9/9	100%
CPS_{II} – indirect	9/9	100%

Results in Table 6.10 show that II – Kumar et al. (2010) ranking method obtains the least number of consistent ranking results where the method ranks three out of nine (33.33%) cases of benchmark examples provided in this study. II – Cheng (1998) and II – Yu et al. (2013) with α = 0 and 0.5 share the same number of consistent ranking results with four out of nine cases which is equivalence to 44.44%. II – Zhang et al. (2014) with α = 0.5and 0.5 ranking methods successfully rank five out of nine (55.55%) benchmark examples. II – Dat et al. (2012) and II – Zhang et al. (2014) with α = 1 ranking methods achieve seven out nine cases while II – Yu et al. (2013) ranking method ranks eight out of nine cases of benchmarking examples prepared in this study. Among all ranking methods considered in this evaluation, only the CPS_{II} ranking method for both direct and indirect ranking, perfectly rank all nine (100%) cases of benchmarking examples with correct ranking order such that all results obtained are 100% consistent with human intuition. Therefore, this evaluation clearly indicates that the CPS_{II} ranking method is considered as a ranking method that correctly ranks all interval type – II fuzzy numbers such that the ranking results are 100% consistent with human intuition.

6.3.2 Evaluation of Efficiency

This subsection discusses the efficiency evaluations of all the ranking methods considered in this study including the CPS_{II} ranking method. It is intentionally prepared as a separate subsection from the summary of the consistency evaluation because all ranking methods considered in this study and the CPS_{II} ranking method, perform similar efficiency capability when ranking three interval type – II fuzzy numbers. This is because the efficiency result of a ranking method is the same for all benchmarking examples provided in this study even if the consistency evaluations are different. Therefore, without loss of generality of Section 4.5, the efficiency evaluations of all ranking methods considered in this study including the CPS_{II} ranking method are summarised in Table 5.11.

Table 6.11: Summary of Efficiency Evaluation

Methods	Efficiency Evaluation
Mitchell (2006)	Slightly Efficient
Wu & Mendel (2009)	Slightly Efficient
II – Cheng (1998)	Slightly Efficient
II – Kumar et al. (2010)	Slightly Inefficient
II – Dat et al. (2012)	Slightly Efficient
II – Yu et al. (2013) for α = 0	Slightly Efficient
II – Yu et al. (2013) for α = 0.5	Slightly Efficient
II – Yu et al. (2013) for α = 1	Very Inefficient
II – Zhang et al. (2014) for α = 0	Very Inefficient
II – Zhang et al. (2014) for α = 0.5	Very Inefficient
II – Zhang et al. (2014) for α = 1	Very Inefficient
CPS_{II} – direct	Very Efficient
CPS_{II} – indirect	Very Efficient

In Table 6.11, II – Zhang et al. (2014) ranking method with α = 0, 0.5 and 1, is classified as a very inefficient ranking method as this method is a pairwise ranking method and needs additional operation to ranking interval type – II fuzzy numbers appropriately. II – Dat et al. (2012) ranking method is evaluated as a slightly inefficient ranking method because it is a pairwise ranking method but does not need additional operation when ranking interval type – II fuzzy numbers appropriately. Mitchel (2006), Wu & Mendel (2009), II – Cheng (1998) and II – Yu et al. (2012) ranking methods are considered as slightly efficient ranking methods in this evaluation as both simultaneously rank the interval type – II fuzzy numbers but incorporate additional operation in obtaining the final ranking order. In this evaluation, the CPS_{II} ranking method for both direct and indirect ranking are regarded as a very efficient ranking methods as these methods rank interval type – II fuzzy numbers correctly such that the ranking result is consistent with human intuition using simultaneous ranking without incorporating any additional operation. Therefore, this evaluation signifies that the CPS_{II} ranking method is capable to rank three interval type – II fuzzy numbers simultaneously without incorporating additional operation.

6.4 SUMMARY

In this chapter, the capability of the *CPSII* ranking method to ranking interval type – II fuzzy numbers is provided. Two main empirical validations namely the consistency and efficiency of the *CPSII* ranking method are also highlighted in this chapter. In the validation, the capability of the *CPSII* ranking method on correctly ranks all cases of interval type – II fuzzy numbers such that the ranking results are consistent with human intuition is addressed. The efficiency of the *CPSII* ranking method on ranking three interval type – II fuzzy numbers simultaneously is also demonstrated in this chapter where the method is capable to ranking three interval type – II fuzzy numbers simultaneously without incorporating additional operation. In this respect, the *CPSII* ranking method is considered as a ranking method that is capable to ranking interval type – II fuzzy numbers consistently and efficiently when both direct and indirect ways of ranking are used. In Chapter 7, the thesis discusses the applicability of the *CPS* ranking methodology in ranking Z – fuzzy numbers.

CHAPTER SEVEN

RANKING OF Z – UMBERS

7.1 INTRODUCTION

This chapter discusses details on validation of the novel methodology for ranking Z – fuzzy numbers based on centroid point and spread, CPS_Z. Theoretical and empirical validation defined in Section 4.4 and 4.5 respectively are demonstrated in this chapter. These validation which are associated with properties of ranking fuzzy quantities as well as consistency and efficiency evaluation of ranking operations are described in detail here. Therefore, without loss of generality of Section 4.4 and 4.5, details on those aforementioned both validation are extensively discussed in sections and subsections of this chapter.

7.2 THEORETICAL VALIDATION

This subsection validates theoretically the novel CPS_Z ranking method using theoretical properties adopted from Wang & Kerre (2001, 2002). These properties justify the capability of the CPS_Z ranking method to ranking interval Z – numbers appropriately by proofs provided which are applicable to CPS_Z ranking method. Therefore, with no loss of generality, theoretical ordering properties by Wang & Kerre (2001, 2002) which are prepared for CPS_Z ranking method are presented as follows.

Let $Z_{\tilde{A}_1}$ and $Z_{\tilde{A}_2}$ be two standardised generalised Z – numbers where $Z_{\tilde{A}_1}$ and $Z_{\tilde{A}_2}$ are of any types of Z – numbers.

***Property 1*:** If $Z_{\tilde{A}_1} \succcurlyeq Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_2} \succcurlyeq Z_{\tilde{A}_1}$, then $Z_{\tilde{A}_1} \approx Z_{\tilde{A}_2}$

***Proof*:**

Since, $Z_{\tilde{A}_1} \succcurlyeq Z_{\tilde{A}_2}$ implies that $CPS_Z\left(Z_{\tilde{A}_1}\right) \geq CPS_Z\left(Z_{\tilde{A}_2}\right)$, and $Z_{\tilde{A}_2} \succcurlyeq Z_{\tilde{A}_1}$ implies that $CPS_Z\left(Z_{\tilde{A}_2}\right) \geq CPS_I\left(Z_{\tilde{A}_1}\right)$, hence indicates that, $CPS_Z\left(Z_{\tilde{A}_1}\right) = CPS_Z\left(Z_{\tilde{A}_2}\right)$, which is $Z_{\tilde{A}_1} \approx Z_{\tilde{A}_2}$.

***Property 2*:** If $Z_{\tilde{A}_1} \succcurlyeq Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_2} \succcurlyeq Z_{\tilde{A}_3}$, then $Z_{\tilde{A}_1} \succcurlyeq Z_{\tilde{A}_3}$

***Proof*:**

For CPS_Z ranking method, $Z_{\tilde{A}_1} \succcurlyeq Z_{\tilde{A}_2}$ implies that $CPS_Z\left(Z_{\tilde{A}_1}\right) \geq CPS_Z\left(Z_{\tilde{A}_2}\right)$ and $Z_{\tilde{A}_2} \succcurlyeq Z_{\tilde{A}_3}$ implies that $CPS_Z\left(Z_{\tilde{A}_2}\right) \geq CPS_Z\left(Z_{\tilde{A}_3}\right)$. This indicates that $CPS_Z\left(Z_{\tilde{A}_1}\right) \geq CPS_Z\left(Z_{\tilde{A}_3}\right)$, which is $Z_{\tilde{A}_1} \succcurlyeq Z_{\tilde{A}_3}$.

***Property 3*:** If $Z_{\tilde{A}1} \cap Z_{\tilde{A}_2} = \emptyset$ and $Z_{\tilde{A}_1}$ is on the right side of $Z_{\tilde{A}_2}$, then $Z_{\tilde{A}_1} \succcurlyeq Z_{\tilde{A}_2}$

***Proof*:**

Since, $Z_{\tilde{A}1} \cap Z_{\tilde{A}_2} = \emptyset$ and $Z_{\tilde{A}_1}$ is on the right side of $Z_{\tilde{A}_2}$, hence, implies that $CPS_Z\left(Z_{\tilde{A}_1}\right) \geq CPS_Z\left(Z_{\tilde{A}_2}\right)$, thus, $Z_{\tilde{A}_1} \succcurlyeq Z_{\tilde{A}_2}$.

***Property 4*:** The order of $Z_{\tilde{A}_1}$ and $Z_{\tilde{A}_2}$ is not affected by the other fuzzy numbers under comparison.

***Proof*:**

Since, the ordering of $Z_{\tilde{A}_1}$ and $Z_{\tilde{A}_2}$, is completely determined by $CPS_Z\left(Z_{\tilde{A}_1}\right)$ and $CPS_Z\left(Z_{\tilde{A}_2}\right)$ respectively, hence indicates that the ordering of $Z_{\tilde{A}_1}$ and $Z_{\tilde{A}_2}$ is not affected by the other fuzzy numbers under comparison.

The above theoretical validation clearly indicates that the CPS_Z ranking method is capable to ranking fuzzy numbers appropriately. This is signified through proof based – properties fulfilment by the CPS_Z ranking method on all theoretical validations considered in this subsection. In the next section, a generic empirical validation for any ranking fuzzy numbers methods is thoroughly discussed

7.3 EMPIRICAL VALIDATION

This section discusses empirical validation of the CPS_Z ranking method on ranking Z – fuzzy numbers. The empirical validation provided is a comparative – based ranking order analysis between the CPS_Z ranking method and established ranking methods under consideration on their consistency and efficiency to ranking Z – fuzzy numbers. All of the established ranking methods considered in this validation are methods for ranking type – I fuzzy numbers as there is no method for ranking Z – fuzzy numbers found in the literature of fuzzy sets. Thus, it is worth mentioning here that all established ranking methods used in this section are added 'Z' (e.g. Z – Cheng (1998)) to indicate that the method is applied to ranking Z – fuzzy number for the first time. Therefore, without loss of generality in terms of information in Section 4.4, the consistency and efficiency evaluations of the CPS_Z ranking method are given as follows.

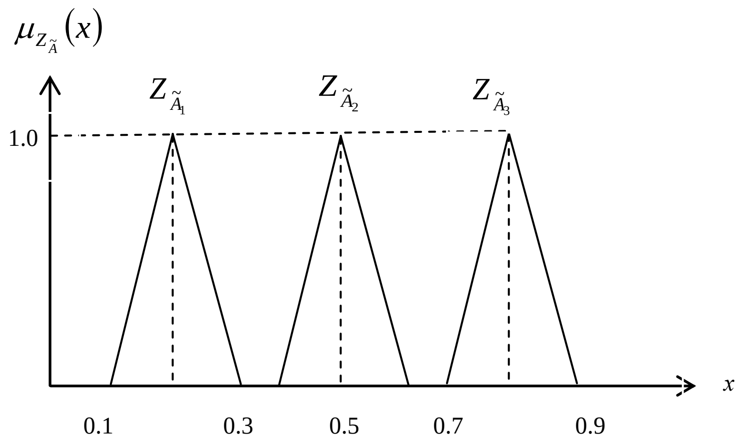
7.3.1 Evaluation of Consistency

In this subsection, 9 benchmarking sets of Z – fuzzy numbers are introduced for the first time in this study. This is because there is no benchmark example for empirical validation found in literature of fuzzy sets. Since, this is the first time CPSZ is applied to ranking Z – fuzzy numbers and the Z – fuzzy numbers are in the form of standardised generalised type – I fuzzy numbers, hence all established methods for ranking type – I fuzzy numbers considered in this study are also applicable to ranking Z – fuzzy numbers.

Trivial Case

Trivial Case 1

Trivial case 1 involves three triangular Z – fuzzy numbers of similar shapes and not overlapped which is illustrated in Figure 7.1.



$$Z_{\tilde{A}_1} = (0.1,0.2,0.2,0.3;1.0),(0.1,0.2,0.2,0.3;1.0) \quad Z_{\tilde{A}_2} = (0.4,0.5,0.5,0.6;1.0),(0.4,0.5,0.5,0.6;1.0)$$

$$Z_{\tilde{A}_3} = (0.7,0.8,0.8,0.9;1.0),(0.7,0.8,0.8,0.9;1.0)$$

Fig 7.1: Trivial Case 1

Using the CPS_Z ranking method, the ranking order for $Z_{\tilde{A}_1}, Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ in this case is as follows:

Step 1: Compute the centroid points for $Z_{\tilde{A}}$ by finding the horizontal – x value for $Z_{\tilde{A}_1}$, $x^*_{Z_{\tilde{A}_1}}$ using equation (4.4) as

$$x^*_{Z_{\tilde{A}_1}} = \frac{1}{3}\left[0.1+0.2+0.2+0.3-\frac{(0.06-0.02)}{(0.5-0.3)}\right]$$

$$= 0.2$$

whereas, using equation (4.5), the value of $y^*_{Z_{\tilde{A}_1}}$ is obtained as

$$y^*_{Z_{\tilde{A}_1}} = \frac{1}{3}\left[1 + \frac{0}{(0.5-0.3)}\right]$$

$$= 0.3333$$

Hence, centroid point for $Z_{\tilde{A}_1}$ is (0.2, 0.3333).

It has to be noted here that since $Z_{\tilde{A}_1}$ consists of two equivalence type – I fuzzy numbers, hence the other value of $x_{Z_{\tilde{A}_1}}$ is also 0.2. Thus, centroid point for $Z_{\tilde{A}_1}$ is expressed as

$$(x_{Z_{\tilde{A}_1}}, y_{Z_{\tilde{A}_1}}) = \{(0.2, 0.3333), (0.2, 0.3333)\}$$

Using same techniques as shown above, centroid points of $Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are calculated accordingly and the results are as follows:

$$(x_{Z_{\tilde{A}_2}}, y_{Z_{\tilde{A}_2}}) = \{(0.5, 0.3333), (0.5, 0.3333)\}$$
$$(x_{Z_{\tilde{A}_3}}, y_{Z_{\tilde{A}_3}}) = \{(0.8, 0.3333), (0.8, 0.3333)\}$$

Step 2: Spread values of $Z_{\tilde{A}_1}, Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are calculated such that spread of $Z_{\tilde{A}_1}$ is

$$s(Z_{\tilde{A}_1}) = 0.2 \times 0.3333$$
$$= 0.0667$$

Similarly as in Step 1, two values of spreads are also obtained in this step.

Thus, spread of $Z_{\tilde{A}_1}$ is

$$s(Z_{\tilde{A}_1}) = \{(0.0667), (0.0667)\}$$

while for $Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$, their spread values are

$$s(Z_{\tilde{A}_2}) = \{(0.0667),(0.0667)\}$$
$$s(Z_{\tilde{A}_3}) = \{(0.0667),(0.0667)\}$$

Step 3: Ranking values of $Z_{\tilde{A}_1}, Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are computed whereby ranking value for $Z_{\tilde{A}_1}$ is

$$CPS_Z\left(Z_{\tilde{A}_1}\right) = \left(\frac{0.2+0.2}{2}\right) \times \left(\frac{0.3333+0.3333}{2}\right) \times \left(\frac{(1-0.0667)+(1-0.0667)}{2}\right)$$

$$= 0.0662$$

and ranking values for $Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are

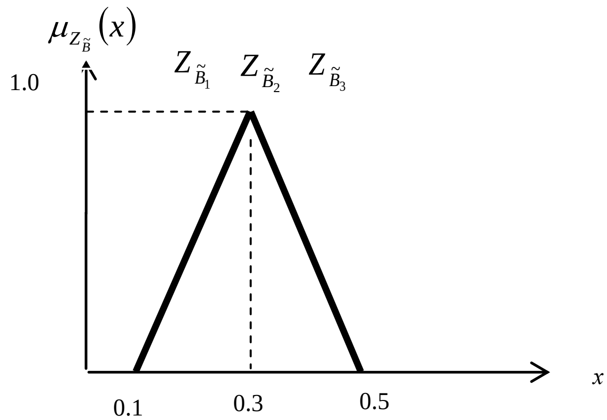
$$CPS_Z\left(Z_{\tilde{A}_2}\right) = 0.1555$$
$$CPS_Z\left(Z_{\tilde{A}_3}\right) = 0.2489$$

Since $CPS_Z\left(Z_{\tilde{A}_3}\right) > CPS_Z\left(Z_{\tilde{A}_2}\right) > CPS_Z\left(Z_{\tilde{A}_1}\right)$, hence ranking order result for for Z – numbers $Z_{\tilde{A}_1}, Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ is $Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$.

