Area-width scaling in generalised Motzkin paths

Haug, N; Prellberg, T; Siudem, G

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Scaling in area-weighted generalized Motzkin paths

Nils Haug*, Thomas Prellberg* and Grzegorz Siudem†

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Abstract

We consider a generalized version of Motzkin paths, where horizontal steps have length ℓ, with ℓ being a fixed positive integer. We first give the general functional equation for the area-length generating function of this model. Using a heuristic ansatz, we derive the area-length scaling behaviour in terms of a scaling function in one variable for the special cases of Dyck, (standard) Motzkin and Schröder paths, before generalizing our approach to arbitrary ℓ. We then derive an expression for the generating function of Schröder paths and analyse the scaling behaviour of this function rigorously in the vicinity of the tri-critical point of the model by applying the method of steepest descents for the case of two coalescing saddles. Our results show that for Dyck and Schröder paths, the heuristic scaling ansatz reproduces the rigorous results.

1 Introduction

In this paper, we consider a generalized version of Motzkin paths, called ℓ-Motzkin paths, with steps $(1,1)$, $(1,-1)$ and $(0,\ell)$, where ℓ is a fixed positive integer. This model has been studied previously in the combinatorics literature with a focus on bijections [1]. The special cases $\ell = 1$ and 2 correspond to (standard) Motzkin paths and Schröder paths, respectively, whereas Dyck paths can be identified with the limiting case $\ell = \infty$ [2, 3]. Motzkin paths are closely related to RSOS configurations [4].

The area-length generating functions for Dyck and (standard) Motzkin paths have been given in [5] and [4], and for Dyck paths, the scaling form of the generating function has been derived rigorously in [6]. Due to the nature of the exact solution of the generating function of Motzkin paths, no corresponding scaling form has been derived yet. The area-length generating

*School of Mathematical Sciences, Queen Mary University of London, London, E1 4NS, United Kingdom

†Faculty of Physics, Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland

function for general ℓ satisfies a functional equation, from which a continued fraction expression can be obtained by iteration.

We derive the scaling behaviour for Dyck, Motzkin and Schröder paths by heuristically inserting a single-variable scaling ansatz into the functional equation for the generating function and generalize this approach to arbitrary ℓ.

We then show that the generating function for Schröder paths, weighted with respect to their length and their area, can be expressed in terms of a quotient of basic hypergeometric series, similar to the well-known expression for Dyck paths. From this we derive the associated scaling form by rigorous saddle point analysis.

Our results show that the heuristic scaling ansatz reproduces the rigorous results for Dyck and Schröder paths. Moreover, we obtain the same scaling form for all values of ℓ, and therefore in particular for Motzkin paths.

2 The model

Given $\ell \in \mathbb{N}$, we define an ℓ-Motzkin path of length m to be a lattice walk $(x_i, y_i)_{i=0}^m$ on $\mathbb{N}_0^2$ such that $(x_0, y_0) = (0, 0)$ and from any point (x, y) on the path, the walker can either step towards $(x+1, y+1)$, $(x+1, y-1)$ or towards $(x+\ell, y)$, which we call an up-, down- or horizontal step, respectively. Moreover, the path needs to end on the horizontal line $y = 0$. Fig. 1 shows an example trajectory for the case $\ell = 2$. Since we will only consider ℓ-Motzkin paths in this paper, we will shortly refer to them as ℓ-paths from now on.

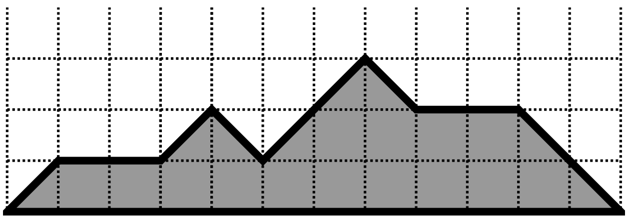

Figure 1: A Schröder path of length 12 with two horizontal steps of length 2, four pairs of up/down steps, and total area 18.

