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We determine the full post-Newtonian limit of theories of gravity that extend general relativity by replacing the Ricci scalar, R, in the generating Lagrangian by some analytic function, $f(R)$. We restrict ourselves to theories that admit Minkowski space as a suitable background, and perform a perturbative expansion in the manner prescribed by the parameterised post-Newtonian formalism. Extra potentials are found to be present that are not accounted for in the usual treatment, and a discussion is provided on how they may be used to observationally distinguished these theories from general relativity at the post-Newtonian level.

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I. INTRODUCTION

There exists an extensive literature on relativistic theories that generalise Einstein's theory of general relativity (GR), and that reduce to GR in the appropriate limits. A particularly appealing class of these generalisations are the fourth-order theories. These are theories derived from a Lagrangian density that is a scalar function of contractions of the Riemann tensor only. Considerations of fourth-order theories of gravity have a long history, having been first considered by Eddington in as early as the 1920's [1]. One frequently considered generalisation is to replace the Ricci curvature scalar, R, in the Einstein-Hilbert action with some analytic function, $f(R)$ (see e.g. [2, 3, 4, 5]). It is the post-Newtonian limit of such theories that we will be interested in here.

There are a variety of reasons why one may wish to consider these generalised theories. Strong motivation comes the from their renormalization properties in the presence of matter fields [6]. Other motivation can be found from cosmological considerations where it has been found that generalising the Einstein-Hilbert action can be of use for better understanding the late-time acceleration of the universe [7, 8], early universe inflation [9, 10, 11] and the nature of the initial singularity [12, 13, 14, 15]. Whatever the motivation for considering generalised fourth-order theories, it is essential that we sufficiently understand their weak-field limit, and that they conform with the ever increasing body of observational data.

The usual frame-work for considering the weak-field effects of modified theories is the parameterised post-Newtonian (PPN) formalism. The first steps towards this formalism were again made by Eddington [1], together with Robertson [16] and Schiff [17], who treated the planets as test bodies moving in the gravitational field of the sun. In their formalism they introduced the test metric

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{2\beta m^2}{r^2}\right) dt^2 + \left(1 + \frac{2\gamma m}{r}\right) d\mathbf{x^2}$$

where $d\mathbf{x^2}$ is the three dimensional Euclidean line-element, r is a radial coordinate and m is the mass of the central gravitating object. The parameters β and γ are to be determined by experiment. By constraining them it is possible to verify, or potentially disprove, GR and its alternatives. For example, GR predicts $\beta = \gamma = 1$, whilst Brans-Dicke theory [18] predicts $\beta = 1$ and $\gamma = (1+\omega)/(2+\omega)$, where ω is the Brans-Dicke coupling constant. Using radio communications from the Cassini spacecraft the constraint $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$ has been obtained [19]. This is evidently in good agreement with GR, and can be used to constrain the Brans-Dicke parameter to be $\omega \gtrsim 40\,000$, to 2σ.

Our goal here is to calculate the post-Newtonian limit of fourth-order theories, so that they can be the subject of observation in a similar manner. To achieve this we will consider them in the context of the full PPN formalism. This formalism is a generalisation of the Eddington, Robertson, Schiff parameterisation outlined above, and allows for more general configurations of matter fields than a single point-like gravitating source. The PPN approach was developed primarily by Nordvedt and Will [20], and is explained in detail in [21]. We will give a brief explanation of the PPN approach below, in so far as will be required for coherence of this work. For a more complete exposition the reader is referred to [21].

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Absent in previous studies is a rigorous exemplification of how fourth-order theories fit into the PPN formalism. Here we will remedy this by extending the PPN formalism to include fourth-order $f(R)$ theories. In doing so we will attempt to maintain, to the highest degree possible, the principles and spirit of the PPN formalism, as expounded in [21]. Previous attempts have been made in this direction by Capozziello and Troisi [22] and Olmo [23]. These authors attempt to derive the post-Newtonian limit of $f(R)$ theories by appealing to their equivalence with scalar-tensor theories [24]. Here we work directly with the fourth-order theory, and by direct integration of the field equations find results that explicity state their post-Newtonian limit. In doing so we restrict our attention to the subset of theories that admit Minkowski space as a suitable background, and find that it is necessary to introduce a number of new post-Newtonian potentials. Theories with other backgrounds will be investigated elsewhere. We note that the post-Newtonian limit of Gauss-Bonnet gravity has been found by Sotiriou and Barausse [25].

In section II we introduce theories that are derivable from a Lagrangian density of the form $\mathcal{L} = f(R)$. The field equations are derived, and a discussion is given of the condition that Minkowski space be an appropriate background. Section III gives a brief introduction to the PPN formalism, in as much as is required for the self-consistency of this article. In sections IV and V we find the Newtonian and post-Newtonian limits, respectively. These calculations are performed in the presence of a perfect fluid, and result in many new potentials that are not usually present. In section VI the results found in the previous two sections are transformed into the standard post-Newtonian gauge, in which the spatial part of the metric is diagonal and derivatives of quantities associated with the matter fields are removed. Section VII gives a discussion of the results obtained, and in particular gives the relevant post-Newtonian limit if the usual Newtonian potential is to dominate at the Newtonian level of approximation. In section VIII we conclude, and indicate the principle ways in which the $f(R)$ theories we are considering may be observationally distinguishable from GR at the post-Newtonian level. The appendices give some details of the more lengthy calculations.

II. FOURTH-ORDER THEORIES

The Lagrangian density for the theories we will be considering is

$$\mathcal{L} = f(R). \tag{1}$$

The action associated with this density is then given by

$$S = \int \sqrt{-g}\mathcal{L} + S_m, \tag{2}$$

where S_m denotes the action associated with the matter fields. Extremizing this action with respect to the metric results in the field equations

$$f'R_{\mu\nu} - \frac{1}{2}f g_{\mu\nu} + f'_{;}{}^{\sigma\rho}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho}) = 8\pi T_{\mu\nu} \tag{3}$$

where primes denote differentiation with respect to R and $T_{\mu\nu}$ is the energy-momentum tensor, defined in terms of S_m and $g_{\mu\nu}$ in the usual way. Throughout we us Greek letters to run over all space-time indices and choose units so that $c = G = 1$. The Lagrangian formulation of these theories guarantees the covariant conservation of energy-momentum.