It is worth mentioning here that the entire steps utilised by the CPS_Z ranking method in ranking Z – fuzzy numbers are only demonstrated in Trivial Case 1. This is because these steps are also applied to the remaining eight cases of benchmarking examples considered in this study, thus repeating the entire steps in the thesis are redundant. Therefore, only definitions, illustration, ranking results and discussions of results on each case considered are provided.

Trivial Case 2

Trivial case 2 involves three identical triangular Z – fuzzy numbers which are embedded with each other. The following Figure 7.2 illustrates Z – fuzzy numbers of trivial case 2.



$$Z_{\tilde{B}_1} = (0.1,0.3,0.3,0.5;1.0),(0.1,0.3,0.3,0.5;1.0) \quad Z_{\tilde{B}_2} = (0.1,0.3,0.3,0.5;1.0),(0.1,0.3,0.3,0.5;1.0)$$

$$Z_{\tilde{B}_3} = (0.1,0.3,0.3,0.5;1.0),(0.1,0.3,0.3,0.5;1.0)$$

Fig 7.2: Trivial Case 2.

Results and Validation

Comparisons of ranking order for trivial case 1 and 2 between the CPS_Z ranking method and established ranking methods considered in this study are shown in Table 7.1 and 7.2 respectively.

Table 7.1: Ranking Results for Trivial Case 1

Methods	Z – Numbers $Z_{\tilde{A}_1}$	$Z_{\tilde{A}_2}$	$Z_{\tilde{A}_3}$	Ranking Results	Level of Consistency (%)
Z – Cheng (1998)	0.680	0.726	0.746	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100
Z – Kumar et al. (2010)	0.300	0.500	0.700	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100
Z – Dat et al. (2012)	0.000 / 0.040	0.040 / 0.400	0. 400 / 0.000	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100
Z – Yu et al. (2013) for α = 0	0.300	0.500	0.700	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100
Z – Yu et al. (2013) for α = 0.5	0.300	0.500	0.700	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100

Z – Yu et al. (2013) for $\alpha = 1$	0.500	0.7200	0.969	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100
Z – Zhang et al. (2013) for $\alpha = 0$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100
Z – Zhang et al. (2013) for $\alpha = 0.5$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100
Z – Zhang et al. (2013) for $\alpha = 1$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100
CPS$_Z$	0.066	0.155	0.245	$Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$	100

Table 7.2: Ranking Results for Trivial Case 2

Methods	Z – Numbers			Ranking Results	Level of Consistency (%)
	$Z_{\tilde{B}_1}$	$Z_{\tilde{B}_2}$	$Z_{\tilde{B}_3}$		
Z – Cheng (1998)	0.680	0.680	0.680	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
Z – Kumar et al. (2010)	0.300	0.300	0.300	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
Z – Dat et al. (2012)	0.040 / 0.040	0.040 / 0.040	0. 040 / 0.040	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
Z – Yu et al. (2013) for $\alpha = 0$	0.300	0.300	0.300	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
Z – Yu et al. (2013) for $\alpha = 0.5$	0.300	0.300	0.300	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
Z – Yu et al. (2013) for $\alpha = 1$	0.300	0.300	0.300	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 0$	0.040 / 0.040	0.040 / 0.040	0. 040 / 0.040	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 0.5$	0.040 / 0.040	0.040 / 0.040	0. 040 / 0.040	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 1$	0.040 / 0.040	0.040 / 0.040	0. 040 / 0.040	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100
CPS$_Z$	0.119	0.119	0.119	$Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$	100

Discussions

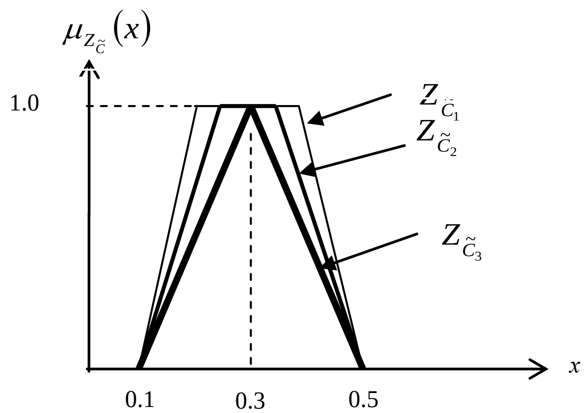
For trivial case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $Z_{\tilde{A}_3} \succ Z_{\tilde{A}_2} \succ Z_{\tilde{A}_1}$. This is because $Z_{\tilde{A}_3}$ is located at the farthest right among them, followed by $Z_{\tilde{A}_2}$ and then $Z_{\tilde{A}_1}$. all established ranking methods considered in this study including the *CPS*$_Z$ ranking method produce correct ranking order for this case such that the ranking result is consistent with human intuition. This indicates that the *CPS*$_Z$ ranking method is capable to deal with Z – fuzzy numbers of different locations.

For trivial case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $Z_{\tilde{B}_1} \approx Z_{\tilde{B}_2} \approx Z_{\tilde{B}_3}$. This is due to the fact that all Z – fuzzy numbers under consideration are the same in term of their shapes, spreads, heights and centroids. Shown in Table 7.2, all ranking results obtained by all established ranking methods considered in this study and the CPS_Z ranking method are the correct ranking order whereby the results are consistent with human intuition. This points out that the CPS_Z ranking method is capable to give same ranking value for each Z – fuzzy numbers even if same Z – fuzzy numbers are compared.

Embedded Case

Embedded Case 1

Embedded case 1 involves three embedded Z –fuzzy numbers where two of them are in trapezoidal Z – fuzzy numbers while the other is a triangular Z – fuzzy number. All of these Z – fuzzy numbers are of same height but differed in centroid point and spread as shown in Figure 7.3.



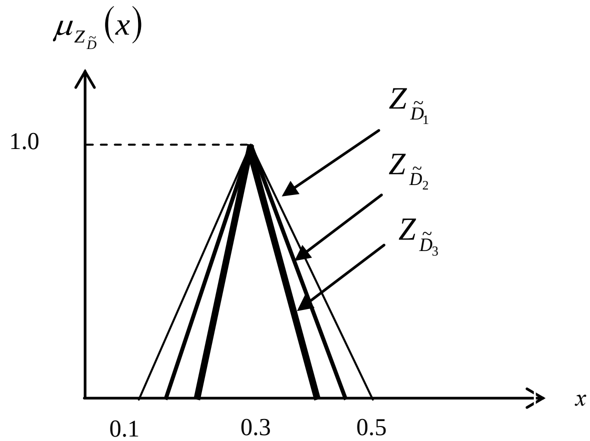
$Z_{\tilde{C}_1} = (0.1,0.2,0.4,0.5;1.0),(0.1,0.2,0.4,0.5;1.0)$ $Z_{\tilde{C}_2} = (0.1,0.25,0.35,0.5;1.0),(0.1,0.25,0.35,0.5;1.0)$

$Z_{\tilde{C}_3} = (0.1,0.3,0.3,0.5;1.0),(0.1,0.3,0.3,0.5;1.0)$

Fig 7.3: Embedded Case 1

Embedded Case 2

Embedded case 2 involves three triangular Z – fuzzy numbers where they are embedded with each other, same height and same centroid point but different in term of their spread. Figure 7.4 best is the illustration for this case.



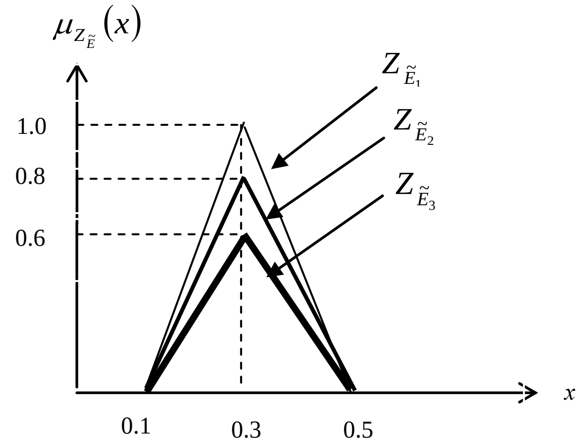
$Z_{\tilde{D}_1} = (0.1,0.3,0.3,0.5;1.0),(0.1,0.3,0.3,0.5;1.0)$ $Z_{\tilde{D}_2} = (0.15,0.3,0.3,0.45;1.0),(0.15,0.3,0.3,0.45;1.0)$

$$Z_{\tilde{D}_3} = (0.2,0.3,0.3,0.4;1.0),(0.2,0.3,0.3,0.4;1.0)$$

Fig 7.4: Embedded Case 2

Embedded Case 3

Embedded case 3 shown in Figure 7.5 involves three triangular Z – fuzzy numbers that are embedded with each other and having the same centroid point and spread but different in normality.



$$Z_{\tilde{E}_1} = (0.1,0.3,0.3,0.5;1.0),(0.1,0.3,0.3,0.5;1.0) \quad Z_{\tilde{E}_2} = (0.1,0.3,0.3,0.5;0.8),(0.1,0.3,0.3,0.5;0.8)$$

$$Z_{\tilde{E}_3} = (0.1,0.3,0.3,0.5;0.6),(0.1,0.3,0.3,0.5;0.6)$$

Fig 7.5: Embedded Case 3

Results and Validation

Comparisons of ranking order for embedded case 1, 2 and 3 between the CPS_Z ranking method and established ranking methods considered in this study are illustrated in Table 7.3, 7.4 and 7.5 respectively.

Table 7.3: Ranking Results for Embedded Case 1

Methods	Z – Numbers			Ranking Results	Level of Consistency (%)
	$Z_{\tilde{C}_1}$	$Z_{\tilde{C}_2}$	$Z_{\tilde{C}_3}$		
Z – Cheng (1998)	0.583	0.583	0.583	$Z_{\tilde{C}_1} \approx Z_{\tilde{C}_2} \approx Z_{\tilde{C}_3}$	0
Z – Kumar et al. (2010)	0.300	0.300	0.300	$Z_{\tilde{C}_1} \approx Z_{\tilde{C}_2} \approx Z_{\tilde{C}_3}$	0
Z – Dat et al. (2012)	0.333 / 0.333	0.333 / 0.333	0.222 / 0.222	$Z_{\tilde{C}_1} \approx Z_{\tilde{C}_2} \succ Z_{\tilde{C}_3}$	50
Z – Yu et al. (2013) for α = 0	1.00	1.00	1.00	$Z_{\tilde{C}_1} \prec Z_{\tilde{C}_2} \prec Z_{\tilde{C}_3}$	0
Z – Yu et al. (2013) for α = 0.5	1.00	1.00	1.00	$Z_{\tilde{C}_1} \approx Z_{\tilde{C}_2} \approx Z_{\tilde{C}_3}$	0
Z – Yu et al. (2013) for α = 1	1.00	1.00	1.00	$Z_{\tilde{C}_1} \succ Z_{\tilde{C}_2} \succ Z_{\tilde{C}_3}$	100
Z – Zhang et al. (2013) for α = 0	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$Z_{\tilde{C}_1} \prec Z_{\tilde{C}_2} \prec Z_{\tilde{C}_3}$	0
Z – Zhang et al. (2013) for α = 0.5	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$Z_{\tilde{C}_1} \approx Z_{\tilde{C}_2} \approx Z_{\tilde{C}_3}$	0
Z – Zhang et al. (2013) for α = 1	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$Z_{\tilde{C}_1} \succ Z_{\tilde{C}_2} \succ Z_{\tilde{C}_3}$	100

CPS_Z	0.119	0.107	0.089	$Z_{\tilde{C}_1} \succ Z_{\tilde{C}_2} \succ Z_{\tilde{C}_3}$	100

Table 7.4: Ranking Results for Embedded Case 2

Methods	Z – Numbers			Ranking Results	Level of Consistency (%)
	$Z_{\tilde{D}_1}$	$Z_{\tilde{D}_2}$	$Z_{\tilde{D}_3}$		
Z – Cheng (1998)	0.583	0.583	0.583	$Z_{\tilde{D}_1} \approx Z_{\tilde{D}_2} \approx Z_{\tilde{D}_3}$	0
Z – Kumar et al. (2010)	0.300	0.300	0.300	$Z_{\tilde{D}_1} \approx Z_{\tilde{D}_2} \approx Z_{\tilde{D}_3}$	0
Z – Dat et al. (2012)	0.333 / 0.333	0.333 / 0.333	0.333 / 0.333	$Z_{\tilde{D}_1} \approx Z_{\tilde{D}_2} \approx Z_{\tilde{D}_3}$	0
Z – Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$Z_{\tilde{D}_1} \succ Z_{\tilde{D}_2} \succ Z_{\tilde{D}_3}$	0
Z – Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$Z_{\tilde{D}_1} \approx Z_{\tilde{D}_2} \approx Z_{\tilde{D}_3}$	0
Z – Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$Z_{\tilde{D}_1} \prec Z_{\tilde{D}_2} \prec Z_{\tilde{D}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 0$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$Z_{\tilde{D}_1} \succ Z_{\tilde{D}_2} \succ Z_{\tilde{D}_3}$	0
Z – Zhang et al. (2013) for $\alpha = 0.5$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$Z_{\tilde{D}_1} \approx Z_{\tilde{D}_2} \approx Z_{\tilde{D}_3}$	0
Z – Zhang et al. (2013) for $\alpha = 1$	0.500 / 0.500	0.500 / 0.500	0.500 / 0.500	$Z_{\tilde{D}_1} \prec Z_{\tilde{D}_2} \prec Z_{\tilde{D}_3}$	100
CPS_Z	0.089	0.107	0.119	$Z_{\tilde{D}_1} \prec Z_{\tilde{D}_2} \prec Z_{\tilde{D}_3}$	100

Table 7.5: Ranking Results for Embedded Case 3

Methods	Z – Numbers			Ranking Results	Level of Consistency (%)
	$Z_{\tilde{E}_1}$	$Z_{\tilde{E}_2}$	$Z_{\tilde{E}_3}$		
Z – Cheng (1998)	0.583	0.461	0.346	$Z_{\tilde{E}_1} \succ Z_{\tilde{E}_2} \succ Z_{\tilde{E}_3}$	100
Z – Kumar et al. (2010)	0.240	0.240	0.240	$Z_{\tilde{E}_1} \approx Z_{\tilde{E}_2} \approx Z_{\tilde{E}_3}$	0
Z – Dat et al. (2012)	0.266 / 0.133	0.133 / 0.067	0.067 / 0.266	$Z_{\tilde{E}_1} \succ Z_{\tilde{E}_2} \succ Z_{\tilde{E}_3}$	100
Z – Yu et al. (2013) for $\alpha = 0$	1.00	1.00	1.00	$Z_{\tilde{E}_1} \prec Z_{\tilde{E}_2} \prec Z_{\tilde{E}_3}$	0
Z – Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	1.00	$Z_{\tilde{E}_1} \approx Z_{\tilde{E}_2} \approx Z_{\tilde{E}_3}$	0
Z – Yu et al. (2013) for $\alpha = 1$	1.00	1.00	1.00	$Z_{\tilde{E}_1} \succ Z_{\tilde{E}_2} \succ Z_{\tilde{E}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 0$	x	x	x	-	N/A
Z – Zhang et al. (2013) for $\alpha = 0.5$	x	x	x	-	N/A
Z – Zhang et al. (2013) for $\alpha = 1$	x	x	x	-	N/A
CPS_Z	0.119	0.107	0.089	$Z_{\tilde{E}_1} \succ Z_{\tilde{E}_2} \succ Z_{\tilde{E}_3}$	100

Note: 'x' denotes method as unable to calculate the ranking value.

'-' denotes no ranking order is obtained.

Discussions

For embedded case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $Z_{\tilde{C}_1} \succ Z_{\tilde{C}_2} \succ Z_{\tilde{C}_3}$. This is because the vertical – y centroid of $Z_{\tilde{C}_1}$ is the largest among the three, followed by $Z_{\tilde{C}_2}$ and then $Z_{\tilde{C}_3}$. In Table 7.3, Z – Cheng (1998) and Z – Kumar et al. (2010) ranking methods produce incorrect ranking order such that the ranking result is 0% consistent with human intuition for this case where both methods give equal ranking, $Z_{\tilde{C}_1} \approx Z_{\tilde{C}_2} \approx Z_{\tilde{C}_3}$ as they treat all Z – fuzzy numbers under consideration as having the same area. A partially correct ranking order such that the ranking result is 50% consistent with human intuition is obtained by Z – Dat et al. (2012) where this method is incapable to differentiate $Z_{\tilde{C}_1}$ and $Z_{\tilde{C}_2}$ appropriately. Different ranking orders are produced by Z – Yu et al. (2013) and Z – Zhang et al. (2013) as both ranking methods depend on decision maker's opinion to ranking fuzzy numbers. The CPS_Z ranking method on the other hand, ranks this case with correct ranking order such that the ranking result is 100% consistent with human intuition which emphasises that this method is capable to deal with embedded Z – fuzzy numbers of different shapes.

