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A NOTE ON THE CHERN-SIMONS-DIRAC EQUATIONS IN THE COULOMB GAUGE

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ABSTRACT. We prove that the Chern-Simons-Dirac equations in the Coulomb gauge are locally well-posed from initial data in H^s with $s > \frac{1}{4}$. To study nonlinear Wave or Dirac equations at this regularity generally requires the presence of null structure. The novel point here is that we make no use of the null structure of the system. Instead we exploit the additional elliptic structure in the Coulomb gauge together with the bilinear Strichartz estimates of Klainerman-Tataru.

1. INTRODUCTION

Chern-Simons gauge theories form an important component of the relativistic theory of planar physics. In particular they are used to model physical phenomena such as the fractional quantum hall effect, and have been well studied by physicists, see for instance [15, 5, 4] and the references therein. Mathematically, Chern-Simons terms were first introduced in [3] in connection with certain geometric invariants. More recently a number of results have appeared studying the properties of various partial differential equations arising in connection with Chern-Simons theories, for instance the Chern-Simons-Higgs equations [1, 9, 16], the Chern-Simons-Schrödinger equations [14, 17], and the Chern-Simons-Dirac equations [2, 4, 6, 10].

In the current article we study the local well-posedness of the Chern-Simons-Dirac (CSD) equations which are given by

$$
\begin{aligned}
i\gamma^\mu D_\mu \psi &= m\psi \\
\frac{1}{2}\epsilon^{\mu\nu\rho} F_{\nu\rho} &= -J^\mu
\end{aligned}
\tag{1}
$$

where the unknowns are the spinor $\psi : \mathbb{R}^{1+2} \longrightarrow \mathbb{C}^2$, and the gauge $A_\mu : \mathbb{R}^{1+2} \longrightarrow \mathbb{R}$, $\mu = 0, 1, 2$. Repeated indices are summed over $\mu = 0, ..., 2$ and raised and lowered using the Minkowski metric $g = \text{diag}(1, -1, -1)$, $\epsilon^{\mu\nu\rho}$ is the completely anti-symmetric tensor with $\epsilon^{012} = 1$, $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative, and $F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$ denotes the curvature of the connection A_μ. The equations are coupled using the Dirac current $J^\nu = \overline{\psi}\gamma^\nu \psi$ where $\overline{\psi} = \psi^\dagger \gamma^0$ is the Dirac adjoint, and $\psi^\dagger$ denotes the conjugate transpose. The Gamma matrices γ^μ are 2×2 complex matrices which satisfy the relations

$$
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I_{2\times 2}, \qquad \left(\gamma^j\right)^\dagger = -\gamma^j, \qquad \left(\gamma^0\right)^\dagger = \gamma^0.
$$

We take the representation

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

The CSD equations are derived from the Lagrangian

$$\mathcal{L}_{CSD} = \frac{1}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + \overline{\psi}(\gamma^\mu D_\mu - m)\psi$$

and solutions (ψ, A_μ) are *gauge invariant*. Namely if (ψ, A_μ) is a solution, then $(\psi e^{i\theta}, A_\mu + \partial_\mu\theta)$ is also a solution for any sufficiently regular map $\theta : \mathbb{R}^{1+2} \to \mathbb{R}$. Thus to obtain a well-posed Cauchy problem, we need to couple the CSD system (1) with a choice of gauge. Common choices are the Coulomb gauge $\partial_1 A_1 + \partial_2 A_2 = 0$, the Lorenz gauge $\partial^\mu A_\mu = 0$, and the Temporal gauge $A_0 = 0$.

Solutions to the CSD equation also satisfy conservation of charge

$$\|\psi(t)\|_{L^2_x} = \|\psi(0)\|_{L^2_x} \tag{2}$$

and if $m = 0$, are invariant under the rescaling $(\psi, A_\mu)(t, x) \mapsto \lambda(\psi, A_\mu)(\lambda t, \lambda x)$. This rescaling leaves the L^2 norm unchanged, and so the CSD equation is *charge critical*. Thus ideally we would like to prove local well-posedness from initial data in L^2. This would be particulary interesting in view of the conservation of charge (2).

