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DOUBLE BUBBLES IN THE 3-TORUS

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Abstract. We present a conjecture, based on computational results, on the area minimizing way to enclose and separate two arbitrary volumes in the flat cubic three-torus $\mathbf{T}^3$. For comparable small volumes, we prove that an area minimizing double bubble in $\mathbf{T}^3$ is the standard double bubble from $\mathbf{R}^3$.

1. Introduction

Our Central Conjecture 2.1 states that the ten different types of two-volume enclosures pictured in Figure 1 comprise the complete set of surface area minimizing *double bubbles* in the flat cubic three-torus $\mathbf{T}^3$. Our numerical results, summarized in Figure 2, indicate the volumes for which we conjecture that each type of double bubble minimizes surface area. Our main theorem, Theorem 4.1, states that given any fixed ratio of volumes, for small volumes, the minimizer is the standard double bubble. This result applies to any smooth flat Riemannian manifold of dimension three or four with compact quotient by its isometry group.

The *double bubble problem* is a two-volume generalization of the famous isoperimetric problem. The isoperimetric problem seeks the least-area way to enclose a single region of prescribed volume. About 200 BC, Zenodorus argued that a circle is the least-perimeter enclosure of prescribed area in the plane (see [8]). In 1884, Schwarz [22] proved by symmetrization that a round sphere minimizes perimeter for a given volume in $\mathbf{R}^3$. Isoperimetric problems arise naturally in many areas of modern mathematics. Ros [21] provides a beautiful survey.

Soap bubble clusters seek the least-area (least-energy) way to enclose and separate several given volumes. Bubble clusters have served as models for engineers, architects, and material scientists (the chapter by Emmer in [6] is a nice survey of architectual applications, and the text by Weaire and Hutzler [26], an introduction to the physics of space-filling bubble clusters or *foams*, discusses numerous other applications).

Existence and regularity. The surface area minimizing property of bubble clusters can be codified mathematically in various useful ways, using the rectifiable currents, varifolds, or $(\mathbf{M}, \epsilon, \delta)$-minimal sets of geometric measure theory (see [13]). In three dimensions, mathematically idealized bubble clusters consist of constant-mean-curvature surfaces meeting smoothly in threes at 120° along smooth curves, which meet in fours at a fixed angle of approximately 109° ([24, Theorems II.4,

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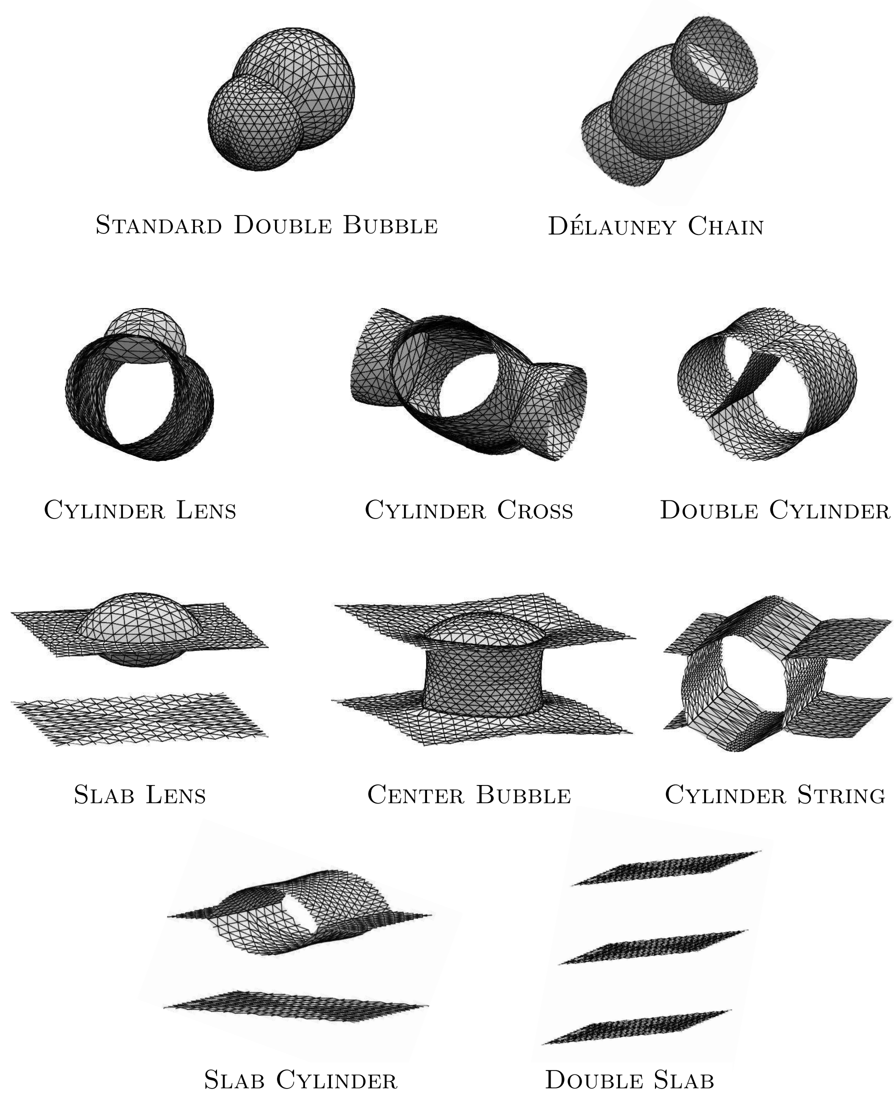

