Perturbation Theory in Lemaitre-Tolman-Bondi Cosmology

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Perturbation Theory in Lemaître-Tolman-Bondi Cosmology

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The Lemaître-Tolman-Bondi solution has received much attention as a possible alternative to Dark Energy, as it is able to account for the apparent acceleration of the Universe without any exotic matter content. However, in order to make rigorous comparisons between these models and cosmological observations, such as the integrated Sachs-Wolfe effect, baryon acoustic oscillations and the observed matter power spectrum, it is absolutely necessary to have a proper understanding of the linear perturbation theory about them. Here we present this theory in a fully general, and gauge-invariant form. It is shown that scalar, vector and tensor perturbations interact, and that the natural gauge invariant variables in Lemaître-Tolman-Bondi cosmology do not correspond straightforwardly to the usual Bardeen variables, in the limit of spatial homogeneity. We therefore construct new variables that reduce to pure scalar, vector and tensor modes in this limit.

I. INTRODUCTION

The concordance model of cosmology is a result of assuming that the Universe is approximately the same at all points in space, and that at every point it appears isotropic. The observations we make locally can then be extrapolated to the Universe at large, and the resulting Friedmann-Lemaître-Robertson-Walker (FLRW) model can be used to interpret all cosmological observations. The FLRW models are, of course, very well understood, and have proved to be highly successful in very many different capacities. Unfortunately, despite its aesthetic appeal and mathematical simplicity, the adoption of the FLRW cosmology has led to the undesirable requirement that the Universe must be filled with a smoothly distributed and gravitationally repulsive exotic substance, known as Dark Energy. The conceptual problems associated with Dark Energy are well known, and are distressing enough that it now seems worth investigating alternative, less appealing models.

One such alternative is the spherically symmetric, dust dominated cosmological solution of Lemaître, Tolman and Bondi [1]. This model is isotropic about a single point in space, in agreement with local galaxy surveys and observations of the Cosmic Microwave Background, but is anisotropic everywhere else [2]. As such, we are required to live very near the centre of symmetry, indicating a break with the Copernican Principle that *"the Earth is not in a central, specially favoured position"* [3]. The Lemaître-Tolman-Bondi (LTB) model allows the Hubble rate to vary with radial distance from the centre of symmetry, as well as with time, and so allows for the possibility of apparent acceleration (if observations within them are interpreted in an FLRW frame-work), while no part of them need undergo any accelerating expansion.

One may then try and argue on aesthetic grounds, or on the grounds of presumed early Universe physics, which of these models is more likely. Clearly there are good reasons to disfavour either, depending on which one abhors most: the Earth being in a special position in the Universe, or the existence of Dark Energy, which requires us to live at a special time when the acceleration is just starting to kick in. However, a more objective (and scientific) approach to the problem would be to try and distinguish between the two models with direct observations. This has been attempted a number of different times in the literature already [4]. However, in order to make comparisons to different types of astrophysical data it is absolutely necessary to have a proper understanding of how structures form and behave in these models, and this means understanding their perturbation theory. An alternative approach is to try to test the Copernican principle directly [5], but this is very difficult to achieve at present through direct tests.

For observations such as the integrated Sachs-Wolfe effect, Rees-Sciama effect, or matter power spectrum it is obvious that a proper interpretation of the data is dependent on understanding the relevant perturbation theory. However, other observations, that do not necessarily require perturbation theory in FLRW, will also require a knowledge of perturbation theory in the LTB case. An example is the use of baryon acoustic oscillations as a probe of cosmology. In FLRW it suffices to consider comoving points in an unperturbed fluid, to see how length scales at recombination are evolved onto distance measures on our past light cone. In an LTB universe, however, the rate of growth of structure can be different at different places, and the peculiar velocity of perturbations need not cancel when averaged over suitable scales (in fact, one may naively expect otherwise). Likewise, when considering light reflecting off a galaxy cluster in FLRW it may suffice to consider that on average the motions of galaxy clusters will cancel out, so that approximating galaxy clusters as being comoving with the background fluid may be sufficient to get accurate results. In an LTB universe this is unlikely to be true, as gradients in gravitational potentials will develop due to different

growth rates at different positions.

As well as the requirement of understanding linear perturbation theory in order to accurately account for already established probes of cosmology, there is also the possibility of finding new observational phenomena that could be used to distinguish the LTB and FLRW cosmologies, or constrain the type of LTB models that are observationally viable. An example of this is the potential coupling of scalar gravitational potentials to tensor gravitational waves. In an FLRW background (to linear order) the decomposition theorem ensures that no such coupling occurs. On a homogeneous background scalar, vector and tensor modes then evolve independently. In an inhomogeneous LTB background, however, the decomposition theorem will only be valid on the two dimensional surfaces of spherical symmetry. In the three dimensional space there will be no such theorem, and in general one should expect coupling to occur between what we conventionally think of as 'scalar', 'vector' and 'tensor' modes. Though complicated, such effects could potentially be used as probes of the geometry of the universe.

In section II we introduce the formalism required for perturbation theory about a spherically symmetric background. In section III A and III B we present the perturbation equations, and in section III C and IV we discuss their solutions and how they relate to perturbation theory in FLRW. Finally, in section V we conclude. The Appendices contain information about gauge invariant quantities for the metric perturbation, and the perturbation equations for some special cases.

We use units in which $c = G = 1$, throughout.

II. PERTURBATION THEORY ON A SPHERICALLY SYMMETRIC BACKGROUND

Perturbations on a spherically symmetric space-time have been considered a number of times in the literature, mostly in the context of modelling static and stationary stars [6]. A time dependent formalism was first developed by Seidel [7], and later specialised to a space-time containing a perfect fluid by Gundlach and Martín-García (GMG) [8]. Perturbations of self similar models have been investigated in [9], and have the advantage that all PDEs reduce to ODEs in the system of perturbation equations. A limited class of perturbations in LTB cosmology have also been studied by Zibin [10].

Different formalisms exist in the literature for constructing both the system of perturbation equations, and the gauge-invariant variables. Gerlach and Sengupta [11] (GS) developed a formalism based on a 2+2 covariant split of the metric and energy-momentum tensor. This turns the field equations into a system of second-order PDEs, and was made explicit by GMG, who found a closed set of master equations describing all perturbations. Their approach was used in the study of self-similar LTB perturbations performed in [9]. Alternatively, a covariant 1+1+2 formalism has been developed

in [12], which builds on the covariant 1+3 formalism that has been usefully applied in cosmology [13]. Here the Bianchi and Ricci identities, plus the Ricci rotation coefficients for the semi-tetrad introduced, are covariantly split into a system of first-order differential equations. This formalism has not yet been reduced to a tractable set of master equations for a general space-time, but has been used by [10] to study LTB perturbations. Here we apply the GS formalism developed by GMG to the case of an LTB space-time. This results in a simple set of coupled second-order PDEs that describe general perturbations to the space-time, in a gauge invariant way.

A. The LTB background

The unperturbed LTB line-element can be written

$$ds^2 = -dt^2 + \frac{a_\parallel^2(t,r)}{(1 - \kappa r^2)}dr^2 + a_\perp^2(t,r)r^2 d\Omega^2, \quad (1)$$

where $a_\parallel = (ra_\perp)_{,r}$ and $\kappa = \kappa(r)$ is a free function of r. The FLRW scale factor, a, has been replaced here by two new scale factors, $a_\parallel$ and $a_\perp$, describing expansion parallel and perpendicular to the radial direction, respectively. For future use we will define the radial and azimuthal Hubble rates to be

$$H_\parallel \equiv \frac{\dot{a}_\parallel}{a_\parallel} \quad \text{and} \quad H_\perp \equiv \frac{\dot{a}_\perp}{a_\perp}, \quad (2)$$

where an over-dot denotes partial differentiation with respect to t. The analogue of the Friedmann equation in this space-time is then given by

$$H_\perp^2 = \frac{M}{a_\perp^3} - \frac{\kappa}{a_\perp^2}, \quad (3)$$

where $M = M(r)$ is another free function of r, and the locally measured energy density is

$$8\pi\rho = \frac{(Mr^3)_{,r}}{a_\parallel a_\perp^2 r^2}, \quad (4)$$

which obeys the conservation equation

$$\dot{\rho} + (2H_\perp + H_\parallel)\rho = 0. \quad (5)$$

The acceleration equations in the perpendicular and parallel directions are

$$\frac{\ddot{a}_\perp}{a_\perp} = -\frac{M}{2a_\perp^3} \quad \text{and} \quad \frac{\ddot{a}_\parallel}{a_\parallel} = -4\pi\rho + \frac{M}{a_\perp^3}. \quad (6)$$

For what follows it will also be useful to define the radial derivative

$$X' \equiv \frac{\sqrt{1 - \kappa r^2}}{a_\parallel}X_{,r}. \quad (7)$$

This derivative does not commute with the time derivative, but instead obeys

$$(\dot{X})' - (X')\dot{} = H_\parallel X'. \tag{8}$$

We also define the curvature function

$$W \equiv \frac{\sqrt{1 - \kappa r^2}}{a_\perp r}. \tag{9}$$

The following relations are then obeyed

$$H'_\perp = W(H_\parallel - H_\perp), \tag{10}$$

$$\dot{W} = -H_\perp W, \tag{11}$$

$$W' = -W^2 - 4\pi\rho + H_\perp H_\parallel + \frac{M}{2r^3}. \tag{12}$$

In the perturbation equations that follow we will choose to eliminate κ in favour of W, so that the equations take their simplest form.