For given ℓ, we define the generating function

$$G(s, t, p, q) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{k,l,m,n}\, s^k\, t^l\, p^m\, q^n, \tag{1}$$

where $c_{k,l,m,n}$ is the number of paths with k horizontal steps, l pairs of up- and down-steps, m $(\ell \times 1)$-rectangles under all the horizontal steps, and n unit squares under all the up- and down-steps (including the half unit squares directly underneath these steps). Note that by introducing separate area generating variables p and q, there is no explicit ℓ-dependence in G.

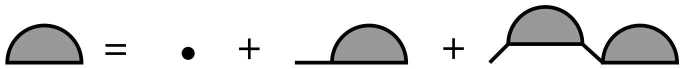

Figure 2: Graphical interpretation of Eq.(2).

A functional equation for $G(s,t,p,q)$ can be obtained by noting that a path either consists of zero steps, or it starts with a horizontal step followed by a path, or it starts with an up-step followed by a path, followed by a down-step, followed by another path – see Fig. 2 for an illustration. From this we get

$$G(s,t,p,q) = 1 + s\,G(s,t,p,q) + qt^2 G(ps,qt,p,q)\,G(s,t,p,q). \tag{2}$$

By iteration of Eq.(2), we obtain the continued fraction representation

$$G(s,t,p,q) = \cfrac{1}{1 - s - \cfrac{qt^2}{1 - ps - \cfrac{q^3t^2}{1 - p^2 s - \cfrac{q^5 t^2}{1 - p^3 s - \ldots}}}}. \tag{3}$$

We now choose the parametrization

$$G^{(\ell)}(a,t,q) \equiv G(t^\ell, \sqrt{a}t, q^\ell, q) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} p_{k,m,n} a^k t^m q^n, \tag{4}$$

which we refer to as the area-length generating function of ℓ-paths, since $p_{k,m,n}$ is the number of paths with $2k$ diagonal steps, total length m and an area n enclosed between the trajectory and the bottom line, counted in units of lattice cells. Substituting Eq.(4) into Eq.(2), we obtain the functional equation

$$G^{(\ell)}(a,t,q) = 1 + t^\ell G^{(\ell)}(a,t,q) + aqt^2 G^{(\ell)}(a,qt,q) G^{(\ell)}(a,t,q). \tag{5}$$

For $q=1$, Eq.(5) is solved by

$$G^{(\ell)}(a,t,1) = \frac{1 - t^\ell - \sqrt{(1-t^\ell)^2 - 4at^2}}{2at^2}. \tag{6}$$

Setting $a=1$ in Eq.(6), we obtain the generating functions of the Motzkin numbers for $\ell = 1$ and the large Schröder numbers for $\ell = 2$ (A001006 and A006318 in [7]).

For given ℓ and a, we denote the smallest positive value for which the discriminant $(1-t^\ell)^2 - 4at^2$ vanishes by t_c and define $G_c = G^{(\ell)}(a, t_c, 1)$. From Eq.(6) it follows that

$$G_c = \frac{1-t_c^\ell}{2at_c^2} = \frac{1}{\sqrt{a}t_c}. \tag{7}$$

For $|t| < 1$, the function $G^{(\infty)}(a,t,q) = \lim_{\ell\to\infty} G^{(\ell)}(a,t,q)$ satisfies the equation

$$G^{(\infty)}(a,t,q) = 1 + aqt^2 G^{(\infty)}(a,qt,q) G^{(\infty)}(a,t,q). \tag{8}$$

The parameter a becomes redundant in this case and can be set to one for convenience. We consequently write $G^{(\infty)}(t,q) \equiv G^{(\infty)}(1,t,q)$. Eq.(8) is then readily identified as the functional equation for the area-length generating function of Dyck paths. If $q = 1$, it is solved by the generating function of the Catalan numbers (A000108 in [7]).

