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Interplay between consensus and coherence in a model of interacting opinions

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The formation of agents' opinions in a social system is the result of an intricate equilibrium among several driving forces. On the one hand, the social pressure exerted by peers favours the emergence of local consensus. On the other hand, the concurrent participation of agents to discussions on different topics induces each agent to develop a coherent set of opinions across all the topics in which he is active. Moreover, the prevasive action of external stimuli, such as mass media, pulls the entire population towards a specific configuration of opinions on different topics. Here we propose a model in which agents with interrelated opinions, interacting on several layers representing different topics, tend to spread their own ideas to their neighbourhood, strive to maintain internal coherence, due to the fact that each agent identifies meaningful relationships among its opinions on the different topics, and are at the same time subject to external fields, resembling the pressure of mass media. We show that the presence of heterogeneity in the internal coupling assigned by agents to their different opinions allows to obtain states with mixed levels of consensus, still ensuring that all the agents attain a coherent set of opinions. Furthermore, we show that all the observed features of the model are preserved in the presence of thermal noise up to a critical temperature, after which global consensus is no longer attainable. This suggests the relevance of our results for real social systems, where noise is inevitably present in the form of information uncertainty and misunderstandings. The model also demonstrates how mass media can be effectively used to favour the propagation of a chosen set of opinions, thus polarising the consensus of an entire population.

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I. INTRODUCTION

The increasing availability of data sets about social relationships, such as friendship, collaboration, competition, and opinion formation, has recently spurred a renewed interest for the basic mechanisms underpinning human dynamics [1]. Aside with the classical studies in social sciences and social network analysis [2–4], some interesting contributions to the understanding of social dynamics have lately come from statistical physics [5], which has brought in the field new tools and analytical methods to study systems consisting of many interacting agents. In such wider context, much effort has been devoted to the study of the dynamics responsible for opinion formation in populations of interacting agents, and in particular to a more in-depth understanding of the elementary mechanism allowing the emergence of global consensus and of the role of endogenous and exogenous driving forces, including social pressure and mass media. As a result of this investigation, a plethora of models of opinion formation have been proposed and studied [6–13].

Although the majority of those models originally made the simplifying assumption of considering homogeneous interaction patterns (basically, regular lattices), the rise of network science [14–17] provided the tools to overcome this limitation, featuring more realistic interaction patterns. More recently, also the role of mass media in the formation of global consensus has attracted a lot of interest [18–23].

An aspect of social relationships that has been mostly discarded in the study of the emergence of consensus is the fact that agents usually interact in a variety of different contexts, making the interaction pattern effectively multilayered and multi-faceted. As a matter of fact, the urge to maintain a certain level of coherence among opinions on different but related subjects might actually play a crucial role in determining the reaction of each agent to external pressure and in facilitating (or hindering) the emergence of global consensus. Moreover, the balance between the internal tendency towards coherence and the necessity to adequately respond to social pressure is naturally dependent on each person's attitude, thus implying a certain level of heterogeneity. Some individuals may be more prone to align more closely to the opinions of their neighbors in each of the different contexts where they interact, putting little or no importance to the overall coherence of their profile. On the contrary, some other agents may indeed be more reluctant to change their opinion on a topic, in spite of being urged by other individuals or media, if such a change results in a contradiction with another of their opinions on a different but related subject.

In this paper we propose a model of opinion formation that takes into account *i)* the concurrent participation of agents to distinct yet connected interaction levels (representing discussion topics or social spheres), *ii)* the presence of social pressure and *iii)* the exogenous action of mass media. Our analysis can be naturally cast in the framework of multiplex networks [24–27], which has recently proven successful for a more realistic modeling of different social dynamics [28–33]. According to this framework, agents are represented by nodes connected by links of different nature, where links of the same kind belong to the same layer of the system. Each layer thus represents the interaction pattern of individuals discussing a given topic. Different layers are in general endowed with different topologies, to mimic multi-layer real-world social systems where distinct interaction patterns are present at different levels. Peer social pressure occurs on each topic through intra-layer links. The opinions of an individual on the different topics are also driven towards a specific state by the tension towards internal agent's coherence, represented by a preferred configuration of opinions on different topics. Mass media are introduced as fields acting uniformly on all the agents at the level of each single topic.

The resulting model is a natural extension of the traditional Ising model of magnetic interaction [34] and of more recent variations introduced to take into account the effect of external forces on the emergence of consensus [35], in the spirit of less and more recent work connecting statistical mechanics of disordered systems and opinion dynamics [36–38]. The key ingredient of heterogeneous distributed couplings between opinions lead to interesting equilibrium states, where agents can remain fully coherent while a variable level of global consensus is attained, depending on the strength of the pressure exerted by mass media. This clearly resembles the dynamics observed in real societies, thereby supporting the relevance of our approach.

