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A few remarks on Colour-Flavour Transformations, truncations of random unitary matrices, Berezin reproducing kernels and Selberg type integrals

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Abstract

We investigate diverse relations of the colour-flavour transformations (CFT) introduced by Zirnbauer in [41, 42] to various topics in random matrix theory and multivariate analysis, such as measures on truncations of unitary random matrices, Jacobi ensembles of random matrices, Berezin reproducing kernels and a generalization of the Selberg integral due to Kaneko, Kadell and Yan involving the Schur functions. Apart from suggesting explicit formulas for bosonic CFT for the unitary group in the range of parameters beyond that in [42], we also suggest an alternative variant of the transformation, with integration going over an unbounded domain of a pair of Hermitian matrices. The latter makes possible evaluation of certain averages in random matrix theory.

Short title: A few remarks on Colour-Flavour Transformations

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1 Introduction

Let $(\boldsymbol{x}_j)_{j=1}^m$ and $(\boldsymbol{y}_j)_{j=1}^m$ be two sets of column-vectors in $\mathbb{C}^N$. By convention, the column-vectors are regarded as matrices consisting of a single column, and we shall use the star to denote the Hermitian conjugate (complex conjugate transpose) of a matrix. Let $U(N)$ stand for the group of unitary matrices of size $N \times N$ equipped with the Haar measure $d\mu_H(U)$ fixed by the normalization condition $\int_{U(N)} d\mu_H(U) = 1$. Further, consider the matrix ball $Q^*Q \leq I_m$ in $\mathbb{C}^{m\times m}$, the space of complex $m \times m$ matrices, equipped with the unit mass measure

$$d\mu^B_{N,m}(Q) = \text{const.}\ \det(I_m - Q^*Q)^{N-2m}\ (dQ), \qquad N \geq 2m, \tag{1}$$

where I_m is identity matrix and (dQ) is the cartesian volume element in $\mathbb{C}^{m\times m}$,

$$(dQ) = \prod_{j,k=1}^{m} d\,\mathrm{Re}\, Q_{jk} d\,\mathrm{Im}\, Q_{jk}.$$

With these notations in mind, the following remarkable identity, called the bosonic Colour-Flavour Transformation (bCFT),

$$\int_{U(N)} e^{\sum_{j=1}^{m}(\boldsymbol{y}_j^* U \boldsymbol{x}_j + \boldsymbol{x}_j^* U^* \boldsymbol{y}_j)}\, d\mu_H(U) = \int_{Q^*Q\leq I_m} e^{\sum_{j,k=1}^{m}(Q_{jk}\boldsymbol{x}_k^*\boldsymbol{x}_j + (Q^*)_{jk}\boldsymbol{y}_k^*\boldsymbol{y}_j)}\, d\mu_{N,m}^{B}(Q), \tag{2}$$

is known to hold for $N \geq 2m$. Similarly, the fermionic Colour-Flavour transformation (fCFT) asserts that

$$\int_{U(N)} e^{\sum_{j=1}^{m}(\boldsymbol{\chi}_j^* U \boldsymbol{\psi}_j + \boldsymbol{\psi}_j^* U^* \boldsymbol{\chi}_j)}\, d\mu_H(U) = \int_{\mathbb{C}^{m\times m}} e^{\sum_{j,k=1}^{m}(Q_{jk}\boldsymbol{\chi}_k^*\boldsymbol{\chi}_j - (Q^*)_{jk}\boldsymbol{\psi}_k^*\boldsymbol{\psi}_j)}\, d\mu_{N,m}^{F}(Q), \tag{3}$$

where now $\boldsymbol{\chi}_j$, $\boldsymbol{\psi}_j$, $\boldsymbol{\chi}_j^*$ and $\boldsymbol{\psi}_j^*$ are vectors with anti-commuting components. The $\boldsymbol{\chi}_j$ and $\boldsymbol{\psi}_j$ are column-vectors and the $\boldsymbol{\chi}_j^*$ and $\boldsymbol{\psi}_j^*$ are row-vectors. In contrast to (2), there is no restriction on m and N in (3). The integration on the right-hand side in (3) is over the entire space of complex $m \times m$ matrices and

$$d\mu_{N,m}^{F}(Q) = \mathrm{const.}\, \det{}^{-N-2m}(I_m + Q^*Q)\ (dQ). \tag{4}$$

Both identities, together with the unifying supersymmetric variant of Colour-Flavour Transformation (CFT) and extensions to other classical groups, were originally discovered by Zirnbauer in 1996 [41, 42] and proved by a skillful use of the machinery of generalized coherent states [33]. Following Zirnbauer's approach, variants of the CFT were also obtained for the special unitary group in [7, 34, 39]. Later on it was realized, again by Zirnbauer, see e.g [44, 10], that actually all forms of the CFT are just manifestations of a very deep algebraic fact related to the so-called Howe duality [20]. Since their introduction, the Colour-Flavour Transformations proved to be a very useful tool, finding diverse applications in such areas of physics as lattice gauge theory [42, 7, 36], random network models [1, 43], quantum chaos models [2, 3], and the random matrix theory [41, 16].

In this paper we revisit the simplest case of the unitary group. Our inspiration comes from noticing a certain similarity between a few results known in the random matrix theory and the bosonic and fermionic versions of the CFT. The central role in our investigation is played by certain integrals involving the so-called Schur functions $s_\lambda(A)$. The latter are explicitly defined for any $m \times m$ matrix A in terms of its eigenvalues $a_1, \ldots, a_m$ as

$$s_\lambda(A) = s_\lambda(a_1, \ldots, a_m) = \frac{\det\left(a_i^{m+\lambda_j-j}\right)_{i,j=1}^{m}}{\det\left(a_i^{m-j}\right)_{i,j=1}^{m}}, \qquad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0, \tag{5}$$

with λ being a partition, i.e. a non-increasing sequence of non-negative integers λ_j. The Schur functions $s_\lambda(A)$ are symmetric polynomials in the eigenvalues of $A = (A_{ij})$, and are also polynomials in A_{ij}. A concise introduction to the theory of symmetric functions can be found in the first chapter of [27], see also [8].

The usefulness of Schur functions for our purposes can be traced to the fact that they are characters of irreducible representations of the general linear group and its unitary subgroup and, as a consequence, possess certain properties of orthogonality, see, in particular, equations (51) and (52). This makes Schur function expansions a powerful tool for evaluating integrals over unitary groups as has been already demonstrated in [26, 4, 36, 39, 31, 32, 19]. In this paper, we use the Schur function expansion technique to extend the bosonic version of CFT (2) to the range $N \leq 2m \leq 2N$. We also reveal the relation of bCFT and fCFT to several important matrix integrals due to Berezin, and also to certain generalizations of the famous Selberg integral due to Kaneko, Kadell and Yan. Finally, we also derive a new variant of the bosonic CFT, replacing the integration on the right-hand side in (2) with one going over an unbounded matrix domain parameterized by a pair of Hermitian matrices. Such a representation is useful for studying regularized inverse spectral determinants of complex random matrices, the subject of our earlier work [16]. The latter work has in fact quite a few points of intersection with some of the topics discussed in the present paper.

The fact that the Schur functions play central role in our way of understanding and extending both bCFT and fCFT can be traced back to the above mentioned Howe duality, although we do not exploit the latter explicitly in the present paper. Without going into any detailed discussion, we would like to mention that in one of its incarnations Howe duality can be looked at, see e.g. [9] or Chapter 43 in book [8], as the ultimate reason for the validity of the so-called Cauchy identities

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-t_ix_j} = \sum_{\lambda} s_\lambda(t_1,\ldots,t_m)s_\lambda(x_1,\ldots,x_n) \tag{6}$$

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1+t_ix_j) = \sum_{\lambda} s_\lambda(t_1,\ldots,t_m)s_{\lambda'}(x_1,\ldots,x_n), \tag{7}$$

where λ' stands for the conjugate partition. Being "a point of very direct connection between representation theory and combinatorics" [8], it is therefore no surprise that both identities (6) and (7) play an important role in obtaining the results of this paper.

The organization of the paper is as follows. In the next section we will provide, for the reader's convenience, an overview and short discussion of the most important results following from our approach to CFT. The subsequent sections will be devoted to technical derivations of the main formulas.

2 An overview of the main results.

Our first observation relates the bCFT to truncations of random unitary matrices. Truncations of random unitary matrices emerged recently in a different context of quantum chaotic scattering. Various matrix distributions arising from such truncations were the subject of a few recent works, see [45, 13, 37, 11].

The relation between the bCFT and truncations of unitary matrices becomes more apparent if one writes (2) using matrix notation in the exponential. Introducing two $N \times m$ matrices X and Y with columns $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m$ and $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m$ respectively, one can write

$$\sum_{j=1}^{m}(\boldsymbol{y}_j^*U\boldsymbol{x}_j+\boldsymbol{x}_j^*U^*\boldsymbol{y}_j) = \mathrm{Tr}(Y^*UX+X^*U^*Y) = \mathrm{Tr}(XY^*U+U^*YX^*)$$

and

$$X^*X = (\boldsymbol{x}_j^*\boldsymbol{x}_k)_{j,k=1}^m \quad \text{and} \quad Y^*Y = (\boldsymbol{y}_j^*\boldsymbol{y}_k)_{j,k=1}^m.$$

Then the bCFT (2) takes the following form

$$\int_{U(N)} e^{\mathrm{Tr}(XY^*U+U^*YX^*)}\, d\mu_H(U) = \int_{Q^*Q\le I_m} e^{\mathrm{Tr}(X^*XQ+Q^*Y^*Y)}\, d\mu_{N,m}^B(Q). \tag{8}$$

Note that the left-hand side in (8) is well defined for any m whilst the integration measure on the right-hand side is singular if $2m > N$ and care must be taken if one wants to interpret the above formula for $2m > N$. We argue in Section 3 that for $m \le N$ one can rewrite the right-hand side in a form free from singularities. Indeed, the matrix XY^* and its Hermitian conjugate Y^*X have rank $m \le N$, or, to be more precise, at most m. In view of the invariance of the Haar measure, this means that the integral on the left-hand side in (8) goes effectively over the principal (top left) $m \times m$ sub-block Q of the unitary matrix U. With this observation in hand, a straightforward application of the Schur function expansion for the exponential function $\exp \mathrm{Tr}\, M$ allows one to recast the bCFT in the following form

$$\int_{U(N)} e^{\mathrm{Tr}(XY^*U+U^*YX^*)}\, d\mu_H(U) = \int_{Q^*Q\le I_m} e^{\mathrm{Tr}(X^*XQ+Q^*Y^*Y)}\, d\rho_{N,m\times m}(Q), \tag{9}$$

where $d\rho_{N,m\times m}(Q)$ is the image of the Haar measure under the truncation $U \mapsto Q$. It is known from [13] and [11], see also [30], that $d\rho_{N,m\times m}(Q) \propto \det(I_m - Q^*Q)^{N-2m}(dQ)$ for $N \ge 2m$. Thus, as one would expect, (9) reverts back to the original version of the bCFT in the interval $N \ge 2m$.

Identity (9) which holds for any $m \le N$ (note that $d\rho_{N,m\times m}(Q)$ is a unit mass measure and is free from singularities) provides the basis for our extension of the bCFT to the range $m \le N$. The boundary case $m = N$ is straightforward. Indeed, if $m = N$ then no truncation is involved and (9) takes the form

$$\int_{U(N)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)} d\mu_H(U) = \int_{U(N)} e^{\mathrm{Tr}(X^*XU+U^*Y^*Y)} d\mu_H(U).$$

The intermediate range $N < 2m < 2N$ requires additional calculations which are of interest on their own. Though in this range the measure $d\rho_{N,m\times m}(Q)$ can still be found explicitly, the resulting expression is too complicated to be practically used. However, for our purposes we only need to know the radial part of $d\rho_{N,m\times m}(Q)$ and this latter can be found in the explicit form and then used to rewrite the right-hand side of (9).

As we discuss in Section 3, in the range $N < 2m < 2N$ the measure $d\rho_{N,m\times m}(Q)$ is supported by the set Υ on the boundary of the matrix ball $Q^*Q \le I_m$ defined by the condition that $\mathrm{rank}(I_m - Q^*Q) = N - m$. This set can be parametrized by the matrices

$$Q_{UZV^*} = U \begin{pmatrix} Z & 0 \\ 0 & I_{2m-N} \end{pmatrix} V^* \tag{10}$$

where Z runs through the matrix ball $Z^*Z < I_{N-m}$ in $\mathbb{C}^{(N-m)\times(N-m)}$, the space of complex $(N-m)\times(N-m)$ matrices, and U and V run through the unitary group $U(m)$. However, this parametrization is not one-to-one: different Q_{UZV^*} may represent the same point Q in Υ. In the terminology of [21] (page 118) the set of matrices Q_{UZV^*} is a covering of Υ. Exploiting such a parametrization, we obtain the following variant of the bCFT in the range $N < 2m < 2N$

$$\int_{U(N)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)} d\mu_H(U) = \tag{11}$$
$$\int_{U(m)}\int_{U(m)}\int_{Z^*Z\le I_{N-m}} e^{\mathrm{Tr}(X^*XQ_{UZV^*}+Q^*_{UZV^*}Y^*Y)}\ d\mu^B_{N,N-m}(Z)d\mu_H(U)d\mu_H(V),$$

where Q_{UZV^*} is as defined in (10) and $d\mu^B_{N,N-m}(Z)$ is the unit mass measure on the matrix ball $Z^*Z \le I_{N-m}$ defined in (1),

$$d\mu^B_{N,N-m}(Z) = \mathrm{const.}\det(I_{N-m} - Z^*Z)^{2m-N}(dZ), \qquad N \le 2m < 2N. \tag{12}$$

It has to be mentioned that another variant of extension of the bCFT to the range $m \le N$,

$$\int_{U(N)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)} d\mu_H(U) = \int_{U(m)} \det(VY^*X)^{m-N} e^{\mathrm{Tr}(X^*XV^*+VY^*Y)} d\mu_H(V), \tag{13}$$

was obtained recently in [39]. Clearly, apart from the boundary case of $N = m$ this formula is different from ours. Interestingly, as was observed in [39], in the range $m < N$, the integral on the left-hand side in (13) does not change if the integration over the unitary group is replaced by the integration over the special unitary group:

$$\int_{U(N)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)} d\mu_H(U) = \int_{SU(N)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)} d\mu_H(U), \qquad m < N. \tag{14}$$

This means that for $m < N$ our formula (11) holds without further changes if one replaces $U(N)$ by $SU(N)$ in the integral on the left-hand side.

