# Regularity for energy-minimizing area-preserving deformations

**Citation for published version:**
Karakhanyan, A 2014, 'Regularity for energy-minimizing area-preserving deformations' Journal of Elasticity, vol. 114, no. 2, pp. 213-223. DOI: 10.1007/s10659-013-9436-3

**Digital Object Identifier (DOI):**
10.1007/s10659-013-9436-3



**Published In:**
Journal of Elasticity

# REGULARITY FOR ENERGY-MINIMIZING AREA-PRESERVING DEFORMATIONS

ARAM L. KARAKHANYAN

ABSTRACT. In this paper we establish the square integrability of the nonnegative hydrostatic pressure $p$, that emerges in the minimization problem

$$\inf_{\mathcal{K}} \int_\Omega |\nabla \mathbf{v}|^2, \qquad \Omega \subset \mathbb{R}^2$$

as the Lagrange multiplier corresponding to the incompressibility constraint $\det \nabla \mathbf{v} = 1$ a.e. in $\Omega$. Our method employs the Euler-Lagrange equation for the mollified Cauchy stress $\mathbf{C}$ satisfied in the image domain $\Omega^\star = \mathbf{u}(\Omega)$. This allows to construct a convex function $\psi$, defined in the image domain, such that the measure of the normal mapping of $\psi$ controls the $L^2$ norm of the pressure. As a by-product we conclude that $\mathbf{u} \in C^{\frac{1}{2}}_{\mathrm{loc}}(\Omega)$ if the dual pressure (introduced in [6]) is nonnegative.

## 1. INTRODUCTION

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$ and $\mathcal{K} = \{\mathbf{v} \in W^{1,2}(\Omega, \mathbb{R}^2), \det \nabla \mathbf{v} = 1 \text{ a.e. in } \Omega\}$. For $\mathbf{v} \in \mathcal{K}$ we define the stored energy as

$$E[\mathbf{v}] = \int_\Omega |\nabla \mathbf{v}|^2, \qquad \mathbf{v} \in \mathcal{K}. \tag{1.1}$$

Let us recall the definition of local minimizers [1], [2], [6].

**Definition 1.1.** We say that an area-preserving deformation $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^2)$ is a *local minimizer* if for all area preserving (or incompressible) deformations $\mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^2)$ with $\mathrm{supp}(\mathbf{w} - \mathbf{u}) \subset \Omega$ the following holds

$$\int_\Omega |\nabla \mathbf{u}|^2 \le \int_\Omega |\nabla \mathbf{w}|^2. \tag{1.2}$$

Our primary interest is to analyze the properties of the local minimizers of $E[\cdot]$ and the integrability of the hydrostatic pressure $p$ sought as the Lagrange multiplier corresponding to the incompressibility constraint $\det \nabla \mathbf{v} = 1$. The sufficiently regular local minimizers solve the system

$$\begin{cases} \mathrm{div}\, \mathbf{T} = 0 & \text{in } \Omega, \\ \det \nabla \mathbf{u} = 1 & \text{a.e. in } \Omega, \end{cases} \tag{1.3}$$

where $\mathbf{T} = \nabla \mathbf{u} + p(\nabla \mathbf{u})^{-t}$ is the first Piola-Kirchhoff tensor and $(\nabla \mathbf{u})^{-t}$ is the transpose of the inverse matrix, see [7], pages 371 and 379. Since $\det \nabla \mathbf{u} = 1$ we have

$$(\nabla \mathbf{u})^{-1} = \begin{pmatrix} u^2_2 & -u^1_2 \\ -u^2_1 & u^1_1 \end{pmatrix}, \qquad (\nabla \mathbf{u})^{-t} = \begin{pmatrix} u^2_2 & -u^2_1 \\ -u^1_2 & u^1_1 \end{pmatrix}. \tag{1.4}$$

From (1.4) we deduce that (1.3) is equivalent to the system

$$\begin{cases} \mathrm{div}[\nabla u^1 - p \mathscr{J} \nabla u^2] = 0, \\ \mathrm{div}[\nabla u^2 + p \mathscr{J} \nabla u^1] = 0, \\ \det \nabla \mathbf{u} = 1. \end{cases} \tag{1.5}$$

Here $\mathscr{J}$ is the $90^\circ$ counterclockwise rotation

2010 *Mathematics Subject Classification.* Primary 35Q74, 35J96, 70B20.
*Keywords.* Incompressible, area-preserving, deformations, Monge-Ampère equation.

$$\mathscr{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.6}$$

For $\mathbf{u} \in W^{1,2}(\Omega)$ the equations (1.3) or (1.5) cannot be justified. In fact the term $p(\nabla\mathbf{u})^{-t}$ is not well-defined unless $\nabla\mathbf{u}$ is better than $L^2$ integrable, see [2]. The lack of higher integrability of $\nabla\mathbf{u}$ produces a number of technical difficulties, see [6]. To circumvent them author and N. Chaudhuri succeeded to compute the first variation of the energy (1.6) in the image domain $\Omega^\star = \mathbf{u}(\Omega)$ under very weak assumptions (note that $\mathbf{u}$ is open map [10]). For $\mathbf{u} \in W^{s,l}(\Omega)$ with $s > \frac{2}{l} + 1$ this was done in [8], Theorem 5.1. Below we formulate one of the main results from [2] relevant to the present work.

