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Modulation theory, dispersive shock waves and Gerald Beresford Whitham

A.A. Minzoni[a], Noel F. Smyth[b,*]

[a] *Department of Mathematics and Mechanics, I.I.M.A.S., National Autonomous University of Mexico, 01000 Mexico, D.F., Mexico*

[b] *School of Mathematics, University of Edinburgh, Edinburgh EH9 3FD, Scotland, U.K.*

Abstract

Gerald Beresford (GB) Whitham, FRS, (13th December, 1927 to 26th January, 2014) was one of the leading applied mathematicians of the twentieth century whose work over forty years had a profound, formative impact on research on wave motion across a broad range of areas. Many of the ideas and techniques he developed have now become the standard tools used to analyse and understand wave motion, as the papers of this special issue of *Physica D* testify. Many of the techniques pioneered by GB Whitham have spread beyond wave propagation into other applied mathematics areas, such as reaction-diffusion, and even into theoretical physics and pure mathematics, in which Whitham modulation theory is an active area of research. GB Whitham's classic textbook *Linear and Nonlinear Waves*, published in 1974, is still the standard reference for the applied mathematics of wave motion. In honour of his scientific achievements, GB Whitham was elected a Fellow of the American Academy of Arts and Sciences in 1959 and a Fellow of the Royal Society in 1965. He was awarded the Norbert Wiener Prize for Applied Mathematics in 1980.

Keywords: Modulation theory; Dispersive shock wave; Undular bore; Averaged Lagrangian; Biography

1. Academic Career

Gerald Beresford (GB) Whitham was born on the 13th December, 1927 in Halifax, North Yorkshire, in the United Kingdom and remained a proud Yorkshireman all his life. He was educated at Elland Grammar School, Elland, Yorkshire. GB then attended the University of Manchester in the neighbouring county of Lancashire, England, obtaining a B.Sc. honours degree in Mathematics in 1948, an M.Sc. degree in 1949 and a Ph.D. degree in 1953. His thesis advisor was Professor M.J. Lighthill, FRS, another great applied mathematician of the twentieth century. The period after the Second World

*Corresponding author
Email address: N.Smyth@ed.ac.uk (Noel F. Smyth)

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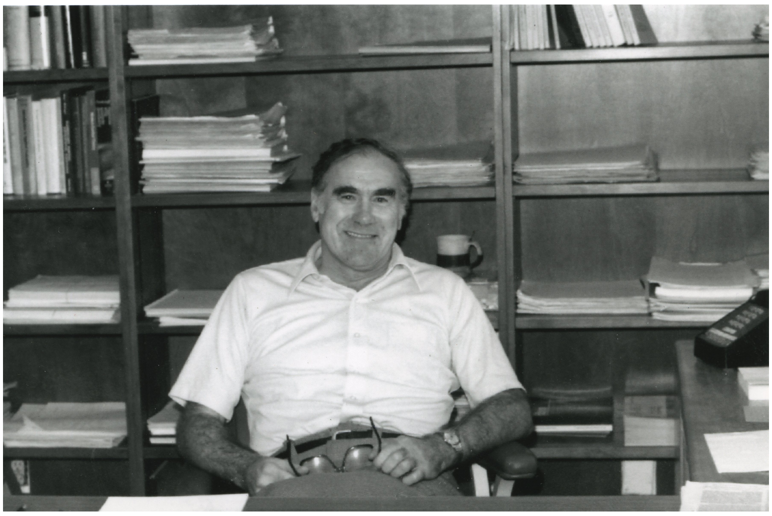

Figure 1: G.B. Whitham in his office at the California Institute of Technology. Photograph courtesy of Sheila Shull of the California Institute of Technology.

War saw the development of the jet engine which enabled supersonic flight. There was then much mathematical interest in supersonic flow and shock waves, so that Lighthill suggested research in this area to GB, even though he initially wanted to work on general relativity. GB's thesis title was "Propagation of a spherical blast." Lighthill was later to state [1]

"I feel some pride at having, nineteen years ago, chosen as the problem to give to the first really good research student I had, the estimation of pressure pulses transmitted to great distances by bodies moving at supersonic speed."

