The self-representing Universe

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THE SELF-REPRESENTING UNIVERSE

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ABSTRACT. We revisit our approach to Fourier and observer-observed duality as the origin of the structure of physical reality. We explain how Fourier duality on the exterior algebra recovers Hodge duality. We describe recent joint work on duality of quantum differential structures on quantum groups and speculate on the interpretation of de Morgan duality and bi-Heyting algebras in this context. We describe our recent work on the algebraic description of a classical Riemannian geometry as a certain type of Batalin-Vilkovisky algebra. Using this we explain how classical Riemannian geometry emerges out of nothing but the Leibniz rule, as a kind of 2-cocycle that governs the possible ways that a noncommutative differential algebra can extend a given classical manifold. This explains how classical Riemannian geometry emerges out of noncommutative algebra under very minimal assumptions and in some sense explains why there is Riemannian geometry in the first place. These matters are discussed within our general philosophy of 'Relative Realism' whereby physical reality is in some sense created by the assumptions of Science.

1. INTRODUCTION

In our philosophy works [15, 2] we have argued for a position which we called 'relative realism'. This takes the view that there both isn't and is such a thing as a hard and solid physical reality governed by the equations of physics and that if this paradox could be properly understood then we would in fact *understand* the origin of the equations of physics. The philosophy comes out of the nature of mathematics and says basically that how we look at the world and our assumptions going into that in some sense 'create' the reality that we perceive. In particular we have argued that scientific or more precisely *physical reality* should be characterised by a kind of meta-equation that picks out its structure out as subset of the tableau of all mathematics. This meta-equation would then be the ultimate theory of physics just with colourful names for the structures involved. Moreover, we have formulated what we believe to be the essence of this meta-equation as the *principle of representation theoretic self-duality*[15]:

> The search for the ultimate theory of physics is the search for a self-dual structure in a self-dual category, i.e. physical reality is characterised as being self-representing.

Key words and phrases. Relative realism, quantum theory, gravity, quantum gravity, noncommutative geometry, quantum groups, representation, Fourier theory, Born reciprocity, Hodge duality, buddhism, Heyting algebra.

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Like many philosophical statements, if you make this too precise you will probably be making it wrong, i.e. one needs to treat this only as the 'principle' and not the detail of what is going on.

In this article I want to explore new concrete mechanisms for how physical reality as we know it 'emerges' from the assumptions of Science, such as[22, 23] how Riemannian geometry emerges out of the axioms of differential calculus and a breakdown of associativity. And if such ideas are indeed how physical reality emerges then what are the moral implications? I will argue as I have in the past[2] that these implications are not unlike the philosophical implications of buddhism.

2. Relative Realism and the nature of Reality

It is hardly a new idea that somehow reality as we perceive it might be an illusion, for example inside someone else's computer programme as we might say in the 21st century, or someone's dream according to some ancient beliefs. But what this really means is that some other being sets the structures within which we find ourselves embedded, be it a computer programme or the dreamer in whose dream we are, which only postpones and does not answer what *their* reality is. Do they have an absolute reality or is there perhaps an infinite hierarchy of dreamers?

Relative realism is a new approach to this conundrum in that there is no need for an external dreamer although there is still a hierarchical structure to physical reality. Instead of a dreamer, we the observer are the source of the 'hardness' of reality through the choices that we consciously or unconsciously make in understanding the world. This hardness is important particularly for the 'hard sciences' because it means that reality is in some sense absolute and it is not the case that everything is arbitrary. On the other hand being dependent on the observer one could also say it is relative to the observer. In that case the question of reality becomes: what choices does the observer have to chose from. In relative realism these possible observer choices are themselves seem as a 'hard reality' but as a fact at a higher level of understanding. The hardness of those choices is in turn sourced from deeper choices that we the observer made, hence the hierarchical structure. In summary:

(1) Reality is a collective phenomenon that we all generate together through choices or assumptions.
(2) The things we generate are not arbitrary but constrained by their own hard reality dictating the possible choices.
(3) There is an absolute reality but it consists of the hierarchical structure of all the structures that we have to chose from.
(4) There *is* a freedom in that we can choose or not choose to make those choices to determine where we are in the hierarchy.

In [15, 2] I have given a number of examples, such as the 'reality' created by the decision to agree and play by the rules of chess. So long as you accept the rules and are playing chess, you can experience the reality and anguish of being checkmated in a game. But you are also free to become aware that it is 'just a game' and then the reality of your checkmate dissolves. It is still the case that chess is a good game for which the rules work well, and that is an element of reality at a higher level whereby there are not that many games to chose from, a kind of meta-reality