Recently the local and global well-posedness of Chern-Simons systems has received considerable attention, see for instance [1, 2, 9, 8, 7, 10, 14, 16]. In particular, it was shown by Huh-Oh [10] that if we couple the Chern-Simons-Dirac equations with the Lorenz gauge condition $\partial_\mu A^\mu = 0$, then we have local well-posedness for initial data in H^s with $s > \frac{1}{4}$. This improved earlier work of Huh [6] where local well-posedness was obtained for $s > \frac{1}{2}$ in the Coulomb gauge, $s > \frac{5}{8}$ Lorenz gauge, and $s > \frac{3}{4}$ in the Temporal gauge.

A crucial component in the proof of local well-posedness of Huh-Oh in [10] was the presence of *null structure*. Here null structure refers to the fact that, from the point of view of bilinear estimates, the nonlinear terms in (1) behave better than generic bilinear terms such as $|\psi|^2$. More precisely, if we consider a nonlinear wave equation of the general form

$$\Box u = u\nabla u \tag{3}$$

then in general, we have ill-posedness if $s < \frac{3}{4}$ due to the counterexamples of Lindblad [13]. On the other hand, if we replace the nonlinearity $u\nabla u$ with a null form such as $Q_{ij}(|\nabla|^{-1}u, u)$ where

$$Q_{ij}(u, v) = \partial_i u \partial_j v - \partial_j u \partial_i v,$$

then we have well-posedness for $s > \frac{1}{4}$, see for instance [11]. Note that the nonlinearities $u\nabla u$ and $Q_{ij}(|\nabla|^{-1}u, u)$ are roughly of the same "strength" in terms of derivatives. Now if we write the CSD equations as a system of nonlinear wave equations, then schematically the CSD equations are of the form (3). Thus, at least at first glance, it appears that null structure is essential to get LWP below $\frac{3}{4}$.

In the current article we show that, if we couple the system (1) with the Coulomb Gauge condition

$$\partial_1 A_1 + \partial_2 A_2 = 0, \tag{4}$$

then LWP holds for $s > \frac{1}{4}$. This extends the recent results of Huh-Oh from the Lorenz gauge to the Coulomb gauge. The advantage of the Coulomb gauge is that *no null structure is needed*. This is somewhat surprising in light of the schematic form of the CSD equation, and the counterexamples of Lindblad

mentioned above.

Our main result is the following.

Theorem 1. *Let $\frac{1}{4} < s < 1$ and assume $\psi(0) \in H^s$. Then there exists a $T = T(\|\psi(0)\|_{H^s}) > 0$ and a solution $(\psi, A) \in C([0,T], H^s \times \dot{H}^{2s})$ to the CSD equations in the Coulomb gauge. Moreover the solution depends continuously on the initial data, and if we let $I = [0,T]$ then we have*

$$\|\psi\|_{L^\infty_t H^s_x(I \times \mathbb{R}^2)} + \|\gamma^\mu \partial_\mu \psi\|_{L^1_t H^s_x(I \times \mathbb{R}^2)} + \|A_\mu\|_{L^\infty_t \dot{H}^{2s}_x(I \times \mathbb{R}^2)} \lesssim \|\psi(0)\|_{H^s(\mathbb{R}^2)}$$

and the solution is unique in this class.

Remark 1. In the result of Huh [6], local well-posedness in the Coulomb gauge was obtained under the conditions $\psi \in H^{\frac{1}{2}+\epsilon}$ and $A_\mu(0) \in L^2_x$ where the initial data should satisfy the constraints

$$\partial_1 A_1 + \partial_2 A_2 = 0, \qquad \partial_1 A_2 - \partial_2 A_1 = J^0.$$

This is in contrast to Theorem 1 where we only provide initial data for the spinor ψ. This apparent ambiguity is reconciled by the fact that the initial data for ψ, completely determines $A_\mu(0)$ via the constraint equations. Thus there is no need to specify data for the gauge $A_\mu(0)$. See Section 2 below.