The original CFT (2) and its extension (9) almost hide the fact that both integrals, the one on the left-hand side and the one on the right hand side, depend only on the eigenvalues of the matrix product X^*XY^*Y. In fact, the bosonic CFT can be stated in the following, slightly more abstract form. If A and B are two $N \times N$ matrices of rank $m \le N$ then

$$\int_{U(N)} e^{\mathrm{Tr}(AU+U^*B)} d\mu_H(U) = \int_{Q^*Q\le I_m} e^{\mathrm{Tr}(CQ+Q^*D)}\ d\rho_{N,m\times m}(Q) \tag{15}$$

for any pair of $m \times m$ matrices C and D such that the eigenvalues of CD coincide with the non-zero eigenvalues of AB. In the range $N < 2m < 2N$ the integration over matrices Q on the right-hand side can be replaced by integration over the matrices Q_{UZV^*} as in (11).

Our next observation is that the CFT is related to several interesting matrix integrals.

It will be apparent from our calculations in Section 3 that the bosonic CFT (9) implies the following identity for the Schur functions

$$\int_{Q^*Q\le I_m} s_\lambda(Q^*Q)\ d\rho_{N,m\times m}(Q) = \frac{s^2_\lambda(I_m)}{s_\lambda(I_N)}, \qquad m \le N, \tag{16}$$

and vice versa, (16) implies (9). In fact, (16) is a particular case (corresponding to $m = n$) of a more general relation

$$\int_{Q^*Q\le I_m} s_\lambda(Q^*Q)\ d\rho_{N,n\times m}(Q) = \frac{s_\lambda(I_m)s_\lambda(I_n)}{s_\lambda(I_N)}, \qquad m, n \le N. \tag{17}$$

Here the matrices Q are $n \times m$ and $d\rho_{N,n\times m}(Q)$ is the image of the Haar measure under the truncation of unitary matrix to its principal $n \times m$ sub-block, $U \mapsto Q$. This relation and its generalization

$$\int_{Q^*Q\leq I_m} s_\lambda(XQYQ^*)\ d\rho_{N,n\times m}(Q) = \frac{s_\lambda(X)s_\lambda(Y)}{s_\lambda(I_N)}, \qquad m,n \leq N, \tag{18}$$

are simple corollaries of the invariance of the measure $d\rho_{N,n\times m}(Q)$ with respect to the right and left multiplication by unitary matrices. Another corollary of this invariance (and of the orthogonality of the Schur functions, see (52)) are the following orthogonality relations

$$\int_{Q^*Q\leq I_m} s_\lambda(LQ)\overline{s_\mu(MQ)}\ d\rho_{N,n\times m}(Q) = \delta_{\lambda,\mu}\ \frac{s_\lambda(M^*L)}{s_\lambda(I_N)},, \qquad m,n \leq N, \tag{19}$$

which hold for arbitrary $m \times n$ matrices L and M. Identities (18) and (19) are derived in Section 3.

We show that the integration formulas (17)–(19) imply several non-trivial matrix integrals, some of which we believe to be new. In particular, by making use of the Schur function expansion (6), we derive the following identity

$$\int_{U(N)} \frac{d\mu_H(U)}{\det(I_N - AU)^m \det(I_N - U^*B^*)^n} = \tag{20}$$
$$\text{const.} \int_{Z^*Z\leq I_{\min(m,n)}} \frac{1}{\det(I - Z^*Z \otimes B^*A)}\ \det(Z^*Z)^{|n-m|} \det(I - Z^*Z)^{N-m-n}\ (dZ)$$

which reduces the group integral on the left to an average over the Jacobi ensemble of random matrices Z of size $k \times k$, $k = \min(m,n)$. Identity (20) holds for $N \geq n+m$ and $A^*A < I_N$, $B^*B < I_N$ and generalizes our earlier result [16] from $n = m$ to the case $n \neq m$. A similar identity holds in the range $N < m+n < 2N$.

In [16] we obtained an identity which is dual to (16),

$$\int_{\mathbb{C}^{m\times m}} s_\lambda(Z^*Z)\ d\mu^F_{N,m}(Z) = \frac{s^2_\lambda(I_m)}{s_{\lambda'}(I_N)}. \tag{21}$$

Here $d\mu^F_{N,m}$, see (4), is the measure which appears on the right-hand side in the fermionic version of the CFT (3). By making use of the Schur function expansion (7), identity (21) is equivalent to the matrix integral

$$\int_{U(N)} \det(I_N + AU)^m \det(I_N + U^*B^*)^m\ d\mu_H(U) = \int_{\mathbb{C}^{m\times m}} \det(I + Z^*Z \otimes B^*A)\ d\mu^F_{N,m}(Z) \tag{22}$$

which is dual to the $m = n$ version of (20). The emergence of $d\mu^F_{N,m}(Z)$ on the right-hand side in (22) is not coincidental. In fact, we show in Section 4 that the fermionic version of the CFT (3) implies (22) directly. We expect that identity (21) (or, equivalently, the matrix integral (22)) should in turn imply the fermionic CFT, by analogy to the relation between the bosonic CFT and identity (16). Unfortunately, we have succeeded in verifying such equivalence only for the simplest case $m = 1$.

In the same way as identity (16) allows for an extension to rectangular matrices, identity (21) allows for a similar extension:

$$\int_{\mathbb{C}^{n\times m}} s_\lambda(Q^*Q)\; d\mu^F_{N,n\times m}(Q) = \frac{s_\lambda(I_n)s_\lambda(I_m)}{s_{\lambda'}(I_N)}, \tag{23}$$

where $d\mu^F_{N,n\times m}(Q)$ is the unit mass measure

$$d\mu^F_{N,n\times m}(Q) = \text{const. } \det(I_m + Q^*Q)^{-N-m-n}\;(dQ)$$

on the space of complex $n \times m$ matrices. Identity (23) holds for any positive integers N, m and n, and in turn, implies the identity dual to (20):

$$\int_{U(N)} \det(I_N + AU)^m \det(I_N + U^*B^*)^n\; d\mu_H(U) = \tag{24}$$

$$\text{const.} \int_{\mathbb{C}^{\min(m,n)\times\min(m,n)}} \det(I + Z^*Z \otimes B^*A)\; \frac{\det(Z^*Z)^{|n-m|}}{\det(I + Z^*Z)^{N+m+n}}\;(dZ),$$

again reducing evaluation of the integral over the unitary group on the left to evaluation of an integral over a Jacobi ensemble of random matrices. This is a generalization of our earlier result from [16].

Identities (16) and (21) yield another pair of matrix integrals, again by the way of the Schur function expansions (6)–(7):

$$\int_{Q^*Q\le I_m} \frac{d\rho_{N,n\times m}(Q)}{\det(I_n - Z_1Q^*)^N \det(I_n - QZ_2^*)^N} = \frac{1}{\det(I_n - Z_1Z_2^*)^N}, \tag{25}$$

where $Z_i^*Z_i < I_m$, and

$$\int_{\mathbb{C}^{n\times m}} \det(I_m + Z_1Q^*)^N \det(I_m + QZ_2^*)^N\; d\mu^F_{N,n\times m}(Q) = \det(I_m + Z_1Z_2^*)^N. \tag{26}$$

These matrix integrals are not new. They are a variant of integrals obtained by Berezin in his work on quantization in complex symmetric spaces [5, 6]. Berezin proved (26) for integer N and (26) for a range of real N that includes $N \ge n+m$. In Section 4 we discuss this link and quote some of Berezin's results.

Since identities (17) and (23) are so useful in the context of Schur function expansions, we think it is worth to have a closer look at them. Without loss of generality we can assume that $n \ge m$. In the bosonic case we also assume that $N \ge n+m$, so that the integration measure $d\rho_{N,n\times m}(Q)$ in (17) is replaced by $\text{const.} \det(I_m - Q^*Q)^{N-m-n}(dQ)$. In the range $N < n+m < 2N$ one can obtain slightly different formulas by using parametrization (10), see especially the integration formula (50).

The integration in (17) and (23) is effectively over the eigenvalues of Q^*Q. By making the corresponding change of variables (see, [21] or [12]) one brings the matrix integral in (17) to

$$\frac{1}{c^N_{n,m}} \int_0^1 \cdots \int_0^1 s_\lambda(x_1,\ldots,x_m) \prod_{j=1}^m x_j^{n-m}(1-x_j)^{N-m-n} \prod_{1\le i<j\le m}(x_i - x_j)^2\; \prod_{j=1}^m dx_j = \frac{s_\lambda(1_n)s_\lambda(1_m)}{s_\lambda(1_N)} \tag{27}$$

and the one in (23) to

$$\frac{1}{k_{n,m}^N}\int_0^\infty\!\!\cdots\int_0^\infty s_\lambda(x_1,\ldots,x_m)\prod_{j=1}^m \frac{x_j^{n-m}}{(1+x_j)^{N+m+n}}\prod_{1\le i<j\le m}(x_i-x_j)^2\prod_{j=1}^m dx_j=\frac{s_\lambda(1_n)s_\lambda(1_m)}{s_{\lambda'}(1_N)}. \tag{28}$$

The normalization constants in these formulas can be computed with the help of the celebrated Selberg integral, and are given in (87) and (88).

Interestingly, the integral in (27) is the $\gamma=1$ case of the following extension of the Selberg integral due to Kaneko [25] (see also related works by Kadell [24] and Yan [40])

$$\begin{aligned}&\int_0^1\!\!\cdots\int_0^1 J_\lambda^{\frac{1}{\gamma}}(x_1,\ldots,x_m)\prod_{j=1}^m x_j^{p-1}(1-x_j)^{q-1}\prod_{1\le i<j\le m}|x_i-x_j|^{2\gamma}\prod_{j=1}^m dx_j \\ &= J_\lambda^{\frac{1}{\gamma}}(1_m)\prod_{i=1}^m\frac{\Gamma(i\gamma+1)\Gamma(\lambda_i+p+\gamma(m-i))\Gamma(q+\gamma(m-i))}{\Gamma(1+\gamma)\Gamma(\lambda_i+p+q+\gamma(2m-i-1))}.\end{aligned} \tag{29}$$

Here $J_\lambda^{\frac{1}{\gamma}}(x)$ are the Jack symmetric functions [38], $J_\lambda^{\frac{1}{\gamma}}(x)$ are proportional to the Schur functions if $\gamma=1$, $J_\lambda^1(x)=H_\lambda s_\lambda(x)$, where H_λ is a coefficient independent of x. Similarly, (28) is the $\gamma=1$ case of the integral

$$\int_0^\infty\!\!\cdots\int_0^\infty J_\lambda^{\frac{1}{\gamma}}(t_1,\ldots,t_m)\prod_{j=1}^m\frac{t_j^{p-1}}{(1+t_j)^{p+q+2(m-1)\gamma}}\prod_{1\le i<j\le m}|t_i-t_j|^{2\gamma}\prod_{j=1}^m dt_j. \tag{30}$$

To the best of our knowledge, the latter integral is not evaluated yet for $\gamma\neq 1$ and we consider its computation as an interesting open problem. In particular, we note that although for the zero partition $\lambda_1=\lambda_2=\ldots=\lambda_m=0$ the substitution $x=1/(1-t)$ reduces the integral in (30) to the one in (29), this substitution does not preserve the Jack symmetric functions, and hence the integral in (30) for non-zero λ requires a separate evaluation.

Surprisingly, the $\gamma=1$ case is simple in the sense that the integral in (30) (and the one in (29)) can be evaluated by elementary means, as we demonstrate in the end of Section 4.

Section 5 of our paper is devoted to yet another alternative ("deformed") version of the bosonic CFT with a different integration manifold in the integral on the right-hand side of (2). Consider the manifold of matrices (Q_1,Q_2) in $\mathbb{C}^{m\times m}\times\mathbb{C}^{m\times m}$ parametrized as follows

$$Q_1=TPT^*,\qquad Q_2=(T^*)^{-1}PT^{-1}, \tag{31}$$

where T runs through the general linear group, $T\in GL_m(\mathbb{C})$, and P runs through the set of diagonal matrices

$$P=\mathrm{diag}(p_1,\ldots,p_m);\ \ -1\le p_j\le 1,\ \ j=1,\ldots,m, \tag{32}$$

and *introduce* the integration "measure"

$$(dQ_1dQ_2)=\prod_{1\le j<k\le m}(p_j^2-p_k^2)^2\prod_{j=1}^m p_jdp_j\ d\mu_H(T), \tag{33}$$

where $d\mu_H(T)=(dT)/\det(T^*T)^m$ is the invariant measure on $GL_m(\mathbb{C})$. Note that p_j change sign, and therefore (dQ_1dQ_2) is not a proper positive measure, see [15] for a discussion. Then we claim the validity of the following variant of the bosonic CFT ($N \geq 2m$)

$$\int_{U(N)} e^{-i\,\mathrm{Tr}(Y^*UX+X^*U^*Y)} d\mu_H(U) = \int \det(I_m - Q_1Q_2)^{N-2m} e^{-i\,\mathrm{Tr}(Q_1Y^*Y+Q_2X^*X)} (dQ_1 dQ_2). \tag{34}$$

Note the most important features of this formula are: (i) integration domain in (34) is unbounded, in contrast to the bounded domain $Q^*Q \leq I_m$ as in the standard bCFT (8) and (ii) the matrices Q_1 and Q_2 are Hermitian. We finish that section by demonstrating that the last property makes the new version of bCFT indispensable when evaluating expectations values of negative powers of certain (regularized) spectral determinants.