3 Heuristic scaling ansatz

In analogy to similar models [6,8–10], one expects that in vicinity of the point $(a,t,q) = (a,t_c,1)$, the area-length generating function of ℓ-paths satisfies the scaling relation

$$G^{(\ell)}\big(a, t_c - z\,\epsilon^\phi, 1-\epsilon\big) = G^{(\ell)}_{sc}(a,z,\epsilon) + \mathcal{O}(\epsilon) \tag{9}$$

as $\epsilon \to 0^+$, where $\phi = 2/3$ and

$$G^{(\ell)}_{sc}(a,z,\epsilon) = G_c + \epsilon^\theta F(z) \tag{10}$$

with $\theta = 1/3$. Moreover,

$$F(z) = \mu \frac{\mathrm{Ai}'(\lambda z)}{\mathrm{Ai}(\lambda z)}, \tag{11}$$

where the Airy function is defined as

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_{c_-\infty}^{c_+\infty} \exp\left(\frac{u^3}{3} - zu\right) du \tag{12}$$

with $c_\pm = \exp(\pm i\pi/3)$ and the factors μ and λ depend on a and ℓ.

Note that for $a > 0$ it follows from the positivity of the coefficients of the generating function $G^{(\ell)}(a,t,z)$ that

$$\lim_{z\to\infty} F(z) = -\infty. \tag{13}$$

Following [10], we now set

$$t(z,\epsilon) = t_c - z\epsilon^{\phi} \tag{14}$$

and assume that $G^{(\ell)}(a, t(z,\epsilon), 1-\epsilon)$ satisfies Eq.(9) for $\epsilon \to 0^+$. The function $F(z)$ and the critical exponents ϕ and θ are then determined by inserting this ansatz into the functional equation (5). We begin by treating the special cases of Dyck, Motzkin and Schröder paths, before giving the results for the general case.

3.1 Dyck paths ($\ell = \infty$)

The area-length generating function of Dyck paths satisfies Eq.(8) for $a = 1$. From the solution for $q = 1$ given in Eq.(6) we obtain the critical values

$$t_c = \frac{1}{2} \text{ and } G_c = 2. \tag{15}$$

We now define

$$\Phi_\infty(z,\epsilon) = 1 - G_{sc}^{(\infty)}(z,\epsilon) + (1-\epsilon)\, t(z,\epsilon)^2\, G_{sc}^{(\infty)}(z + t_c\epsilon^{1-\phi}, \epsilon)\, G_{sc}^{(\infty)}(z,\epsilon).$$

Under the assumption that Eq.(9) holds, it follows from Eq.(8) that

$$\Phi_\infty(z,\epsilon) = \mathcal{O}(\epsilon) \tag{16}$$

as $\epsilon \to 0^+$. Expanding $\Phi_\infty(z,\epsilon)$ into a series in the variable ϵ, we see that for Eq.(16) to hold we necessarily have

$$2\theta - \phi = 0 \text{ and } \theta - 2\phi = 1,$$

and therefore $\theta = 1/3$ and $\phi = 2/3$. Inserting these exponents into the expansion, we obtain

$$\Phi_\infty(z,\epsilon) = \left[\frac{1}{8}F'(z) + \frac{1}{8}F^2(z) - 2z\right]\epsilon^{2/3} + \mathcal{O}(\epsilon),$$

which gives the Riccati type ODE

$$F'(z) = Az - BF(z)^2, \tag{17}$$

where

$$A = 16 \text{ and } B = 1. \tag{18}$$

Eq.(17) can be linearized by using the ansatz

$$F(z) = \mu \frac{f'(\lambda z)}{f(\lambda z)}$$

and the general solution is then obtained as

$$F(z) = \left(\frac{A}{B^2}\right)^{1/3} \frac{(C-1)\,\mathrm{Ai}'\left((AB)^{1/3}z\right) + (C+1)\,\mathrm{Bi}'\left((AB)^{1/3}z\right)}{(C-1)\,\mathrm{Ai}\left((AB)^{1/3}z\right) + (C+1)\,\mathrm{Bi}\left((AB)^{1/3}z\right)}, \qquad (19)$$

