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Global aspects of double geometry

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Abstract: We consider the concept of a generalised manifold in the $O(d,d)$ setting, i.e., in double geometry. The conjecture by Hohm and Zwiebach for the form of finite generalised diffeomorphisms is shown to hold. Transition functions on overlaps are defined. Triple overlaps are trivial concerning their action on coordinates, but non-trivial on fields, including the generalised metric. A generalised manifold is an ordinary manifold, but the generalised metric on the manifold carries a gerbe structure. We show how the abelian behaviour of the gerbe is embedded in the non-abelian T-duality group. We also comment on possibilities and difficulties in the U-duality setting.

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1 Introduction

Double field theory is by now becoming a well developed subject. After the interesting early work of [1, 2] constructed a version of the string in a doubled space. Siegel then showed that there was a new type of geometry which may be used to describe supergravity in a duality covariant way [3–5]. Ref. [6] showed that T-folds have a natural formulation as a geometric doubled space. It also introduced a doubled sigma model and applied it to such cases. Ref. [7] developed this further and introduced a lot of the ideas behind what would become double field theory. Then in 2009 double field theory (DFT) was introduced with the seminal work of Hull and Zwiebach [8]. DFT is much more than duality covariant versions of supergravity since it allows dynamics in all doubled dimensions. This was then followed up and developed in various directions in works such as [9–12] and also with the work of Park and collaborators [13–15]. For recent reviews on this subject see refs. [16–18].

In double field theory and in the duality covariant forms of supergravity, the metric may be constructed through the non-linear realisation [19] of the $O(d,d)/O(d) \times O(d)$ coset. This same construction is central to the E_{11} programme of West and others [20–25]. As such the duality manifest form of the Bosonic sector of supergravity is somewhat of a truncation of the E_{11} theory, a fact used to construct the RR sector of double field theory in ref. [26, 27]. Of course, the E_{11} programme is much more ambitious and also provides many other important results such as amongst others an explanation of all the gauged maximal supergravities in five dimensions [28].

There have been many interesting developments in numerous directions, see e.g. refs. [29–34], and a full listing of all the interesting and relevant work is outside the scope of this brief note and so we refer the reader to the three reviews for a review of developments and a fuller list of citations of this rapidly maturing subject.

Hull and Zwiebach [35], explored the local symmetries of double field theory infinitesimally and related them to the local symmetries in ordinary string theory and in the processes uncovered an interesting Courant algebroid structure. Given the study of the infinitesimal symmetries the immediate question was what are the finite local transformations. The question was answered by Hohm and Zwiebach in ref. [36]. Finite transformations have also been considered in ref. [37]. We will further study the finite local symmetries looking at various global questions.

2 Local symmetries in double field theory: an exegesis

Double field theory was brought into being in order to make manifest the hidden $O(d, d; \mathbb{Z})$ T-duality symmetry of string theory. It does so by doubling the space and then combining the usual metric and NS-NS two-form into a single "generalised metric" that transforms linearly under a global $O(d, d; \mathbb{R})$ transformation through conjugation. The action of double field theory is manifestly invariant under this transformation. The connection to the normal spacetime is made through the so called *section condition* or *strong constraint* whereby the coordinate dependence of half the dimensions is removed and hence takes the $2d$ doubled space back to d-dimensions. More formally the section condition restricts the theory to live on a maximally isotropic subspace which is the normal spacetime. Different choices of solution to the section condition then give the duality related theories. The section condition is crucial in all that follows with most statements only being true using this condition.

One is immediately confronted with several questions.

- First, shouldn't the theory only possess the $O(d, d)$ symmetry when compactified on a torus?
- Second, how come it is the continuous $O(d, d; \mathbb{R})$ group and not the restricted arithmetic subgroup $O(d, d; \mathbb{Z})$?
- Third, what about the local symmetries of double field theory?
- Finally, a slightly more sophisticated issue. Given that double field theory should describe the background of a string it cannot possess any global symmetries that are not the global part of a local symmetry. That is, in string theory all global symmetries must be gauged [38]. There are some important caveats to this and indeed ideally one should rework the proof in ref. [38] for the string in the doubled background with the action and vertex operators described in ref. [2]. Nevertheless, let us assume that this lore should be true also in double field theory. And so how can we have a global continuous $O(d, d)$ symmetry?

All the answers to these questions are related. Given that the action possesses a global continuous symmetry one should simply hope that it is actually a local symmetry. In fact one can show that this is the case! The double field theory action is invariant under a "local $O(d, d; \mathbb{R})$ transformation" of the generalised metric. This transformation is just the usual combination of d-dimensional diffeomorphisms and 2-form gauge transformations (up to section condition). In double field theory the infinitesimal version of this transformation is then given by the so called *generalised Lie derivative*. Note, it is not the Lie derivative of the generalised space. And so we see why the *generalised Lie derivative* is somewhat of a slight misnomer. It provides an infinitesimal, local, $O(d, d; \mathbb{R})$ transformation, in the same sense that ordinary diffeomorphisms induce local $GL(d; \mathbb{R})$ transformations. To be more concrete, it is not any local $O(d, d)$ transformation but one constructed from the "usual" tensor transformation matrix, $M_M{}^N = \frac{\partial X^N}{\partial X'^M}$ and the $O(d, d)$ structure, η. (The details of the induced transformation are given in section 3). This then answers the third question. The local symmetries are the usual ones but described as a particular realisation of a local $O(d, d)$ transformation. This also fits nicely with recent work by Lee and Park on double field theory as a gauged spacetime [39].