3 bCFT from truncations of random unitary matrices

In this section we first address the problem of evaluation of integrals of the form

$$\int_{Q^*Q\leq I_m} f(Q^*Q)\ d\rho_{N,n\times m}(Q), \tag{35}$$

where $d\rho_{N,n\times m}(Q)$ is the image of the Haar measure under the truncation of unitary matrices as defined below. Then we apply the obtained formulas to derive the bCFT in the form of equations (9) and (11).

Let U be an $N \times N$ unitary matrix and m and n be positive integer numbers, $m \leq n \leq N$. Partition U into the four blocks

$$U = \begin{pmatrix} Q & R \\ P & S \end{pmatrix} \tag{36}$$

where the top left block Q is $n \times m$. Partition (36) defines the map $\omega : U \to Q$ from the unitary group into the matrix ball $Q^*Q \leq I_m$ in $\mathbb{C}^{n\times m}$, the space of complex $n \times m$ matrices. Under this map the Haar measure $d\mu_H(U)$ on $U(N)$ induces a measure on the matrix ball $Q^*Q \leq I_m$ which we shall denote by $d\rho_{N,n\times m}(Q)$. The unitarity of U imposes constraints on its sub-blocks. In particular,

$$Q^*Q + P^*P = I_m. \tag{37}$$

If $N \geq m + n$ then, generically, the matrix P^*P has rank m, and the image of $U(N)$ under the map ω is the entire matrix ball $Q^*Q \leq I_m$. In this case the measure $d\rho_{N,n\times m}(Q)$ has been previously computed in [30, 13] for square matrices Q and in [11] for rectangular matrices,

$$d\rho_{N,n\times m}(Q) = \mathrm{const.} \det(I_m - Q^*Q)^{N-m-n} (dQ), \qquad N \geq n + m, \tag{38}$$

where (dQ) is the cartesian volume element in $\mathbb{C}^{n\times m}$. By making an appropriate change of variables, see e.g. [12] or one of the calculations below, the integral in (35) can then be reduced to the familiar form

$$\int_{Z^*Z\leq I_m} f(Z^*Z) \det(Z^*Z)^{n-m} \det(I_m - Z^*Z)^{N-m-n} (dZ) \tag{39}$$

of an average over the so-called Jacobi ensemble [12] of random $m \times m$ matrices Z and can be evaluated by the standard tools of the random matrix theory.

If $N < m + n$, i.e. $N - n < m$, then, generically, the matrix P^*P in (37) has rank $N - n$ and, as a consequence, $z = 1$ is an eigenvalue of Q^*Q of multiplicity $m + n - N$. Therefore, in this case the image of $U(N)$ under the map ω is a submanifold of the boundary of the matrix ball $Q^*Q \leq I_m$. This submanifold is defined by the equations

$$\frac{d^k}{dz^k}\det(zI_m - Q^*Q)\bigg|_{z=1} = 0, \qquad k = 0, 1, \ldots, m+n-N-1. \tag{40}$$

An expression for $d\rho_{N,n\times m}$ in this case can be obtained by following the method of calculation of $d\rho_{N,n\times m}$ for $N \geq n+m$ that was suggested in [13] and extended in [11]. It makes use of the matrix integral [14]

$$\int \frac{e^{i\,\mathrm{Tr}\,FX}(dF)}{\det^l(F - zI_k)} = c_{l,k} e^{iz\,\mathrm{Tr}\,X} \det X^{l-k}, \quad l \geq k,\ \mathrm{Im}\, z > 0 \text{ and } X > 0$$

over Hermitian $k \times k$ matrices F. Extension of this calculation to the case of $N < n+m$ requires evaluation of the above integral for positive semi-definite matrices X of rank $l < k$. Such evaluation was given in [23]. Using this result one can calculate $d\rho_{N,n\times m}$ for $N < n+m$, however, the resulting expression contains many delta-functional factors and is not very useful for direct applications. Fortunately, evaluation of integrals (35) is a simpler task and can be accomplished with the help of the following standard calculation from multivariate analysis.

Consider $\mathbb{C}^{l\times k}$, the space of complex $l \times k$ matrices. If $l \geq k$ then any matrix $P \in \mathbb{C}^{l\times k}$ of rank k can be uniquely written as $P = HT$ where $T = (T_{ij}) \in \mathbb{C}^{k\times k}$ is upper-triangular with positive diagonal elements and $H \in \mathbb{C}^{l\times k}$ is such that $H^*H = I_k$. Correspondingly, the cartesian volume element (dP) in $\mathbb{C}^{l\times k}$ transforms as follows, see e.g. [29, 28],

$$(dP) = \mathrm{const} \times \det(TT^*)^{l-k} \prod_{i=1}^{k} T_{ii}^{2(k-i)+1} (dT)(H^*dH) \tag{41}$$

where

$$(dT) = \prod_{1\leq i\leq k} T_{ii} \prod_{1\leq i<j\leq k} d\,\mathrm{Re}\, T_{ij} d\,\mathrm{Im}\, T_{ij}$$

and (H^*dH) is the invariant volume element[1] on the Stiefel manifold $V_k(\mathbb{C}^l) \cong U(l)/U(l-k)$ of complex $l \times k$ matrices with orthonormal columns. It follows from (41) that if f is a function on $\mathbb{C}^{k\times k}$ then

$$\int_{\mathbb{C}^{l\times k}} f(P^*P)(dP) = \mathrm{const.} \int_{\mathbb{C}^{k\times k}} f(Z^*Z)\det(Z^*Z)^{l-k}(dZ), \tag{42}$$

where we have used (41) twice, at first making the substitution $P = HT$ and then making the reverse substitution $T = V^{-1}Z$ with $Z \in \mathbb{C}^{k\times k}$ and $V \in U(k)$. Similarly, if g is a function on $\mathbb{C}^{l\times l}$ then

$$\int_{\mathbb{C}^{l\times k}} g(PP^*)(dP) = \mathrm{const.} \int_{V_k(\mathbb{C}^l)} \int_{\mathbb{C}^{k\times k}} g(HZZ^*H^*)\det(ZZ^*)^{l-k}(dZ)(H^*dH). \tag{43}$$

The integration rules (42) and (43) hold for $l \geq k$. If $l < k$ then

$$\int_{\mathbb{C}^{l\times k}} f(P^*P)(dP) = \mathrm{const.} \int_{V_l(\mathbb{C}^k)} \int_{\mathbb{C}^{l\times l}} f(HZ^*ZH^*)\det(Z^*Z)^{k-l}(dZ)(H^*dH) \tag{44}$$

[1]For details of its construction, see [22, 29]

and

$$\int_{\mathbb{C}^{l\times k}} g(PP^*)(dP) = \text{const.} \int_{\mathbb{C}^{l\times l}} g(ZZ^*)\det(ZZ^*)^{k-l}(dZ). \tag{45}$$

Returning to integrals (35), let us consider, alongside with ω, another map defined by partition (36),

$$\tau : U \to H = \begin{pmatrix} Q \\ P \end{pmatrix}.$$

It maps the unitary group $U(N)$ onto the Stiefel manifold of $N\times m$ matrices H with orthonormal columns, $H^*H = I_m$. Obviously, the image of the Haar measure $d\mu_H(U)$ under this map is invariant with respect to the right and left multiplications by unitary matrices. Since the Stiefel manifold is a coset space of the unitary group, such invariant measure is unique up to a multiplicative constant. Therefore,

$$\int_{U(N)} g(\tau(U))\ d\mu_H(U) = \int_{U(N)} g(H)\ d\mu_H(U) = \text{const.} \int_{V_m(\mathbb{C}^N)} g(H)(H^*dH). \tag{46}$$

Sometimes it is convenient to write the invariant measure on $V_m(\mathbb{C}^N)$ as a singular measure on $\mathbb{C}^{N\times m}$

$$\int_{V_m(\mathbb{C}^N)} g(H)(H^*dH) = \text{const.} \int_{C^{N\times m}} g(H)\delta(H^*H - I_m)(dH), \tag{47}$$

where (dH) is the cartesian volume element in $\mathbb{C}^{N\times m}$ and $\delta(H^*H - I_m)$ is the matrix delta function on the space of Hermitian matrices,

$$\delta(A) = \prod_{j=1}^{m} \delta(A_{jj}) \prod_{1\le j<k\le m} \delta(\mathrm{Re}\, A_{jk})\delta(\mathrm{Im}\, A_{jk}).$$

Thinking of the truncation $\omega : U \to Q$, where $U \in U(N)$ and $Q \in \mathbb{C}^{n\times m}$, as a composition of the two successive truncations

$$U \mapsto \begin{pmatrix} Q \\ P \end{pmatrix} \mapsto Q,$$

we have, by (46)–(47),

$$\int_{Q^*Q\le I_m} f(Q^*Q)d\rho_{N,n\times m}(Q) = \text{const.} \int_{\mathbb{C}^{n\times m}} \left(\int_{\mathbb{C}^{(N-n)\times m}} \delta(Q^*Q + P^*P - I_m)(dP) \right) f(Q^*Q)(dQ). \tag{48}$$

If $N \ge n + m$ then, by making use of the integration rule (42), one can replace the integration over Q and P in the integral on the right-hand side by integrations over $m\times m$ matrices Z and F, respectively, thus reducing the integral to the following one

$$\int_{\mathbb{C}^{m\times m}} \left(\int_{\mathbb{C}^{m\times m}} \delta(Z^*Z + F^*F - I_m)\det(F^*F)^{N-n-m}\ (dF) \right) f(Z^*Z)\det(Z^*Z)^{n-m}\ (dZ)\,.$$

Performing the integration over F one obtains

$$\int_{Q^*Q\le I_m} f(Q^*Q)d\rho_{N,n\times m}(Q) = \text{const.} \int_{Z^*Z\le I_m} f(Z^*Z)\det(I_m - Z^*Z)^{N-n-m}\det(Z^*Z)^{n-m}(dZ), \tag{49}$$

in agreement with (39).

If $N < n + m$, i.e. $N - n < m$, then by making use of the integration rule (42) to replace integration over Q by integration over Z as above and the integration rule (44) to replace integration over P by integration over FH^*, where $H \in V_{N-n}(\mathbb{C}^m)$ and $F \in \mathbb{C}^{(N-n)\times(N-n)}$, one arrives, after carrying out the integration over Z, at

$$\int_{Q^*Q\leq I_m} f(Q^*Q)d\rho_{N,n\times m}(Q) = \text{const.}\times$$
$$\int_{V_{N-n}(\mathbb{C}^m)}\int_{F^*F\leq I_{N-n}} f(I_m - HF^*FH^*)\det(F^*F)^{n+m-N}\det(I_{N-n} - F^*F)^{n-m}\ (dF)(H^*dH).$$

If the function f is invariant with respect to the conjugation by unitary matrices, i.e. $f(UAU^*) = f(A)$ for unitary U, then

$$f(I_m - HF^*FH^*) = f\left(I_m - \begin{pmatrix} F^*F & 0 \\ 0 & 0 \end{pmatrix}\right) \quad \text{for any } H \text{ such that } H^*H = I_m,$$

and the integral on the right-hand side simplifies,

$$\int_{Q^*Q\leq I_m} f(Q^*Q)d\rho_{N,n\times m}(Q) = \text{const.}\times$$
$$\int_{Z^*Z\leq I_{N-n}} f\begin{pmatrix} Z^*Z & 0 \\ 0 & I_{m+n-N} \end{pmatrix}\det(I_{N-n} - Z^*Z)^{n+m-N}\det(Z^*Z)^{n-m}\ (dZ) \qquad (50)$$

If the function $f(A)$ is invariant under the conjugation by unitary matrices then it is effectively a function of the eigenvalues $a_1, \ldots, a_m$ of $A = (A_{jk})_{j,k=1}^m$. There is one class of such functions for which the integral in (35) can be easily computed, see (17). These are the Schur functions s_λ (5). The Schur functions are the characters of irreducible polynomial representations of the general linear group. Such representations remain irreducible when restricted to the unitary subgroup of the general linear group. The orthogonality of matrix elements of irreducible representations as functions on the unitary group then implies the following integration formulae, see e.g. [27], p.445,

$$\int_{U(m)} s_\lambda(AUBU^*)d\mu_H(U) = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(I_m)} \qquad (51)$$

and

$$\int_{U(m)} s_\lambda(AU)\overline{s_\mu(BU)}d\mu_H(U) = \delta_{\lambda,\mu}\frac{s_\lambda(AB^*)}{s_\lambda(I_m)}. \qquad (52)$$

Our calculation of the integral in (35) for $f = s_\lambda$ rests on the integration formula (51) and the invariance of the measure $d\rho_{N,n\times m}$ with respect to the left and right multiplications by unitary matrices.

Let X be an $n \times n$ matrix and Y be an $m \times m$ matrix. Consider the integral

$$\int_{U(N)} s_\lambda(AUBU^*)\ d\mu_H(U) = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(I_N)} \qquad (53)$$

where A and B are the block diagonal matrices $A = \mathrm{diag}(X, 0)$, and $B = \mathrm{diag}(Y, 0)$. It is apparent from (5) that

$$s_\lambda(a_1, a_2, \ldots, a_m, 0, \ldots, 0) = \begin{cases} s_\lambda(a_1, a_2, \ldots, a_m) & \text{if } l(\lambda) \le m \\ 0 & \text{if } l(\lambda) > m \end{cases} \tag{54}$$

where $l(\lambda)$ is the length of λ (the number of non-zero parts λ_j)[2]. Therefore the integral in (53) vanishes for $l(\lambda) > m$ (recall that $m \le n$) and is equal to $s_\lambda(X)s_\lambda(Y)/s_\lambda(I_N)$ if $l(\lambda) \le m$.