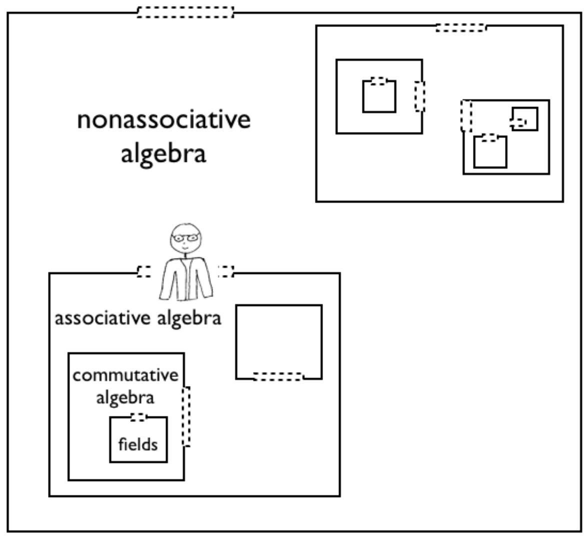

FIGURE 1. Mathematical reality as a series of rooms within rooms and the observer within it.

that you move into when you decide that chess is just a game you did not have to play. Above that there are meta-meta realities and so forth. Also note that this conception of reality is relative specifically to what the observer is conscious of and hence the latter has a central role.

All of this should make sense to a pure mathematician, where 'mathematical reality' is all about writing down axioms or assumptions that create a field of study. At the same time the possible axioms of interest are highly constrained and part of a higher mathematical reality about what interesting axioms you could have chosen. For this reason mathematics is generally hierarchically structured where you move up the hierarchy as you drop axioms or assumptions. On the one hand pure mathematicians enjoy the freedom to make or not make such assumptions so they can move up and down this hierarchy. It also suggests that mathematics is invented and hence arbitrary yet many mathematicians take the view that the 'right axioms' or mathematical structures are an absolute (Platonic) reality 'out there' waiting to be discovered in some cross section or other. So these are the two faces of relative realism. I have proposed in [2] to think of the structure here as a series of 'rooms within rooms' where when you take on some axioms to study a field of mathematics you enter that room. Now at the threshold of a room you can look two ways: facing into the room you see all the elements of reality of that branch of mathematics. Facing out of the room you turn your back on that as a mere product of assumptions and look into and around the bigger room in which the room you began with is just one of many structures.

As the reader may not be a pure mathematician, let me try now to give a more everyday example. Consider a day in this your fictional life. You wake up. You are sleepy and want to stay in bed but are you have to face the hard reality that you have a job that requires you to clock in at 9am every weekday. You reflect on how did your life come to this? Unfortunately, when you were looking around for jobs a few years ago your choices were limited by your decision not to complete high school was well by your interests. And how did *that* come about? You did not complete high school in order to look after your sick parents and as for your interests, they were formed in your childhood and you could hardly help that. So here you are, your life is a product of all the limited choices you made to get here and if you want to pay the mortgage the hard reality is that you have to get up and go to work.

Now of course you *could* chuck it all and think laterally. You could step out of the box and reinvent your life. You could in some sense rewind the clock and make a whole different set of choices. But what is available to you, that's still the 'real world' and where did *that* come from. How did it get to be that society works the way it does and that these are your choices? Why do bankers and footballers get the high salaries and could society be different? Here again there were choices but this time buried in the history and mechanisms of societal development, sometimes shaped by choices of key leaders at the time and perhaps if we believe Tolstoy in *War and Peace* subject to their own immutable laws of history. Could we think laterally and reject all that too? Possibly by setting up on a tropical island but we would be divorced from the rest of society and we would have a different set of limitations. Moreover, the rest of society would still be there and does not disappear in absolute terms so long as many people still think that way.

Could you go further? Most people would assume that there is ultimately a *physical reality* which is not a product of any assumptions or choices and within which the limits to society operate. But what if the process continues and physical reality which provided the backdrop to our tale is also a product of choices and assumptions made now not by society but by the development of language and Science? As we probe into these more fundamental matters we are digging deeper and probably earlier both into the development of our own psychology and into the development of Science. Maybe we do have choices and assumptions but thinking laterally and rejecting these deepest assumptions would require such originality of thought as to be on the edge of sanity. On the other hand it's on this cutting edge that revolutions in Science do take place from time to time. We can sometimes in Science think laterally and completely rethink what is going on. In the context of our tale above we could imagine that we were born on some different planet or if you believe string theorists then some different multiverse with different values for the constants of Nature and perhaps even different rules. Suddenly our philosophy of reality is on the cutting edge of modern physics and given that we do not have a complete theory of physics that reconciles quantum theory and gravity, there really *are* assumptions and choices to be explored and that we perhaps did not have to make.

Now there is one big difference here when we talk about the physical world rather than social realities and this is a kind of convergence phenomenon whereby when there is a paradigm shift the experimental facts are not changed just their interpretation. It is not usually felt by scientists that reality is being changed, only our view and precision of it, but I would argue that this is a different use of 'real' as pertaining just to experimental facts. In fact the reality in any paradigm is an experience that is and interaction between theory (what we think is going on) and so-called fact. In that sense every scientific fact is still within a paradigm or set of choices about this or that structure and is more in the nature of a measurement on that structure. In other words, even 'hard science' depends on theoretical or structural choices. Rejecting those choices does not change the facts any more than going to a tropical island changes the rest of society, but nor are those 'facts' any more real in that they take place within a series of assumptions about how we look at the world.