B. Harmonic functions

A natural way to split this space-time is in a 2+2 decomposition, so that the space-time manifold becomes $M^4 = M^2 \times S^2$, where S^2 indicates the 2 dimensional spherically symmetric surfaces. We will use lower case Latin indices $a, b, c, \ldots$ to denote coordinates in S^2, upper case Latin indices $A, B, C, \ldots$ to denote coordinates in M^2, and Greek indices $\mu, \nu, \xi, \ldots$ to denote coordinates that run over all 4 space-time dimensions.

In FLRW cosmology, any perturbation can be split into scalar, vector and tensor (SVT) modes that decouple from each other, and so evolve independently (to first order). This classification is based on how they transform on the homogeneous and isotropic spatial hyper-surfaces, and is essentially just a generalisation of Helmholtz's theorem [14].

Such a split cannot usefully be performed in the same way here, as the background is no longer spatially homogeneous, and modes written in this way would couple together (as we shall see). However, one can perform an analogous classification based on how the perturbations transform on the surfaces of spherical symmetry. This results in a decoupling of the perturbations into two independent modes, called 'polar' (or even) and 'axial' (or odd), which are analogous, but not equivalent, to scalar and vector modes in FLRW. Unlike the FLRW case, however, there is no further decomposition into tensor modes as no non-trivial symmetric, transverse and trace-free rank 2 tensors can exist on S^2. Therefore, only two distinct sectors exist. Scalars (rank 0 tensors) on S^2 can then be expanded as a sum of polar modes, and higher rank tensors on S^2 can be expanded in sums over both the polar and axial modes. Only a scalar can contain spherical perturbations (given by $\ell = 0$, defined below), and only scalars and vectors (i.e., tensors of rank 0 and 1) can contain a dipole term ($\ell = 1$). Higher multipoles can be present in all tensors.

An appropriate family of basis functions for this split are tensor spherical harmonics. These are derived from the usual spherical harmonic functions, $Y^{(\ell m)}(x^a)$, that obey

$$\bar{\nabla}^2 Y^{(\ell m)} = -\ell(\ell + 1) Y^{(\ell m)}, \tag{13}$$

where the Natural number, ℓ, gives the angular scale of the perturbation. The Laplacian, $\bar{\nabla}^2$ here, is on the surface of spherical symmetry, and is given by $\bar{\nabla}^2 \phi = \gamma^{ab} \phi_{:ab}$, where the colon subscript indicates a covariant derivative with respect to the metric on the unit sphere, γ_{ab}. Scalar perturbations on S^2 can then be written with their angular dependence given in terms of the solutions to this equation, and expanded as

$$\phi(x^A, x^a) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \phi^{(\ell m)}(x^A) Y^{(\ell m)}(x^a). \tag{14}$$

It is now possible to construct a basis for all higher rank tensors from $Y^{(\ell m)}$, its covariant derivatives, and the contractions of those derivatives with the fundamental antisymmetric tensor, ϵ_{ab}. Modes that can be described without requiring ϵ_{ab} are called polar, while those that require ϵ_{ab} are called axial.

We can now form harmonic functions for higher rank tensor perturbations with polar degrees of freedom by first defining the vector, for $\ell \geq 1$:

$$Y_a^{(\ell m)} \equiv Y_{:a}^{(\ell m)}. \tag{15}$$

We define also the trace-less tensor, for $\ell \geq 2$:

$$Y_{ab}^{(\ell m)} \equiv Y_{:ab}^{(\ell m)} + \frac{\ell(\ell+1)}{2} Y^{(\ell m)} \gamma_{ab}. \tag{16}$$

Taking divergences of $Y_a^{(\ell m)}$ and $Y_{ab}^{(\ell m)}$ reduces these expressions to equations involving $Y^{(\ell m)}$.

For axial perturbations on S^2 we define a divergence-free vector harmonic, for $\ell \geq 1$,

$$\bar{Y}_a^{(\ell m)} \equiv \epsilon_a{}^b Y_{:b}^{(\ell m)}. \tag{17}$$

We can then construct a symmetric and trace-free rank-2 axial harmonic function by defining, $\ell \geq 2$,

$$\bar{Y}_{ab}^{(\ell m)} \equiv 2\bar{Y}_{(a:b)}^{(\ell m)} = -2\epsilon^d{}_{(a} Y_{:b)d}^{(\ell m)}, \tag{18}$$

where round brackets around indices denote symmetrisation, as usual. Taking covariant derivatives, this tensor harmonic can be reduced to an expression involving $\bar{Y}_a^{(\ell m)}$.

For both parities, the 'vector harmonics' obey

$$\bar{\nabla}^2 Y_a^{(\ell m)} = [1 - \ell(\ell+1)] Y_a^{(\ell m)} \tag{19}$$

(similarly for $\bar{Y}_a^{(\ell m)}$), and the 'tensor harmonics' obey

$$\bar{\nabla}^2 Y_{ab}^{(\ell m)} = [4 - \ell(\ell+1)] Y_{ab}^{(\ell m)}. \tag{20}$$

where the fundamental anti-symmetric tensor $\epsilon_{AB} = n_A u_B - u_A n_B$ and a pipe denotes the covariant derivative on M^2. The only axial perturbation that can then occur to the matter content is a perturbation to the matter four velocity, such that

$$u_\mu = (\hat{u}_A, \bar{v}\bar{Y}_a), \qquad (33)$$

where $\hat{u}^A$ is defined as in the polar case, and $\bar{v}$ equals one of the gauge invariant variables of GMG, in Appendix B.

1. 'Scalar-Vector-Tensor' variables

The GI variables defined above give a very concise set of governing equations (see below). However, in the FLRW limit we shall see that they mix up the normal SVT modes in a complicated way – for example, the variable φ contains all three types of perturbations. By taking combinations of the variables defined in the perturbed metric we may form variables which do reduce to scalars, vectors, or tensors in the FLRW limit, and so may be used to identify generalised SVT modes. Such variables are:

Tensors

$$\text{polar:} \quad \hat{\chi} = \chi \qquad (34)$$

$$\text{axial:} \quad \hat{\Upsilon} \equiv \Pi'' + 6W\Pi' + \left(8W^2 - \frac{\ell(\ell+1)+2}{a_\perp^2 r^2}\right)\Pi$$

$$+16\pi\frac{(\rho\bar{v})'}{a_\perp^2 r^2}, \qquad (35)$$

Vectors

$$\text{polar:} \quad \hat{\xi} \equiv \frac{3a_\perp}{2W}\left[\frac{1}{r^3}\left(r^2\dot{\chi}\right)' + \left(\frac{\varsigma}{r}\right)'' + 2W\left(\frac{\varsigma}{r}\right)'\right.$$

$$\left. - \left(\frac{\ell(\ell+1)-3}{a_\perp^2 r^2} + 3W^2\right)\frac{\varsigma}{r}\right] \qquad (36)$$

$$\text{axial:} \quad \hat{\bar{v}} = \bar{v}, \qquad (37)$$

Scalars

$$\hat{\zeta} \equiv \hat{\lambda}'' + 2W\hat{\lambda}' - \frac{\ell(\ell+1)}{a_\perp^2 r^2}\hat{\lambda}$$

$$+ rW\hat{\xi}' + r\left(3W^2 - \frac{1}{a_\perp^2 r^2}\right)\hat{\xi}, \quad (38)$$

where

$$\hat{\lambda} \equiv 8\pi\rho a_\perp\left[H_\perp^{-1}\Delta - 3v\right]. \qquad (39)$$

We shall justify these variables in Sec. IV C.

III. MASTER EQUATIONS

The equations of motion governing the dynamics may be reduced to a coupled system of evolution equations and constraints. For the polar sector the equations come in the form of three coupled PDEs for χ, φ and ς, with all other non-trivial gauge invariant variables determined from the solution of this system. For the axial modes, the dynamics are determined by a much simpler system of equations for the variables Π and $\bar{v}$.