In order to define a perturbative expansion we must first decide the appropriate background to expand about. In the usual PPN treatment this background is taken to be Minkowski space, and the metric is then expanded as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4}$$

where $h_{\mu\nu} \ll 1$. Such an expansion is well motivated in GR where it is known that in the absence of any matter fields Birkhoff's theorem ensures staticity of spherically symmetric space-times, and that these space-times will be asymptotically flat. Adding small amounts of matter to such a background is then well modelled by the perturbative expansion given by (4), and one need only be concerned with the effects of matching this region to a suitable cosmological solution at large distances.

More care is required with fourth-order theories, where staticity and asymptotic flatness cannot be so easily assumed. Birkhoff's theorem is not valid in these theories, and so staticity, if it is required, must be imposed as an extra condition on spherically symmetric vacua. However, the imposition of this extra symmetry does *not*, in general, result in Minkowski space as a suitable background. It was shown explicitly in [26] that if any single power of the gravitational Lagrangian dominates, other than the Einstein-Hilbert one, then spherically symmetric vacuum space-times are not asymptotically attracted to Minkowski form. Instead, the line-element asymptotically approaches

$$ds^2 \to -r^n dt^2 + dr^2 + r^2 d\Omega^2$$

as $r \to \infty$, where $n = 0$ only if the Einstein-Hilbert term dominates. Such asymptotic behaviour can be readily shown to be incompatible with observations, unless n is very small [26].

An alternative prescription to the imposition of a static background was investigated in [27]. Here the method of imposing time independence, as outlined above, was contrasted with the method of assuming asymptotic homogeneity and isotropy of the background vacuum. The assumption of homogeneity and isotropy removes the need for r dependence in the asymptotic form of the metric, but at the expense of introducing time dependence. It was shown in [27], using exact solutions as well as perturbative expansions, that the choice of symmetries for the background has a demonstrable effect on the weak-field expansions that are performed within them.

In short, one must make a choice of background to expand about, and this choice can have important consequences for the expansion itself. If any single power of the gravitational Lagrangian other than the Einstein-Hilbert one dominates at asymptotically large distances, then Minkowski space may not be an appropriate, stable choice of background. One is then forced to recant either the time independence or the homogeneity of the background metric, with non-trivial consequences.

Here we will avoid these difficulties by considering only theories in which the Einstein-Hilbert term dominates in the low curvature regime. We are then justified in performing a perturbative expansion about Minkowski space, which allows a more direct comparison with the usual PPN approach. The post-Newtonian limit of other $f(R)$ theories will be dealt with in a future study. Fourth-order theories that admit a Minkowski background, and are an analytic functions of R, can then be written as

$$f(R) = \sum_{i=1}^{\infty} c_i R^i \tag{5}$$

where the c_i are a set of real, positive valued constants. The post-Newtonian analysis that is to follow will show how the c_i are manifest in the weak-field limit, and hence how they can be potentially observed with gravitational experiments and observations.

III. THE PPN APPROACH

This section is a recapitulation of the PPN formalism, as propounded in [21], and as is necessary for coherence of this article. The PPN formalism is a perturbative treatment of weak-field gravity. Such an expansion requires a small parameter to expand in. An "order of smallness" is therefore defined by

$$U \sim v^2 \sim \frac{p}{\rho} \sim \Pi \sim O(2)$$

where U is the Newtonian potential, v is the velocity a fluid element, p is the pressure of the fluid, ρ is its rest-mass density and Π is the ratio of energy density to rest-mass density. Time derivatives are also taken to have an order of smallness associated with them, relative to spatial derivatives:

$$\frac{|\partial/\partial t|}{|\partial/\partial x|} \sim O(1).$$

(Recall that we have chosen to set $c = 1$). The PPN formalism now proceeds as an expansion in this order of smallness.

The equations of motion show that for time-like particles propagating along geodesics the level of approximation required to recover the Newtonian limit is g_{00} to $O(2)$, with no other knowledge of the metric components beyond the background level being necessary. The post-Newtonian limit for time-like particles requires a knowledge of

$$\begin{array}{lll} g_{00} & \text{to} & O(4) \\ g_{0i} & \text{to} & O(3) \\ g_{ij} & \text{to} & O(2). \end{array}$$

Latin letters are used to denote spatial indices. To obtain the Newtonian limit of null particles we only need to know the metric to background order: Light follows straight lines, to Newtonian accuracy. The post-Newtonian limit of null particles requires a knowledge of g_{00} and g_{ij} both to $O(2)$.

Using the expansion (4) we can now calculate the Ricci and energy-momentum tensors to the appropriate orders. However, before doing so it is worth recognising that we have four gauge freedoms, associated with four coordinate

choices. Specifying the four gauge conditions

$$h_{i0,i} = \frac{1}{2}h_{ii,0} + O(5) \tag{6}$$

$$h_{ij,j} = \frac{1}{2}h_{jj,i} - \frac{1}{2}h_{00,j} + O(4) \tag{7}$$

allows the components of the Ricci tensor to be written

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00} - \frac{1}{2}|\nabla h_{00}|^2 + \frac{1}{2}h_{jk}h_{00,jk} + O(6) \tag{8}$$

$$R_{0i} = -\frac{1}{2}\nabla^2 h_{0i} - \frac{1}{4}h_{00,i0} + O(5) \tag{9}$$

$$R_{ij} = -\frac{1}{2}\nabla^2 h_{ij} + O(4) \tag{10}$$

where $\nabla^2 = \partial^i\partial_i$ is the Laplacian on three dimensional Euclidean space. We still have the freedom to make gauge transformations of the form

$$x^\mu \to x^\mu + \xi^\mu$$

and we will use this freedom in the following analysis to transform to a "standard post-Newtonian gauge" in which the spatial part of the metric is diagonal, and terms containing time derivatives are removed. The components of the stress-energy tensor, to the relevant order, are

$$T_{00} = \rho(1 + \Pi + v^2 - h_{00}) \tag{11}$$

$$T_{0i} = -\rho v_i \tag{12}$$

$$T_{ij} = \rho v_i v_j + p\delta_{ij}. \tag{13}$$

We can now substitute these expressions for $R_{\mu\nu}$ and $T_{\mu\nu}$ into the field equations (3), together with (5), and solve the equations order by order in perturbations. Transforming to an appropriate gauge will then yield the PPN limit of these fourth-order theories.

