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THE ENDPOINT MULTILINEAR KAKEYA THEOREM VIA THE BORSUK–ULAM THEOREM

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Abstract. We give an essentially self-contained proof of Guth's recent endpoint multilinear Kakeya theorem which avoids the use of somewhat sophisticated algebraic topology, and which instead appeals to the Borsuk–Ulam theorem.

1. Introduction

The multilinear Kakeya problem was introduced in [1], and its study began in earnest in [2], where the natural conjecture was established up to the endpoint. Working in $\mathbb{R}^n$, we suppose that we are given n transverse families $\mathcal{T}_1, \dots, \mathcal{T}_n$ of 1-tubes, which means that each $T \in \mathcal{T}_j$ is a 1-neighbourhood of a doubly-infinite line in $\mathbb{R}^n$ with direction $e(T) \in \mathbb{S}^{n-1}$, and that the directions $e(T)$ for $T \in \mathcal{T}_j$ all lie within a small fixed neighbourhood (depending only on the dimension n) of the j'th standard basis vector e_j.

The question is whether for each $q \geq 1/(n-1)$ we have the inequality

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathcal{T}_1} a_{T_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathcal{T}_n} a_{T_n} \chi_{T_n}(x) \right)^q dx \leq C_{n,q} \left(\sum_{T_1 \in \mathcal{T}_1} a_{T_1} \cdots \sum_{T_n \in \mathcal{T}_n} a_{T_n} \right)^q$$

for nonnegative coefficients a_{T_j}. In [2] this was proved for each $q > 1/(n-1)$ using a heat-flow technique which, because of certain error terms arising, did not apply at the endpoint $q = 1/(n-1)$. (For further background on this problem consult [2].)

More recently, Guth in [10] established the endpoint case $q = 1/(n-1)$ using completely different techniques motivated in part by the polynomial method used by Dvir [9] to solve the finite field Kakeya set problem, but which also relied upon a fairly heavy dose of algebraic topology, and which were therefore perhaps a little intimidating to the analyst or combinatorialist. In particular, Guth used the technology of cohomology classes, cup products and the Lusternik–Schnirelmann vanishing lemma in establishing his result. We believe that the endpoint multilinear Kakeya theorem is of such significance and importance that a proof free of these techniques should be available, and so the purpose of this paper is to provide an argument leading to Guth's result which does not rely upon such sophisticated algebraic topology, but whose input is instead the Borsuk–Ulam theorem. It is hoped therefore that this paper might lead to further exploitation of Guth's techniques in a more flexible setting. (For some recent works applying the multilinear Kakeya perspective in other contexts, see [3], [4] and [6].)

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The Borsuk–Ulam theorem, while topological in nature, nevertheless has many proofs accessible to the analyst – see for example [12], and also [7] for a recent such proof. (See Section 5 below for its statement.) The use of the Borsuk–Ulam theorem in the context of Kakeya theorems is by now natural, as it can be considered as a topological analogue of the elementary linear-algebraic statement that there are no linear injections $T : V \to W$ if V and W are finite-dimensional vector spaces with $\dim V > \dim W$; this was a key element of Dvir's solution [9] of the finite field Kakeya problem. It also features explicitly in Guth's warm-up discussion to the full result of [10].

In order to proceed, we place matters in a more general context which does not impose conditions on the directions of the tubes, nor requires the level of multilinearity to equal the dimension of the underlying euclidean space. Thus we now suppose that we are given d arbitrary families of 1-tubes $\mathcal{T}_1, \dots, \mathcal{T}_d$ in $\mathbb{R}^n$, where $d \leq n$. For $v_1, \dots v_d \in \mathbb{R}^n$ let $v_1 \wedge \cdots \wedge v_d$ denote the unsigned (i.e. nonnegative) d-dimensional volume of the parallelepiped whose sides are given by the vectors $v_1, \dots, v_d$.

Theorem 1 (The Multilinear Kakeya Theorem). *Let $2 \leq d \leq n$. Then there exists a constant $C_{d,n}$ such that if $\mathcal{T}_1, \dots, \mathcal{T}_d$ are families of 1-tubes in $\mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathcal{T}_1} a_{T_1} \chi_{T_1}(x) \cdots \sum_{T_d \in \mathcal{T}_d} a_{T_d} \chi_{T_d}(x)\, e(T_1) \wedge \cdots \wedge e(T_d) \right)^{1/(d-1)} dx \tag{1}$$
$$\leq C_{d,n} \left(\sum_{T_1 \in \mathcal{T}_1} a_{T_1} \cdots \sum_{T_d \in \mathcal{T}_d} a_{T_d} \right)^{1/(d-1)}.$$

(The case $d = 2$ is of course trivial.)

The situation where the level of multilinearity is less than the ambient euclidean dimension was already addressed in [2], where once again the result was established up to the endpoint. The incorporation of the factor $e(T_1) \wedge \cdots \wedge e(T_d)$ on the left-hand side is natural in view of the affine-invariant formulation of the Loomis–Whitney inequality, and was considered in Section 7 of [5], where Theorem 1 was first proved. Indeed, when $d = n$, the statement of Theorem 1 is affine-invariant.[1] A variant of Theorem 1 where lines are replaced by algebraic curves of bounded degree was also proved in [5] (and can likewise be established by replacing Guth's original argument for Theorem 2 below by that of the current paper). On the other hand, the results of [2] have a somewhat more general scope in so far as they apply

[1]The multilinear Kakeya theorem can also be cast in the following equivalent form when $d = n$. For a unit vector $\omega \in \mathbb{R}^n$ let Π_ω denote the hyperplane in $\mathbb{R}^n$ which is perpendicular to ω and which contains the origin. Let $\pi_\omega : \mathbb{R}^n \to \Pi_\omega$ be the orthogonal projection map. Then for nonnegative g_j we have

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \left(\prod_{j=1}^n g_j(\omega_j, \pi_{\omega_j} x) \right) \omega_1 \wedge \cdots \wedge \omega_n \, d\sigma(\omega_1) \dots d\sigma(\omega_n) \right)^{1/(n-1)} dx$$
$$\leq C_n \prod_{j=1}^n \left(\int_{\mathbb{S}^{n-1}} \int_{\Pi_{\omega_j}} g_j(\omega_j, \xi) d\xi d\sigma(\omega_j) \right)^{1/(n-1)}.$$

to 1-neighbourhoods of k-planes for arbitrary k, rather than just 1-neighbourhoods of lines, i.e. tubes, as in the present discussion.

The principal notion that Guth employs in proving the endpoint theorem is that of the *visibility* $\mathrm{vis}\,(Z)$ of a hypersurface $Z \subseteq \mathbb{R}^n$ – see Section 3 below for the definition, which differs from Guth's in so far as in our treatment it (roughly) scales as does $(n-1)$-dimensional Hausdorff measure $\mathcal{H}_{n-1}$ – and the centrepiece of Guth's argument is the following result:

Theorem 2. *Given a nonnegative function M defined on the lattice $\mathcal{Q}$ of unit cubes of $\mathbb{R}^n$, there exists a non-zero polynomial p such that*

$$\deg p \leq C_n \left(\sum_{Q \in \mathcal{Q}} M(Q)^n \right)^{1/n}$$

and such that if we set $Z = Z_p = \{x \in \mathbb{R}^n \ : \ p(x) = 0\}$, then for all $Q \in \mathcal{Q}$ we have

$$\mathrm{vis}\,(Z \cap Q) \geq C_n M(Q).$$

It is in the proof of this result that Guth uses algebraic-topological techniques, and the main contribution of the present paper is to provide a proof of Theorem 2 which does not use such topological machinery, but is instead a consequence of the Borsuk–Ulam theorem. (In fact, in our proof of Theorem 2, we do not use the Borsuk–Ulam theorem *per se* but instead an equivalent Lusternik–Schnirelmann type covering statement. See Section 5.) On the other hand, we must acknowledge that many of the arguments and constructions of the present paper are inspired by those of Guth's approach.

In view of the connection between visibility and $(n-1)$-dimensional Hausdorff measure, and as a warm-up to our proof of Theorem 2, we indicate how the Borsuk–Ulam theorem can be used to establish the following morally weaker variant of Theorem 2.