In the above discussion there is no mention as to any toroidal compactification; the $O(d, d)$ is simply a continuous symmetry such that when made local it can describe the known local symmetries on the space. This leads to the first two questions on the list. What then if we do compactify the theory on a torus? As usual we will be left with the mapping class group. Normally the mapping class group arises from the global diffeomorphisms not connected to the identity that preserve the manifold. Here it is the global $O(d, d)$ transformations not connected to the identity. This then breaks the local $O(d, d; \mathbb{R})$ to global $O(d, d; \mathbb{Z})$ giving the normal T-duality group.

The use of this local $O(d, d)$ symmetry is to construct T-folds [6], exotic branes [18, 40–43] and other non-geometric backgrounds. To do so we require a non-contractible one-cycle. We then have non-trivial holonomies of the generalised metric of the $2d$ doubled space around this one cycle. This will correspond to an element of $O(d, d; \mathbb{Z})$. This is normally identified as an element of the T-duality group. The T-duality group though is a global symmetry and so it does not really make sense to have a holonomy for a global symmetry. In fact it is a holonomy in the local $O(d, d)$ symmetry which due to the topology of the space we can then identify with the arithmetic T-duality group. The reduction to the arithmetic subgroup is due to having a torus fibred over the one cycle. The above discussion indicates that one could also search for other solutions in double field theory where the holonomies are other subgroups of $O(d, d; \mathbb{R})$. Obviously, the fibres must then have different topology to the torus. The construction of such objects is currently not known. This maybe the way to realise so called mirrorfolds [6] where we have a Calabi-Yau space as a fibre that becomes identified with its mirror upon going round the one cycle of the base, see also [44].

And so we have a manifold whose coordinates transform as normal under general coordinate transformations. The generalised metric though (and all other generalised tensors) transform under local $O(d, d)$. (The induced $O(d, d)$ transformation for a given coordinate transformation while be described later). Is double field theory then some sort of $O(d, d)$ fibre bundle? The answer is no.

The doubled manifold is a manifold equipped with an $O(d, d)$ structure. We have a set of coordinate patches $\{U_\alpha\}$ such that in each patch the space is isomorphic to $\mathbb{R}^{2d}$ and so we may introduce coordinates $X^I_{(\alpha)}$, $I = 1, \ldots, 2d$. (The signature is dependent on whether one wishes to double time or not; the subject of which would require a more involved discussion than we wish to get into in this paper. For now we will just consider doubling Euclidean space). The $O(d, d)$ structure normally denoted by η_{IJ} then acts as a polarisation and so permits the coordinates on each patch to be decomposed into pairs so that $X^I = (x^i, \tilde{x}_i)$ (with $i = 1 \ldots d$) and where $X^I \eta_{IJ} X^J = x^i \delta_i^j \tilde{x}_j$.

On the overlap of any two patches $U_\alpha \cap U_\beta$ we have transition functions which transform the coordinates from one patch into the coordinates of the other. These transition functions are simply diffeomorphisms. That is $X^I_{(\alpha)}(X_{(\beta)})$ is invertible. As with everything though we only allow diffeomorphisms that obey the section condition. Since coordinates are not tensors we will state more explicitly what this means. On an overlap we may define the difference $\delta X^I_{(\alpha\beta)} = X^I_{(\alpha)} - X^I_{(\beta)}$. This is a tensor and it should obey the section condition. The composition of transition functions on triple overlaps is called the cocycle condition and for a manifold it must be trivial. It remains somewhat mysterious that the transformations in $O(d,d;\mathbb{Z})$ applied specifically to the overlaps in the geometries named "genuinely non-geometric" in ref. [18], seem not to obey the strong section condition, and thus can not be formed as finite generalised diffeomorphisms, as defined in ref. [36] and the present paper.

Now, we wish to put a metric on the manifold and so we introduce the tangent bundle, F. As such we proceed as with the usual construction where we introduce a set of local trivialisations, denoted by $\{U_\alpha\}$ which maps the tangent bundle to the local product, of the manifold, M^{2d} (the base) and its tangent space TM (the fibre). The presence of the $O(d,d)$ structure on M then also allows us the decompose the tangent space of the 2d dimensional manifold into a direct sum of the tangent space and cotangent space of a single d dimensional manifold i.e., $TM^{2d} = TM^d \oplus T^*M^d$. The generalised metric lives in this space. Subsequently applying the section condition on F so that we restrict to the d-dimensional maximally isotropic submanifold then leads to so called "generalised geometry". For a more mathematically complete description of the geometry of double field theory the reader is referred to refs. [45–47].

The bundle is then equipped with transition functions $f_{\alpha\beta}$ which act on the fibres and so the generalised metric. In usual geometry these would be the normal diffeomorphisms acting on a tensor induced from the coordinate transformations between patches i.e., $\frac{\partial X^I_{(\alpha)}}{\partial X^J_{(\beta)}}$.