II. MODEL

We consider a population of N individuals interacting through M different layers, representing different topics or subjects. The network of each layer $\alpha = 1, \ldots, M$ represents the pattern of interactions among agents on a specific topic, which is in general distinct from those of the other layers, and is encoded by the adjacency matrix $A^{[\alpha]} = \{a_{ij}^{[\alpha]}\}_{i,j=1,\ldots,N}$, whose element $a_{ij}^{[\alpha]} = 1$ only if agent i and agent j are neighbors on layer α, and equal to zero otherwise. The structure of the overall interaction pattern is thus concisely represented by the vector of adjacency matrices $\mathcal{A} = \{A^{[1]}, \ldots, A^{[M]}\}$, where all the matrices $A^{[\alpha]}$ are in general distinct. Each agent $i = 1, \ldots, N$ expresses a binary opinion $s_i^{[\alpha]} = \pm 1$ on each subject $\alpha = 1, \ldots, M$. An example with $M = 2$ is shown in Fig. 1(a), where upwards and downwards spins represent the two possible values of $s_i^{[\alpha]}$. We assume that agent opinions evolve over time due to two concurrent mechanisms. On the one hand, agents are subject to social pressure from their peers on each layer (denoted by the red and blue links in Fig. 1), so that the opinion of agent i on node α will tend to remain aligned with the opinions of its neighbors on the same layer. This mechanism, based on the elimination of conflicting opinions on a microscopic scale, has been widely observed in many real-world social systems [40], and is responsible for the attainment of local consensus on each layer. On the other hand, we assume that the opinions of agent i at the different layers are not independent from each other but are instead interacting, so that for each agent there exists a preferred configuration of opinions at the different layers which is considered *coherent*. For instance, the political orientation of a person is often related to his/her ideas about economy and welfare, so that the emergence of consensus with its neighbors on one subject should remain coherent with its current opinions on the other layers. Moreover, we imagine that agents are exposed, on each layer, to the action of mass-media, a mean-field external force which preferentially drives their opinions towards either $+1$ or -1.

We formalize the interplay of these concurrent dynamics by defining the functional:

$$f_i^{[\alpha]} = J \sum_{j=1}^{N} a_{ij}^{[\alpha]} s_j^{[\alpha]} + h^{[\alpha]} + \gamma \frac{\chi_i}{J} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{M} s_i^{[\beta]} \qquad (1)$$

for each agent i and each topic α. The first sum on the rhs of Eq. (1) represents the social pressure exerted on i by its neighbors on layer α, and is weighted by the coefficient J, which models its intrinsic permeability to social pressure. The variables $h^{[\alpha]}$ represent the external effect of mass-media on the formation of agents' opinions, which are considered in this case as a mean-field force acting homogeneously on all the agents of a layer. Finally, the second sum represents the tendency of agent i towards internal coherence, where the global parameter γ sets the relative importance of internal coherence and social pressure. Specifically, when $\gamma \simeq 0$ the opinions of the agents are mainly driven by peer and external pressure, whereas when $\gamma \to \infty$ they are determined by the internal coherence, such that coherent configurations are strongly favoured.

This setup is depicted in Fig. 1(b) for the case $M = 2$, where links with different colors indicate the connections of an agent at the two layers. In practice, J is the strength of the interaction of each agent with its neighbors, while χ_i determines the importance (and sign) of internal agent coherence. In this case, as shown in Fig. 1(c), the preferred configuration of agent's i spins is concordant if $\chi_i > 0$ and discordant if $\chi_i < 0$. We notice that the actual value of χ_i, which in the following always lie in the interval $[-1, 1]$, is a measure of how much agent i is flexible towards a change of one of its opinions, eventually leading to configurations which do not agree with what it would consider a coherent configuration of its spins. In other words, agents for which $|\chi_i| \simeq 0$ assign less importance to internal coherence and more relevance to social pressure, while the opposite happens when $|\chi_i| \simeq 1$.

In our model, the opinions of each agent evolve towards configurations which maximize the function $F_i^{[\alpha]} = s_i^{[\alpha]} f_i^{[\alpha]}$, in order to attain, at the same time, internal coherence and local consensus with their neighbors on each layer. As a consequence, agents will naturally prefer configurations of spins at all layers which ensure a balanced trade-off between social pressure and coherence, depending on the respective values of the parameters γ, J, χ_i, and of the external fields $h^{[\alpha]}$. Although being a somehow simplified model of real-life interactions, where not just binary but also intermediate opinions between two extremes are possible and agents might respond dif-

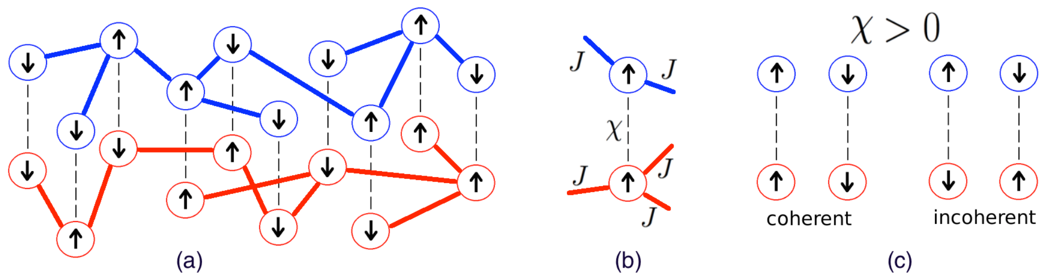

Figure 1. (a) Model of interacting opinions for $M = 2$ different topics. Every agent in the population expresses a binary opinion on each subject $\alpha = 1, 2$ (upward or downward arrows) and may interact with other individuals. The pattern of intra-layer interactions (blue/red lines for each α respectively) may in general be different. (b) Opinions may change according to both social pressure (weighted by the parameter J) and external fields, e.g. mass media (not shown). Individuals are also differently prone to change one or more of their opinions, according to their internal coherence. Such effect is taken into account by identifying natural couplings between the opinions of an agent at the two layers, weighted by the agent-dependent parameter χ_i. (c) The internal coherence/incoherence of an agent is determined by the sign of its opinions (orientation of the arrows): when $\chi_i > 0$, coherent configurations are those with both opinions of the same sign (left couple), whereas incoherent ones present opinions of different sign (right couple). The opposite holds if $\chi_i < 0$.

ferently to social pressure and to the external effect of mass-media, this model turns out to be already general enough to investigate the elementary mechanisms driving interacting opinions.