A. Polar perturbation equations for $\ell \geq 2$

Substituting the polar perturbed metric tensor (26), and the perturbed matter quantities, (27) and (28), into the field equations results in a system of three coupled master evolution equations for the three variables χ, φ and ς. The remaining variables associated with the fluid can then be determined directly from the solution to this system.

The three evolution equations are

$$-\ddot{\chi} + \chi'' - 3H_\parallel\dot{\chi} - 2W\chi' + \left[16\pi\rho - \frac{6M}{a_\perp^3} - 4H_\perp(H_\parallel - H_\perp) - \frac{(\ell-1)(\ell+2)}{a_\perp^2 r^2}\right]\chi \qquad (40)$$

$$= -2(H_\parallel - H_\perp)\varsigma' - 2\left[H_\parallel' - 2(H_\parallel - H_\perp)W\right]\varsigma + 4(H_\parallel - H_\perp)\dot{\varphi} - 2\left[8\pi\rho - \frac{3M}{a_\perp^3} - 2H_\perp(H_\parallel - H_\perp)\right]\varphi,$$

and

$$\ddot{\varphi} + 4H_\perp\dot{\varphi} - 2\left(\frac{1}{a_\perp^2 r^2} - W^2\right)\varphi = -H_\perp\dot{\chi} + W\chi' - \left[2W^2 - \frac{\ell(\ell+1)+2}{2a_\perp^2 r^2}\right]\chi + 2W(H_\parallel - H_\perp)\varsigma, \qquad (41)$$

and

$$\dot{\varsigma} + 2H_\parallel\varsigma = -\chi', \qquad (42)$$

together with the constraint

$$\eta = 0. \tag{43}$$

These four equations are the LTB version of Equations (GMG87), (GMG88), (GMG89) and (GMG82) from [8].

The three constraint equations can then be written as

$$8\pi\rho w = (\dot{\varphi})' - (H_\| - 2H_\perp)\varphi' - W\dot{\chi} + H_\perp\chi' + \left[\frac{\ell(\ell+1)+2}{2a_\perp^2 r^2} + H_\perp^2 + 2H_\perp H_\| - W^2 - 4\pi\rho\right]\varsigma, \tag{44}$$

$$8\pi\rho\Delta = -\varphi'' - 2W\varphi' + (H_\| + 2H_\perp)\dot{\varphi} + W\chi' + H_\perp\dot{\chi} + \left[\frac{\ell(\ell+1)}{a_\perp^2 r^2} + 2H_\perp^2 + 4H_\| H_\perp - 8\pi\rho\right](\chi + \varphi) \tag{45}$$

$$- \frac{(\ell-1)(\ell+2)}{2a_\perp^2 r^2}\chi + 2H_\perp\varsigma' + 2(H_\| + H_\perp)W\varsigma,$$

$$8\pi\rho v = \dot{\varphi} + \frac{\dot{\chi}}{2} + H_\|(\chi + \varphi) + \frac{\varsigma'}{2}. \tag{46}$$

These are the LTB versions of the equations (GMG93), (GMG94) and (GMG95) from [8]. It is also useful to consider the evolution equations that result from differentiating (44), (45) and (46). These take the particularly simple form

$$\dot{v} = \frac{\chi}{2} + \frac{\varphi}{2}, \tag{47}$$

$$\dot{w} = \frac{\varphi'}{2} - H_\| w - \frac{H_\|}{2}\varsigma, \tag{48}$$

$$\dot{\Delta} + \left(w + \frac{\varsigma}{2}\right)' = -\frac{\dot{\chi}}{2} - \frac{3\dot{\varphi}}{2} + \frac{\ell(\ell+1)}{a_\perp^2 r^2}v - \frac{\rho'}{\rho}\left(w + \frac{\varsigma}{2}\right) - 2W\left(w + \frac{\varsigma}{2}\right), \tag{49}$$

and are the LTB versions of (GMG96), (GMG97) and (GMG98). There are then six coupled equations for the six variables χ, φ, ς, w, Δ and v.

Equations (40)-(49) are clearly more complicated than their FLRW counterparts, but the fact that they can be written as concisely as they are above is quite remarkable.

For very large angle fluctuations, with $\ell = 0$ or 1, the field equations no longer give $\eta = 0$, which has been used to simplify the equations in this section. Instead, there are additional gauge freedoms that can be used to simplify (40)-(49). These are discussed in Appendix C.

B. Axial perturbation equations

The axial perturbation equations take a simpler form than their polar counterparts. Because we are considering dust dominated cosmologies, the field equations give us $\dot{\bar{v}} = 0$, and so $\bar{v} = \bar{v}(r)$ and must be set by initial conditions. For $\ell \geq 2$ the metric perturbation Π can be shown to obey the wave equation

$$-\ddot{\Pi} + \Pi'' - (6H_\perp + H_\|)\dot{\Pi} + 6W\Pi' \tag{50}$$

$$- \left[16\pi\rho + \frac{(\ell-2)(\ell+3)}{a_\perp^2 r^2}\right]\Pi = -16\pi\frac{(\rho\bar{v})'}{a_\perp^2 r^2},$$

which is the LTB version of (GMG69). This is the only dynamical equation that needs solving, for the single variable Π. Once this equation is solved it gives the axial metric perturbations, k_A, via

$$(\ell-1)(\ell+2)k_A = 16\pi\rho a_\perp^2 r^2\bar{v}u_A - \epsilon_{AB}(a_\perp^4 r^4\Pi)^{|B}. \tag{51}$$

Clearly $\ell = 1$ is a special case. For large angle perturbations of this type we have $(a_\perp^4 r^4\Pi) \dot{} = 0$ and $(a_\perp^4 r^4\Pi)' = -16\pi a_\perp^2 r^2\rho\bar{v}$. The solution for Π is then

$$\Pi = -\frac{2}{a_\perp^4 r^4}\int \frac{\bar{v}(Mr^3)_{,r}}{\sqrt{1 - \kappa r^2}}dr. \tag{52}$$

The metric perturbations k_A must then be obtained from inverting Eq. (32). This will involve an extra degree of freedom in k_A that means it cannot be determined uniquely.

There are no axial perturbations with $\ell = 0$.

C. Structure of the equations and solutions

It can be seen from the master equations, (40)-(49), that χ and Π contain gravitational wave degrees of freedom, as their leading derivatives are of the form $-(\ddot{\ }) + (\ )''$ (i.e. a wave equation with a characteristic speed of unity). It can also be seen that ς and $\bar{v}$ look rather like vectors in FLRW cosmology, while φ appears to govern density perturbations like the Newtonian potential. Unfortunately, this roughly analogous behaviour does not hold completely, as we will show below when we consider the FLRW limit. What is clear, however, is that

there exist complicated couplings between the variables in this system. We can therefore no longer expect gravitational waves to decouple from density perturbations, as the varying curvature of the background serves to couple these perturbations intimately.

The solutions to the perturbation equations in sections III A and III B will, in general, need to be found numerically. However, if we consider $\ell \geq 2$ and neglect the contribution of the gravitational waves described by χ, we can make progress. With $\eta = \chi = 0$, Equation (42) can be integrated to find

$$\varsigma \propto \frac{1}{a_\parallel^2}.$$

Clearly, $\varsigma \to 0$ as the Universe expands, and $a_\parallel \to \infty$. Assuming χ to be negligible can then be seen to lead to a negligible ς in the late Universe. With $\chi = \varsigma = 0$, Equation (41) becomes

$$\ddot{\varphi} + 4H_\perp \dot{\varphi} - \frac{2\kappa}{a_\perp^2}\varphi = 0. \tag{53}$$

This equation can be solved parametrically, together with the analogue of the Friedmann equation, (3). There are three solutions, depending on the sign of κ.

For $\kappa < 0$ we find the solution

$$a_\perp = \frac{M(1 - \cosh 2\Theta)}{2\kappa} \tag{54}$$

$$\varphi = \frac{\cosh \Theta}{\sinh^5 \Theta}[c_1 + c_2(\sinh 2\Theta - 6\Theta \tag{55}$$
$$+ 4\tanh \Theta)]$$

$$t - t_0 = \frac{M(\sinh 2\Theta - 2\Theta)}{2(-\kappa)^{3/2}}, \tag{56}$$

where $c_1 = c_1(r)$, $c_2 = c_2(r)$ and $t_0 = t_0(r)$ are free functions that must be specified as part of the initial conditions. The function $t_0(r)$ is the 'bang time'.