However, we now have an $O(d,d)$ structure, η that must be preserved; in other words it is globally defined. This means that to preserve η_{IJ} a coordinate transformation must crucially only induce an $O(d,d)$ transformation on tensors. This is exactly what happens, under a coordinate transformation in the doubled space, all tensors transform under a local $O(d,d)$ transformation and so the transition functions, $f_{\alpha\beta}$ on the fibres now lie in $O(d,d)$. The precise form of the induced $O(d,d)$ transformation is given in the following section.

We will show in section 4 that the transition functions between patches, $f_{\alpha\beta} \in O(d,d)$ need not obey the cocycle condition on triple intersections. That is:

$$\text{On} \qquad U_\alpha \cap U_\beta \cap U_\gamma : \quad f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1 \tag{2.1}$$

may **not** hold (even up to section condition) as it must in order to be a fibre bundle. (The cocycle condition on quadruple intersections will be trivial however.) This implies double field theory possesses a gerbe structure [48] and leads to the following question: if it is a gerbe and valued in $O(d,d)$ then is it a non-abelian gerbe [49]? The answer as we will show is no for a slightly subtle reason. The section condition effectively abelianises the gerbe in a way described later in this paper. This makes perfect sense if one thinks in terms of the ordinary symmetries of the NSNS sector. The diffeomorphisms have trivial gerbe structure

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i.e., obey the cocycle condition on triple intersections but the NS two form potential is the connection on a local trivialisation for an abelian gerbe and its gauge transformations need not. The $O(d,d)$ transformation on the doubled metric combines these two and so indeed one expects there to be a gerbe structure but only an abelian part. For a previous discussion on gerbes and their relation to T-duality see refs. [50, 51]. The presence of the gerbe structure on triple overlaps may be connected to the results of Blumenhagen et al., Lüst et al. and Szabo et al. on non-associative deformations in string theory, see refs. [52–55] and references therein). An appropriate mathematical structure for understanding double field theory may be that of the Lie algebroid as discussed in refs. [56, 57].

There is a related issue that can further obfuscate matters. Given a non-trivial coordinate transformation the resulting $O(d,d)$ tensor transformation may be trivial. In other words, there are coordinate transformations that are trivial for the generalised metric. One may see this already at the infinitesimal level with the generalised Lie derivative [36, 37, 58]. It means there is effectively an equivalence class of coordinate transformations for each $O(d,d)$ transformation. The consequences were explored extensively in the seminal work of ref. [36] where the exponential of the generalised Lie derivative was first compared with a conjectured induced $O(d,d)$ transformation on tensors. We follow this work closely in the next section and show that one can identify the exponentiated generalised Lie derivative with the induced $O(d,d)$ transformation of ref. [36] up to an, $O(d,d)$ trivial, coordinate transformation. This is shown to be true to all orders in the infinitesimal parameter describing the coordinate transformation. This identification [59] was previously done explicitly up to quartic order, although a different argument [37] has also been given for its complete consistency.

In the section that follows, we then look at the triple overlap structure and show how the gerbe structure arises and how the section condition abelianises the gerbe.

Finally we end with some comments on the extension of these ideas to the extended, exceptional geometries that occur in M-theory where the full U-duality groups are made a manifest symmetry [58, 60–73].

3 The limitations of this analysis

This paper does not describe how one may patch together the total doubled space to allow a geometrisation of an arbitrary gerbe. This is a notorious problem and one that is explored in great detail in the paper [74]. In [74] (which appeared while this paper was in preprint form on the arXiv) a global obstruction in double field theory is discussed that would indicate that a construction of the 3rd cohomology classes associated to three form fluxes requires a highly constrained topology on the additional dual space. The transformations described in this paper, for example in section five, (see equation 5.9) are really for cohomologically trivial fluxes since the gauge transformations of B described in section five are exact. (In other words we are missing the transformations corresponding to closed but non exact two forms.) This along with an absence of discussing the topology in the extended space limits our analysis. Such an analysis is beyond this paper and is an important open question in this area.

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Note, even more recently a paper [75] has appeared (several months after this paper appeared on the arXiv), that analyses some of these global questions in detail and the proposals we describe here).

4 Infinitesimal and finite generalised diffeomorphisms

Generalised diffeomorphisms are generated by the generalised Lie derivative, or Dorfman bracket, and are parametrised by a doubled vector ξ^M, encoding diffeomorphisms as well as B-field gauge transformations. We will adopt the notation of ref. [36], where ξ denotes the vector field $\xi^M\partial_M$ and where the matrix a is defined as

$$a_M{}^N = \partial_M \xi^N \,. \tag{4.1}$$

The $O(d,d)$ structure allows for a globally defined metric η_{MN}, which we may use to define a "transposition" of matrices given by

$$(a^t)_M{}^N = \eta_{MP}\eta^{NQ}a_Q{}^P \,. \tag{4.2}$$

In what follows we will always assume that the derivatives satisfy the strong section condition,

$$\eta^{MN}\partial_M A \partial_N B = 0 \,, \tag{4.3}$$

for all A and B.