Numerical implementation of this dynamical evolution is obtained through extensive Monte Carlo simulations, adopting an appropriately modified version of the Glauber algorithm [39]. In particular, at each step we update all the spins $s_i^{[\alpha]}$, $i = 1, \ldots, N$, $\alpha = 1, 2, \ldots, M$ in a random order. The update is performed by proposing a flip of the current spin $s_i^{[\alpha]} \to -s_i^{[\alpha]}$ and accepting the flip only when the new configuration leads to a larger value of the function $F_i^{[\alpha]}$. Every time a flip is accepted, $F_i^{[\alpha]}$ is also updated according to the new configuration. Clearly, the form of $f_i^{[\alpha]}$ captures both the contributions of intra-topic and inter-topic couplings and those of the existing external fields, so that larger values of $F_i^{[\alpha]}$ correspond to preferred configurations for node i.

Clearly, these rules imply a deterministic evolution of the opinions, which is not observed in real social systems. We then need to account for the presence of stochastic noise. Its simulation is realized by introducing a parameter $T \geq 0$, which may be regarded as a social temperature in analogy with magnetic systems, induced by all those mechanisms which drive the system out of its deterministic dynamics, such as partial information or misunderstandings. We include such thermal noise in the dynamics of our model in a standard way: when $T > 0$, an agent i may change its opinion on the topic α even if it leads to configurations with a smaller $F_i^{[\alpha]}$ with probability $e^{\frac{\Delta F_i^{[\alpha]}}{T}}$, with $\Delta F_i^{[\alpha]}$ being the variation in $F_i^{[\alpha]}$ due to the flip of thew spin $s_i^{[\alpha]}$.

We are interested in understanding how the presence of three concurrent factors, namely the response to social pressure, the tension towards internal coherence and the presence of an external mean-field force on each layer, affects the emergence of consensus in the population. We consider three scenarios, namely *i)* the case in which $\chi_i = 1, \forall i$ (homogeneous agents); *ii)* the case in which χ_i is a random variable sampled from a certain probability distribution (heterogeneous agents); and finally *iii)* the case in which the dynamics is affected by noise. For the sake of simplicity, in the following we focus on the case of two interconnected layers, i.e. $M = 2$, and we set $J = 1$, so that the relative importance of internal coherence and social pressure is determined, for each agent, by the product $\gamma\chi_i$. The model is numerically investigated in the three different setups described above, where the parameters γ, $h^{[1]}$, and $h^{[2]}$ play the role of control parameters.

We study the emergence of consensus at each layer $\alpha = 1, 2$ through the order parameter:

$$M^{[\alpha]} = \frac{1}{N} \sum_{i=1}^{N} s_i^{[\alpha]}, \tag{2}$$

which satisfies $-1 \leq M^{[\alpha]} \leq 1$, with $\left|M^{[\alpha]}\right|$ denoting the strength of consensus and $\text{sgn}\left(M^{[\alpha]}\right)$ indicating which of the opinion is prevalent among the population. We also define the average internal coherence of the agents as follows:

$$C = \frac{1}{N} \sum_{i}^{N} \text{sgn}(\chi_i)\, s_i^{[1]} s_i^{[2]}. \tag{3}$$

Notice that, when $\gamma > 0$, $C = +1$ if the two spins of each agent are coherent with their preferred configuration, while $C = -1$ if they are incoherent for every agent. The opposite holds when $\gamma < 0$.

An interesting remark is that the global function $H = -\sum_{\alpha=1}^{M} \sum_{i=1}^{N} F_i^{[\alpha]}$ (which we do not consider in this study) effectively is the Hamiltonian of a multi-layer Ising model, where the population evolves equivalently towards configurations that minimise H. In this sense, our model can be considered as a generalization of the coupled Ising model on lattices [41]. Following this analogy, we note that the order parameters $M^{[\alpha]}$ can be interpreted as the magnetization of the different layers of the system.

III. RESULTS

We discuss in this section the transition towards coherence and consensus and the equilibrium properties of the model, focusing on the dependence of the order parameters C and $M^{[\alpha]}$ in Eqs. (2-3) on the parameter γ and on the external fields $h^{[\alpha]}$. In details, we investigate in Sec. III A the case of $\chi_i = 1\ \forall i$ and $T = 0$, i.e. a population of homogeneous agents in the absence of social noise. In Sec. III B we consider a population of heterogeneous agents (χ_i not fixed), while keeping $T = 0$. Finally, in Sec. III C we study the effect of social noise by investigating the dependence on T.

Simulations of the Glauber dynamics described in the previous section are realized by varying the global parameter γ adiabatically. The initial configuration is obtained by setting $s_i^{[1]} = 1$ and $s_i^{[2]} = -1\ \forall i$. We let the system perform two complete hysteresis cycles before recording the resulting configurations. This procedure eliminates possible effects due to the specific initial conditions.

The results presented here are obtained by simulating the dynamics on a multiplex of two uncorrelated Barabasi-Albert networks [42] with the same average degree $\langle k \rangle$=6. Nevertheless, we remark that analogous qualitative results have been found for different interaction patterns, such as random graphs with the same density or systems with different values of inter-layer degree-correlation [43], suggesting that the only topological parameter playing a major role in the long-term behavior of the dynamics is the average degree of the networks at the two layers.