On the other hand, the non-zero eigenvalues of the $N \times N$ matrix $AUBU^*$ coincide with those of the $m \times m$ matrix $XQYQ^*$ where Q is the principal $n \times m$ block of U, see (36). Therefore

$$\int_{U(N)} s_\lambda(AUBU^*)\, d\mu_H(U) = \int_{Q^*Q \le I_m} s_\lambda(XQYQ^*)\, d\rho_{N,n\times m}(Q). \tag{55}$$

The measure $d\rho_{N,n\times m}(Q)$ is invariant with respect to the right multiplication by unitary matrices. Hence

$$\int_{Q^*Q \le I_m} s_\lambda(XQYQ^*)\, d\rho_{N,n\times m}(Q) = \int_{U(m)} \int_{Q^*Q \le I_m} s_\lambda(Q^*XQVYV^*)\, d\rho_{N,n\times m}(Q) d\mu_H(V),$$

where we have used the invariance of $s_\lambda(M_1 M_2 \ldots)$ under the cyclic permutations of the matrices M_j. Reverting the order of integrations and applying the integration rule (51), one obtains

$$\int_{Q^*Q \le I_m} s_\lambda(XQYQ^*)\, d\rho_{N,n\times m}(Q) = \frac{s_\lambda(Y)}{s_\lambda(I_m)} \int_{Q^*Q \le I_m} s_\lambda(XQQ^*)\, d\rho_{N,n\times m}(Q).$$

The measure $d\rho_{N,n\times m}(Q)$ is also invariant with respect to the left multiplications by unitary matrices. Repeating the above procedure, one decouples X and QQ^* thus obtaining

$$\int_{Q^*Q \le I_m} s_\lambda(XQYQ^*)\, d\rho_{N,n\times m}(Q) = \frac{s_\lambda(X)s_\lambda(Y)}{s_\lambda(I_m)s_\lambda(I_n)} \int_{Q^*Q \le I_m} s_\lambda(Q^*Q)\, d\rho_{N,n\times m}(Q).$$

On comparing this to (53) and (55), one concludes that

$$\int_{Q^*Q \le I_m} s_\lambda(XQYQ^*)\, d\rho_{N,n\times m}(Q) = \frac{s_\lambda(X)s_\lambda(Y)}{s_\lambda(I_N)} \tag{56}$$

of which (17) is a particular case of $X = I_n$, $Y = I_m$. It is worth mentioning that the quotient $s_\lambda(I_n)s_\lambda(I_m)/s_\lambda(I_N)$ can be easily evaluated in terms of λ_j's by recalling Weyl's dimension formula

$$s_\lambda(I_n) = \left\{ \prod_{1 \le i < j \le m} (\lambda_i - i - \lambda_j + j) \right\} \times \prod_{j=1}^{m} \frac{(n + \lambda_j - j)!}{(m + \lambda_j - j)!(n - j)!} \tag{57}$$

which holds for any integer $n \ge m \ge l(\lambda)$.

By repeating the argument which was used to evaluate the integral in (56), one can extend the integration formulae (51) and (52) from integrals over unitary group to integrals over complex matrices provided that the integration measure is invariant with respect to the left *and* right

[2] By convention, one does not distinguish between partitions which differ merely by the number of zero parts, i.e. $(\lambda_1, \ldots, \lambda_m) = (\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)$

multiplication by unitary matrices. For example, if L and M are $m \times n$ matrices and $d\rho(Q)$ is a measure on $\mathbb{C}^{n\times m}$ invariant with respect to the right and left multiplication by unitary matrices and such that the integral below converges absolutely then

$$\int_{\mathbb{C}^{n\times m}} s_\lambda(LQ)\overline{s_\mu(MQ)}\, d\rho(Q) = \delta_{\lambda,\mu}\, \frac{s_\lambda(M^*L)}{s_\lambda(I_n)s_\lambda(I_m)} \int_{\mathbb{C}^{n\times m}} s_\lambda(Q^*Q)d\rho(Q). \tag{58}$$

In particular, if $d\rho$ is the projection of the Haar measure on the matrix ball $Q^*Q \leq I_m$, i.e., $d\rho = d\rho_{N,n\times m}$, then (58) and (56) imply the orthogonality relation (19).

We would like to make two remarks at this point. One is that equations (49) and (50) effectively give the joint probability distribution of the singular values of the truncations of random unitary matrices. This distribution has an interesting symmetry: Consider two square truncations, Q_1 of size $m \times m$ and Q_2 of size $(N-m)\times(N-m)$. Assuming that $2m \leq N$, $x = 1$ is the singular value of Q_2 of multiplicity $N-2m$ and the remaining m singular values of Q_2 have the same distribution as the m singular values of Q_1.

The other is that some of our calculations can be repeated, almost verbatim, for truncations of random orthogonal matrices. In particular, let Q be the top left block of size $n\times m$ of random orthogonal matrix O of size $N \times N$, $m \leq n \leq N$. Denote by $d\rho_{O(N),n\times m}$ the image of the Haar measure under the map $O \mapsto Q$. Then, by repeating the steps of the above derivation of (49) one obtains the integration formula

$$\int_{Q^tQ\leq I_m} f(Q^tQ)d\rho_{O(N),n\times m}(Q) = \text{const.}$$
$$\int_{X^tX\leq I_m} f(X^tX)\det(I_m - X^tX)^{\frac{1}{2}(N-n-m+1)}\det(X^tX)^{\frac{1}{2}(n-m)}(dX), \qquad N \geq n+m,$$

where the integration on the right is over real $m\times m$ matrices X. This integration formula is not new and was previously obtained in [11] by a different method. If $N < n+m$ and $f(A)$ is a function on $\mathbb{R}^{m\times m}$ which is invariant under the conjugation by orthogonal matrices then a similar formula holds

$$\int_{Q^tQ\leq I_m} f(Q^*Q)d\rho_{O(N),n\times m}(Q) = \text{const.}\times$$
$$\int_{X^tX\leq I_{N-n}} f\begin{pmatrix} X^*X & 0 \\ 0 & I_{m+n-N}\end{pmatrix}\det(I_{N-n} - X^*X)^{\frac{1}{2}(n+m-N+1)}\det(X^tX)^{\frac{1}{2}(n-m)}\,(dX).$$

Formula (51) also has its analogue for orthogonal matrices

$$\int_{O(m)} C_\lambda(XOYO^t)\, d\mu_H(O) = \frac{C_\lambda(X)C_\lambda(Y)}{C_\lambda(I_m)},$$

where C_λ are the so-called zonal polynomials. For the definition of zonal polynomials and their properties see [29]. As a consequence, formula (56) also has its analogue for real matrices: one just replaces Schur functions in (56) by zonal polynomials C_λ. However, the orthogonality relations (52) and (58) do not seem to have analogues for real matrices.

Now we are in a position to derive the bosonic CFT formula in the range $0 \leq m \leq N$. Our approach is based on the Schur function expansion for the exponential $e^{\operatorname{Tr} M}$ combined with the orthogonality relation (58). This yields the bCFT in the form (9). The integration formulas (49) and (50) then lead to the specialization (2) in the interval $2m \leq N$ and (11) in the interval $N < 2m < 2N$.

Recall that the matrices X and Y are $N \times m$. The singular value decomposition for XY^* reads

$$XY^* = V\tilde{D}W^*, \qquad \tilde{D} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \tag{59}$$

where V and W are unitary matrices of size $N \times N$ and D is a diagonal matrix of size $m \times m$. The entries of D are exactly the square roots of the eigenvalues of the matrix X^*XY^*Y, so that

$$s_\lambda(D^2) = s_\lambda(X^*XY^*Y) \tag{60}$$

In view of (59) the left-hand side of (9) takes the form of an integral over truncations of unitary matrices:

$$\int\limits_{U(N)} e^{\mathrm{Tr}(XY^*U+U^*YX^*)} d\mu_H(U) = \int\limits_{U(N)} e^{\mathrm{Tr}(\tilde{D}U+U^*\tilde{D})} d\mu_H(U) = \int\limits_{Q^*Q\leq I_m} e^{\mathrm{Tr}(DQ+Q^*D)} d\rho_{N,m\times m}(Q), \tag{61}$$

where Q is the top left $m \times m$ block of U and $d\rho_{N,m\times m}(Q)$ is the image of the Haar measure under the map $U \mapsto Q$.

The exponential function $e^{\mathrm{Tr}\, M}$ is a symmetric function of the eigenvalues of M and as such can be expanded in Schur functions, see e.g. [4]:

$$e^{\mathrm{Tr}\, M} = \sum_\lambda c_\lambda s_\lambda(M). \tag{62}$$

The sum above is over all partitions λ of length $l(\lambda) \leq N$ where N is the matrix dimension of M. However, if the matrix M has only $m \leq N$ non-zero eigenvalues then, in view of (54), the sum is effectively over partitions λ of length $l(\lambda) \leq m$. The coefficients c_λ in (62) can be computed in terms of λ:

$$c_\lambda = \det\left(\frac{1}{(\lambda_j - j + i)!}\right)_{i,j=1}^N = \frac{\prod_{1\leq i<j\leq l}(\lambda_i - i - \lambda_j + j)}{\prod_{j=1}^l (l + \lambda_j - j)!}, \tag{63}$$

where l is the length of partition λ and, by convention, $\frac{1}{k!} = 0$ for negative k. Note that c_λ are independent of the matrix dimension N. In Appendix we give an analytic method of computing the coefficients of Schur function expansion for multiplicative functionals of eigenvalues of M, to complement the algebraic method of [4].

Let us now expand $e^{\mathrm{Tr}\, DQ}$ and $e^{\mathrm{Tr}\, Q^*D}$ in the integral on the left in (61) in the Schur functions and apply the orthogonality relation (58) (recall that $d\rho_{N,m\times m}(Q)$ is invariant with respect to the right and left multiplications by unitary matrices). This yields

$$\int_{Q^*Q\leq I_m} e^{\mathrm{Tr}(DQ+Q^*D)} d\rho_{N,m\times m}(Q) = \sum_\lambda c_\lambda^2 \frac{s_\lambda(X^*XY^*Y)}{s_\lambda^2(I_m)} \int_{Q^*Q\leq I_m} s_\lambda(Q^*Q) d\rho_{N,m\times m}(Q), \tag{64}$$

where we have used (60). Applying (58) again, now in the opposite direction, and folding the Schur function expansions for the exponential functions, one obtains

$$\sum_\lambda c_\lambda^2 \frac{s_\lambda(X^*XY^*Y)}{s_\lambda^2(I_m)} \int_{Q^*Q\le I_m} s_\lambda(Q^*Q) d\rho_{N,m\times m}(Q) = \int_{Q^*Q\le I_m} e^{\mathrm{Tr}(X^*XQ+Q^*Y^*Y)} d\rho_{N,m\times m}(Q), \tag{65}$$

and hence the bosonic CFT for $m \le N$ in the form of equation (9).

The right-hand side of (9) takes different forms depending on m. If $N \ge 2m$ then $d\rho_{N,m\times m}(Q) = \text{const.} \det(I_m - Q^*Q)^{N-2m}$, see (38), and we are back to formula (8). In the range $N < 2m < 2N$ the measure $d\rho_{N,m\times m}(Q)$ is not very handy in its explicit form. However, by (65), the integral on the right-hand side in (9), is effectively written in terms of the integrals

$$\int_{Q^*Q\le I_m} s_\lambda(Q^*Q)\, d\rho_{N,m\times m}(Q), \tag{66}$$

which makes it possible to evaluate it explicitly by making use of (66).

Recall, that in the range $N < 2m < 2N$ the measure $d\rho_{N,m\times m}$ is supported by the set Υ of $m\times m$ matrices Q such that the rank of $I_m - Q^*Q$ is $N-m$ and $Q^*Q \le I_m$. This set can be parameterized by matrices Q_{UZV^*}, see (10), and the integral in (66) is effectively over matrices Z, see (50). Such parametrization allows one to rewrite the right-hand side of (9) in a more explicit form. Indeed, it is apparent that

$$s_\lambda \begin{pmatrix} Z^*Z & 0 \\ 0 & I_{2m-N} \end{pmatrix} = s_\lambda(Q^*_{UZV^*} Q_{UZV^*}).$$

Hence, by (50),

$$\int_{Q^*Q\le I_m} s_\lambda(Q^*Q)\; d\rho_{N,m\times m}(Q) = \int_{U(m)^2} \int_{Z^*Z<I_{N-m}} s_\lambda(Q^*_{UZV^*} Q_{UZV^*})\; d\mu^B_{N,N-m}(Z) d\mu_H(U) d\mu_H(V), \tag{67}$$

where $d\mu^B_{N,N-m}(Z)$ is the measure defined in (12). Substituting the obtained expression for integral (66) into (64), one obtains the variant of the CFT in the range $N < 2m < 2N$ as presented in (11).

We would like to finish this section with two observations. Firstly, instead of reducing the group integral on the left-hand side in (9) to an integral over truncations of unitary matrices, one can directly expand the exponentials in the group integral and then apply integration formulas (52) and (17), the latter in the right-to-left direction. On this way one easily obtains the bCFT in the form of equation (15).