The generalised Lie derivative on a vector V^M is

$$\mathcal{L}_\xi V^M = \xi V^M - V^N(a-a^t)_N{}^M \,, \tag{4.4}$$

and for a covector W_M

$$\mathcal{L}_\xi W_M = \xi W_M + (a-a^t)_M{}^N W_N \,. \tag{4.5}$$

It will be useful to write this in a compact form as follows,

$$\mathcal{L}_\xi = \xi + a - a^t = L_\xi - a^t \,, \tag{4.6}$$

where L_ξ is the ordinary Lie derivative. The generalised Lie derivatives are derivatives in that they obey the Leibniz rule for products. The generalised Lie derivative on a scalar is the same as the Lie derivative, i.e., it is simply the so called translation term, ξ acting on the scalar. (Note that the generalised Lie derivative on η vanishes).

The algebra of generalised diffeomorphisms is (up to section condition)

$$[\mathcal{L}_\xi, \mathcal{L}_\chi] = \mathcal{L}_{[\![\xi,\chi]\!]} \,, \tag{4.7}$$

with $[\![\xi,\chi]\!] = \frac{1}{2}(\mathcal{L}_\xi\chi - \mathcal{L}_\chi\xi)$.

A short calculation shows that we can equally well express the commutator in terms of the ordinary Lie bracket of vector fields (Lie derivatives) together with an extra term, Δ. We then have

$$[\mathcal{L}_\xi, \mathcal{L}_\chi] = \mathcal{L}_{[\xi,\chi]} + \Delta_{\xi,\chi} \,, \tag{4.8}$$

where $\Delta_{\xi,\chi}$ has the simple form

$$\Delta_{\xi,\chi} = -ab^t + ba^t\,, \tag{4.9}$$

with $a_M{}^N = \partial_M \xi^N$, $b_M{}^N = \partial_M \chi^N$.

Since $\mathcal{L}_{[\xi,\chi]}$ in the right hand side of eq. (4.8) is a gauge transformation, so is Δ. Δ represents a generalised coordinate transformation with parameter $\zeta^M = -\xi^N(b^t)_N{}^M$. Crucially the translation term vanishes for Δ. Thus it has the important property that it affects neither coordinates nor derivatives. This is because

$$a^t\partial = 0 \tag{4.10}$$

thanks to the section condition. Note that, the multiplication of any two matrices of the form (4.9) gives zero. Eq. (4.8) is the infinitesimal version of what we will later, for finite transformations, associate to a gerbe. Note that exponentiation of these non-translating transformations is simple, the nilpotent property of Δ immediately gives

$$e^{\Delta} = 1 + \Delta\,. \tag{4.11}$$

Hohm and Zwiebach [36] conjectured an explicit expression for the finite transformation of tensors under a generalised diffeomorphism in doubled geometry. When $X \to X'(X)$, a covector transforms as

$$W'_M(X') = F_M{}^N(X', X) W_M(X)\,, \tag{4.12}$$

where

$$F_M{}^N(X', X) = \frac{1}{2}\left(M(M^{-1})^t + (M^{-1})^t M\right)\,, \tag{4.13}$$

$M_M{}^N$ being the "usual" transformation matrix,

$$M_M{}^N = \frac{\partial X^N}{\partial X'^M}\,. \tag{4.14}$$

The section condition ensures, quite non-trivially, that the conjectured transformation matrix F is a group element of $O(d,d)$. We will now demonstrate that the expression (4.13) provides the correct expression for the exponentiation of the generalised Lie derivative. More precisely, we will show that the finite transformation obtained from the exponentiated Lie derivative lies in the same equivalence class as F modulo non-translating transformations.

When working with expressions in terms of the matrices a and a^t, it is important to recall that they satisfy $a^t a = 0$ due to the section condition. One can therefore replace any matrix that is produced of a and a^t by its ordered expression with all the a's to the left and all the a^t's to the right, in a way reminiscent of normal ordering of operators in a quantum mechanics. Any matrix which is obtained only from matrix multiplication of the basic ingredients a, a^t can be put on such a form. We may identify the form of a matrix $f(a, a^t)$ given by this ordering prescription with a "symbol" $S[f](x, y)$ of the matrix. Here, x and y are not matrices but commuting formal variables. This identification of course

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yields $S[a] = x$, $S[a^t] = y$, and e.g. $S[aa^t] = xy$, $S[a^t a] = S[0] = 0$. The matrix product is reproduced by an associative star product of symbols:

$$S[fg] = S[f] \star S[g] \,, \tag{4.15}$$

where

$$(A \star B)(x, y) = A(x, y)B(0, y) + A(x, 0)B(x, y) - A(x, 0)B(0, y) \,, \tag{4.16}$$

allowing for index-free calculations.

We can as a warm up exercise exponentiate the matrix part of the generalised Lie derivative,

$$e^{a-a^t} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{1}{n!} a^{n-i} (-a^t)^i \,, \tag{4.17}$$

so we find its symbol to be

$$\begin{aligned} S[e^{a-a^t}](x, y) &= \frac{1}{x+y} \sum_{n=0}^{\infty} \frac{1}{n!} \left(x^{n+1} - (-y)^{n+1} \right) \\ &= \frac{xe^x + ye^{-y}}{x+y} \,, \end{aligned} \tag{4.18}$$

The seemingly singular behaviour of the denominator is of course compensated by the nominator.