Proposition 1. *Given a finitely supported function M defined on the lattice $\mathcal{Q}$ of unit cubes of $\mathbb{R}^n$ and taking nonzero values in $[1, \infty)$, there exists a non-zero polynomial p such that*

$$\deg p \leq C_n \left(\sum_{Q \in \mathcal{Q}} M(Q)^n \right)^{1/n}$$

and such that for all $Q \in \mathcal{Q}$

$$\mathcal{H}_{n-1}\,(Z \cap Q) \geq C_n M(Q).$$

Proof. Break up each Q into $\sim M(Q)^n$ congruent subcubes S; note that altogether we have $\sim \sum_Q M(Q)^n$ small cubes S of various sizes. Consider the map

$$F : p \mapsto \left\{ \int_{\{p>0\}\cap S} 1 - \int_{\{p<0\}\cap S} 1 \right\}_S$$

defined on the vector space $\mathcal{P}_k$ of polynomials of degree at most k in n real variables, which has dimension $\sim k^n$. Clearly F is continuous, homogeneous of degree 0 and odd.

So we can think of F as

$$F : \mathbb{S}^N \to \mathbb{R}^J$$

where $N \sim k^n$ and $J \sim \sum_Q M(Q)^n$.

So provided $N \geq J$ – which we can arrange if $k \sim \left(\sum_Q M(Q)^n\right)^{1/n}$ – the Borsuk–Ulam theorem tells us that F vanishes at some p. This means that the zero set Z of p exactly bisects each S.

Now if S is a subcube of Q, S will have volume $\sim M(Q)^{-n}$ and diameter $\sim M(Q)^{-1}$ and hence any bisecting surface will meet it in a set of $(n-1)$-dimensional measure $\gtrsim M(Q)^{-(n-1)}$. This will be true for each of the $M(Q)^n$ disjoint S's whose union is Q, so Z will meet Q in a set of $(n-1)$-dimensional measure $\gtrsim M(Q)^n \times M(Q)^{-(n-1)} = M(Q)$, as was needed. $\square$

In the proof we used the "geometrically obvious" fact that a hypersurface bisecting the unit cube must have large surface area inside the cube. For a discussion of this in the context of the unit ball, see Lemma 11 in the Appendix. Note that in the statements of both Theorem 2 and Proposition 1, a polynomial has the desired properties if and only if any non-zero scalar multiple of it does; for this reason we may choose to search for a suitable polynomial within the unit sphere of the class of polynomials of a given degree.

Proposition 1 is morally weaker than Theorem 2 because not only does it place stronger conditions on M, but more importantly, in many situations of interest, we have vis $(Z \cap Q) \leq C_n \mathcal{H}_{n-1}\,(Z \cap Q)$ – see (7) below.

On an informal level, the fundamental difference between the proof of Theorem 2 and that of Proposition 1 is that, roughly speaking, we no longer chop each cube Q into $\sim M(Q)^n$ congruent *subcubes*, but we instead select, for each Q, an *ellipsoid* $E(Q)$ of volume $\sim M(Q)^{-n}$, so that $\sim M(Q)^n$ translates of $E(Q)$ essentially tessellate Q. However the *shape* and *orientation* of the ellipsoid $E(Q_0)$ will depend not only on the value of $M(Q_0)$ but on the whole ensemble $\{M(Q)\}_Q$, and is in effect an output of the Borsuk–Ulam theorem at the same time as it produces the desired polynomial. At the risk of over-simplifying matters, we now give an informal example which illustrates why, if we want the broad thrust of the proof of Proposition 1 to work in the context of Theorem 2, the shape of the ellipsoid selected *must* depend on the totality of the function $M(Q)$. This example may be safely ignored on a first reading of the paper.

Informal example. Let $n = 2$ and consider the function $M(Q)$ which is supported on a row of N unit cubes centred at $(k - 1/2, 1/2)$ for $1 \leq k \leq N$, and takes the value $N^{1/2}$ on each of these cubes. Then $\left(\sum_Q M(Q)^2\right)^{1/2} = N$. Consider the polynomial

$$p(x) = (x_1 - 1) \ldots (x_1 - N) \cdot (x_2 - 1/2N)(x_2 - 3/2N) \ldots (x_2 - (2N-1)/2N)$$

which has degree $2N$, and let Z denote its zero set. For a subset $Z' \subseteq Z$ of Z, and for each Q[2] in the support of M, consider the projections counted with multiplicities

[2]In this example we assume that the right-hand edge of a rectangle belongs to the rectangle while the left-hand edge does not.

of $Z' \cap Q$, in the directions of the two standard basis vectors e_1 and e_2; let their total lengths be $a_1(Z' \cap Q)$ and $a_2(Z' \cap Q)$ respectively, and let

$$W(Z' \cap Q) := \{a_1(Z' \cap Q)a_2(Z' \cap Q)\}^{1/2}$$

be their geometric mean. Now it transpires that the quantity $W(Z' \cap Q)$ is closely related to $\mathrm{vis}(Z' \cap Q)$ – see Section 3 below – and we shall pretend (for the rest of this example) that W really is the visibility. Note that if Z_1 and Z_2 are disjoint subsets of $Z \cap Q$ we have[3]

$$W(Z_1 \cup Z_2) \geq W(Z_1) + W(Z_2).$$

Now $a_1(Z \cap Q) = 1$ and $a_1(Z \cap Q) = N$ so that $W(Z \cap Q) = N^{1/2}$. We consider whether it is possible to break up each Q into rectangles R_j of area $1/N$ so that if p bisects each rectangle then we can deduce that $N^{1/2} = W(Z \cap Q)$ by using $W(Z \cap Q) \geq \sum_j W(Z \cap R_j)$.

Firstly, we could break up Q into subcubes R_j of side $N^{-1/2}$. Now for all R_j except those which meet the right-hand edge of Q we shall have $W(Z \cap R_j) = 0$, while for those which do meet the right hand edge of Q we have $W(Z \cap R_j) = N^{-1/4}$, which only gives $W(Z \cap Q) \geq \sum_j W(Z \cap R_j) = N^{1/4}$; this is not adequate.

Next, we could try breaking up Q into vertical rectangles R_j of sides $1/N \times 1$. Only the rectangle R_0 meeting the right-hand edge of Q will have a non-zero value of $W(Z \cap R_j)$, and $W(Z \cap R_0) = 1$, giving $W(Z \cap Q) \geq \sum_j W(Z \cap R_j) = 1$, which is even worse.

Finally, we could try breaking up Q into horizontal rectangles R_j of sides $1 \times 1/N$. In this case each $W(Z \cap R_j) = N^{-1/2}$, resulting in the desired $W(Z \cap Q) \geq \sum_j W(Z \cap R_j) \geq N^{1/2}$.

So only the third decomposition into horizontal rectangles is compatible with our needs. Once we have accepted that the polynomial p above is more or less "canonical" for this M, we are essentially forced to break up each Q into horizontal rectangles of sides $1 \times 1/N$ in order for our strategy to be successful. Crucially, observe that this decomposition reflects the global shape of the function M: if the support of M had been along the x_2-axis we would have had to instead decompose each Q into vertical rectangles. The decomposition must therefore be aligned with the "global profile" of M. □

To simplify the constructions in the proof we actually stop just short of fully developing the moral outline given above. In fact we do not spend any time constructing ellipsoids at the scale $\sim M(Q)^{-n}$ which would be needed to really nail down the zero set of the polynomial we get from the Borsuk–Ulam theorem. Instead we let ourselves be satisfied with finding a *good* polynomial p with zero set Z_p which satisfies a given lower bound on the visibility $\mathrm{vis}\,(Z_p \cap Q)$ for all cubes Q. This we get by constructing ellipsoids for all *bad* polynomials, which are those polynomials whose zero sets have visibility less than desired on some cube. Using these ellipsoids we show that the bad polynomials cannot cover the unit sphere in the space of polynomials and in this way we see that there must be a good polynomial, which gives the lower bounds mentioned above. Herein lies the reason for using the covering

[3]The corresponding property for visibility fails.

statement instead of the Borsuk–Ulam theorem itself. This very informal outline is developed more fully in Section 6.

For completeness, we also indicate in the following sections how Theorem 1 follows from Theorem 2, so that we give what is in essence a fully self-contained proof of Theorem 1 (subject to the appeal to the Borsuk–Ulam theorem). Throughout, C and c will denote generic constants which depend only on the dimension n and the degree of multilinearity $d \leq n$; $P \lesssim Q$ and $P \gtrsim Q$ mean $P \leq CQ$ and $P \geq CQ$ respectively, and $P \sim Q$ means both $P \lesssim Q$ and $P \gtrsim Q$.

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2. A Preliminary reduction

Recall that we have collections $\mathcal{T}_j$, $1 \leq j \leq d$, of 1-tubes T in $\mathbb{R}^n$ with directions $e(T) \in \mathbb{S}^{n-1}$. Let $\mathcal{Q}$ denote the lattice of unit cubes in $\mathbb{R}^n$.