A. Transition towards full coherence in the case of homogeneous agents

We consider here the case of homogeneous agents $\chi_i = 1\ \forall i$, in the absence of social noise, i.e., the case $T = 0$. The effects induced by the external forces, e.g., the mass media, are studied by choosing fields with opposite signs and relative strength according to the two typical cases: $|h^{[1]}| = |h^{[2]}|$ or $|h^{[1]}| > |h^{[2]}|$. We remark that the qualitative behaviour observed does not depend on the specific values of $|h^{[1]}|$ and $|h^{[2]}|$. First, we study the transition in coherence as a function of γ: for fields of both equal and different intensity, we provide evidence of the existence of a sharp transition along with a hysteresis loop. We are also able to propose an empirical relation to estimate the transition points $\gamma_\pm$, given the intensity of the fields and the density of the layers. We note that the case of fields with equal signs is somehow trivial, since the opinions on both layers are pulled in the same direction and global consensus emerges easily. Second, we find that a coherent population, i.e. in the regime $\gamma \to \infty$, exhibits either states of full or null consensus and that states of partial consensus cannot be attained in a population of homogenous agents.

We show examples of the steep transitions that the system exhibits by plotting C as a function of γ in the top panels of Fig. 2(a1) for $|h^{[1]}| > |h^{[2]}|$ and of Fig. 2(a2) for $|h^{[1]}| = |h^{[2]}|$. The behavior of the coherence is robust with respect to the relative strength of the external fields: we always observe a sharp transition from $C = -1$ to $C = +1$ characterised by a marked hyseresis loop. However, the actual values of $h^{[1]}$ and $h^{[2]}$ deeply affect the corresponding level of consensus emerging in the population. This is shown in the bottom panels of Figs. 2(a-b), where we plot the corresponding value of $M^{[2]}$ as a function of γ. If the external fields have the same intensity $|h^{[1]}| = |h^{[2]}|$, we have $M^{[2]} = 0$ for $\gamma > \gamma_+$, while $M^{[2]} = -1$ when $\gamma_- \leq \gamma \leq \gamma_+$. As the transition is sharp, we can always infer the value of $M^{[1]}$ from the corresponding values of C and $M^{[2]}$. In fact, we respectively have $M^{[1]} = \pm M^{[2]}$ when $C = \pm 1$.

This result has a clear interpretation. When γ is increased, the second term in the rhs of Eq. (1) becomes dominant, meaning that the agents give more importance to internal coherence than to social pressure. At the same time, however, none of the two external fields, which have opposite signs, is able to force a flip of opinions on the the other layer. This leads naturally to states of vanishing magnetization on each layer, i.e., no consensus. The opposite situation is observed when we decrease γ. Indeed, the agents become more flexible, so that different opinions on different topics can coexist. Of course, this tendency gradually increases the effect of the external fields on their own topic. At the transition point, the population becomes globally incoherent, whereas the external fields induce full consensus separately on each layer, with their sign determining the dominant opinion.

In the case where one of the two fields is larger than the other, i.e. $|h^{[1]}| > |h^{[2]}|$, the situation is radically different. We indeed find $M^{[2]} = +1$ for $\gamma > \gamma_+$, $M^{[2]} = -1$ when γ increases in the interval $[\gamma_-, \gamma_+]$, and $M^{[2]} = +1$ when γ decreases in $[\gamma_-, \gamma_+]$. The interpretation follows straightforwardly with a reasoning similar to the one reported above for the case $|h^{[1]}| = |h^{[2]}|$. As γ increases, the agents become more and more inflexible, thus favoring opinions of the same sign throughout the different topics. Moreover, since $|h^{[1]}|$ is larger than $|h^{[2]}|$, states of non-vanishing consensus are favored. In particular, one of the opinions ends up prevailing not just on layer 1 but, through the internal agent coherence, also on the other layer. Thus, the concurrent effect of these two mechanisms causes a steep transition towards a state of both full coherence and full consensus on a single opinion on