where $C \in \mathbb{R}$, and the functions $\mathrm{Ai}(z)$ (Eq.(12)) and

$$\mathrm{Bi}(z) = e^{-i\pi/6}\,\mathrm{Ai}(ze^{-2i\pi/3}) + e^{i\pi/6}\,\mathrm{Ai}(ze^{2i\pi/3}) \qquad (20)$$

are two independent solutions of the Airy differential equation

$$f''(z) - zf(z) = 0. \qquad (21)$$

From the asymptotic behaviour of the Airy functions [11] and boundary condition (13) it follows that necessarily $C = -1$. Inserting the coefficients from Eq.(18), we obtain Eq.(11) with $\mu = \lambda = 2^{4/3}$.

3.2 Motzkin paths ($\ell = 1$)

Setting $\ell = 1$ and $q = 1$ in Eq.(6), we get the critical values for standard Motzkin paths as

$$G_c = \frac{1 + 2\sqrt{a}}{\sqrt{a}} \text{ and } t_c = \frac{1}{1 + 2\sqrt{a}}. \qquad (22)$$

As in the previous case, we define $\Phi_1(z, \epsilon)$ from Eq.(5) as

$$\begin{aligned}\Phi_1(a, z, \epsilon) =& 1 - G_{sc}^{(1)}(a, z, \epsilon) + t(z, \epsilon) G_{sc}^{(1)}(a, z, \epsilon) + \\ &+ a(1-\epsilon)\, t(z, \epsilon)^2\, G_{sc}^{(1)}(a, z + t_c \epsilon^{1-\phi}, \epsilon)\, G_{sc}^{(1)}(a, z, \epsilon).\end{aligned}$$

Again, assumption (9) requires $\theta = 1/3$ and $\phi = 2/3$. From the expansion

$$\Phi_1(a, z, \epsilon) = \left[aG_c t_c^4 F'(z) + a t_c^3 F(z)^2 - z(2aG_c^2 t_c^2 + G_c)\right] \epsilon^{2/3} + \mathcal{O}(\epsilon),$$

we are lead to the same ODE (17) as for Dyck paths, with the coefficients now being

$$A = \frac{2G_c}{t_c^2} + \frac{1}{a t_c^3} \text{ and } B = \sqrt{a}. \qquad (23)$$

The final form of the scaling function is given by Eq.(11) with

$$\mu = \left(\frac{2\sqrt{a} + 1}{a^2 t_c^3}\right)^{1/3} \text{ and } \lambda = \sqrt{a}\,\mu. \qquad (24)$$

3.3 Schröder paths ($\ell = 2$)

For Schröder paths, the critical values are given by

$$t_c = \sqrt{1 + 2\,a - 2\sqrt{a(a+1)}} \text{ and } G_c = \frac{1}{\sqrt{a}t_c}. \tag{25}$$

As for Dyck and Motzkin paths, we define

$$\begin{aligned}\Phi_2(a,z,\epsilon) =& 1 - G_{sc}^{(2)}(a,z,\epsilon) + t(z,\epsilon)^2\, G_{sc}^{(2)}(a,z,\epsilon) + \\ &+ a(1-\epsilon)\, t(z,\epsilon)^2\, G_{sc}^{(2)}(a, z + t_c\,\epsilon^{1-\phi}, \epsilon)\, G_{sc}^{(2)}(a,z,\epsilon).\end{aligned}$$

and assumption (9) determines $\theta = 1/3$ and $\phi = 2/3$. Expanding $\Phi_2(a,z,\epsilon)$ in ϵ gives

$$\Phi_2(a,z,\epsilon) = \left[aG_c t_c^4 F'(z) + a t_c^3 F^2(z) - z\left(2aG_c^2 t_c^2 + 2G_c t_c\right)\right]\epsilon^{2/3} + \mathcal{O}(\epsilon),$$

which again leads to Eq.(17), where the coefficients are now

$$A = \frac{2G_c}{t_c^2} + \frac{2}{a t_c^2} \text{ and } B = \sqrt{a}. \tag{26}$$