Figure 1. Catalog of Conjectured Minimizers.

IV.5, IV.8], or see [13, Section 13.9]).

Recent results. The existence and regularity of solutions to the double bubble problem played a key role in the proof by Hutchings, Morgan, Ritoré, and Ros ([11], [10]; see [13, Chapter 14]) that the standard double bubble, the familiar shape consisting of three spherical caps meeting one another at 120 degree angles, provides the least-area way to separate two volumes in $\mathbf{R}^3$. The proof relied on a component bound that had been developed by Hutchings [9, Theorem 4.2], and a detailed stability argument to rule out the possible remaining candidates. Reichardt *et al.* [18] extended these results to $\mathbf{R}^4$.

Contemporaneously with the development of results presented here, Corneli, Holt, Leger, and Schoenfeld [5] produced a more or less complete solution to the $\mathbf{T}^2$ version of the problem. Their proofs also rely on regularity theory, which in two dimensions implies that bubbles are bounded by circular arcs. A variational bound on the number of components of bubble clusters in surfaces due to Wichiramala [17] provided considerable added simplifications. A sequence of proofs using the techniques of plane geometry then eliminated all but five candidates (of which only four are expected to appear). Examining the $\mathbf{T}^2$ candidates was helpful to us in our work on the $\mathbf{T}^3$ project.

Bubbles in the three-torus. In comparison with $\mathbf{R}^3$ and $\mathbf{T}^2$, the double bubble problem in $\mathbf{T}^3$ appears to be more difficult. In the torus it is not possible to push through a component bound like Hutchings', since a key step in his proof is to show that the double bubble has an axial symmetry. Nor is a variational bound after Wichiramala forthcoming, due to the additional topological complications in three dimensions. Until some new approach provides a component bound, there probably will be no definitive results. Indeed, the single bubble for the three-torus is not yet completely understood, although there are partial results. The smallest enclosure of half of the volume of the torus was shown by Barthe and Maurey [2, Section 3] to be given by two parallel two-tori. Morgan and Johnson [15, Theorem 4.4] show that the least-area enclosure of a small volume is a sphere. Spheres, tubes around geodesics, and pairs of parallel two-tori are shown to be the only types of area minimizing enclosures for most tori by the work of Ritoré and Ros ([19, Theorem 4.2], [20]).

Theorem about small volumes. Theorem 4.1 states that any sequence of area minimizing double bubbles of decreasing volume and fixed volume ratio has a tail consisting of standard double bubbles. The central difficulty is to bound the curvature. This accomplished, we show that the bubble lies inside some small ball that lifts to $\mathbf{R}^3$, where a minimizer is known to be standard [11]. The result extends to any flat 3- or 4-manifold with compact quotient by the isometry group.

Our proof goes roughly as follows. From the original sequence of double bubbles we generate a new sequence by rescaling the manifold at each stage so that one of the volumes is always equal to one. We can then apply compactness arguments and area estimates to the rescaled sequence to show that certain subsequences of translates have non-trivial limits. These limits are used to obtain a curvature bound on the original sequence. With such a bound, we can apply monotonicity to conclude that if the volumes are small, the double bubble is contained in a small ball. We conclude that it must be the same as the minimizer in $\mathbf{R}^3$ or $\mathbf{R}^4$, *i.e.* that it must be the standard double bubble by [11] or [18]).