The key observation in the proof of Theorem 1 is that, the equations for the gauge, coupled with the Coulomb gauge condition, mean that A_μ satisfies elliptic equations of the form

$$\Delta A_\mu = \nabla \psi^2.$$

Note that this is peculiar to the Chern-Simons action, if we have instead couple the Dirac equation with the Maxwell equations, then in the Coulomb gauge we only have an elliptic equation for a *component* of the gauge A_μ. On the other hand, the Chern-Simons action gives sufficiently good control over the curvature of the gauge A_μ, that we have an elliptic equation for the *whole* gauge. The proof is completed by using the bilinear Strichartz estimates of Klainerman-Tataru [12].

2. Elliptic Structure

We start by examining the equations for the gauge A_μ, for this we need a little preliminary notation. Define the "curl" $\nabla^\perp = (-\partial_2, \partial_1)$ and recall the identity

$$\Delta B = \nabla(\nabla \cdot B) + \nabla^\perp(\nabla^\perp \cdot B) \tag{5}$$

where $B : \mathbb{R}^2 \to \mathbb{C}^2$. Define the projections $\mathcal{P}_{cf}, \mathcal{P}_{df}$ by

$$\mathcal{P}_{cf} B = \frac{1}{\Delta} \nabla(\nabla \cdot B), \qquad \mathcal{P}_{df} B = \frac{1}{\Delta} \nabla^\perp(\nabla^\perp \cdot B).$$

It is easy to see that $\mathcal{P}_{cf}$ and $\mathcal{P}_{df}$ are orthogonal projections on $L^2(\mathbb{R}^2)$ and $\nabla \cdot \mathcal{P}_{df} = \nabla^\perp \cdot \mathcal{P}_{cf} = 0$. Let $A = (A_1, A_2)$ denote the spatial component of the gauge A_μ. Then the gauge equations in (1) can be written as

$$\partial_t A - \nabla A_0 = N$$
$$\nabla^\perp A = -J^0$$

with $N = (-J^2, J^1)^T$. Decompose $A = \mathcal{P}_{cf}A + \mathcal{P}_{df}A = A^{cf} + A^{df}$ into divergence free and curl free components. Then the previous equations are equivalent to

$$\partial_t A^{cf} - \nabla A_0 = \mathcal{P}_{cf}N$$
$$\Delta A^{df} = -\nabla^\perp J_0.$$

Note that, unlike in the Maxwell or Yang-Mills gauge theories, we have an elliptic component *independent of the choice of gauge.* If now enforce the Coulomb gauge condition

$$\nabla \cdot A = \nabla \cdot A^{cf} = 0$$

we see that we must have $A^{cf} = 0$ and therefore, the equations for the gauge (A_0, A) are

$$\nabla A_0 = -\mathcal{P}_{cf}N$$
$$\Delta A^{df} = -\nabla^\perp J^0.$$

Taking $\nabla^\perp$ of both sides of the equation for A_0, and adding the equations for the spinor ψ, we see that the CSD equations in the Coulomb gauge are

$$\begin{aligned}
i\gamma^\mu \partial_\mu \psi &= m\psi - A_\mu \gamma^\mu \psi \\
\Delta A_0 &= \partial_1 J_2 - \partial_2 J_1 \\
\Delta A^{df} &= -\nabla^\perp J^0 \\
A^{cf} &= 0.
\end{aligned} \tag{6}$$