GB met his wife, Nancy Whitham (née Lord), while he was a Ph.D. student at the University of Manchester. The story which GB told was that they met when GB was a tutor for a mathematics course which Nancy was taking. Nancy Whitham also obtained an honours B.Sc. degree in Mathematics at the University of Manchester. This background enabled her to help edit GB's classic textbook *Linear and Nonlinear Waves* [2]. Nancy Lord was born on the 29th December, 1929 in Oldham, Lancashire, United Kingdom, the daughter of Frank and Amy Lord (née Buckley). She and GB married on the 1st September, 1951 in Manchester. They had three children, Ruth, Michael and Susan.

Richard Courant visited Manchester University in 1950. GB was invited by Courant to be a research associate at the Institute of Mechanics and Applied Mathematics, now the Courant Institute of Mathematical Sciences, of New York University, where he was from 1951 to 1953. The work for GB's thesis was completed in 1951, but the actual degree was awarded in 1953 due to GB being in the United States at the Institute of Mechanics and Applied Mathematics during the intervening period. GB felt that the emphasis at the Courant Institute was too pure for his tastes. He also felt that he should return to the United Kingdom. He then returned to the University of Manchester as a lecturer in applied mathematics in 1953, which position he held until 1956. However, he missed the atmosphere of the Courant Institute and the higher standard of living in the United States, so moved back to the Courant Institute as an associate professor in applied mathematics in 1956 and remained until 1959. The old feeling that at the Courant Institute there was too much of an emphasis on theorems and that there were few people who had physical type reasoning returned, so he became restless. Through the influence of George Carrier and Sydney Goldstein at Harvard University, the latter at the University of Manchester when GB was a student, and C.C. Lin at the Massachusetts Institute of Technology (MIT), Whitham moved to Boston to become a professor of mathematics at MIT, were he stayed from 1959 to 1962. The more applied bent of the mathematics there appealed to GB, but he found MIT to be very large and lacking in intimacy as compared to the Courant Institute.

Hans Liepmann of the Graduate Aeronautical Laboratories of the California Institute of Technology (GALCIT) visited Harvard University in the summer of 1960. He suggested that GB visit GALCIT and the California Institute of Technology (Caltech) as a visiting Professor of Applied Mechanics, with an eye to a future appointment. This GB did for a year from 1961 to 1962. He enjoyed Caltech immensely as it had the small, intimate size of the Courant Institute and the people with the physical applied mathematics orientation he could interact with. These included such famous names as Clark Millikan, H. Liepmann, L. Lees, P.A. Lagerstrom, J.D. Cole, D.E. Coles, F.E. Marble and A. Roshko. GB accepted an appointment as a professor of Aeronautics and Mathematics at Caltech in 1962, and so began his association with Caltech and the eventual Department of Applied Mathematics. He remained a professor of Aeronautics and Mathematics until 1967. Applied Mathematics at Caltech was then a part of the Department of Aeronautics. GB was the leading force in setting up a separate Department of Applied Mathematics at Caltech in 1967, becoming professor of applied mathematics from 1967 to 1983, serving as chairman of the Committee on Applied Mathematics from 1962 to 1971 and as the department's executive officer from 1971 to 1980. Under GB's guidance, the Department of Applied Mathematics at Caltech became one of the leading applied mathematics departments in the world. In 1983 he was appointed the Charles Lee Powell Professor, which position he retained until his retirement in 1998.

During his life, GB Whitham obtained scientific honours for his outstanding contributions to applied mathematics. The highest honour was being elected a Fellow of the Royal Society in 1965. GB was also elected a Fellow of the American Academy of Arts and Sciences in 1959. In 1980, he obtained one of the highest awards in applied mathematics, the Wiener Prize in Applied Mathematics. The description of this award states

"The Wiener Prize is awarded for an outstanding contribution to "applied mathematics in the highest and broadest sense." Awarded jointly by the American Mathematical Society and the Society for Industrial and Applied Mathematics. This prize was established in 1967 in honour of Professor Norbert Wiener and was endowed by a fund from the Department of Mathematics of the Massachusetts Institute of Technology."

After his retirement, GB moved with his wife Nancy to Portland, Oregon to be nearer to their three children. He passed away there on the 26th January, 2014. For a complete obituary of G.B. Whitham, see the *Biographical Memoirs of Fellows of the Royal Society* [3].