Plan of the Paper. Section 2 reviews the methods leading to our Central Conjecture 2.1 and to Figures 1 and 2. Section 3 surveys some subconjectures. Section 4 focuses on the proof of Theorem 1 on small volumes. Section 5 shifts from the cubic to other tori and discusses other conjectures and candidates, including a "Hexagonal Honeycomb."

Acknowledgments. This paper has its origins in a problem given by Frank Morgan to the audience of his two week course on Geometric Measure Theory and the

Proof of the Double Bubble Conjecture, at the Clay Mathematics Institute (CMI) Summer School on the Global Theory of Minimal Surfaces, held at the Mathematical Sciences Research Institute (MSRI) in June and July of 2001.

We owe our deepest gratitude to Frank Morgan for introducing us to this problem and taking the time to guide us along as we have worked on it, by many helpful discussions and constructive comments. CMI and MSRI are to be thanked for making the Summer School possible, and for providing research and travel support to the authors. We should like to extend our thanks to Joel Hass, David Hoffman, Arthur Jaffe, Antonio Ros, Harold Rosenberg, Richard Schoen, and Michael Wolf, who organized the Summer School. We benefited tremendously from the outstanding lectures and informal discussions, for which more people are owed thanks than we can name here. Three faculty participants, Michael Dorff, Denise Halverson, and Gary Lawlor, of Brigham Young University, are to be singled out and thanked for helping us get our computations started. We thank Manuel Ritoré and John Sullivan for helpful conversations. Other participants who made notable contributions include Baris Coskunuzer, Brian Dean, Dave Futer, Tom Fleming, Jim Hoffman, Jon Hofmann, Paul Holt, Matt Kudzin, Jesse Ratzkin, Eric Schoenfeld, and Jean Steiner.

Yair Kagan of New College of Florida created the picture of the Hexagonal Honeycomb in Figure 4. The authors would like thank him for this contribution.

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2. The Conjecture

Generating Candidates. Many possibilities for double bubbles in the three-torus $\mathbf{T}^3$ were proposed in brainstorming sessions by participants in the Clay/MSRI Summer School. In order to classify the candidates we used the following method. Starting with a standard double bubble, we imagined one of the two volumes growing until the bubble enclosing it wrapped around the torus and encountered an obstruction. Following the principals of regularity for bubble clusters, if a bubble collided with itself we opened the walls up, whereas if two different bubbles collided we allowed them to stick together. We then repeated this procedure for these new double bubbles, sometimes changing the perspective slightly slightly or becoming a bit more fanciful (e.g. the Center Cylinder of Figure 3 or Gary Lawlor's Fire Hydrant of Figure 3, also known as "Scary Gary").

Producing the Phase Diagram. Brakke's *Surface Evolver* [3] was used to closely approximate the minimal area that a double bubble of each type needs to enclose specified volumes. Our initial simulations gave us the approximate surface area for each candidate double bubble, tested on partitions $v_1 : v_2 : v_3$ of a unit volume taken in increments of 0.01. From the data obtained in these simulations, we found the least-area competitor for each volume triple. Figure 1 shows the candidates we found to be minimizing for some set of volumes. The phase diagram appearing in Figure 2 is the result of refining our initial computations along the boundaries with a 0.005 increment.

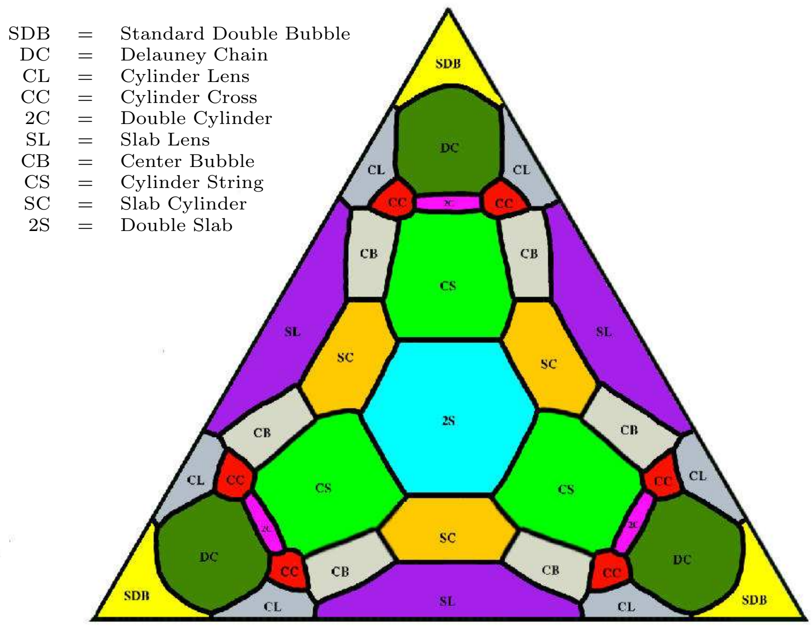

FIGURE 2. Phase portrait: volumes and corresponding double bubble. In the center both regions and the complement have one third of the total volume; along the edges one volume is small; in the corners two volumes are small.