For $\kappa > 0$ we find

$$a_\perp = \frac{M(1 - \cos 2\Theta)}{2\kappa} \tag{57}$$

$$\varphi = \frac{\cos \Theta}{\sin^5 \Theta}[c_1 + c_2(\sin 2\Theta - 6\Theta \tag{58}$$
$$+ 4\tan \Theta)]$$

$$t - t_0 = \frac{M(2\Theta - \sin 2\Theta)}{2(\kappa)^{3/2}}. \tag{59}$$

Again, c_1, c_2 and t_0 are free functions of r.

Finally, for $\kappa = 0$, a parametric solution is not required, and we find

$$a_\perp = \frac{(18M)^{1/3}(t - t_0)^{2/3}}{2} \tag{60}$$

$$\varphi = \frac{c_1}{(t - t_0)^{5/3}} + c_2. \tag{61}$$

Once more, with c_1, c_2 and t_0 as free functions of r.

In the $\kappa \neq 0$ cases, Θ is a monotonically increasing function of t, and in all three cases c_1 corresponds to a decaying mode, and c_2 to a growing mode. In the cases with $\kappa \leq 0$ the 'growing mode' is itself either a constant (for $\kappa = 0$) or a decreasing function of t (for $\kappa < 0$), and in those cases we mean growing with respect to the other mode, which is decaying faster.

For $\kappa = 0$ the power-law forms of $a_\perp(t)$ and φ mean that the asymptotic form of both the growing and decaying modes is clear. It can also be seen for $\kappa \neq 0$ that both modes, together with $a_\perp$, behave as if $\kappa = 0$ in the limit $t \to t_0$ (as expected, due to their dynamical evolution being dominated by ρ, and not κ, at early times). In the case of $\kappa < 0$ the growing mode can be seen to decay as $\sim 1/(t - t_0)$ when $t \to \infty$, and the scale factor, $a_\perp$, behaves like an open Milne universe. For $\kappa > 0$ the scale factor initially grows, reaches a maximum of expansion, and then collapses to zero in finite time, t. As this future singularity is approached, both growing and decaying modes of φ diverge to infinity.

These solutions may serve as the basis for calculating the matter power spectrum. However, they will only remain so as long as the evolution equation for χ remains approximately satisfied.

IV. THE FLRW LIMIT

From the perturbed LTB equations above it is not clear what each gauge invariant variable corresponds to in terms of the more familiar Bardeen potentials, and so forth, of standard cosmology. Here we derive the standard FLRW perturbation equations in the GMG formalism, and re-express the GS and GMG variables in terms of FLRW variables.

In the FLRW limit $\kappa \to$ constant, and we have that $a_\perp$ and $a_\parallel \to a(t)$. The master equations in the polar sector (40)-(42) then become

$$\left[\partial_\tau^2 + 2\mathcal{H}\partial_\tau - \vec{\nabla}^2 + \frac{4(1 - \kappa r^2)}{r}\partial_r - \frac{2}{r^2}\right]\chi = 0,$$

and

$$\left[\partial_\tau^2 + 3\mathcal{H}\partial_\tau - 2\kappa\right]\varphi$$
$$= \left[-\mathcal{H}\partial_\tau + \frac{(1 - \kappa r^2)}{r}\partial_r + \frac{(\ell(\ell+1) - 2 + 4\kappa r^2)}{2r^2}\right]\chi,$$

and

$$[\partial_\tau + 2\mathcal{H}]\varsigma = \sqrt{1 - \kappa r^2}\partial_r\chi,$$

while in the axial sector we have

$$\left[\partial_\tau^2 + 6\mathcal{H}\partial_\tau - \vec{\nabla}^2 - \frac{4(1 - \kappa r^2)}{r}\left(\partial_r + \frac{3}{2r}\right) + 6\mathcal{H}^2\right]\Pi$$
$$= 16\pi\rho\frac{\sqrt{1 - \kappa r^2}}{ar^2}\partial_r\bar{v}, \tag{62}$$

and

$$\partial_\tau \bar{v} = 0, \tag{63}$$

where $d\tau \equiv dt/a$ is conformal time, and $\mathcal{H} \equiv a_{,\tau}/a$ is the conformal Hubble rate which obeys the Friedmann and Raychaudhuri equations

$$\mathcal{H}^2 = \frac{8\pi a^2 \rho}{3} - \kappa \quad \text{and} \quad \partial_\tau \mathcal{H} = -\frac{1}{2}\left(\mathcal{H}^2 + \kappa\right). \tag{64}$$

Throughout this section $\vec{\nabla}^2$ will always refer to the Laplacian acting on a 3-scalar, so that

$$\vec{\nabla}^2 = (1 - \kappa r^2)\partial_r^2 + \frac{(2 - 3\kappa r^2)}{r}\partial_r - \frac{\ell(\ell+1)}{r^2}.$$

It is evident from the equations above that χ is a gravitational wave, as its characteristics are null. However, χ can also be seen to act as a source for φ even though the homogeneous part of the evolution equation looks like very similar to the Bardeen equation. Similarly, ς looks like it almost obeys the vector decay law, but is also coupled to gravitational waves through χ. The fact that these variables do not decouple in the FLRW limit means that their interpretation in terms of FLRW gauge invariants will not be straightforward.

Let us now consider the perturbed FLRW line-element in the longitudinal, or conformal Newtonian, gauge:

$$ds^2 = -a^2(1 + 2\Phi)d\tau^2 - 2a^2 V_i d\tau dx^i \tag{65}$$
$$+ a^2[(1 - 2\Psi)\gamma_{ij} + h_{ij}]dx^i dx^j,$$

where $a = a(\tau)$ is the scale factor, and $\gamma_{ij} = \text{diag}[(1 - \kappa r^2)^{-1}, r^2 \gamma_{ab}]$ is the background spatial metric in spherical coordinates of curvature κ. $\vec{\nabla}_i$ is the covariant derivative with respect to γ_{ij}. The metric (65) is split in the standard way into two 3-scalars (Φ, Ψ), a 3-vector (V_i) and a 3-tensor (h_{ij}), where the '3' is not usually emphasised. All of these quantities are gauge invariant variables (see Appendix D). The vector V_i is divergence-free, and the tensor h_{ij} is divergence and trace-free. Note that the coordinates used in the metric in the longitudinal gauge, given by Eq. (65), and the metric in the RW gauge, Eqs. (26) and (30), are not the same.

In Appendix D we show that, in the FLRW limit, the LTB gauge invariants (φ, ς, χ, η, Π and $\bar{v}$) can be written in terms of the usual FLRW invariants (Φ, Ψ, V_i and h_{ij}) in the following way:

Polar:

$$\varphi = -2\Psi - 2\mathcal{H}V - 2\frac{(1 - \kappa r^2)}{r}h_r + \frac{1}{r^2}h^{(T)} + \left[-\mathcal{H}\partial_\tau + \frac{(1 - \kappa r^2)}{r}\partial_r + \frac{\ell(\ell+1) - 4(1 - \kappa r^2)}{2r^2}\right]h^{(TF)}, \tag{66}$$

$$\varsigma = \sqrt{1 - \kappa r^2}\left\{V_r - \partial_r V + \partial_\tau h_r - \left(\partial_r - \frac{1}{r}\right)\partial_\tau h^{(TF)}\right\}, \tag{67}$$

$$\chi = (1 - \kappa r^2)h_{rr} + 2\left[-(1 - \kappa r^2)\partial_r + \frac{1}{r}\right]h_r - \frac{1}{r^2}h^{(T)}$$
$$+ \left[(1 - \kappa r^2)\partial_r^2 - \frac{(3 - 2\kappa r^2)}{r}\partial_r - \frac{\ell(\ell+1) - 8 + 4\kappa r^2}{2r^2}\right]h^{(TF)}, \tag{68}$$

$$\eta = \Phi - \Psi - (\partial_\tau + 2\mathcal{H})V + \frac{1}{2}(1 - \kappa r^2)h_{rr} + \left[-(1 - \kappa r^2)\partial_r + \kappa r\right]h_r$$
$$+ \frac{1}{2}\left[-\partial_\tau^2 + (1 - \kappa r^2)\partial_r^2 - 2\mathcal{H}\partial_\tau - \frac{(2 - \kappa r^2)}{r}\partial_r + \frac{2}{r^2}\right]h^{(TF)}. \tag{69}$$

Axial:

$$\Pi = \frac{\sqrt{1 - \kappa r^2}}{a^2 r^2}\left[\left(\partial_r - \frac{2}{r}\right)\bar{V} + \partial_\tau \bar{h}_r\right] \tag{70}$$

$$16\pi \rho a \bar{v} = \left[-\vec{\nabla}^2 + \frac{(2 - 4\kappa r^2)}{r}\partial_r - 4\kappa\right]\bar{V} \tag{71}$$

where, in order to compare with the GMG formalism, we have split the 3-vector V_i and the 3-tensor h_{ij} into their radial and angular parts, and then these into their polar and axial components, as described in Appendix D. This makes explicit the mixing of SVT modes in the

LTB gauge invariants. We have substituted Eq. (D10) to remove the tensor contribution to $\bar{v}$ in Eq. (71).