IV. THE NEWTONIAN LIMIT

The Newtonian limit of these theories will now be investigated. This limit has been found before for a point-like mass at the origin, originally by [28], and again several times since. Here we find the general solution for a space-time containing a perfect fluid.

Beginning with the trace of the field equations (3), we have to $O(2)$

$$\nabla^2 R^{(2)} - \frac{c_1}{6c_2}R^{(2)} = -\frac{4\pi}{3c_2}\rho \tag{14}$$

where $R^{(2)}$ denotes the Ricci scalar to $O(2)$. This is an inhomogeneous Helmholtz equation which has the solution

$$R^{(2)} = \frac{1}{3c_2}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{15}$$

We have ignored here the other possible root in the exponential, which is an equally valid solution of the Helmholtz equation, but does not give an appropriate limit at asymptotically large distances. It can be seen that (15) is an exponentially decaying Yukawa potential if c_1 and c_2 have the same signs, and is a damped oscillatory function if they have opposite signs.

The $0-0$ and trace field equations, given by (3), can now be written to $O(2)$ as

$$\nabla^2\left(\frac{1}{4}c_1 h_{00}^{(2)} + \frac{1}{4}c_1 h_{ii}^{(2)} + 2c_2 R^{(2)}\right) = -8\pi\rho.$$

and

$$\nabla^2\left(h_{ii}^{(2)} + 5h_{00}^{(2)}\right) = -\frac{64\pi}{c_1}\rho$$

where the $O(2)$ parts of (8) and (11) have been used. Solving these two inhomogeneous Poisson equations simultaneously we find

$$h_{00}^{(2)} = \frac{2}{c_1}(U + c_2 R^{(2)}) \tag{16}$$

where

$$U \equiv \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{17}$$

is the Newtonian potential. From (16) it can be seen that the Ricci scalar itself acts as a Newtonian level potential.

V. THE POST-NEWTONIAN LIMIT

In this section we investigate the Post-Newtonian limit of the these theories. Expressions for h_{ij} to $O(2)$, h_{0i} to $O(3)$ and h_{00} to $O(4)$ are found. These expressions will not be in the standard post-Newtonian gauge, discussed above. We perform a transformation into this gauge in the subsequent section.

A. The h_{ij} terms

We begin by evaluating the terms h_{ij} to $O(2)$. These quantities, together with (16) above, are sufficient to determine the post-Newtonian limit of null geodesics.

The $i-j$ field equation (3) can now be written

$$\nabla^2 \left(\frac{c_1}{2} h_{ij}^{(2)} + c_2 \delta_{ij} R^{(2)} + 12 \frac{c_2^2}{c_1} R_{,ij}^{(2)} - 4 \frac{c_2}{c_1} U_{,ij} \right) = -4\pi\rho\delta_{ij}$$

where we have made use of the expressions above for R_{ij} and T_{ij}, (10) and (13), the trace equation (14) and the definition of U, (17). This equation can be integrated to give

$$h_{ij}^{(2)} = \frac{2}{c_1} \left(U\delta_{ij} - c_2 \delta_{ij} R^{(2)} - 12 \frac{c_2^2}{c_1} R_{,ij}^{(2)} + 4 \frac{c_2}{c_1} U_{,ij} \right). \tag{18}$$

Equation (18) is not diagonal, and so is not in the standard post-Newtonian gauge. In the next section will remove the off-diagonal components with the appropriate transformation.

B. The h_{0i} terms

As discussed above, the first non-zero contribution to the h_{0i} terms is at $O(3)$. The $0-i$ field equation (3) can now be written

$$\nabla^2 \left(c_1 h_{0i}^{(3)} + 30 \frac{c_2^2}{c_1} R_{,0i}^{(2)} - 10 \frac{c_2}{c_1} U_{,0i} - \frac{1}{2} V_i + \frac{1}{2} W_i \right) = 16\pi\rho v_i$$

where we have used the expressions for R_{0i} and T_{0i}, (9) and (12), the expression for h_{00} to $O(2)$, (16), the trace equation, (14), and the two new potentials V_i and W_i, that are defined as

$$V_i \equiv \int \frac{\rho(\mathbf{x}') v_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{19}$$

$$W_i \equiv \int \frac{\rho(\mathbf{x}')(\mathbf{v}(\mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}'))(x - x')_i}{|\mathbf{x} - \mathbf{x}'|^3} d^3x' \tag{20}$$

as in the usual PPN treatment, so that $\nabla^2(W_i - V_i) = 2U_{,0i}$. Use has also been made of the conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{21}$$

Integrating the field equation above we now have

$$h_{0i}^{(3)} = -\frac{7}{2c_1} V_i - \frac{1}{2c_1} W_i + 10 \frac{c_2}{c_1^2} U_{,0i} - 30 \frac{c_2^2}{c_1^2} R_{,0i}^{(2)} \tag{22}$$

which, again, will be subject to a gauge transformation in the next section.