Central Conjecture 2.1. *The ten double bubbles pictured in Figure 1 represent each type of surface area minimizing two-volume enclosure in a flat, cubic three-torus, and these types are minimizing for the volumes illustrated in Figure 2.*

Comments. One might expect that minimizers would be found among the various regularity-satisfying conglomerations of topological spheres and products of spheres and homotopically non-trivial tori, other possibilities being excessively complex. This was borne out in our computations. It is interesting to note, however, that not all of the simple possibilities along appeared as minimizers, for example the Transverse Cylinders pictured in Figure 3. The various double bubbles of Figure 3, while stable for a certain range of volumes, are never area minimizing. A challenging unsolved problem is to find all of the stable non-minimizing bubbles. Note that a given type from Figure 1 might be stable for a much wider range of volumes than those for which it actually minimizes surface area.

It is worth observing that all of the conjectured minimizers for the double bubble problem on $\mathbf{T}^2$ (a Standard Double Bubble, Band with Lens, Symmetric Chain, and Double Band) are echoed here in at least two ways. The Double Cylinder, the Slab Cylinder, the Double Slab, and the Cylinder String are $\mathbf{T}^2$ minimizers $\times$ $\mathbf{T}^1$. There are also more direct analogues, as is seen by comparing, for example, the three- and two-dimensional Standard Double Bubbles, or the Délauney and Symmetric Chains. See Corneli *et al.* [5] for more on the T^2 minimizers.

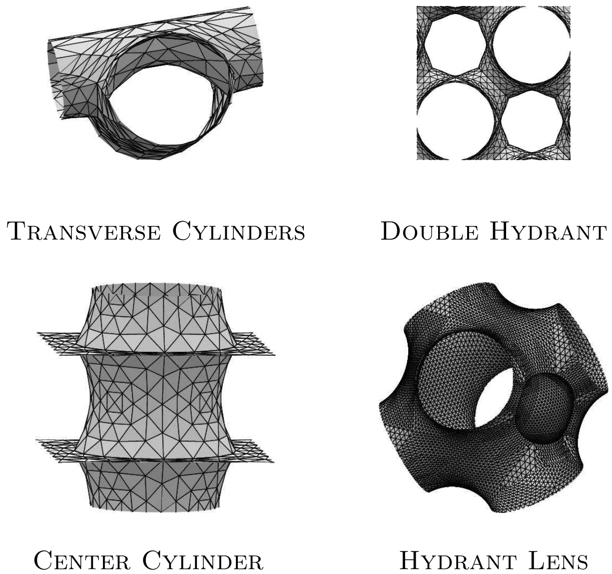

Figure 3. Inefficient double bubbles

3. Subconjectures

We now present a list of natural subconjectures, suggested either by the phase diagram or by examination of the pictures made with *Surface Evolver*.

One immediate observation is that the edges of our phase diagram appear to characterize single bubbles in $\mathbf{T}^3$.

Conjecture 3.1 (Ritoré and Ros [19], [20], [21])**.** *The optima for the isoperimetric problem in a cubic $\boldsymbol{T}^3$ are the sphere, cylinder, and slab.*

We still do not know proofs for the following intuitive conjectures about the double bubble problem (although Theorem 4.1 gives partial results on Conjecture 3.3):

Conjecture 3.2. *An area-minimizing double bubble in $\boldsymbol{T}^3$ has connected regions and complement.*

Conjecture 3.3. *For two small volumes the standard double bubble is optimal.*

Conjecture 3.4. *For one very small volume and two moderate volumes, the Slab Lens is optimal.*

Conjecture 3.5. *The first phase transition as small equal, or close to equal, volumes grow is from the standard double bubble to a chain of two bubbles bounded by Délaunay surfaces.*

Délaunay surfaces are constant-curvature surfaces of rotation, and as such have full rotational symmetry. From the *Surface Evolver* pictures, it appears that the conjectured surface area minimizers always have the maximial symmetry, given the constraints , a fact which leads to the following natural conjecture.