Secondly, our derivation of the bCFT does not use the explicit expression (63) for the coefficients c_λ of the Schur function expansion for the exponential function. All that is needed of c_λ's is the property

$$c_{(\lambda_1,\ldots,\lambda_m,0,\ldots,0)} = c_{(\lambda_1,\ldots,\lambda_m)}. \tag{68}$$

Therefore our calculation will go through for any convergent series $g(A) = \sum_\lambda c_\lambda S_\lambda(A)$ provided that the coefficients c_λ satisfy (68) yielding the following generalization of the bCFT formula (9)

$$\int_{U(N)} |g((XY^*U)|^2 d\mu_H(U) = \int_{QQ^*\le I_m} g(X^*XQ)\overline{g(Y^*YQ)}\, d\rho_{N,m\times m}(Q). \tag{69}$$

4 CFT, reproducing kernels and Selberg integrals

In this section we investigate relations between the CFTs and several interesting matrix integrals.

It is apparent from the calculations in the previous section that the bosonic CFT (9) implies the identity (16) and vice versa.

Consider now the matrix integral

$$\int_{U(N)} \frac{d\mu_H(U)}{\det(I_N - AU)^m \det(I_N - U^*B^*)^m} = \sum_{\lambda} \frac{s_\lambda^2(I_m)}{s_\lambda(I_N)}\, s_\lambda(B^*A). \tag{70}$$

where $A^*A < I_N$ and $B^*B < I_N$. One can see that the integral on the left-hand side coincides with the series on the right-hand side by recalling the Cauchy identity (6). It is a natural generalization of the well-known expansion of the inverse determinant $1/\det(I-M)$ in terms of the complete symmetric functions h_r of the eigenvalues $(z_1,\ldots,z_m)$ of M,

$$\frac{1}{\det(I-M)} = \prod_{j=1}^{m} \frac{1}{1-z_j} = \sum_{r=0}^{\infty} h_r(z_1,\ldots,z_m).$$

Expanding each of the two determinants in (70) with the help of (6) and then applying the orthogonality relation (52) one obtains the series on the right-hand side in (70). By making use of (16) and (6), one can fold this series back to a matrix integral, now over matrices Q:

$$\int_{U(N)} \frac{d\mu_H(U)}{\det(I_N - AU)^m \det(I_N - U^*B^*)^m} = \int_{Q^*Q\le I_m} \frac{d\rho_{N,m\times m}(Q)}{\det(I_{mN} - Q^*Q\otimes B^*A)}, \qquad m \le N. \tag{71}$$

Thus, the bosonic CFT (9) implies (71) and vice versa.

In the range $2m \le N$ the measure $d\rho_{N,m\times m}(Q)$ coincides with $d\mu^B_{N,m}(Q)$ of (1), and (71) reads

$$\int_{U(N)} \frac{d\mu_H(U)}{\det(I_N - AU)^m \det(I_N - U^*B^*)^m} = \int_{Q^*Q\le I_m} \frac{d\mu^B_{N,m}(Q)}{\det(I - Q^*Q\otimes B^*A)}.$$

This identity was obtained in our earlier work [16].

In the range $N < 2m < 2N$ one can replace the integration over matrices Q by integration over matrices Q_{UZV^*}, see (10) and (67), and (71) takes this form

$$\int\limits_{U(N)} \frac{d\mu_H(U)}{\det(I_N - AU)^m \det(I_N - U^*B^*)^m} = \int\limits_{Z^*Z\le I_{N-m}} \frac{d\mu^B_{N,N-m}(Z)}{\det(I - Z^*Z\otimes B^*A)\det(I_N - B^*A)^{2m-N}}.$$

The integration on the right-hand side is over complex $(N-m)\times(N-m)$ matrices Z and the integration measure is given in (12). This identity is new.

If $m = N$ then $d\rho_{N,m\times m}$ is the Haar measure on $U(N)$, and (71) reads

$$\int_{U(N)} \frac{d\mu_H(U)}{\det(I_N - AU)^N \det(I_N - U^*B^*)^N} = \frac{1}{\det(I_N - B^*A)^N}.$$

This identity is almost apparent in view of the orthogonality relation (52).

In the previous section we verified identity (17), of which (16) is a special case ($m = n$). Identity (17) implies

$$\int_{U(N)} \frac{d\mu_H(U)}{\det(I_N - AU)^m \det(I_N - U^*B^*)^n} = \int_{Q^*Q \le I_m} \frac{d\rho_{N,n\times m}(Q)}{\det(I_{mN} - Q^*Q \otimes B^*A)}, \qquad m, n \le N. \tag{72}$$

in the same way as (16) implies (71). In the range $m+n \le N$ one can use the explicit expression (38) for $d\rho_{N,n\times m}(Q)$. Replacing the integration over the $n \times m$ matrices Q by integration over $k \times k$ matrices Z, $k = \min(m, n)$, as in (39), one obtains the identity which was claimed in (20).

Consider now the matrix integral

$$\int_{U(N)} \det(I_N + AU)^m \det(I_N + U^*B^*)^m \; d\mu_H(U) = \sum_\lambda \frac{s_\lambda^2(I_m)}{s_{\lambda'}(I_N)} \; s_{\lambda'}(B^*A) \tag{73}$$

where λ' is the partition conjugate to λ. This integral is dual to the one in (70). The equality in (73) is a straightforward consequence of the orthogonality relations (52) and the dual Cauchy identity (7) which is a generalization of the well known expansion of the determinant $\det(I+M)$ in elementary symmetric functions e_r of the eigenvalues $z_1, \dots, z_m$ of M,

$$\det(I + M) = \prod_{j=1}^m (1 + z_j) = \sum_{r=0}^m e_r(z_1, \dots, z_m).$$

In [16] we proved identity (21) which is dual to (16). It is apparent that (21) implies the matrix integral claimed in (22) and vice versa. We shall now give an independent derivation of (22) from the fermionic version (3) of the CFT and thus showing that the fCFT implies (21).

Recall that $\det M$ can be written as a Gaussian integral over anticommuting variables φ_j, φ_j^* with Berezin's integration rules $\int (1, \varphi_j) d\varphi_j = (0, 1)$, $\int (1, \varphi_j^*) d\varphi_j^* = (0, 1)$,

$$\det M = \int e^{\sum_{i,j} \varphi_i^* M_{ij} \varphi_j} \prod_j d\varphi_j d\varphi_j^* = \int e^{\;\boldsymbol{\varphi}^* M \boldsymbol{\varphi}} \; (d\varphi).$$

By doubling the dimension,

$$\det(I_N + AU)\det(I_N + U^*B^*) = -\det \begin{pmatrix} 0 & U^* + A \\ U + B^* & 0 \end{pmatrix}$$

and

$$\det(I_N + AU)^m \det(I_N + U^*B^*)^m = \pm \int\!\!\int e^{\mathrm{Tr}[\Psi^*(U+B^*)\Phi + \Phi^*(U^*+A)\Psi]} (d\Phi)(d\Psi). \tag{74}$$

Here Φ and Ψ are $N \times m$ and Φ^* and Ψ^* are $m \times N$ matrices with anticommuting entries and $(d\Phi)$ (correspondingly $(d\Psi)$) is the product of the "differentials" of the matrix entries of Φ and Φ^* (correspondingly, $d\Psi$ and $d\Psi^*$). The sign in front of the integral in (74) depends on the particular ordering of terms in these products. It is not essential for our calculation (one can always verify the right sign at the end of calculation) and will be omitted.

On substituting (74) in the integral on the left-hand side in (73) and applying the fCFT (3) one reduces this integral to the following one

$$\int\!\!\int \left(\int_{\mathbb{C}^{m\times m}} e^{\mathrm{Tr}(\Psi^* B^* \Phi + \Phi^* A \Psi + \Psi^* \Psi Q - Q^* \Phi^* \Phi)} d\mu_{N,m}^F(Q) \right) (d\Phi)(d\Psi) \tag{75}$$

The quadratic form in the exponential,

$$\begin{aligned}\mathrm{Tr}\,(\Psi^* B^* \Phi + \Phi^* A \Psi + \Psi^* \Psi Q - Q^* \Phi^* \Phi) = \\ \sum_{j=1}^{m} \left(\boldsymbol{\psi^*}_j B^* \boldsymbol{\varphi}_j + \boldsymbol{\varphi^*}_j A \boldsymbol{\psi}_j \right) + \sum_{i,j=1}^{m} \left(Q_{kj} \boldsymbol{\psi^*}_j \boldsymbol{\psi}_k - Q^*_{jk} \boldsymbol{\varphi^*}_j \boldsymbol{\varphi}_k \right),\end{aligned}$$

where $\boldsymbol{\varphi}_j$ and $\boldsymbol{\psi}_j$ are the columns of Φ and Ψ, and $\boldsymbol{\varphi^*}_j$ and $\boldsymbol{\psi^*}_j$ are the rows of Φ^* and Ψ^*, is the one defined by the matrix

$$M = \begin{pmatrix} -Q^* \otimes I_N & I_m \otimes A \\ I_m \otimes B^* & Q \otimes I_N \end{pmatrix}.$$

Therefore, $\int\int \dots (d\Psi)(d\Phi)$ in (75) yields $\det M = \det(Q^*Q \otimes I_N + I_m \otimes B^* A)$ and one arrives at the identity

$$\int_{U(N)} \det(I_N + AU)^m \det(I_N + U^* B^*)^m \, d\mu_H(U) = \int_{\mathbb{C}^{m\times m}} \det(Q^*Q \otimes I_N + I_m \otimes B^* A) d\mu^F_{N,m}(Q). \quad (76)$$

On making the substitution

$$Q = Z^{-1}, \quad (dQ) = \det(Z^*Z)^{2m} (dZ),$$

in the integral on the right-hand side in (76), one obtains the matrix integral (22).

Identity (16) and its dual version (21) are rather useful in the context of Schur function expansions. For example, the matrix integrals (25) and (26) are straightforward corollaries of these identities. Consider, for example, the matrix integral in (25). Expanding each of the determinants on the left-hand side with the help of (6), one arrive by the way of the orthogonality relation (58) at

$$\int_{Q^*Q \le I_m} \frac{d\rho_{N,m}(Q)}{\det(I_m - Z_1 Q^*)^N \det(I_m - Q Z_2^*)^N} = \sum_{\lambda} \frac{s^2_\lambda(I_N)}{s^2_\lambda(I_m)} \, s_\lambda(Z_1 Z_2^*) \int_{Q^*Q \le I_m} s_\lambda(Q^*Q) d\rho_{N,m\times m}(Q).$$

In view of (16), the series on the right-hand side folds to

$$\sum_{\lambda} \, s_\lambda(I_N) \, s_\lambda(Z_1 Z_2^*) = \frac{1}{\det(I_m - Z_1 Z_2^*)^N}.$$

Hence (16) implies (25) and vice versa. Similarly, (21) implies (26) and vice versa.

If $N \ge 2m$ then $d\rho_{N,m\times m} = d\mu^B_{N,m}$ and (25) reads

$$\int_{Q^*Q < I_m} \frac{d\mu^B_{N,m}(Q)}{\det(I_m - Z_1 Q^*)^N \det(I_m - Q Z_2^*)^N} = \frac{1}{\det(I_m - Z_1 Z_2^*)^N}. \quad (77)$$

The matrix integrals (77) and (26) are variants of integrals obtained by Berezin in [5, 6]. We would like to elaborate more on this link and quote some of Berezin's results. Consider complex rectangular matrices Z and define

$$\Omega_B = \{ Z \in \mathbb{C}^{n\times m} : \; Z^*Z < I_m \} \quad , \quad \Omega_F = \mathbb{C}^{n\times m}, \quad (78)$$

and, (cf. (1) and (4))

$$\begin{aligned} d\mu^B_{N,n\times m}(Z) &= \text{const.}\det(I - Z^*Z)^{-n-m+N}\ (dZ), \quad Z\in\Omega_B, \quad N\geq n+m, \\ d\mu^F_{N,n\times m}(Z) &= \text{const.}\det(I + Z^*Z)^{-n-m-N}\ (dZ), \quad Z\in\Omega_F, \quad N\geq 0, \end{aligned}$$

where (dZ) is the cartesian volume element in $\mathbb{C}^{n\times m}$. The multiplicative constants are fixed by the normalisation $\int_\Omega d\mu_{N,n\times m}(Z) = 1$.