Proposition 2. *In order to prove Theorem 1, it suffices to establish the following assertion: for every finitely supported nonnegative function $M : \mathcal{Q} \to \mathbb{R}$ satisfying $\sum_Q M(Q)^n = 1$, there exist nonnegative functions $S_j : \mathcal{Q} \times \mathcal{T}_j \to \mathbb{R}$ such that for all $T_j \in \mathcal{T}_j$ with $T_j \cap Q \neq \emptyset$,*

$$e(T_1) \wedge \cdots \wedge e(T_d)\, M(Q)^n \leq C S_1(Q, T_1) \ldots S_d(Q, T_d) \tag{2}$$

and, for all j and all $T_j \in \mathcal{T}_j$

$$\sum_{Q \,:\, T_j \cap Q \neq \emptyset} S_j(Q, T_j) \leq C. \tag{3}$$

Proof. Firstly, if we can find S_j as in the statement of the proposition, homogeneity dictates that for every finitely supported nonnegative function $M : \mathcal{Q} \to \mathbb{R}$ there exist nonnegative functions $S_j : \mathcal{Q} \times \mathcal{T}_j \to \mathbb{R}$ such that for all $T_j \in \mathcal{T}_j$ with $T_j \cap Q \neq \emptyset$,

$$e(T_1) \wedge \cdots \wedge e(T_d)\, M(Q)^n \leq C S_1(Q, T_1) \ldots S_d(Q, T_d) \left(\sum_Q M(Q)^n \right)^{(n-d)/n} \tag{4}$$

and, for all j and all $T_j \in \mathcal{T}_j$

$$\sum_{Q \,:\, T_j \cap Q \neq \emptyset} S_j(Q, T_j) \leq C \left(\sum_Q M(Q)^n \right)^{1/n}. \tag{5}$$

Secondly, we note that by the l^1 nature of the right hand side in Theorem 1, we may assume that the sets $\mathcal{T}_j$ are finite and that all the coefficients a_{T_j} are equal to 1.

For a unit cube Q let

$$F(Q) = \sum_{T_j \in \mathcal{T}_j \text{ with } T_j \cap Q \neq \emptyset} e(T_1) \wedge \cdots \wedge e(T_d).$$

It then suffices to prove

$$\sum_Q F(Q)^{1/(d-1)} \le C \left(\#\mathcal{T}_1 \dots \#\mathcal{T}_d\right)^{1/(d-1)}.$$

Let $M(Q)^n = F(Q)^{1/(d-1)} = F(Q)^{(1/(d-1)-1/d)d}$, so that

$$\begin{aligned}
&\sum_Q F(Q)^{1/(d-1)} = \sum_Q F(Q)^{1/d} M(Q)^{n/d}\\
&= \sum_Q \left(\sum_{T_j \in \mathcal{T}_j \text{ with } T_j \cap Q \neq \emptyset} e(T_1) \wedge \dots \wedge e(T_d) \right)^{1/d} M(Q)^{n/d}\\
&= \sum_Q \left(\sum_{T_j \in \mathcal{T}_j \text{ with } T_j \cap Q \neq \emptyset} e(T_1) \wedge \dots \wedge e(T_d)\, M(Q)^n \right)^{1/d}\\
&\le C \sum_Q \left(\sum_{T_j \in \mathcal{T}_j \text{ with } T_j \cap Q \neq \emptyset} S_1(Q,T_1) \dots S_d(Q,T_d) \left(\sum_Q M(Q)^n \right)^{(n-d)/n} \right)^{1/d}\\
&= C \sum_Q \left(\prod_{j=1}^d \sum_{T_j \in \mathcal{T}_j \text{ with } T_j \cap Q \neq \emptyset} S_j(Q,T_j) \right)^{1/d} \left(\sum_Q M(Q)^n \right)^{(n-d)/dn}\\
&\le C \prod_{j=1}^d \left(\sum_Q \sum_{T_j \in \mathcal{T}_j \text{ with } T_j \cap Q \neq \emptyset} S_j(Q,T_j) \right)^{1/d} \left(\sum_Q M(Q)^n \right)^{(n-d)/dn}\\
&= C \prod_{j=1}^d \left(\sum_{T_j \in \mathcal{T}_j} \sum_{Q \text{ with } T_j \cap Q \neq \emptyset} S_j(Q,T_j) \right)^{1/d} \left(\sum_Q M(Q)^n \right)^{(n-d)/dn}\\
&\le C \prod_{j=1}^d (\#\mathcal{T}_j)^{1/d} \left(\sum_Q M(Q)^n \right)^{1/n} \left(\sum_Q M(Q)^n \right)^{(n-d)/dn}\\
&= C \prod_{j=1}^d (\#\mathcal{T}_j)^{1/d} \left(\sum_Q F(Q)^{1/(d-1)} \right)^{1/d}
\end{aligned}$$

where the inequalities follow from (4), Hölder's inequality and (5) respectively. Rearranging, we obtain

$$\left(\sum_Q F(Q)^{1/(d-1)} \right)^{(d-1)/d} \le C \prod_{j=1}^d (\#\mathcal{T}_j)^{1/d},$$

from which the result follows. □

Interestingly, the line of argument here can be reversed in certain circumstances: assuming that the special case of the multilinear Kakeya theorem for transverse families of tubes $\mathcal{T}_j$ holds, it follows that for all M one can find S_j satisfying (4) and (5). See [8] for more details.

3. Directional surface area and visibility

We follow Guth [10] and Bourgain–Guth [5] in defining the functions S_j and establishing their desired properties (2) and (3). In order to do this some geometric notions are required. We first recall the notion of directional surface area (termed "directed volume" by Guth) of a hypersurface $Z \subseteq \mathbb{R}^n$ in the direction of a unit vector e. If the element of surface area of Z is denoted by $dS = d\mathcal{H}_{n-1}|_S$, and e is a unit vector, the element of the component of surface area of Z perpendicular to e is $|e \cdot n(x)| dS(x)$ where $n(x)$ is the unit normal at x (which is assumed to make sense for $\mathcal{H}_{n-1}$–almost every $x \in Z$). Thus the **directional surface area of Z in the direction** $e \in \mathbb{S}^{n-1}$ is defined as

$$\mathrm{surf}_e(Z) = \int_Z |e \cdot n(x)| \, dS(x).$$

If Z is given by the graph of a function $\Gamma : \Omega \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ above the hyperplane $x_n = 0$, then its directional surface area in the direction e_n is simply the $(n-1)$-dimensional area of Ω. If Z is given by disjoint graphs of functions above the hyperplane $x_n = 0$ then its directional surface area in the direction e_n is just $\int_{\mathbb{R}^{n-1}} J(y) \, dy$ where $J(y)$ is the number of times the line through y parallel to e_n passes through Z. These considerations lead immediately to Guth's "cylinder estimate":

Lemma 1 (Guth's cylinder estimate)**.** *If T is a 1-tube in $\mathbb{R}^n$ and $Z = \{x \, : \, p(x) = 0\}$ is the zero hypersurface of a non-zero polynomial p of degree at most k, then*

$$\mathrm{surf}_{e(T)}(Z \cap T) \leq Ck.$$

Secondly, we associate a fundamental centrally-symmetric convex body $K(Z)$ to a hypersurface Z. Indeed, with $\mathbb{B}$ denoting the unit ball of $\mathbb{R}^n$, define

$$K(Z) := \{u \in \mathbb{B} \, : \, \mathrm{surf}_{\widehat{u}}(Z) \leq 1/|u|\}. \tag{6}$$

Here $\widehat{u}$ is the unit vector in the direction of u. (Notice that if Z is such that $\mathrm{surf}_e(Z) \geq 1$ for all unit vectors e, then the requirement that u lie in $\mathbb{B}$ is superfluous.) It is clear that $K(Z)$ is symmetric. To see that it is in fact convex, note that u satisfies $\mathrm{surf}_{\widehat{u}}(Z) \leq 1/|u|$ if and only if $\int_Z |u \cdot n| \, dS \leq 1$; this condition is clearly retained under convex combinations of u's. We then define [4] the **visibility** of Z as

$$\mathrm{vis}(Z) := (\mathrm{vol}\ K(Z))^{-1/n}.$$

Note that since $K(Z) \subseteq \mathbb{B}$ we always have $\mathrm{vis}(Z) \geq C$.

The next lemma allows us to relate visibilty to geometric means of directional surface areas.