Conjecture 3.6. *A minimizer is as symmetric as possible, given its topological type.*

We will conclude this section with a proposition which represents the first step towards establishing a symmetry property. Specifically, we prove that for any double bubble, there is a pair of parallel planes (actually two-tori) that cut both regions in half. Hutchings [9, Theorem 2.6] used the fact that in $\mathbf{R}^n$ a double bubble has two perpendicular planes that divide both regions in half. This is an easy step in a difficult proof that the function that gives the least-area to enclose two given volumes is concave. Before proving a basic result of a similarly elementary flavor for the torus, we mention that we conjecture that the much deeper concavity result also holds:

Conjecture 3.7. *The least area to enclose and separate two given volumes in the three-torus is a concave function of the volumes.*

If one could prove Conjecture 3.7, one would then be able to apply other ideas in Hutchings' paper [9, Section 4] to obtain a functional bound on the number of components of a minimizing bubble.

Proposition 3.8 (Deluxe Ham Sandwich Theorem*)**.** *In a rectangular torus, if a double bubble lies inside a cylinder $S^1 \times D^2$, then there is a plane that cuts both volumes in half.*

Proof. This is a generalization of the standard argument for the ham sandwich theorem in Euclidean space. Assume that the identification occurs in the vertical direction. Take a circle in the xy plane indexed by $\alpha \in [0, \pi]$. Then for each α, rotate the surface by α in the xy plane and consider the family of pairs of flat tori parallel to the xz plane that are at distance $1/2$ from each other. Then there is at least one such pair of planes that cuts the volume V_1 in half, for each α. There may be an interval of such pairs for a given α. However, some one parameter family of these planes may be chosen which varies continuously as a function of α, and one such plane cuts V_2 in half. □

4. Small Volumes

Conjecture 3.3 stated that small volumes are best enclosed by a standard double bubble. Theorem 4.1 proves that for any fixed volume ratio, the standard double becomes optimal when the volumes are sufficiently small. This result holds for many flat three- and four-dimensional manifolds (and see Remark 4.4).

The standard double bubble consists of three spherical caps meeting at 120 degrees. (If the volumes are equal, the middle surface is planar.) It is known to be minimizing in $\mathbf{R}^3$ [11] and $\mathbf{R}^4$ [18].

Theorem 4.1. *Let M be a smooth flat Riemannian manifold of dimension three or four, such that M has compact quotient by the isometry group. Fix $\lambda \in (0, 1]$. Then there is an $\epsilon > 0$, such that if $0 < v < \epsilon$, a perimeter-minimizing double bubble of volumes $v, \lambda v$ is standard.*

Remark 4.2. Solutions to the double bubble problem exist for all volume pairs in manifolds with compact quotient by their isometry group. The proof is the same as in Morgan [13, Section 13.7].

For the proof we will regard a double bubble as a pair of 3- or 4−dimensional rectifiable currents, R_1 and R_2, each of multiplicity one, of volumes $V_1 = \mathbf{M}(R_1)$ and $V_2 = \mathbf{M}(R_2)$. The total perimeter of such a double bubble is $\frac{1}{2}(\mathbf{M}(\partial R_1) + \mathbf{M}(\partial R_2) + \mathbf{M}(\partial(R_1 + R_2)))$. Here $\mathbf{M}$ denotes the mass of the current, which can be thought of as the Hausdorff measure of the associated rectifiable set (counting multiplicities). For a review of the pertinent definitions, see notes from Morgan's course at the Summer School [16] or the texts by Morgan [13] or Federer [7].

Before proceeding it is helpful to fix some notation. By the Nash and subsequent embedding theorems, we may assume M is a submanifold of some fixed $\mathbf{R}^N$. We will consider a sequence of perimeter-minimizing double bubbles in M, containing the volumes v and λv, as $v \to 0$. For each v, M_v will denote $s_v(M)$ in $s_v(\mathbf{R}^N)$, where s_v is the scaling map that takes regions with volume v to similar regions with volume 1. In particular, s_v maps our perimeter-minimizing double bubble containing volumes v and λv to a double bubble which we call S_v which contains volumes 1 and λ, and which is of course perimeter-minimizing for these volumes.

We focus on the case of dimension three; the proof for dimension four is essentially identical.