3. Proof of Well-posedness

Here we prove Theorem 1. By taking the equations for the gauge A_μ, and substituting them into the Dirac component, we see that to prove Theorem 1, it is enough to prove well-posedness for the cubic Dirac equation

$$\begin{aligned}
i\gamma^\mu \partial_\mu \psi &= m\psi - N(\psi, \psi)\psi \\
\psi(0) &= f
\end{aligned} \tag{7}$$

where N is the bilinear operator

$$N(\psi, \phi) = \frac{1}{\Delta}\left[\left(\partial_1(\overline{\psi}\gamma^2\phi) - \partial_2(\overline{\psi}\gamma^1\phi)\right)\gamma^0 + \partial_2(\overline{\psi}\gamma^0\phi)\gamma^1 - \partial_1(\overline{\psi}\gamma^0\phi)\gamma^2\right]. \tag{8}$$

Once we have the solution ψ to (7), we then reconstruct the gauge A_μ by solving the elliptic equations

$$\begin{aligned}
\Delta A_0 &= \partial_1 J_2 - \partial_2 J_1 \\
\Delta A^{df} &= -\nabla^\perp J_0 \\
A^{cf} &= 0.
\end{aligned} \tag{9}$$

The proof of local well-posedness for (7) will rely on the following bilinear refinement of the classical Strichartz estimates for the wave equation due to Klainerman-Tataru [12].

Proposition 2 ([12]). *Let $u = e^{it|\nabla|}f$, $v = e^{\pm it|\nabla|}g$ and $\frac{1}{q} + \frac{1}{2r} = \frac{1}{2}$ with $r < \infty$. Then*

$$\left\||\nabla|^{-a}(uv)\right\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s}\|g\|_{\dot{H}^s}$$

provided $s = \frac{3}{4}\left(1 - \frac{1}{r}\right) - \frac{a}{2}$ and $0 \leqslant a < 1 - \frac{1}{r}$.

This has the following useful consequence.

Corollary 3. *Let $\frac{1}{4} < s \leqslant \frac{1}{2}$ and $I \subset \mathbb{R}$ with $|I| < \infty$. Let B be as in (8) and assume that $\psi = e^{\pm it|\nabla|} f$, $\phi = e^{\pm it|\nabla|} g$. Then*

$$\|N(\psi, \phi)\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^2)} + \||\nabla|^{s+\frac{1}{2}} N(\psi, \phi)\|_{L_t^4 L_x^2(I \times \mathbb{R}^2)} \lesssim \|f\|_{H^s} \|g\|_{H^s}.$$

Proof. To obtain the $L_t^4 L_x^2$ bound we just note that an application of Proposition 2 with $q = 4$, $r = 2$ gives[1]

$$\||\nabla|^{s+\frac{1}{2}} N(\psi, \phi)\|_{L_t^4 L_x^2(I \times \mathbb{R}^2)} \lesssim \||\nabla|^{s-\frac{1}{2}}(\psi\phi)\|_{L_t^4 L_x^2} \lesssim \|f\|_{H^s} \|g\|_{H^s}.$$

On the other hand, for the $L_t^2 L_x^\infty$ bound, we start by writing

$$N(\psi, \phi) = P_{<1} N(\psi, \phi) + P_{>1} N(\psi, \phi)$$

where $P_{<1}$ is the projection onto frequencies $|\xi| < 1$. To deal with the low frequency component we use the assumption $|I| < \infty$ to obtain

$$\|P_{<1} N(\psi, \phi)\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^2)} \lesssim \sum_{\lambda \leqslant 1} \lambda^2 \|P_\lambda N(\psi, \phi)\|_{L_t^2 L_x^1(I \times \mathbb{R}^2)}$$

$$\lesssim_I \sum_{\lambda \leqslant 1} \lambda \|\psi\|_{L_t^\infty L_x^2} \|\phi\|_{L_t^\infty L_x^2}$$

$$\lesssim_I \|f\|_{H^s} \|g\|_{H^s}$$

where the sum is over dyadic $\lambda \in 2^{\mathbb{Z}}$, $\lambda \leqslant 1$, and the P_λ are the standard Littlewood-Paley projections onto frequencies $|\xi| \approx \lambda$.