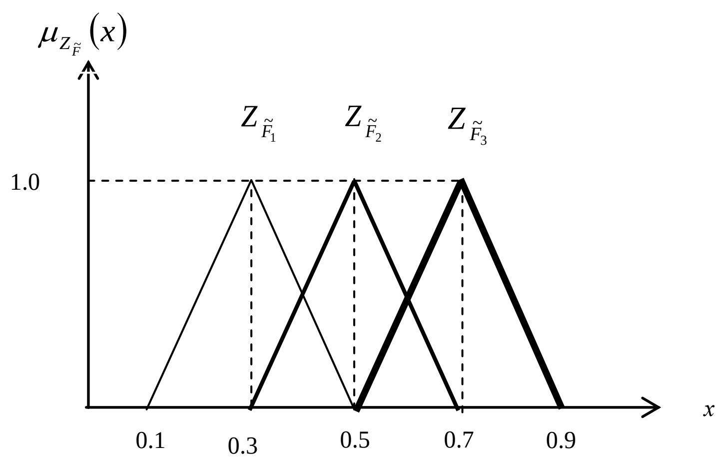
For embedded case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $Z_{\tilde{D}_1} \prec Z_{\tilde{D}_2} \prec Z_{\tilde{D}_3}$. This is due to the fact that the spread value for $Z_{\tilde{D}_3}$ is the smallest among the three, followed by $Z_{\tilde{D}_2}$ and then $Z_{\tilde{D}_1}$. Clearly indicated in Table 7.4, the incorrect ranking results by Z – Cheng (1998), Z – Kumar et al. (2010) and Z – Dat et al. (2012) such that the results are 0% consistent with human intuition. All of them give equal ranking for this case, $Z_{\tilde{D}_1} \approx Z_{\tilde{D}_2} \approx Z_{\tilde{D}_3}$ because Z – Cheng (1998) and Z – Kumar et al. (2010) ranking methods treat all Z – fuzzy numbers under consideration as the same area whereas Z – Dat et al. (2012) ranking method produces same distance for all Z – fuzzy numbers in this case. Z – Yu et al. (2013) and Z – Zhang et al. (2013) ranking methods produce many ranking results for this case since both take into account decision makers' opinion when ranking fuzzy numbers. Only the CPS_Z ranking method obtains the correct ranking order such that the ranking result is 100% consistent with human intuition for this case which signaling that this method is capable to differentiate Z – fuzzy numbers with different spread appropriately.

For embedded case 3, correct ranking order such that the ranking result is 100% consistent with human intuition is $Z_{\tilde{E}_1} \succ Z_{\tilde{E}_2} \succ Z_{\tilde{E}_3}$. $Z_{\tilde{E}_1}$ is considered as the greatest Z –fuzzy number among the three because height of $Z_{\tilde{E}_1}$ is the largest followed by $Z_{\tilde{E}_2}$ and then $Z_{\tilde{E}_3}$. In Table 7.5, ranking method by Z – Kumar et al. (2010) treats this case with equal ranking, $Z_{\tilde{E}_1} \approx Z_{\tilde{E}_2} \approx Z_{\tilde{E}_3}$ as this method considers all Z – fuzzy numbers under consideration as the same area. Z – Yu et al. (2013) ranking method produces different ranking order for different decision makers' opinions while Z – Zhang et al. (2013) ranking method is incapable to come out with any ranking order as the method is not applicable to non – normal Z – fuzzy numbers. Nonetheless, correct ranking orders such that the ranking result is 100% consistent with human intuition are obtained by Z – Cheng (1998), Z – Dat et al. (2012) and the CPS_Z ranking method. This result implies that the CPS_Z ranking method is capable to deal with Z – fuzzy numbers of different heights effectively.

Overlapping Case

Overlapping Case 1

Overlapping case 1 illustrates in Figure 7.6 involves three overlapping identical triangular Z – fuzzy numbers which are same in spread and height. Nevertheless, they are differed in terms of their positions.



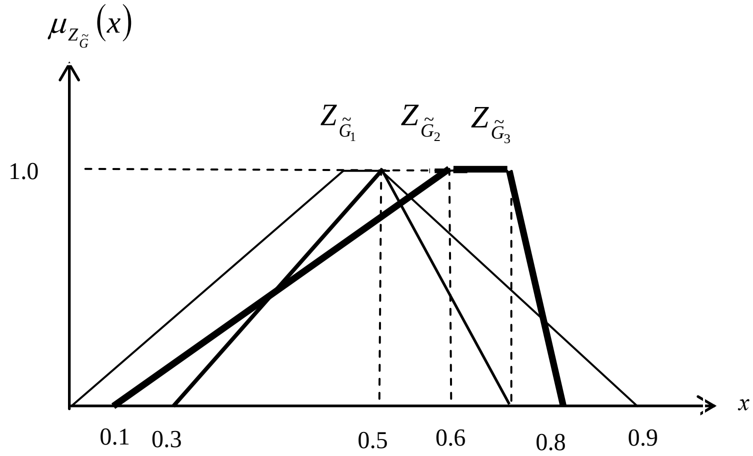
$Z_{\tilde{F}_1} = (0.1,0.3,0.3,0.5;1.0),(0.1,0.3,0.3,0.5;1.0)$ $Z_{\tilde{F}_2} = (0.3,0.5,0.5,0.7;1.0),(0.3,0.5,0.5,0.7;1.0)$

$Z_{\tilde{F}_3} = (0.5,0.7,0.7,0.9;1.0),(0.5,0.7,0.7,0.9;1.0)$

Fig 7.6: Overlapping Case 1

Overlapping Case 2

Overlapping case 2 involves three overlapping Z – fuzzy numbers comprise two trapezoidal Z – fuzzy numbers and a triangular Z – fuzzy number as illustrate in Figure 7.7. All of them are same of height but different of centroid point and spread.



$$Z_{\tilde{G}_1} = (0.0,0.4,0.5,0.7;1.0),(0.0,0.4,0.5,0.7;1.0) \quad Z_{\tilde{G}_2} = (0.2,0.5,0.5,0.9;1.0),(0.2,0.5,0.5,0.9;1.0)$$

$$Z_{\tilde{G}_3} = (0.1,0.6,0.7,0.8;1.0),(0.1,0.6,0.7,0.8;1.0)$$

Fig 7.7: Overlapping Case 2

Results and Validation

Comparisons of ranking order for overlapping case 1 and 2 between the CPS_Z ranking method and established ranking methods considered in this study are illustrated in Table 7.6 and 7.7 respectively.

Table 7.6: Ranking Results for Overlapping Case 1

Methods	Z – Numbers			Ranking Results	Level of Consistency (%)
	$Z_{\tilde{F}_1}$	$Z_{\tilde{F}_2}$	$Z_{\tilde{F}_3}$		
Z – Cheng (1998)	0.583	0.583	0.583	$Z_{\tilde{F}_1} \approx Z_{\tilde{F}_2} \approx Z_{\tilde{F}_3}$	0
Z – Kumar et al. (2010)	0.088	0.088	0.088	$Z_{\tilde{F}_1} \approx Z_{\tilde{F}_2} \approx Z_{\tilde{F}_3}$	0
Z – Dat et al. (2012)	0 .000 / 0.040	0.040 / 0.400	0. 400 / 0.000	$Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$	100
Z – Yu et al. (2013) for $\alpha = 0$	0.300	0.500	0.700	$Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$	100
Z – Yu et al. (2013) for $\alpha = 0.5$	0.300	0.500	0.700	$Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$	100
Z – Yu et al. (2013) for $\alpha = 1$	0.300	0.500	0.700	$Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 0$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 0.5$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 1$	0.500 / 0.720	0.720 / 0.969	0.969 / 0.500	$Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$	100
CPS_Z	0.089	0.107	0.119	$Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$	100

Table 7.7: Ranking Results for Overlapping Case 2

Methods	Z – Numbers			Ranking Results	Level of Consistency (%)
	$Z_{\tilde{G}_1}$	$Z_{\tilde{G}_2}$	$Z_{\tilde{G}_1}$		
Z – Cheng (1998)	0.746	0.726	0.680	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
Z – Kumar et al. (2010)	0.700	0.500	0.300	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
Z – Dat et al. (2012)	0.400 / 0.040	0.040 / 0.000	0.000 / 0.4000	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
Z – Yu et al. (2013) for $\alpha = 0$	0.700	0.500	0.300	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
Z – Yu et al. (2013) for $\alpha = 0.5$	0.700	0.500	0.300	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
Z – Yu et al. (2013) for $\alpha = 1$	0.500	0.7200	0.969	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 0$	0.969 / 0.720	0.720 / 0.500	0.500 / 0.969	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 0.5$	0.969 / 0.720	0.720 / 0.500	0.500 / 0.969	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
Z – Zhang et al. (2013) for $\alpha = 1$	0.969 / 0.720	0.720 / 0.500	0.500 / 0.969	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100
CPS_Z	0.119	0.107	0.089	$Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$	100

Discussions

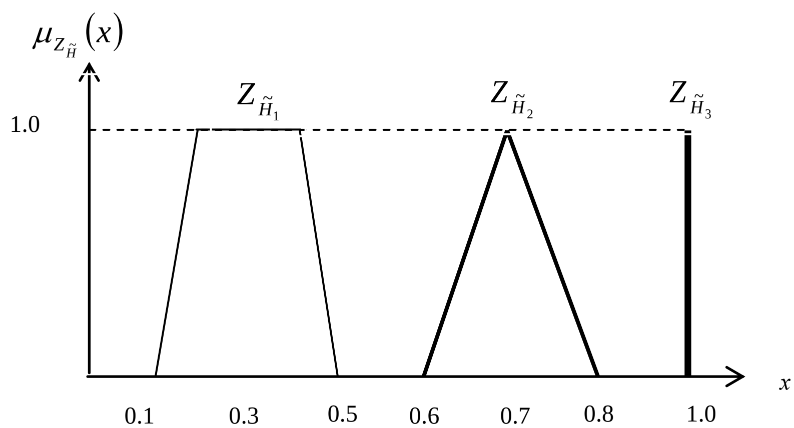
For overlapping case 1, the correct ranking order such that the ranking results is 100% consistent with human intuition is $Z_{\tilde{F}_1} \prec Z_{\tilde{F}_2} \prec Z_{\tilde{F}_3}$. This is because $Z_{\tilde{F}_3}$ is situated on the farthest right among the three, followed by $Z_{\tilde{F}_2}$ and then $Z_{\tilde{F}_1}$. Table 7.6, indicates that only Z – Cheng (1998) and Z – Kumar et al. (2010) ranking methods produce incorrect ranking order for this such that the result is 0% consistent with human intuition where they give equal ranking, $Z_{\tilde{F}_1} \approx Z_{\tilde{F}_2} \approx Z_{\tilde{F}_3}$ for this case. For other ranking methods considered in this study including the CPS_Z ranking method, all of them produce correct ranking order such that the ranking results are 100% consistent with human intuition. The result of the CPS_Z ranking method obtained in this case indicates that this method is capable to appropriately differentiate partial overlapping Z – fuzzy numbers.

For overlapping case 2, the correct ranking order such that the ranking results is 100% consistent with human intuition is $Z_{\tilde{G}_1} \succ Z_{\tilde{G}_2} \succ Z_{\tilde{G}_3}$. This is due to the fact that when combining both values of centroid point and spread of each Z – fuzzy number under consideration, $Z_{\tilde{G}_1}$ is the greatest followed by $Z_{\tilde{G}_2}$ and then $Z_{\tilde{G}_3}$. Table 7.7 shows all ranking methods under consideration including the CPS_Z ranking method produce the same correct ranking order such that the ranking result is 100% consistent with human intuition. This signifies that the CPS_Z ranking method is capable to appropriately deal with overlapping case of Z – fuzzy numbers like other established ranking methods.

Non - Overlapping Case

Non - Overlapping Case 1

Non - overlapping Case 1 involves different types of Z – fuzzy numbers namely trapezoidal, triangular and singleton that are not overlapped as shown in Figure 7.8. In this case, all of the Z – fuzzy numbers considered are differed in terms of the centroid point and spread but are the same of height.



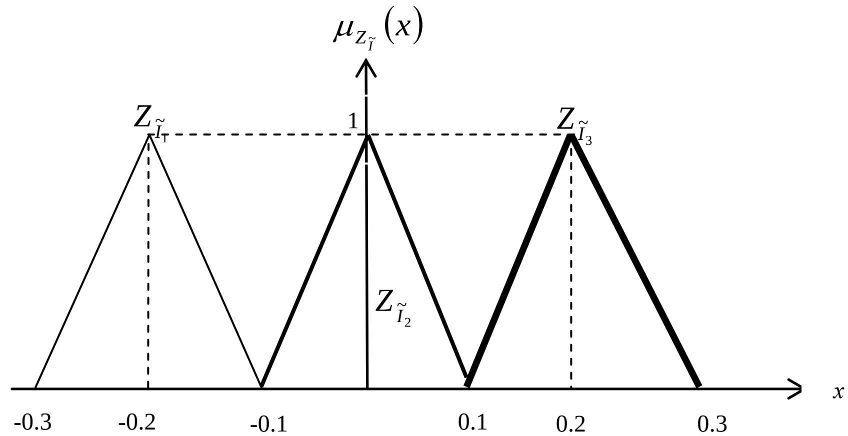
$Z_{\tilde{H}_1} = (0.1,0.3,0.3,0.5;1.0),(0.1,0.3,0.3,0.5;1.0)$ $Z_{\tilde{H}_2} = (0.6,0.7,0.7,0.8;1.0),(0.6,0.7,0.7,0.8;1.0)$

$Z_{\tilde{H}_3} = (1.0,1.0,1.0,1.0;1.0),(1.0,1.0,1.0,1.0;1.0)$

Fig 7.8: Non - Overlapping Case 1

Non - Overlapping Case 2

Non – overlapping case 2 considers three identical triangular Z – numbers of position. One of them is situated on the negative side, one is on positive side and the other is on both sides. This case is classified as the mirror image situation or reflection case of Z –numbers. Figure 7.9 is the illustration for Z – numbers of non – overlapping case 2.



$$Z_{\tilde{I}_1} = (-0.3,-0.2,-0.2,-0.1;1,0),(-0.3,-0.2,-0.2,-0.1;1,0)$$

$$Z_{\tilde{I}_2} = (-0.1,0.0,0.0,0.1;1,0),(-0.1,0.0,0.0,0.1;1,0)$$

$$Z_{\tilde{I}_3} = (0.1,0.2,0.2,0.3;1,0),(0.1,0.2,0.2,0.3;1,0)$$

Fig 7.9: Non – Overlapping Case 2

Results and Validation

Comparisons of ranking order for non – overlapping Case 1 and 2 between the CPS_Z ranking method and other established ranking methods considered in this study are illustrated in Table 7.8 and 7.9 respectively.

Table 7.8: Ranking Results for Non – Overlapping Case 1.

Methods	Z - Numbers			Ranking Results	Level of Consistency (%)
	$Z_{\tilde{H}_1}$	$Z_{\tilde{H}_2}$	$Z_{\tilde{H}_3}$		
Z – Cheng (1998)	0.424	0.583	x	-	N/A
Z – Kumar et al. (2010)	0.300	0.300	x	-	N/A
Z – Dat et al. (2012)	0.000 / 0.333	0.333 / 0.600	0.600 / 0.000	$Z_{\tilde{H}_1} \prec Z_{\tilde{H}_2} \prec Z_{\tilde{H}_3}$	100
Z – Yu et al. (2013) for $\alpha = 0$	0.700	0.300	x	-	N/A
Z – Yu et al. (2013) for $\alpha = 0.5$	1.00	1.00	x	-	N/A

Z – Yu et al. (2013) for α = 1	0.300	0.700	x	-	N/A
Z – Zhang et al. (2013) for α = 0	1.00	1.00	x	-	N/A
Z – Zhang et al. (2013) for α = 0.5	1.00	1.00	x	-	N/A
Z – Zhang et al. (2013) for α = 1	1.00	1.00	x	-	N/A
CPS CPS_Z	0.089	0.107	0.119	$Z_{\tilde{H}_1} \prec Z_{\tilde{H}_2} \prec Z_{\tilde{H}_3}$	100

Note: ‘x’ denotes method as unable to calculate the ranking value.

‘-’ denotes no ranking order is obtained.