C. The h_{00} term to $O(4)$

At this order of perturbation the equations become more unsightly, and so we choose to relegate the majority of them to appendices, stating here only the results. Hence, in the gauge specified by (6) and (7), the h_{00} term to $O(4)$ is

$$\begin{aligned}
h_{00}^{(4)} =& -\frac{2}{c_1^2}U^2 + 2\frac{c_2^2}{c_1^2}R^2 - \frac{16c_2}{3c_1^2}UR - 36\frac{c_2^2}{c_1^2}R_{,00} + 12\frac{c_2}{c_1^2}U_{,00} + 8\frac{c_2}{c_1^3}|\nabla U|^2 - 24\frac{c_2^3}{c_1^3}|\nabla R|^2 - 16\frac{c_2^2}{c_1^3}\nabla U\cdot\nabla R \\
&- \frac{7}{18\pi c_1}\mathcal{V}(UR) + \frac{3c_2}{4\pi c_1}\mathcal{V}(R^2) + \frac{64}{9c_1^2}\mathcal{V}(\rho U) - \frac{44c_2}{3c_1^2}\mathcal{V}(\rho R) - \frac{40c_2}{3c_1^3}\mathcal{V}(\nabla\rho\cdot\nabla U) + \frac{40c_2^2}{c_1^3}\mathcal{V}(\nabla\rho\cdot\nabla R) \\
&+ \frac{2}{c_1}\mathcal{V}(\rho\Pi) + \frac{4}{c_1}\mathcal{V}(\rho v^2) + \frac{6}{c_1}\mathcal{V}(p) - \frac{1}{4\pi}\left(\frac{c_2}{c_1} - \frac{c_3}{2c_2}\right)X(R^2) + \frac{1}{6\pi c_1}X(UR) - \frac{4}{3c_1^2}X(\rho U) + \frac{8c_2}{3c_1^2}X(\rho R) \\
&+ \frac{8c_2}{3c_1^3}X(\nabla\rho\cdot\nabla U) - \frac{8c_2^2}{c_1^3}X(\nabla\rho\cdot\nabla R) - \frac{2}{c_1}X(p) + \frac{2}{3c_1}X(\rho\Pi) - \sqrt{\frac{2c_2}{3c_1^3}}\hat{\chi}_{,00}.
\end{aligned} \tag{23}$$

The derivation of this result can be found in appendix A, where it is given there as equations (A3) and (A4). The new potentials $\mathcal{V}$, X and $\hat{\chi}$ are defined as

$$\mathcal{V}(Q) \equiv \int \frac{Q'}{|\mathbf{x}-\mathbf{x}'|}dx'^3 \tag{24}$$

$$X(Q) \equiv \int \frac{Q' e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{25}$$

$$\hat{\chi} \equiv \int \rho' e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{26}$$

Primes here label quantities that are functions of $\mathbf{x}'$. It should be noted that this definition of $\mathcal{V}$ is degenerate with some of the usual PPN parameters: For example, $\mathcal{V}(\rho v^2)$ is identical to the potential Φ_2 of [21]. We use this definition of $\mathcal{V}$ as it is convenient for expressing the new potentials.

Again, this result is not in the standard post-Newtonian gauge. In the following section we will transform it so that the terms proportional to $R_{,00}$, $U_{,00}$, $|\nabla U|^2$, $|\nabla R|^2$, $\nabla U\cdot\nabla R$ and $\hat{\chi}_{,00}$ are eliminated.

VI. GAUGE TRANSFORMING

In the preceding section we used the gauge specified by conditions (6) and (7). This has been a convenient choice, and has allowed integration of the field equations to post-Newtonian accuracy. However, it is desirable to transform the results found above to a gauge in which the spatial part of the metric is diagonal, and in which the metric takes it simplest form. By making the coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ the metric is transformed in such a way that

$$h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu} + O(\xi^2).$$

Then by making the choices

$$\begin{aligned}
\xi_0 &= 6\frac{c_2}{c_1^2}U_{,0} - 18\frac{c_2^2}{c_1^2}R_{,0} - \sqrt{\frac{c_2}{6c_1^3}}\hat{\chi}_{,0} \\
\xi_i &= 4\frac{c_2}{c_1^2}U_{,i} - 12\frac{c_2^2}{c_1^2}R_{,i}
\end{aligned}$$

the metric perturbations transform as

$$\begin{aligned}
h_{ij}^{(2)} &\to h_{ij}^{(2)} + 24\frac{c_2^2}{c_1^2}R_{,ij} - 8\frac{c_2}{c_1^2}U_{,ij} \\
h_{0i}^{(3)} &\to h_{0i}^{(3)} - 10\frac{c_2}{c_1^2}U_{,0i} + 30\frac{c_2^2}{c_1^2}R_{,0i} + \sqrt{\frac{c_2}{6c_1^3}}\hat{\chi}_{,0i} \\
h_{00}^{(4)} &\to h_{00}^{(4)} - 12\frac{c_2}{c_1^2}U_{,00} + 36\frac{c_2^2}{c_1^2}R_{,00} + \sqrt{\frac{2c_2}{3c_1^3}}\hat{\chi}_{,00} - 8\frac{c_2}{c_1^3}|\nabla U|^2 + 24\frac{c_2^3}{c_1^3}|\nabla R|^2 + 16\frac{c_2^2}{c_1^3}\nabla U\cdot\nabla R
\end{aligned}$$

whilst $h_{00}^{(2)}$ is unchanged. These transformations are exactly what is required to diagonalize the spatial part of the metric, and to remove unwanted terms from the other metric components. The final form of the perturbed metric can now be written to the required order as

$$\begin{aligned} g_{00} =& -1+\frac{2}{c_1}\left(U+c_2R\right)-\frac{2}{c_1^2}U^2+2\frac{c_2^2}{c_1^2}R^2-\frac{16c_2}{3c_1^2}UR-\frac{7}{18\pi c_1}\mathcal{V}(UR)+\frac{3c_2}{4\pi c_1}\mathcal{V}(R^2)+\frac{64}{9c_1^2}\mathcal{V}(\rho U)\\ &-\frac{44c_2}{3c_1^2}\mathcal{V}(\rho R)-\frac{40c_2}{3c_1^3}\mathcal{V}(\nabla\rho\cdot\nabla U)+\frac{40c_2^2}{c_1^3}\mathcal{V}(\nabla\rho\cdot\nabla R)+\frac{2}{c_1}\mathcal{V}(\rho\Pi)+\frac{4}{c_1}\mathcal{V}(\rho v^2)+\frac{6}{c_1}\mathcal{V}(p)\\ &+\frac{1}{6\pi c_1}X(UR)-\frac{1}{4\pi}\left(\frac{c_2}{c_1}-\frac{c_3}{2c_2}\right)X(R^2)-\frac{4}{3c_1^2}X(\rho U)+\frac{8c_2}{3c_1^2}X(\rho R)+\frac{8c_2}{3c_1^3}X(\nabla\rho\cdot\nabla U)\\ &-\frac{8c_2^2}{c_1^3}X(\nabla\rho\cdot\nabla R)-\frac{2}{c_1}X(p)+\frac{2}{3c_1}X(\rho\Pi) \end{aligned} \tag{27}$$