On the other hand, for the high frequency piece we use Sobolev embedding followed by an application of Holder in time to deduce that

$$\|P_{>1} N(\psi, \phi)\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^2)} \lesssim \||\nabla|^{-a}(\psi\phi)\|_{L_t^2 L_x^r(I \times \mathbb{R}^2)} \lesssim \||\nabla|^{-a}(\psi\phi)\|_{L_t^q L_x^r(I \times \mathbb{R}^2)}$$

where $a = 1 - \frac{3}{r} < 1 - \frac{2}{r}$ (so we can apply Sobolev embedding) and $q > 2$ such that $\frac{1}{q} + \frac{1}{2r} = \frac{1}{2}$ where $r < \infty$ is to be chosen later. An application of Proposition 2 then gives

$$\|P_{>1} N(\psi, \phi)\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^2)} \lesssim \|f\|_{H^{s'}} \|g\|_{H^{s'}}$$

where

$$s' = \frac{3}{4}\left(1 - \frac{1}{r}\right) - \frac{a}{2} = \frac{1}{4} + \frac{3}{4r}.$$

Result now follows by taking r sufficiently large. $\qquad \square$

We also require the following version of the product rule for H^s.

Proposition 4. *Let $s > 0$ and $\alpha \geqslant 0$. Then*

$$\|fg\|_{\dot{H}^s} \lesssim \|f\|_{L^\infty} \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s+\alpha}} \||\nabla|^{-\alpha} g\|_{L^\infty}. \tag{10}$$

Proof. See the Appendix. $\qquad \square$

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[1]Whenever we multiply two spinors together, i.e. $\psi\phi$, we really mean $\sum_{i,j} \psi_i \phi_j$ where ψ_i, ψ_j are the components of the spinor.

The intuition here is that when g is higher frequency than f, we should have $|\nabla|^s(fg) \approx f|\nabla|^s g$, which is essentially the first term. On the other hand, when f is higher frequency than g, we should be able to shift derivatives from g onto f, or $f|\nabla|^\alpha g \lesssim (|\nabla|^\alpha f)g$, since it is much worse to have a derivative fall on a high frequency piece rather than a low frequency term. To make this more precise requires a straightforward application of Littlewood-Paley theory.

Fix $T > 0$. The proof of Theorem 1 will proceed by the standard iteration argument using the Duhamel norm

$$\|\psi\|_{Y_T^s} = \|\psi\|_{L_t^\infty H_x^s(I \times \mathbb{R}^2)} + \|\gamma^\mu \partial_\mu \psi\|_{L_t^1 H_x^s(I \times \mathbb{R}^2)}$$

where $I = [0, T]$. It is easy to see that we have the energy inequality

$$\|\psi\|_{Y_T^s} \lesssim \|\psi(0)\|_{H_x^s} + \|\gamma^\mu \partial_\mu \psi\|_{L_t^1 H_x^s(I \times \mathbb{R}^2)}.$$

Moreover we have the following version of the transference principle.

Lemma 5. *Let $s \in \mathbb{R}$ and $1 \leqslant q, r \leqslant \infty$. Suppose that we have*

$$\left\| M\left(e^{\pm it|\nabla|} f\right) \right\|_{L_t^q L_x^r(I \times \mathbb{R}^2)} \lesssim \|f\|_{H^s} \tag{11}$$

for any $f \in H^s$ where M is a Fourier multiplier acting only on the spatial variable $x \in \mathbb{R}^2$. Then for any $\psi \in Y_T^s$ we have

$$\|M\psi\|_{L_t^q L_x^r(I \times \mathbb{R}^2)} \lesssim \|\psi\|_{Y_T^s}.$$

Proof. Let $U(t)f$ denote the solution operator for the Dirac equation $i\gamma^\mu \partial_\mu \psi = 0$ with initial data $\psi(0) = f$. An easy computation shows that $U(t - s) = U(t)U(s)$ and

$$U(t) = e^{it|\nabla|}L_+ + e^{-it|\nabla|}L_-$$

where $L_\pm$ are bounded, time-independent, Fourier multipliers on H^s for all $s \in \mathbb{R}$. Now given any $\psi \in L_t^\infty H_x^s(I \times \mathbb{R}^2)$ we can write