The transformation matrix M can be expressed as

$$M = e^{-\xi} e^{\xi + a} \,, \tag{4.19}$$

where the first factor translates back to the original coordinate. In these expressions, the operator ξ acts to everything on the right. Some care has to be taken when transposing an operator. In what follows we will also need the transpose of M, which is

$$M^t = e^{-\xi + a^t} e^{\xi} \,. \tag{4.20}$$

The analogous expression to M for the generalised Lie derivative is,

$$G = e^{-\xi} e^{\xi + a - a^t} \,. \tag{4.21}$$

This is what we now want to compare to F in eq. (4.13). Using the section condition, we may rewrite the matrix F in the normal ordered form as follows:

$$\begin{aligned} F &= \frac{1}{2} \left(M(M^{-1})^t + M + (M^{-1})^t - 1 \right) \\ &= \frac{1}{2} \left(e^{-\xi} e^{\xi + a} e^{-\xi} e^{\xi - a^t} + e^{-\xi} e^{\xi + a} + e^{-\xi} e^{\xi - a^t} - 1 \right) \,. \end{aligned} \tag{4.22}$$

(Here, and in the following, we have chosen not to use the symbols introduced earlier in this section, but use explicit a's and a^t's. When needed, subscripts R and L are used to denote right and left multiplication.) To compare to F, we need to put the matrix G also in normal ordered form. We therefore write

$$e^{\xi+a-a^t} = \sum_{n=0}^{\infty} \frac{1}{n!}(\xi + a - a^t)^n \,, \tag{4.23}$$

and use

$$\begin{aligned} (\xi + a - a^t)^n &= (\xi + a)^n + \sum_{i=1}^{n} (\xi + a)^{n-i}(-a^t)(\xi - a^t)^i \\ &= (\xi + a)^n - \frac{(\xi + a)_L^n - (\xi - a^t)_R^n}{(\xi + a)_L - (\xi - a^t)_R}\, a^t \\ &= \frac{(\xi + a)^n a + a^t(\xi - a^t)^n}{(\xi + a)_L - (\xi - a^t)_R} \,, \end{aligned} \tag{4.24}$$

where the first step is easily shown by induction, and where R and L denote that the operators stand on the far right and left. Therefore, the exponentiated transformation takes the form

$$e^{\xi+a-a^t} = \frac{e^{\xi+a} a + a^t e^{\xi - a^t}}{(\xi + a)_L - (\xi - a^t)_R} \,. \tag{4.25}$$

We now use the expression (4.16) for products of "functions" of a and a^t. The operator ξ is a scalar in the sense that it commutes with a^t, after using the section condition, and so eq. (4.16) for the product of ordered expressions still holds. Then after some calculations one find

$$G^{-1}F = 1 + \frac{1}{2} \frac{e^{-(\xi+a)}(a - a^t)e^{\xi-a^t} - (a - a^t)}{(\xi + a)_L - (\xi - a^t)_R} + \frac{1}{2}(e^{-\xi}e^{\xi-a^t} - e^{-(\xi+a)}e^{\xi}) \,. \tag{4.26}$$

Despite the appearance of negative powers of operators, the series expansion contains only positive powers and is well defined. It is not yet obvious to why this is of the form we want it to be, namely a transformation which is a product of factors of the form (4.11). Note that the second term in eq. (4.26) can be written on the form $f(\mathrm{ad}_\xi + a_L - a^t_R) \cdot (a - a^t)$, where $f(z) = z^{-1}(e^{-z} - 1)$, and that the rôle of the last term is to remove from this the parts that contain only a and only a^t (i.e., the parts obtained by formally setting a^t or a to zero). These properties will be essential for the proof.

In order to better understand what this implies, we investigate the expression $(\mathrm{ad}_\xi + a_L + a^t_R)^n \cdot a$ and its transpose (here, the dot denotes the operator acting only on the following matrix/operator, not on everything on the right), occurring in an expansion of the function f. For any operator b, it is straightforward to show by induction that

$$(\mathrm{ad}_\xi + a_L + a^t_R)^n \cdot b = (\mathrm{ad}_\xi + a)^n \cdot b + \sum_{k=0}^{n-1} \binom{n}{k} ((\mathrm{ad}_\xi + a)^k \cdot b)((\mathrm{ad}_\xi + a^t)^{n-k-1} \cdot a^t) \,, \tag{4.27}$$

so eq. (4.26) may be written as

$$\begin{aligned} G^{-1}F &= 1 + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^{n+1}}{(n+1)!} \binom{n}{k} ((\mathrm{ad}_\xi + a)^k \cdot a)((\mathrm{ad}_\xi + a)^{n-k-1} \cdot a)^t - (\ldots)^t \\ &= 1 + \frac{1}{2} \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^n(n - 2k - 1)}{(n+1)(k+1)!(n-k)!} ((\mathrm{ad}_\xi + a)^k \cdot a)((\mathrm{ad}_\xi + a)^{n-k-1} \cdot a)^t \,. \end{aligned} \tag{4.28}$$