**Proposition 1.2.** *Let $\boldsymbol{u} \in \mathcal{K}$ be a local minimizer of (1.1). Consider the matrix*

$$\sigma_{ij}(y) = \sum_m u^i_m(\boldsymbol{u}^{-1}(y))u^j_m(\boldsymbol{u}^{-1}(y)) \tag{1.7}$$

*where $y \in \boldsymbol{u}(\Omega) = \Omega^\star$ and $\boldsymbol{u}^{-1}$ is the inverse of $\boldsymbol{u}$ ($\boldsymbol{u}^{-1}$ is well-defined see Remark 3.3* [10]*). If $\rho_\varepsilon$ is a mollification kernel and $\sigma^\varepsilon = \sigma * \rho_\varepsilon$ then there is a $C^\infty$ function $q^\varepsilon$ such that*

$$\operatorname{div} \sigma^\varepsilon(y) + \nabla q^\varepsilon(y) = 0 \qquad y \in \Omega^\star, \tag{1.8}$$

The regularized equation (1.8) in the image domain plays the crucial role in the proof of Theorem A (see below), notably it links (1.3) to the Monge-Ampère equation and from there we infer that $\{q^\varepsilon\}$ is uniformly bounded in $L^2_{\rm loc}(\Omega^\star)$.

**Theorem A.** *Let $\boldsymbol{u} \in \mathcal{K}$ be a local minimizer of $E[\cdot]$. If there is a sequence of $q^{\varepsilon_j} \geq 0$ solving (1.8) such that $q^{\varepsilon_j}$ converges to a nonnegative Radon measure in $B_1 \subset \Omega^\star$, then there is a convex function $\psi^\varepsilon$ defined in $B_1$ such that*

$$D^2\psi^\varepsilon = adj\sigma^\varepsilon + q^\varepsilon\mathbb{I}$$

*where $adj\sigma^\varepsilon = (\sigma^\varepsilon)^{-1}\det\sigma^\varepsilon$ and $\mathbb{I}$ is the identity matrix. Moreover,*

- *there is a subsequence $q^{\varepsilon_{j(m)}}$ and $q \in L^2_{\rm loc}(\Omega^\star)$ such that $q^{\varepsilon_{j(m)}} \to q$ strongly in $L^2_{\rm loc}(\Omega^\star)$,*
- *there is a convex function $\psi : B_1 \mapsto \mathbb{R}$ such that $\psi^{\varepsilon_{j(m)}} \to \psi$ uniformly on the compact subsets of $B_1$.*

In [2] the authors found a representation for $q^\varepsilon$ given by a sum of Calderón-Zygmund type singular integrals of $\sigma^\varepsilon_{ij}(y)$. As a result $q^\varepsilon$ inherits the "half" of the integrability of $\nabla\mathbf{u}$. In other words $\{q^\varepsilon\}$ is uniformly bounded in $L^{1+\frac{\delta}{2}}_{\rm loc}(\Omega^\star)$ if $\nabla u \in L^{2+\delta}(\Omega), \delta > 0$ and in $L^1_{\rm loc}(\Omega^\star)$ if $|\nabla\mathbf{u}|^2 \in L\log(2+L)(\Omega)$. This observation gives rise to the following question: Does the higher integrability of the pressure $q$ translate to $\nabla\mathbf{u}$?

Theorem A gives a partial answer to this question: if $B_1 \subset \Omega^\star$, $q \in L^{2+\delta}(B_1), \delta > 0$ and $\sigma \in L^2(B_1)$ then it follows from Lemma 7.1 $\mathbf{1}^\circ$ that $D^2\psi = \mathrm{adj}\sigma + q\mathbb{I}$ and $D^2\psi \in L^2(B_{\frac{7}{8}})$. Since by (1.7) $\sigma(y) = [\nabla\mathbf{u}(\nabla\mathbf{u})^t] \circ \mathbf{u}^{-1}(y), y \in \Omega^\star$ we infer that $\det\mathrm{adj}\sigma = 1$, which is equivalent to the Monge-Ampère equation

$$\det\left[D^2\psi - q\mathbb{I}\right] = 1$$

satisfied a.e. in $B_1$. Hence from the regularity theory available for the Monge-Ampére equation we will conclude higher integrability for $D^2\psi$ in $B_{\frac{1}{2}}$, which translates to $\nabla\mathbf{u}$ in $\Omega$ through the equation $D^2\psi = \mathrm{adj}\sigma + q\mathbb{I}$ and the inverse mapping theorem.

As one can observe from (1.8), the pressure $q^\varepsilon$ is defined modulo a constant. The assumption $q^{\varepsilon_j} \geq 0$ seems a natural one since from a purely physical point of view the pressure must be nonnegative. From Theorem A we can conclude that the first equation in (1.3) is well defined in $\Omega$. Moreover applying the duality argument from [6] we infer that there is a function $P : \Omega^\star \mapsto \mathbb{R}$ such that the pair $(\mathbf{u}^{-1}, P)$ is a solution the corresponding Euler-Lagrange equations in $\Omega^\star$, see Theorem 2 [6]. Combining Theorem A with this observation we obtain

**Theorem B.** *Let* $\boldsymbol{u} : \Omega \mapsto \mathbb{R}^n$ *and* $q \in L^2(\Omega^\star)$ *be as in Theorem A.*

**1°** *Then* $p(x) = q(\boldsymbol{u}(x)), x \in \Omega$ *is locally* $L^2$ *integrable in* $\Omega, p(x)(\nabla \boldsymbol{u})^{-t} \in L^2_{loc}(\Omega)$ *and the pair* $(\boldsymbol{u}, p)$ *solves the equation*

$$\operatorname{div}[\nabla \boldsymbol{u} + p(\nabla \boldsymbol{u})^{-t}] = 0 \qquad in \ \ \Omega$$

*in the weak sense.*

**2°** *Let* $\boldsymbol{v} = \boldsymbol{u}^{-1}$ *and* $Q$ *be the dual pressure in* $\Omega$ *corresponding to* $\boldsymbol{v}, Q(\boldsymbol{v}(z)) = P(z)$. *If* $Q \geq 0$ *then* $\boldsymbol{u} \in C^{\frac{1}{2}}_{loc}(\Omega)$.