Table 7.9: Ranking Results for Non – Overlapping Case 2

Methods	Z – Numbers			Ranking Results	Level of Consistency (%)
	$Z_{\tilde{I}_1}$	$Z_{\tilde{I}_2}$	$Z_{\tilde{I}_3}$		
Z – Cheng (1998)	0.583	0.583	0.583	$Z_{\tilde{I}_1} \approx Z_{\tilde{I}_2} \approx Z_{\tilde{I}_3}$	0
Z – Kumar et al. (2010)	0.300	0.300	0.300	$Z_{\tilde{I}_1} \approx Z_{\tilde{I}_2} \approx Z_{\tilde{I}_3}$	0
Z – Dat et al. (2012)	0.000 / 0.333	0.333 / 0.600	0.600 / 0.000	$Z_{\tilde{I}_1} \prec Z_{\tilde{I}_2} \prec Z_{\tilde{I}_3}$	100
Z – Yu et al. (2013) for α = 0	751	0.000	0.001	$Z_{\tilde{I}_1} \succ Z_{\tilde{I}_3} \succ Z_{\tilde{I}_2}$	0
Z – Yu et al. (2013) for α = 0.5	1.00	1.000	1.00	$Z_{\tilde{I}_1} \approx Z_{\tilde{I}_2} \approx Z_{\tilde{I}_3}$	0
Z – Yu et al. (2013) for α = 1	0.001	0.000	751	$Z_{\tilde{I}_1} \prec Z_{\tilde{I}_2} \prec Z_{\tilde{I}_3}$	100
Z – Zhang et al. (2013) for α = 0	0.969 / 0.720	0.720 / 0.500	0.500 / 0.969	$Z_{\tilde{I}_1} \succ Z_{\tilde{I}_2} \succ Z_{\tilde{I}_3}$	0
Z – Zhang et al. (2013) for α = 0.5	0.969 / 0.720	0.720 / 0.500	0.500 / 0.969	$Z_{\tilde{I}_1} \succ Z_{\tilde{I}_2} \succ Z_{\tilde{I}_3}$	0
Z – Zhang et al. (2013) for α = 1	0.969 / 0.720	0.720 / 0.500	0.500 / 0.969	$Z_{\tilde{I}_1} \succ Z_{\tilde{I}_2} \succ Z_{\tilde{I}_3}$	0
CPS_Z	0.089	0.107	0.119	$Z_{\tilde{I}_1} \prec Z_{\tilde{I}_2} \prec Z_{\tilde{I}_3}$	100

Discussion

For non – overlapping case 1, the correct ranking order such that the ranking result is 100% consistent with human intuition is $Z_{\tilde{H}_1} \prec Z_{\tilde{H}_2} \prec Z_{\tilde{H}_3}$. This is because $Z_{\tilde{H}_3}$ is situated on the farthest right among the three, followed by $Z_{\tilde{H}_2}$ and then $Z_{\tilde{H}_1}$. Table 7.8 clearly signified that only Z – Dat et al. (2012) and the CPS_Z ranking methods are capable to rank this case correctly such that the ranking result is 100% consistent with human intuition. For other ranking methods considered in this study, all of them are incapable to rank singleton Z – fuzzy numbers appropriately, thus all of them are not applicable for ranking Z – fuzzy numbers. This shows that the CPS_Z ranking method is capable to appropriately deal with non – overlapping Z – fuzzy numbers and singleton Z – fuzzy numbers.

For non – overlapping case 2, the correct ranking order such that the ranking result is 100% consistent with human intuition is $Z_{\tilde{I}_1} \prec Z_{\tilde{I}_2} \prec Z_{\tilde{I}_3}$. This is due to the fact that $Z_{\tilde{I}_3}$ is located on the farthest right which is on the positive side, followed by $Z_{\tilde{I}_2}$ and then $Z_{\tilde{I}_1}$. In Table 7.9, Z – Cheng (1998) and Z – Kumar et al. (2010) ranking methods obtain equal ranking, $Z_{\tilde{I}_1} \approx Z_{\tilde{I}_2} \approx Z_{\tilde{I}_3}$ for this case, which is incorrect such that the ranking result is 0% consistent with human intuition. Z – Yu et al. (2013) and Z – Zhang et al. (2013) ranking methods also come out with many ranking orders for this case as they depend on decision makers' opinions when ranking fuzzy numbers. Only Z – Dat et al. (2012) and the CPS_Z ranking methods give correct ranking order for this case such that the ranking result is 100% consistent with human intuition. This directly emphasises that the CPS_Z ranking method is capable to effectively deal with negative and positive Z – fuzzy numbers simultaneously.

Summary of Consistency Evaluation

This subsection covers the summary on the consistency evaluations for all ranking methods considered in section 7.2.1 including the CPS_Z ranking method. The summary provides clear observation in terms of number of consistent ranking result produced by all ranking methods considered in this study and their performance percentage. Using Section 4.4 as guideline and information obtained from Table 7.1 until Table 7.9, the following Table 7.10 summaries the consistency evaluation of all ranking methods considered in this study including the CPS_Z ranking method on ranking Z – fuzzy numbers.

Table 7.10: Summary of Consistency Evaluation

Methods	Consistency Evaluation	
	Proportion of Result with 100% Level of Consistency	Percentage of Result with 100% Level of Consistency
Z – Cheng (1998)	4/9	44.44%
Z – Kumar et al. (2010)	3/9	33.33%
Z – Dat et al. (2012)	7/9	77.75%
Z – Yu et al. (2013) for $\alpha = 0$	4/9	44.44%
Z – Yu et al. (2013) for $\alpha = 0.5$	4/9	44.44%
Z – Yu et al. (2013) for $\alpha = 1$	4/9	44.44%
Z – Zhang et al. (2014) for $\alpha = 0$	4/9	55.55%
Z – Zhang et al. (2014) for $\alpha = 0.5$	4/9	55.55%
Z – Zhang et al. (2014) for $\alpha = 1$	4/9	55.55%
CPS_Z	9/9	100%

Results in Table 7.10 show that Z – Kumar et al. (2010) ranking method obtains the least number of consistent ranking results where the method ranks three out of nine (33.33%) cases of benchmark examples provided in this study. Z – Cheng (1998) and Z – Yu et al. (2013) with α = 0 and 0.5 share the same number of consistent ranking results with four out of nine cases which is equivalence to 44.44%. Z – Zhang et al. (2014) with α = 0 and 0.5 ranking methods successfully rank five out of nine (55.55%) benchmark examples. Z – Dat et al. (2012) and Z – Zhang et al. (2014) with α = 1 ranking methods achieve seven out nine cases while Z – Yu et al. (2013) ranking method ranks eight out of nine cases of benchmarking examples prepared in this study. Among all ranking methods considered in this evaluation, only the CPS_Z ranking method perfectly ranks all nine (100%) cases of benchmarking examples with correct ranking order such that all results obtained are 100% consistent with human intuition. Therefore, this evaluation clearly indicates that the CPS_Z ranking method is considered as a ranking method that correctly ranks all Z – fuzzy numbers such that the ranking results are 100% consistent with human intuition.

7.3.2 Evaluation of Efficiency

This subsection discusses the efficiency evaluations of all the ranking methods considered in this study including the CPS_Z ranking method. It is intentionally prepared as a separate subsection from the summary of the consistency evaluation because all ranking methods considered in this study and the CPS_Z ranking method, perform similar efficiency capability when ranking three Z – fuzzy numbers. This is because the efficiency result of a ranking method is the same for all benchmarking examples provided in this study even if the consistency evaluations are different. Therefore, without loss of generality of Section 4.5, the efficiency evaluations of all ranking methods considered in this study including the CPS_Z ranking method are summarised in Table 7.11.

Table 7.11: Summary of Efficiency Evaluation

Methods	Efficiency Evaluation
Z – Cheng (1998)	Slightly Efficient
Z – Kumar et al. (2010)	Slightly Efficient
Z – Dat et al. (2012)	Slightly Inefficient
Z – Yu et al. (2013) for $\alpha = 0$	Slightly Efficient
Z – Yu et al. (2013) for $\alpha = 0.5$	Slightly Efficient
Z – Yu et al. (2013) for $\alpha = 1$	Slightly Efficient
Z – Zhang et al. (2014) for $\alpha = 0$	Very Inefficient
Z – Zhang et al. (2014) for $\alpha = 0.5$	Very Inefficient
Z – Zhang et al. (2014) for $\alpha = 1$	Very Inefficient
CPS_Z	Very Efficient

In Table 7.11, Z – Zhang et al. (2014) ranking method with α = 0, 0.5 and 1, is classified as a very inefficient ranking method as this method is a pairwise ranking method and needs additional operation to ranking Z – fuzzy numbers appropriately. Z – Dat et al. (2012) ranking method is evaluated as a slightly inefficient ranking method because it is a pairwise ranking method but does not need additional operation when ranking Z – fuzzy numbers appropriately. Z – Cheng (1998) and Z – Yu et al. (2012) ranking methods are considered as slightly efficient ranking methods in this evaluation as both simultaneously rank the Z – fuzzy numbers but incorporate additional operation in obtaining the final ranking order. In this evaluation, the CPS_Z ranking method is regarded as a very efficient ranking method as this method ranks fuzzy numbers correctly such that the ranking result is consistent with human intuition using simultaneous ranking without incorporating any additional operation. Therefore, this evaluation signifies that the CPS_Z ranking method is capable to rank three Z – fuzzy numbers simultaneously without incorporating additional operation when ranking Z – fuzzy numbers.

7.3 SUMMARY

In this chapter, the capability of the CPS_Z ranking method on ranking Z – fuzzy numbers is provided. Two main empirical validation namely the consistency and efficiency of the CPS_Z ranking method are also highlighted in this chapter. In the validation, the capability of the CPS_Z ranking method to correctly rank all cases of Z – fuzzy numbers such that the ranking results are consistent with human intuition is addressed. The efficiency of the CPS_Z ranking method on ranking three Z – fuzzy numbers simultaneously is also demonstrated in this chapter where the method is capable on ranking three Z – fuzzy numbers simultaneously without incorporating additional operation. In this respect, the CPS_Z ranking method is considered as a ranking method that is capable on ranking Z – fuzzy numbers consistently and efficiently. In Chapter 8, the thesis discusses the applicability of the CPS_I, CPS_{II} and CPS_Z ranking methods in solving respective case studies in the literature of fuzzy sets.

CHAPTER EIGHT

CASE STUDIES

8.1 INTRODUCTION

This chapter covers applications of the proposed methodology for ranking fuzzy numbers based on centroid point and spread, *CPS*, on case studies found in the literature of fuzzy sets. The *CPS* ranking methodology which consists of CPS_I (Chapter 5), CPS_{II} (Chapter 6) and CPS_Z (Chapter 7) ranking methods are applied to established case studies on risk analysis (Chen & Chen, 2009), footprint of uncertainty (Wu & Mendel, 2009) and vehicle selection under uncertain environment (Kang et al., 2012) respectively. These case studies are considered and discussed in this study as they utilise type – I fuzzy numbers, type – II fuzzy numbers and Z – fuzzy numbers in the analysis. Therefore, the applicability of the *CPS* ranking methodology in solving those aforementioned case studies is discussed in sections and subsections of this chapter.

8.2 CASE STUDY 1: RISK ANALYSIS

8.2.1 Overview

In this investigation by Chen & Chen (2009), three manufactories which are represented by three manufacturers, C_1 ,C_2 and C_3 produce the same product A_i , i = 1, 2, 3 where A_1 is the product of C_1, A_2 for C_2 and A_3 for C_3. For every product A_i produces by each manufactory, each consists of sub – components A_{i1} A_{i2} and A_{i3}, where the sub – components are evaluated based on two criteria namely the probability of failure, S_i and severity of loss, W_i . In the following Figure 8.1, the structure of fuzzy risk analysis for all manufactories under consideration is given.

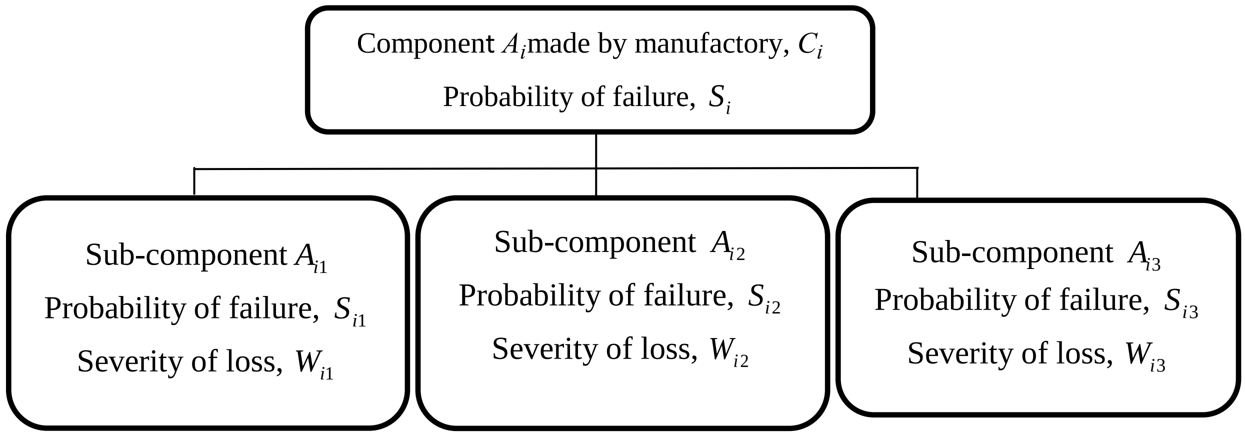


Fig 8.1: Fuzzy Risk Analysis Structure (Chen & Chen, 2009)

This study by Chen & Chen (2009), defines the level of risk faced by each manufacturer under consideration using nine distinct linguistic terms where all of linguistic terms are represented by nine respective generalised trapezoidal type – I fuzzy numbers as described in Table 8.1.

Table 8.1: Linguistic Terms and Their Corresponding Generalised Type – I Fuzzy Numbers (Chen & Chen, 2009)

Linguistic terms	Generalised Type - I Fuzzy Numbers
Absolutely – low	(0.0, 0.0, 0.0, 0.0; 1.0)
Very – low	(0.0, 0.0, 0.02, 0.07; 1.0)
Low	(0.04, 0.10, 0.18, 0.23; 1.0)
Fairly – low	(0.17, 0.22, 0.36, 0.42; 1.0)
Medium	(0.32, 0.41, 0.58, 0.65; 1.0)
Fairly – high	(0.58, 0.63, 0.80, 0.86; 1.0)
High	(0.72, 0.78, 0.92, 0.97; 1.0)
Very – high	(0.93, 0.98, 1.0, 1.0; 1.0)
Absolutely – high	(1.0, 1.0, 1.0, 1.0; 1.0)

With no loss of generality of Table 8.1, Chen & Chen (2009) gives the linguistic evaluating values of sub – components made by manufacturers C_1, C_2 and C_3 as in Table 8.2.

Table 8.2: Linguistic Evaluating Values of Sub – Components Made By Manufacturers C_1, C_2 and C_3 (Chen & Chen, 2009).

	Sub-component	Linguistic value of the severity of loss	Linguistic values of the probability of failure
	A_{11}	W_{11} = low	S_{11} = fairly – low $\left(w_{\tilde{R}_{11}} = 0.9\right)$
C_1	A_{12}	W_{12} = fairly – high	S_{12} = medium $\left(w_{\tilde{R}_{12}} = 0.7\right)$
	A_{13}	W_{13} = very – low	S_{13} = fairly – high $\left(w_{\tilde{R}_{33}} = 0.8\right)$
	A_{21}	W_{21} = low	S_{21} = very – high $\left(w_{\tilde{R}_{21}} = 0.85\right)$
C_2	A_{22}	W_{22} = fairly – high	S_{22} = fairly – high $\left(w_{\tilde{R}_{22}} = 0.95\right)$
	A_{23}	W_{23} = very – low	S_{23} = medium $\left(w_{\tilde{R}_{23}} = 0.9\right)$
	A_{31}	W_{31} = low	S_{31} = fairly – low $\left(w_{\tilde{R}_{31}} = 0.95\right)$
C_3	A_{32}	W_{32} = fairly – high	S_{32} = high $\left(w_{\tilde{R}_{32}} = 0.8\right)$
	A_{33}	W_{33} = very – low	S_{33} = fairly – high $\left(w_{\tilde{R}_{33}} = 1.0\right)$

Using information provided in Table 8.2, evaluation on level of the risk for each manufacturer C_1, C_2 and C_3 are determined using the following aggregation method (Chen & Chen, 2009).

$$R_i = \frac{\sum_{k=1}^{p} S_{ik} \times W_{ik}}{\sum_{k=1}^{p} W_{ik}}$$

$$= \left(r_{i1}, r_{i2}, r_{i3}, r_4; w_{R_i}\right) \tag{8.1}$$

where R_i is a type – I fuzzy number for $1 \le i \le n$.