Thus, also for Schröder paths, the scaling function is given by Eq.(11), with

$$\mu = \left(\frac{2\sqrt{a} + 2\,t_c}{a^2 t_c^3}\right)^{1/3} \text{ and } \lambda = \sqrt{a}\,\mu. \tag{27}$$

3.4 The case of general ℓ

Now we assume ℓ to be any positive integer. For general ℓ, it is not possible to give an expression for the critical value t_c as a function of a.
As in the special cases, we define

$$\begin{aligned}\Phi_\ell(a,z,\epsilon) =& 1 - G_{sc}^{(\ell)}(a,z,\epsilon) + t(z,\epsilon)^\ell G_{sc}^{(\ell)}(a,z,\epsilon) + \\ &+ a(1-\epsilon) t(z,\epsilon)^2\, G_{sc}^{(\ell)}\left(a, z + t_c\epsilon^{1-\phi}, \epsilon\right) G_{sc}^{(\ell)}(a,z,\epsilon)\,,\end{aligned} \tag{28}$$

and from the assumption that $\Phi_\ell(a,z,\epsilon) = \mathcal{O}(\epsilon)$ one obtains $\theta = 1/3$ and $\phi = 2/3$. Expanding the RHS of Eq.(28) in ϵ we obtain

$$\Phi_\ell(s,z,\epsilon) = \left[aG_c t_c^4 F'(z) + a t_c^3 F(z)^2 - (2aG_c^2 t_c^2 + \ell\, G_c t_c^{\ell-1})z\right]\epsilon^{2/3} + \mathcal{O}(\epsilon),$$

which leads to Eq.(17) with

$$A = \frac{2G_c}{t_c^2} + \frac{\ell\, t_c^{\ell-4}}{a} \text{ and } B = \sqrt{a}. \tag{29}$$

The solution of this equation is given in Eq.(11) with parameters

$$\mu = \left(\frac{2\sqrt{a} + \ell\, t_c^{\ell-1}}{a^2 t_c^3}\right)^{1/3} \text{ and } \lambda = \sqrt{a}\,\mu. \tag{30}$$

4 Solution for $\ell = 2$

For $\ell = 2$, the parameter t only appears squared in the functional Eq.(5), which makes it possible to obtain a more compact solution as opposed to the case $\ell = 1$, which has been solved in [4].
Inserting the ansatz

$$G^{(2)}(a,t,q) = \frac{H(a,qt^2,q)}{H(a,t^2,q)} \tag{31}$$

into Eq.(5) for $\ell = 2$, we get the linear functional equation

$$aqt^2 H(q^2t^2) + (t^2-1)H(qt^2) + H(t^2) = 0, \tag{32}$$

where we have abbreviated $H(t^2) \equiv H(a,t^2,q)$ for convenience. Eq.(32) is solved by the basic hypergeometric series defined as

$$H(t) = \sum_{n=0}^{\infty} \frac{(-a;q)_n}{(q;q)_n} q^{\binom{n}{2}} (-t)^n = {}_1\phi_1 \left(\begin{matrix} -aq \\ 0 \end{matrix} ; q, t \right), \tag{33}$$

where $(z;q)_n = (1-z)\cdot(1-q\,z)\cdots(1-q^{n-1}\,z)$. Note that $G^{(2)}(1,t,q)$ generates a q-deformation of the large Schröder numbers as defined in [12].

In order to validate our results from Section 3.3, we will now analyse the scaling behaviour of $G^{(2)}(a,t,z)$ by carrying out rigorous saddle point analysis. The same technique has been applied before to area-perimeter weighted staircase polygons and area-length-weighted Dyck paths [6, 8].

As an aside, we note that

$$G^{(2)}\left(\frac{1}{(h-1)\,q}, t\,\sqrt{q(h-1)}, q\right) = G_p^{(\infty)}(h,t,q), \tag{34}$$

where $G_p^{(\infty)}(h,t,q)$ is the generating function of Dyck paths, with weights h, t and q associated to the number of peaks, length and area, respectively.