Since the more fundamental choices are for an individual made earlier in life, in [2] I have taken this to its logical conclusion that our experience of reality can be traced back to the day we first became conscious and the choices we took on. In that case the deepest aspects of the hard Sciences should have their roots in the very earliest assumptions that we, an otherwise clean slate, take on. And the search for the as yet elusive theory of quantum gravity (the still missing foundation of modern physics) should probably require us to think laterally about a very very basic assumption, i.e. about something we have taken on very early in this process. It is impossible for me to say what and when exactly was that assumption since quantum gravity is not known, but for both quantum theory and gravity one can identify key moments in a child's development which we have to wind back to in order to make the step to each of those theories. Here are just a couple of examples, from[2]:

(1) for quantum theory children go through a container phase in which they enjoy putting things into containers. This is their first exposure to Boolean logic or set theory, however there is anecdotal evidence (from my own kids experience) that initially at least kids to not assume that if you put something in a box and then open the box, that it is still there. Similarly children at this age (around 2) can also be comfortable with someone being at two places at the same time. These are elements of quantum logic where the full rules of classical logic are not yet assumed.

(2) for gravity we need geometry and of course children learn the basics of geometry as they learn to walk and play. But they also learn language and in particular the use of relative coordinates or 'pronouns'. Thus 'me' and 'you' are learned in place of absolute third person references just as we eventually learn to describe geometrical points by coordinates relative to a choice of frame rather than simply pointing, which ultimately leads to the modern notion of a classical manifold. For quantum gravity however, it is my own view that we can and should rewind the clock back to an absolute coordinate free (and symbolic) description of geometry.

3. Relative Realism and morality

Although I suspect my mathematician colleagues might not agree, as far as I am concerned this section is about the moral implications of the world view known as Pure Mathematics. Personally I think that scientists of all kinds can and should see their work in the bigger picture and conversely relative realism says that the bigger picture can't be separated from Science.

So, at a mathematics conference on, say, differential geometry, you will generally find two kinds of talks. There will be talks working within the axioms of differential geometry and people arguing about the implications of these. These can even be heated arguments resolved by logical analysis. Then there will be talks about the axioms of differential geometry and their placement within a wider context, i.e. generalisations thereof. From this point of view the reality of differential geometry is a game that you don't have to play. These are the two faces of relative realism. If people don't realise this distinction then arguments can get cross-purpose.

Similarly, I suppose that many disputes, even wars, have at their root assumptions of a particular ideology or set of assumptions and the enforcement of those on other people. In relative realism this should be a matter of individual freedom as to whether a person wants to accept such a world view or adopt a different one. Certainly differential geometers do not war with group theorists about who is 'right', each group is equally entitled to work within its own paradigm. Just if they want to call themselves mathematicians (and they do not have to of course) then they should work within the meta-rules of logic and elegance that govern mathematics. Similarly in a conflict between two parties the first thing to ask should be *are they making the same assumptions?*, i.e. living in the same relative reality and if not then this should be pointed out and the focus shifted 'out of the box' to look at the assumptions themselves. Perhaps one party of another can accept that some of the assumptions it was making were not really essential and could be replaced.

The same thinking can be useful at a personal level. Whenever faced with a difficult choice we can ask if those choices are really the only ones or if we are allowed to think laterally or 'out of the box' and change the underlying premise to the something more suitable. Sometimes the answer is that the 'rules' are being dictated from above but quite often those 'above' had not really thought about it and could be flexible if there was a good reason to be.

We surely are all familiar with such situations but if we believe that all of reality is assumption-based then this should become a way of life or philosophy about the Universe and our place in it. We would both enjoy the restrictions that we have assumed at a certain point in the same way as we may enjoy a game of chess, or we may choose to transcend those restrictions to move up the hierarchy to a more elevated viewpoint. Regarding assumptions that we are not aware of, we may seek to elucidate hidden assumptions to clarify the relative nature of our experience and in so doing we would be gaining understanding of *why* something was true, i.e. a theoretical explanation. This process of becoming self-aware of hidden assumptions is a particular type of revelation and my own vision of the ultimate goal of theoretical physics is to travel so far up this hierarchy as to understand that all of physics has to be the way it is as a consequence of very basic assumptions about the nature of Science. We can even go beyond that and seek to understand why the wider world is the way it is and our place within it, and meanwhile on the way moving with greater equanimity from 'room' to 'room' in our increasingly uncovered hierarchy of assumptions.

Such a philosophy of life is a humanist position that is the opposite of organised religion in the sense that it dictates that each person engages in a personal journey to analyse and transcend their own hidden assumptions. To me this seems closer to how Science should be done but our outlook also has striking similarities to the philosophy of buddhism. The central tenets of buddhism revolving around compassion and transcendence from unnecessary assumptions are core to our philosophy too. The difference is that whereas in buddhism the material world is regarded as a delusory product of assumptions, for us the structure of these assumptions is

FIGURE 2. Was the Buddha a mathematician?

our object of study and *is* the 'real world' that we seek to understand. So it's a different spin but the mechanism of perceived reality as a product of assumptions that we can and should be able to transcend through analysis or reflection is the same. In that sense I would say that the Buddha was inspired by the same thing that inspires mathematics itself.