A. Polar Perturbations

As we are dealing with FLRW perturbations we may separate the scalar, vector and tensor parts of these equations, as we know they evolve independently.

1. Scalars

From $\eta = 0$ we have $\Phi = \Psi$, as is usual in a dust dominated FLRW cosmology. From the φ-equation we then find

$$\partial_\tau^2 \Psi + 3\mathcal{H}\partial_\tau \Psi - 2\kappa\Psi = 0, \qquad (72)$$

which is the usual Bardeen equation for the Newtonian potential. The ς- and χ-equations contain no scalar modes. We then note that the scalar part of the gauge invariant matter perturbations can be written

$$4\pi a^2 \rho \, \Delta = \vec{\nabla}^2 \Psi - 3\mathcal{H}\partial_\tau \Psi - 3(\mathcal{H}^2 - \kappa)\Psi, \quad (73)$$
$$4\pi a \rho \, v = -(\partial_\tau + \mathcal{H})\Psi, \qquad (74)$$
$$4\pi a^2 \rho \, w = -\sqrt{1 - \kappa r^2}(\partial_\tau + \mathcal{H})\partial_r \Psi. \qquad (75)$$

It can be seen from (73) that the scalar part of the energy density perturbation is just the gauge invariant density fluctuation $\delta\rho^{(GI)} \equiv \delta\rho + \partial_\tau\rho(B - \partial_\tau E)$ [16]. However, we will see below that we cannot simply identify $\Delta = \delta\rho^{(GI)}/\rho$, as Δ has non-zero vector and tensor parts.

2. Vectors

From $\eta = 0$, and the evolution equation for ς, we find that

$$\partial_\tau V = -2\mathcal{H}V \quad \text{and} \quad \partial_\tau V_r = -2\mathcal{H}V_r. \qquad (76)$$

Thus, these two equations give the usual a^{-2} law for vector modes. The vector part of the evolution equation for φ automatically follows from these two, while the χ-equation contains no vectors. The vector parts of the gauge invariant matter perturbations are then given by

$$\Delta = 3\mathcal{H}V \qquad (77)$$
$$16\pi a \rho \, v = \left[(1 - \kappa r^2)\partial_r - \kappa r\right] V_r \qquad (78)$$
$$\qquad - \left[(1 - \kappa r^2)\partial_r^2 - \kappa r\partial_r - 2(3\mathcal{H}^2 + \kappa)\right] V,$$
$$16\pi a^2 \rho \, w = \sqrt{1 - \kappa r^2}\bigg\{ \left[\frac{\ell(\ell+1)}{r^2} + 3\mathcal{H}^2 - \kappa\right] V_r \quad (79)$$
$$\qquad + \left[\frac{\ell(\ell+1)}{r^2} - 3(\mathcal{H}^2 + \kappa)\right]\partial_r V \bigg\}.$$

It is a curiosity that the vector part of $\Delta = 3\mathcal{H}V$ is non-zero, so that while Δ looks like it is just a gauge invariant density perturbation, it actually contains vectors.

3. Tensors

The GMG formalism is rather less tidy when trying to describe 3-tensor modes. The equation which looks like it should be describing gravitational waves is that for χ. But while this equation consists only of tensor modes, it is in a very ugly form when evaluated in terms

of the FLRW perturbed metric, as the χ equation contains fourth derivatives of $h^{(TF)}$. In fact, as all four GMG variables contain 3-tensors, all four field equations contain tensor modes, and showing that they reduce to the usual FLRW wave equation for tensors is non-trivial. The four equations form a set of coupled PDEs for the variables $h_{rr}, h_r, h^{(T)}, h^{(TF)}$, the simplest of which may be read off from $\eta = 0$ above. Recall, however, that there is only one degree of freedom in the polar part of h_{ij}, with the trace and divergence free conditions removing the other three. If we choose this degree of freedom to be $h^{(T)}$ then it can be shown from the GMG equations for η, φ and ς, when combined with the conditions given by Eqs. (D7), (D8) and (D9), reduce to the single master equation

$$[\partial_\tau^2 + 2\mathcal{H}\partial_\tau - \vec{\nabla}^2]h^{(T)} = 0, \qquad (80)$$

which is just the trace of the usual FLRW wave equation. The χ equation then follows automatically.

The tensor parts of the variables associated with the fluid can then be written

$$\Delta = \frac{3}{2}\mathcal{H}\partial_\tau h^{(TF)}, \qquad (81)$$
$$v = \frac{1}{2}a\partial_\tau h^{(TF)}, \qquad (82)$$
$$w = \frac{1}{2}\sqrt{1 - \kappa r^2}\left(\partial_\tau h_r + \frac{1}{r}\partial_\tau h^{(TF)}\right). \qquad (83)$$

B. Axial Perturbations

There are no scalar modes in the axial sector.

For vector modes it can be seen from $\partial_\tau \bar{v} = 0$, and the radial equation of (51), that the usual vector decay law, $\partial_\tau \bar{V} = -2\mathcal{H}\bar{V}$, is obeyed. The Π-equation is then automatically satisfied.

Combining the radial equation of (51) and Eq. (D10), we find the wave equation

$$[\partial_\tau^2 + 2\mathcal{H}\partial_\tau - \vec{\nabla}^2 + 2\kappa r\partial_r + 3\kappa]\bar{h}_r = 0, \qquad (84)$$

where the extra curvature terms are due to $\vec{\nabla}^2$ acting here on a scalar that is actually part of a 3-tensor. The equation for Π then follows.

C. SVT Variables

We have found that what appear as gauge invariant fluid perturbations in the GMG formalism – namely Δ, v and w – are not exclusively fluid modes, as they are usually understood in FLRW cosmology, as they can be excited by tensors (FLRW models need a tensor part in the anisotropic stress for fluid perturbations to couple to tensor modes, which we have not included here). It is also the case that the GMG metric perturbations do not correspond in a straightforward way to the SVT decomposition we are familiar with from FLRW cosmology. For

example, it is clear that χ represents gravitational waves, and in the FLRW limit it is a pure tensor mode: but what about scalars and vectors? Can we identify combinations of gauge invariant GMG variables which will represent purely scalar or vector modes in the FLRW limit, and so be useful physical variables in LTB cosmology?

In order to find these, let us first note that the combination

$$\lambda \equiv 8\pi a^2 \mathcal{H}^{-1} \rho \left[\Delta - 3\frac{\mathcal{H}}{a} v \right] \qquad (85)$$

does not contain any tensor modes in the FLRW limit (as can be verified from Eqs. (81) and (82)). Also, note that we can construct from ς and χ a quantity that contains only vector modes:

$$\xi \equiv \frac{3r}{2\sqrt{1-\kappa r^2}} [\vec{\nabla}^2 + 3\kappa] r^{-1} \varsigma + \frac{3}{2r^2} \partial_r (r^2 \partial_\tau \chi)$$
$$= \frac{3}{2} \left[\vec{\nabla}^2 + 3\kappa - \frac{2}{r} \partial_r \right] (V_r - \partial_r V). \qquad (86)$$

From λ and ξ we can now construct a variable that is a function of scalar modes only:

$$\zeta \equiv a^{-2} \vec{\nabla}^2 \lambda + a^{-2} \left[(1-\kappa r^2) \partial_r + \frac{2-3\kappa r^2}{r} \right] \xi$$
$$= \frac{2}{a^2 \mathcal{H}} \left[\vec{\nabla}^2 + 3\kappa \right] \vec{\nabla}^2 \Psi. \qquad (87)$$

This is related to the gauge invariant density fluctuation $\delta\rho^{(GI)}$ and the curvature perturbation, $\mathcal{R} = \Psi - \mathcal{H}(\mathcal{H}\Phi + \partial_\tau \Psi)/(\partial_\tau \mathcal{H} - \mathcal{H}^2 - \kappa)$ [17], by

$$\frac{\mathcal{H}\zeta}{8\pi\rho} = \vec{\nabla}^2 \left[\frac{\delta\rho^{(GI)}}{\rho} + 3\mathcal{R} - 3\Psi \right]. \qquad (88)$$

For the axial perturbations we already see that $\bar{v}$ is a pure vector mode, but that Π is an awkward mixture of vectors and tensors. Defining the variable

$$\Upsilon \equiv a^{-2} \left[\vec{\nabla}^2 + \frac{4(1-\kappa r^2)}{r} \partial_r + \frac{2(3-4\kappa r^2)}{r^2} \right] \Pi$$
$$+ 16\pi\rho \frac{\sqrt{1-\kappa r^2}}{a^3 r^2} \partial_r \bar{v} \qquad (89)$$
$$= \frac{\sqrt{1-\kappa r^2}}{a^4 r^2} \left[\vec{\nabla}^2 - 2\kappa r \partial_r - \kappa \right] \partial_\eta \bar{h}_r \qquad (90)$$

we are able to define a pure tensor mode using the GMG gauge invariant variables.

