



Schur function averages for the real Ginibre ensemble

Sommers, HJ; Khoruzhenko, BA

“The final publication is available at <http://iopscience.iop.org/article/10.1088/1751-8113/42/22/222002/meta>”

For additional information about this publication click this link.

<http://qmro.qmul.ac.uk/xmlui/handle/123456789/13462>

Information about this research object was correct at the time of download; we occasionally make corrections to records, please therefore check the published record when citing. For more information contact scholarlycommunications@qmul.ac.uk

Schur function averages for the real Ginibre ensemble

Hans-Jürgen Sommers¹ and Boris A Khoruzhenko²

¹ Fachbereich Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany

² Queen Mary University of London, School of Mathematical Sciences, London E1 4NS, UK

E-mail: H.J.Sommers@uni-due.de, b.khoruzhenko@qmul.ac.uk

Abstract.

We derive an explicit simple formula for expectations of all Schur functions in the real Ginibre ensemble. It is a positive integer for all entries of the partition even and zero otherwise. The result can be used to determine the average of any analytic series of elementary symmetric functions by Schur function expansion.

PACS numbers: 02.10Yn, 02.50.-r, 05.40.-a, 75.10.Nr

1. Introduction

Real asymmetric random matrices, and in particular statistics of their (complex) eigenvalues, have many interesting applications in modelling of a wide range of physical phenomena. They appeared in the studies of stability of complex biological networks [1], dynamics of neural networks with asymmetric synaptic couplings [2], directed quantum chaos in randomly pinned superconducting vortices [3], delayed time series in financial markets [4], quantum chromodynamics with a real representation of the Dirac operator [5], or random quantum operations in quantum information theory [6].

All statistical information about eigenvalues of random matrices is encoded in their joint probability density function (pdf). For real asymmetric random matrices the joint pdf of eigenvalues was derived in [7] almost 20 years ago. It gave insights about the generic properties of the eigenvalue distribution, like the level repulsion, but had a complex structure due to singularities which made the precise statistical inference about eigenvalues difficult at that time. It suffices to say that the eigenvalue counting measures for complex [8] and real eigenvalues [9] were initially obtained without the aid of knowing the joint pdf. Very recently the interest in the spectral properties of real asymmetric random matrices has been revived [10] and the Pfaffian representation of n -point eigenvalue correlation functions has been derived directly from the joint pdf by various methods [11, 12, 13, 14]. This progress made it possible to compute the bulk and edge scaling limits of the eigenvalue correlation functions [15] and, in

particular, to recover the results of [8, 9] about the eigenvalue counting measures in the limit of infinite matrix dimension [12, 13]. Interestingly, the kernel that determines the eigenvalue correlation functions can be written in terms of averages of the characteristic polynomial of random matrices of lower dimension, much in the spirit of [8, 9], see also [16] for complex matrices, and hence can be computed in a very simple way [17]. One intriguing open question is the relation of real random matrices to integrability theory such as τ -functions which are generating functions for the cumulants of power sums of eigenvalues.

The eigenvalues of the real asymmetric matrices are either real or pairwise complex conjugate. This implies a complicated and singular structure of the eigenvalue correlation functions. It seems that in many aspects the elementary symmetric functions of eigenvalues, e_n , or their first N power sums, t_n , form a more convenient set of independent real variables. The Jacobian of the transformation from this set to the eigenvalues is just the Vandermonde determinant appearing in the joint pdf of eigenvalues [7]. Since symmetric functions of eigenvalues can be expanded in products of e_n 's (or t_n 's), averages of symmetric functions of eigenvalues can be written in terms of the joint moments of elementary symmetric functions (or power sums). Equivalent to all these moments is the set of all averages of Schur functions which form another basis in the space of symmetric functions. As Schur functions are determinants, divided by the Vandermonde their average can again be reduced to a Pfaffian. This Pfaffian can be evaluated explicitly as we will show in this paper. The result is very simple. If the partition characterizing the Schur function is odd then the Schur function average is zero. For even partitions the Schur function average is a positive integer expressible in terms of Gamma functions. This gives an alternative method of evaluating averages of symmetric functions of eigenvalues by the way of Schur function expansions.