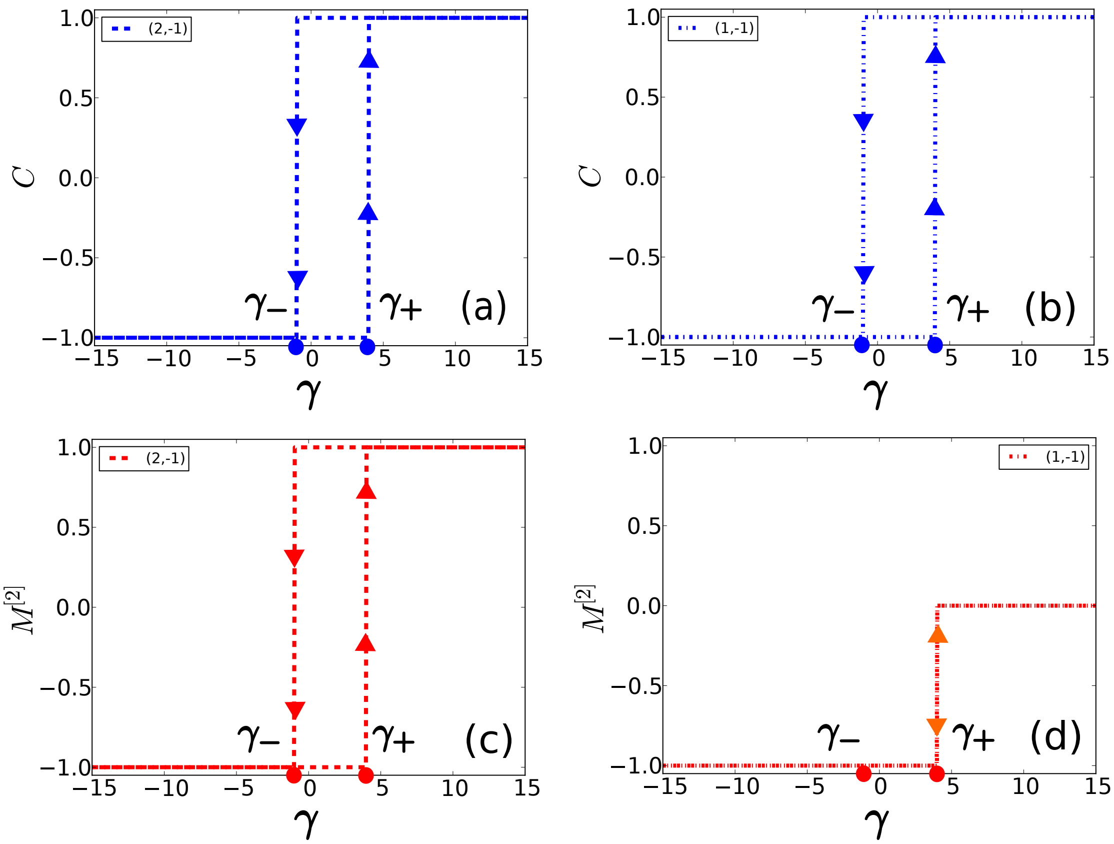

Figure 2. Values of the average agent coherence C (top panels) and of the consensus on the second topic $M^{[2]}$ (bottom panels) as a function of γ for a population of homogeneous agents ($\chi_i = 1$, $\forall i$). External fields are chosen with opposite signs and either different ($|h^{[1]}| > |h^{[2]}|$) (left panels) or equal ($|h^{[1]}| = |h^{[2]}|$) relative intensity (right panels). A sharp transition towards full coherence ($C = 1$), characterised by an hysteresis cycle delimited by the transition points γ_+ and γ_-, is observed in both cases (panels a-b), independently from the relative strength of the media. Conversely, the corresponding value of $M^{[2]}$ after the transition, i.e., when agents are coherent, differs significantly: if $|h^{[1]}| = |h^{[2]}|$ (panel d), we have $M^{[2]} = 0$, while if $|h^{[1]}| > |h^{[2]}|$ (panel c), we have $M^{[2]} = 1$. The presence of a stronger media pressure on a specific topic indeed influences also the other one. We note that when agents are homogeneous no states of partial consensus are allowed on either layer.

both the topics, which is determined by the leading external field. The same dynamical explanation of the previous case can instead be given for decreasing values of γ beyond γ_-.

As suggested before, these qualitative patterns are robust with respect to the strengths of the external fields, which only determine the exact transition points γ_+ and γ_-, as shown in Fig. 3(a). We find that the transitions points γ_+ and γ_- where the hysteresis loop starts and ends respectively are given by the following empirical non-linear relation:

$$\gamma_\pm = \pm \langle k \rangle /2 - \mathrm{sign}(h^{[1]}h^{[2]}) \min\left(|h^{[1]}|, |h^{[2]}|\right), \quad (4)$$

where $\langle k \rangle = \frac{1}{2N}\sum_{\alpha=1}^{2}\sum_{i,j=1}^{N} a_{ij}^{[\alpha]}$ is the average degree of the two layers. The actual values of $h^{[1]}, h^{[2]}$ only determine a shift of the metastable region, whereas they do not modify the width of the hysteresis cycle. We support this conjecture by showing in Fig. 3(b) the values of $(\gamma_+ + \gamma_-)/2$ (i.e. the center of the hysteresis cycle) obtained from the simulations as a function of $h^{[1]}$ and $h^{[2]}$, confirming the validity of the relation expressed in Eq. (4).

We conclude that in the case of homogeneous agents the system always reaches configurations of full consensus on both layers, where the dominant opinion on each layer is determined by the sign of the strongest external field (phase diagram in Fig. 4, top panel a). The only exception is given by the critical line $h_1 = -h_2$ where we find $M^{[2]} \approx 0$. As expected, the assumption of homogeneity of the agents imposes a strong constraint on the dynamics of the model, leading only to unrealistic patterns of perfect (or null in the specified particular case) consensus always accompanied by perfect coherence, but not allowing intermediate configurations. These sharp scenarios are different from those observed for real-world