[4] Guth's definition of visibility is the n'th power of the one given here, but we find the current definition more natural for three reasons: firstly it allows us to emphasise the "L^n"-aspect of the statement of Theorem 2 which, at least when $n = d$, is no coincidence, and is a reflection of the fact that the optimal L^p estimate for the linear Kakeya problem in $\mathbb{R}^n$ is conjectured to be at $p = n$; secondly it scales roughly as does $(n-1)$-dimensional Hausdorff measure which permits the comparison with Proposition 1; and thirdly, in the theory of finite-dimensional Banach spaces, if K is a convex body in isotropic position, the quantity $(\mathrm{vol}\ K)^{-1/n}$ arises naturally as its isotropic constant.

Lemma 2. *Suppose that for all unit vectors $e \in \mathbb{R}^n$ we have $1 \lesssim \mathrm{surf}_e(Z) \lesssim D$. If $v_1, \dots, v_d$, $(1 \le d \le n)$ are unit vectors, then*

$$(v_1 \wedge \cdots \wedge v_d)^{1/n} \operatorname{vis}(Z) \le C D^{(n-d)/n} \left(\mathrm{surf}_{v_1}(Z) \dots \mathrm{surf}_{v_d}(Z)\right)^{1/n}.$$

Proof. We may assume that $\{v_1, \dots, v_d\}$ is linearly independent and we extend it to a basis $\{v_1, \dots, v_n\}$ where $v_{d+1}, \dots, v_n$ are mutually orthogonal unit vectors which are also orthogonal to the span of $\{v_1, \dots, v_d\}$.

Since $\mathrm{surf}_e(Z) \gtrsim 1$ for all e, we have that $\pm c v_j / \mathrm{surf}_{v_j}(Z) \in K(Z)$ for all j, so that by convexity of $K(Z)$

$$\begin{aligned}
\mathrm{vol}\, K(Z) &\ge C\, v_1 \wedge \cdots \wedge v_n \prod_{j=1}^{n} \mathrm{surf}_{\,v_j}(Z)^{-1} \\
&= C\, v_1 \wedge \cdots \wedge v_d \prod_{j=1}^{d} \mathrm{surf}_{\,v_j}(Z)^{-1} \prod_{j=d+1}^{n} \mathrm{surf}_{\,v_j}(Z)^{-1} \\
&\ge C D^{-(n-d)}\, v_1 \wedge \cdots \wedge v_d \prod_{j=1}^{d} \mathrm{surf}_{\,v_j}(Z)^{-1},
\end{aligned}$$

from which the result follows. □

It is not hard to show that under the assumption that $\mathrm{surf}_e(Z) \gtrsim 1$ for all e,

$$\operatorname{vis}(Z) \sim \prod_{j=1}^{n} \mathrm{surf}_{e_j}(Z)^{1/n}$$

where $e_1, \dots, e_n$ are the principal directions of the John ellipsoid associated to $K(Z)$ (i.e. the ellipsoid of maximal volume contained in $K(Z)$ – see [11]) and hence

$$\operatorname{vis}(Z) \sim \inf_{\{f_j\} \text{ approx orthonormal}} \prod_{j=1}^{n} \mathrm{surf}_{f_j}(Z)^{1/n} \tag{7}$$

where we say that the unit vectors $f_1, \dots, f_n$ are "approximately orthonormal" if their wedge product satisfies $f_1 \wedge \cdots \wedge f_n \ge c_n$ for a suitable dimensional constant c_n. By the arithmetic-geometric mean inequality the right-hand side of (7) is in turn dominated by $\mathcal{H}_{n-1}(Z)$. This shows in particular that Theorem 2 is morally stronger than Proposition 1.

The John ellipsoid E of a symmetric convex body K satisfies $E \subseteq K \subseteq n^{1/2} E$, and combining the latter inclusion with Lemma 1 we obtain:

Lemma 3. *Let p be a non-zero polynomial such that for some unit vector e, $\mathrm{surf}_e(Z_p \cap Q) \lesssim 1$. Then*

$$\operatorname{vis}(Z_p \cap Q)^{n/(n-1)} \le C \deg p.$$

Proof. Let E be the John ellipsoid associated to $K(Z_p \cap Q)$ and let $l_1 \ge l_2 \ge \cdots \ge l_n$ be the lengths of the principal axes of E. Let $A = \operatorname{vis}(Z_p \cap Q)$. By hypothesis and the fact that $K(Z_p \cap Q) \subseteq n^{1/2} E$, we have $l_1 \gtrsim 1$. Moreover we have $(l_1 \dots l_n)^{-1/n} \sim A$, so $l_2 \dots l_n \lesssim A^{-n}$ and therefore $l_n \lesssim A^{-n/(n-1)}$. So if e_n is the direction with which l_n is associated, we have that $\mathrm{surf}_{e_n}(Z_p \cap Q) \gtrsim A^{n/(n-1)}$. The cylinder estimate now gives $\deg p \gtrsim A^{n/(n-1)}$. □

In order to deal with a continuity issue later in the argument (in Lemma 10 of Section 9), we need (as does Guth) to define variants of the directional surface area and visibility which are continuous functionals of $Z = Z_p$ when the polynomial p is allowed to vary. In view of the fact that the class of polynomials with the desired properties for Theorem 2 is invariant under multiplication by non-zero scalars, it is natural to consider the unit sphere of the class $\mathcal{P}_k$ of polynomials of degree at most k in n real variables. Indeed, $\mathcal{P}_k$ is a vector space of dimension $\sim k^n$, and so its unit sphere $\mathcal{P}_k^*$ is homeomorphic to $\mathbb{S}^N$ where $N = N(k) \sim k^n$. So with k fixed, we allow p to vary within $\mathcal{P}_k^*$.[5] The continuity property needed is most simply achieved by replacing $\mathrm{surf}_e(Z)$ for surfaces of the form $Z = Z_p \cap U$ (where $p \in \mathbb{S}^N$ and U is open in $\mathbb{R}^n$) by $\mathrm{surf}_{e,\varepsilon}(Z)$ which we define as the average of $\mathrm{surf}_e(Z')$ with $Z' = Z_{p'} \cap U$ over p' in a ball of radius ε centred at p in $\mathbb{S}^N$. From this we define $K_\varepsilon(Z)$ and $\mathrm{vis}_\varepsilon(Z)$ in analogy to $K(Z)$ and $\mathrm{vis}(Z)$. In the argument we will have to choose ε sufficiently small so that these entities behave in certain ways similarly to the unmollified versions.

It is a routine matter to verify that $K_\varepsilon(Z)$ is convex and that the three lemmas of this section hold with these mollified variants. To be precise, fixing k and the associated definitions of $\mathrm{surf}_{e,\varepsilon}(Z)$, $K_\varepsilon(Z)$ and $\mathrm{vis}_\varepsilon(Z)$ as above for $\mathcal{P}_k^*$, we have, (with implicit constants independent of $\varepsilon > 0$):

Lemma 4. *If T is a 1-tube in $\mathbb{R}^n$ and $Z = \{x : p(x) = 0\}$ is the zero hypersurface of a polynomial $p \in \mathcal{P}_k^*$, then for all $\varepsilon > 0$,*

$$\mathrm{surf}_{e(T),\varepsilon}(Z \cap T) \leq Ck.$$

Lemma 5. *Suppose that $p \in \mathcal{P}_k^*$ and that $Z = Z_p \cap U$ as above. Also suppose that for some $\varepsilon > 0$ and all unit vectors $e \in \mathbb{R}^n$ we have $1 \lesssim \mathrm{surf}_{e,\varepsilon}(Z) \lesssim D$. If $v_1, \dots, v_d$, $(1 \leq d \leq n)$ are unit vectors, then,*

$$(v_1 \wedge \cdots \wedge v_d)^{1/n} \,\mathrm{vis}_\varepsilon(Z) \leq CD^{(n-d)/n} \left(\mathrm{surf}_{v_1,\varepsilon}(Z) \dots \mathrm{surf}_{v_d,\varepsilon}(Z)\right)^{1/n}.$$

Lemma 6. *Suppose $p \in \mathcal{P}_k^*$ is such that for some $\varepsilon > 0$ and some unit vector e, $\mathrm{surf}_{e,\varepsilon}(Z_p \cap Q) \lesssim 1$. Then*

$$\mathrm{vis}_\varepsilon(Z_p \cap Q)^{n/(n-1)} \leq Ck.$$

The reader may wish to proceed with the unmollified variants in mind on a first reading.