2. Modulation Theory

This special issue of *Physica D* on *Dispersive Hydrodynamics,* dedicated to G.B. Whitham, is especially appropriate as the year 2015 is the 50th anniversary of the publication of GB Whitham's seminal papers "Non-linear dispersive waves" [4] and "A general approach to linear and non-linear dispersive waves using a Lagrangian" [5]. These ground-breaking papers introduced the method of averaged Lagrangians, or modulation theory, or Whitham modulation theory, as it is also referred to. GB's original research area was hyperbolic partial differential equations, compressible flow and shock waves, to which he made major contributions [2, 6–8], including his famous modelling of flood waves [9] and traffic flow [10].

In the 1960's GB moved into the area of dispersive waves, which was to be his major research emphasis for the rest of his career. GB was concerned with analysing wavetrains whose properties, such as amplitude, wavenumber and mean height, can vary in space and time, that is modulate. These variations were assumed to have scales much longer than the variation of the phase of the wavetrain, so that the phase was a "fast" variable and the wave parameters, amplitude, wavenumber, mean height etc., were "slow" variables, or slowly varying. Mathematically this meant that the fast phase oscillation of the wavetrain could be decoupled from the slow variation of its parameters, resulting in partial differential equations for these slowly varying parameters in the slow space and time variables, the famous modulation equations. In the process, GB showed that a slick way to derive these modulation equations was via an "averaged Lagrangian," that is integrating the Lagrangian for the equation(s) governing the wavetrain over a period of the fast phase variable, leaving an averaged Lagrangian in the slow space and time variables. Variations of this averaged Lagrangian with respect to the slowly varying wavetrain parameters yielded the modulation equations. This averaged Lagrangian technique is related to the standard method of multiple scales from perturbation theory [11]. The modulation equations are the conditions from multiple scales theory needed to eliminate secular terms.

One of the major motivations behind Whitham developing modulation theory was the extension of the key concept of linear dispersive wave theory, group velocity, to nonlinear dispersive wave equations [5, 12]. The expectation was then that the modulation equations would form a system of hyperbolic equations with the characteristic velocities the generalisation of the linear concept of group velocity. Indeed, when modulation theory was applied to linear dispersive wave equations this was exactly the case [2]. However, when calculations were performed for familiar nonlinear dispersive wave systems Whitham found that the modulation equations were often elliptic, which seemed counter-intuitive and which puzzled Whitham. The classic example of this is the modulation equations for deep water surface gravity waves [2, 13]. At this time, GB heard of wave tank experiments by Benjamin for which he had trouble generating the classic Stokes wave [14, 15], so that Benjamin suspected that the Stokes wavetrain was unstable ([16] as related by Alan Newell). "The penny then dropped" ([16] as related by Alan Newell) and Whitham made the connection between the classification of the modulation equations and the stability of the underlying wavetrain [2, 13], with hyperbolic modulation equations corresponding to stable wavetrains and elliptic modulation equations with unstable wavetrains. Whitham modulation theory then gives a clear and accessible derivation of the Benjamin-Fier sideband instability [2, 13–15, 17]. Whitham modulation theory, or the method of averaged Lagrangians, is an elegant method which made possible the full nonlinear analysis of the slow modulations of wavetrains. When the modulation equations are hyperbolic, the wavetrain is stable and modulations flow along the characteristics of these equations, so that the characteristic velocities are the nonlinear counterparts of linear group velocity [2]. In contrast, if the modulation equations are elliptic, the wavetrain is unstable and the modulations grow, so that higher order nonlinearity and dispersion are needed to study the long term evolution of the wavetrain.