5 Saddle point asymptotics

Using a contour integral expression for the function $H(t^2)$ and inserting an asymptotic expression for the involved q-products, one can show that for $q \to 1^-$,

$$H(t^2) = \frac{D}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \exp\left(\frac{1}{\epsilon}f(z)\right) g(z)\big(1+\mathcal{O}(\epsilon)\big)\,dz, \tag{35}$$

where $\epsilon = -\ln(q)$; $D = (q;q)_\infty/(-a;q)_\infty$ and

$$f(z) \;=\; 2\ln(t)\ln(z) + \mathrm{Li}_2(z) + \mathrm{Li}_2(-a/z), \tag{36}$$

$$g(z) \;=\; \sqrt{\frac{z}{(1-z)(z+a)}}. \tag{37}$$

Since the derivation of Eq.(35) is exactly analogous to the one for other basic hypergeometric series [6,8], it will not be carried out here in more detail.

It is now possible to apply the method of steepest descents to the integral on the RHS of Eq.(35). The function $f(z)$ has the two saddle points

$$z_1 = \frac{1}{2}\left(1 - t^2 - \sqrt{d}\right) \quad ; \quad z_2 = \frac{1}{2}\left(1 - t^2 + \sqrt{d}\right) \tag{38}$$

where $d = (1-t^2)^2 - 4\,a\,t^2$. For $d = 0$, the saddles coalesce in $z_m = (1-t^2)/2$.

It follows from a theorem which was proven in [13] that there exists a transformation $u \mapsto z(u)$ which is regular in a domain containing z_1 and z_2 if d is sufficiently close to zero, such that

$$f(z) = \frac{1}{3}\,u^3 - \alpha\,u + \beta = p(u). \tag{39}$$

Since the transformation is regular, it maps the two saddle points $u_{1,2} = \pm\sqrt{\alpha}$ of the polynomial $p(u)$ onto the saddle point of $f(z)$. From this we can infer

$$\alpha = \left(\frac{3}{4}\big(f(z_1) - f(z_2)\big)\right)^{2/3} \quad \text{and } \beta = \frac{1}{2}\big(f(z_1) + f(z_2)\big). \tag{40}$$

Applying the transformation defined by Eq.(39), we can rewrite Eq.(35) as

$$H(t^2) = \frac{D}{2\pi\,i}\int_{c_-\infty}^{c_+\infty} \exp\left(\frac{1}{\epsilon}p(u)\right)\, S(u)\big(1 + \mathcal{O}(\epsilon)\big)\,du, \tag{41}$$

with $S(u) = g(z(u))\,dz/du$ and $c_\pm = \exp(\pm\,i\,\pi/3)$. Using the expansion

$$S(u) = p^{(0)} + u\,q^{(0)} + (u^2 - \alpha)S_1(u), \tag{42}$$

where $S_1(u)$ is a regular function of u, we arrive at

$$H(t^2) = D\,\exp\left(\frac{\beta}{\epsilon}\right)\left(\epsilon^{1/3}p^{(0)}\,\mathrm{Ai}\left(\frac{\alpha}{\epsilon^{2/3}}\right) - \epsilon^{2/3}\,q^{(0)}\,\mathrm{Ai}'\left(\frac{\alpha}{\epsilon^{2/3}}\right) + \mathcal{O}(\epsilon)\right), \tag{43}$$

with the coefficients being given by

$$p^{(0)} = \sqrt{\frac{\sqrt{\alpha}}{2}}\left(\frac{g(z_2)}{\sqrt{f''(z_2)}} + \frac{g(z_1)}{\sqrt{-f''(z_1)}}\right), \tag{44}$$

$$q^{(0)} = \sqrt{\frac{1}{2\sqrt{\alpha}}}\left(\frac{g(z_2)}{\sqrt{f''(z_2)}} - \frac{g(z_1)}{\sqrt{-f''(z_1)}}\right). \tag{45}$$