4. Application of the main result to multilinear Kakeya

The version of Theorem 2 that we will actually need is:

Theorem 3. *Given a nonnegative function $M : \mathcal{Q} \to \mathbb{R}$, there exists a nonnegative integer k, a polynomial $p \in \mathcal{P}_k^*$ and an $\varepsilon > 0$ such that*

$$k \leq C \left(\sum_Q M(Q)^n \right)^{1/n}$$

[5]From now on we shall use the notations $\mathcal{P}_k^*$ and $\mathbb{S}^N$ where $N = N(k)$ interchangeably, the former when we are thinking of individual polynomials, the latter when continuity and topological considerations are foremost.

and such that for all $Q \in \mathcal{Q}$

$$\mathrm{vis}_\varepsilon(Z_p \cap Q) \geq CM(Q).$$

(Note that since it is always the case that $\mathrm{vis}_\varepsilon(Z_p \cap Q) \gtrsim 1$, the latter condition has content only for those Q with $M(Q) \gtrsim 1$.)

In this section we show how this result implies the conditions of Proposition 2, that is, given a finitely supported nonnegative function $M : \mathcal{Q} \to \mathbb{R}$ satisfying $\sum_Q M(Q)^n = 1$, there exist nonnegative functions $S_j : \mathcal{Q} \times \mathcal{T}_j \to \mathbb{R}$ such that (2) and (3) hold.

Given such a finitely supported nonnegative function $M(Q)$ with $\sum_Q M(Q)^n = 1$, we define $M_0(Q) = \lambda M(Q)$ for some $\lambda \gg 1$ which is required to satisfy $\lambda \geq M(Q)^{-n}$ for all Q in the support of M. Apply Theorem 3 with data M_0 to obtain a k, a $p \in \mathcal{P}_k^*$ and an $\varepsilon > 0$ such that

$$k \leq C\lambda \tag{8}$$

and

$$\mathrm{vis}_\varepsilon(Z_p \cap Q) \geq C\lambda M(Q). \tag{9}$$

Using (8), (9) and our requirement on λ, we have

$$\begin{aligned} k \leq C\lambda &= C\lambda^{n/(n-1)}\lambda^{-1/(n-1)} \\ &\leq C\left(\frac{\mathrm{vis}_\varepsilon(Z_p \cap Q)}{M(Q)}\right)^{n/(n-1)}\left(M(Q)^{-n}\right)^{-1/(n-1)} \\ &= C\,\mathrm{vis}_\varepsilon(Z_p \cap Q)^{n/(n-1)}. \end{aligned}$$

Using Lemma 6 we deduce that for all cubes Q in the support of M and all unit vectors e we have $\mathrm{surf}_{e,\varepsilon}(Z_p \cap Q) \gtrsim 1$.[6] This in turn will permit us to apply Lemma 5 (relating visibility to geometric means of directional surface areas) below.

We turn to the verification of (3). By Lemma 4 and (8) we have, for all $e \in \mathbb{S}^{n-1}$,

$$\mathrm{surf}_{e,\varepsilon}(Z_p \cap Q) \leq C\lambda \tag{10}$$

and moreover

$$\sum_{Q\,:\,Q\cap T_j \neq \emptyset} \mathrm{surf}_{e(T_j),\varepsilon}(Z_p \cap Q) \leq C\ \mathrm{surf}_{e(T_j),\varepsilon}(Z_p \cap \tilde{T}_j) \leq Ck \leq C\lambda, \tag{11}$$

(where $\tilde{T}$ denotes the expansion of a tube T about its axis by a dimensional factor).

We now define

$$S_j(Q, T_j) := \lambda^{-1}\ \mathrm{surf}_{e(T_j),\varepsilon}(Z_p \cap Q)$$

and observe that (11) immediately implies

$$\sum_{Q\,:\,Q\cap T_j \neq \emptyset} S_j(Q, T_j) \leq C,$$

which establishes (3).

[6]Another way of achieving this is to multiply the polynomial which Theorem 3 produces with a polynomial whose zero set consists of hyperplanes parallel to the coordinate hyperplanes which pass through the cubes in the support of M. This has an insignificant effect on the degree of the polynomial provided λ is large enough. However, some care must be taken when considering how this augmentation interacts with the mollification.

On the other hand, to see that (2) is satisfied, note that (9) and (10) together with Lemma 5 give

$$C\lambda M(Q) \leq \mathrm{vis}_\varepsilon(Z_p \cap Q) \leq \frac{C\lambda^{(n-d)/n}\left(\mathrm{surf}_{e(T_1),\varepsilon}(Z_p \cap Q)\dots\mathrm{surf}_{e(T_d),\varepsilon}(Z_p \cap Q)\right)^{1/n}}{\left(e(T_1)\wedge\cdots\wedge e(T_d)\right)^{1/n}},$$

and so

$$S_1(Q,T_1)\dots S_d(Q,T_d) = \lambda^{-d}\prod_{j=1}^{d}\mathrm{surf}_{e(T_j),\varepsilon}(Z_p \cap Q)$$

is at least

$$C\lambda^{-d}\lambda^n M(Q)^n\lambda^{d-n}e(T_1)\wedge\cdots\wedge e(T_d) = CM(Q)^n e(T_1)\wedge\cdots\wedge e(T_d),$$

and thus (2) is established.

Consequently, the multilinear Kakeya theorem is reduced to proving Theorem 3.

5. The Borsuk–Ulam theorem and a covering lemma

The Borsuk–Ulam theorem is as follows:

Theorem 4 (Borsuk–Ulam)**.** *Suppose that $N \geq J$ and that $F : \mathbb{S}^N \to \mathbb{R}^J$ is continuous and satisfies $F(-x) = -F(x)$ for all $x \in \mathbb{S}^N$. Then there is an $x \in \mathbb{S}^N$ such that $F(x) = 0$.*

For a delightful discussion of this theorem and its applications, see [12]. See also [7] for a recent proof of the Borsuk–Ulam theorem using only point-set topology and Stokes' theorem. Included in [12] there is a discussion of various equivalent forms of this theorem, some of which (known as Lusternik–Schnirelmann results) take the form of covering statements for the sphere. In this section we formulate another such equivalent covering statement which we shall use in our proof of Theorem 3.

Lemma 7. *Suppose that $A_i \subseteq \mathbb{S}^N$ for $1 \leq i \leq J$, and suppose that for each i, $A_i \cap (\overline{-A_i}) = \emptyset$. If $J \leq N$, then the $2J$ sets A_i and $-A_i$ do not cover $\mathbb{S}^N$.*

Note that no topological hypothesis on the sets A_i is needed.

Proof. Let $F_i(x) = d(x, -A_i) - d(x, A_i)$ for $1 \leq i \leq J$. Then, with $F : \mathbb{S}^N \to \mathbb{R}^J$ defined by $F(x) = (F_1(x), \dots, F_J(x))$, we have that F is continuous and $F(-x) = -F(x)$ for all x, so by the Borsuk–Ulam theorem there is an x with $F(x) = 0$. We claim that this x does not belong to any A_i or to $-A_i$. For if $x \in A_i$ we have $d(x, A_i) = 0$, hence $d(x, -A_i) = 0$, hence $x \in \overline{-A_i}$, a contradiction. Likewise, since by hypothesis $\overline{A_i} \cap (-A_i) = \emptyset$, $x \in -A_i$ gives $x \in \overline{A_i}$, another contradiction. □

Remark. The converse argument also holds: if we assume the assertion of the lemma, *but only for open sets* U_i, we can recover the Borsuk–Ulam theorem. Indeed, suppose $F : \mathbb{S}^N \to \mathbb{R}^J$ is continuous, $F(-x) = -F(x)$ and $N \geq J$. Let U_i be the open set $\{x : F_i(x) > 0\}$. Then $-U_i = \{x : F_i(x) < 0\}$, so that $\overline{-U_i} \subseteq \{x : F_i(x) \leq 0\}$ and so $U_i \cap (\overline{-U_i}) = \emptyset$. By assumption there is an x which is not in any of the U_i or $-U_i$. So $F_i(x) \leq 0$ for all i and $F_i(x) \geq 0$ for all i. Hence $F_i(x) = 0$ for all i, that is, $F(x) = 0$.[7]

[7]That the assertion of Lemma 7 for open sets U_i logically implies the same statement for sets A_i with no topological restrictions can easily be seen directly from the fact that the metric space

6. Outline of the proof of Theorem 3

We now describe the scheme of the proof of Theorem 3. The function M is given, and we will be working with the class $\mathcal{P}_k^* = \mathbb{S}^N$ of normalised polynomials $p : \mathbb{R}^n \to \mathbb{R}$ of degree bounded by some $k \in \mathbb{N}$. Recall that $N \sim k^n$. For each such polynomial p, its zero set is the algebraic hypersurface $Z_p = \{x : p(x) = 0\}$, and we let

$$S(Q) = \{p \in \mathcal{P}_k^* \ : \ \mathrm{vis}_\varepsilon(Z_p \cap Q) \le M(Q)\}.$$

Following Guth [10], the aim is to show that if we take a suitable $k \sim (\sum_Q M(Q)^n)^{1/n}$, and a suitable $\varepsilon > 0$, then we can find a polynomial in $\mathcal{P}_k^*$ which is not in any of the $S(Q)$. (Note that $S(Q) = \emptyset$ for those Q such that $M(Q) \lesssim 1$.) Our method to establish this diverges somewhat from that of Guth, but there are of course many points of contact between the two lines of argument.