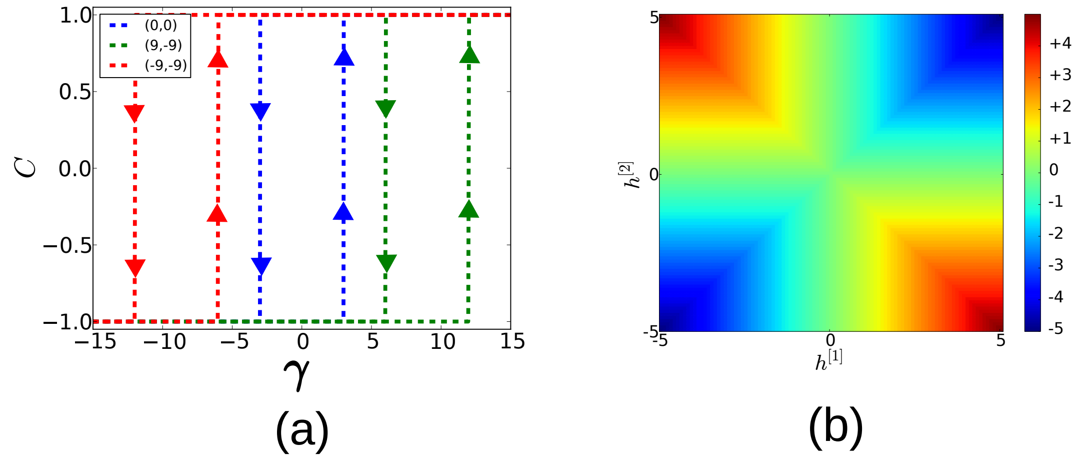

Figure 3. (a) Plot of the hysteresis cycle of the average internal coherence C for different values of ther external fields. Even if the qualitative behaviour of the system does not depend on the actual values of $h^{[1]}$ and $h^{[2]}$) (i.e., the system is characterised by a sharp transition in C with a marked hysteresis loop, whose width is determined solely by the average degree on the two layers $\langle k \rangle$), the exact positions of the transition points γ_+ and γ_- change according to the intensity and sign of the external fields. (b) Simulated values of $(\gamma_+ + \gamma_-)/2$ as a function of $h^{[1]}$ and $h^{[2]}$. These numerical results support the validity of the empirical relation of Eq. (4).

systems, where states of partial consensus are often observed. The heterogeneity in the relative weight and sign assigned to internal coherence indeed plays a crucial role by favoring the influence of the mass media over the attainment of full coherence in the population. Thus, we expect that the relaxation of the homogeneity hypothesis in our model could lead to milder patterns, with different levels of consensus at equilibrium, thus better resembling the observed dynamics of real-world societies.

B. Heterogeneous agents and the emergence of partial consensus states

We here consider the case of a population of heterogeneous agents, i.e. χ_i may be different for each agent i. As in the previous section, we set $T = 0$, meaning that we neglect the effect of social noise. Such realistic scenarios break the steep transition of the average internal coherence C and allow for the emergence of states of partial consensus in populations of coherent agents. We support this claim by reporting in Fig. 4 both the phase diagrams of the consensus on the second layer $M^{[2]}$ for $\gamma \gg 1$, or equivalently $C = 1$, (top panels) and the plot of C as a function of γ for a typical choice of the external fields ($h^{[1]} = 5$, $h^{[2]} = -3$ specifically) for a few simple but explanatory cases.

We first consider the simplest possible setup where half of the population is assigned $\chi_i = 1/2$, whereas the other one is assigned $\chi_i = 1$ [Fig. 4(b)], meaning respectively that 50% of the population is flexible with respect to internal consensus ($\chi_i = 1/2$) while the remaining agents are intransigent ($\chi_i = 1$). Even if in this case the phase diagram looks similar to the one in Fig. 4(a) for a population of homogeneous agents, we can already observe the emergence of states of partial consensus close to the diagonal, i.e., for $|h^{[1]}|, |h^{[2]}| > 2.5$. The breaking of the steep transition in C is also confirmed in the bottom panel of Fig. 4(b).

We then consider in Fig. 4(c) the case of an heterogeneous population with $\chi_i \in U(0, 1)$, i.e., uniformly distributed in the interval $[0, 1]$. In this case, the qualitative behaviour of both $M^{[2]}$ and C is similar to the one shown in Fig. 4(b). However, as expected due to the increase of the level of heterogeneity of the population, the regions of partial consensus are wider and characterized by lower values of $M^{[2]}$ with respect to the previous case.

Thus, we may expect to find even richer phase diagrams and smoother transitions in C with respect to the cases presented before if we further increase the heterogeneity of the population. Indeed, when χ_i is sampled uniformly in $[-1, 1]$, the phase diagram looks qualitatively different: $M^{[2]}$ smoothly increases from -1 to $+1$ for increasing values of $h^{[2]}$ and fixed $h^{[1]}$. Furthermore, the consensus attained in the region $|h^{[2]}| < 2.5$ with $|h^{[1]}| > 2.5$ is significantly smaller than in the other cases. These results suggest that one can smoothly tune the level of consensus on each topic by choosing the relative strength of the media acting on the two layers, and yet obtain states in which the majority of the agents are internally coherent. We also recall that in all the non-homogeneous cases (Fig. 4(b-d)) the system reaches full coherence, but the transition is not sharp. We conclude by highlighting that our model, even if simplified, is nev-