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Finally, we need to reinterpret the factors $(\mathrm{ad}_\xi + a)^k \cdot a$, in order to ensure that the terms are of the desired form (4.9), with both factors being derivatives of vectors. Explicit expansion yields unwieldy expressions, of which the lowest ones are

$$
\begin{aligned}
(\mathrm{ad}_\xi + a)^0 \cdot a &= a\,, \\
(\mathrm{ad}_\xi + a)^1 \cdot a &= a^2 + [\xi, a]\,, \\
(\mathrm{ad}_\xi + a)^2 \cdot a &= a^3 + [\xi, a]a + 2a[\xi, a] + [\xi, [\xi, a]]\,, \\
(\mathrm{ad}_\xi + a)^3 \cdot a &= a^4 + [\xi, a]a^2 + 2a[\xi, a]a + 3a^2[\xi, a] \\
&\quad + [\xi, [\xi, a]]a + 3[\xi, a][\xi, a] + 3a[\xi, [\xi, a]] + [\xi, [\xi, [\xi, a]]]\,.
\end{aligned}
\tag{4.29}
$$

Inspection of the first of these expressions leads to the guess

$$
(\mathrm{ad}_\xi + a)^n a = \partial(\xi^n \cdot \vec{\xi})\,, \tag{4.30}
$$

where the vector notation denotes $(\partial \vec{v})_M{}^N = \partial_M v^N$. It is straightforward to then prove this by induction. This implies that $G^{-1}F$ is the product of matrices of the form (4.11), and thus is a finite non-translating generalised diffeomorphism connected to the identity. Explicitly,

$$
G^{-1}F = \prod_{n=2}^{\infty} \prod_{k=0}^{n-1} \left(1 + \frac{1}{2}\frac{(-1)^n(n-2k-1)}{(n+1)(k+1)!(n-k)!}\partial(\xi^k \cdot \vec{\xi})(\partial(\xi^{n-k-1} \cdot \vec{\xi}))^t\right)\,. \tag{4.31}
$$

Observe that the sum (4.26) may be replaced by a product as has been done above since the product of any two terms vanishes via the section condition. Each term in $G^{-1}F$ is of the form given in (4.9) and therefore does not involve a translation. Thus $G^{-1}F$ is in the same equivalence class of transformations as the identity.

In order to compare to previous results it is useful to explicitly evaluate this expression as a series expansion. To do this, first the expression f or $G^{-1}F$ is written as a function of two variables: let x represent the first ξ and y the second. Then the deviation from unity is encoded in the function:

$$
\begin{aligned}
T(x,y) - 1 &= \frac{(e^{-x}+1)(e^{-y}-1)x - (e^{-x}-1)(e^{-y}+1)y}{2(x+y)} \\
&= -\frac{1}{12}(x^2y - xy^2) + \frac{1}{24}(x^3y - xy^3) - \frac{1}{80}(x^4y - xy^4) - \frac{1}{120}(x^3y^2 - x^2y^3) \\
&\quad + \frac{1}{360}(x^5y - xy^5) + \frac{1}{288}(x^4y^2 - x^2y^4) \\
&\quad - \frac{1}{2016}(x^6y - xy^6) - \frac{1}{1120}(x^5y^2 - x^2y^5) - \frac{1}{2016}(x^4y^3 - x^3y^4) + \ldots
\end{aligned}
\tag{4.32}
$$

A series expansion of eq. (4.26) to third order in the parameter confirms the earlier results of ref. [36]:

$$
G^{-1}F = 1 - \frac{1}{12}(a^2a^t + [\xi, a]a^t - (\ldots)^t) + O(\xi^4) \tag{4.33}
$$

This is a non-translating transformation with parameter

$$
\zeta_3^M = -\frac{1}{12}\xi^N (aa^t)_N{}^M\,, \tag{4.34}
$$

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which relies on the appearance of the combination $a^2a^t + [\xi, a]a^t$. The above demonstrates that this happens to all orders for finite transformations connected to the identity.

5 Overlaps and gerbes

The matrix F may be more clearly associated to a specific finite change of coordinates by writing it $F(M)$. It is a group element in $O(d,d)$, and $F(M)F(M^{-1}) = 1$.

Composition of transformations is less straightforward. Remember that $M = \frac{\partial X'}{\partial X}$. From the transformation of the coordinates one might expect the chain rule to hold,

$$\frac{\partial X''}{\partial X} = \frac{\partial X'}{\partial X}\frac{\partial X''}{\partial X'}\,, \tag{5.1}$$

but F generically fails to respect the expected corresponding composition rule, i.e.,

$$F(M)F(N) \neq F(MN)\,. \tag{5.2}$$

That is, the map $F : GL(2d) \to O(d,d)$ is not a group homomorphism. This has led to attempts [18, 36] to modify the product of two F's in terms of a "$*$-product", which turns out to be non-associative.