$$g_{0i} = -\frac{7V_i}{2c_1}-\frac{W_i}{2c_1}+\frac{X(\rho v_i)}{6c_1}-\frac{Y_i}{6c_1}-\frac{Z_i}{6\sqrt{6c_1c_2}} \tag{28}$$

$$g_{ij} = \left(1+\frac{2}{c_1}\left(U-c_2R\right)\right)\delta_{ij} \tag{29}$$

where we have introduced the new potentials Y_i and Z_i, which are defined as

$$Y_i \equiv \int \frac{\rho'\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')(x-x')_i}{|\mathbf{x}-\mathbf{x}'|^3}e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{30}$$

$$Z_i \equiv \int \frac{\rho'\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')(x-x')_i}{|\mathbf{x}-\mathbf{x}'|^2}e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{31}$$

and where use has again been made of the conservation equation (21). The reader will notice in equation (27) potentials that are functions of gradients of ρ, U and R, such as $\mathcal{V}(\nabla\rho\cdot\nabla U)$. This type of term is not of the usual PPN form, where the metric contains functionals of rest mass, energy, pressure and velocity, but not their gradients [21]. In appendix B we re-express the offending terms in a more proper PPN form, where gradients are absent.

VII. DISCUSSION

We have found the PPN limit of analytic $f(R)$ theories of gravity that allow an asymptotically Minkowski background. The weak-field metric for these theories, in the presence of a perfect fluid and in the standard post-Newtonian gauge, is given by equations (27), (28) and (29). We shall now proceed to investigate their form.

Firstly, we will consider the limit where the higher order contributions to the action are vanishing, so that $c_3 \to c_2 \to 0$. In such a limit it can be seen from the trace equation, (14), that $R \to 8\pi\rho/c_1$, and that the metric specified by (27), (28) and (29) then reduces to

$$\begin{aligned} g_{00} &\to -1+2U-2U^2+4\mathcal{V}(\rho U)+2\mathcal{V}(\rho\Pi)+4\mathcal{V}(\rho v^2)+6\mathcal{V}(p)\\ g_{0i} &\to -\frac{7}{2}V_i-\frac{1}{2}W_i\\ g_{ij} &\to (1+2U)\,\delta_{ij} \end{aligned}$$

where the two terms $\mathcal{V}(\rho U)$ and $\mathcal{V}(RU)$ in (27) have contributed to $\mathcal{V}(\rho U)$ in the expression above. Here we have set $c_1 = 1$. This metric is the PPN limit of GR (see [21] for details).

In order to be considered viable for non-zero c_2 and c_3 it is necessary for these theories to reduce to Newtonian gravity in the appropriate limit. From the $O(2)$ term of the g_{00} component of the metric, (27), it can be seen that this can occur iff either

$$(i)\ 3c_2R \sim U \qquad \text{or} \qquad (ii)\ 3c_2R \sim 0$$

on observable length scales. Condition (i) is met if

$$|\sqrt{c_1/6c_2}|L \ll 1$$

for the largest length-scales on which Newtonian gravity has been observed, L. Alternatively, condition (ii) can be met if R is a decaying exponential with

$$\sqrt{c_1/6c_2}l \gg 1,$$

where l is the smallest length scale on which Newtonian gravity has been observed. In both cases we can now find simple expressions for the post-Newtonian limit, which will allow us to relate our results to the relevant observations.

Let us first consider case (i). This case can be easily dismissed by considering only the Newtonian limit of g_{00} and the post-Newtonian limit of g_{ij}, equations (27) and (29) above. If $\sqrt{c_1/6c_2}|\mathbf{x}-\mathbf{x}'| \ll 1$ then $3c_2R \simeq U$, and we must set $c_1 = 4/3$ to obtain the appropriate Newtonian term

$$g_{00} = -1 + 2U + O(4).$$

However, substitution of this value of c_1 into (29) in the same limit gives

$$g_{ij} = (1+U)\delta_{ij} + O(4),$$

as previously found by Chiba, Smith and Erickcek [29]. This result is entirely incompatible with many observations of null geodesics (see e.g. [19]) and so we will not consider this case any further.

Let us now consider case (ii). Here the potentials containing exponentials are expected to be sub-dominant to those without, on observable length scales, so that the R potential in $g_{00}^{(2)}$ is only effective at very small distances. Setting $c_1 = 1$, and discarding potentials which are exponentially suppressed with regards to others, we then find that the metric in this case is given to the appropriate order as

$$\begin{aligned} g_{00} \simeq & -1 + 2U - 2U^2 + \frac{64}{9}\mathcal{V}(\rho U) - \frac{7}{18\pi}\mathcal{V}(UR) + 2\mathcal{V}(\rho\Pi) + 4\mathcal{V}(\rho v^2) \\ & + 6\mathcal{V}(p) - \frac{40c_2}{3}\psi_1 + \frac{40}{3}\sqrt{\frac{c_2}{6}}\psi_2 + \frac{c_3}{8\pi c_2}X(R^2) \end{aligned} \tag{32}$$

$$g_{0i} \simeq -\frac{7}{2}V_i - \frac{1}{2}W_i - \frac{Z_i}{6\sqrt{6c_2}} \tag{33}$$

$$g_{ij} \simeq (1+2U)\,\delta_{ij} \tag{34}$$

where ψ_1 and ψ_2 are defined, as in appendix B, by

$$\psi_1 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3|\mathbf{x}'-\mathbf{x}''|^3} d^3x' d^3x'' \tag{35}$$

$$\psi_2 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3|\mathbf{x}'-\mathbf{x}''|^2} e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}'-\mathbf{x}''|} d^3x' d^3x''. \tag{36}$$

In the metric above we have retained the term proportional to Z_i in the g_{0i} components. Although this term contains an exponential suppression factor, it is of a different form to V_i and W_i and so cannot necessarily be assumed to be negligibly small in comparison to them.