Using these new variables, we can now generalise to the inhomogeneous case. The resulting functions will have the useful feature that they reduce to pure scalar or vector modes in the FLRW limit. An example set of variables is given by Eqs. (34)-(39), in section II C 1. These reduce to the quantities above in the FLRW limit. The variables $\hat{\zeta}$, $\hat{\xi}$, $\hat{\bar{v}}$, $\hat{\chi}$ and $\hat{\Upsilon}$, are then useful gauge invariant variables in the LTB perturbation equations as they encode generalised scalar, vector and tensor perturbations, respectively, as they are normally referred to in cosmology.

V. DISCUSSION

We have presented here a full system of master equations that represent the general perturbations to LTB space-times, in terms of the gauge invariant variables $\varphi, \varsigma, \chi, \Pi, \bar{v}$ (and η, strictly speaking). This formalism can now be used to investigate, in a fully consistent fashion, the growth of linear structure in LTB models. As such, predictions for phenomena such as baryon acoustic oscillations, the integrated Sachs-Wolfe effect, and the observed matter power spectrum can now be made, and used to compare with astrophysical data. This will allow us to establish what deviations should be expected from the usual FLRW predictions, and whether or not such deviations can be observed in our Universe.

However, while $\varphi, \varsigma, \chi, \Pi, \bar{v}$ and η appear completely naturally, and produce an elegant system of equations in the form of an initial value formulation, they do not reduce to anything intuitive in the FLRW limit. Instead, they are a cumbersome mixture of scalar, vector and tensor perturbations – a situation made worse still by the additional couplings that exist in the evolution equations in the more general, inhomogeneous case.

To address these problems we have defined a new set of variables in the polar sector, $\hat{\zeta}$, $\hat{\xi}$ and $\hat{\chi}$, that can be thought of as generalised scalar, vector and tensor perturbations, respectively. Similarly, we propose new variables in the axial sector, $\hat{\bar{v}}$ and $\hat{\Upsilon}$, that correspond to generalised vector and tensor modes. In the FLRW limit these new variables become pure scalars, vectors and tensors. We expect these new variables to prove useful in future studies where they can be used to set the initial conditions for the evolution of perturbations. For example, in an LTB model with homogeneous bang time, $t_0(r) =$constant, the surfaces of constant t will be almost homogeneous at early times, and will then be well approximated by an FLRW description. The new variables can then be used to match the FLRW initial conditions onto the LTB evolution equations in a straightforward way. This will be considered further in future studies.

APPENDIX A: THE GERLACH-SENGUPTA FORMALISM

The general form of perturbations to the metric and stress-energy tensors can be written, in terms of the harmonic functions outlined in Section II, as $g_{\mu\nu} \to g_{\mu\nu} + \Delta g_{\mu\nu}$ and $T_{\mu\nu} \to T_{\mu\nu} + \Delta T_{\mu\nu}$, where [11]

$$\Delta g_{\mu\nu} \equiv \begin{pmatrix} 0 & h_A^{\text{axial}} \bar{Y}_a \\ h_A^{\text{axial}} \bar{Y}_a & h \, \bar{Y}_{ab} \end{pmatrix} \qquad (A1)$$

$$\Delta T_{\mu\nu} \equiv \begin{pmatrix} 0 & \Delta t_A^{\text{axial}} \bar{Y}_a \\ \Delta t_A^{\text{axial}} \bar{Y}_a & \Delta t^{(1)} \, \bar{Y}_{ab} \end{pmatrix}, \qquad (A2)$$

for axial perturbations, and

$$\Delta g_{\mu\nu} \equiv \begin{pmatrix} h_{AB}Y & h_A^{\text{polar}}Y_a \\ h_A^{\text{polar}}Y_a & a_\perp^2 r^2(KY\gamma_{ab} + GY_{:ab}) \end{pmatrix} \quad \text{(A3)}$$

$$\Delta T_{\mu\nu} \equiv \begin{pmatrix} \Delta t_{AB}Y & \Delta t_A^{\text{polar}}Y_a \\ \Delta t_A^{\text{polar}}Y_a & \Delta t^{(2)}Y_{ab} + a_\perp^2 r^2 \Delta t^{(3)}Y\gamma_{ab} \end{pmatrix}, \quad \text{(A4)}$$

for polar perturbations. The new variables h_A^{axial}, h, $\Delta t_A^{\text{axial}}$, $\Delta t^{(1)}$, h_{AB}, h_A^{polar}, K, G, Δt_{AB}, $\Delta t_A^{\text{polar}}$, $\Delta t^{(2)}$ and $\Delta t^{(3)}$ are all functions of the two coordinates x^A.

From these quantities we can then construct the gauge invariant variable [11]

$$k_A \equiv h_A^{\text{axial}} - h_{|A} + 2hv_A, \quad \text{(A5)}$$

for axial perturbations, and

$$k_{AB} \equiv h_{AB} - p_{A|B} - p_{B|A} \quad \text{(A6)}$$
$$\varphi \equiv K - 2v^A p_A \quad \text{(A7)}$$
$$t_{AB} \equiv \Delta t_{AB} - t_{AB|C}p^C - t_{AC}p^C_{|B} - t_{BC}p^C_{|A} \quad \text{(A8)}$$
$$t_A \equiv \Delta t_A^{\text{polar}} - t_{AC}p^C, \quad \text{(A9)}$$

for scalar perturbations. The quantities $\Delta t_A^{\text{axial}}$, $\Delta t^{(1)}$, $\Delta t^{(2)}$ and $\Delta t^{(3)}$ are already gauge invariant for a dust dominated universe. Here a pipe indicates a covariant derivative with respect to the metric on M^2, and we have defined $p_A \equiv h_A^{\text{polar}} - a_\perp^2 r^2 G_{|A}/2$, along with GS, for concision. All of these expressions are constructed so as to be invariant under infinitesimal coordinate transformations of the form $x^\mu \to x^\mu + \xi^\mu$.

There exists here a useful gauge in which $h = h_A^{\text{polar}} = G = 0$. This is the Regge-Wheeler gauge [15], where the remaining perturbation variables are all equal to gauge invariant quantities, as can be seen from the expressions above.

APPENDIX B: THE GUNDLACH-MARTÍN-GARCÍA FORMALISM

The background quantities used by GMG can be written in the LTB case, using our notation, as

$$U = H_\perp \quad \text{(B1)}$$
$$V = \frac{H_\perp}{W} \quad \text{(B2)}$$
$$m = \frac{M}{2} \quad \text{(B3)}$$
$$\mu = H_\parallel \quad \text{(B4)}$$
$$\nu = 0. \quad \text{(B5)}$$

GMG then write their perturbed fluid four velocity as $u_\mu \to u_\mu + \Delta u_\mu$, where

$$\Delta u_\mu = (0, \bar v\, \bar Y_a) \quad \text{(B6)}$$

for axial perturbations, and

$$\Delta u_\mu = \left[\left(\tilde w n_A + \frac{1}{2}h_{AB}u^B\right)Y, \tilde v\, Y_a\right] \quad \text{(B7)}$$

for scalar perturbations. Here $n_A \equiv -\epsilon_{AB}u^B$ is a unit space-like vector, and $\tilde v$, $\bar v$ and $\tilde w$ are all functions of x^A. Density perturbations are then parametrised by $\rho \to \rho + \tilde\Delta Y\rho$, and the following gauge invariant quantities can be constructed:

$$v \equiv \tilde v - p^B u_B \quad \text{(B8)}$$
$$w \equiv \tilde w - n^A u_{A|B}p^B + \frac{1}{2}n^A u^B(p_{A|B} - p_{B|A}) \quad \text{(B9)}$$
$$\Delta \equiv \tilde\Delta - p^A(\ln\rho)_{|A} \quad \text{(B10)}$$

with $\bar v$ already gauge invariant. Again, the Regge-Wheeler gauge can be seen to have a special significance. The gauge invariants above can be related to those in Appendix A via

$$L_A = \bar v u_A \quad \text{(B11)}$$
$$t_A = v\rho u_A \quad \text{(B12)}$$
$$t_{AB} = \rho\Big[w(u_A n_B + n_A u_B) + \Delta u_A u_B \quad \text{(B13)}$$
$$+ \frac{1}{2}(k_{AC}u_B + u_A k_{BC})\Big] \quad \text{(B14)}$$

together with $L = t^{(2)} = t^{(3)} = 0$. GMG then continue to decompose k_{AB} into the 'fluid frame' via

$$k_{AB} \equiv \eta(n_A n_B - u_A u_B) + \phi(n_A n_B + u_A u_B) \quad \text{(B15)}$$
$$+ \varsigma(u_A n_B + n_A u_B), \quad \text{(B16)}$$

and to define the new variable

$$\chi \equiv \phi - \varphi + \eta. \quad \text{(B17)}$$

In these variables the perturbation equations take a particularly simple form [8].