The paper is organized as follows: Section 2 is devoted to the construction of the family of functions $\psi^\varepsilon$. Then we prove uniform estimates for this family using some geometric ideas and the Poincaré-Wirtinger's theorem for the functions of bounded variation (or $BV-$functions, see [4]). This is contained in Section 3. A lower estimate for the $\det \operatorname{adj} \sigma^\varepsilon$ is established in Section 4. Next, in order to prove Theorem A, we recall the notion of generalized solution of the Monge-Ampère equation and define the corresponding normal mapping in Section 5. The proof of Theorem A is given in Section 6. Section 7 contains a brief discussion of the properties of the convex function $\psi$ and its Legendre-Fenchel transformation. Finally, Section 8 contains the proof of Theorem B.

## 2. The Euler-Lagrange equation in image domain

In this section we construct a convex function $\psi^\varepsilon$ such that the mollification of the Cauchy stress tensor $\mathbf{C}_{ij} = \sigma_{ij} + q\delta_{ij}$ is the Hessian of $\psi^\varepsilon$.

We start by recalling that if $\mathbf{w}$ is $C^\infty$ divergence free vectorfield in 2D then there is a scalar $C^\infty$ function $\varphi$ such that $\mathbf{w} = \mathscr{J} D\varphi = (-D_2\varphi, D_1\varphi)$.

Suppose that $B_1 \subset \Omega^\star$. From the mollified equation (1.8) it follows that the vectorfields $(\sigma^\varepsilon_{11} + q^\varepsilon, \sigma^\varepsilon_{12})$ and $(\sigma^\varepsilon_{21}, \sigma^\varepsilon_{22} + q^\varepsilon)$ are divergence free in $\Omega^\star$. Hence there are two scalar functions $\varphi^\varepsilon_1, \varphi^\varepsilon_2$ such that $\varphi^\varepsilon_i \in C^\infty(B_1), i = 1, 2$ and

$$\begin{aligned} (\sigma^\varepsilon_{11} + q^\varepsilon, \sigma^\varepsilon_{12}) &= \mathscr{J} D\varphi^\varepsilon_1 = (-\partial_2\varphi^\varepsilon_1, \partial_1\varphi^\varepsilon_1), \\ (\sigma^\varepsilon_{21}, \sigma^\varepsilon_{22} + q^\varepsilon) &= \mathscr{J} D\varphi^\varepsilon_2 = (-\partial_2\varphi^\varepsilon_2, \partial_1\varphi^\varepsilon_2). \end{aligned} \tag{2.1}$$

Since

$$[\sigma_{ij}(z)] = \begin{pmatrix} |\nabla u^1(\mathbf{u}^{-1}(z))|^2 & \nabla u^1(\mathbf{u}^{-1}(z)) \cdot \nabla u^2(\mathbf{u}^{-1}(z)) \\ \nabla u^1(\mathbf{u}^{-1}(z)) \cdot \nabla u^2(\mathbf{u}^{-1}(z)) & |\nabla u^2(\mathbf{u}^{-1}(z))|^2 \end{pmatrix} \tag{2.2}$$

and $\sigma^\varepsilon_{ij} = \sigma_{ij} * \rho_\varepsilon$, where $\rho_\varepsilon$ is a mollifying kernel, we conclude that $\sigma^\varepsilon_{ij}$ is symmetric. Moreover the gradient matrix of the mapping $\Phi^\varepsilon = (\varphi^\varepsilon_1, \varphi^\varepsilon_2)$ is

$$\nabla \Phi^\varepsilon = \begin{pmatrix} \partial_1\varphi^\varepsilon_1 & \partial_2\varphi^\varepsilon_1 \\ \partial_1\varphi^\varepsilon_2 & \partial_2\varphi^\varepsilon_2 \end{pmatrix} = \begin{pmatrix} \sigma^\varepsilon_{12} & -\sigma^\varepsilon_{11} - q^\varepsilon \\ \sigma^\varepsilon_{22} + q^\varepsilon & -\sigma^\varepsilon_{21} \end{pmatrix}. \tag{2.3}$$

Therefore the mapping $\Phi = (\varphi^\varepsilon_1, \varphi^\varepsilon_2)$ is divergence free, because

$$\operatorname{div} \Phi^\varepsilon = \partial_1\varphi^\varepsilon_1 + \partial_2\varphi^\varepsilon_2 = \sigma^\varepsilon_{12} - \sigma^\varepsilon_{21} = 0$$

and the matrix $\sigma^\varepsilon_{ij}$ is symmetric.

Thus, there is a scalar function $\psi^\varepsilon$ such that $\Phi^\varepsilon = \mathscr{J}\nabla\psi^\varepsilon$. In other words $\varphi_1^\varepsilon = -\partial_2\psi^\varepsilon, \varphi_2^\varepsilon = \partial_1\psi^\varepsilon$, which in view of (2.1) implies the following identity for the Hessian of $\psi^\varepsilon$

$$D^2\psi^\varepsilon(y) = \begin{pmatrix} \sigma_{22}^\varepsilon(y) + q^\varepsilon(y) & -\sigma_{21}^\varepsilon(y) \\ -\sigma_{21}^\varepsilon(y) & \sigma_{11}^\varepsilon(y) + q^\varepsilon(y) \end{pmatrix}. \tag{2.4}$$

Furthermore, $\det D^2\psi^\varepsilon = \det \mathrm{adj}\sigma^\varepsilon + (q^\varepsilon)^2 + q^\varepsilon \mathrm{Tr}\sigma^\varepsilon$ and $\det(D^2\psi - q^\varepsilon\mathbb{I}) = \det\mathrm{adj}\sigma^\varepsilon$, where $\mathbb{I} = \delta_{ij}$ is the identity matrix.