$$\psi = U(t)\psi(0) + \int_0^t U(t - s)F(s)ds$$

where $F(s) = i\gamma^\mu \partial_\mu \psi$. Hence using (11) we obtain

$$\|M\psi\|_{L_t^q L_x^r(I \times \mathbb{R}^2)} \lesssim \sum_\pm \left\| M\left(e^{\pm it|\nabla|}L_\pm \psi(0)\right) \right\|_{L_t^q L_x^r(I \times \mathbb{R}^2)} + \int_0^T \left\| M\left(e^{\pm i(t-s)|\nabla|}L_\pm F(s)\right) \right\|_{L_t^q L_x^r(I \times \mathbb{R}^2)} ds$$

$$\lesssim \sum_\pm \|L_\pm \psi(0)\|_{H_x^s} + \int_0^T \|L_\pm F(s)\|_{H_x^s} ds$$

$$\leqslant \|\psi\|_{Y_T^s}.$$

$\square$

Remark 2. A similar argument shows that a multi-linear version of Lemma 5 also holds. Thus an estimate of the form

$$\left\| M\left(e^{\pm_1 it|\nabla|}f_1, ..., e^{\pm_m it|\nabla|}f_m\right) \right\|_{L_t^q L_x^r(I \times \mathbb{R}^2)} \lesssim \Pi_{j=1}^m \|f_j\|_{H_x^s}$$

immediately implies that

$$\left\| M\left(\psi_1, ..., \psi_m\right) \right\|_{L_t^q L_x^r(I \times \mathbb{R}^2)} \lesssim \Pi_{j=1}^m \|\psi_j\|_{Y_T^s}.$$

We now come to the proof of local well-posedness for the cubic Dirac equation (7). A standard iteration argument using the energy inequality, followed by Holder in time, and Lemma 5, shows that to prove local well-posedness for (7) it is enough to prove the estimate

$$\|N(\psi_1, \psi_2)\psi_3\|_{L_t^2 H_x^s(I \times \mathbb{R}^2)} \lesssim_I \Pi_{j=1}^3 \|f_j\|_{H_x^s} \tag{12}$$

where we assume $\psi_j = e^{\pm_j it|\nabla|} f_j$ is a homogeneous wave with data $f_j \in H_x^s$. To prove (12) we start by considering the low frequency case $|\xi| < 1$. Then by Corollary 3 we obtain

$$\|P_{\leqslant 1} N(\psi_1, \psi_2)\psi_3\|_{L_t^2 H_x^s} \lesssim \|N(\psi_1, \psi_2)\psi_3\|_{L_{t,x}^2} \lesssim \|N(\psi_1, \psi_2)\|_{L_t^2 L_x^\infty} \|\psi_3\|_{L_t^\infty L_x^2} \lesssim \|f_1\|_{H_x^s} \|f_2\|_{H_x^s} \|f_3\|_{H_x^s}.$$

We can now replace the H_x^s norm on the left hand side of (12) with the homogeneous version $\dot{H}^s$ and hence, via an application of the Sobolev product rule in Proposition 4 (with $\alpha = \frac{1}{2}$), we deduce that

$$\|N(\psi_1, \psi_2)\psi_3\|_{L_t^2 \dot{H}_x^s(I \times \mathbb{R}^2)} \lesssim \|N(\psi_1, \psi_2)\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^2)} \||\nabla|^s \psi_3\|_{L_t^\infty L_x^2(I \times \mathbb{R}^2)}$$
$$+ \||\nabla|^{s+\frac{1}{2}} N(\psi_1, \psi_2)\|_{L_t^4 L_x^2(I \times \mathbb{R}^2)} \||\nabla|^{-\frac{1}{2}} \psi_3\|_{L_t^4 L_x^\infty(I \times \mathbb{R}^2)}$$
$$\lesssim \|f_1\|_{H_x^s} \|f_2\|_{H_x^s} \|f_3\|_{H_x^s}$$

where we used the bilinear estimates in Corollary 3, together with the linear $L_t^4 L_x^\infty$ Strichartz estimate.