This short history of the development of Whitham modulation theory leads to the subject of the workshop "Dispersive Hydrodynamics: the Mathematics of Dispersive Shock Waves and Applications," which is dispersive shock waves, or undular bores as they are referred to in hydrodynamics. Undular bores are a common phenomenon in fluid flow [18], ranging from tidal bores in coastal regions with strong tidal flows to atmospheric phenomena such as Morning Glories [19, 20]. Dispersive shock waves arise due to the dispersive resolution of some initial discontinuity or near discontinuity. In water waves, there are also viscous bores in which viscosity balances dispersion, leading to a steady bore [21]. These are not the concern of the workshop as the term dispersive shock waves refers to non-viscous bores. Dispersive shock waves are unsteady and continually expand, adding more waves as they develop. For stable wavetrains, the associated modulation equations are hyperbolic, as already discussed. In an impressive piece of work [2, 4] Whitham derived the modulation equations for the Korteweg-de Vries (KdV) equation. The periodic wavetrain solution of the KdV equation is given in terms of the Jacobi elliptic cosine function cn. GB essentially used the properties of elliptic functions and their derivatives and integrals to derive these modulation equations, although in an implicit form. The modulation equations for the KdV equation form a hyperbolic system, so that the cnoidal wave is stable. The modulation equations for the single phase cnoidal wave solution of the KdV equation were then generalised to N phase wavetrains using similar techniques [22] and using functional analysis coupled with the inverse scattering solution of the KdV equation [23], the latter a general method which can be employed to derive the modulation equations for any equation with an inverse scattering solution. With his background in hyperbolic equations, GB realised that the modulation equations for the KdV equation possessed an obvious centred, simple wave solution. However, such solutions correspond to a jump initial condition, so that he did not think that such a solution was physically valid for a slowly varying theory. This simple wave solution was subsequently published by Gurevich and Pitaevskii [24], who showed that it was the solution for an undular bore. The undular bore consisted of a modulated wavetrain with solitons at its leading edge and linear dispersive waves at its trailing edge. GB always regretted not having more faith in modulation theory and publishing this solution with his original work. The work of Gurevich and Pitaevskii did prompt GB to collaborate with Bengt Fornberg on a numerical and theoretical investigation of a wide range of nonlinear wave equations and phenomena, including a comparison between the undular bore solution of the KdV modulation equations and the numerical bore solution [25]. This work showed that the modulation equations gave excellent agreement with the numerical undular bore.

The work of GB Whitham, particularly that for the KdV equation, showed the great utility of modulation theory for nonlinear wave problems and prompted the derivation of the modulation equations for other integrable equations, for instance the nonlinear Schrödinger (NLS) equation [26] and the Benjamin-Ono equation [27]. In tandem with this, it was found that the undular bore, or dispersive shock wave, solution of modulation equations were useful to describe waves arising in flow over topography [28, 29], water waves [30, 31], oceanography [32], meteorology [33], geophysics [34], nonlinear optics [35, 36] and Fermi gases [37]. Undular bores were found to arise in solutions of initial-boundary value problems for the KdV equation [29, 38, 39]. Rather than the finite number of solitons generated from a square integrable initial condition for the KdV equation on the infinite line, on the semi-infinite line the boundary acts as a source, with waves being continually generated there as a bore, these waves developing into solitons as they propagate.

A key recent discovery was the realisation that components of undular bore solutions could be developed without the full modulation equations themselves being derived [40, 41]. This enabled undular bore solutions to be found for non-integrable nonlinear wave equations, for which it is usually very difficult to determine these full modulation equations. It was shown that with only a knowledge of the leading solitary wave and trailing linear wave solutions, the amplitudes and velocities of the leading and trailing fronts of the bore could be determined. The undular bore at these two limits was connected by an ingenious argument. As it is usually straightforward to determine the solitary wave and linear travelling wave solutions of many nonlinear dispersive wave equations, El's method can be easily employed [30, 34, 36, 40, 41]. Furthermore, in experimental and observational contexts the leading edge of a bore is the most easily distinguished and measured [32, 33].

Figure 2: Memorial bench to Professor G.B. Whitham and Mrs. Nancy Whitham outside the Firestone Laboratory at the California Institute of Technology. Photograph by author Noel Smyth.

Whitham modulation theory has found application in areas far removed from its original derivation for and application to nonlinear dispersive wave theory. In recent years Whitham's modulation theory has been shown to unify several aspects of isomonodromy, Seiberg-Witten maps and renormalisation groups. This allows an understanding of supersymmetry breaking in a non-perturbative manner using the Whitham equations to modulate the parameters in the supersymmetric Lagrangian, which then leads to averaged theories. Whitham's approach also provides a way of resumming the steps in the renormalisation procedure. These applications of Whitham modulation theory to theoretical physics is discussed in great detail in the review paper by Carroll [42].

The papers in this special issue of *Physica D* on *Dispersive Hydrodynamics* continues this train of research on modulation equations and dispersive shock waves (undular bores) pioneered by G.B. Whitham, with many of the leading current researchers in the field being represented.

The authors also acknowledge the use of the extensive notes on his life and career which G.B. Whitham wrote before his passing. A.A. Minzoni and N.F. Smyth are former graduate students of G.B. Whitham, from 1972–1976 and 1980–1984, respectively.

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