We will treat both even and odd matrix dimensions N in a coherent way. We will reveal a further interesting structure of the theory. The basic building block is a skew symmetric kernel $\mathcal{K}_N(z_1, z_2)$, which is a polynomial of degree $N - 1$ in both eigenvalues z_1, z_2 , thus building a skew symmetric form with a matrix A^{-1} . If one knows this matrix A^{-1} in all even dimensions, one is able to determine all correlation functions and all moments. Since A^{-1} has the same form in all dimensions, which just can be considered as a submatrix of one infinite dimensional matrix, the averages of general Schur functions are given by Pfaffians of finite submatrices of this infinite dimensional matrix. For odd dimension one has to modify this procedure slightly.

2. The real Ginibre ensemble

In this paper we consider real random $N \times N$ matrices H with probability measure

$$d\mu(H) = \exp\left(-\frac{1}{2} \sum_{i,j=1}^N H_{ij}^2\right) \prod_{i,j=1}^N \frac{dH_{ij}}{\sqrt{2\pi}}. \quad (1)$$

This ensemble of random matrices is known under the name of real Ginibre ensemble [18]. The matrix distribution (1) induces a probability distribution on the eigenvalues of H which is known to be of the form [7, 8, 19]

$$d\mu(z_1, z_2, \dots, z_N) = C_N \cdot \prod_{i < j} (z_i - z_j) \cdot \prod_k f(z_k) \cdot dz_1 \dots dz_N, \quad (2)$$

with positive function $f(z) = f(\bar{z}) > 0$, $f^2(z) = e^{-\text{Re}(z^2)} \text{erfc}(|\text{Im } z|\sqrt{2})$. The eigenvalues z_j of H are real or pairwise complex conjugate and are ordered in such a way that $d\mu \geq 0$.

Introducing the skew symmetric weight function ($z = x + iy$, $d^2z = dx dy$)

$$\mathcal{F}(z_1, z_2) = f(z_1)f(z_2)(2i\delta^2(z_1 - \bar{z}_2)\text{sgn}(y_1 - y_2) + \delta(y_1)\delta(y_2)\text{sgn}(x_2 - x_1))$$

with $\delta^2(z_1 - \bar{z}_2) = \delta(x_1 - x_2)\delta(y_1 + y_2)$, all correlation functions are determined with the help of the skew symmetric kernel

$$\mathcal{K}_N(z_1, z_2) = \sum_{k,l=1}^N A_{kl}^{-1} z_1^{k-1} z_2^{l-1} \quad (3)$$

with the skew symmetric matrix

$$A_{kl} = \int d^2z_1 \int d^2z_2 \mathcal{F}(z_1, z_2) z_1^{k-1} z_2^{l-1}. \quad (4)$$

The eigenvalue correlations are Pfaffians with combinations of \mathcal{K}_N and \mathcal{F} as entries [19]. Comparing the 1-point density $R_1(z) = \int d^2u \mathcal{F}(z, u) \mathcal{K}_N(u, z)$ with Edelman's expression [8] for the density of complex eigenvalues,

$$R_1(z) = f(z)f(\bar{z}) \frac{z - \bar{z}}{i\sqrt{2\pi}} \sum_{n=0}^{N-2} \frac{(z\bar{z})^n}{n!}, \quad \text{Im } z > 0,$$

one finds

$$\mathcal{K}_N(z_1, z_2) = \frac{z_1 - z_2}{2\sqrt{2\pi}} \sum_{n=0}^{N-2} \frac{(z_1 z_2)^n}{n!}. \quad (5)$$

This result can independently be found by a simple supersymmetric (Grassmannian) calculation of the average of a product of two characteristic polynomials. This calculation can be extended to partly symmetric or chiral counterparts [17].