To complete the proof of Theorem 1, it only remains to reconstruct the gauge A_μ by using (9). To compute the correct regularity for the gauge, note that from (9) we have

$$\|A_\mu\|_{L_t^\infty \dot{H}^r} \lesssim \|\psi^2\|_{L_t^\infty \dot{H}^{r-1}}$$

and since we are assuming the spinor $\psi \in L_t^\infty H_x^s$ we need the product estimate

$$\|\psi^2\|_{\dot{H}^{r-1}} \lesssim \|\psi\|_{\dot{H}^s}^2. \tag{13}$$

The required conditions for product estimates in $\dot{H}^s$ to hold, are given by the following.

Proposition 6. *Assume $s_1 + s_2 + s_3 = \frac{n}{2}$ with $s_j + s_k > 0$ for $j \neq k$. Then*

$$\|fg\|_{\dot{H}^{-s_1}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^{s_2}(\mathbb{R}^n)} \|g\|_{\dot{H}^{s_3}(\mathbb{R}^n)}.$$

We omit the proof of Proposition 6 since it is well known. However for the special case that we use below, namely $s_1 = 1 - 2s$, $s_2 = s_3 = s$, we note that, provided $0 < s < \frac{1}{2}$, the estimate follows by a simple application of Sobolev embedding

$$\|fg\|_{\dot{H}^{2s-1}} \lesssim \|fg\|_{L^p} \lesssim \|f\|_{L^q} \|g\|_{L^q} \lesssim \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^s}$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1-2s}{2}$ and $\frac{1}{p} = \frac{2}{q}$.

It we now return to estimating the gauge A_μ, we observe that if we want to put $A_\mu \in L_t^\infty \dot{H}^r$, in light of (13) and Proposition 6, we need

$$r - 1 + 1 = 2s, \qquad \Longrightarrow \qquad r = 2s$$

and consequently the correct regularity for the gauge is $A_\mu \in L_t^\infty \dot{H}^{2s}$. Note that this required the assumption $s < 1$, if $s \geqslant 1$, then the same argument puts the gauge $A_\mu \in \dot{H}^r$ for $0 < r < s + 1$.

Appendix - Proof of Proposition 4

Proof. The first step is to write

$$fg \approx \sum_\lambda f_\lambda g_{\ll \lambda} + \sum_\lambda f_\lambda g_\lambda + \sum_\lambda f_{\ll \lambda} g_\lambda$$

where $f_\lambda = P_\lambda f$ (with $\lambda \in 2^{\mathbb{Z}}$) and $f_{\ll \lambda} = \sum_{\mu \ll \lambda} P_\mu f$. Note that we can write $f_{\ll \lambda} = \phi_\lambda * f$ with $\phi \in \mathcal{S}$, supp $\phi \subset \{|\xi| \leqslant 2\}$ and $\phi_\lambda(x) = \lambda^2 \phi(\lambda x)$.

For the high-low piece, we use the fact that the Fourier support of $f_\lambda g_{\ll \lambda}$ is contained inside the annulus $|\xi| \approx \lambda$ and so

$$\|\sum_\lambda f_\lambda g_{\ll \lambda}\|_{\dot{H}^s}^2 \approx \sum_\lambda \left(\lambda^s \|f_\lambda g_{\ll \lambda}\|_{L^2}\right)^2 \lesssim \sum_\lambda \lambda^{2s} \|f_\lambda\|_{L^2}^2 \|g_{\ll \lambda}\|_{L^\infty}^2.$$