Replacing $g(z)$ by $h(z) = g(z)/z$ in the above analysis, we obtain the leading asymptotic behaviour of of $H(qt^2)$. We name the associated expansion coefficients $p^{(1)}$ and $q^{(1)}$. With this we arrive at the following result.

$$G^{(2)}(a,t,q) = \frac{p^{(1)}\,\mathrm{Ai}(\alpha\epsilon^{-2/3}) - q^{(1)}\,\mathrm{Ai}'(\alpha\epsilon^{-2/3})\epsilon^{1/3}}{p^{(0)}\,\mathrm{Ai}(\alpha\epsilon^{-2/3}) - q^{(0)}\,\mathrm{Ai}'(\alpha\epsilon^{-2/3})\epsilon^{1/3}} + \mathcal{O}\left(\epsilon\right), \tag{46}$$

as $\epsilon = -\ln(q) \to 0^+$. Note that this expression is uniform for a range of values of t and a including the critical point $d = 0$. In particular, setting $t = t(z,\epsilon)$ as in Eq.(14), Eq.(46) gives for $\epsilon \to 0^+$,

$$G^{(2)}(a,t(z,\epsilon),q) = \frac{1}{z_m}\left(1 + \left(\frac{q^{(0)}}{p^{(0)}} - \frac{q^{(1)}}{p^{(1)}}\right)\frac{\mathrm{Ai}'(\alpha\epsilon^{-2/3})}{\mathrm{Ai}(\alpha\epsilon^{-2/3})}\,\epsilon^{1/3} + \mathcal{O}\left(\epsilon\right)\right). \tag{47}$$

Expanding the coefficients $\left(q^{(0)}/p^{(0)} - q^{(1)}/p^{(1)}\right)$ and α up to linear order around the critical point, we obtain the coefficients given in Eq.(27), thereby confirming the validity of the heuristic scaling ansatz.

Figure 3 shows the remarkable agreement of scaling function and partition function asymptotics for q close to one.

6 Summary

We analyzed the scaling behaviour of the generating function of area-weighted Dyck, Motzkin and Schröder paths around the tri-critical point by using a heuristic ansatz and generalized this approach to ℓ-Motzkin paths with arbitrary ℓ. We then derived an expression for the area-length generating function of Schröder paths and analyzed the result by applying rigorous saddle point analysis, thereby confirming the validity of the heuristic ansatz.

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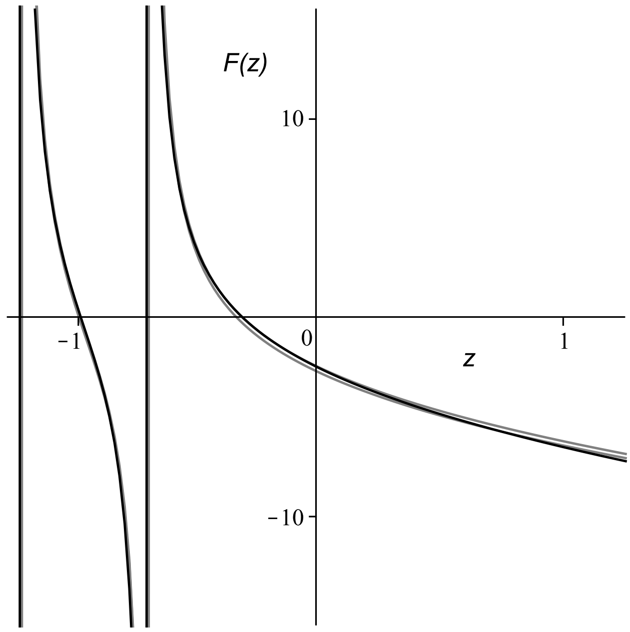

Figure 3: Plot of the scaling function $F(z)$ given by Eq.(11) with coefficients (27) for $a = 1$ (black) against the approximation of the scaling function obtained directly from the generating function $G(1, t, q)$ and fixed values $\epsilon = 10^{-3}$ and $\epsilon = 10^{-4}$ (gray).

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