With the help of the duplication formula for the Gamma function

$$2\sqrt{2\pi}\Gamma(N-1) = 2^{(N-1)/2}\Gamma((N-1)/2) \cdot 2^{N/2}\Gamma(N/2) \quad (6)$$

(3) and (5) imply that

$$A_{kl}^{-1} = a_k^{-1} \epsilon_{kl} a_l^{-1} \quad (7)$$

with $a_k = 2^{k/2}\Gamma(k/2)$ and ϵ_{kl} being the tridiagonal skew symmetric matrix with -1 in the upper diagonal. The structure of the Vandermonde determinant in (2) and (or) the results of paper [19] imply that this expression can be used for odd dimension too, if we extend the matrix A^{-1} to dimension $N+1$ by putting $a_{N+1} = 1$ in (7). In this case the skew symmetric matrix ϵ can be inverted and the inverse yields (4) from (7). However,

in that case formula (3) holds with the sum over k, l running only from 1 to N . The normalization constant C_N in (2) is given by

$$1/C_N = \text{Pfaff}(A) \quad (8)$$

that is the Pfaffian of the skew symmetric matrix A .

3. Schur function averages

The description of the system by the complex eigenvalues might not be the most convenient one. A more suitable set of independent real variables is the set of power sums

$$t_n = \text{Tr}(H)^n = \sum_{i=1}^N z_i^n \quad (9)$$

or the set of elementary symmetric functions e_1, \dots, e_N which are the coefficients in the expansion of the characteristic polynomial in powers of x ,

$$\det(1 + xH) = \sum_{n=0}^N x^n e_n(z_1, \dots, z_N) . \quad (10)$$

Sometimes complete symmetric functions h_n come in handy. These are the coefficients in the expansion of the reciprocal characteristic polynomial in powers of x ,

$$1/\det(1 - xH) = \sum_{n=0}^{\infty} x^n h_n(z_1, \dots, z_N) . \quad (11)$$

Homogeneous symmetric polynomials in z_i of degree ≥ 1 can be expressed as sums of products of power sums t_n , or equally as sums of products of e_n 's or, equally, of h_n 's. Hence, when calculating averages of the symmetric polynomials in eigenvalues of H , or analytic functions, one is led to the problem of finding the joint moments of the eigenvalues. A full description of all symmetric moments of eigenvalues can be given by the so-called Schur polynomials or Schur functions. These functions are indexed by sequences, called partitions, $\lambda = (\lambda_1, \dots, \lambda_N)$ of ordered non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$, called parts, and can be conveniently written as the ratio of two determinants

$$\sigma_\lambda(z_1, \dots, z_N) = \det(z_m^{N-n+\lambda_n}) / \det(z_m^{N-n}) . \quad (12)$$

We will follow the convention of not showing zero parts, i.e.

$$(\lambda_1, \lambda_2, \dots, \lambda_l, 0, \dots, 0) \equiv (\lambda_1, \lambda_2, \dots, \lambda_l)$$

and when referring to specific partitions will adopt the notation $\lambda = (k^{a_k}, \dots, 2^{a_2}, 1^{a_1})$ to indicate that there are precisely a_1 of 1's, a_2 of 2's, etc., among the parts of partition λ .

It can be verified directly from (12) that if partition λ consists of one part, $\lambda = (n)$, then the corresponding Schur function is just the complete symmetric function, $\sigma_{(n)} = h_n$ and if partition λ has only zeros or ones among its parts, $\lambda = (1^n)$ then the

corresponding Schur function is just the elementary symmetric function, $\sigma_{(1^n)} = e_n$. Also, by convention, $\sigma_\lambda = 0$ if the number of non-zero parts of λ is greater than the number of indeterminants z_j .