Therefore, the aggregation value for R_1 is

$$R_1 = \left\{ \frac{\begin{array}{l}[(0.17,0.22,0.36,0.42;0.9)\otimes(0.04,0.10,0.18,0.23;1.0)]+\\ [(0.32,0.41,0.58,0.65;0.7)\otimes(0.58,0.63,0.80,0.86;1.0)]+\\ [(0.58,0.63,0.80,0.86;0.8)\otimes(0,0,0.02,0.07;1.0)]\end{array}}{\begin{array}{l}(0.04,0.10,0.18,0.23;1.0)+(0.58,0.63,0.80,0.86;1.0)+\\ (0,0,0.02,0.07;1.0)\end{array}} \right\}$$

$$= (0.1659,0.2803,0.7463,1.1545;0.7)$$

and aggregation values for R_2 and R_3 are

$$R_2 = (0.3221,0.4949,1.1392,1.6373;0.85)$$
$$R_3 = (0.3659,0.5134,1.1189,1.5984;0.8)$$

When the aggregation process for all type – I fuzzy numbers R_i, $i = 1,2,3$ completes, all values of R_i are next transformed into standardised generalised type – I fuzzy numbers $\tilde{R}$ as in Figure 8.2 using equation (3.2). Based on equation (3.2), transformation of R_i into $\tilde{R}$ is as follows.

$$\tilde{R}_1 = \left(\frac{0.1659}{1.6373}, \frac{0.2803}{1.6373} \frac{,0.7463}{1.6373}, \frac{1.154}{1.6373} 5;0.7 \right)$$
$$= (0.1013,0.1712,0.4558,0.7051;0.7)$$

Similarly, the standardised generalised type – I fuzzy numbers for $\tilde{R}_2$ and $\tilde{R}_3$ are

$$\tilde{R}_2 = (0.1967,0.3023,0.6958,1.0;0.85)$$
$$\tilde{R}_3 = (0.2235,0.3136,0.6834,0.9762;0.8)$$

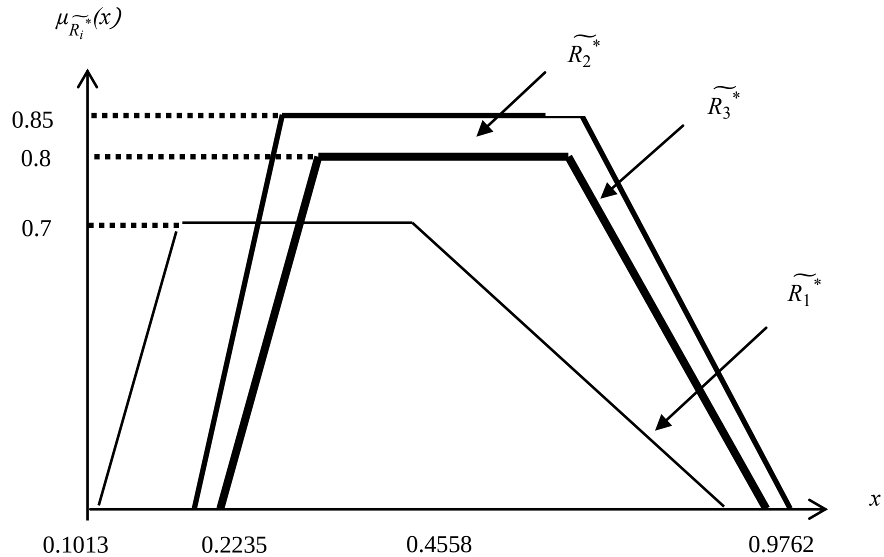


Fig 8.2: Standardised Generalised Fuzzy Number for $\widetilde{R_1}^*$, $\widetilde{R_2}^*$ and $\widetilde{R_3}^*$

8.2.1 Application

Since, it is noted that Chen & Chen (2009) utilised type – I fuzzy numbers in represented the level of risk faced by the manufacturers, hence the CPS_I ranking method is applied to this case as the CPS_I ranking method is developed for type – I fuzzy numbers. Therefore, levels of risk for manufacturers C_1, C_2 and C_3 evaluated by the CPS_I ranking method are as follows

Step 1: Centroid points (x^*, y^*) for $\tilde{R}_1, \tilde{R}_2$ and $\tilde{R}_3$ are obtained such that value of $x^*_{\tilde{R}_1}$ is calculated using formula in equation (3.10) as

$$x^*_{\tilde{R}_1} = \frac{1}{3}\left[0.1013 + 0.1712 + 0.4558 + 0.7051 - \frac{(0.3214 - 0.0173)}{(1.1609 - 0.2725)}\right]$$

$$= 0.3637$$

whereas, value of $y^*_{\tilde{R}_1}$ is calculated as

$$y^*_{\tilde{R}_1} = \frac{0.7}{3}\left[1 + \frac{0.2846}{(1.1609 - 0.2725)}\right]$$

$$= 0.3081$$

Hence, centroid point for $\tilde{R}_1$ is (0.3637, 0.3081).

While centroid points of $\tilde{R}_2$ and $\tilde{R}_3$ are calculated and shown as follows:

$$(x^*_{\tilde{R}_2}, y^*_{\tilde{R}_2}) = (0.5544, 0.3765)$$
$$(x^*_{\tilde{R}_3}, y^*_{\tilde{R}_3}) = (0.5549, 0.3545)$$

Step 2: Spread values of $\tilde{R}_1, \tilde{R}_2$ and $\tilde{R}_3$ are calculated such that spread of $\tilde{R}_1$ is

$$s(\tilde{R}_1) = 0.6038 \times 0.3081$$
$$= 0.1860$$

While, spread values for $\tilde{R}_2$ and $\tilde{R}_3$ are

$$s(\tilde{A}_2) = 0.3024$$
$$s(\tilde{A}_3) = 0.2668$$

Step 3: Ranking values of $\tilde{R}_1, \tilde{R}_2$ and $\tilde{R}_3$ are computed whereby ranking value for $\tilde{R}_1$ is

$$CPS(\tilde{R}_1) = 0.3637 \times 0.3081 \times (1 - 0.1860)$$

$$= 0.0912$$

and ranking values for $\tilde{R}_2$ and $\tilde{R}_3$ are

$$CPS(\tilde{R}_2) = 0.1456$$
$$CPS(\tilde{R}_3) = 0.1442$$

Since $CPS_I(\tilde{R}_2) > CPS_I(\tilde{R}_3) > CPS_I(\tilde{R}_1)$, hence ranking order result for $\tilde{R}_1, \tilde{R}_2$ and $\tilde{R}_3$ is $\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1$. Therefore, the level of risk evaluations for manufacturer from the most risky to the least risky is $C_2 > C_3 > C_1$.

8.3 CASE STUDY 2: WORD CLASSIFICATION

8.3.1 Overview

In a research done by Wu & Mendel (2009), 32 words which are randomly ordered are compiled as a dataset where some of them are changed with more commonly used words. All of these words are classified in Table 8.3 into three groups namely small – sounding words, (little, low amount, somewhat small, very tiny amount, none to very little, very small, very little, teeny-weeny, small amount and tiny), medium – sounding words (fair amount, modest amount, moderate amount, medium, good amount, a bit, some to moderate and some), and large – sounding words (sizeable, large, quite a bit, humongous amount, very large, extreme amount, considerable amount, a lot, very sizeable, high amount, maximum amount, very high amount and substantial amount).

Table 8.3: 32 words and respective interval type – II fuzzy numbers with modification (Wu & Mendel, 2009)

Type – II fuzzy numbers	Word	Upper Membership Function	Lower Membership Function
A'_1	None to very little	[0, 0, 0.14, 1.97; 1]	[0, 0, 0.05, 0.66; 1.00]
A'_2	Teeny – weeny	[0, 0, 0.14, 1.97; 1]	[0, 0, 0.01, 0.13; 1.00]
A'_3	Tiny	[0, 0, 0.26, 2.63; 1]	[0, 0, 0.05, 0.63; 1.00]
A'_4	Very Tiny amount	[0, 0, 0.36, 2.63; 1]	[0, 0, 0.05, 0.63; 1.00]
A'_5	Very small	[0, 0, 0.64, 2.47; 1]	[0, 0, 0.10, 1.16; 1.00]
A'_6	Very little	[0, 0, 0.64, 2.63; 1]	[0, 0, 0.99, 0.99; 1.00]
A'_7	A bit	[0.59, 1.50, 2.00, 3.41; 1]	[0.79, 1.68, 1.68, 2.21; 0.74]
A'_8	Little	[0.38, 1.50, 2.50, 4.62; 1]	[1.09, 1.83, 1.83, 2.21; 0.53]
A'_9	Low amount	[0.09, 1.25, 2.50, 4.62; 1]	[1.67, 1.92, 1.92, 2.21; 0.30]
A'_{10}	Small	[0.09, 1.50, 3.00, 4.62; 1]	[1.79, 2.28, 2.28, 2.81; 0.40]
A'_{11}	Somewhat small	[0.59, 2.00, 3.25, 4.41; 1]	[2.29, 2.70, 2.70, 3.21; 0.42]
A'_{12}	Some	[0.38, 2.50, 5.00, 7.83; 1]	[2.88, 3.61, 3.61, 4.21; 0.35]
A'_{13}	Some to moderate	[1.17, 3.50, 5.50, 7.83; 1]	[4.09, 4.65, 4.65, 5.41; 0.40]
A'_{14}	Moderate amount	[2.59, 4.00, 5.50, 7.62; 1]	[4.29, 4.75, 4.75, 5.21; 0.38]
A'_{15}	Fair amount	[2.17, 4.25, 6.00, 7.83; 1]	[4.79, 5.29, 5.29, 6.02; 0.41]
A'_{16}	Medium	[3.59, 4.75, 5.50, 6.91; 1]	[4.86, 5.03, 5.03, 5.14; 0.27]
A'_{17}	Modest amount	[3.59, 4.00, 6.00, 7.41; 1]	[4.79, 5.30, 5.30, 5.71; 0.42]

A'_{18}	Good amount	[3.38, 5.50, 7.50, 9.62; 1]	[5.79, 6.50, 6.50, 7.21; 0.41]
A'_{19}	Sizeable	[4.38, 6.50, 8.00, 9.41; 1]	[6.79, 7.38, 7.38, 8.21; 0.49]
A'_{20}	Quite a bit	[4.38, 6.50, 8.00, 9.41; 1]	[6.79, 7.38, 7.38, 8.21; 0.49]
A'_{21}	Considerable amount	[4.38, 6.50, 8.25, 9.62; 1]	[7.19, 7.58, 7.58, 8.21; 0.37]
A'_{22}	Substantial amount	[5.38, 7.50, 8.75, 9.81; 1]	[7.79, 8.22, 8.22, 8.81; 0.45]
A'_{23}	A lot	[5.38, 7.50, 8.75, 9.83; 1]	[7.69, 8.19, 8.19, 8.81; 0.47]
A'_{24}	High amount	[5.38, 7.50, 8.75, 9.81; 1]	[7.79, 8.30, 8.30, 9.21; 0.53]
A'_{25}	Very sizeable	[5.38, 7.50, 9.00, 9.81; 1]	[8.29, 8.56, 8.56, 9.21; 0.38]
A'_{26}	Large	[5.98, 7.75, 8.60, 9.52; 1]	[8.03, 8.36, 8.36, 9.17; 0.57]
A'_{27}	Very large	[7.37, 9.41, 10, 10; 1]	[8.72, 9.91, 10, 10; 1.00]
A'_{28}	Very large amount	[7.37, 9.82, 10, 10; 1]	[9.74, 9.98, 10, 10; 1.00]
A'_{29}	Huge amount	[7.37, 9.59, 10, 10; 1]	[8.95, 9.93, 10, 10; 1.00]
A'_{30}	Very high amount	[7.37, 9.73, 10, 10; 1]	[9.34, 9.95, 10, 10; 1.00]
A'_{31}	Extreme amount	[7.37, 9.82, 10, 10; 1]	[9.37, 9.95, 10, 10; 1.00]
A'_{32}	Maximum amount	[8.68, 9.91, 10, 10; 1]	[9.61, 9.97, 10, 10; 1.00]

In order to ensure that all 32 words in Table 8.3 and their interval type – II fuzzy numbers representations are reliable in decision making, all of them are first transformed into standardised generalised interval type – II fuzzy numbers using Definition (3.8). In Table 8.4, words of uncertainty with respective standardised generalised interval type – II fuzzy numbers are tabulated.

Table 8.4: 32 words with respective standardised generalised interval type – II fuzzy numbers

Type – II fuzzy numbers	Upper Membership Function	Lower Membership Function
A'_1	[0, 0, 0.014, 0.197; 1]	[0, 0, 0.005, 0.066; 1.00]
A'_2	[0, 0, 0.014, 0.197; 1]	[0, 0, 0.001, 0.013; 1.00]
A'_3	[0, 0, 0.026, 0.263; 1]	[0, 0, 0.005, 0.063; 1.00]
A'_4	[0, 0, 0.036, 0.263; 1]	[0, 0, 0.005, 0.063; 1.00]
A'_5	[0, 0, 0.064, 0.247; 1]	[0, 0, 0.010, 0.116; 1.00]
A'_6	[0, 0, 0.064, 0.263; 1]	[0, 0, 0.099, 0.099; 1.00]
A'_7	[0.059, 0.150, 0.200, 0.341; 1]	[0.079, 0.168, 0.168, 0.221; 0.74]
A'_8	[0.038, 0.150, 0.250, 0.462; 1]	[0.109, 0.183, 0.183, 0.221; 0.53]
A'_9	[0.009, 0.125, 0.250, 0.462; 1]	[0.167, 0.192, 0.192, 0.221; 0.30]

A'_{10}	[0.009, 0.150, 0.300, 0.462; 1]	[0.179, 0.228, 0.228, 0.281; 0.40]
A'_{11}	[0.059, 0.200, 0.325, 0.441; 1]	[0.229, 0.270, 0.270, 0.321; 0.42]
A'_{12}	[0.038,0.250, 0.500, 0.783; 1]	[0.288, 0.361, 0.361, 0.421; 0.35]
A'_{13}	[0.117, 0.350, 0.550, 0.783; 1]	[0.409, 0.465, 0.465, 0.541; 0.40]
A'_{14}	[0.259, 0.400, 0.550, 0.762; 1]	[0.429, 0.475, 0.475, 0.521; 0.38]
A'_{15}	[0.217, 0.425, 0.600, 0.783; 1]	[0.479, 0.529, 0.529, 0.602; 0.41]
A'_{16}	[0.359, 0.475, 0.550, 0.691; 1]	[0.486, 0.503, 0.503, 0.514; 0.27]
A'_{17}	[0.359, 0.400, 0.600, 0.741; 1]	[0.479, 0.530, 0.530, 0.571; 0.42]
A'_{18}	[0.338, 0.550, 0.750, 0.962; 1]	[0.579, 0.650, 0.650, 0.721; 0.41]
A'_{19}	[0.438, 0.650, 0.800, 0.941; 1]	[0.679, 0.738, 0.738, 0.821; 0.49]
A'_{20}	[0.438, 0.650, 0.800, 0.941; 1]	[0.679, 0.738, 0.738, 0.821; 0.49]
A'_{21}	[0.438, 0.650, 0.825, 0.962; 1]	[0.719, 0.758, 0.758, 0.821; 0.37]
A'_{22}	[0..538, 0.750, 0.875, 0.981; 1]	[0.779, 0.822, 0.822, 0.881; 0.45]
A'_{23}	[0.538, 0.750, 0.875, 0.983; 1]	[0.769, 0.819, 0.819, 0.881; 0.47]
A'_{24}	[0.538, 0.750, 0.875, 0.981; 1]	[0.779, 0.830, 0.830, 0.921; 0.53]
A'_{25}	[0.538, 0.750, 0.900, 0.981; 1]	[0.829, 0.856, 0.856, 0.921; 0.38]
A'_{26}	[0.598, 0.775, 0.860, 0.952; 1]	[0.803, 0.836, 0.836, 0.917; 0.57]
A'_{27}	[0.737, 0.941, 1.000, 1.000; 1]	[0.872, 0.991, 1.000, 1.000; 1.00]
A'_{28}	[0.737, 0.982, 1.000, 1.000; 1]	[0.974, 0.998, 1.000, 1.000; 1.00]
A'_{29}	[0.737, 0.959, 1.000, 1.000; 1]	[0.895, 0.993, 1.000, 1.000; 1.00]
A'_{30}	[0.737, 0.973, 1.000, 1.000; 1]	[0.934, 0.995, 1.000, 1.000; 1.00]
A'_{31}	[0.737, 0.982, 1.000, 1.000; 1]	[0.937, 0.995, 1.000, 1.000; 1.00]
A'_{32}	[0.868, 0.991, 1.000, 1.000; 1]	[0.961, 0.997, 1.000, 1.000; 1.00]

8.3.2 Application

Main concern in this decision making problem is to come out with a reasonable ranking order in terms of all interval type – II fuzzy numbers defined in Table 8.3 with their respective words based on meanings. Thus, application of the CPS_{II} ranking method for ranking interval type – II fuzzy numbers for this case is as follows.