APPENDIX C: POLAR $\ell = 0, 1$ PERTURBATIONS

For polar perturbations with $\ell = 0, 1$ it is no longer the case that the field equations give $\eta = 0$. The general system of equations given in Sec III A now reads:

For $\ell \geq 1$:

$$-\ddot{\chi} + \chi'' + 2(H_\parallel - H_\perp)\varsigma' - 2\eta'' \tag{C1}$$
$$= -2\left[8\pi\rho - \frac{3M}{a_\perp^3} - 2H_\perp(H_\parallel - H_\perp)\right](\chi + \varphi) + \frac{(\ell-1)(\ell+2)}{a_\perp^2 r^2}\chi + 3H_\parallel\dot{\chi} + 4(H_\parallel - H_\perp)\dot{\varphi} + 2W\chi'$$
$$-2\left[H_\parallel' - 2(H_\parallel - H_\perp)W\right]\varsigma - 2(H_\parallel - H_\perp)\dot{\eta} - 6W\eta' - \left[\frac{\ell(\ell+1) + 8}{a_\perp^2 r^2} + 8H_\parallel H_\perp + 4H_\perp^2 - 16W^2 - 32\pi\rho\right]\eta,$$

$$-\ddot{\varphi} = 4H_\perp\dot{\varphi} + \left[2W^2 - \frac{\ell(\ell+1) + 2}{2a_\perp^2 r^2}\right](\chi - 2\eta) - 2W(H_\parallel - H_\perp)\varsigma \tag{C2}$$
$$+H_\perp(\dot{\chi} - 2\dot{\eta}) - W(\chi' - 2\eta') - 2W^2\eta - 2\left(\frac{1}{a_\perp^2 r^2} - W^2\right)\varphi,$$

$$-\dot{\varsigma} = 2H_\parallel\varsigma + \chi' + 2W\eta - 2\eta'. \tag{C3}$$

$$8\pi\rho v = \frac{\varsigma'}{2} + H_\parallel(\chi + \varphi) + \frac{\dot{\chi}}{2} + \dot{\varphi} - (H_\parallel + H_\perp)\eta. \tag{C4}$$

$$\dot{v} = \frac{\chi}{2} + \frac{\varphi}{2} - \eta, \tag{C5}$$

and for $\ell \geq 0$:

$$8\pi\rho w = (\dot{\varphi})' - W\dot{\chi} + H_\perp\chi' - (H_\parallel - 2H_\perp)\varphi' - 2H_\perp\eta' + \left[\frac{\ell(\ell+1) + 2}{2a_\perp^2 r^2} + H_\perp^2 + 2H_\perp H_\parallel - W^2 - 4\pi\rho\right]\varsigma, \tag{C6}$$

$$8\pi\rho\Delta = \left[\frac{\ell(\ell+1)}{a_\perp^2 r^2} + 2H_\perp^2 + 4H_\parallel H_\perp - 8\pi\rho\right](\chi + \varphi) + 2H_\perp\varsigma' - \frac{(\ell-1)(\ell+2)}{2a_\perp^2 r^2}\chi \tag{C7}$$
$$+2(H_\parallel + H_\perp)W\varsigma + H_\perp\dot{\chi} - \varphi'' + (H_\parallel + 2H_\perp)\dot{\varphi} + W\chi' - 2W\varphi' - 2H_\perp(H_\perp + 2H_\parallel)\eta,$$

$$\dot{w} = \frac{\varphi'}{2} - H_\parallel w - \frac{H_\parallel}{2}\varsigma - W\eta, \tag{C8}$$

$$\dot{\Delta} + \left(w + \frac{\varsigma}{2}\right)' = \frac{\ell(\ell+1)}{a_\perp^2 r^2}v - \left(w + \frac{\varsigma}{2}\right)\frac{\rho'}{\rho} - \frac{\dot{\chi}}{2} - \frac{3\dot{\varphi}}{2} - 2W\left(w + \frac{\varsigma}{2}\right). \tag{C9}$$

1. $\ell = 1$

For $\ell = 1$ there is an additional gauge freedom which can be used to eliminate one of the perturbation variables. The obvious choice may be to use this to set $\eta = 0$, but GMG showed this to lead to some ambiguity. We therefore follow them in using the less ambiguous choice $\varphi = 0$. The evolution and conservations equations above can then be combined to give

$$W\eta' - H_\perp\dot{\eta} = 4\pi\rho\Delta - 16\pi\rho H_\perp v - \left(\frac{2}{a_\perp^2 r^2} - 3W^2 + H_\perp^2\right)\eta - \left[W^2 + H_\perp^2 - 4\pi\rho\right]\chi - 2H_\perp W\varsigma, \tag{C10}$$

$$W\chi' - H_\perp\dot{\chi} = 8\pi\rho\Delta - 32\pi\rho H_\perp v - \left[\frac{2}{a_\perp^2 r^2} + 2H_\perp^2 - 8\pi\rho\right]\chi - 2(H_\parallel + H_\perp)W\varsigma - 2H_\perp^2\eta, \tag{C11}$$

$$W\varsigma' - H_\perp\dot{\varsigma} = 8\pi\rho(w + 2Wv) + 2H_\parallel W(\eta - \chi) + 4H_\perp W\eta - \left[H_\perp^2 - W^2 + \frac{2}{a_\perp^2 r^2} - 4\pi\rho\right]\varsigma. \tag{C12}$$

These are the LTB equivalents of (GMGA12), (GMGA13) and (GMGA14) from [8]. These three equations can be solved, along with the three conservation equations (47)-(49), with $\varphi = 0$, for the six remaining perturbation variables, η, χ, ς, v, w and Δ.

<div align="center">

2. $\quad \ell = 0$

</div>

Polar perturbations with $\ell = 0$ are spherical. As such one could conceive of absorbing such perturbations into the background parameters, which themselves describe the most general spherically symmetric dust dominated solution of Einstein's equations. Despite this being the case, it still seems worth considering $\ell = 0$ perturbations in a spherical background, as one can then separate them out more easily from a smooth background.

In this case we again have a non-zero η, but now have sufficient gauge freedom to set $\varphi = 0$ and

$$\varsigma = \frac{2H_\perp W}{H_\perp^2 + W^2}(\eta - \chi), \tag{C13}$$

as done by GMG. The constraint equations then give

$$W\eta' - H_\perp \dot{\eta} = 4\pi\rho \left(\chi + \frac{2H_\perp^2}{(W^2 - H_\perp^2)}\eta \right) + 16\pi\rho \frac{H_\perp W}{(W^2 - H_\perp^2)}w + 4\pi\rho \frac{(W^2 + H_\perp^2)}{(W^2 - H_\perp^2)}\Delta, \tag{C14}$$

and

$$\begin{aligned}
W\chi' - H_\perp \dot{\chi} = {} & \frac{4H_\perp W}{(H_\perp^2 + W^2)}\left[H_\perp W + 4\pi\rho \frac{H_\perp W}{(W^2 - H_\perp^2)} \right](\chi - \eta) + 16\pi\rho \frac{H_\perp W}{(W^2 - H_\perp^2)}w \\
& + 8\pi\rho \frac{(W^2 + H_\perp^2)}{(W^2 - H_\perp^2)}\Delta + \left(8\pi\rho - \frac{1}{a_\perp^2 r^2} \right)\left(\chi + \frac{2H_\perp^2}{(W^2 - H_\perp^2)}\eta \right).
\end{aligned} \tag{C15}$$