Let, for $r \ge 0$,

$$S^{(r)}(Q) = \{p \in \mathcal{P}_k^* \ : \ \mathrm{vis}_\varepsilon(Z_p \cap Q) \sim 2^{-r} M(Q)\}.$$

Then

$$S(Q) = \bigcup_{1 \lesssim 2^r \lesssim M(Q)} S^{(r)}(Q)$$

since $S^{(r)}(Q) = \emptyset$ for r such that $2^r \gtrsim M(Q)$.

We shall introduce a collection $\mathcal{C}$ of "colours" Θ whose cardinality is bounded by C. For each colour Θ we shall define subsets $S^{(r),\Theta}(Q)$ of $S^{(r)}(Q)$ which have the property that

$$S^{(r)}(Q) = \bigcup_{\Theta \in \mathcal{C}} S^{(r),\Theta}(Q). \tag{12}$$

For each fixed Q and r such that $1 \lesssim 2^r \lesssim M(Q)$ we will define an indexing set $\mathcal{A}_{Q,r}$ of cardinality $C 2^{-rn} M(Q)^n$, and for each $\alpha \in \mathcal{A}_{Q,r}$, a subset $S_\alpha^{(r),\Theta}(Q)$ of $S^{(r),\Theta}(Q)$ such that

$$S^{(r),\Theta}(Q) = \bigcup_{\alpha \in \mathcal{A}_{Q,r}} S_\alpha^{(r),\Theta}(Q). \tag{13}$$

To ensure that this decomposition is well-defined, it will transpire that ε must be taken to be small.

Finally we shall decompose each $S_\alpha^{(r),\Theta}(Q)$ as

$$S_\alpha^{(r),\Theta}(Q) = S_\alpha^{(r),\Theta+}(Q) \cup S_\alpha^{(r),\Theta-}(Q),$$

where

$$S_\alpha^{(r),\Theta-}(Q) = -S_\alpha^{(r),\Theta+}(Q) \tag{14}$$

in such a way that for all Q, r, Θ and α,

$$S_\alpha^{(r),\Theta+}(Q) \cap \overline{S_\alpha^{(r),\Theta-}(Q)} = \emptyset. \tag{15}$$

(The closure here refers to the natural topology of $\mathcal{P}_k^* = \mathbb{S}^N$.)

$\mathbb{S}^N$ satisfies appropriate separation axioms. More precisely, if two subsets A and B of a metric space are separated in the sense that the closure of either does not meet the other, then there are open sets U and V with the same property such that $A \subseteq U$ and $B \subseteq V$.

The reason for the introduction of colours is to ensure that there is sufficient separation between the sets $S_\alpha^{(r),\Theta\pm}(Q)$ and their antipodes for (15) to hold.

In summary then,

$$\bigcup_Q S(Q) = \bigcup_Q \bigcup_{1\lesssim 2^r \lesssim M(Q)} \bigcup_{\Theta\in\mathcal{C}} \bigcup_{\alpha\in\mathcal{A}_{Q,r}} \left(S_\alpha^{(r),\Theta+}(Q) \cup S_\alpha^{(r),\Theta-}(Q)\right), \tag{16}$$

where $S_\alpha^{(r),\Theta+}(Q)$ and $S_\alpha^{(r),\Theta-}(Q)$ satisfy (14) and (15).

Lemma 7 then implies that if the cardinality of the set indexing the union on the right hand side of (16) is less than or equal to N, then the sets in the union cannot cover $\mathbb{S}^N = \mathcal{P}_k^*$.

Now the number of terms indexing the union is at most

$$C\sum_Q \sum_{r\geq 0} \sum_{\Theta\in\mathcal{C}} \sum_{\alpha\in\mathcal{A}_{Q,r}} 1 \leq C \sum_Q \sum_{r\geq 0} 2^{-rn} M(Q)^n \leq C\sum_Q M(Q)^n.$$

So provided that $N \gtrsim \sum_Q M(Q)^n$, amongst the polynomials in $\mathbb{S}^N = \mathcal{P}_k^*$, there will exist one which is not in any of the $S(Q)$. Since $N \sim k^n$, we can therefore take k with $k \sim \left(\sum_Q M(Q)^n\right)^{1/n}$ and a $p \in \mathcal{P}_k^*$ which, for suitable $\varepsilon > 0$, satisfies $\mathrm{vis}_\varepsilon(Z_p \cap Q) > M(Q)$ for all Q, as was needed.

It remains now to define the various decompositions introduced above, and establish the assertions we have made concerning them.

7. Colours

In this section we describe how to establish (12) in such a way that the indexing set $\mathcal{C}$ has cardinality at most C.

Let $\mathcal{E}$ denote the class of centred ellipsoids in $\mathbb{R}^n$, that is images of the unit ball $\mathbb{B}$ by affine linear maps A. Each ellipsoid $A(\mathbb{B})$ is determined by an orthonormal basis of principal axes or directions given by the normalised eigenvectors of A^tA, and corresponding semiaxes, the squares of whose lengths are the eigenvalues of A^tA. Thus $\mathcal{E}$ is a manifold of dimension $n(n-1)/2 + n = n(n+1)/2$.

Let $\mathcal{K}$ denote the class of centrally symmetric convex bodies in $\mathbb{R}^n$. By the John ellipsoid theorem [11], every member K of $\mathcal{K}$ is close to some ellipsoid E in the sense that $n^{-1/4}E \subseteq K \subseteq n^{1/4}E$.

There is a natural metric (the Banach–Mazur metric) to put on the class $\mathcal{K}$, given by

$$d(K, L) = \log\inf\{\alpha \geq 1 \,:\, \alpha^{-1}K \subseteq L \subseteq \alpha K\}.$$

The John ellipsoid theorem asserts that every convex body is distant at most $(\log n)/4 \lesssim 1$ from some ellipsoid. An ellipsoid with semiaxes of lengths $2^{k_1}, \dots, 2^{k_n}$ where $k_1 + \cdots + k_n = 0$ will be distant $\lesssim \max|k_j|$ from the unit ball. Two congruent ellipses in $\mathbb{R}^2$ with semiaxes of lengths 1 and N and principal directions differing by θ will be distant $\lesssim \theta N$ apart.

To set the scene for the covering property of ellipsoids that we need, note that in $\mathbb{R}^N$, given a scale $\rho > 0$ and a pre-assigned number $\gamma > 1$, we can find a set $\mathcal{X}$ of ρ-separated points $x_i \in \mathbb{R}^N$ such that every point of $\mathbb{R}^N$ is in some $B(x_i, \rho)$, and such that $\mathcal{X}$ can be partitioned into $O_N(\gamma^N)$ families (colours), so that points of $\mathcal{X}$

of the same colour are distant at least $\gamma\rho$ from each other. This property expresses the idea that the dimensionality of $\mathbb{R}^N$ as a metric space is N. We can then assign to each $x \in \mathbb{R}^N$ one or more colours according to whether $d(x, x_i) < \rho$ for some $x_i \in \mathcal{X}$ of that particular colour.

Similarly it is not hard to verify that given $\rho > 0$ and $\gamma > 1$, there exists a ρ-separated subset $\mathcal{E}_0$ of $\mathcal{E}$ such that $\{B(E,\rho) : E \in \mathcal{E}_0\}$ covers $\mathcal{E}$ and such that we can partition $\mathcal{E}_0$ into at most $O_n(\gamma^{n(n+1)/2})$ families (colours) such that any two ellipsoids in $\mathcal{E}_0$ of the same colour are distant at least $\gamma\rho$ from each other.

Choosing $\rho = 1$ and γ sufficiently large depending only on the dimension n, and using the John ellipsoid theorem, we obtain the following:

Lemma 8. *Supose $\alpha_n > 1$ is sufficiently large. Then there exists a set $\mathcal{E}_0 \subseteq \mathcal{E}$ with the property that for every $K \in \mathcal{K}$ there is an $E \in \mathcal{E}_0$ such that*

$$\alpha_n^{-1} K \subseteq E \subseteq \alpha_n K \tag{17}$$

and such that the set $\mathcal{E}_0$ can be partitioned into at most $C = C(n, \alpha_n)$ colours in such a way that every $K \in \mathcal{K}$ satisfies (17) *for at most one $E \in \mathcal{E}_0$* of a given colour.