Let $\mathcal{A}_B$ be the Hilbert space of analytic (in the nm variables Z_{ij}) functions on Ω_B with the scalar product

$$(f,g) = \int_{\Omega_B} f(Z)\overline{g(Z)}\ d\mu^B_{N,n\times m}(Z) \tag{79}$$

and $\mathcal{A}_F$ be the Hilbert space of analytic (in the nm variables Z_{ij}) functions on Ω_F with the scalar product

$$(f,g) = \int_{\Omega_F} f(Z)\overline{g(Z)}\ d\mu^F_{N,n\times m}(Z). \tag{80}$$

For (f,f) in (80) to be finite, $f(Z)$ cannot grow at infinity faster than a certain power of $\|Z\|$. Hence $\mathcal{A}_F$ consists of polynomials in Z_{ij} and is a finite-dimensional subspace of $L^2(\Omega_F, d\mu^F_{N,n\times m})$. Let us choose a basis $f_j(Z)$ in $\mathcal{A}_F$. Given a vector f in $L^2(\Omega_F, d\mu^F_{N,n\times m})$, its orthogonal projection on $\mathcal{A}_F$ is given by

$$(K_F f)(Z) = \sum_j (f, f_j) f_j(Z) = \int_{\Omega_F} K_F(Z,\bar{Q}) f(Q) d\mu^F_{N,n\times m}(Q),$$

where $K_F(Z,\bar{Q}) = \sum_j f_j(Z) f_j(\bar{Q})$. As $K_F(Z,\bar{Q})$ is the kernel of the operator of orthogonal projection onto $\mathcal{A}_F$, it is independent of the choice of basis there. Similarly, if f_j is a basis in $\mathcal{A}_B$ then $K_B(Z,\bar{Q}) = \sum_j f_j(Z) f_j(\bar{Q})$ is the kernel of the orthogonal projection from $L^2(\Omega_B, d\mu^B_{N,n\times m})$ onto $\mathcal{A}_B$. The Hilbert space $\mathcal{A}_B$ is infinite dimensional and, hence, there is a question of convergence of the series for $K_B(Z,\bar{Q})$. It can be shown that this series converges absolutely and uniformly on compact sets, see [5]. The kernels $K_i(Z,\bar{Q})$ define coherent states in $\mathcal{A}_i$, $i = B, F$. Indeed, for each $Q\in\Omega_i$, $K_i(Z,\bar{Q})$ as a function in Z, $f_{\bar{Q}}(Z) = K_i(Z,\bar{Q})$, belongs to $\mathcal{A}_i$ and for any f

$$(f, f_{\bar{Q}}) = \sum_j (f, f_j) f_j(Q) = f(Q). \tag{81}$$

Hence, one has the resolution of identity

$$(f,g) = \int_{\Omega_i} (f, f_{\bar{Q}})(f_{\bar{Q}}, g) d\mu^i_{N,n\times m}(Q), \qquad i = B, F.$$

The vectors $f_{\bar{Q}}$ are not orthogonal. Putting $f_{\bar{Z}}$ for f in (81), one obtains $(f_{\bar{Z}}, f_{\bar{Q}}) = f_{\bar{Z}}(Q)$, or, equivalently

$$\int_{\Omega_i} K_i(Q,\bar{R}) K_i(R,\bar{Z}) d\mu^i_{N,n\times m}(R) = K_i(Q,\bar{Z}) \qquad i = B, F, \tag{82}$$

which means that the kernels $K_i(Z,\bar{Q})$ are reproducing. The coherent states which are defined by these kernels played an important role in Berezin's construction of quantization in symmetric spaces. For some parameter values these kernels can be found in the explicit form [6]

$$\begin{aligned} K_B(Z,\bar{Q}) &= \det(I_m - Q^*Z)^{-N} \quad \text{for real } N\geq n+m, \\ K_F(Z,\bar{Q}) &= \det(I_m - Q^*Z)^{N} \quad \text{for } N = 0,1,2,\ldots, \end{aligned}$$

and identity (82) expressing the reproducing property of these kernels takes the form of the matrix integrals

$$\int_{\Omega_B} \frac{d\mu^B_{N,n\times m}(Q)}{\det(I_m - Q^* Z_1)^N \det(I_m - Z_2^* Q)^N} = \frac{1}{\det(I_m - Z_2^* Z_1)^N} \tag{83}$$

and

$$\int_{\Omega_F} \det(I_m + Q^* Z_1)^N \det(I_m + Z_2^* Q)^N \; d\mu^F_{N,n\times m}(Q) = \det(I_m + Z_2^* Z_1)^N, \tag{84}$$

of which (25) and (26) are particular cases. Correspondingly, the matrix integrals (83) and (84) allows one to extend identities (16) and (21) to rectangular matrices as claimed in (17) and (23). Indeed, assuming that N is a non-negative integer and expanding the integrands in (83) and (84) in Schur functions by the means of the Cauchy identities (6)–(7), and then applying the orthogonality relation (cf. (58))

$$\int_{\Omega_i} s_\lambda(AQ) s_\mu(Q^* B^*) \; d\mu^i_{N,n\times m}(Q) = \delta_{\lambda,\mu} \frac{s_\lambda(AB^*)}{s_\lambda(I_n) s_\lambda(I_m)} \int_{\Omega_i} s_\lambda(Q^* Q) \; d\mu^i_{N,n\times m}(Q), \qquad i = B, F, \tag{85}$$

one obtains Schur function expansions for the determinantal powers on the right-hand side in (83) and (84). On comparing the coefficients in these expansions with those in (6)–(7) one arrives at (17) and (23).

Identities (17) and (23) can also be derived in an elementary way and independently of (83) and (84) and then used to prove (83) and (84). We shall demonstrate this at the end of this section.

With identity (23) in hand, we can revisit the matrix integral (73) and consider unequal powers of determinants

$$\int_{U(N)} \det(I_N + AU)^m \det(I_N + U^* B^*)^n \; d\mu_H(U) = \sum_\lambda \frac{s_\lambda(I_m) s_\lambda(I_n)}{s_{\lambda'}(I_N)} \; s_{\lambda'}(B^* A). \tag{86}$$

By making use of (23), one can fold the sum on the right,

$$\sum_\lambda \frac{s_\lambda(I_m) s_\lambda(I_n)}{s_{\lambda'}(I_N)} \; s_{\lambda'}(B^* A) = \int_{\mathbb{C}^{n\times m}} \det(I + Z^* Z \otimes B^* A) \; d\mu^F_{N,n\times m}(Z).$$

Replacing the integration over the $n \times m$ matrices Z by integration over $k \times k$ matrices Z, $k = \min(m, n)$, as in (39), one obtains the identity claimed in (24).

Now we turn to identities (17) and (23). The integration on the left-hand side in (17) and (23) goes effectively over the eigenvalues of $Q^* Q$. Consider $N \geq 2m$ assuming without loss of generality that $m \leq n$. Recalling the singular value decomposition for Q, $Q = H\sqrt{X}V^*$, where V is $m \times m$ unitary, i.e. $V \in U(m)$, H is $n \times m$ unitary, i.e. $H \in V_m(\mathbb{C}^n)$ (see Section 2), and X is diagonal $m \times m$ matrix of the eigenvalues $x_1, \ldots, x_m$ of $Q^* Q$, one can make the substitution $Q = H\sqrt{X}V^*$ in the integrals in (17) and (23). The corresponding Jacobian is well known, see e.g. [29, 28],

$$(dQ) = \text{const.} \prod_{1 \leq i < k \leq m} (x_i - x_j)^2 \prod_{i=1}^m x_j^{(n-m)} \prod_{j=1}^m dx_j (H^* dH)(V^* dV),$$

where (H^*dH) and (V^*dV) are the invariant volume elements in, respectively $V_m(\mathbb{C}^n)$ and $U(m)$. This substitution reduces the matrix integral in (17) to (27) and the one in (23) to (28). The normalization constants are given by

$$\begin{aligned} c_{n,m}^N &= \int_0^1 \cdots \int_0^1 \prod_{j=1}^m x_j^{n-m}(1-x_j)^{N-m-n} \prod_{1\le i<j\le m}(x_i-x_j)^2 \prod_{j=1}^m dx_j \\ &= \prod_{j=0}^{m-1} \frac{(1+j)!\,(n-m+j)!\,\Gamma(N-n-m+j+1)}{\Gamma(N-m+j+1)} \end{aligned} \tag{87}$$

and

$$\begin{aligned} k_{n,m}^N &= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^m \frac{x_j^{n-m}}{(1+x_j)^{N+m+n}} \prod_{1\le i<j\le m}(x_i-x_j)^2 \prod_{j=1}^m dx_j \\ &= \prod_{j=0}^{m-1} \frac{(1+j)!\,(n-m+j)!\,\Gamma(N+j+1)}{\Gamma(N+n+j+1)}. \end{aligned} \tag{88}$$

The integrals in (87) and (88) are particular cases of the Selberg integral

$$\begin{aligned} \int_0^1 \cdots \int_0^1 \prod_{j=1}^m x_j^{p-1}(1-x_j)^{q-1} \prod_{1\le i<j\le m}|x_i-x_j|^{2\gamma} \prod_{j=1}^m dx_j &= \prod_{j=0}^{m-1} \frac{\Gamma(p+j\gamma)\Gamma(q+j\gamma)\Gamma(1+(1+j)\gamma)}{\Gamma(p+q+(m+j-1)\gamma)\Gamma(1+\gamma)} \\ &= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^m \frac{t_j^{p-1}}{(1+t_j)^{p+q+2(m-1)\gamma}} \prod_{1\le i<j\le m}|t_i-t_j|^{2\gamma} \prod_{j=1}^m dt_j, \end{aligned}$$

which is a multivariate generalization of the Euler beta integral

$$\int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \int_0^\infty \frac{t^{p-1}\,dx}{(1+t)^{p+q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p,q).$$

Here Γ and B are the Gamma and Beta functions, respectively.

The rest of this section is devoted to explicit evaluation of the integral in (27) and the one in (28). To the best of our knowledge, this evaluation is new.

Let

$$S_\lambda^B(p,q;m) = \int_0^1 \cdots \int_0^1 s_\lambda(x_1,\ldots,x_m) \prod_{j=1}^m x_j^{p-1}(1-x_j)^{q-1} \prod_{1\le i<j\le m}(x_i-x_j)^2 \prod_{j=1}^m dx_j$$

and

$$S_\lambda^F(p,q;m) = \int_0^\infty \cdots \int_0^\infty s_\lambda(x_1,\ldots,x_m) \prod_{j=1}^m \frac{x_j^{p-1}}{(1+x_j)^{p+q+2(m-1)}} \prod_{1\le i<j\le m}(x_i-x_j)^2 \prod_{j=1}^m dx_j$$

In view of (5),

$$s_\lambda(x_1,\ldots,x_m) \prod_{1\le i<j\le m}(x_i-x_j)^2 = \det\left(x_i^{m+\lambda_j-j}\right)_{i,j=1}^m \det\left(x_i^{m-j}\right)_{i,j=1}^m.$$

By making use of the Gram identity

$$\int \cdots \int \det\left(F_i(x_j)\right)_{i,j=1}^m \det\left(G_i(x_j)\right)_{i,j=1}^m \prod_{i=1}^m dx_i = m! \det\left(\int F_i(x)G_j(x)dx\right)_{i,j=1}^m , \qquad (89)$$

we have

$$S_\lambda^B(p,q;m) = m! \det\left(\int_0^1 x^{m+p+f_j-i-1}(1-x)^{q-1}dx\right)_{i,j=1}^m = m! \det\left(B(m+p+f_j-i,q)\right)_{i,j=1}^m ,$$

where we have introduced $f_j = m + \lambda_j - j$, and

$$\begin{aligned} S_\lambda^F(p,q;m) &= m! \det\left(\int_0^\infty \frac{x^{m+p+f_j-i-1}dx}{(1-x)^{p+q+2(m-1)}}\right)_{i,j=1}^m \\ &= m! \det\left(B(m+p+f_j-i, q+m-f_j-2+i)\right)_{i,j=1}^m \end{aligned} \qquad (90)$$

In [16] we proved the following identity for binomial determinants

$$\det\left(B(p_j-i,q_j+i)\right)_{i,j=1}^m = \det\left(B(p_j-i,q_j+1)\right)_{i,j=1}^m . \qquad (91)$$

By making use of this identity,

$$\begin{aligned} S_\lambda^B(p,q;m) &= m! \det\left(B(m+p+f_j-i,q+i-1)\right)_{i,j=1}^m \\ &= m! \left\{\prod_{j=1}^m \frac{\Gamma(q+i-1)}{\Gamma(m+p+q+f_j-1)}\right\} \det\left(\Gamma(m+p+f_j-i)\right)_{i,j=1}^m \end{aligned}$$

Now, recalling the standard determinant

$$\det\left(\Gamma(p_j+m-i)\right)_{i,j=1}^m = \prod_{j=1}^m \Gamma(p_j) \prod_{1\le i<j\le m}(p_i-p_j), \qquad (92)$$

one arrives at

$$S_\lambda^B(p,q;m) = m! \left\{\prod_{j=1}^m \frac{\Gamma(q+j-1)\Gamma(p+f_j)}{\Gamma(m+p+q+f_j-1)}\right\} \prod_{1\le i<j\le m}(f_i-f_j), \qquad f_j = m+\lambda_j-j. \qquad (93)$$

Similarly, applying (90) to the determinant on the right in (90), one obtains

$$\begin{aligned} S_\lambda^F(p,q;m) &= m! \det\left(B(m+p+f_j-i,q+m-f_j-1)\right)_{i,j=1}^m \\ &= m! \left\{\prod_{j=1}^m \frac{\Gamma(q+m-f_j-1)}{\Gamma(p+q+2m-j-1)}\right\} \det\left(\Gamma(m+p+f_j-i)\right)_{i,j=1}^m , \end{aligned}$$

and, by (92),

$$S_\lambda^F(p,q;m) = m! \left\{\prod_{j=1}^m \frac{\Gamma(p+f_j)\Gamma(q+m-f_j-1)}{\Gamma(p+q+2m-j-1)}\right\} \prod_{1\le i<j\le m}(f_i-f_j), \qquad f_j = m+\lambda_j-j. \qquad (94)$$

We would like to finish this section with a calculation showing that (93) implies (83). As has been mentioned above, for integer N this claim follows from the Cauchy identity (6). The case of non-integer N can be handled with the help of the following identity where M is $m \times m$:

$$\frac{1}{\det(I_m - M)^N} = \sum_\lambda \beta_\lambda^B s_\lambda(I_m) s_\lambda(M), \quad \beta_\lambda^B = \prod_{j=1}^m \frac{\Gamma(N+\lambda_j - j + 1)(m-j)!}{\Gamma(N-j+1)(m+\lambda_j - j)!}, \qquad (95)$$

This identity is due to Hua [21]. In Appendix we derive it and its dual version with the help of integration over the unitary group, see examples 4 and 5.

It can be verified from the Selberg integral (93) that for real $N \geq n+m$ and $n \geq m$

$$\int_{Q^*Q \leq I_m} s_\lambda(Q^*Q)\, d\mu^B_{N,n\times m}(Q) = \frac{s_\lambda(I_n)}{\beta^B_\lambda}. \qquad (96)$$

By making use of the Schur function expansion in (95) and orthogonality relation (85), the left-hand side in (83) can be expanded as follows

$$\sum_\lambda \frac{(\beta_\lambda^B)^2 s_\lambda(I_m)}{s_\lambda(I_n)} \left(\int_{Q^*Q \leq I_m} s_\lambda(Q^*Q)\, d\mu^B_{N,n\times m}(Q) \right) s_\lambda(Z_2^* Z_1).$$

Applying now (96) and then folding the series with the help of (95), one gets the right-hand side of (83). Hence (95) implies (83) for real $N \geq n+m$.