**Lemma 2.1.** *If $q^\varepsilon \geq C$ for some $C \in \mathbb{R}$, independent of $\varepsilon$, then $\psi^\varepsilon(y) - \frac{C}{2}|y|^2$ are convex for any $\varepsilon > 0$.*

**Proof:** Let $e = (a, b) \in \mathbb{R}^2$ and $\partial_e = a\partial_1 + b\partial_2$. Then using (2.2) and (2.4) we conclude

$$\begin{aligned} \partial_{ee}\psi^\varepsilon(z) &= a^2\partial_{11}\psi^\varepsilon + 2ab\partial_{12}\psi^\varepsilon + b^2\partial_{22}\psi^\varepsilon \\ &= a^2\sigma_{22}^\varepsilon + 2ab\sigma_{12}^\varepsilon + b^2\sigma_{11}^\varepsilon + q^\varepsilon(z)(a^2+b^2) \\ &= \left|a\nabla_x u^2(\mathbf{u}^{-1}(z) + b\nabla_x u^1(\mathbf{u}^{-1}(z))\right|^2 + q^\varepsilon(z)(a^2+b^2) \\ &\geq C(a^2+b^2). \end{aligned}$$

Therefore $\psi(z) - \frac{C}{2}|z|^2$ is convex. □

**Remark 2.2.** *The pressure $q^\varepsilon(z)$ is defined modulo a constant as it is seen from the equation (1.8). In particular, $\psi^\varepsilon$ is determined modulo a quadratic polynomial. Thus if $q_0^\varepsilon(z) = q^\varepsilon(z) - C$ then $\psi_0^\varepsilon(z) = \psi^\varepsilon(z) - \frac{C}{2}|z|^2$ solves $\det(D^2\psi_0^\varepsilon - q_0^\varepsilon(z)\mathbb{I}) = \det\mathrm{adj}\sigma^\varepsilon$ and (2.4) holds with $\psi^\varepsilon$ and $q^\varepsilon$ replaced by $\psi_0^\varepsilon$ and $q_0^\varepsilon$ respectively.*

## 3. Uniform estimates for $\psi^\varepsilon$

**Lemma 3.1.** *Suppose that the sequence $q^\varepsilon$ converges to a nonnegative Radon measure $q$. Then there is a positive constant $C$ such that $\sup\limits_{\partial B_1}|\psi^\varepsilon| \leq C$.*

**Proof:** By Helmholtz-Weyl decomposition [3], $\Phi^\varepsilon = Dh^\varepsilon + \mathscr{J}D\eta^\varepsilon$ where $h^\varepsilon$ solves the Neumann problem

$$\begin{cases} \Delta h^\varepsilon = 0 & \text{in } B_1, \\ Dh^\varepsilon\cdot\nu = \Phi^\varepsilon\cdot\nu & \text{on } \partial B_1. \end{cases} \tag{3.1}$$

Moreover $-\Delta\eta^\varepsilon = \mathrm{curl}\Phi^\varepsilon = \sigma_{11}^\varepsilon + \sigma_{22}^\varepsilon + 2q^\varepsilon$ and $\eta^\varepsilon = 0$ on $\partial B_1$.

By Poincaré-Wirtinger's theorem $\widetilde{\Phi}^\varepsilon = \Phi^\varepsilon - \fint_{B_1}\Phi^\varepsilon \in BV(B_1, \mathbb{R}^2)$, i.e. $\varphi_i^\varepsilon - \fint_{B_1}\varphi_i^\varepsilon \in BV(B_1), i = 1, 2$. Since $\Phi^\varepsilon$ is defined modulo a constant (see (2.3)), in what follows, we take $\widetilde{\Phi}^\varepsilon = \Phi^\varepsilon - \fint_{B_1}\Phi^\varepsilon$. Thus the estimate

$$\|\widetilde{\Phi}^\varepsilon\|_{L^1(B_1)} = \left\|\Phi^\varepsilon - \fint_{B_1}\Phi^\varepsilon\right\|_{L^1(B_1)} \leq C\sup\left\{\left|\int_{B_1}\Phi^\varepsilon \operatorname{div}\xi\right|, \forall\xi \in C_0^1(B_1, \mathbb{R}^2), |\xi| \leq 1\right\} \tag{3.2}$$

is true, with $C > 0$ independent from $\varepsilon$.

On the other hand after integration by parts we get

$$\int_{B_1}\widetilde{\Phi}^\varepsilon \operatorname{div}\xi = \int_{B_1}\Phi^\varepsilon\operatorname{div}\xi = -\int_{B_1}\xi\nabla\Phi^\varepsilon \tag{3.3}$$

for any $\xi \in C_0^1(B_1, \mathbb{R}^2)$ which in conjunction with (2.3) gives

$$\left|\int_{B_1}\varphi_1^\varepsilon \operatorname{div}\xi\right| = \left|-\int_{B_1}\xi D\varphi_1^\varepsilon\right| = \left|\int_{B_1}\xi^1\sigma_{12}^\varepsilon-\xi^2(\sigma_{11}^\varepsilon+q^\varepsilon)\right| \leq \int_{B_1}[|\sigma_{11}^\varepsilon|+|\sigma_{12}^\varepsilon|+q^\varepsilon]. \tag{3.4}$$

Similarly, one can check that $\left|\int_{B_1}\varphi_2^\varepsilon \operatorname{div}\xi\right| \leq \int_{B_1}[|\sigma_{12}^\varepsilon|+|\sigma_{22}^\varepsilon|+q^\varepsilon]$. Because $\sigma_{ij}\in L^1$ and $q^\varepsilon$ converges to a nonnegative Radon measure it follows that

$$\|\widetilde{\Phi}^\varepsilon\|_{BV(B_1)} \leq C\left(\|\sigma_{ij}\|_{L^1(B_1)}+\|q\|_{\mathscr{M}(B_1)}\right),$$

where $\mathscr{M}(B_1)$ is the space of measures in $B_1$.