Given n we now fix α_n sufficiently large, and fix our palette $\mathcal{C}$ consisting of at most $C(n, \alpha_n)$ colours once and for all so that the conclusion of Lemma 8 holds. We say that two convex bodies E and K are **close** if (17) holds. So every $K \in \mathcal{K}$ is close to some member of $\mathcal{E}_0$, but there is at most one $E \in \mathcal{E}_0$ of a given colour to which it is close. For a colour $\Theta \in \mathcal{C}$ let

$$\mathcal{E}_0^\Theta = \{E \in \mathcal{E}_0 \ : \ E \text{ is of colour } \Theta\}.$$

Finally, given Q and $r \geq 0$, let

$$S^{(r),\Theta}(Q) = \{p \in S^{(r)}(Q) \ : \ K_\varepsilon(Z_p \cap Q) \text{ is close to a member of } \mathcal{E}_0^\Theta\};$$

then we have

$$S^{(r)}(Q) = \bigcup_{\Theta \in \mathcal{C}} S^{(r),\Theta}(Q),$$

and (12) is established.

8. Translates

We now fix Q, $r \geq 0$ and a colour $\Theta \in \mathcal{C}$. In this section we establish (13) for suitable subsets $S_\alpha^{(r),\Theta}(Q) \subseteq S^{(r),\Theta}(Q)$ which are indexed by $\alpha \in \mathcal{A}_{Q,r}$, where $\mathcal{A}_{Q,r}$ has cardinality $\sim 2^{-rn} M(Q)^n$. We can assume that $S^{(r),\Theta}(Q) \neq \emptyset$.

If $p \in S^{(r),\Theta}(Q)$, the convex body $K_\varepsilon(Z_p \cap Q) \subseteq \mathbb{B}$ has volume $\sim 2^{rn}/M(Q)^n \lesssim 1$, and it is close to a unique member $E(p)$ of $\mathcal{E}_0^\Theta$ of comparable volume. Hence we can fit $\sim 2^{-rn} M(Q)^n$ disjoint parallel translates of $E(p)$ inside Q, with the translations along the principal directions of $E(p)$. Likewise, if $\eta < 1$ is a numerical scaling factor, we can fit $\sim \eta^{-n} 2^{-rn} M(Q)^n$ disjoint parallel translates of $\eta E(p)$ inside Q, with the translations again along the principal directions of $E(p)$. Indeed, if the lengths of the semiaxes of $E(p)$ are $l_1, \ldots, l_n \leq c$, and the principal directions are $e_1, \ldots, e_n$, we can place the centres of the translated copies of $\eta E(p)$ at the points

$x_Q + \eta \sum_j m_j l_j e_j$ for $m_j \in 2\mathbb{Z}$ and $|m_j| \leq c\eta^{-1} l_j^{-1}$; here x_Q is the centre of Q. In this construction the number of translated copies equals the product

$$c\eta^{-n}(l_1 \dots l_n)^{-1} = c\eta^{-n} 2^{-rn} M(Q)^n. \tag{18}$$

Lemma 9. *There is a dimensional constant C_n such that if $p \in S^{(r),\Theta}(Q)$ and $\eta < 1$, then Z_p bisects at most $C_n \eta^{-(n-1)} 2^{-rn} M(Q)^n$ disjoint translates of $\eta E(p)$ in Q.*

Proof. Suppose that $E(p)$ has principal directions $\{e_j\}$ and corresponding semiaxes with lengths $\{l_j\}$. If Z_p bisects a translate $\eta E(p) + \xi$ of $\eta E(p)$, then for at least one j we will have[8]

$$\operatorname{surf}_{e_j}(Z_p \cap (\eta E(p) + \xi)) \geq C_n \operatorname{vol}(\eta E(p) + \xi)/\eta l_j = C_n \eta^{n-1} \operatorname{vol} E(p)/l_j.$$

This is just the affine-invariant formulation of the fact that a hypersurface which bisects the unit ball must have large $(n-1)$-dimensional Hausdorff measure inside the ball – see Lemma 11 in the Appendix. So

$$\sum_{j=1}^n l_j \operatorname{surf}_{e_j}(Z_p \cap (\eta E(p) + \xi)) \geq C_n \eta^{n-1} \operatorname{vol} E(p).$$

If now Z_p bisects as many as $A 2^{-rn} M(Q)^n$ disjoint copies of $\eta E(p)$ in Q, we will have

$$\sum_{j=1}^n l_j \operatorname{surf}_{e_j}(Z_p \cap Q) \geq C_n A \eta^{n-1} \operatorname{vol} E(p)\ 2^{-rn} M(Q)^n.$$

If p' is another polynomial in $\mathbb{S}^N$ sufficiently close to p, and if p bisects an ellipsoid E, we can conclude that p' approximately bisects E in the sense that p' is positive on at least 40% of E and negative on at least 40% of E. Since we are only interested in a finite number of ellipsoids here, namely the translates $\eta E + \xi$ for $E \in \mathcal{E}_0$ with $M(Q)^{-n} \lesssim \operatorname{vol} E \lesssim 1$ for the relevant Q, then by choosing ε small enough we will have this approximate bisection property for all polynomials which affect the value of $\operatorname{surf}_{e,\varepsilon}$ in the expressions above. Therefore we have the estimate

$$\sum_{j=1}^n l_j \operatorname{surf}_{e_j,\varepsilon}(Z_p \cap Q) \geq C_n A \eta^{n-1} \operatorname{vol} E(p)\ 2^{-rn} M(Q)^n.$$

The definition (cf. (6)) of $K_\varepsilon(Z_p \cap Q)$ together with the fact that $K_\varepsilon(Z_p \cap Q)$ and $E(p)$ are close implies that

$$l_j \operatorname{surf}_{e_j,\varepsilon}(Z_p \cap Q) \leq C_n$$

for each j, and moreover, as $p \in S^{(r),\Theta}(Q)$,

$$\operatorname{vol} E(p) \sim 2^{rn}/M(Q)^n.$$

Therefore A must satisfy $A \leq C_n \eta^{-(n-1)}$. □

[8] Here and in the next two displayed equations we are using the *unmollified* notion of directional surface area surf_e.

Choose η sufficiently small so that $C_n \eta < c$. Here C_n is the constant from Lemma 9 and c is the constant in (18). Then for each $p \in S^{(r),\Theta}(Q)$, Z_p can bisect only a proportion $C_n\eta/c < 1$ of the $c\eta^{-n}2^{-rn}M(Q)^n$ disjoint copies of $\eta E(p)$ in Q. In particular, Z_p will *not* bisect *all* of the available disjoint copies of $\eta E(p)$ in Q.

For each $E(p) \in \mathcal{E}_0^{\Theta}$ of volume $\sim 2^{rn}/M(Q)^n$, the set of translated ellipsoids which are placed into Q in the construction above is of cardinality $c2^{-rn}M(Q)^n$. We label these ellipsoids with an index α from $\mathcal{A}_{Q,r} = \{1, \dots, c2^{-rn}M(Q)^n\}$. Now take a polynomial $p \in S_\alpha^{(r),\Theta}(Q)$. Then p is close to a unique member $E(p) \in \mathcal{E}_0^{\Theta}$ of volume comparable to $2^{rn}/M(Q)^n$ and we can ask whether or not p bisects the α'th translate of $\eta E(p)$. Note that for this question to be meaningful the uniqueness in the previous sentence is important and if we had not already restricted attention to a single colour then the uniqueness would not hold.

For $\alpha \in \mathcal{A}_{Q,r}$ we can thus define

$S_\alpha^{(r),\Theta}(Q) := S^{(r),\Theta}(Q) \cap \{Z_p \text{ does not bisect the } \alpha\text{'th translate of } \eta E(p) \text{ in } Q\}$

Then, since Z_p cannot bisect all of the translates of $\eta E(p)$ in Q, we have

$$S^{(r),\Theta}(Q) = \bigcup_{\alpha \in \mathcal{A}_{Q,r}} S_\alpha^{(r),\Theta}(Q)$$

as required, where there are $c2^{-rn}M(Q)^n$ terms in the indexing set $\mathcal{A}_{Q,r}$. Thus (13) is established.

9. Antipodes

In this section we establish (15).