A similar calculation shows that (94) implies (84) for integer $N \geq 0$. One only needs to recall the dual Cauchy identity (7).

5 Deformed version of the bosonic CFT.

In order to introduce the deformation of the bosonic CFT that we are going to derive, we would like to demonstrate an evaluation of the right-hand side of the bosonic CFT in terms of the eigenvalues of the matrix X^*XY^*Y. We only consider the range $2m \leq N$, however, a similar calculation gives an answer in the range $N < 2m < 2N$. As we shall eventually replace the integration over the matrix ball which is a bounded domain by integration over a hyperbolic domain which is unbounded, we first rewrite (8) by changing to complex exponentials

$$\int_{U(N)} e^{-i\,\mathrm{Tr}(Y^*UX + X^*U^*Y)} d\mu_H(Q) = \text{const.} \int_{Q^*Q \leq I_m} e^{-i\,\mathrm{Tr}(X^*XQ + Q^*Y^*Y)} \det(I - Q^*Q)^{N-2m} (dQ). \qquad (97)$$

Denote $Q_A = X^*X$ and $Q_R = Y^*Y$ and let D is diagonal matrix of the square roots of the eigenvalues of the matrix product Q_AQ_R. Then one can always find a non-degenerate matrix T such that

$$Q_A = TDT^* \qquad \text{and} \qquad Q_R = (T^*)^{-1} D T^{-1}. \qquad (98)$$

Such parametrisation is possible for any two positive definite matrices [17]. This can be seen by diagonalising the matrix $Q_R^{1/2} Q_A Q_R^{1/2}$. Writing $Q_R^{1/2} Q_A Q_R^{1/2} = UDU^*$, we have $T = Q_R^{-1/2} U D^{1/2}$. It is apparent from (98) that T is defined up to the right multiplication by diagonal unitary matrices.

By making use of (98) and singular value decomposition for Q

$$Q = U\,\mathrm{diag}(q_1,\ldots,q_m)V, \qquad (dQ) \propto (U^*dU)(V^*dV)\prod_{1\le j<k\le m}(q_j^2-q_k^2)^2\prod_{j=1}^{m}d(q_j^2),$$

one rewrites the integral on the right-hand side in (97) as follows

$$\int_0^1\cdots\int_0^1\prod_{j=1}^{m}dq_j\,q_j\,(1-q_j^2)^{n-2m}\prod_{1\le j<k\le m}(q_j^2-q_k^2)^2\times$$
$$\int_{U(m)}d\mu_H(U)\int_{U(m)}d\mu_H(V)\ \exp\left[-i\,\mathrm{Tr}(U\hat{q}VTDT^*+(T^*)^{-1}DT^{-1}V^*\hat{q}U^*\right], \qquad (99)$$

where $\hat{q}=\mathrm{diag}(q_1,\ldots q_m)$. The integral in (99) was computed in [36],

$$\int_{U(m)}d\mu_H(U)\int_{U(m)}d\mu_H(V)\ e^{\frac{1}{2}\,\mathrm{Tr}(UAVB+CV^*DU^*)}\propto\frac{\det\left(I_0(x_jy_k)\right)_{j,k=1}^{m}}{\prod_{1\le j<k\le m}(x_j^2-x_k^2)\prod_{1\le j<k\le m}(y_j^2-y_k^2)}$$

where x_j^2 and y_j^2 are the eigenvalues of AB and BC, respectively, and I_0 is the modified Bessel function of zero order. In our case $AD=-4\hat{q}^2$ and $BC=TD^2T^{-1}$, and collecting everything together we obtain

$$\int_{U(N)}e^{-i\,\mathrm{Tr}(Y^*UX+X^*U^*Y)}d\mu_H(Q)=$$
$$\frac{\mathrm{const.}}{\prod\limits_{1\le j<k\le m}(d_j^2-d_k^2)}\int_0^1\cdots\int_0^1\prod_{j=1}^{m}dq_jq_j(1-q_j^2)^{n-2m}\prod_{1\le j<k\le m}(q_j^2-q_k^2)\det\left(J_0(2q_jd_k)\right)_{j,k=1}^{m},$$

where $J_0(z)$ is the Bessel function of zero order. By making use of the Gram formula (89),

$$\int_{U(N)}e^{-i\,\mathrm{Tr}(Y^*UX+X^*U^*Y)}d\mu_H(Q)\ \propto\ \frac{\det\left(\int\limits_0^1 J_0(2qd_k)q^{2(m-j)+1}(1-q^2)^{N-2m}dq\right)_{j,k=1}^{m}}{\prod\limits_{1\le j<k\le m}(d_j^2-d_k^2)}. \qquad (100)$$

This formula can also be derived directly from the character expansion (65), see Lemma 5 in [16]. Such calculation is standard, and in fact the above two-fold integral over unitary group was evaluated in [36] using the character expansion method [4]. Formula (100) can also be extended to the range $N<2m<2N$. In this case one gets the Bessel function J_0 and its derivatives in the determinant on the right-hand side in (100).

Formula (100) allows one to obtain an alternative version of the bosonic CFT with the integration manifold in the right-hand side parametrized by the matrices Q_1 and Q_2 defined in Eq.(31). Now, consider the integral

$$\begin{aligned}F_m(Q_A,Q_R) &= \int(dQ_1dQ_2)\det(I_m-Q_1Q_2)^{N-2m}\exp\left[-i\,\mathrm{Tr}(Q_1Q_R+Q_2Q_A)\right]\\ &= \mathrm{const.}\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{1\le j<k\le m}(p_j^2-p_k^2)^2\prod_{j=1}^{m}p_j(1-p_j^2)^{N-2m}\prod_{j=1}^{m}dp_j\\ &\quad\times\int_{GL_m(\mathbb{C})}d\mu_H(T)\exp\left\{-i\left[TPT^*Q_R+(T^*)^{-1}PT^{-1}Q_A\right]\right\} \qquad (101)\end{aligned}$$

Recalling the $Q_A = T_0 D T_0^*$ and $Q_R = (T_0^*)^{-1} D T_0^{-1}$ for some $T_0 \in GL_m(\mathbb{C})$, see (98), we can rewrite the integral in (101) in terms of P, D and T. The matrix T_0 disappears because of the invariance of $d\mu_H(T)$. The resulting integral is known [17, 15]

$$\int_{GL_m(\mathbb{C})} d\mu_H(T) \exp\left\{-i\left[T^* D T + T^{-1} D (T^*)^{-1}\right] P\right\} = \frac{\mathrm{const.}\det\left(K_0(2ip_j d_k)\right)_{j,k=1}^m}{\prod_{1\le j<k\le m}(p_j^2 - p_k^2)\prod_{1\le j<k\le m}(d_j^2 - d_k^2)}$$

where $K_0(z)$ is the Macdonald function. Note that because the matrices D and P are diagonal, the above integral is effectively going over the right cosets $GL_m(\mathbb{C})/U(1)\times\ldots\times U(1)$. On substituting this expression back in (101) and using the Gram formula (89), one obtains

$$F_m(Q_A, Q_R) \;\propto\; \frac{\det\left(\int\limits_{-1}^{1} K_0(2pd_k) p^{2(m-j)+1}(1-p^2)^{N-2m} dp\right)_{j,k=1}^m}{\prod\limits_{1\le j<k\le m}(d_j^2 - d_k^2)}. \qquad (102)$$

Recall the identity ([18], Equations 8.405 and 8.421)

$$K_0(iu) = -\frac{\pi}{2}\left[Y_0(|u|) + i\,\mathrm{sgn}(u) J_0(|u|)\right]$$

where Y_0 is the Neumann function. It follows from this identity that

$$\int_{-1}^{1} K_0(2pd) p^{2(m-j)+1}(1-p^2)^{N-2m} dp = 2i\int_0^1 J_0(2pd) p^{2(m-j)+1}(1-p^2)^{N-2m} dp.$$

Therefore, the determinants in (100) and (102) differ only by a multiplicative constant, and we arrive at the variant of the bosonic CFT , Eq.(34).

It is instructive to write the formula (34) for the simplest but yet non-trivial case of $m = 1$. In this case Q_1 and Q_2 are just real numbers,

$$Q_1 = |t|^2 p, \quad Q_2 = \frac{p}{|t|^2}, \quad p \in [-1, 1], \quad t = Re^{i\theta} \in \mathbb{C},$$

the integration measure is

$$(dQ_1 dQ_2) = p(1-p^2)^{N-2} dp\,\frac{dt d\bar{t}}{|t|^2} = p(1-p^2)^{N-2}\; dp\;\frac{dR}{R}\; d\theta,$$

and (34) reads

$$\int_{U(N)} \exp\left[-i\,\mathrm{Tr}(\boldsymbol{y}^* U \boldsymbol{x} + \boldsymbol{x}^* U^* \boldsymbol{y})\right] d\mu_H(U) \;\propto\; \int_0^1 p(1-p^2)^{N-2} dp \int_{-\infty}^{\infty} \frac{dR}{R}\,\exp\left[-ip\left(R|\boldsymbol{x}|^2 + \frac{|\boldsymbol{y}|^2}{R}\right)\right].$$

This should be compared to the original version of the bosonic CFT. For $m = 1$ it reads

$$\begin{aligned}\int_{U(N)} \exp\left[-i\,\mathrm{Tr}(\boldsymbol{y}^* U \boldsymbol{x} + \boldsymbol{x}^* U^* \boldsymbol{y})\right] d\mu_H(U) \;&\propto\; \int_{|z|^2\le 1} dz d\bar{z}(1-|z|^2)^{N-2}\exp\left[-i(z|\boldsymbol{x}|^2 + \bar{z}|\boldsymbol{y}|^2)\right] \\ &\propto\; \int_0^1 p(1-p^2)^{N-2} dp\; J_0(2p|\boldsymbol{x}||\boldsymbol{y}|).\end{aligned}$$

Therefore the transition from the original version of the bosonic CFT to its deformed version amounts to replacing the Bessel function by an integral,

$$J_0(2pab) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dR}{R}\,\exp\left[-ip\left(Ra^2+\frac{b^2}{R}\right)\right], \qquad p,a,b>0.$$

We would like to demonstrate the usefulness of the deformed CFT (102) on the example of the matrix integral

$$R_\varepsilon(BB^*) = \int_{U(N)}\frac{d\mu_H(U)}{\det[\varepsilon^2 I_N + (I_N - BU)(I_N - U^*B^*)]^m}. \tag{103}$$

For $\varepsilon > 0$ this integral is well defined for any matrix B. By doubling matrix dimension,

$$\det[\varepsilon^2 I + (I-UB)(I-U^*B^*)] = \begin{vmatrix} \varepsilon I & i(U^*-B) \\ i(U-B^*) & \varepsilon I \end{vmatrix}$$

The quadratic form $\boldsymbol{f}^*M\boldsymbol{f}$, $\boldsymbol{f}^* = (\boldsymbol{x}^*,\boldsymbol{y}^*)$, corresponding to the $2N\times 2N$ matrix on the right-hand side is

$$\varepsilon(\boldsymbol{x}^*\boldsymbol{x}+\boldsymbol{y}^*\boldsymbol{y}) + i\boldsymbol{x}^*B\boldsymbol{y} + i\boldsymbol{y}^*B^*\boldsymbol{x} - i\boldsymbol{x}^*U^*\boldsymbol{y} - i\boldsymbol{y}^*U\boldsymbol{x}.$$

By making use of the Gaussian integral

$$\frac{1}{\det M} = \frac{1}{\pi^{2N}}\int_{\mathbb{C}^{2N}} e^{-\boldsymbol{f}^*M\boldsymbol{f}}\,(d\boldsymbol{f}),$$

we have

$$R_\varepsilon(BB^*) \;\propto\; \int_{\mathbb{C}^{N\times m}}(dX)\int_{\mathbb{C}^{N\times m}}(dY)e^{-\varepsilon\,\mathrm{Tr}(X^*X+Y^*Y-iX^*BY-iY^*B^*X)}J_{cf}(X^*X,Y^*Y)$$

where

$$J_{cf}(X^*X,Y^*Y) = \int_{U(N)} d\mu_H(U)e^{-i\,\mathrm{Tr}(Y^*UX+X^*U^*Y)}$$

This integral is exactly the one appearing on the left-hand side in the bosonic CFT (2). However, if one mindlessly applies (2), one gets a diverging integral and the use of CFT for evaluation of integral (103) appears to be problematic. The deformed version (34), as we shall show below, is free of this problem.