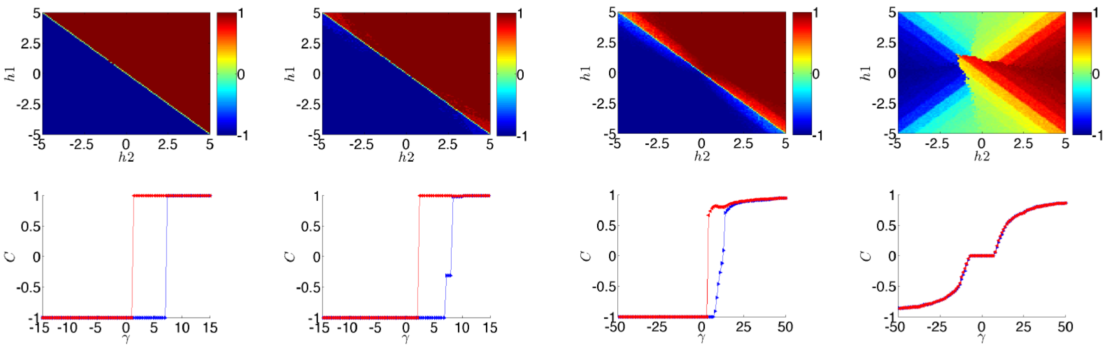

Figure 4. Phase diagrams of $M^{[2]}$ (consensus on the second layer) with respect to the external fields (top panels) for a population of coherent agents, i.e., for γ large enough to have average coherence $C = 1$ ($\gamma = 15$ for panels (a-b) and $\gamma = 50$ for panels (c-d)), and the corresponding transition of C (bottom panels) as a function of γ (forward/backward branch of the hysteresis cycle respectively for blue/red lines) for $h^{[1]} = 5$, $h^{[2]} = -3$. The inter-layer coupling χ_i for each panel is sampled from different distributions, namely (a) $\chi_1 = 1\ \forall i$, (b) half of the agents with $\chi_i = 1$ and the remaining half with $\chi_i = 1/2$, (c) χ_i uniformly distributed in $[0, 1]$, and (d) χ_i uniformly distributed in $[-1, 1]$. We recall that in this regime, where $C = 1$, we have $M^{[1]} = M^{[2]}$.

ertheless able to generate non trivial states of partial consensus across the layers due to the driving effect of mass media, while at the same time ensuring that each agent will still find itself coherent.

C. The effect of social noise

We here consider the case with social noise, i.e. $T > 0$. For simplicity, we investigate its effect in a population of homogeneous agents ($\chi_i = 1\ \forall i$). We find that the system exhibits the same qualitative behaviour described in the case $T = 0$ for all temperatures below a non-null critical temperature T_c, whereas for $T \geq T_c$ it does have absorbing states for finite values of γ, thus lying in a paramagnetic phase dominated by noise.

This is shown in Fig. 5(a) where we plot C as a function of both T and γ (forward branch of the hysteresis cycle) for an exemplary choice of the external fields $h^{[1]}$, $h^{[2]}$ with opposite signs. We note that different values of $|h^{[1]}|$, $|h^{[2]}|$ do not change qualitatively the results presented. Indeed, for $T < T_c$ the system exhibits steep transitions to states of full coherence and consensus. However, when T increases, i.e. as the noise becomes stronger, the jump of the transition becomes less pronounced and the hysteresis cycle shrinks considerably, eventually disappearing at $T = T_c$. For $T > T_c$ only states of partial coherence and consensus can be obtained, and $|C| \simeq 0$ for $T \gg T_c$. Only in the limit $\gamma \to \infty$, the population is able to recover full coherence.

This scenario is confirmed by Fig. 5(b), where we report projections of the phase diagram of Fig. 5(a) for different exemplary values of T. For $T = 0$ the hysteresis cycle is wide and the jump in C goes from -1 to 1. For $T = 10$, slightly below T_c, the hysteresis cycle has almost disappeared and the jump in C is consistently reduced, though still present. For $T = 20$, i.e. beyond the critical level of noise, the transition in C becomes continuous and the hysteresis loop disappears. We note that the noise similarly affects the system in the case of a population of heterogeneous agents, such that a paramagnetic phase appears beyond T_c also in this case. Furthermore, we stress that T_c depends non trivially on the set of parameters of the system. However, deriving such functional relation is beyond the scope of the present work.

We conclude by recalling that opinion evolution in real social systems is inevitably affected by noise as already suggested by recent works on the subject (see for instance [44]). In this section, we have shown that the behavior of the system for $T = 0$ does not change qualitatively in the presence of noise below some critical value T_c for both a population of homogeneous agents and one of heterogeneous agents. This ultimately suggests that our finding that heterogeneity is necessary in population of coherent agents in order to exhibit realistic states of partial consensus, found for noise-free setups of our model, may still be relevant for real social systems.

IV. DISCUSSION

Understanding the elementary mechanisms responsible for the emergence of consensus in social systems is a fascinating problem that has stimulated research in several different fields, from sociology to mathematics, from computer science to theoretical physics, for more than a