Another significant difference in the metric above is that the coefficient of the term $\mathcal{V}(\rho U)$ is 64/9, instead of its usual value of 4 in GR, and the inclusion of the new potential $\mathcal{V}(UR)$. We have already seen that in the limit $c_2 \to 0$ that the $\mathcal{V}(UR)$ term reduces to $8\pi\mathcal{V}(\rho U)/c_1$, which is exactly sufficient to recover the GR limit of $4\mathcal{V}(\rho U)$ in g_{00}. The fact that this potential is significant in the limit of small c_2, even though it contains an exponential suppression term, is due to the factor of c_2 in its denominator, as R is present in the integrand. This causes it to approach a finite value, instead of zero, when c_2 is small. This potential cannot therefore be considered negligible, as it is not necessarily exponentially smaller than any other.

There are three further potentials in g_{00} that are not present in GR: ψ_1, ψ_2 and $X(R^2)$. The ψ_1 term contains no exponential suppression factor, while the ψ_2 and $X(R^2)$ terms do. However, as before, these potentials are included none the less as they are not directly suppressed with respect to any other. The existence of ψ_1 in the post-Newtonian limit is of particular interest as it is the first new potential that is not exponentially suppressed, with obvious significance for constraining the theory with observations. The $X(R^2)$ term is also of special interest as it is the only term with a dependence on c_3. In the limit $c_2 \to 0$ this term reduces to

$$\frac{c_3}{8\pi c_2}X(R^2) \to 3c_3\rho^2,$$

which is non-zero when c_3 and $\rho \neq 0$. This term therefore provides an opportunity to test for deviations from GR at the level of R^3 in the generating Lagrangian.

One may now wish to compare the metric obtained to the standard PPN metric, and to read off the relevant parameters. The standard PPN metric is given in the present notation by

$$g_{00}^{(PPN)} = -1 + 2U - 2\beta U^2 + (2\gamma + 2 + \zeta_1)\mathcal{V}(\rho v^2) + 2(3\gamma - 2\beta + 1 + \zeta_2)\mathcal{V}(\rho U) + 2(1 + \zeta_3)\mathcal{V}(\rho\Pi) + 6(\gamma + \zeta_4)\mathcal{V}(p) \tag{37}$$

$$g_{0i}^{(PPN)} = -\frac{1}{2}(4\gamma + 3 + \zeta_1)V_i - \frac{1}{2}(1 - \zeta_1)W_i \tag{38}$$

$$g_{ij}^{(PPN)} = (1 + 2\gamma U)\delta_{ij} \tag{39}$$

where β, γ and ζ_i are the post-Newtonian parameters, to be set for a particular gravitational theory. We have excluded here the preferred location and preferred frame terms, as they are of no relevance for the present study. Comparison of the metric (32), (33) and (34) with the above allows one to read off the following values

$$\beta = 1, \;\; \gamma = 1, \;\; \zeta_1 = 0, \;\; \zeta_3 = 0 \qquad \text{and} \qquad \zeta_4 = 0,$$

as in GR. The value of ζ_2 is not so straightforwardly determined, and a naive comparison would yield the result $\zeta_2 = 14/9$ instead of the usual value of zero in GR. However, we have seen above that the $\mathcal{V}(UR)$ term approaches $\mathcal{V}(\rho U)$ for small c_2, and gives the GR result in the limit. Care must therefore be taken with the value of this parameter, and in the present case it seems more appropriate to consider two contributions towards ζ_2 - one coming from the usual $\mathcal{V}(\rho U)$ term, and the other coming from $\mathcal{V}(UR)$. The terms in (32) and (33) proportional to ψ_1, ψ_2, $X(R^2)$ and Z_i are also inadequately accounted for in the PPN metric above. Clearly, new terms are required if these potentials are to be included.

VIII. CONCLUSIONS

We have determined here the post-Newtonian limit of fourth-order theories of gravity that are analytic functions of the Ricci tensor, and that admit Minkowski space as a background. In the Newtonian limit we have recovered the well known result that an exponentially suppressed Yukawa potential is present. These deviations from Newton's law should be expected to be observed at small distance scales, and a number of experimental efforts have been made to find them (see e.g. [30]). These searches have not yet detected any deviations from Newton's law at small distances, and so we must consider terms containing exponential factors to be heavily suppressed.

To determine the post-Newtonian limit of null geodesics we require knowledge of the g_{ij} components of the metric to an accuracy of $O(2)$, as is given in equation (29). We again find the well known result that the only correction to this term at the post-Newtonian level of accuracy is in the form of a Yukawa potential, which must be heavily suppressed. Comparison with the PPN metric (39) then gives us that we should expect $\gamma = 1$ for these theories, and hence that experiments involving observations of null geodesics should be unable to distinguish them from GR.

We then proceeded to determine the full post-Newtonian limit to $O(4)$ in the g_{00} component, (27), and to $O(3)$ in the g_{0i} component, (28). It is found that there exists a large number of new potentials at this order of perturbations. Using our knowledge that terms containing exponential factors are heavily suppressed it is possible to neglect a number of these potentials, resulting in equations (32) and (33). A comparison of these results with (37) and (38) shows that the PPN parameters β, ζ_1, ζ_3 and ζ_4 all take the same values in these $f(R)$ theories as they do in GR. Experiments which are designed to determine these parameters will therefore be unable to distinguish between the two. However, there are differences between (32) and (33), and the PPN limit of GR, that may be potentially observable.