Now we observe that[2]

$$|g_{\ll \lambda}| = |(|\nabla|^\alpha \phi_\lambda) * (|\nabla|^{-\alpha} g)|$$
$$\lesssim \||\nabla|^\alpha \phi_\lambda\|_{L^1} \||\nabla|^{-\alpha} g\|_{L^\infty}$$
$$\approx \lambda^\alpha \||\nabla|^{-\alpha} g\|_{L^\infty}$$

and consequently

$$\|\sum_\lambda f_\lambda g_{\ll \lambda}\|_{H^s} \lesssim \left(\sum_\lambda \lambda^{2s} \|f_\lambda\|_{L^2}^2 \|g_{\ll \lambda}\|_{L^\infty}^2\right)^{\frac{1}{2}}$$
$$\lesssim \||\nabla|^{-\alpha} g\|_{L^\infty} \left(\sum_\lambda \lambda^{2(s+\alpha)} \|f_\lambda\|_{L^2}^2\right)^{\frac{1}{2}}$$
$$\lesssim \|f\|_{\dot{H}^{s+\alpha}} \||\nabla|^{-\alpha} g\|_{L^\infty}.$$

The low-high piece follows an identical argument, essentially just repeat the previous reasoning but replace f with g, g with f and take $\alpha = 0$.

Thus it only remains to deal with the high-high case. The key trick is to write

$$\sum_\lambda P_\mu(f_\lambda g_\lambda) = \sum_{\lambda \geqslant \frac{\mu}{4}} P_\mu(f_\lambda g_\lambda) = \sum_{\lambda \geqslant \frac{1}{4}} P_\mu(f_{\lambda\mu} g_{\lambda\mu})$$

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[2] We use the fact that $|\nabla|^\alpha \phi \in L^1$ provided $\alpha \geqslant 0$. This is obvious in the case $\alpha = 0$. On the other hand if $\alpha > 0$, by Holder's inequality, followed by the Hausdorff-Young inequality,

$$\||\nabla|^\alpha \phi\|_{L_x^1} \lesssim \sum_{|\kappa| \leqslant n} \|x^\kappa (|\nabla|^\alpha \phi)\|_{L_x^q} \lesssim \sum_{|\kappa| \leqslant n} \|\partial_\xi^\kappa (|\xi|^\alpha \widehat{\phi}(\xi))\|_{L_\xi^{q'}}$$

where we are free to choose any $2 < q < \infty$. Thus it suffices to prove $\partial_\xi^\kappa (|\xi|^\alpha \widehat{\phi}(\xi)) \in L_\xi^{q'}$ for $|\kappa| \leqslant n$. Now using the fact that $\widehat{\phi} \in C_0^\infty$ and $|\partial^\kappa |\xi|^\alpha| \lesssim |\xi|^{\alpha-n}$, we see that we require $|\xi|^{\alpha-n} \in L^{q'}(|\xi| \leqslant 1)$ which holds provided $(\alpha - n)q' > -n$. Rearranging we obtain $\alpha > n(1 - \frac{1}{q'}) = \frac{n}{q}$ which holds provided q is sufficiently large.

where to obtain the last equality we just relabeled our sequence to start at $\frac{1}{4}$ instead of μ ($\in 2^{\mathbb{Z}}$). Now using a similar argument to before

$$\begin{aligned}
\|\sum_\lambda f_\lambda g_\lambda\|_{H^s} &= \Big(\sum_\mu \mu^{2s}\|\sum_\lambda P_\mu(f_\lambda g_\lambda)\|_{L^2}^2\Big)^{\frac{1}{2}} \\
&\lesssim \sum_{\lambda \geqslant 1}\Big(\sum_\mu \mu^{2s}\|f_{\lambda\mu}g_{\lambda\mu}\|_{L^2}^2\Big)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^\infty}\sum_{\lambda\geqslant 1}\Big(\sum_\mu \mu^{2s}\|g_{\lambda\mu}\|_{L^2}^2\Big)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^\infty}\sum_{\lambda\geqslant 1}\lambda^{-s}\Big(\sum_\mu (\lambda\mu)^{2s}\|g_{\lambda\mu}\|_{L^2}^2\Big)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^\infty}\|g\|_{\dot{H}^s}
\end{aligned}$$

where the last line follows by again relabeling the sequence.

$\square$

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