4. From Relative Realism to Self-Duality

Here I will recap my ideas from [15, 2] about self-duality but let me stress that you do not need to accept this in order to accept the wider vision above. It's merely an attempt to formulate a key assumption but is not necessarily the whole picture.

The idea is that, within the general scheme of relative realism, the specific axiom that characterises Science is the idea of experimental measurement. Something is hypothecated to be true and an experiment measures or illustrates this. The outcomes of the measurements are the 'real' things that are independent of any theory in the sense that theories come and go but the outcomes of experiments are what they are within their error bars. Newtonian mechanics is nowadays seen to be 'wrong' but it lives on within its domain of applicability in that its predictions are amply verified in everyday life. Those facts have not changed just because our point of view has. Finally, theory is typically the positing of some kind of abstract structure and the notion of experimental measurement is some kind representation of it. By representation we mean loosely some kind of realisation of the abstract structure in something that is more concrete and self-evident.

Now let X denote some abstract mathematical structure, such as the circle group. Now for any mathematical structure there is usually some notion of representation f of that structure in something more primitive, so for each $x \in X$ we have the value $f(x)$ of x in representation f. *However*, mathematics has an intrinsic 'observer-observed' duality in which the roles can be reversed: we can regard f as a member of some collection $\hat{X}$ of all such representations, with its own structure, and regard the same value $f(x)$ as $x(f)$, the value of x seen as a representation of $\hat{X}$ on $f \in \hat{X}$. *The observed can always be seen as instead the observer, observing what had previously been the observer as an element of the set of all possible observers.* In mathematical terms one usually has $X \subseteq \hat{\hat{X}}$ and in nice cases this is an isomorphism of structures. In the case of the circle group, its set of representations is labelled by the integers and we can think of an angle in the circle as a representation of the additive group of integers. So next time you see an innocent expression $f(x)$ ask yourself is it really the value of f at x or is it the value of x at f? This was my answer to the Plato's cave parable in [15, 2] – one could assume that the shadows on the cave wall are a representation of the puppet show reality going on behind the prisoners heads but one could equally well regard the different shadow-sets (as determined by the location of the fire light source behind the puppets) as the 'reality' and the different puppet locations as determining different representations of that. Turning the tables like this can be very hard to 'see' since we generally have to transcend a lot of assumptions to do this, but it makes sense within relative realism in that neither point of view is absolute but relative to our choice to regard X or $\hat{X}$ as the 'real thing'.

So, mathematics and hence the mathematical structure of physical reality has this somewhat arbitrary division into some bits which are assume or chose as the 'reality' and other bits which are representations of those first bits but which we could equally well have regarded as 'real'. We have argued that this is the deep origin of *Born reciprocity* that position and momentum in physics are somewhat interchangeable. The weak principle of self-duality in [15] is that one has this division and it is somewhat arbitrary, much as the division into x, p on a symplectic manifold is somewhat arbitrary. This more or less comes out of the nature of mathematics. The much stronger 'strong principle' goes further and states that the ultimate theory of physics is self-dual in that the dual picture in which x, p are interchanged looks much the same. This comes out of an analysis of the nature of Science, namely that theorists and experimentalists will always have different views that X or $\hat{X}$ are real and hence will only agree when we accept both. But of we have both $X, \hat{X}$ then we also won't be happy with the product $X \times \hat{X}$ because of the feeling that physics should be irreducible and not disconnected. Hence we will tend to extend such structures to some X_1 and then build its representations $\hat{X}_1$ and so forth. We have proposed this in [15] as a kind of semantic engine that drives the evolution of Science and which 'stalls' or has a fixed point when the total structure of the theory if self-dual in the sense of the strong principle of self-duality. For if $X \cong \hat{X}$ then we do not need to take a direct product and can instead agree that the two camps are talking about the same thing with different notations. This last step is somewhat extreme and in general we are content if $X, \hat{X}$ are at least the same type of object, i.e. objects in a self-dual category.

5. Self-duality in quantum gravity

The search for self-dual algebraic structures as toy models of quantum gravity led to one of the two main classes of examples of true quantum groups in the 1980s, namely the bicrossproduct ones[16]. This is one of the two origins of quantum groups. The simplest example, the 'Planck scale Hopf algebra', has two parameters which can be seen as $\hbar$ and the gravitational constant G and is the solution E to the extension problem

$$\mathbb{C}[p] \to E \to \mathbb{C}[x]$$

of position space in one variable x by momentum space in one variable p. Details of this story are recounted in [1, 2] but the main idea is that the dual Hopf algebra has the roles of x, p swapped and the roles of quantum and gravity swapped but is otherwise isomorphic (it is a self-dual Hopf algebra). In some sense quantum theory and gravity arise out of the possible solutions or *choices* for E that are possible once we have assumed position and momentum. This was a toy version of relative realism at work.