Step 1: Centroid points (x^*, y^*) for A'_1 until A'_{32} are obtained such that value of $x_{A'_1}$ is calculated using formula in equation (6.3) given as

$$x^*_{A'_1} = \frac{1}{3}\left[\left(0+0+0.014+0.197-\frac{(0.003-0)}{(0.211-0)}\right),\left(0+0+0.005+0.066-\frac{(0.0003-0)}{(0.071-0)}\right)\right]$$

$$= (0.0660, 0.0221)$$

Whereas, using equation (6.4), value of $y^*_{A'_1}$ is obtained as

$$y^*_{A'_1} = \frac{1}{3}\left[\left(1+\frac{0.014}{(0.211-0)}\right),\left(1+\frac{0.005}{(0.071-0)}\right)\right]$$

$$= (0.3555, 0.3568).$$

Hence, centroid point for A'_1 is $(0.0660, 0.3555)$ and $(0.0221, 0.3568)$.

Note that, final result of centroid point is in terms of (x^*, y^*). Using equations (6.3) and (6.4), the remaining centroid points of A'_2 until A'_{32} are as follows:

$(x^*_{A'_2}, y^*_{A'_2}) = [(0.0660, 0.3555), (0.0044, 0.3571)]$ $(x^*_{A'_{18}}, y^*_{A'_{18}}) = [(0.6500, 0.4142), (0.6500, 0.1367)]$

$(x^*_{A'_3}, y^*_{A'_3}) = [(0.0884, 0.3633), (0.0211, 0.3578)]$ $(x^*_{A'_{19}}, y^*_{A'_{19}}) = [(0.7041, 0.4099), (0.7460, 0.1633)]$

$(x^*_{A'_4}, y^*_{A'_4}) = [(0.0884, 0.3633), (0.0211, 0.3578)]$ $(x^*_{A'_{20}}, y^*_{A'_{20}}) = [(0.7041, 0.4099), (0.7460, 0.1633)]$

$(x^*_{A'_5}, y^*_{A'_5}) = [(0.0867, 0.4019), (0.0389, 0.3598)]$ $(x^*_{A'_{21}}, y^*_{A'_{21}}) = [(0.7156, 0.4168), (0.7660, 0.1233)]$

$(x^*_{A'_6}, y^*_{A'_6}) = [(0.0918, 0.2986), (0.0495, 0.5000)]$ $(x^*_{A'_{22}}, y^*_{A'_{22}}) = [(0.7811, 0.4067), (0.8277, 0.1500)]$

$(x^*_{A'_7}, y^*_{A'_7}) = [(0.1904, 0.3835), (0.1560, 0.2467)]$ $(x^*_{A'_{23}}, y^*_{A'_{23}}) = [(0.7794, 0.4018), (0.8230, 0.1567)]$

$(x^*_{A'_8}, y^*_{A'_8}) = [(0.2302, 0.3969), (0.1710, 0.1767)]$ $(x^*_{A'_{24}}, y^*_{A'_{24}}) = [(0.7811, 0.4067), (0.8433, 0.1767)]$

$(x^*_{A'_9}, y^*_{A'_9}) = [(0.2160, 0.4), 054(0.1933, 0.1000)]$ $(x^*_{A'_{25}}, y^*_{A'_{25}}) = [(0.7869, 0.4177), (0.8687, 0.1267)]$

$(x^*_{A'_{10}}, y^*_{A'_{10}}) = [(0.2311, 0.4163), (0.2293, 0.1333)]$ $(x^*_{A'_{26}}, y^*_{A'_{26}}) = [(0.7919, 0.3979), (0.8520, 0.1900)]$

$$\left(x^*_{A'_{11}}, y^*_{A'_{11}}\right) = \left[(0.2552, 0.4155), (0.2733, 0.1400)\right] \quad \left(x^*_{A'_{27}}, y^*_{A'_{27}}\right) = \left[(0.9087, 0.3944), (0.9571, 0.3552)\right]$$
$$\left(x^*_{A'_{12}}, y^*_{A'_{12}}\right) = \left[(0.3957, 0.4171), (0.3567, 0.1167)\right] \quad \left(x^*_{A'_{28}}, y^*_{A'_{28}}\right) = \left[(0.9119, 0.3547), (0.9913, 0.3571)\right]$$
$$\left(x^*_{A'_{13}}, y^*_{A'_{13}}\right) = \left[(0.4500, 0.4103), (0.4717, 0.1333)\right] \quad \left(x^*_{A'_{29}}, y^*_{A'_{29}}\right) = \left[(0.9105, 0.3783), (0.9649, 0.3542)\right]$$
$$\left(x^*_{A'_{14}}, y^*_{A'_{14}}\right) = \left[(0.4959, 0.4099), (0.4750, 0.1267)\right] \quad \left(x^*_{A'_{30}}, y^*_{A'_{30}}\right) = \left[(0.9115, 0.3644), (0.9779, 0.3568)\right]$$
$$\left(x^*_{A'_{15}}, y^*_{A'_{15}}\right) = \left[(0.5052, 0.4121), (0.5367, 0.1367)\right] \quad \left(x^*_{A'_{31}}, y^*_{A'_{31}}\right) = \left[(0.9119, 0.3547), (0.9789, 0.3578)\right]$$
$$\left(x^*_{A'_{16}}, y^*_{A'_{16}}\right) = \left[(0.5201, 0.3948), (0.5010, 0.0900)\right] \quad \left(x^*_{A'_{32}}, y^*_{A'_{32}}\right) = \left[(0.9558, 0.3546), (0.9869, 0.3571)\right]$$
$$\left(x^*_{A'_{17}}, y^*_{A'_{17}}\right) = \left[(0.5276, 0.4479), (0.5267, 0.1400)\right]$$

Step 2: Spread values of A'_1 until A'_{32} are calculated such that spread of A'_1 is

$$\begin{aligned} s(A'_1) &= \left[(0.1970 \times 0.3555), (0.0660 \times 0.3568)\right] \\ &= \left[(0.0700), (0.0235)\right] \end{aligned}$$

While for the remaining spread values, s of A'_2 until A'_{32}, all are shown as follows.

$$s(A'_2) = \left[(0.0700), (0.0046)\right] \quad s(A'_{18}) = \left[(0.2585), (0.0194)\right]$$
$$s(A'_3) = \left[(0.0956), (0.0225)\right] \quad s(A'_{19}) = \left[(0.2062), (0.0232)\right]$$
$$s(A'_4) = \left[(0.0982), (0.0225)\right] \quad s(A'_{20}) = \left[(0.2062), (0.0232)\right]$$
$$s(A'_5) = \left[(0.0993), (0.0417)\right] \quad s(A'_{21}) = \left[(0.2184), (0.0126)\right]$$
$$s(A'_6) = \left[(0.1048), (0.0495)\right] \quad s(A'_{22}) = \left[(0.1802), (0.0155)\right]$$
$$s(A'_7) = \left[(0.1082), (0.0350)\right] \quad s(A'_{23}) = \left[(0.1788), (0.0175)\right]$$
$$s(A'_8) = \left[(0.1683), (0.0198)\right] \quad s(A'_{24}) = \left[(0.1802), (0.0251)\right]$$
$$s(A'_9) = \left[(0.1837), (0.0054)\right] \quad s(A'_{25}) = \left[(0.1850), (0.0117)\right]$$
$$s(A'_{10}) = \left[(0.1886), (0.0136)\right] \quad s(A'_{26}) = \left[(0.1408), (0.0217)\right]$$
$$s(A'_{11}) = \left[(0.1587), (0.0129)\right] \quad s(A'_{27}) = \left[(0.1037), (0.0455)\right]$$
$$s(A'_{12}) = \left[(0.3107), (0.0155)\right] \quad s(A'_{28}) = \left[(0.0933), (0.0093)\right]$$
$$s(A'_{13}) = \left[(0.2733), (0.0176)\right] \quad s(A'_{29}) = \left[(0.0995), (0.0372)\right]$$
$$s(A'_{14}) = \left[(0.2062), (0.0117)\right] \quad s(A'_{30}) = \left[(0.0958), (0.0235)\right]$$
$$s(A'_{15}) = \left[(0.2332), (0.0168)\right] \quad s(A'_{31}) = \left[(0.0933), (0.0225)\right]$$
$$s(A'_{16}) = \left[(0.1311), (0.0025)\right] \quad s(A'_{32}) = \left[(0.0468), (0.0139)\right]$$
$$s(A'_{17}) = \left[(0.1711), (0.0129)\right]$$

Step 3: Ranking values of A_1' until A_{32}' are computed whereby ranking value for A_1' is

$$CPS_{II}(A_1') = \left(\frac{0.0660 + 0.0211}{2}\right) \times \left(\frac{0.3555 + 0.3568}{2}\right) \times \left(\frac{(1-0.0700) + (1-0.0236)}{2}\right)$$

$$= 0.0150$$

and ranking values for A_2' until A_{32}' are

$CPS_{II}(A_2') = 0.0121$
$CPS_{II}(A_3') = 0.0186$
$CPS_{II}(A_4') = 0.0189$
$CPS_{II}(A_5') = 0.0222$
$CPS_{II}(A_6') = 0.0293$
$CPS_{II}(A_7') = 0.0507$
$CPS_{II}(A_8') = 0.0521$
$CPS_{II}(A_9') = 0.0468$
$CPS_{II}(A_{10}') = 0.0569$
$CPS_{II}(A_{11}') = 0.0671$
$CPS_{II}(A_{12}') = 0.0840$
$CPS_{II}(A_{13}') = 0.1070$
$CPS_{II}(A_{14}') = 0.1161$
$CPS_{II}(A_{15}') = 0.1251$
$CPS_{II}(A_{16}') = 0.1155$
$CPS_{II}(A_{17}') = 0.1407$
$CPS_{II}(A_{18}') = 0.1542$
$CPS_{II}(A_{19}') = 0.1840$
$CPS_{II}(A_{20}') = 0.1840$
$CPS_{II}(A_{21}') = 0.1770$
$CPS_{II}(A_{22}') = 0.2020$
$CPS_{II}(A_{23}') = 0.2018$
$CPS_{II}(A_{24}') = 0.2126$
$CPS_{II}(A_{25}') = 0.2031$
$CPS_{II}(A_{26}') = 0.2220$
$CPS_{II}(A_{27}') = 0.3236$
$CPS_{II}(A_{28}') = 0.3213$
$CPS_{II}(A_{29}') = 0.3199$
$CPS_{II}(A_{30}') = 0.3203$
$CPS_{II}(A_{31}') = 0.3173$
$CPS_{II}(A_{32}') = 0.3352$

Since,

$$CPS_{II}\left(A'_{32}\right) > CPS_{II}\left(A'_{27}\right) > CPS_{II}\left(A'_{28}\right) > CPS_{II}\left(A'_{30}\right) > CPS_{II}\left(A'_{29}\right) > CPS_{II}\left(A'_{31}\right) >$$
$$CPS_{II}\left(A'_{26}\right) > CPS_{II}\left(A'_{24}\right) CPS_{II}\left(A'_{25}\right) > CPS_{II}\left(A'_{22}\right) > CPS_{II}\left(A'_{23}\right) > CPS_{II}\left(A'_{19}\right) >$$
$$CPS_{II}\left(A'_{20}\right) > CPS_{II}\left(A'_{21}\right) > CPS_{II}\left(A'_{18}\right) > CPS_{II}\left(A'_{17}\right) > CPS_{II}\left(A'_{15}\right) > CPS_{II}\left(A'_{14}\right) >$$
$$CPS_{II}\left(A'_{16}\right) > CPS_{II}\left(A'_{13}\right) > CPS_{II}\left(A'_{12}\right) > CPS_{II}\left(A'_{11}\right) > CPS_{II}\left(A'_{10}\right) > CPS_{II}\left(A'_{8}\right) >$$
$$CPS_{II}\left(A'_{7}\right) > CPS_{II}\left(A'_{9}\right) > CPS_{II}\left(A'_{6}\right) > CPS_{II}\left(A'_{5}\right) > CPS_{II}\left(A'_{4}\right) > CPS_{II}\left(A'_{3}\right) >$$
$$CPS_{II}\left(A'_{1}\right) > CPS_{II}\left(A'_{2}\right)$$

hence ranking order result for A'_1 until A'_{32} is

$$A'_{32} \succ A'_{27} \succ A'_{28} \succ A'_{30} \succ A'_{29} \succ A'_{31} \succ A'_{26} \succ A'_{24} \succ A'_{25} \succ A'_{22} \succ A'_{23} \succ A'_{19} \succ A'_{20} \succ A'_{21} \succ$$
$$A'_{18} \succ A'_{17} \succ A'_{15} \succ A'_{14} \succ A'_{16} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{8} \succ A'_{7} \succ A'_{9} \succ A'_{6} \succ A'_{5} \succ A'_{4} \succ$$
$$A'_{3} \succ A'_{1} \succ A'_{2}$$

8.4 CASE STUDY 3: VEHICLE SELECTION

8.4.1 Overview

Based on Kang et al. (2012), there are three options of vehicles that are available to select. They are car, taxi and train. In order to choose for an appropriate vehicle for a journey, three main criteria are taken into consideration namely price (*P*), journey time (*T*) and comfort, (*C*). In this investigation, *P* is classified as the most important criterion as compared to the other two criteria, which affects on the direction of decision made. According to Kang et al. (2012), all criteria considered are described in terms of Z – numbers which are reflected as linguistic terms Z_P = (very high, very high), Z_T = (high, very high) and Z_C = (medium, very high) for price, journey time and comfort respectively. All of these linguistic terms are defined using triangular membership function in Definition (3.2) and are shown in the following Table 8.5.

Table 8.5: Linguistic terms with respective triangular linguistic values

Linguistic term	Triangular linguistic value
Very low, *VL*	(0.00, 0.00, 0.25; 1.00)
Low, *L*	(0.00, 0.25, 0.50; 1.00)
Medium, *M*	(0.25, 0.50, 0.75; 1.00)
High, *H*	(0.50, 0.75, 1.00; 1.00)
Very high, *VH*	(0.75, 1.00, 1.00; 1.00)

Based on Table 8.5, Kang et al. (2012) come out with a decision matrix with Z – numbers based linguistic values for all vehicles with respective criteria which is illustrated in Table 8.6.

Table 8.6: Decision matrix with Z – numbers based linguistic values for all vehicles with respective criteria (Kang et al. (2012a).

Vehicle	Criteria		
	P (pounds)	*T* (minutes)	*C*
	(*VH*, *VH*)	(*H*, *VH*)	(*M*, *VH*)
Car	[(9, 10, 12), *VH*]	[(70, 100, 120), *M*]	[(4, 5, 6), *H*]
Taxi	[(20, 24, 25), *H*]	[(60, 70, 100), *VH*]	[(7, 8, 10), *H*]
Train	[(15, 15, 15), *H*]	[(70, 80, 90), *H*]	[(1, 4, 7), *H*]

Using Table 8.5 and 8.6 as guidelines, the following Table 8.7 on decision matrix with numerical values is constructed.

Table 8.7: Decision matrix with Z – numbers based numerical values for all vehicles with respective criteria (Kang et al. (2012a).

Vehicle	Criteria		
	P (pounds)	*T* (minutes)	*C*
	[(0.75, 1.00, 1.00), (0.75, 1.00, 1.00)]	[(0.50, 0.75, 1.00), (0.75, 1.00, 1.00)]	[(0.25, 0.50, 0.75), (0.75, 1.00, 1.00)]
Car	[(9, 10, 12), (0.75, 1.00, 1.00)]	[(70, 100, 120), (0.25, 0.50, 0.75)]	[(4, 5, 6), (0.50, 0.75, 1.00)]
Taxi	[(20, 24, 25), (0.50, 0.75, 1.00)]	[(60, 70, 100), (0.75, 1.00, 1.00)]	[(7, 8, 10), (0.50, 0.75, 1.00)]
Train	[(15, 15, 15), (0.50, 0.75, 1.00)]	[(70, 80, 90), (0.50, 0.75, 1.00)]	[(1, 4, 7), (0.50, 0.75, 1.00)]

Before the CPS_Z ranking method is applied to solving this decision making problem, all of those numerical values in Table 8.7 are first transformed into standardised generalised Z – numbers for easy computation and reliable (Chen & Chen, 2007). Note that, only numerical values which are not in the form of standardised generalised Z – numbers are transformed, others are remained the same. The following Table 8.8 illustrates decision matrix with standardised generalised Z – numbers using Definition (3.8).