Lemma 4.3. *There is a $\gamma > 0$ such that if R is a region in an open Euclidean 3-cube K and* $\mathrm{vol}(R) \leq \mathrm{vol}(K)/2$*, then*

$$\mathrm{area}(\partial R) \geq \gamma(\mathrm{vol}(R))^{2/3}.$$

Proof. Let γ_0 be such an isoperimetric constant for a cubical 3-torus, so that $\mathrm{area}(\partial P) \geq \gamma_0(\mathrm{vol}(P))^{2/3}$ for all regions $P \subseteq \mathbf{T}^3$. Such a γ_0 exists by the isoperimetric inequality for compact manifolds [13, Section 12.3]. Make the necessary reflections and identifications of the cube K to obtain a torus containing a region R' with eight times the volume and eight times the boundary area of R. The claim follows, with $\gamma = \gamma_0/2$. □

Proof of Theorem 4.1. The first step is to show that the sequence S_v, suitably translated and rotated, has a subsequence that converges as $v \to 0$ and has $V_1 \neq 0$ in the limit. Our argument also shows that there is a subsequence that converges to a limit with $V_2 \neq 0$, but does not show that there is a subsequence where both volumes are non-zero in the limit. This is because while we are exerting ourselves trapping the first volume in a ball, the second one may wander off to infinity.

We first show the existence of a covering $\mathscr{K}_v$ of M_v with bounded multiplicity, consisting of cubes contained in M_v, each of side-length L. Lemma 1 will give us a positive lower bound on the volume of the part of R_1 that is inside one of these cubes for each S_v. We will then apply a standard compactness theorem to show that a subsequence of the S_v's converges.

Take a maximal packing by balls of radius $\frac{1}{4}L$. Enlargements of radius $\frac{1}{2}L$ cover M_v. Circumscribed cubes of edge-length L provide the desired covering $\mathscr{K}_v$. To see that the multiplicity of this covering is bounded, consider a point $p \in M_v$. The ball centered at p with radius $2L$ contains all the cubes that might cover it, and the number of balls of radius $\frac{1}{4}L$ that can pack into this ball is bounded, implying that the multiplicity of $\mathscr{K}_v$ is also bounded by some $m > 0$.

Now consider some such covering, with $L = 2$. By Lemma 1, there is an isoperimetric constant γ such that

$$\mathrm{area}(\partial(R_1 \cap K_i)) \geq \gamma(\mathrm{vol}(R_1 \cap K_i))^{2/3},$$

and therefore, since $\max_k \operatorname{vol}(R_1 \cap K_k) \geq \operatorname{vol}(R_1 \cap K_i)$ for any i,

$$\operatorname{area}(\partial(R_1 \cap K_i)) \geq \gamma \frac{\operatorname{vol}(R_1 \cap K_i)}{(\max_k \operatorname{vol}(R_1 \cap K_k))^{1/3}}. \tag{4.1}$$

Note that the total area of the surface is greater than $1/m$ times the sum of the areas in each cube, and the total volume enclosed is less than the sum of the volumes, so summing Equation 4.1 over all the cubes K_i in the covering $\mathscr{K}_v$ yields

$$\operatorname{area}(S_v) \geq \operatorname{area}(\partial R_1) \geq m\gamma \frac{V_1}{(\max_k \operatorname{vol}(R_1 \cap K_k))^{1/3}}$$

and

$$(\max_k(\operatorname{vol}(R_1 \cap K_k)))^{1/3} \geq m\gamma \frac{V_1}{\operatorname{area}(S_v)} \geq \delta > 0,$$

because $V_1 = 1$ and it is easy to show that $\operatorname{area}(S_v)$ is bounded (since there is a bounded way of enclosing the volumes). Translate each M_v so that the cube where the maximum occurs is centered at the origin of $\mathbf{R}^N$, and rotate so that the tangent space of each M_v at the origin is equal to a fixed $\mathbf{R}^3$ in $\mathbf{R}^N$. The limit of the M_v will be equal to this $\mathbf{R}^3$. Since a cube with edge-length L centered at the origin fits inside a ball of radius $2L$ centered at the origin, we have

$$\operatorname{vol}(R_1 \cap B(0, 2L)) \geq \delta^3$$

for every S_v. By the compactness theorem for locally integral currents ([13, pp. 64,88], [23, Section 27.3, 31.2, 31.3]) we know that a subsequence of the S_v has a limit, which we will call D, with the property that $\operatorname{vol}(R_1) \geq \delta^3$. This completes the first step.