Using Theorems 2.10 and 2.11 from [4] we conclude that the trace $\Phi_0^\varepsilon\in L^1(\partial B_1)$ of $\widetilde{\Phi}^\varepsilon$ is well-defined and satisfies the following uniform estimate

$$\|\widetilde{\Phi}_0^\varepsilon\|_{L^1(\partial B_1)} \leq C\|\widetilde{\Phi}^\varepsilon\|_{BV(B_1)} \leq C\left(\|\sigma_{ij}\|_{L^1(B_1)}+\|q\|_{\mathscr{M}(B_1)}\right). \tag{3.5}$$

In particular (3.5) implies that the Neumann problem (3.1) for $h^\varepsilon$ is well-defined.

Next we have that $\Phi^\varepsilon=\mathscr{J}\nabla\psi^\varepsilon=\nabla h^\varepsilon+\mathscr{J}\nabla\eta^\varepsilon$ or equivalently

$$\nabla\psi^\varepsilon-\nabla\eta^\varepsilon=-\mathscr{J}\nabla h^\varepsilon.$$

In particular $\psi^\varepsilon-\eta^\varepsilon$ is harmonic in $B_1$. We want to estimate the tangential component of $\nabla\psi^\varepsilon$ on the boundary $\partial B_1$. Let $\tau$ be a unit tangent vector to $\partial B_1$, then

$$\nabla\psi^\varepsilon\cdot\tau=\nabla\eta^\varepsilon\cdot\tau-\mathscr{J}\nabla h^\varepsilon\cdot\tau=\nabla h^\varepsilon\cdot\nu,$$

where $\nu=\mathscr{J}\tau$ is a unit vector normal to $\partial B_1$. Using polar coordinates $(r,\theta),\theta\in(0,2\pi)$, we obtain that

$$\psi^\varepsilon(\theta)=\psi^\varepsilon(0)+\int_0^\theta\nabla h\cdot\nu d\theta=\psi^\varepsilon(0)+\int_0^\theta\Phi_0^\varepsilon\cdot\nu d\theta. \tag{3.6}$$

Without loss of generality we assume that $\psi^\varepsilon(0)=0$ (see Remark 2.2). Thus

$$|\psi^\varepsilon(\theta)|\leq C\|\Phi_0^\varepsilon\|_{L^1(\partial B_1)},\qquad \forall\theta\in(0,2\pi).$$

The desired result now follows from (3.5). □

**Lemma 3.2.** *Retain the assumptions of previous lemma. Then there is a constant $C$, such that $\inf_{B_1}\psi^\varepsilon\geq C$ uniformly in $\varepsilon$.*

**Proof:** It suffices to prove that $\nabla\psi^\varepsilon\in L^1(\partial B_1)$ uniformly in $\varepsilon$. Indeed, $\psi^\varepsilon$ is convex hence if $\psi^\varepsilon$ tends to $-\infty$ then the $\nabla\psi^\varepsilon$ becomes uniformly large on $\partial B_1$.

From Lemma 3.5 we have that

$$\nabla\psi^\varepsilon=\nabla\eta^\varepsilon-\mathscr{J}\nabla h^\varepsilon=\mathscr{J}(-\mathscr{J}\nabla\eta^\varepsilon-\nabla h^\varepsilon)=-\mathscr{J}\widetilde{\Phi}^\varepsilon$$

implying the estimate

$$\|\nabla\psi^\varepsilon\|_{L^1(\partial B_1)}\leq\|\widetilde{\Phi}_0^\varepsilon\|_{L^1(\partial B_1)}.$$

The proof now follows if we recall (3.5). □

## 4. Lower estimate for $\det(\mathrm{adj}\ \sigma^\varepsilon)$

**Lemma 4.1.** *Let $\sigma^\varepsilon = \sigma * \rho_\varepsilon$, where $\sigma(z) = [\nabla \boldsymbol{u}(\nabla \boldsymbol{u})^t] \circ \boldsymbol{u}^{-1}(z), z \in \Omega^\star$ then for any $\varepsilon > 0$*

$$\det(\mathrm{adj}\ \sigma^\varepsilon(z)) \geq 1 \qquad z \in \Omega^\star.$$

**Proof:** Using the definition of $\sigma^\varepsilon(z)$ and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \det(\mathrm{adj}\ \sigma^\varepsilon) &= \sigma^\varepsilon_{11}\sigma^\varepsilon_{22} - \sigma^\varepsilon_{12}\sigma^\varepsilon_{21} \\ &= \int_{B_1} \sigma_{11}\rho_\varepsilon \int_{B_1} \sigma_{22}\rho_\varepsilon - \left(\int_{B_1} \sigma_{12}\rho_\varepsilon\right)^2 \\ &\geq \left(\int_{B_1} \sqrt{\sigma_{11}\sigma_{22}}\rho_\varepsilon\right)^2 - \left(\int_{B_1} \sigma_{12}\rho_\varepsilon\right)^2 \\ &= \int_{B_1} (\sqrt{\sigma_{11}\sigma_{22}} - \sigma_{12})\rho_\varepsilon \int_{B_1} (\sqrt{\sigma_{11}\sigma_{22}} + \sigma_{12})\rho_\varepsilon. \end{aligned}$$