Other bicrossproduct quantum groups can be seen similarly but they also arise slightly differently in 3D quantum gravity (and hence conjecturally in 4D) as solutions E to the extension problem

$$C(M) \to E \to U(so_n)$$

where $C(M)$ is the algebra of functions on a classical momentum group M, i.e. we consider the choices for all possible 'Poincare quantum groups' for the given momentum and projecting onto the chosen rotations algebra $U(so_n)$. This quantum group then acts on the enveloping algebra $U(\mathfrak{m})$ viewed as the noncommutative coordinate algebra of quantum spacetime, where $\mathfrak{m}$ is the Lie algebra of M. Let's call this the A-model. What is more relevant in 3D quantum gravity is a semidualisation process in which we swap the roles of $C(M)$ and $U(\mathfrak{m})$ as mutually dual Hopf algebras i.e. position and momentum – a process which we have called *quantum Born reciprocity*. In the resulting semi-dual or B-model, M becomes the classical spacetime with classical isometry group $SO_n {\bowtie} M$ locally factorising into SO_n, M. Such factorisations and the above extensions are equivalent data[16, 3]. We see[24] that this part of quantum gravity on quantum spacetime (the A-model) semidualises to a classical particle on a classical spacetime (the B-model) which previously had been momentum, a duality which extends the observable-state duality in [16] and also has a flavour similar in some respects to mirror symmetry. In particular, 3D quantum gravity with point sources and zero cosmological constant semidualises to a classical particle on $S^3 = SU_2$. The q-deformation of the A-model here is quantum gravity *with* cosmological constant approximated by quantum spacetime $U_q(su_2)$ and its semidual (the B-model) is a q-deformed particle moving on the quantum spacetime $C_q(SU_2)$. These A and B models have different algebras but remarkably they are related by a twisting equivalence as long as $q \neq 1$. In this sense 3D quantum gravity is self-dual up to twisting *iff* the cosmological constant is not zero[24]. We see that the search for self-dual systems and quantum Born reciprocity leads in the specific context of 3D quantum gravity to requiring a nonzero cosmological constant. As explained in [24], what happens is that the Planck scale parameter and the cosmological length scale parameter are interchanged under semidualisation. We refer to [2, 3, 24] for more details.

Here we refocus more broadly on the mathematical structures relevant to our attempt to apply the above philosophical ideas to quantum gravity. We think of Riemannian geometry, which is the basis of gravity, as consisting of three layers: the topological, the differential manifold and the Riemannian structure.

(1) Let X be a compact topological space, $C(X)=A$ the unital commutative algebra of continuous functions on X. But we could equally well regard A as the real thing and $X=\mathrm{SpecA}$ the set of irreducible representations of A with an inherited topology. This Gelfand-Naimark theorem or some algebraic version of it is the basis of algebraic geometry, as well as of noncommutative geometry when A is allowed to be noncommutative. Going beyond this, (projective) modules of A are, by the Serre-Swan theorem, sections of vector bundles. So in physics one can say that matter fields are elements of the dual of A or the double dual of X in some sense. We normally assume that algebras are associative.

(2) Let X be a manifold, Ω the graded-commutative algebra of smooth differential forms on X equipped with a degree 1 map d axiomatized as a graded-derivation in the sense

$$\mathrm{d}(\omega\eta)=(\mathrm{d}\omega)\eta+(-1)^{|\omega|}\omega\mathrm{d}\eta,\quad \forall\omega,\eta\in\Omega$$

and obeying $\mathrm{d}^2=0$. We can essentially recover X as a space as in setting (1) from the degree zero part $A=C^\infty(X)$, and the differential structure on X from d. The latter at the level of X also appears as the boundary map ∂. We can let A be other kinds of commutative algebra, which is the basis of algebraic geometry, and we can also allow A to be noncommutative and Ω non-graded-commutative as an approach to noncommutative differential geometry.

(3) Let X be a Riemannian manifold, $(\Omega,\mathrm{d},\delta)$ its differential algebra as above but further equipped now with the codifferential δ. This can typically be defined via $*^{-1}\mathrm{d}*$ using the Hodge structure or by a trace of the Levi-Civita connection. The properties of δ have recently been axiomatised in [23] as a certain (nondegenerate and symmetric) type of Batalin-Vilkovisky algebra. Specifically, we axiomatise the requirements for δ as a degree -1 map with $\delta^2=0$, the '6-term relation'

$$\delta(a\omega\eta)-(\delta(a\omega))\eta-a\delta(\omega\eta)-(-1)^{|\omega|}\omega\delta(a\eta)+a(\delta\omega)\eta+(-1)^{|\omega|}a\omega\delta\eta=0$$

for all $a\in C^\infty(X),\ \omega,\eta\in\Omega$, and the 'symmetry condition'

$$\delta(a\mathrm{d}b)-a\delta\mathrm{d}b=\delta(b\mathrm{d}a)-b\delta\mathrm{d}a,\quad\forall a,b\in C^\infty(X).$$