Since D is contained in the limit of the M_v, namely, the copy of $\mathbf{R}^3$ chosen above, and each S_v is minimizing for its volumes, a standard argument shows that the limit D is the perimeter-minimizing way to enclose and separate the given volumes in $\mathbf{R}^3$ (cf. [13, 13.7]). In the limit, V_2 could be zero, in which case D is a round sphere. If both volumes are non-zero, D is the standard double bubble ([11] and [18], or see [13, Chapter 14]).

Our goal is now to prove that the double bubble of volumes v, λv lies inside a trivial ball, when v is small enough. This can be accomplished using the monotonicity theorem for mass ratio [1, Section 5.1(1)], which implies that for a perimeter-minimizing bubble cluster, a small ball around any point on the surface contains some substantial amount of area. This will limit the number of disjoint balls we can place on the surface. The monotonicity theorem applies only to surfaces for which the mean curvature is bounded, *i.e.*, for which there is a C such that for smooth variations

$$\frac{dA}{dV} \leq C.$$

Accordingly, the second major step in the proof is to obtain such bound on the curvature, for v small. It suffices to show that all smooth variation vector fields have the property that changes in the volume of S_v and in the area of ∂S_v are controlled. Take a smooth variation vector field F in $\mathbf{R}^n$ such that for D,

$$dV_1/dt = \int_{\partial R1} (F \cdot n)\, dA = c > 0$$

and

$$dV_2/dt = 0.$$

(Note that we need the first volume to be non-zero, or its variation could be zero.) For v small enough the subsequence of S_v headed towards the limit D has the property that dV_1/dt is approximately c and dV_2/dt is approximately 0.

By the argument in the first step, we can translate each S_v similarly, so that a subsequence of the subsequence above converges to a minimizer D' in $\mathbf{R}^3$ where the second volume is non-trivial. This time we take a smooth variation vector field F', such that for v small enough the subsequence of S_v headed to D' has the property that dV_1/dt is approximately 0 and dV_2/dt is approximately $c' > 0$. This proves that for this subsequence the change in volume is bounded below.

Now we need to show that the change in area is bounded above. This follows from the fact that every rectifiable set can be thought of as a varifold [13, Section 11.2]. By compactness for varifolds [1, Section 6], the S_v, situated so that the first volume does not disappear, converge as varifolds to some varifold, J. The first variation of the varifolds also converge, *i.e.*, $\delta S_v \to \delta J$, see [1]. The first variation of a varifold is a function representing the change in area. Therefore, far enough out in the sequence the change in area of the S_v under F is bounded close to the change in area of J under F, which is finite. Similarly, the change in area of the S_v under F' is bounded.

We conclude that the mean curvature of S_v is bounded for two independent directions in the two-dimensional space of volume variations, and hence for all variations. This completes the second step.

The third and final step is to show that all of the surface area is contained in some ball in M_v, of fixed radius for all v. Eventually, as v shrinks and M_v grows, this ball will have to be trivial in M_v. We will then use the result that the optimal double bubble in $\mathbf{R}^3$ is standard to show that our double bubble is standard as well.

As the S_v are approaching the minimizer in $\mathbf{R}^3$, there must be a bound A on the perimeters of the S_v. By monotonicity of mass ratio [1, Section 5.1(1)], every unit ball centered at a point of S_v contains perimeter $\delta > 0$. Therefore there are at most A/δ such disjoint balls.

We claim that the S_v are eventually connected. There is an upper bound on the diameter of any component, $2A/\delta$. Since we are controlling curvature, our components cannot become too small. Since every unit ball contains at least δ area, we also have a lower bound on the area of each component, when unit balls are trivial. Unless eventually the S_v are connected, you can arrange to get in the limit a disconnected minimizer in $\mathbf{R}^3$, a contradiction.

Hence S_v is contained in a ball of radius $2A/\delta$ for all v. Since our original manifold has compact quotient by its isometry group, there is a radius such that balls in the original manifold of that radius or smaller are topologically trivial. Hence, as we expand the manifold, eventually balls of radius $2A/\delta$ can be lifted to $\mathbf{R}^3$, which means that they are Euclidean. Hence, S_v is eventually contained in a Euclidean ball, and is therefore the standard double bubble ([18], [11], [13, Chapter 14]).

Finally, since $S_v \subset M_v$ is simply a scaled version of the original double bubble in M, we conclude that the original double bubble is standard as desired. $\square$

Remark 4.4. Given n, m, similar arguments show that for any smooth n-dimensional Riemannian manifold with compact quotient by the isometry group, given $0 \leq \lambda \leq$

1, there are C, $\epsilon > 0$, such that for any $0 < v < \epsilon$, a minimizing cluster with m prescribed volumes between λv and v lies inside a ball of diameter at most $Cv^{1/n}$.