Table 8.8: Decision matrix with standardised generalised Z – numbers.

Vehicle	Criteria		
	P (pounds)	T (minutes)	C
	[(0.75, 1.00, 1.00), (0.75, 1.00, 1.00)]	[(0.50, 0.75, 1.00), (0.75, 1.00, 1.00)]	[(0.25, 0.50, 0.75), (0.75, 1.00, 1.00)]
Car	[(0.36, 0.40, 0.48), (0.75, 1.00, 1.00)]	[(0.58, 0.83, 1.00), (0.25, 0.50, 0.75)]	[(0.40, 0.50, 0.60), (0.50, 0.75, 1.00)]
Taxi	[(0.80, 0.96, 1.00), (0.50, 0.75, 1.00)]	[(0.50, 0.58, 0.83), (0.75, 1.00, 1.00)]	[(0.70, 0.80, 1.00), (0.50, 0.75, 1.00)]
Train	[(0.60, 0.60, 0.60), (0.50, 0.75, 1.00)]	[(0.58, 0.67, 0.75), (0.50, 0.75, 1.00)]	[(0.10, 0.40, 0.70), (0.50, 0.75, 1.00)]

It is clearly noted that in Table 8.8, both vehicles and criteria components are in Z – numbers, $Z = (A, B)$. Next, numerical values in terms of Z – numbers for both components are aggregated so that a single Z – number is obtained for car, taxi and train. Aggregation process for Car is as follows.

$$Z_{car} = \left[\left(\frac{\begin{array}{l}((0.75,1.00,1.00)\otimes(0.36,0.40,0.48)) + ((0.50,0.75,1.00)\otimes(0.58,0.83,1.00)) + \\ ((0.25,0.50,0.75)\otimes(0.40,0.50,0.60))\end{array}}{(0.75,1.00,1.00)+(0.50,0.75,1.00)+(0.25,0.50,0.75)} \right), \right.$$

$$\left. \left(\frac{\begin{array}{l}((0.75,1.00,1.00)\otimes(0.75,1.00,1.00)) + ((0.50,0.75,1.00)\otimes(0.25,0.50,0.75)) + \\ ((0.25,0.50,0.75)\otimes(0.50,0.75,1.00))\end{array}}{(0.75,1.00,1.00)+(0.50,0.75,1.00)+(0.25,0.50,0.75)} \right) \right]$$

$$= [(0.2400,0.5656,1.2867),(0.3750,0.7500,1.2222)].$$

while, for taxi and train, their aggregation values are

$$Z_{taxi} = [(0.3727,0.7978,1.7200),(0.4375,0.8333,1.3333)]$$
$$Z_{train} = [(0.2782,0.5789,1.2500),(0.3750,0.7500,1.3333)]$$

Since, all vehicles are not in the form of standardised generalised Z – numbers, hence all of them are transformed into standardised generalised Z – numbers using equation (3.6) shown as follows. Standardised generalised Z – number for car is

$$Z_{car} = \left[\left(\frac{0.2400}{1.2867}, \frac{0.5656}{1.2867}, \frac{1.2867}{1.2867} \right), \left(\frac{0.3750}{1.2222}, \frac{0.7500}{1.2222}, \frac{1.2222}{1.2222} \right) \right]$$

$$= [(0.1865,0.4396,1.0000),(0.3068,0.6136,1.0000)]$$

While, for taxi and train, their standardised generalised Z – numbers are

$$Z_{taxi} = [(0.2167,0.4638,1.0000),(0.3281,0.6250,1.0000)]$$
$$Z_{train} = [(0.2226,0.4631,1.0000),(0.2813,0.5625,1.0000)]$$

For easy computation, this study defines Z_{car}, Z_{taxi} and Z_{train} as $Z_{\tilde{A}_1}, Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ respectively. Notice that, all Z – numbers used are first reduced into Z – fuzzy numbers as suggested by Kang et al. (2012). Thus, using the conversion of Z – numbers into Z – fuzzy numbers introduced in this study, the following Z – numbers are obtained.

$$Z_{car} = (0.1492, 0.3517, 0.8000; 1.0000)$$
$$Z_{taxi} = (0.1730, 0.3703, 0.7985; 1.0000)$$
$$Z_{train} = (0.1745, 0.3631, 0.7854; 1.0000)$$

8.4.2 Application

This, utilising the CPS_Z ranking method, flow on solving this problem is as follows.

Step 1: Centroid points (x^*, y^*) for $Z_{\tilde{A}_1}, Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are obtained such that

value of $x_{Z_{\tilde{A}_1}}$ is calculated using formula in equation (7.3) as

$$x_{Z_{\tilde{A}_1}} = \frac{1}{3}\left[0.1492 + 0.3517 + 0.3517 + 0.8000 - \frac{(0.2814 - 0.0525)}{(1.1517 - 0.5009)}\right]$$

$$= 0.4336$$

whereas, using equation (7.4), value of $y_{Z_{\tilde{A}_1}}$ is obtained as

$$y_{Z_{\tilde{A}_1}} = \frac{1}{3}\left[1 + \frac{0}{(1.4396 - 0.6261)}\right]$$

$$= 0.3333$$

Hence, centroid point for $Z_{\tilde{A}_1}$ is (0.4336, 0.3333).

Using same techniques as shown above, centroid points for $Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are calculated accordingly and the results are as follows.

$$(x_{Z_{\tilde{A}_2}}, y_{Z_{\tilde{A}_2}}) = (0.4473, 0.3333)$$
$$(x_{Z_{\tilde{A}_3}}, y_{Z_{\tilde{A}_3}}) = (0.4406, 0.3333)$$

Step 2: Spread values of $Z_{\tilde{A}_1}, Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are calculated such that spread of $Z_{\tilde{A}_1}$ is

$$\begin{aligned} s(Z_{\tilde{A}_1}) &= 0.6508 \times 0.3333 \\ &= 0.2169 \end{aligned}$$

while for $Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$, their spread values are

$$s(Z_{\tilde{A}_2}) = 0.2085$$
$$s(Z_{\tilde{A}_3}) = 0.2031$$

Step 3: Ranking values of $Z_{\tilde{A}_1}, Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are computed whereby ranking value for $Z_{\tilde{A}_1}$ is

$$\begin{aligned} CPS_Z\left(Z_{\tilde{A}_1}\right) &= 0.4336 \times 0.3333 \times (1 - 0.2169) \\ &= 0.1132 \end{aligned}$$

and ranking values for $Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ are

$$CPS_Z\left(Z_{\tilde{A}_2}\right) = 0.1180$$
$$CPS_Z\left(Z_{\tilde{A}_3}\right) = 0.1170$$

Since $CPS_Z\left(Z_{\tilde{A}_2}\right) > CPS_Z\left(Z_{\tilde{A}_3}\right) > CPS_Z\left(Z_{\tilde{A}_1}\right)$, hence ranking order result for for Z – numbers $Z_{\tilde{A}_1}$, $Z_{\tilde{A}_2}$ and $Z_{\tilde{A}_3}$ is $Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$. This implies that taxi is the best vehicle to select, followed by train and then car.

8.5 DISCUSSION

This section discusses the applications of the *CPS* ranking methodology in subsections 8.2.1, 8.2.2 and 8.2.3. It is worth mentioning here that all decision making case studies namely risk analysis, words classification and vehicle selection which are prepared in subsection 8.3.1, 8.3.2 and 8.3.3 respectively are analysed using established ranking methods considered in this study and the *CPS* ranking methodology. These ranking methods evaluate all the aforementioned case studies based on their consistency and efficiency in ranking fuzzy numbers. Therefore, the discussion on the application of established ranking methods considered in this study including the *CPS* ranking methodology is as follows.

8.5.1 CASE STUDY 1

This subsection illustrates the consistency and efficiency of the CPS_I ranking method and established ranking methods considered in this study in solving risk analysis case study by Chen & Chen (2009). The following Table 8.9 signifies the consistency and efficiency evaluation of the CPS_I ranking method and established ranking methods considered in this study.

Table 8.9: Consistency and Efficiency Evaluation

Method	Type – I Fuzzy Numbers			Ranking Results	Evaluation	
	$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$		Level of Consistency	Level of Efficiency
Cheng (1998)	x	x	x	-	N/A	Slightly Efficient
Kumar et al. (2010)	0.300	0.300	0.300	$\tilde{R}_1 \approx \tilde{R}_2 \approx \tilde{R}_3$	0%	Slightly Efficient
Dat et al. (2012)	0.000/ 0.600	0.600/ 0.300	0.300/ 0.000	$\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1$	100%	Slightly Efficient
Yu et al. (2013) for $\alpha = 0$	0.700	0.300	x	-	N/A	Slightly Efficient
Yu et al. (2013) for $\alpha = 0.5$	1.000	1.000	x	-	N/A	Slightly Efficient
Yu et al. (2013) for $\alpha = 1$	0.300	0.700	x	-	N/A	Slightly Efficient
Zhang et al. (2013) for $\alpha = 0$	1.000	1.000	x	-	N/A	Slightly Inefficient
Zhang et al. (2013) for $\alpha = 0.5$	1.000	1.000	x	-	N/A	Slightly Inefficient
Zhang et al. (2013) for $\alpha = 1$	1.000	1.000	x	-	N/A	Slightly Inefficient
CPS_I	0.091	0.145	0.144	$\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1$	100%	Very Efficient

In this case study, the correct ranking order such that the ranking result is 100% consistent with human intuition is $\tilde{R}_2 \succ \tilde{R}_3 \succ \tilde{R}_1$. $\tilde{R}_2$ is considered as the greatest type – I fuzzy numbers among the three because it has the largest value of centroid point and height, followed by $\tilde{R}_3$ and then $\tilde{R}_1$. In Table 8.9, ranking method by Kumar et al. (2010) treats this case study as equal ranking, $\tilde{R}_1 \approx \tilde{R}_2 \approx \tilde{R}_3$, such that the result is 0% consistent with human intuition as this ranking method considers all type – I under consideration as the same area. Other established ranking methods considered in this study except Dat et al. (2012), produce no ranking result for this case study. On contrary, Dat et al. (2012) and the CPS_I ranking method obtain correct ranking order for this case study such that the result is 100% consistent with human intuition. This result implies that the CPS_I ranking method is applicable to deal with any case studies involving type – I fuzzy numbers.

In terms of efficiency, Zhang et al. (2014) ranking method is classified as very inefficient ranking method in this evaluation because this method is a pairwise ranking method and needs additional operation to rank correctly type – I fuzzy numbers in this case study. On the other hand, Dat et al. (2012) ranking method is graded as a slightly inefficient ranking method because it is a pairwise ranking method but does not need additional operation to rank correctly type – I fuzzy numbers of this case study. On the other hand, Cheng (1998) and Yu et al. (2013) ranking methods are considered as slightly efficient ranking methods as both rank type – I fuzzy numbers of this case study simultaneously but incorporate additional operation in obtaining the final ranking order. The CPS_I ranking method in this case, is classified as a very efficient ranking method as this method ranks correctly all type – I fuzzy numbers considered in this case study using simultaneous ranking without incorporate any additional operation.

8.5.2 CASE STUDY 2

This subsection illustrates the consistency and efficiency of the CPS_{II} ranking method and established ranking methods considered in this study in words classification case study by Wu & Mendel (2009). The following Table 8.10 signifies the consistency and efficiency evaluation of the CPS_{II} ranking method and established ranking methods considered in this study.

Table 8.10: Consistency and Efficiency Evaluation

Method	Type – II Fuzzy Numbers							
	A'_1	A'_2	A'_3	A'_4	A'_5	A'_6	A'_7	A'_8
[1] Mitchell (2006)	0.066	0.067	0.068	0.069	0.071	0.072	0.073	0.074
[2] Wu & Mendel (2009)	0.470	0.560	0.630	0.640	0.660	0.670	1.750	2.130
[3] II – Kumar et al. (2010)	0.331	0.363	0.394	0.425	0.456	0.488	0.519	0.550
[4] II – Dat et al. (2012)	0.000/ 0.066	0.066/ 0.067	0.067/ 0.068	0.068/ 0.069	0.069/ 0.071	0.071/ 0.072	0.072/ 0.073	0.073/ 0.074
[5] II – Yu et al. (2013) for $\alpha = 0$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[6] II – Yu et al. (2013) for $\alpha = 0.5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[7] II – Yu et al. (2013) for $\alpha = 1$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[8] II – Zhang et al. (2013) for $\alpha = 0$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[9] II – Zhang et al. (2013) for $\alpha = 0.5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[10] II – Zhang et al. (2013) for $\alpha = 1$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[11] CPS_{II} – direct	0.089	0.117	0.119	0.125	0.129	0.134	0.141	0.147
[12] CPS_{II} – indirect	0.089	0.117	0.119	0.125	0.129	0.134	0.141	0.147

Table 8.10: Consistency and Efficiency Evaluation (continue)

Method	Type – II Fuzzy Numbers										
	A'_9	A'_{10}	A'_{11}	A'_{12}	A'_{13}	A'_{14}	A'_{15}	A'_{16}	A'_{17}	A'_{18}	A'_{19}
[1]	0.075	0.076	0.077	0.078	0.079	0.080	0.081	0.082	0.083	0.084	0.085
[2]	2.190	2.320	2.590	3.900	4.560	4.950	5.130	5.190	5.410	6.500	7.160
[3]	0.581	0.613	0.644	0.675	0.706	0.738	0.769	0.800	0.831	0.863	0.894
[4]	0.074/ 0.075	0.075/ 0.076	0.076/ 0.077	0.077/ 0.078	0.078/ 0.079	0.079/ 0.080	0.080/ 0.081	0.081/ 0.082	0.082/ 0.083	0.083/ 0.084	0.084/ 0.085
[5]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[6]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[7]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[8]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[9]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[10]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[11]	0.153	0.155	0.162	0.166	0.174	0.181	0.190	0.195	0.205	0.216	0.222
[12]	0.153	0.155	0.162	0.166	0.174	0.181	0.190	0.195	0.205	0.216	0.222

Table 8.10: Consistency and Efficiency Evaluation (continue)

Method	Type – II Fuzzy Numbers										
	A'_{20}	A'_{21}	A'_{22}	A'_{23}	A'_{24}	A'_{25}	A'_{26}	A'_{27}	A'_{28}	A'_{29}	A'_{30}
[1]	0.086	0.087	0.088	0.089	0.090	0.091	0.092	0.094	0.095	0.096	0.097
[2]	7.160	7.250	7.900	7.910	8.010	8.030	8.120	9.300	9.310	9.340	9.370
[3]	0.925	0.956	0.961	0.964	0.968	0.969	0.972	0.978	0.981	0.982	0.985
[4]	0.085/ 0.086	0.086/ 0.087	0.087/ 0.088	0.088/ 0.089	0.089/ 0.090	0.090/ 0.091	0.091/ 0.092	0.092/ 0.094	0.094/ 0.095	0.095/ 0.096	0.096/ 0.097
[5]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[6]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[7]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[8]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[9]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[10]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
[11]	0.230	0.235	0.246	0.255	0.261	0.272	0.281	0.289	0.296	0.301	0.310
[12]	0.230	0.235	0.246	0.255	0.261	0.272	0.281	0.289	0.296	0.301	0.310

Table 8.10: Consistency and Efficiency Evaluation (continue)

Method	Type – II Fuzzy Numbers		Ranking Result
	A'_{20}	A'_{21}	
[1]	0.098	0.099	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[2]	9.380	9.690	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[3]	0.989	00901	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[4]	0.097/ 0.098	0.098/ 0.099	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[5]	1.000	1.000	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[6]	1.000	1.000	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[7]	1.000	1.000	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[8]	1.000	1.000	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[9]	1.000	1.000	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[10]	1.000	1.000	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[11]	0.321	0.333	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$
[12]	0.321	0.333	$A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20}$

Table 8.10: Consistency and Efficiency Evaluation (continue)

Method	Ranking Result
[1]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[2]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[3]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[4]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$

[5]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[6]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[7]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[8]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[9]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[10]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[11]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$
[12]	$\succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{11} \succ A'_{10} \succ A'_{9} \succ A'_{8} \succ A'_{7} \succ A'_{6}$

Table 8.10: Consistency and Efficiency Evaluation (continue)

Method	Ranking Result	Evaluation	
		Level of Consistency	Level of Efficiency
[1]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Slightly Efficient
[2]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Slightly Efficient
[3]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Slightly Efficient
[4]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Slightly Inefficient
[5]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Slightly Efficient
[6]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Slightly Efficient
[7]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Slightly Efficient
[8]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Very Inefficient
[9]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Very Inefficient
[10]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Very Inefficient
[11]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Very Efficient
[12]	$\succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$	100%	Very Efficient

In this case study, the correct ranking order such that the ranking result is 100% consistent with human intuition is $A'_{32} \succ A'_{31} \succ A'_{30} \succ A'_{29} \succ A'_{28} \succ A'_{27} \succ A'_{26} \succ A'_{25} \succ A'_{24} \succ A'_{23} \succ A'_{22} \succ A'_{21} \succ A'_{20} \succ A'_{19} \succ A'_{18} \succ A'_{17} \succ A'_{16} \succ A'_{15} \succ A'_{14} \succ A'_{13} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_{12} \succ A'_{11} \succ A'_{10} \succ A'_9 \succ A'_8 \succ A'_7 \succ A'_6 \succ A'_5 \succ A'_4 \succ A'_3 \succ A'_2 \succ A'_1$. In Table 8.10, all established ranking methods including the CPS_{II} ranking method for direct and indirect approaches, rank correctly interval type – II fuzzy numbers of this case study such that the result is 100% consistent with human intuition. This is because all interval type – II fuzzy numbers considered in this is case study are trivial and easy to rank. The result implies that the CPS_{II} ranking method for both direct and indirect approaches are applicable to deal with any case studies involving interval type – II fuzzy numbers.