The way we would rather understand composition, is that $F(M)$ is one representative in a class of transformations, all transforming the coordinates identically, but differing by non-translating transformations as in the previous section. Composition of transformations $F(M)$ and $F(N)$ will result in a third transformation which is in the equivalence class dictated by the composition (5.1). It will however not be the representative given by $F(MN)$, but differ from it by some non-translating transformation of the special type discussed in the previous section. In other words, the composition rule holds up to a non-translating transformation. Schematically, eq. (5.2) is replaced by a "twisting" by a non-translating transformation:

$$F(M)F(N) = F(MN)\exp\left(\sum_i \Delta_i\right). \tag{5.3}$$

Composition of equivalence classes is obeyed, but we need to make the detailed structure clearer. The behaviour points towards a description in terms of gerbes, where e^{Δ} represents the cocycle.

Let us think of the matrix $F(M)$ as providing the transformation relating fields (e.g. a generalised metric) on the intersection of two patches. In order to isolate the deviation from the chain rule, we consider instead the overlap as defined by[1]

$$H(M,N) = F(M)F(M^{-1}N)F(N^{-1})\,. \tag{5.4}$$

In standard language, this would be identified with an overlap $f_{\alpha\beta}$ between patches α and β. Clearly, eq. (5.4) satisfies

$$H(M,N)H(N,M) = 1\,, \tag{5.5}$$

corresponding to $f_{\alpha\beta}f_{\beta\alpha} = 1$.

[1]This is in itself a triple overlap, with one patch fixed [49]. The preference of this form over F lies in the fact that the matrices around $F(M^{-1}N)$ compensate for the indices belonging to different frames.

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A triple overlap is locally written as $\lambda_{\alpha\beta\gamma} = f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha}$. This amounts to forming

$$\begin{aligned}\Lambda(M,N,P) &= H(M,N)H(N,P)H(P,M)\\ &= F(M)F(M^{-1}N)F(N^{-1}P)F(P^{-1}M)F(M^{-1})\,,\end{aligned} \tag{5.6}$$

We want to evaluate this expression, using the basic form (4.13) of F. This is straightforward but somewhat tedious work. The section condition will, as usual, be necessary, and tricks similar to eq. (4.16) are helpful. The matrix $H(M,N)$ turns out to have the form

$$H(M,N) = 1 - \frac{1}{2}(mn^t - nm^t) = 1 + Q(M,N)\,, \tag{5.7}$$

where $M = 1+m$, $N = 1+n$. This is a group element of $O(d,d)$ of a special form, 1 plus an antisymmetric matrix Q, which is nilpotent, $Q^2 = 0$, thanks to the section condition, leading to $HH^t = (1+Q)(1-Q) = 1-Q^2 = 1$. Furthermore, in the multiplication of any two H's, their "Q terms" will simply add, as the product of the Q's is zero. The matrix takes the same form as the ones encountered earlier in eq. (4.11). Therefore, the rest of the calculation leading to the triple overlap Λ is simple. We find

$$\Lambda(M,N,P) = 1 - \frac{1}{2}(mn^t + np^t + pm^t - nm^t - pn^t - mp^t)\,. \tag{5.8}$$

Now Λ behaves as an overlap should, with manifest properties under permutation. In a certain sense, the matrices H, although being group elements of $O(d,d)$, behave in an abelian way. So a cocycle condition on Λ can be written down without any problem. This is slightly problematic had the gerbe turned out to be non-abelian [49]. Let us investigate this gerbe structure by explicitly picking a solution to the section condition.

Divide the index M into two groups of light-like ones, $(m,\bar{m})$, with $\eta_{m\bar{n}} = \delta_{mn}$. Then solve the section condition by letting the derivative carry unbarred indices only. The form of the matrix H is then

$$H_M{}^N = \begin{pmatrix} H_m{}^n & H_m{}^{\bar{n}} \\ H_{\bar{m}}{}^n & H_{\bar{m}}{}^{\bar{n}} \end{pmatrix} = \begin{pmatrix} \delta_m{}^n & -\frac{1}{2}(\partial_m\xi^P\partial^{\bar{n}}\chi_P - \partial_m\chi^P\partial^{\bar{n}}\xi_P) \\ 0 & \delta_{\bar{m}}{}^{\bar{n}} \end{pmatrix}\,. \tag{5.9}$$

Here it is even more explicit that the non-trivial terms on the off-diagonal are additive, and that they vanish for transformations involving x^m only. This shows that the gerbe structure is confined to gauge transformations of the B-field. This can be seen explicitly by acting with H on the generalised metric through conjugation and seeing that it transforms B-field. The gerbe structure connected to B-field gauge transformation was obtained in generalised (not doubled) Kähler geometry in ref. [51].

This shows how a doubled manifold, before it is equipped with a generalised metric structure, is an ordinary manifold (subject to some additional constraints due to the presence of a global $O(d,d)$ structure), and may be patched together with ordinary overlaps. Once a solution to the section condition is specified, the topology in fact becomes that of a fibre bundle. Introduction of a generalised metric (or other tensors) refines the structure and introduces a gerbe over the doubled manifold. Earlier discussion on non-associative structures are a reflection of the presence of a non-trivial triple overlap.