If λ is a partition of n , i.e. $\lambda_1 + \dots + \lambda_N = n$, then the Schur function σ_λ is a homogeneous symmetric polynomial of degree n . In fact the set of all such Schur functions form a basis in the space of homogeneous symmetric polynomials of degree n . In particular, for $n \geq 1$, see, e.g., [20],

$$t_n = \sum_{k=1}^n (-1)^{n-k} \sigma_{(k, 1^{n-k})}, \quad (13)$$

where the sum is over all hook partitions $\lambda = (k, 1^{n-k})$ of n . Vice versa, as already mentioned above, every Schur function, as any other symmetric polynomial, can be expressed in terms of products of power sums. One consequence of this is that the Schur function of matrix argument defined by $\sigma_\lambda(H) = \sigma_\lambda(z_1, \dots, z_N)$ with z_j being the eigenvalues of H , is a polynomial in the matrix entries of H and $\sigma_\lambda(H) = \sigma_\lambda(THT^{-1})$ for any non-degenerate matrix T .

The Schur functions are the characters of irreducible representations of the unitary group and as such are orthogonal with respect to integration over the Haar measure. This property is quite useful for finding the coefficients in Schur function expansions. For example, expanding the product of characteristic polynomials in Schur functions,

$$\prod_{j=1}^n \prod_{k=1}^N (1 + x_j z_k) = \sum_{\lambda} c_{\lambda}(x_1, \dots, x_n) \sigma_{\lambda}(z_1, \dots, z_N),$$

one can find the ‘‘Fourier coefficients’’ $c_{\lambda}(x_1, \dots, x_n)$ by integration over the unitary group

$$c_{\lambda}(x_1, \dots, x_n) = \int \prod_{j=1}^n \det(I + x_j U) \overline{\sigma_{\lambda}(U)} dU, \quad (14)$$

where the integration is over the unitary group $U(N)$ with respect to the normalized Haar measure dU . The group integral in (14) can be easily evaluated yielding $c_{\lambda} = \det(e_{\lambda_k - k + j})$ which, by the Jacobi-Trudi formula is again a Schur function, $\det(e_{\lambda_k - k + j}) = \sigma_{\lambda'}$, with λ' being the partition conjugate to λ †. Thus one arrives at the Schur function expansion

$$\prod_{j=1}^n \prod_{k=1}^N \det(1 + x_j H) = \sum_{\lambda} \sigma_{\lambda'}(x_1, \dots, x_n) \sigma_{\lambda}(H). \quad (15)$$

This expansion is well known under the name of dual Cauchy identity and can be derived by purely algebraic means, see e.g. [21]. Other known Schur function expansions can be obtained via (14), see, e.g., Appendix in [22].

The Schur function averages over the real Ginibre ensemble (1) can be obtained by the same reasoning as in (7) – (8) or following the calculations in [19]. The only

† The conjugate to partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ with λ'_j being the number of λ_k that are greater than or equal to j .

It is easy to see that $\text{Pfaff}(-\epsilon) = \int d\chi_1 \dots d\chi_N (-\chi_1\chi_2)(-\chi_3\chi_4)\dots = 1$. The corresponding inverse ϵ^{-1} has the form

$$\epsilon^{-1} = \begin{pmatrix} 0 & +1 & 0 & +1 & 0 & +1 & \dots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & +1 & 0 & +1 & \dots \\ -1 & 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & +1 & \dots \\ -1 & 0 & -1 & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (24)$$

Again we see that $\text{Pfaff}(\epsilon^{-1}) = \int d\chi_1 \dots d\chi_N (-\chi_1\chi_2)(-\chi_3\chi_4)\dots = 1$. Now let us calculate the Pfaffian in (22). We first try to calculate the determinant which is simpler (remember $(\text{Pfaff}A)^2 = \det A$), then we determine the sign. To calculate subdeterminants of ϵ^{-1} we consider the generating function

$$D_N = \det(x_k \delta_{k,l} + (\epsilon^{-1})_{k,l}) . \quad (25)$$

From this we can calculate subdeterminants by differentiating with respect to x_k to cut row and column number k and then put $x = 0$. Now we transform using $\det \epsilon^{-1} = \det \epsilon = 1$