Fix Q, $r \geq 0$, $\Theta \in \mathcal{C}$ and $\alpha \in \mathcal{A}_{Q,r}$. For $p \in S_\alpha^{(r),\Theta}(Q)$ let $E(p)$ be as before the unique member of $\mathcal{E}_0^{\Theta}$ to which $K_\varepsilon(Z_p \cap Q)$ is close. Let $E_\alpha = E_\alpha(p)$ denote the α'th translate of $\eta E(p)$ in Q. Since Z_p does not bisect E_α, we have either

$$\text{vol}\,(\{p > 0\} \cap E_\alpha) > \text{vol}(\{p < 0\} \cap E_\alpha)$$

(in which case we say that $p \in S_\alpha^{(r),\Theta+}(Q)$), or

$$\text{vol}\,(\{p > 0\} \cap E_\alpha) < \text{vol}(\{p < 0\} \cap E_\alpha),$$

(in which case we say that $p \in S_\alpha^{(r),\Theta-}(Q)$).

Then

$$S_\alpha^{(r),\Theta}(Q) = S_\alpha^{(r),\Theta+}(Q) \cup S_\alpha^{(r),\Theta-}(Q).$$

Moreover

$$S_\alpha^{(r),\Theta-}(Q) = -S_\alpha^{(r),\Theta+}(Q),$$

and so to establish (15) we wish to show that for all α,

$$S_\alpha^{(r),\Theta+}(Q) \cap \overline{S_\alpha^{(r),\Theta-}(Q)} = \emptyset.$$

To see this, suppose for a contradiction that for some $\alpha \in \mathcal{A}_{Q,r}$ there is a $p \in S_\alpha^{(r),\Theta+}(Q)$ and a sequence of $p_m \in S_\alpha^{(r),\Theta-}(Q)$ which converges to p in $\mathbb{S}^N$. That is, we suppose that

$$\text{vol}\,(\{p > 0\} \cap E_\alpha(p)) > \text{vol}(\{p < 0\} \cap E_\alpha(p)) \tag{19}$$

and

$$\mathrm{vol}\,(\{p_m > 0\} \,\cap\, E_\alpha(p_m)) < \,\mathrm{vol}(\{p_m < 0\} \,\cap\, E_\alpha(p_m))$$

where p_m converges to p in $\mathbb{S}^N$.

Lemma 10. *Fix Q, r and Θ. Suppose that $p \in S^{(r),\Theta}(Q)$, $p_m \in S^{(r),\Theta}(Q)$ for $m \in \mathbb{N}$ and that p_m converges to p in $\mathbb{S}^N$. Then for all sufficiently large m we have $E(p_m) = E(p)$. If $\alpha \in \mathcal{A}_{Q,r}$ and in addition $p, p_m \in S_\alpha^{(r),\Theta}(Q)$, then for m sufficiently large, $E_\alpha(p_m) = E_\alpha(p)$.*

Proof. Since we are using the mollified version of the directional surface area and quantities defined in terms of it, the convergence of p_m to p in $\mathbb{S}^N$ implies that the convex bodies $K_\varepsilon(Z_{p_m} \cap Q)$ converge to $K_\varepsilon(Z_p \cap Q)$ as $m \to \infty$[9] and in particular $K_\varepsilon(Z_{p_m} \cap Q)$ and $K_\varepsilon(Z_p \cap Q)$ are close for m sufficiently large. Since p and p_m are members of $S^{(r),\Theta}(Q)$ then $K_\varepsilon(Z_{p_m} \cap Q)$ and $K_\varepsilon(Z_p \cap Q)$ must be close to *some* member of $\mathcal{E}_0^\Theta$ and thus, for m sufficiently large, they are close to the *same* member of $\mathcal{E}_0^\Theta$, which must be $E(p)$. In particular, for m sufficiently large, we have $E(p_m) = E(p)$ and consquently $E_\alpha(p_m) = E_\alpha(p)$ for all α. $\square$

(It is at the end of the proof of this lemma, and in the construction of the sets $S_\alpha^{(r),\Theta}(Q)$, where the relevance of the earlier decomposition into colours becomes clear.)

So, for m sufficiently large we have

$$\mathrm{vol}\,(\{p_m > 0\} \,\cap\, E_\alpha(p)) < \,\mathrm{vol}(\{p_m < 0\} \,\cap\, E_\alpha(p))$$

which, upon taking limits and using the fact that $\mathrm{vol}\,(\{p = 0\}) = 0$ as p is non-zero, implies

$$\mathrm{vol}\,(\{p > 0\} \,\cap\, E_\alpha(p)) \leq \,\mathrm{vol}(\{p < 0\} \,\cap\, E_\alpha(p)),$$

which is in contradiction with (19). Hence

$$S_\alpha^{(r),\Theta+}(Q) \cap \overline{S_\alpha^{(r),\Theta-}(Q)} = \emptyset,$$

and we are therefore finished.

10. Appendix – Bisecting balls

In this appendix we indicate a simple proof of the (geometrically obvious) fact that a hypersurface which bisects the unit ball must have large surface area inside the ball. Let $\mathbb{B}$ be the closed unit ball in $\mathbb{R}^n$ and suppose $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial. Let

$$E = \{x \in \mathbb{B} \,:\, p(x) \leq 0\}$$

and

$$F = \{x \in \mathbb{B} \,:\, p(x) \geq 0\}.$$

Lemma 11. *If $\mathrm{vol}\,(E) = a\,\mathrm{vol}\,(\mathbb{B})$ and $\mathrm{vol}\,(F) = b\,\mathrm{vol}\,(\mathbb{B})$ where $a + b = 1$, then*

$$\mathcal{H}_{n-1}(\{x \in \mathbb{B} \,:\, p(x) = 0\}) > \frac{1}{2}\left(a^{(n-1)/n} + b^{(n-1)/n} - 1\right)\mathcal{H}_{n-1}(\mathbb{S}^{n-1}).$$

[9]in the sense that there is a sequence $\gamma_m \geq 1$ with $\gamma_m \to 1$ such that $\gamma_m^{-1} K_\varepsilon(Z_{p_m} \cap Q) \subseteq K_\varepsilon(Z_p \cap Q) \subseteq \gamma_m K_\varepsilon(Z_{p_m} \cap Q)$

Proof. It is easy to see that $\partial E \cup \partial F = \mathbb{S}^{n-1} \cup (E \cap F)$ and $\partial E \cap \partial F = E \cap F$. Since

$$\mathcal{H}_{n-1}(\partial E \cup \partial F) = \mathcal{H}_{n-1}(\partial E) + \mathcal{H}_{n-1}(\partial F) - \mathcal{H}_{n-1}(\partial E \cap \partial F)$$

and

$$\mathcal{H}_{n-1}(\mathbb{S}^{n-1} \cup (E \cap F)) = \mathcal{H}_{n-1}(\mathbb{S}^{n-1}) + \mathcal{H}_{n-1}(E \cap F) - \mathcal{H}_{n-1}(\mathbb{S}^{n-1} \cap E \cap F)$$

we have

$$2\mathcal{H}_{n-1}(E \cap F) = \mathcal{H}_{n-1}(\partial E) + \mathcal{H}_{n-1}(\partial F) - \mathcal{H}_{n-1}(\mathbb{S}^{n-1}) + \mathcal{H}_{n-1}(\mathbb{S}^{n-1} \cap E \cap F),$$

and so

$$\mathcal{H}_{n-1}(E \cap F) \geq \frac{1}{2}\left(\mathcal{H}_{n-1}(\partial E) + \mathcal{H}_{n-1}(\partial F) - \mathcal{H}_{n-1}(\mathbb{S}^{n-1})\right).$$

By the isoperimetric inequality we have

$$\mathcal{H}_{n-1}(\partial E) \geq a^{(n-1)/n}\mathcal{H}_{n-1}(\mathbb{S}^{n-1})$$

and

$$\mathcal{H}_{n-1}(\partial F) \geq b^{(n-1)/n}\mathcal{H}_{n-1}(\mathbb{S}^{n-1}),$$

(with strict inequality in at least one place) so that

$$\mathcal{H}_{n-1}(E \cap F) > \frac{1}{2}\left(a^{(n-1)/n} + b^{(n-1)/n} - 1\right)\mathcal{H}_{n-1}(\mathbb{S}^{n-1})$$

as required. □

Since for $n \geq 2$ and $0 < a, b < 1$ with $a + b = 1$ we have $(a + b)^{(n-1)/n} < a^{(n-1)/n} + b^{(n-1)/n}$, this establishes the desired bound.

The following question may be of interest. Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body, which we can normalise so that its John ellipsoid is the unit ball. Within the class of polynomial hypersurfaces Z which cut K in the proportions $a : b$, how do we minimise the surface area of $Z \cap K$?

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