By definition we have $\sigma_{11} = |\nabla u^1|^2, \sigma_{22} = |\nabla u^2|^2$ and $\sigma_{12} = \sigma_{21} = \nabla u^1 \cdot \nabla u^2$. Let $\alpha$ be the angle between $\nabla u^1$ and $\nabla u^2$. Recall that $\det \nabla \mathbf{u} = |\nabla u^1||\nabla u^2| \sin\alpha = 1$. Then

$$\sqrt{\sigma_{11}\sigma_{22}} - \sigma_{12} = |\nabla u^1||\nabla u^2|(1 - \cos\alpha) = |\nabla u^1||\nabla u^2| 2\sin^2\frac{\alpha}{2} = \tan\frac{\alpha}{2}$$

and similarly have that

$$\sqrt{\sigma_{11}\sigma_{22}} + \sigma_{12} = |\nabla u^1||\nabla u^2|(1 + \cos\alpha) = |\nabla u^1||\nabla u^2| 2\cos^2\frac{\alpha}{2} = \cot\frac{\alpha}{2}.$$

Applying the Cauchy-Schwarz inequality one more time we obtain

$$\det(\mathrm{adj}\ \sigma^\varepsilon) \geq 1.$$

□

## 5. Normal mapping of the convex function $\psi^\varepsilon$

In this section we will employ some basic concepts from the theory of generalized solutions of Monge-Ampère equation. Our notation follow that of the paper [11]. Let $\psi$ be a convex function defined in $B_1 \subset \mathbb{R}^2$. For $x \in B_1$ we let

$$\chi_\psi(x) = \{\xi \in \mathbb{R}^2 : \psi(y) \geq \psi(x) + \xi \cdot (y - x) \quad \forall y \in B_1\}.$$

For a set $E \subset B_1$ we define the mapping

$$\chi_\psi(E) = \bigcup_{x \in E} \chi_\psi(x). \tag{5.1}$$

$\chi_\psi$ is called the normal mapping of $\psi$. For smooth convex $\psi$, $\chi_\psi$ coincides with the gradient mapping of $\psi$.

Let

$$\mathscr{C} = \{E \subset B_1 : \chi_\psi(E) \text{ is Lebesgue measurable}\}.$$

Then $\mathscr{C}$ is a $\sigma-$algebra containing the Borel subsets of $B_1$, see [11]. For each $E \in \mathscr{C}$ we define the set function

$$\omega(E) = |\chi_\psi(E)|$$

i.e. the Lebesgue measure of the normal mapping of $E$. It is easy to verify that for $\psi \in C^2(B_1)$ we have

$$\omega(E) = \int_E \det D^2\psi, \qquad \text{for all Borel } E \in B_1.$$

It follows from Aleksandrov's theorem, see [11], that

$$|\{\xi\in\mathbb{R}^2:\xi\in\chi_\psi(x)\cap\chi_\psi(y),\text{for }x\neq y,x,y\in B_1|=0.$$

As a consequence, we get that $\omega$ is countably additive Radon measure.

Moreover, we have weak convergence for measure $\omega$. Indeed, let $\psi_j$ be a sequence of convex functions and $\psi_j\to\psi$ uniformly on compact subsets of $B_1$. Let $\omega_j$ and $\omega$ be the Radon measures associated with $\psi_j$ and $\psi$ respectively. Then $\omega_j$ converges weakly on $B_1$ to $\omega$ in the space of measures $\mathscr{M}(B_1)$ [11], i.e.

$$\limsup_{j\to\infty}\omega_j(K)\leq\omega(K) \tag{5.2}$$

for any compact set $K\subset B_1$, and

$$\liminf_{j\to\infty}\omega_j(U)\geq\omega(U) \tag{5.3}$$

for any open set $U\subset B_1$.

## 6. Proof of Theorem A

Let $\omega_j$ be the Radon measure corresponding to $\psi^{\varepsilon_j}$, for some sequence $\{\varepsilon_j\}$. By Lemmas 3.1 and 3.2 the sequence of convex functions $\{\psi^{\varepsilon_j}\}$ is uniformly bounded in $B_1$. Thus for a subsequence, again denoted by $\{\psi^{\varepsilon_j}\}$ we have $\psi^{\varepsilon_j}\to\psi$ uniformly on the compact subsets of $B_1$. Clearly $\psi$ is convex. Let $\omega$ be the Radon measure corresponding to $\psi$. By Lemma 4.1 we have that

$$\begin{aligned}\omega_j(B_r(x_0)) &= \int_{B_r(x_0)}\det D^2\psi^{\varepsilon_j}\\ &= \int_{B_r(x_0)}\det(\text{adj }\sigma^{\varepsilon_j}(z))+q^{\varepsilon_j}(z)\left[|\nabla\mathbf{u}(\mathbf{u}^{-1}(z))|^2*\rho_{\varepsilon_j}\right]+(q^{\varepsilon_j}(z))^2dz\\ &\geq |B_r(x_0)|+\int_{B_r(x_0)}(q^{\varepsilon_j}(z))^2dz\end{aligned} \tag{6.1}$$

for any open ball $B_r(x_0)\subset B_1$.