To further deduce that the cluster smoothly approximates a Euclidean minimizer would require knowing that convergence weakly and in measure, under bounded mean curvature, implies C^1 convergence, as is known for hypersurfaces without singularities ([1, Section 8], see [14, Section 1.2])

5. Special Tori

Changing the shape of the torus, by stretching it or by skewing some or all of its angles, would certainly change the phase diagram of Figure 2:

Conjecture 5.1. *In the special case of a very long T^3 the Double Slab is optimal for most volumes.*

Special tori may have special minimizers:

Conjecture 5.2. *For the special case of a torus based on a relatively short $\frac{\pi}{3}$-rhombic right prism, the Hexagonal Honeycomb prism of Figure 4 is a perimeter-minimizing double bubble for which both regions and the exterior each have one third the volume, or when two volumes are equal and the third is close.*

Indeed, for such volumes the Hexagonal Honeycomb ties the Double Slab, just as in the $\frac{\pi}{3}$-rhombic two-torus a Hexagonal Tiling ties the Double Band [5].

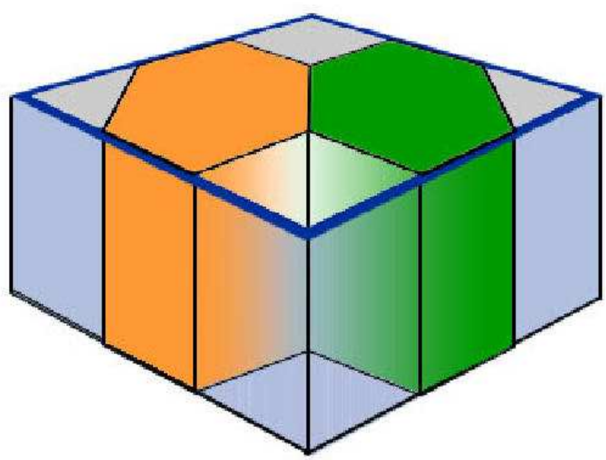

Figure 4. Hexagonal Honeycomb

In the *triple bubble problem* for the Face Centered Cubic (FCC) and Body Centered Cubic (BCC) tori we would expect to find minimizers that lift to $\mathbf{R}^3$ as periodic foams with cells of finite volume. The Weaire-Phelan foam, a counterexample to Kelvin's conjectured best way to divide space into unit volumes ([25], [12]; see [13, Chapter 15], [4]), would be expected to appear as lifts of some solutions to the triple bubble problem in the BCC torus. Kelvin's foam might appear as a solution to the triple bubble problem in the FCC torus. One might also see *eight* Kelvin cells in the BCC torus, or *sixteen* Kelvin cells in the standard cubic torus.

Eight Weaire-Phelan cells also fit in the standard cubic torus. Furthermore, by scaling, higher numbers of cells fit into these tori.

In contrast with the triple bubble problem and problems with more volume constraints, it is extremely unlikely that a solution to the double bubble problem in any torus would lift to a division of $\mathbf{R}^3$ into finite volumes, since by regularity, singular curves meet in fours. This means that singular points look locally like the cone over a tetrahedral frame, and hence have four volumes coming together. Since a region will never be adjacent to another component of the same region (because the dividing wall could be removed to decrease area and maintain volumes), a foam generated using a fundamental domain coming from a double double in the three-torus would have to exhibit the strange property that the singular curves never meet.

Conjecture 5.3. *There are no least-area divisions of $\boldsymbol{T}^3$ into three volumes that lift to a foam in $\boldsymbol{R}^3$.*

This conjecture suggests that it is not likely that there will be are any other special minimizers for the double bubble problem.

Conjecture 5.4. *The double bubbles of Figure 1 together with the Hexagonal Honeycomb of Figure 4 comprise the complete set of area minimizing double bubbles for all three tori.*

As a final remark, in light of the fact that the triple bubble problem in the torus seems likely to produce so many interesting candidates, we would like to mention one final conjecture.

Conjecture 5.5. *For the triple bubble problem in a cubic $\boldsymbol{T}^3$ in the case where one of the volumes is small, the minimizers will look like the double bubbles of Figure 2 with a small ball attached. The phase diagram will look just like our Figure 3.*

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