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6 Some comments on exceptional/extended geometry

It is desirable to perform the corresponding investigation also for the exceptional/extended geometry described in refs. [58, 60–73]. We refer the reader to those papers or the reviews [16, 17] for an introduction. In what follows we will discern some similarities but also some differences when trying to generalise the finite transformations for $O(d, d)$ to the exceptional groups associated with U-duality. This is ongoing work and the final finite form of the induced transformation has not been yet been determined but it is worthwhile showing how some similar structures arise.

In order to make the formalism as close to the one employed in doubled geometry as possible, we note that the generalised Lie derivative takes the form

$$\mathcal{L}_\xi = \xi + a - a^Y = L_\xi - a^Y \,, \tag{6.1}$$

which looks very similar to eq. (4.6), but where the transpose is replaced by the operation

$$a \to a^Y \,, \quad (a^Y)_M{}^N = Y_{MP}{}^{QN} a_Q{}^P \,, \tag{6.2}$$

Y being the $E_{n(n)}$-invariant tensor (in double geometry, $Y_{MN}{}^{PQ} = \eta_{MN}\eta^{PQ}$) responsible for the section condition, $Y_{MN}{}^{PQ}\partial_P \ldots \partial_Q = 0$; and so again we will require:

$$Y_{MN}{}^{PQ}\partial_P A \partial_Q B = 0 \tag{6.3}$$

for all A and B. The Y-tensor also fulfills some algebraic relations, which ensure the closure of the algebra of generalised Lie derivatives (see ref. [58]).

One difference, which seems to lie behind most difficulties in writing a closed form for the finite transformations, is that, unlike the transpose in $O(d, d)$, the operation $a \to a^Y$ is not an involution. It certainly has eigenvalue 1 on the part of a matrix outside the adjoint of $E_{n(n)} \times \mathbb{R}^+$, so that $a - a^Y$ is an element in the Lie algebra, but other eigenvalues are different from -1.

This has so-far prevented us from finding a closed candidate for the transformation F, corresponding to eq. (4.13). Some considerations can be done at the level of infinitesimal transformations however. If we construct a non-translating transformation from a parameter $\vec{\zeta} = \vec{\xi} b^Y = \vec{\xi}(\partial\vec{\eta})^Y$, we get a Lie algebra element

$$\partial\vec{\zeta} - (\partial\vec{\zeta})^Y = ab^Y - (ab^Y)^Y \,. \tag{6.4}$$

(It is important to have a derivative of a vector on the right, otherwise the terms where the derivative acts on that matrix will not go away. This may be the most general form of a non-translating transformation.) This is not the same as $ab^Y - ba^Y$. It does not seem to be true that such Lie algebra elements are in general (quadratically) nilpotent. This may lead to a higher gerbe structure.

Quite non-trivially, an expression for the algebra of generalised Lie derivatives in terms of the usual Lie bracket and a remainder term Δ may be obtained following a similar calculation leading to eq. (4.8). After quite some work one can find that:

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\![\xi,\eta]\!]} = \mathcal{L}_{[\xi,\eta]} + \Delta_{\xi,\eta} \,, \tag{6.5}$$

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where the remainder piece Δ is now given by

$$\begin{aligned}\Delta_{\xi,\eta} &= [a,b]^Y + [a^Y,b^Y] - [a,b^Y] - [a^Y,b] \\ &= -\frac{1}{2}\left([a,b^Y] + [a^Y,b] - [a,b^Y]^Y - [a^Y,b]^Y\right) \\ &= -\frac{1}{2}\left(ab^Y - ba^Y - (ab^Y - ba^Y)^Y\right) .\end{aligned} \tag{6.6}$$

This is a useful step in generalising the results from the $O(d,d)$ case as it encouragingly shows an analogous structure exists in the algebra of generalised Lie derivatives for the exceptional groups. In the above, we have used the "general" identity

$$[a,b]^Y + [a^Y,b^Y] - \frac{1}{2}\left([a,b^Y] + [a,b^Y]^Y + [a^Y,b] + [a^Y,b]^Y\right) = 0 \tag{6.7}$$

where "general" implies it holds even if the section condition is not applied. This is a rewriting in terms of matrices of identities for the Y tensor of ref. [58].

Note, eq. (6.4) is automatically antisymmetric in a and b. This is because of the general identity $[m,m^Y] - [m,m^Y]^Y = 0$. It is easily understood, since $[m,m^Y]$ must be proportional to the commutator between the adjoint projection of m and its complement, which is a module, and therefore the result has vanishing adjoint component. Letting $m \to a + b$, where the first indices fulfill the section condition, shows the antisymmetry.

How can a transformation of the type (6.4) be exponentiated to a finite transformation? What is the final finite form? What are the global gerbe properties? In fact we do expect the presence of higher gerbe structures. The 3-form potential $C_{(3)}$ is naturally the connection on a 3-gerbe which may have a non-trivial 4-overlap. For duality groups beyond SL(5) the 6-form potential $C_{(6)}$ will play a role and it comes with a 6-gerbe structure and may have non-trivial 7-overlap. Some preliminary investigation points in this direction. It remains a key challenge to exceptional/extended geometry to find the finite form of its local symmetries and understand its global properties.

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