Now utilizing the weak convergence of the measures $\omega_j\rightharpoonup\omega$ and (5.2) we obtain the following uniform

$$\int_K(q^{\varepsilon_j}(z))^2dz\leq C+\omega(K)$$

for any compact set $K\subset B_1$. Then a customary compactness argument in $L^2$ finishes the proof. □

## 7. Properties of $\psi$

The convex function $\psi$ enjoys a number of remarkable properties which are summarized in the following

**Lemma 7.1.** *Let $\psi$ be as in Theorem A. Then*

- **1°** *$\psi$ is strictly convex and $\psi\in W^{2,1}_{\text{loc}}(B_1)$,*
- **2°** *$\psi^*\in C^{1,1}$ where $\psi^*$ is the Legendre-Fenchel transformation of $\psi$ in $B_{\frac12}$.*

**Proof:** **1°** Recall that $q^\varepsilon$ is defined modulo a constant summand, see Remark 2.2. Thus, without loss of generality, we assume that $q^\varepsilon\geq1$. Let $y_0$ be an arbitrary point in $B_1$, then by Lemma 4.1 $\det D^2\psi^\varepsilon\geq(q^\varepsilon)^2\geq1$. Thus we conclude that

$$\omega_j(U)\geq|U|,\qquad\forall\text{ open }U\subset B_1.$$

Since $\omega_j \rightharpoonup \omega$ weakly and in view of (5.3) the above inequality implies

$$\omega(U) \geq |U|.$$

Now the strict convexity of $\psi$ follows from Aleksandrov's theorem, see [9], Chapter 2.3 Theorem 2.

The mollified matrices $\sigma_{km}^{\varepsilon_j} \to \sigma_{km}$ strongly in $L^1_{\rm loc}(B_1)$ as $\varepsilon_j \downarrow 0$ and $q^{\varepsilon_j} \to q$ in $L^2_{\rm loc}$ at least for a subsequence. Moreover $\{\psi^{\varepsilon_j}\}$ is uniformly bounded thanks to Lemmas 3.1 and 3.2, hence for a suitable subsequence $\psi^{\varepsilon_j}$ will uniformly converge to a convex function $\psi$ in any compact subset of $B_1$. Let us show that $D^2\psi = {\rm adj}\sigma + q\mathbb{I}$ a.e in $B_1$.

Indeed, let $\eta \in C_0^\infty(B_1)$ and compute

$$\begin{aligned}
\int \partial_k\psi\partial_i\eta &= \int \partial_k\psi^{\varepsilon_j}\partial_i\eta + o(1)\\
&= -\int \partial_{ik}\psi^{\varepsilon_j}\eta + o(1)\\
&= -\int [({\rm adj}\sigma^{\varepsilon_j})_{ik} + q^{\varepsilon_j}\delta_{ik}]\eta + o(1)\\
&\longrightarrow -\int [({\rm adj}\sigma)_{ik} + q\delta_{ik}]\eta.
\end{aligned}$$

Hence $\psi$ has generalized second order derivatives in $L^1_{\rm loc}(B_1)$ and $D^2\psi = {\rm adj}\sigma + q\mathbb{I}$ a.e in $B_1$.

**2°** Recall that the Legendre-Fenchel transformation $\psi^*$ of $\psi$ in $B_{\frac12}$ is given by

$$\psi^*(z) = \sup_{y\in B_{\frac12}} (z\cdot y - \psi(y)), \qquad z \in \chi_\psi(B_{\frac12}).$$

Notice that by part **1°** $\psi$ is strictly convex, hence it can be shown that $\psi^*$ is $C^1$ in the domain of $\psi^*$, see Chapter D of [5].

Let us denote $B = B_{\frac12}$ and $B^* = \chi_\psi(B)$ where $\chi_\psi$ is the normal mapping of $\psi$. Notice that $B^*$ is bounded because $\psi \in C^{0,1}(\overline{B_{\frac12}})$. Denote $(B^\varepsilon)^* = \chi_{\psi^\varepsilon}(B)$, then $(\psi^\varepsilon)^*(z), z \in (B^\varepsilon)^*$ is smooth because $\psi^\varepsilon$ is $C^\infty$. Furthermore from (2.4) we obtain

$$D^2(\psi^\varepsilon)^* = [D^2\psi^\varepsilon]^{-1} = \frac{1}{\det D^2\psi^\varepsilon}(\sigma^\varepsilon + q\mathbb{I})$$

or equivalently

$$\begin{aligned}
\partial_{ij}(\psi^\varepsilon)^* &= \frac{\sigma^\varepsilon_{ij} + q\delta_{ij}}{\det {\rm adj}\sigma + q^\varepsilon{\rm Tr}\sigma^\varepsilon + (q^\varepsilon)^2}\\
&\leq \frac{1}{q^\varepsilon}\frac{\sigma^\varepsilon_{ij} + q\delta_{ij}}{\frac{1}{q^\varepsilon} + {\rm Tr}\sigma^\varepsilon + q^\varepsilon}\\
&\leq \frac{1}{q^\varepsilon} \leq 1, \qquad i = j
\end{aligned}$$

if we assume that $q^\varepsilon \geq 1$, see Remark 2.2.

As for $i \neq j$, we use Lemma 4.1 to conclude

$$|\sigma^\varepsilon_{12}| \leq \sqrt{\sigma^\varepsilon_{11}\sigma^\varepsilon_{22} - 1} \leq \sqrt{\sigma^\varepsilon_{11}\sigma^\varepsilon_{22}} + 1 \leq \frac{\sigma^\varepsilon_{11} + \sigma^\varepsilon_{22}}{2} + 1.$$

Thus $|D^2(\psi^\varepsilon)^*| \leq C$ uniformly in $\varepsilon$.

Next, we extend $(\psi^\varepsilon)^*$ to $B_R$ by the formula $\sup_{z\in B_R}(y\cdot z - \psi^\varepsilon(y))$ with $z \in B_R$ and $R = \sup_\varepsilon \|\nabla\psi^\varepsilon\|_{L^\infty(B_{\frac12})}$. Thus in $B_R$ we have a sequence of convex functions $(\psi^\varepsilon)^*$ with uniformly bounded Hessian matrices. By a customary compactness argument we can show that for at least a subsequence we have $(\psi^{\varepsilon_j})^* \to \overline{\psi}$ for some convex function $\overline{\psi}$. It remains to show that $\psi^* = \overline{\psi}$ in $B^*$.