$$D_N = \left(\prod_{k=1}^N x_k \right) \det\left(\frac{1}{x_k} \delta_{k,l} + \epsilon_{k,l}\right) . \quad (26)$$

which now we may generalize to odd N and deduce due to the simple form of ϵ

$$D_N = D_{N-1} + x_N x_{N-1} D_{N-2} \quad (27)$$

with $D_1 = 1$, $D_2 = 1 + x_1 x_2$. As a result

$$D_N = 1 + x_1 x_2 + x_2 x_3 + \dots + x_1 x_2 x_3 x_4 + \dots \quad (28)$$

is a sum of products of consecutive pairings with all coefficients equal to +1. This implies that we can cut only consecutive pairs. As a result: $|\text{Pfaff}(\epsilon_{N-n+\lambda_n+1, N-m+\lambda_m+1}^{-1})| = 1$, if all λ_n even and = 0 if one or more λ_n odd. To determine the sign we observe that a permutation of rows and columns of the Pfaffian changes it by the sign of the permutation. Let us move in this way all rows and columns, which we don't want to cut, into the upper left corner of the matrix. This yields a sign $(-1)^{|\lambda|}$ of the permutation (which is = +1 if $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_N$ is even). Then we observe, that if we move away only consecutive pairs of rows and columns, the remaining submatrix of ϵ^{-1} in the upper left corner does not change its form. Thus its Pfaffian is +1. As the final result we obtain

$$\langle \sigma_\lambda(H) \rangle_N = \begin{cases} 2^{\frac{|\lambda|}{2}} \prod_{n=1}^N \frac{\Gamma((N-n+\lambda_n+1)/2)}{\Gamma((N-n+1)/2)} & \text{if all } \lambda_n \text{ are even} \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

Here $|\lambda| = \lambda_1 + \dots + \lambda_N$ and the upper limit N of the product can also be replaced by the length of the partition, which is the number of its nonzero parts λ_n . It has to be mentioned that the Schur function average (29), can be also obtained from zonal polynomials [21], however this derivation is rather indirect.

4. Conclusions

We have calculated the average of Schur functions over the real Ginibre ensemble (1) by using the Pfaffian representation (20) for the average and then evaluating the Pfaffian explicitly. It vanishes for odd partitions and is given by a simple expression in terms of Gamma functions (29) for even partitions. This result might be useful for evaluating averages of symmetric polynomials and analytic functions in eigenvalues of real Gaussian matrices. For example, consider the power sums t_n , $n \geq 1$. Their averages can easily be calculated by the way of Schur function expansion (13) and our main result (29). For odd n there are no hook partitions of n with all parts even and therefore the average of t_n vanishes. Of course, this can be verified directly from the symmetries in the matrix distribution (1). If n is even then there is only one hook partition of n with all parts even. This partition is $\lambda = (n)$. Hence, all terms but one on the right-hand side in (13) vanish and one arrives at the relation

$$\langle \text{Tr } H^n \rangle_N = \langle \sigma_{(n)}(H) \rangle_N = \langle h_n(H) \rangle_N$$

where h_n is the complete symmetric function of degree n . Hence, by (29),

$$\langle \text{Tr } H^{2m} \rangle_N = 2^m \frac{\Gamma(N/2 + m)}{\Gamma(N/2)} = \prod_{j=1}^m (N + 2(m - j)), \quad m \geq 1.$$