A potentially significant difference with GR is the value of the PPN parameter ζ_2. A direct comparison of (32) with (37) appears to yield the result $\zeta_2 = 14/9$, which is a significant difference from its value of zero in GR. However, we know that in the limit that GR is approached the new potential $\mathcal{V}(UR)$ in g_{00} makes a contribution that is exactly enough to cancel the value of 14/9 above. This strongly suggests that this potential should be included when observational constraints are applied. The parameter ζ_2 is usually associated with violations of momentum conservation, and the (lack of) self-acceleration of the binary pulsar PSR 1913+16 has led to the bound $\zeta_2 < 4 \times 10^{-5}$ [31]. Furthermore, it appears likely that observations of the binary system PSR J1738+0333 will offer even tighter constraints [32]. However, these previously obtained constraints may not be directly applicable to the current theory. Firstly, we know that $f(R)$ theories of gravity covariantly conserve four-momentum exactly (due to their Lagrangian formulation). Secondly, these constraints have been imposed in the absence of the $\mathcal{V}(UR)$ potential. The extent to which these systems are able to offer constraints on $f(R)$ theories when the $\mathcal{V}(UR)$ term is included remains to be determined.

Further opportunity for observationally constraining these theories comes from the other extra potentials in (32) and (33): ψ_1, ψ_2, $X(R^2)$ and Z_i. These potentials are likely to be small, due to suppressing factors of $c_2/|\mathbf{x}-\mathbf{x}'|^2$. The ψ_1 term is somewhat promising as its suppression is polynomial, and not exponential. The Z_i term is also interesting as it is has polynomial amplification, as well as exponential suppression. Such a term may be potentially observable in experiments that measure vector perturbations, such as gravity probe B [33]. Finally, we will mention that the term proportional to $X(R^2)$ allows for the possibility of constraining any R^3 term that may exist in the generating Lagrangian. Unlike the R^2 term, the R^3 term does not first appear in the perturbative expansion as a Yukawa potential. In fact, in the limit that the R^2 term vanishes the potential $X(R^2)$ reduces to ρ^2, which could be observable in high density environments.

APPENDIX A: SOLVING THE $t-t$ EQUATION TO $O(4)$

In order to determine the $h_{00}^{(4)}$ term, it will first be necessary to determine the Ricci scalar to $O(4)$. This will be achieved by solving the trace equation to the appropriate order. In the gauge defined by equations (6) and (7), the trace of the field equations (3) becomes

$$\nabla^2 R^{(4)} - \frac{c_1}{6c_2}R^{(4)} - R_{,00} + \frac{3c_3}{2c_2}\nabla^2 R^2 + \frac{2}{c_1}U\nabla^2 R - 2\frac{c_2}{c_1}R\nabla^2 R - 24\frac{c_2^2}{c_1^2}R_{,ij}R_{,ij} + 8\frac{c_2}{c_1^2}U_{,ij}R_{,ij} = \frac{4\pi}{c_2}p - \frac{4\pi}{3c_2}\rho\Pi$$

where here we have dropped the superscript on $R^{(2)}$, so that the R above should be implicitly assumed to be of $O(2)$ (as can be recognised from the required order of each term). We now introduce the new potentials

$$\hat{\chi} \equiv \int \rho' e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{A1}$$

$$X(Q) \equiv \int \frac{Q' e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} d^3x', \tag{A2}$$

where a prime now denotes a quantity that is a function of x', so that

$$\left(\nabla^2 - \frac{c_1}{6c_2}\right)\hat{\chi} = -\sqrt{6c_1c_2}R$$
$$\left(\nabla^2 - \frac{c_1}{6c_2}\right)X(Q) = -4\pi Q.$$

Recognising the relations

$$U_{,ij}R_{,ij} = \frac{1}{2}\left(\nabla^2 - \frac{c_1}{6c_2}\right)\nabla U\cdot\nabla R + \frac{2\pi}{3c_2}\nabla\rho\cdot\nabla U + 2\pi\nabla\rho\cdot\nabla R$$
$$R_{,ij}R_{,ij} = \left(\nabla^2 - \frac{c_1}{6c_2}\right)\left(\frac{|\nabla R|^2}{2} - \frac{c_1R^2}{24c_2}\right) + \frac{4\pi}{3c_2}\nabla\rho\cdot\nabla R + \frac{c_1^2}{144c_2^2}R^2 - \frac{\pi c_1}{9c_2^2}\rho R$$

then allows the trace equation above to be integrated, to give

$$\begin{aligned} R^{(4)} = &-\frac{\hat{\chi}_{,00}}{\sqrt{6c_1c_2}} - \left(\frac{3c_3}{2c_2} + \frac{c_2}{c_1}\right)R^2 + 12\frac{c_2^2}{c_1^2}|\nabla R|^2 - 4\frac{c_2}{c_1^2}\nabla U\cdot\nabla R + \frac{X(UR)}{12\pi c_2} - \left(1 - \frac{c_1c_3}{2c_2^2}\right)\frac{X(R^2)}{8\pi} \\ &- \frac{2X(\rho U)}{3c_1c_2} + \frac{4X(\rho R)}{3c_1} + \frac{4X(\nabla\rho\cdot\nabla U)}{3c_1^2} - \frac{4c_2X(\nabla\rho\cdot\nabla R)}{c_1^2} - \frac{X(p)}{c_2} + \frac{X(\rho\Pi)}{3c_2}. \end{aligned} \tag{A3}$$

We are now sufficiently equipped to solve the $t-t$ field equation to $O(4)$, in order to obtain $h_{00}^{(4)}$. Equations (3) give this as

$$\begin{aligned} -\frac{c_1}{2}\nabla^2 h_{00}^{(4)} - \frac{1}{c_1}\nabla^2 U^2 + \left(\frac{3}{2}c_3 - 2\frac{c_2^2}{c_1}\right)\nabla^2 R^2 - 3\frac{c_2}{c_1}\nabla^2 UR - 18\frac{c_2^2}{c_1}\nabla^2 R_{,00} \\ + 6\frac{c_2}{c_1}\nabla^2 U_{,00} + c_2\nabla^2 R^{(4)} + 8\frac{c_2}{c_1^2}U_{,ij}U_{,ij} - 8\frac{c_2^2}{c_1^2}U_{,ij}R_{,ij} - 48\frac{c_2^3}{c_1^2}R_{,ij}R_{,ij} \\ + \frac{5}{6}UR - \frac{c_2}{6}R^2 - \frac{44\pi}{3c_1}\rho U + \frac{52\pi c_2}{3c_1}\rho R = 4\pi\rho\Pi + 8\pi\rho v^2 + 12\pi p \end{aligned}$$