From the definition of $(\psi^\varepsilon)^*$ we have that $(\psi^\varepsilon)^*(z) + \psi^\varepsilon(y) \geq y \cdot z$ and after passing to the limit we obtain $\overline{\psi}(z) + \psi(y) \geq y \cdot z$ implying that $\overline{\psi}(z) \geq \psi^*(z)$. To get the converse inequality we use the uniform convergence

$$\overline{\psi}(z) \longleftarrow (\psi^\varepsilon)^*(z) = \sup_{y\in B}(y \cdot z - \psi^\varepsilon(y)) \leq \sup_{y\in B}(y \cdot z - \psi(y)) + \sup_{y\in B} |\psi(y) - \psi^\varepsilon(y)| \longrightarrow \psi^*(z).$$

This completes the proof. □

**Remark 7.2.** *At each point $z \in intB^*, B^* = \chi_\psi(B_{\frac{1}{2}})$ we can define the lower Gauss curvature* [9]

$$\underline{\omega}^*(z_0) = \liminf_{r\to 0} \frac{|\chi_{\psi^*}(B_r(z_0))|}{|B_r(z_0)|}.$$

*If there is a constant $m > 0$ such that $\underline{\omega}^*(z_0) \geq m > 0$ for a.e. $z_0 \in B^*$ then $\sigma$ and $q$ are bounded in $B_{\frac{1}{2}}$. In particular this will imply that $\mathbf{u}$ is Lipschitz in $\mathbf{u}^{-1}(B_{\frac{1}{2}}) \subset \Omega$.*

## 8. Proof of Theorem B

The part $\mathbf{1}^\circ$ follows from change of variable formula [10] and Theorem A. To prove part $\mathbf{2}^\circ$ we employ the duality principle of $\mathbf{u}$ and its inverse $\mathbf{v} = \mathbf{u}^{-1}$ in [6], i.e. $\mathbf{v}$ is a local minimizer of the dual problem in the image domain $\Omega^\star = \mathbf{u}(\Omega)$. Hence we can apply Theorem A to the pair $(\mathbf{v}, P)$ where $\mathbf{v} = \mathbf{u}^{-1}$. Thus, there is a convex function $\eta^\varepsilon$ such that $D^2\eta^\varepsilon = \mathrm{adj}\widetilde{\sigma}^\varepsilon + Q^\varepsilon\mathbb{I}$ where

$$\widetilde{\sigma}_{ij}(z) = \sum_m v^i_m(\mathbf{v}^{-1}(z))v^j_m(\mathbf{v}^{-1}(z)), \qquad z \in \Omega$$

and $\widetilde{\sigma}^\varepsilon = \widetilde{\sigma} * \rho_\varepsilon$ and $Q^\varepsilon$ are the mollifications of $\widetilde{\sigma}$ and $Q$ respectively. Note that $Q(\mathbf{v}(z)) = P(z), z \in \Omega$. In particular, for any $B_r(x_0) \subset B_1 \subset \Omega$ we have

$$\begin{aligned}\int_{B_r(x_0)} |\nabla \mathbf{u}(x)|^2 dx &= \int_{B_r(x_0)} \mathrm{Tr}\widetilde{\sigma}_{ij}(x)dx \\ &= \int_{B_r(x_0)} \Delta\eta^\varepsilon - 2Q^\varepsilon \\ &\leq \int_{B_r(x_0)} \Delta\eta^\varepsilon \\ &= \int_{\partial B_r(x_0)} \nabla\eta^\varepsilon \cdot \nu \\ &\leq Cr\end{aligned}$$

with some tame constant $C$ depending on the Lipschitz norms of $\eta^\varepsilon$, which is bounded by Lemmas 3.2 and 3.1. Now the result follows from Morrey's estimate. □

## References

[1] Ball J., Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Analysis. 63, 337–403 (1977)

[2] Chaudhuri N., Karakhanyan A.L., On derivation of Euler-Lagrange Equations for incompressible energy-minimizers. Cal. of Variations and PDE, 36, no. 4 (2009), 627–645.

[3] Fujiwara D., Morimoto H., An $L_r$-theorem of the Helmholtz decomposition of vector fields, Tokyo Univ. Fac. Sciences J. 24 (1977), 658–700

[4] Giusti E., Minimal Surfaces and Functions of Bounded Variation, Birkhäuser, 1984

[5] Hiriart-Urruty J.B., Fundamentals of Convex Analysis, 2001.

[6] Karakhanyan A.L., Sufficient conditions for regularity of area-preserving deformations, Manutscripta Math. 138 (2012), 463–476

[7] le Dret H., Constitutive laws and existence questions in incompressible nonlinear elasticity, Journal of Elasticity, Volume 15, Issue 4, (1985), 369–387

[8] LeTallec P., Oden J.T., Existence and characterization of hydrostatic pressure in finite deformations of incompressible elastic bodies, Journal of Elasticity, Volume 11, Issue 4, (1981), 341–357

[9] Pogorelov A.V., Extrinsic geometry of convex surfaces, Translations of Mathematical Monographs. volume 35, AMS, Providence, Rhode Island, 1973

[10] Šverák V., Regularity properties of deformations with finite energy. Archive for Rational Mechanics and Analysis. (1988), 105–127

[11] Urbas J., Regularity of generalized solutions of Monge-Ampre equations. Math. Z. 197 (1988), no. 3, 365393.

School of Mathematics, Maxwell Institute
*E-mail address*: aram.karakhanyan@ed.ac.uk