In terms of efficiency, II – Zhang et al. (2014) ranking method is classified as very inefficient ranking method in this evaluation because this method is a pairwise ranking method and needs additional operation to rank correctly interval type – II fuzzy numbers in this case study. On the other hand, II – Dat et al. (2012) ranking method is graded as a slightly inefficient ranking method because it is a pairwise ranking method but does not need additional operation to rank correctly interval type – II fuzzy numbers of this case study. On the other hand, Mitchel (2006), Wu & Mendel (2009), II – Cheng (1998) and II – Yu et al. (2013) ranking methods are considered as slightly efficient ranking methods as both rank interval type – II fuzzy numbers of this case study simultaneous y but incorporate additional operation in obtaining the final ranking order. The CPS_{II} ranking method for both direct and indirect approaches are in this case classified as a very efficient ranking method as this method ranks correctly all interval type – II fuzzy numbers considered in this case study using simultaneous ranking without incorporate any additional operation.

8.5.3 CASE STUDY 3

This subsection illustrates the consistency and efficiency of the CPS_Z ranking method and established ranking methods considered in this study in vehicle selection case study by Kang et al. (2012a). The following Table 8.11 signifies the consistency and efficiency evaluation of the CPS_Z ranking method and established ranking methods considered in this study.

Table 8.11: Consistency and Efficiency Evaluation

Method	Type – I Fuzzy Numbers			Ranking Results	Evaluation	
	$Z_{\tilde{A}_1}$	$Z_{\tilde{A}_2}$	$Z_{\tilde{A}_3}$		Level of Consistency	Level of Efficiency
Z – Cheng (1998)	0.680	0.746	0.726	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Slightly Efficient
Z – Kumar et al. (2010)	0.300	0.700	0.500	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Slightly Efficient
Z – Dat et al. (2012)	0.000/ 0.600	0.600/ 0.300	0.300/ 0.000	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Slightly Inefficient
Z – Yu et al. (2013) for $\alpha = 0$	0.300	0.700	0.500	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Slightly Efficient
Z – Yu et al. (2013) for $\alpha = 0.5$	0.300	0.700	0.500	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Slightly Efficient
Z – Yu et al. (2013) for $\alpha = 1$	0.500	0.969	0.720	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Slightly Efficient
Z – Zhang et al. (2013) for $\alpha = 0$	1.000	1.000	1.000	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Very Inefficient
Z – Zhang et al. (2013) for $\alpha = 0.5$	1.000	1.000	1.000	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Very Inefficient
Z – Zhang et al. (2013) for $\alpha = 1$	1.000	1.000	1.000	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Very Inefficient
CPS_Z	0.113	0.118	0.117	$Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$	100%	Very Efficient

In this case study, the correct ranking order such that the ranking result is 100% consistent with human intuition is $Z_{\tilde{A}_2} \succ Z_{\tilde{A}_3} \succ Z_{\tilde{A}_1}$. $Z_{\tilde{A}_2}$ is considered as the greatest Z – numbers among the three because it has the largest accumulated value of centroid point and spread, followed by $Z_{\tilde{A}_3}$ and then $Z_{\tilde{A}_1}$. In Table 8.11, all established ranking methods including the CPS_Z ranking method rank correctly Z – numbers of this case study such that the result is 100% consistent with human intuition. This is because all Z – numbers considered in this case study are trivial and easy to rank. The result implies that the CPS_Z ranking method is applicable to deal with any case studies involving Z – numbers.

In terms of efficiency, Z – Zhang et al. (2014) ranking method is classified as very inefficient ranking method in this evaluation because this method is a pairwise ranking method and needs additional operation to rank correctly Z – numbers in this case study. On the other hand, Z – Dat et al. (2012) ranking method is graded as a slightly inefficient ranking method because it is a pairwise ranking method but does not need additional operation to rank correctly Z – numbers of this case study. On the other hand, Z – Cheng (1998) and Z – Yu et al. (2013) ranking methods are considered as slightly efficient ranking methods as both rank Z – numbers of this case study simultaneously but incorporate additional operation in obtaining the final ranking order. The CPS_Z ranking method in this case, is classified as a very efficient ranking method as this method ranks correctly all Z – numbers considered in this case study using simultaneous ranking without incorporate any additional operation.

8.6 SUMMARY

In this chapter, the applicability of the CPS_I, CPS_{II} and CPS_Z ranking methods in solving respective case studies in the literature of fuzzy sets are illustrated. The CPS_I is applied to a risk analysis problem, the CPS_{II} on word classification and the CPS_Z on the vehicle selection problem. All of them are compared in term of their consistency and efficiency with other ranking methods considered in this study. In Chapter 9, the thesis discusses the conclusion part of this study.

CHAPTER NINE

CONCLUSIONS

9.1 INTRODUCTION

This chapter illustrates the contributions of this study, the concluding remarks and recommendations for future works. It discusses a summary of all the works contributed to knowledge in every chapter of the thesis and suggests some significant recommendations towards improving the knowledge of fuzzy sets. Therefore, with no loss of generality of all chapters in the thesis, details on those aforementioned points are intensively discussed in sections and subsections of this chapter.

9.2 CONTRIBUTIONS

As far as this study is concerned, there are three main contributions to knowledge which are underlined in the thesis, namely, contribution to knowledge on literature review, contribution to knowledge on methodology and contribution to knowledge on case studies. These contributions which are underpinned by publication [1] to [4], indicate the strength and novelty of the study in improving and enhance the theory of fuzzy sets. Thus, in this respect, the contributions are highlighted as follows.

9.2.1 Literature Review

The main contribution of this study towards literature of fuzzy sets is the development of a novel ranking methodology for fuzzy numbers based on centroid point and spread, *CPS*. In developing the *CPS* ranking methodology, a novel direction of computing the spread of fuzzy numbers is proposed where it is calculated based on the distance from the centroid point. This kind of spread method is suggested in this study because it enhances the capability of the centroid point in ranking fuzzy numbers as highlighted in Chapter 4 of the thesis. Several theoretical properties of the novel spread method are introduced in this study to strengthen the capability of the method on ranking fuzzy numbers appropriately. Then, the novel ranking methodology is developed using both the novel spread method and an established centroid point approach. Along with this contribution, this study suggets the efficiency evaluation as the validation technique of a ranking fuzzy numbers method together with the established consistency evaluation.

9.2.3 Methodology

As mentioned in Chapter 2, there are three kinds of fuzzy numbers found in the literature of fuzzy sets, they are type – I fuzzy numbers, type – II fuzzy numbers and Z – fuzzy numbers. It is worth reminding here again that the *CPS* ranking methodology which consists of the CPS_I, CPS_{II}, and CPS_Z ranking methods are developed to ranking type – I fuzzy numbers, type – II fuzzy numbers and Z – fuzzy numbers respectively.

In Chapter 5, the CPS_I ranking method is applied to ranking type – I fuzzy numbers. In the analysis, the CPS_I ranking method contributes significant benchmarking examples of type – I fuzzy numbers where it extends cases of type – I fuzzy numbers in the literature of fuzzy sets. The extension covers benchmarking examples with three type – I fuzzy numbers in each case where previous researches on ranking type – I fuzzy numbers consider only two type – I fuzzy numbers. Later in Chapter 6, an extension of the *CPS* ranking methodology on ranking the interval type – II fuzzy numbers, CPS_{II} is developed for the first time. As far as researches on ranking interval type – II fuzzy numbers are concerned, the CPS_{II} ranking method is the third direct ranking method introduced in the literature of fuzzy sets. This is because most ranking methods introduced for interval type – II fuzzy numbers required reduction approach, in other word they utilise the indirect way to ranking interval type – II fuzzy numbers. Main contribution demonstrates by this study on interval type – II fuzzy numbers is the applicability of the CPS_{II} ranking method to ranking interval type – II fuzzy numbers using both ways, direct and indirect. A useful extension of interval type – II fuzzy numbers into standardised generalised interval type – II fuzzy numbers is also introduced in this study as the extension provides generic representations of interval type – II fuzzy numbers. Another extension of the *CPS* ranking methodology is developed for the first time in this study and the literature of fuzzy sets is the development of the method for ranking Z – fuzzy numbers, CPS_Z. This development is considered as new because the concept of Z – fuzzy numbers is relatively new in fuzzy sets which indicate that theoretical aspects with respects to this concept are not yet established. Therefore, the development of the CPS_Z ranking method is a new in fuzzy sets, hence all details on its development, theoretical and empirical frameworks are regarded as other major contributions of this study to knowledge of fuzzy numbers.

9.2.3 Case Studies

Contributions cover under this subsection is described in detailed by Chapter 8 of the thesis. In Chapter 8, the ranking methodology for fuzzy numbers based on centroid point and spread, CPS is applied to three different case studies namely risk analysis, footprint of uncertainty and vehicle selection under uncertain environment. It has to be noted here that, all of these case studies are considered as type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers are used in the investigations. Type – I fuzzy numbers is used on case study involving fuzzy risk analysis, while type – II fuzzy numbers and Z – numbers are utilised in case studies concerning the footprint of uncertainty and vehicle selection under uncertain environment respectively. Consideration of these case studies in this thesis reflects the capability of the *CPS* ranking methodology to not only ranking fuzzy numbers correctly such that the ranking results are consistent with human intuition but also solving any related case studies involving type – I fuzzy numbers, type – II fuzzy numbers and Z – numbers effectively.

Overall, contributions to knowledge by this study are described in detailed by this section. It has to be noted here that some contributions are prepared for knowledge enhancement while some are done for decision making purposes. In the following section, the concluding remarks of this study are provided.

9.3 CONCLUDING REMARKS

This section covers the concluding remarks of this study. There are three main concluding remarks which are exhibited in this study namely the concluding remark on the literature review, concluding remark on the methodology and concluding remark on the case studies. These concluding remarks summarised all works done in chapters provided in the thesis. In this respect, all of these concluding remarks are classified and discussed as follows.

9.3.1 Literature Review

This concluding remark covers with descriptions of established works on ranking fuzzy numbers. In the literature review chapter, gaps of established ranking methods are identified where these are the major concern of this study. Among the gaps mentioned in the literature review chapter are the incapability to ranking the embedded, overlapping and non – overlapping cases of fuzzy numbers with correct ranking order such that the ranking results are consistent with human intuition. These aforementioned gaps by established ranking methods are analysed and solve by the first objective of this study. This indicates that the first objective of this study is successfully accomplished where it caters off all limitations of the established works on ranking fuzzy numbers by developing a ranking methodology for ranking fuzzy numbers.

9.3.2 Methodology

This concluding remark covers description on the development of the ranking methodology for fuzzy numbers based on centroid point and spread. In Chapter 4, a methodology for ranking fuzzy numbers is developed where it consists of ranking method for type – I fuzzy numbers, ranking method for interval type – II fuzzy numbers and ranking for Z – fuzzy numbers. Along with this methodology development, theoretical and empirical validations are outlined in this study in Chapter 5, 6 and 7. The theoretical validation considers relevant established and new properties for ranking fuzzy numbers purposes while the empirical validation takes into account two ranking viewpoints namely the consistency and efficiency evaluations. Based on these descriptions, the second and third objectives of this study are achieved. Furthermore, the ranking methodology developed outperforms other established ranking methods consider in this study.

9.3.3 Case Studies

This concluding remark covers description on the case studies of the thesis. In Chapter 8, three case studies namely fuzzy risk analysis, footprint of uncertainty and vehicle selection under uncertain environment are considered and evaluated using the ranking methodology developed in this study. All of these case studies are considered in this study because type – I fuzzy numbers are used in fuzzy risk analysis case study while, type – II fuzzy numbers Z – numbers are utilised in footprint of uncertainty and vehicle selection under uncertain environment case studies respectively. The ranking methodology developed in this study produces consistent and efficient ranking results for each case study examined. This implies that the last objective of this study is also accomplished.

Overall, the concluding remarks of this study are described in detailed by this section where this reflects by the successfulness in accomplishing all objectives set up by this study. In the following section, recommendations for future work by this study are provided.

9.4 LIMITATIONS

This section discusses limitations of this study where they are figured out from the proposed ranking methodology. The limitations are as follows.

Firstly, the new ranking methodology for fuzzy numbers based on centroid point and spread is not applicable to ranking non – linear fuzzy numbers. This is due to the fact that the ranking methodology considers only linear fuzzy numbers as they are easy to deal with as compared to non – linear fuzzy numbers. Moreover, majority of established ranking methods consider only linear type of fuzzy numbers in their analysis. Thus, consideration of the non – linear fuzzy numbers cases are neglected in this case.

Secondly, with respect to ranking of Z – numbers, this study suggests that Z – number is to first reduce into type – I fuzzy numbers and is then ranked accordingly. This indicates that the ranking methodology incapable to rank Z – numbers simultaneously.

Overall, limitations of this study are described in detailed in this section. It has to be noted here that all limitations mentioned indicate that this research needs further enhancement.

9.5 RECOMMENDATION FOR FUTURE WORK

This section discusses the recommendation of this study for future research work purposes. There are three kinds of recommendations are mentioned here namely recommendation on the literature review, recommendation on fuzzy numbers and recommendation on the case studies. These recommendations focus on improvising the theoretical and empirical qualities in the theory of fuzzy sets. In this respect, recommendations for future work of this study are pointed out as follows.

9.5.1 Literature Review

In this study, a new ranking methodology for fuzzy numbers is developed based on centroid point and spread methods. Although, the ranking methodology gives good theoretical and empirical results, it is recommended for future work that other methods that are capable to effectively capture human intuition are thoroughly explored. This recommendation is purposely suggested by this study because when more detailed investigations on fuzzy numbers are made, more complex cases of fuzzy numbers are figured out, thus indicates that a more commanding ranking methodology is required in this case. Therefore, exploring for suitable methods in the literature of fuzzy sets for ranking fuzzy numbers is necessary as this is crucial for decision making purposes. Another recommendation by this study is on the utilisation of other types of fuzzy numbers apart from linear. As far as researches on ranking fuzzy numbers are concerned, majority of ranking methods use linear type of fuzzy numbers in their analysis. Thus, consideration of the non – linear fuzzy numbers in the future works suggests the representation of fuzzy numbers is more generic and practical as not all cases are well represented by linear type of fuzzy numbers.

9.5.2 Methodology

The chronological evidences suggest that Z – fuzzy numbers are not yet established in the literature of fuzzy sets as compared to type – I fuzzy numbers and interval type – II fuzzy numbers, this study recommends both theoretical and empirical frameworks of Z – fuzzy numbers is extensively explored. This is because Z – fuzzy numbers is more practical than type – I fuzzy numbers and interval type – II fuzzy numbers in terms of representation, thus finding suitable ways to deal with Z – fuzzy numbers is necessary. With respect to ranking methodology, the only way to ranking Z – fuzzy numbers is to reduce them first into type – I fuzzy numbers and then rank them accordingly. This implies that Z – fuzzy numbers are not effectively dealt as this affects the representation of Z – fuzzy numbers. Therefore, this study recommends for future work that methods that are capable to simultaneously rank Z – fuzzy numbers is developed and solve numerous decision making problems.

Overall, recommendations for future work by this study are described in detailed by this section. It has to be noted here that all recommendations provided are prepared for knowledge enhancement and decision making purposes.

9.6 SUMMARY

In this chapter, contributions, the concluding remarks, limitations and recommendation for future works by this study are highlighted. Thus, the thesis ends its discussion by citing all references used throughout the thesis which are provided next after this chapter.

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