where the trace of (3) has been used to eliminate the term proportional to $R^{(4)}$. On recognising

$$U_{,ij}U_{,ij} = \frac{1}{2}\nabla^2|\nabla U|^2 + 4\pi\nabla\rho\cdot\nabla U$$
$$U_{,ij}R_{,ij} = \frac{1}{2}\nabla^2(\nabla U\cdot\nabla R) - \frac{c_1}{24c_2}\nabla^2 UR + \frac{2\pi}{3c_2}\nabla\rho\cdot\nabla U + 2\pi\nabla\rho\cdot\nabla R - \frac{\pi c_1}{6c_2}\rho R - \frac{\pi c_1}{18c_2^2}\rho U + \frac{c_1^2}{144c_2^2}UR$$
$$R_{,ij}R_{,ij} = \frac{1}{2}\nabla^2|\nabla R|^2 - \frac{c_1}{12c_2}\nabla^2 R^2 + \frac{4\pi}{3c_2}\nabla\rho\cdot\nabla R + \frac{c1^2}{36c_2^2}R^2 - \frac{2\pi c_1}{9c_2^2}\rho R$$

this can be integrated to

$$\begin{aligned} h_{00}^{(4)} =& -\frac{2}{c_1^2}U^2 + \left(3\frac{c_3}{c_1} + 4\frac{c_2^2}{c_1^2}\right)R^2 - \frac{16c_2}{3c_1^2}UR - 36\frac{c_2^2}{c_1^2}R_{,00} + 12\frac{c_2}{c_1^2}U_{,00} + 8\frac{c_2}{c_1^3}|\nabla U|^2 - 48\frac{c_2^3}{c_1^3}|\nabla R|^2 \\ &- 8\frac{c_2^2}{c_1^3}\nabla U\cdot\nabla R - \frac{7}{18\pi c_1}\mathcal{V}(UR) + \frac{3c_2}{4\pi c_1}\mathcal{V}(R^2) + \frac{64}{9c_1^2}\mathcal{V}(\rho U) - \frac{44c_2}{3c_1^2}\mathcal{V}(\rho R) - \frac{40c_2}{3c_1^3}\mathcal{V}(\nabla\rho\cdot\nabla U) \\ &+ \frac{40c_2^2}{c_1^3}\mathcal{V}(\nabla\rho\cdot\nabla R) + \frac{2}{c_1}\mathcal{V}(\rho\Pi) + \frac{4}{c_1}\mathcal{V}(\rho v^2) + \frac{6}{c_1}\mathcal{V}(p) + 2\frac{c_2}{c_1}R^{(4)} \end{aligned} \tag{A4}$$

where

$$\mathcal{V}(Q) \equiv \int \frac{Q'}{|\mathbf{x}-\mathbf{x}'|}dx'^3. \tag{A5}$$

Equations (A3) and (A4) now specify h_{00} to $O(4)$.

APPENDIX B: RE-EXPRESSING THE POTENTIALS

The potentials $\mathcal{V}(\nabla\rho\cdot\nabla U)$, $\mathcal{V}(\nabla\rho\cdot\nabla R)$, $X(\nabla\rho\cdot\nabla U)$ and $X(\nabla\rho\cdot\nabla R)$ in equation (27) are not in usual PPN form, as they are written as functions of gradients of ρ, U and R. Here we re-express these terms, with the gradients absent, as

$$\mathcal{V}(\nabla\rho\cdot\nabla U) = 4\pi\mathcal{V}(\rho^2) + \psi_1$$
$$\mathcal{V}(\nabla\rho\cdot\nabla R) = \frac{4\pi}{3c_2}\mathcal{V}(\rho^2) - \frac{c_1}{6c_2}\mathcal{V}(\rho R) + \sqrt{\frac{c_1}{6c_2}}\frac{\psi_2}{3c_2} + \frac{\psi_3}{3c_2}$$
$$X(\nabla\rho\cdot\nabla U) = 4\pi X(\rho^2) + \sqrt{\frac{c_1}{6c_2}}\psi_4 + \psi_5$$
$$X(\nabla\rho\cdot\nabla R) = \frac{4\pi}{3c_2}X(\rho^2) - \frac{c_1}{6c_2}X(\rho R) + \frac{\psi_6}{3c_2} + \sqrt{\frac{c_1}{6c_2}}\frac{(\psi_7+\psi_8)}{3c_2} + \frac{c_1\psi_9}{18c_2^2}.$$

On substituting these expressions back into (27) the terms proportional to $\mathcal{V}(\rho^2)$ and $X(\rho^2)$ cancel exactly. The new potentials, ψ_i, are defined by

$$\psi_1 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3|\mathbf{x}'-\mathbf{x}''|^3}d^3x'd^3x''$$
$$\psi_2 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3|\mathbf{x}'-\mathbf{x}''|^2}e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}'-\mathbf{x}''|}d^3x'd^3x''$$
$$\psi_3 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3|\mathbf{x}'-\mathbf{x}''|^3}e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}'-\mathbf{x}''|}d^3x'd^3x''$$
$$\psi_4 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^2|\mathbf{x}'-\mathbf{x}''|^3}e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|}d^3x'd^3x''$$
$$\psi_5 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3|\mathbf{x}'-\mathbf{x}''|^3}e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|}d^3x'd^3x''$$
$$\psi_6 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3|\mathbf{x}'-\mathbf{x}''|^3}e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}'-\mathbf{x}''|}d^3x'd^3x''$$

$$\psi_7 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3|\mathbf{x}'-\mathbf{x}''|^2} e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}'-\mathbf{x}''|} d^3x' d^3x''$$

$$\psi_8 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^2|\mathbf{x}'-\mathbf{x}''|^3} e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}'-\mathbf{x}''|} d^3x' d^3x''$$

$$\psi_9 \equiv \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^2|\mathbf{x}'-\mathbf{x}''|^2} e^{-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}-\mathbf{x}'|-\sqrt{\frac{c_1}{6c_2}}|\mathbf{x}'-\mathbf{x}''|} d^3x' d^3x''.$$

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