By making use of (34),

$$R_\varepsilon(BB^*) \;\propto\; \int_{\mathbb{C}^{N\times m}}(dX)\int_{\mathbb{C}^{N\times m}}(dY)\exp\left[-(\boldsymbol{x}^*,\boldsymbol{y}^*)\mathcal{M}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix},\right] \tag{104}$$

with the quadratic form in the exponential being

$$\varepsilon\sum_{j=1}^m(\boldsymbol{x}_j^*\boldsymbol{x}_j+\boldsymbol{y}_j^*\boldsymbol{y}_j) - i\sum_{j=1}^m(\boldsymbol{x}_j^*B\boldsymbol{y}_j+\boldsymbol{y}_j^*B^*\boldsymbol{x}_j) + ii\sum_{j,k=1}^m(Q_1)_{jk}\boldsymbol{x}_k^*\boldsymbol{x}_j + i\sum_{j,k=1}^m(Q_2)_{jk}\boldsymbol{y}_k^*\boldsymbol{y}_j.$$

The matrix $\mathcal{M}$ corresponding to this form is

$$\mathcal{M} = \begin{bmatrix} \varepsilon(I_m+iQ_1)\otimes I_N & -iI_m\otimes B \\ -iI_m\otimes B^* & \varepsilon(I_m+iQ_2)\otimes I_N \end{bmatrix} = \varepsilon I + i\begin{bmatrix} Q_1\otimes I_N & -I_m\otimes B \\ -I_m\otimes B^* & Q_2\otimes I_N \end{bmatrix}.$$

Since $\mathcal{M} = \varepsilon + i$ (Hermitian matrix), the Gaussian integral in (104) converges for $\varepsilon > 0$ and is equal to

$$\frac{1}{\det \mathcal{M}} = \frac{1}{\det\{[(\varepsilon I_m + iQ_1)(\varepsilon I_m + iQ_2)] \otimes I_N + I_m \otimes B^*B\}}$$

or, on substituting $Q_1 = TPT^*$, $Q_2 = (T^*)^{-1}PT^{-1}$,

$$\frac{1}{\det \mathcal{M}} = \frac{1}{\det\{[\varepsilon(TT^*)^{-1} + iP][\varepsilon(TT^*) + iP] \otimes I_N + I_m \otimes B^*B\}}.$$

Thus finally

$$\int_{U(N)} \frac{d\mu_H(U)}{\det[\varepsilon^2 I_N + (I_N - BU)(I_N - U^*B^*)]^m} = \int_{[-1,1]^m} \prod_{j=1}^m dp_j \; p_j(1-p_j^2)^{N-2m} \times$$

$$\prod_{1\le j<k\le m} (p_j^2 - p_k^2)^2 \int_{GL_m(\mathbb{C})} \frac{(dT)}{\det(T^*T)^m} \prod_{j=1}^N \frac{1}{\det\left\{[\varepsilon(TT^*)^{-1} + iP][\varepsilon(TT^*) + iP] + b_j^2\right\}},$$

where $P = \text{diag}(p_1, \ldots, p_m)$ and b_j^2 are the eigenvalues of B^*B.

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A Appendix

In this appendix we would like to demonstrate a simple calculation, reminiscent of the one given in Section 3 in [19], to determine coefficients in the Schur function expansions for the class of symmetric functions $g(z_1, \ldots, z_m) = \prod_{j=1}^m h(z_m)$, where $h(z)$ is analytic in a neighbourhood of $|z| = 1$,

$$g(z_1, \ldots, z_m) = \sum_\lambda c_\lambda s_\lambda(z_1, \ldots, z_m). \tag{105}$$

Thinking of the z_j's as of eigenvalues of unitary matrix U, we can rewrite (105) as $g(U) = \sum_\lambda c_\lambda s_\lambda(U)$. Then, because of the orthogonality of the Schur functions on $U(m)$, see (52), the coefficients c_λ are just the "Fourier"-coefficients of the function $g(U)$:

$$c_\lambda = \int_{U(m)} g(U)\overline{s_\mu(U)} \; d\mu_H(U). \tag{106}$$

The unitary matrix U can be brought to diagonal form by a unitary transformation, $U = Ve^{i\Phi}V^*$, where $\Phi = \text{diag}(\phi_1, \ldots, \phi_m), 0 \le \phi < 2\pi$. Correspondingly, the volume element in $U(m)$ transforms as follows (see e.g. [21] or [12])

$$(U^*dU) = \text{const.} \prod_{1\le j<k\le m} |e^{i\phi_j} - e^{i\phi_k}|^2 \prod_{j=1}^m d\phi_j \; (V^*dV),$$

and the integral on the right-hand side in (106) reduces to

$$c_\lambda = \frac{1}{m!(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j=1}^m h(e^{i\phi_j}) \;\det\left(e^{-i\phi_j(m+\lambda_k-k)}\right) \det\left(e^{i\phi_j(m-k)}\right) \prod_{j=1}^m d\phi_j,$$

where we have also used (5). By the Gram identity (89),

$$c_\lambda = \det\left(\frac{1}{2\pi}\int_0^{2\pi} h(e^{i\phi})e^{-i\phi(\lambda_k-k+j)}\;d\phi\right)_{j,k=1}^m.$$

In other words,

$$c_\lambda = \det(\alpha_{\lambda_k-k+j})_{j,k=1}^m, \qquad \alpha_r = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\phi})e^{-i\phi r}\;d\phi. \tag{107}$$

We would like to give several examples.

Example 1. Consider the function $g(z_1,\ldots,z_m) = \exp(\sum_j z_j)$. Then $h(z) = e^z$. Expanding the exponential in the Taylor series, $\alpha_k = 1/k!$, and we recover formula (63).

Example 2. Consider the function $g(z_1,\ldots,z_m) = \prod_{j=1}^m \prod_{k=1}^n 1/(1-t_k z_j)$, as on the left-hand side of the Cauchy identity (6). Then

$$h(z) = \prod_{k=1}^n \frac{1}{1-t_k z} = \sum_{r=0}^\infty h_r(t_1,\ldots,t_n)z^r,$$

where the h_r's are the complete symmetric functions. Thus $\alpha_r = h_r(t_1,\ldots,t_n)$ and $c_\lambda = \det(h_{\lambda_k-k+j})$. In view of the Jacobi-Trudi identity

$$s_\lambda = \det(h_{\lambda_k-k+j}) = \det(e_{\lambda'_k-k+j}),$$

we conclude that $c_\lambda = s_\lambda(t_1,\ldots,t_n)$, thus recovering the Cauchy identity (6).

Example 3. Consider the function $g(z_1,\ldots,z_m) = \prod_{j=1}^m \prod_{k=1}^n (1+t_k z_j)$, as on the left-hand side of the dual Cauchy identity (7). Then

$$h(z) = \prod_{k=1}^n (1+t_k z) = \sum_{r=0}^n e_r(t_1,\ldots,t_n)z^r,$$

where the e_r's are the elementary symmetric functions. Thus $\alpha_r = e_r(t_1,\ldots,t_n)$, and $c_\lambda = \det(e_{\lambda_k-k+j})$. The Jacobi-Trudi identity implies that $c_\lambda = s_{\lambda'}(t_1,\ldots,t_n)$, and we recover the the dual Cauchy identity (7).

Example 4. Consider the function $g(z_1,\ldots,z_m) = \prod_{j=1}^m 1/(1-z_j)^a$, $a \ge 0$. Then

$$h(z) = \frac{1}{(1-z)^a} = \sum_{r=0}^\infty \gamma_r(a)\,z^r, \qquad \gamma_r(a) = \frac{\Gamma(a+r)}{\Gamma(a)r!}. \tag{108}$$

Thus $\alpha_r = \gamma_r(a)$ in (107) and the coefficients in the Schur function expansion (105) of $g(z_1,\ldots,z_m)$ are given by

$$c_\lambda = \det(\gamma_{\lambda_k-k+j}(a))_{j,k=1}^m. \tag{109}$$

Note that one should assume $|z_j| < 1$ to ensure convergence in (108). Since (106) was obtained by integration over $|z_j| = 1$ there arises the question whether one can still use this formula. The answer is positive. For, in view of the homogeneity of the Schur functions

$$s_\lambda(tz_1, \ldots, tz_m) = t^{|\lambda|} s_\lambda(z_1, \ldots, z_m), \qquad |\lambda| = \sum_j \lambda_j,$$

one can always rescale $z_j \to tz_j$ thus extending the domain for z_j to include the unit circle.

The determinant in (109) can be evaluated in terms of λ_k. Introducing the notation $p_k = \lambda_k - k$, we have $c_\lambda = \det(\gamma_{p_k+j}(a))$. By making use of the identity

$$\gamma_{r+1}(a) - \gamma_r(a) = \gamma_{r+1}(a-1)$$

and elementary operations on the columns of the determinant,

$$\begin{aligned} \det(\gamma_{p_k+j}(a)) &= |\gamma_{p_k+1}(a), \gamma_{p_k+2}(a), \ldots, \gamma_{p_k+m}(a)| \\ &= |\gamma_{p_k+1}(a), \gamma_{p_k+2}(a-1), \ldots, \gamma_{p_k+m}(a-1)| \\ &= \ldots \\ &= |\gamma_{p_k+1}(a), \gamma_{p_k+2}(a-1), \ldots, \gamma_{p_k+m}(a-m+1)| = \det(\gamma_{p_k+j}(a-j+1)). \end{aligned}$$

Therefore,

$$c_\lambda = \det\left(\frac{\Gamma(a+p_k+1)}{\Gamma(a-j+1)(p_k+j)!}\right)_{j,k=1}^m = \left\{\prod_{j=1}^m \frac{\Gamma(a+p_j+1)}{\Gamma(a-j+1)}\right\} \det\left(\frac{1}{(p_k+j)!}\right)_{j,k=1}^m.$$

The determinant on the right-hand side can now be easily evaluated,

$$\det\left(\frac{1}{(p_k+j)!}\right)_{j,k=1}^m = \left\{\prod_{j=1}^m \frac{1}{(p_j+m)!}\right\} \prod_{1\le j<k\le m} (p_j - p_k).$$

Recalling that $p_j = \lambda_j - j$, we arrive at

$$\begin{aligned} c_\lambda &= \left\{\prod_{j=1}^m \frac{\Gamma(a+\lambda_j-j+1)}{\Gamma(a-j+1)(m+\lambda_j-j)!}\right\} \prod_{1\le j<k\le m} (\lambda_j - j - \lambda_j + k) \\ &= s_\lambda(1_m) \prod_{j=1}^m \frac{\Gamma(a+\lambda_j-j+1)(m-j)!}{\Gamma(a-j+1)(m+\lambda_j-j)!}, \end{aligned}$$

where the second equality follows from Weyl's dimension formula (57). Thus finally (cf. (6))

$$\prod_{j=1}^m \frac{1}{(1-z_j)^a} = \sum_\lambda \beta_\lambda s_\lambda(1_m) s_\lambda(z_1, \ldots, z_m), \quad \beta_\lambda = \prod_{j=1}^m \frac{\Gamma(a+\lambda_j-j+1)(m-j)!}{\Gamma(a-j+1)(m+\lambda_j-j)!}. \tag{110}$$

This identity folds for $a \ge 0$ and $|z_j| < 1$. It appears in Hua's book (Theorem 1.2.5) where it is derived by a different method.

Example 5. Consider the function $g(z_1,\dots,z_m)=\prod_{j=1}^m(1+z_j)^a$, $a\geq 0$. Then

$$h(z)=(1+z)^a=\sum_{r=0}^{\infty}\gamma_r(a)\,z^r,\qquad \gamma_r(a)=\frac{\Gamma(a+1)}{\Gamma(a-r+1)r!}\tag{111}$$

and the coefficients in the Schur function expansion (105) of $g(z_1,\dots,z_m)$ are given by $c_\lambda=\det(\gamma_{\lambda_k-k+j}(a))_{j,k=1}^m$, with $\gamma_r(a)$ as defined in (111). As in the previous example, this determinant can be evaluated in terms of λ_k. Now for $\gamma_r(a)$ we have

$$\gamma_{r+1}(a)+\gamma_r(a)=\gamma_{r+1}(a+1).$$

By making use of this identity and elementary operations on columns,

$$\det(\gamma_{\lambda_k-k+j}(a))_{j,k=1}^m=\det(\gamma_{\lambda_k-k+m}(a+m-j))_{j,k=1}^m.$$

On substituting the expression in (111) for $\gamma_r(a)$ in the determinant on the right-hand side,

$$\det(\gamma_{\lambda_k-k+m}(a+m-j))_{j,k=1}^m=\left\{\prod_{j=1}^m\frac{\Gamma(a+m-j+1)}{(\lambda_j-j+m)!}\right\}\det\left(\frac{1}{\Gamma(a-j+1-\lambda_k+k)}\right)_{j,k=1}^m.$$

The determinant on the right can be reduced to the Vandermonde determinant by elementary operations on columns,

$$\det\left(\frac{1}{\Gamma(q_k-j)}\right)_{j,k=1}^m=\left\{\prod_{j=1}^m\frac{1}{\Gamma(q_j-1)}\right\}\prod_{1\leq j<k\leq m}(q_k-q_j),$$

and we arrive at

$$\begin{aligned}c_\lambda &= \left\{\prod_{j=1}^m\frac{\Gamma(a+m-j+1)}{\Gamma(a-\lambda_j+j)(m+\lambda_j-j)!}\right\}\prod_{1\leq j<k\leq m}(\lambda_j-j-\lambda_j+k)\\ &= s_\lambda(1_m)\prod_{j=1}^m\frac{\Gamma(a+m-j+1)(m-j)!}{\Gamma(a-\lambda_j+j)(m+\lambda_j-j)!}.\end{aligned}$$

Thus finally

$$\prod_{j=1}^m(1+z_j)^a=\sum_\lambda\beta_\lambda s_\lambda(1_m)s_\lambda(z_1,\dots,z_m),\quad \beta_\lambda=\prod_{j=1}^m\frac{\Gamma(a+m-j+1)(m-j)!}{\Gamma(a-\lambda_j+j)(m+\lambda_j-j)!}.\tag{112}$$

This identity, a companion to (110), holds for $a\geq 0$ and $|z_j|<1$. If a is a positive integer, say $a=N$, then $1/\Gamma(N-\lambda_j+j)=0$ for any partition λ such that $\lambda_1\geq N+1$ so that the sum in (112) is finite. By a direct computation from the Jacobi-Trudi identity,

$$c_\lambda=\left\{\prod_{j=1}^m\frac{(N+m-j)!)}{(m+\lambda_j-j)!(N+j-1-\lambda_j)!}\right\}\prod_{1\leq j<k\leq m}(\lambda_j-j-\lambda_k+k)=s_{\lambda'}(1_N)$$

and we recover a particular case of the dual Cauchy identity (cf. (7))

$$\prod_{j=1}^m(1+z_j)^N=\sum_\lambda s_{\lambda'}(1_N)s_\lambda(z_1,\dots,z_m).$$

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