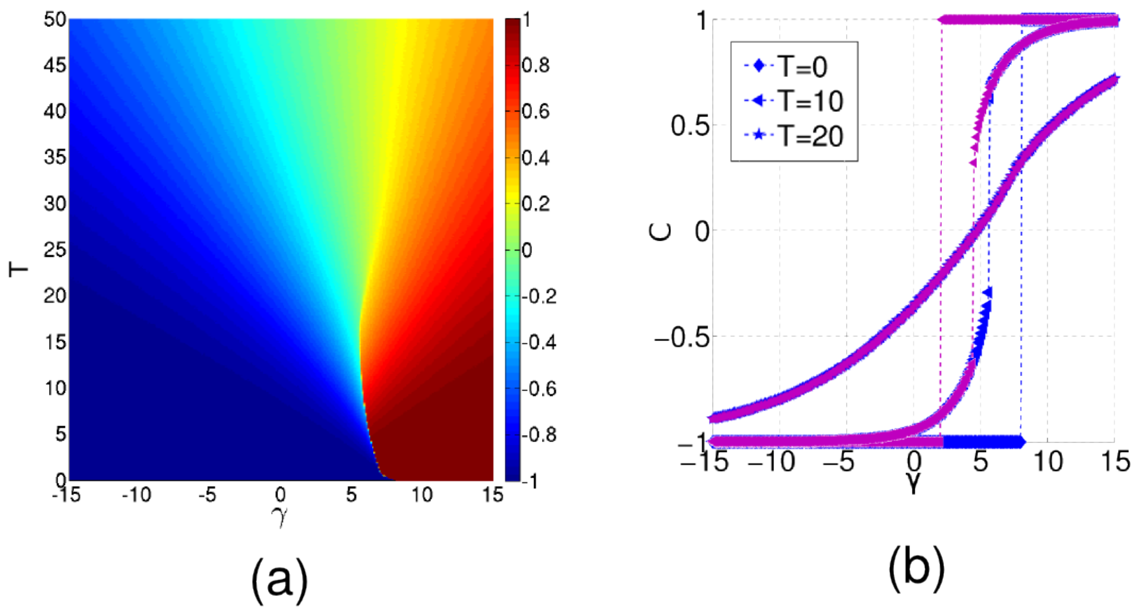

Figure 5. (a) Average internal coherence C as a function of the noise T and of the global parameter γ for $h^{[1]} = 10$, $h^{[2]} = -5$. Simulations are obtained by increasing γ adiabatically for fixed T (forward branches shown). There exists a critical value T_c of T below which the system always attains full coherence ($C = 1$) and consensus ($M^{[1]} = M^{[2]} = +1$) through sharp transitions (consensus not shown). For $T > T_c$ instead the noise becomes dominant and the agents remain incoherent for any γ finite. Eventually for $\gamma \to \infty$ full coherence is obtained via a smooth transition. (b) Projections of C for both increasing and decreasing values of γ (dark/light color respectively) for three different values of T (symbols). The transition between coherent and incoherent states smoothens as T increases, eventually becoming continuous when $T > T_c \approx 10$.

few decades. Nevertheless, traditional models used in the field to describe such systems are still far from capturing the essence of the dynamics of real societies.

Indeed, these models of opinion formation overall underestimate the importance of both (i) the existence of many different contexts where social dynamics may develop, and (ii) the variety of interaction patterns that naturally forms between individuals at each of these different aspects. In details, these models are usually based on the simplifying assumption that the social interactions underpinning consensus are essentially homogeneous, whereas real-world societies are instead intrinsically multilayered and multifaceted, meaning that individuals normally interact with several different neighbourhoods in a number of different yet correlated contexts. Such multilayered structure of social interactions also naturally imply that relationships among each individuals' opinions on many different topics or subjects may exists, thus playing a major role in the formation of an agent's public profile. However, this issue has rarely been addressed in the literature to our knowledge. Overall, these properties of real social systems, force agents to pursue a balanced trade-off between their internal tendency towards providing a coherent image of themselves, corresponding to a coherent set of opinions over the range of contexts in which their social activities develop, and the external pressure towards local homogenization that comes from their concurrent participation to different social circles.

In this work we address the issues (i-ii) thoroughly, and propose a novel, yet simple, model of social opinion dynamics which is capable to account for them all. Our model is obtained by suitably readapting the framework of multilayer networks, which has been developed in the last years in different contexts. Remarkably, the proposed model suggests that the delicate equilibrium between internal agent coherence and responsiveness to external social pressure in a multilayered social environment might indeed be one of the fundamental ingredients responsible for the appearance of non-trivial consensus patterns, such as states of partial consensus emerging from a population of coherent agents. Despite being straightforward in its formulation and relying on rather simple assumptions, the model we proposed allows to take appropriately into account the interplay between each agent's tendency towards coherence, the neighborhood's tendency towards local consensus and the pulling external forces represented by the persistent action of mass media. One of the most interesting findings of the present work is that the introduction of mild heterogeneity in the agents' response to social pressure fosters the emergence of non-trivial states in which internal agent's coherence is always reached at the expenses of a lower level of global consensus. This picture is consistent with what is widely observed in structured societies [45], where a perfect global consensus is never stable while individuals tend to adhere to pre-defined sets of social values which they consider coherent.

Another remarkable effect reproduced by our model is the impact of mass media pressure, especially in the case where the population is heterogeneous. In particular, it is interesting to observe that by an appropriate tuning of the relative strength of the two external fields representing mass media one can indeed set any desired value of consensus on each layer, with the possibility of driving the population from incoherent to more coherent configurations in a continuous way. Finally, the results of the study of the role played by the presence of noise are compatible with real-world scenarios, in which incomplete or inaccurate information about the state of peers is the norm and not an exception.

We highlight that the model discussed in this work is limited to a specific setting, where both the social and mass-media pressure are considered only as a mean-field effect. These assumptions imply that the response of agents to both external fields and interactions with his/her neighbors is homogeneous, which is only a first-order approximation of the real effects of mass media and social pressure on a population of agents. A more realistic approach would require to consider each agent's adaptive response to such influence, i.e., by both considering that the effect of external field on layer α on each node i is a random variable $h_i^{[\alpha]}$ drawn from a certain distribution, and considering an agent-dependent response to interactions with other individuals, i.e. by replacing J with an agent-dependent parameter J_i. However, we purposedly decided to leave the investigation of these generalizations to a future work.

In conclusion, we find it quite intriguing that by taking into account the presence of concurrent interactions on a variety of different topics we were able to provide a simple explanation for the formation of growing patterns of consensus, whose level appears to be dependent on the strength of mass media pressure, as long as the agents acknowledge different couplings between their opinions on the different topics. We believe that the results presented in this work will spur further research towards a better understanding of the implications of interconnected and multilayered interaction patterns on the spreading of opinions and emergence of consensus in real-world social systems.

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