Similarly, on averaging (15) with the help of (29) one obtains

$$\left\langle \prod_{j=1}^n \det(1 + x_j H) \right\rangle_N = \sum_{n/2 \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0} c_\lambda \sigma_{(2\lambda)'}(x_1, \dots, x_n),$$

where $\lambda = (\lambda_1, \dots, \lambda_N)$, $2\lambda = (2\lambda_1, \dots, 2\lambda_N)$ and $c_\lambda = 2^{|\lambda|} \prod_k \frac{\Gamma(\lambda_k + (N - k + 1)/2)}{\Gamma((N - k + 1)/2)}$. If $n = 2$ then the constraints $n/2 \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ imply that $\lambda_j = 1$ or 0 for each j . Correspondingly, $\lambda = (1^k)$, $2\lambda = (2^k)$ and $(2\lambda)' = (k, k)$, where k is the number of non-zero λ_j . Observing that $\sigma_{(k,k)}(x_1, x_2) = (x_1 x_2)^k$ and $c_{(1^k)} = N!/(N - k)!$ one concludes that

$$\langle \det(1 + x_1 H) \det(1 + x_2 H) \rangle_N = N! \sum_{k=0}^N \frac{(x_1 x_2)^k}{(N - k)!}, \quad (30)$$

in agreement with calculations in [8] and [17]. This expansion can also be obtained via (10) directly from the symmetries in the ensemble distribution (1). One only needs to recall the fact that the elementary symmetric polynomial of degree n in the eigenvalues of H is the sum of all principal minors of H of order n . Then (30) follows from the invariance of the distribution (1) with respect to changing the sign of H_{lm} and, also, permutation of columns of H .

Acknowledgments

The first author acknowledges support by SFB/TR12 of the Deutsche Forschungsgemeinschaft.

References

- [1] R.M.May, *Nature* **298** 413 (1972)
- [2] H.Sompolinsky, A.Crisanti and H.-J.Sommers, *Phys. Rev. Lett.* **61** 259 (1988)
- [3] K.B.Efetov, *Phys. Rev. Lett.* **79** 491 (1997)
- [4] J.Kwapien, S. Drozd, A.Z. Gorski and F. Oswiecimka, *Acta Phys. Pol.* **B37**, 3039 (2006)
- [5] M.A.Halasz, J.C. Osborn, and J.J.M. Verbaarschot, *Phys. Rev.* **D56**, 7059(1997)
- [6] W.Bruzda, V.Cappelini, H.-J.Sommers, K.Życzkowski, *Phys Lett A* **373** 320 (2009)
- [7] N.Lehmann and H.-J. Sommers, *Phys. Rev. Lett.* **67**, 941 (1991)
- [8] A.Edelman, *J. Multivariate Anal.* **60**, 203 (1997)
- [9] A.Edelman, E.Kostlan and M.Shub, *J. Amer. Math. Soc.* **7**, 247 (1994)
- [10] E.Kanzieper and G. Akemann, *Phys. Rev. Lett.* **95**, 230201 (2005)
- [11] C.D.Sinclair, *Int. Math. Res. Not.*, **2007**, 1 (2007).
- [12] P.J.Forrester and T. Nagao, *Phys. Rev. Lett.* **99** 050603 (2007)
- [13] H.-J. Sommers, *J. Phys. A* **40**, 671 (2007)
- [14] A.Borodin, C.D.Sinclair, arXiv: math-ph/0706.2670v2
- [15] A.Borodin, C.D.Sinclair, arXiv: math-ph/0805.2986
- [16] Y. V. Fyodorov and B.A. Khoruzhenko, *Comm. Math. Phys.* **273**, 561 (2007).
- [17] G.Akemann, M.J. Phillips, and H.-J. Sommers, *J.Phys.A* **42** 012001 (2009)
- [18] J.Ginibre, *J. Math. Phys.* **6**, 440 (1965)
- [19] H.-J. Sommers and W. Wieczorek, *J.Phys.* **A41**, 405003 (2008)
- [20] B. E. Sagan, *The Symmetric Group*, 2nd ed. (Springer) 1991
- [21] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed. (Clarendon Press) 1995
- [22] Y. V. Fyodorov and B.A. Khoruzhenko, *J. Phys. A* **40** 669 (2007)
- [23] M.L.Mehta, *Random Matrices* (Amsterdam Elsevier) 2004