We have:

Theorem 5.1. [23] *Let $\delta:\Omega\to\Omega$ obey the axioms stated. Then*

$$(\omega,\mathrm{d}a)=\delta(a\omega)-a\delta\omega$$

$$\nabla_\omega\eta=\frac{1}{2}\left(\delta(\omega\eta)-(\delta\omega)\eta+\omega\delta\eta+\mathrm{i}_\omega\mathrm{d}\eta+\mathrm{i}_\eta\mathrm{d}\omega+\mathrm{d}(\omega,\eta)\right)$$

for all $a\in C^\infty(X),\ \omega,\eta\in\Omega^1$, define a possibly-degenerate metric and metric-compatible torsion-free covariant derivative.

Here the covariant derivative ∇ and interior product i are defined along 1-forms but when $(\ ,\ )$ is nondegenerate we obtain a Riemannian metric and a new formula for its Levi-Civita connection on X along the corresponding vector field. In this case one finds[23] the 7-term identity

$$\begin{aligned}\delta(\omega\eta\zeta)=(\delta(\omega\eta))\zeta+(-1)^{|\omega|}\omega\delta(\eta\zeta)+(-1)^{(|\omega|-1)|\eta|}\eta\delta(\omega\zeta)\\ -(\delta\omega)\eta\zeta-(-1)^{|\omega|}\omega(\delta\eta)\zeta-(-1)^{|\omega|+|\eta|}\omega\eta\delta\zeta\end{aligned}$$

for all $\omega,\eta,\zeta\in\Omega$ as an extension of the assumed 6-term identity. A BV algebra is a degree -1 operator δ (usually denoted differently) on a graded-commutative algebra with $\delta^2 = 0$ and this 7-term identity[4]. So a manifold has the structure of a Riemannian manifold iff its exterior algebra has the structure of a BV algebra with δ symmetric and with the associated bilinear nondegenerate. The theory in [23] is slightly more general. First of all we only need δ^2 to be tensorial and, secondly, we can allow δ to have $(\ ,\ )$ degenerate. Our formulation of a covariant derivative along forms still makes sense and gives a generalisation of Riemannian geometry to singular metrics. Theorem 5.1 also meets our long-standing goal of a common language for GR and quantum theory since BV algebras arise naturally in BRST quantisation.

In each case the algebraic structure maps out or in some sense represents the geometric structure but we can also take the view that the algebra is the 'real thing' and the geometry is a 'geometric realisation' of the algebra. In this second point of view we can even let the algebra A be noncommutative and do away with the explicit geometry; this is noncommutative geometry.

We are using the term 'representation' in a loose sense and there is generally more than one way to say what we mean by representation. Thus the relevant duality in practice is sometimes quite loose. One of the variants of the above is to linearise the geometric side, so in place of the space X in the setting of (1) we can take the vector space $\mathbb{C}X$ spanned by X. The pointwise algebra product on A then corresponds on the dual side to a *coalgebra* on $\mathbb{C}X$ with coproduct $\Delta x = x \otimes x$ for all $x \in X$. More generally we can take any coalgebra C not necessarily of the special cocommutative form for an actual space. Similarly, in the setting of (2) the dual concept to a differential structure on Ω means a *graded coderivation* i of degree -1 with respect to a graded coalgebra $\mathcal{C}$ and obeying $i^2 = 0$. Here a graded coderivation means[25]

$$\Delta \circ i = (i \otimes \mathrm{id} + (-1)^D \otimes i) \circ \Delta$$

where D is the degree operator with respect to the grading of $\mathcal{C}$. Finally, in the setting of (3) we have the coalgebra and coderivation i as just discussed but now also a co-BV structure adjoint to δ, as the linearised notion of Riemannian structure.

Next, according to the self-duality principle we should look for objects which are of self-dual type in the sense of both the algebra and coalgebra side at the same time and with some natural compatibility between them. In the above setting (1), this leads to the notion of quantum group or Hopf algebra as an algebra which is at the same time a coalgebra and the two are compatible (and there is an antipode or 'linearised inverse'), see [1]. We have already recounted the story of this for quantum gravity. Going beyond this now, in the above setting (2), we are led in [25] to the notion of a graded super-Hopf algebra Ω (with odd and even parts according to the grading) equipped with a degree 1 graded bi-derivation. The latter means d with $\mathrm{d}^2 = 0$ which is both a graded-derivation as usual with respect to the algebra *and* a graded coderivation with respect to the coalgebra. Let's call this a *differential super-Hopf algebra*. Then

Theorem 5.2. [25] *A differential super-Hopf algebra which is generated by its degree 0 and* d*, is the same thing as a bicovariant differential calculus on the degree 0 quantum group.*

This gives a new point of view on the standard notion of a bicovariant differential calculus on a quantum group[27], and also generalises it. Moreover, we are led[25] to the idea of such a bicovariant calculus *augmented* further by a degree -1 graded bi-derivation i with $i^2 = 0$. In this case the axioms are self-dual and in the finite-dimensional case Ω^* with all maps adjointed is once again an augmented bicovariant differential calculus, but this time on the dual degree 0 quantum group[25]. This extends the duality of quantum groups to duality of quantum groups-with-differential structure. The self-dualisation in the above setting (3) can in principle be done in the same way to give a self-dualisation of the concept of Riemannian geometry. Note, however, that $(\Omega, \mathrm{d}, \delta)$ in (3) by itself already has an aspect of self-duality in having both a differential and codifferential. We turn to this now.