These are the LTB versions of (GMGA18) and (GMGA19). The two remaining conservation equations are then

$$\begin{aligned}
\dot{w} = {} & -\left[H_\parallel + 4\pi\rho \frac{H_\perp}{(W^2 - H_\perp^2)} \right]w - \frac{W}{2a_\perp^2 r^2(W^2 - H_\perp^2)}\eta \\
& + (\eta - \chi)\left[\frac{W(W^2 - H_\perp^2)}{2(W^2 + H_\perp^2)} - \frac{H_\parallel H_\perp W}{(W^2 + H_\perp^2)} + 4\pi\rho \frac{W H_\perp^2}{(H_\perp^4 - W^4)} \right],
\end{aligned} \tag{C16}$$

and

$$\begin{aligned}
\dot{\Delta} + w' = {} & \frac{4\pi\rho H_\perp}{(W^2 - H_\perp^2)}\Delta - \frac{H_\perp}{(W^2 - H_\perp^2)}\left(\frac{1}{2a_\perp^2 r^2} - 4\pi\rho \right)\eta - \left[\frac{\rho'}{\rho} + 2W - 4\pi\rho \frac{W}{(W^2 - H_\perp^2)} \right]w \\
& - (\eta - \chi)\left[\frac{H_\perp W}{(W^2 + H_\perp^2)}\frac{\rho'}{\rho} + H_\perp + H_\parallel + \frac{H_\perp(W^2 - H_\perp^2)}{2(W^2 + H_\perp^2)} + 4\pi\rho \frac{H_\perp W^2}{(W^4 - H_\perp^4)} \right].
\end{aligned} \tag{C17}$$

These equations are the LTB versions of (GMGA15) and (GMGA16). We now have four equations to solve for the four remaining variables η, χ, w and Δ.

APPENDIX D: MATCHING LTB AND FLRW GAUGE INVARIANTS

Here we derive the standard FLRW perturbation equations in the GMG formalism. Unfortunately, the RW gauge is not well adapted to our usual description of FLRW perturbations. We must therefore write both the GMG variables and the perturbed FLRW metric in a general gauge.

In general coordinates an arbitrary perturbation of

FLRW can be written

$$\begin{aligned}
ds^2 = {} & -a^2(1 + 2\phi)d\tau^2 + 2a^2(\vec{\nabla}_i B - S_i)d\tau dx^i \\
& + a^2\left[(1 - 2\psi)\gamma_{ij} + 2\vec{\nabla}_i\vec{\nabla}_j E + 2\vec{\nabla}_i F_j + h_{ij} \right]dx^i dx^j,
\end{aligned} \tag{D1}$$

where $a = a(\tau)$, γ_{ij} is the spatial metric, and $\vec{\nabla}_i$ is the covariant derivative with respect to γ_{ij}. The vectors here are divergence free, $\vec{\nabla}_i S^i = 0$, and the tensors are divergence and trace-free. Gauge invariant metric perturba-

tions are given by

$$
\begin{aligned}
\Phi &= \phi + \mathcal{H}(B - \partial_\tau E) + \partial_\tau(B - \partial_\tau E), \\
\Psi &= \psi - \mathcal{H}(B - \partial_\tau E), \\
V_i &= S_i + \partial_\tau F_i,
\end{aligned}
\tag{D2}
$$

and the perturbations h_{ij} are already gauge-invariant.

To compare with the GMG formalism, we expand this 1+3 split into a 1+1+2 split in spherical coordinates, peeling off the radial parts of each variable. A 3-vector, such as S_i, then splits into a scalar, S_r, and a 2-vector, S_a. These can then be decomposed into spherical harmonics as

$$
\begin{aligned}
S_r &= \sum_{\ell m} S_r^{(\ell m)} Y^{(\ell m)} = S_r Y, \\
S_a &= \sum_{\ell m} S^{(\ell m)} Y_{:a}^{(\ell m)} + \bar{S}^{(\ell m)} \epsilon_a{}^b Y_{:b}^{(\ell m)}, \\
&= S Y_a + \bar{S} \bar{Y}_a,
\end{aligned}
\tag{D3}
$$

where S and $\bar{S}$ are the polar and axial parts of S, respectively. The divergence-free property of the 3-vector, $\vec{\nabla}_i S^i = 0$, then gives us that

$$
(1 - \kappa r^2)\partial_r S_r + \frac{(2 - 3\kappa r^2)}{r} S_r - \frac{\ell(\ell+1)}{r^2} S = 0. \tag{D4}
$$

This reduces the degrees of freedom in the 3-vector to two: one polar and one axial.

For a 3-tensor, the split into radial and angular parts is a bit more messy. The tensor

$$
h_{ij} = \left(\begin{array}{c|c} h_{rr} & h_{ra} \\ \hline h_{ra} & h_{ab} \end{array} \right)
\tag{D5}
$$

splits into a 2-scalar $h_{rr} = h_{rr}^{(\ell m)} Y^{(\ell m)}$ (spherical harmonic sum implied), a 2-vector $h_{ra} = h_r^{(\ell m)} Y_a^{(\ell m)} + \bar{h}_r^{(\ell m)} \bar{Y}_a^{(\ell m)}$ and a remaining part consisting of a 2-scalar (the trace) and polar and axial trace-free 2-tensors:

$$
h_{ab} = h^{(T)} \gamma_{ab} Y + h^{(TF)} Y_{ab} + \bar{h} \bar{Y}_{ab}.
\tag{D6}
$$

This splits the 3-tensor h_{ij} into its polar (h_{rr}, h_r, $h^{(T)}$ and $h^{(TF)}$) and axial ($\bar{h}_r$ and $\bar{h}$) parts.

The trace-free, $\gamma^{ij} h_{ij} = 0$, and divergence-free, $\vec{\nabla}^i h_{ji} = 0$, conditions give:

$$
0 = (1 - \kappa r^2)h_{rr} + \frac{2}{r^2}h^{(T)},
\tag{D7}
$$

$$
0 = (1 - \kappa r^2)\partial_r h_{rr} + \frac{(2 - 4\kappa r^2)}{r}h_{rr} - \frac{2}{r^3}h^{(T)} - \frac{\ell(\ell+1)}{r^2}h_r,
\tag{D8}
$$

$$
0 = (1 - \kappa r^2)\partial_r h_r + \frac{(2 - 3\kappa r^2)}{r}h_r + \frac{1}{r^2}h^{(T)} - \frac{(\ell-1)(\ell+2)}{2r^2}h^{(TF)},
\tag{D9}
$$

$$
0 = (1 - \kappa r^2)\partial_r \bar{h}_r + \frac{(2 - 3\kappa r^2)}{r}\bar{h}_r - \frac{(\ell-1)(\ell+2)}{2r^2}\bar{h}.
\tag{D10}
$$

Again, the degrees of freedom left after applying these relations is two: one per parity.

We can now equate the perturbed FLRW with the perturbed LTB metric, in an arbitrary gauge, to find that the polar perturbations are related by

$$
a^{-2}h_{\tau\tau}^{GMG} = -2\phi,
\tag{D11}
$$

$$
a^{-2}h_{rr}^{GMG} = -\frac{2}{1 - \kappa r^2}\psi + 2\left(\partial_r - \frac{\kappa r}{1 - \kappa r^2}\right)\partial_r E + 2\left(\partial_r - \frac{\kappa r}{1 - \kappa r^2}\right)F_r + h_{rr},
\tag{D12}
$$

$$
a^{-2}h_{r\tau}^{GMG} = -S_r + \partial_r B,
\tag{D13}
$$

$$
a^{-2}h_\tau^{GMG} = -S + B,
\tag{D14}
$$

$$
a^{-2}h_r^{GMG} = 2\left(\partial_r - \frac{1}{r}\right)E + F_r + \left(\partial_r - \frac{2}{r}\right)F + h_r,
\tag{D15}
$$

$$
r^2 G = 2E + 2F + h^{(TF)},
\tag{D16}
$$

$$
K = -2\psi + 2\frac{1 - \kappa r^2}{r}\partial_r E + 2\frac{1 - \kappa r^2}{r}F_r + \frac{1}{r^2}h^{(T)} + \frac{\ell(\ell+1)}{2r^2}h^{(TF)},
\tag{D17}
$$

where everything has been decomposed into spherical harmonics, so that $\phi = \phi^{(\ell m)}(x^A)$ etc. For the axial modes we

have

$$a^{-2}h_\tau^{GMG} = -\bar{S}, \tag{D18}$$

$$a^{-2}h_r^{GMG} = \left(\partial_r - \frac{2}{r}\right)\bar{F} + \bar{h}_r, \tag{D19}$$

$$a^{-2}h^{GMG} = \bar{F} + \frac{1}{2}\bar{h}, \tag{D20}$$

The gauge invariant GMG variables can now be calculated directly, and are given in Section IV.

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