6. Self-duality v Hodge duality

The key expression of representation-theoretic self-duality in physics is Fourier transform. This makes sense for any finite-dimensional Hopf algebra and, with the appropriate care to handle the analysis, a general Hopf algebra A equipped with a, say, right-integral and a coevaluation. The right-integral means a map $\int : A \to \mathbb{C}$ (or whatever the field is) such that $\int a_{(1)} \otimes a_{(2)} = (\int a)1$ where $\Delta a = a_{(1)} \otimes a_{(2)}$ is our notation for the coproduct. It expresses translation in the group in the classical case. If $\{e_I\}$ is a basis of A and $\{f^I\}$ a dual basis then the coevaluation at least in the finite-dimensional case means the canonical element $\underline{\exp} = \sum_\alpha e_I \otimes f^I$. The Fourier transform is then[1]

$$\mathcal{F} : A \to A^*, \quad \mathcal{F}(a) = \sum_I (\int a e_I) f^I, \quad \forall a \in A,$$

where $\underline{\exp}$ formally plays the role of the exponential. One can also say equivalently that $\mathcal{F}(a)(b) = \int ab$ for all $a, b \in A$ and use this in the infinite-dimensional case. The fundamental lemma of Fourier theory is that multiplication on the dual side corresponds to translation $\triangleleft$ on A in the sense

$$\mathcal{F}(a \triangleleft \phi) = \mathcal{F}(a)\phi, \quad a \triangleleft \phi := a_{(1)} \langle S a_{(2)}, \phi \rangle, \quad \forall a \in A,\ \phi \in A^*,$$

where $\langle\ ,\ \rangle$ is the duality pairing and S is the antipode. We have formally applied such ideas to quantum gravity in [11] as a Fourier transform $\mathcal{F} : C(G) \to \overline{U(\mathfrak{g})}$ for suitable types of function on a suitable Lie group and a suitable completion. It implements the quantum Born reciprocity in 3D quantum gravity for the models above[24].

Now, this Fourier theory also makes sense for Hopf algebras in braided categories[13] and in particular in the category of super-vector spaces. If V is a vector space of dimension n then the exterior algebra $\Lambda = \Lambda(V)$ is a a graded-commutative (super) Hopf algebra in this category. Its super-dual is $\Lambda(V)^* = \Lambda(V^*)$.

Proposition 6.1. *The super-Fourier transform $\mathcal{F} : \Lambda(V) \to \Lambda(V^*)$ is the canonical Hodge operator $*$. Moreover, super-Fourier transform converts wedge product in $\Lambda^\star$ to interior product on Λ.*

Proof. This is a matter of working through the definitions. If $\{e_i\}$ is a basis of V then $e_I := e_{i_1} \cdots e_{i_m}$ for $I = (i_1, \cdots, i_m)$, $i_1 < i_2 \cdots < i_m$, $m \le n$ a multiindex, gives a basis of the exterior algebra Λ. The categorical dual super-Hopf algebra $\Lambda^\star$ has the same form with generators a dual basis $\{f^i\}$ of V^* and pairing

$$\langle f^{j_m}\cdots f^{j_1}, e_{i_1}\cdots e_{i_m}\rangle = \delta^{j_1}_{i_1}\cdots\delta^{j_m}_{i_m}, \quad \forall i_1<\cdots<i_m, \quad j_1<\cdots<j_m.$$

$$\underline{\exp} = \sum_{m=0}^{n}\sum_{j_1<\cdots<j_m} e_{j_1}\cdots e_{j_m}\otimes f^{j_m}\cdots f^{j_1} = \sum_{m=0}^{n}\frac{1}{m!}e_{i_1}\cdots e_{i_m}\otimes f^{i_m}\cdots f^{i_1}$$

(summation understood in the last expression) which we see looks like an exponential. For integration we have the usual Berezin integration $\int e_1\cdots e_n = 1$ and zero on other degrees. Then applying the same definition for Fourier transform as for Hopf algebras but now on our super-Hopf algebra gives

$$\mathcal{F}(e_{i_1}\cdots e_{i_m}) = \sum_k \int e_{i_1}\cdots e_{i_m}e_{j_1}\cdots e_{j_k}\otimes f^{j_k}\cdots f^{j_1} = \epsilon_{i_1\cdots i_m j_1\cdots j_{n-m}} f^{j_{n-m}}\cdots f^{j_1}$$

with no sum of the j's, where ϵ is the totally antisymmetric symbol. If we adopt the summation convention as usual, this becomes

$$\mathcal{F}(e_{i_1}\cdots e_{i_m}) = \frac{1}{(n-m)!}\epsilon_{i_1\cdots i_m j_1\cdots j_{n-m}} f^{j_{n-m}}\cdots f^{j_1}, \quad \mathcal{F}:\Lambda^m\to\Lambda^{\star n-m}.$$

This is the standard Hodge * operator aside from a reversal of the order of the f''s and on converting the f's to e's using a metric.

Similarly, right multiplication by f^i in $\Lambda^\star$ corresponds under $\mathcal{F}$ to a right action in Λ which works out as $(-1)^{m+1}$ times

$$(\langle f^i,\ \rangle\otimes\mathrm{id})\Delta(e_{i_1}\cdots e_{i_m}) = (\langle f^i,\ \rangle\otimes\mathrm{id})((e_{i_1}\otimes 1+1\otimes e_{i_1})\cdots(e_{i_m}\otimes 1+1\otimes e_{i_m}))$$
$$= \delta^i_{i_1}e_{i_2}\cdots e_{i_m} - \delta^i_{i_2}e_{i_1}e_{i_3}\cdots e_{i_m}\cdots + (-1)^{m-1}\delta^i_{i_m}e_{i_1}\cdots e_{i_{m-1}}.$$

This expression is the usual interior product converted in our conventions to a right-derivation by the $(-1)^{m+1}$ prefactor that comes from the braiding in the definition in [13]. □

The reversal in the order of the f's in $\mathcal{F}$ and the conventions for the interior product come from the categorical duality pairing between Λ and $\Lambda^\star$ being defined without unnecessary braidings. These conventions are not essential in the present case in that one could rework the Fourier theory for super-Hopf algebras in the same conventions as stated for ordinary Hopf algebras. However, the more general formulation works in any braided category given a suitable left-integral $\int$ on Λ and right-integral $\int^*$ on $\Lambda^\star$ for the inverse Fourier transform[13]. The role of the minus sign in the usual Fourier inverse is now played by the antipode S of the braided-Hopf algebra and the role of 2π is played by

$$\mathrm{Vol} = (\int\otimes\int^*)(\underline{\exp})$$

We need slightly to extend [13] in that $\int$ in practice is not a morphism. One can think that it has values in some non-trivial element of the braided category. Allowing for this, we have

$$\mathcal{F}^* = (\mathrm{id}\otimes\int^*)(\Psi\otimes\mathrm{id})(\mathrm{id}\otimes\underline{\exp})$$

for the adjoint transform in the reverse direction, where Ψ is the braiding. Then c.f.[13] one has

$$\mathcal{F}^*\mathcal{F} = (\int \otimes \mathrm{id} \otimes \int^*)(\Psi^{-1}\otimes \mathrm{id})(S\otimes \underline{\exp}).$$

In our case $\int f^n\cdots f^1 = 1$ so that Vol $= 1$ and $\mathcal{F}^*(f^{i_m}\cdots f^{j_1}) = \epsilon_{i_1\cdots i_m j_1\cdots j_{n-m}} e_{j_1}\cdots e_{j_{n-m}}$ (no summation). Then $\mathcal{F}^*\mathcal{F} = (-1)^{m(n-m)}$ on degree m arises from the antipode $S = (-1)^m$ and a short computation. This is how the square of the Hodge * arises from the super-transposition and antipode.

The above Proposition 6.1 applies locally on a manifold to the exterior algebra of the tangent space of each point. Thus usual bosonic Fourier theory on the symmetric algebra of the tangent space of a point leads to microlocal analysis while the fermionic version is Hodge theory. One could thus say in view of Theorem 5.1 above that Riemannian geometry arises from this fermionic Fourier transform.

This point of view also allows us in nice cases to define the Hodge * operator and interior product in noncommutative geometry, via the braided-Fourier transform. For example, if Ω is a bicovariant differential calculus on a quantum group A as in Theorem 5.2 above, we have $\Omega = A{\triangleright\!\!\!<}\Lambda$ where Λ is a braided-Hopf algebra in the braided category of crossed A-modules (we suppose for this that A has invertible antipode). We can think of the semidual $\Omega^{-*} = A{\triangleright\!\!\!<}\Lambda^\star$ as the exterior tangent bundle and $\mathcal{F}:\Lambda\to\Lambda^\star$ then defines a quantum Hodge operator. Here $V = \Lambda^1$ is the underlying object in the braided category and Λ is a certain quotient of its tensor algebra defined in such a way[17] that $\Lambda(V), \Lambda(V^*)$ are dually paired. This construction of a bicovariant calculus Ω is due to the author but recovers[18] a different construction in [27] using braided-antisymmetrizers and without any braided-Hopf algebra structure. In nice cases we also have $\Lambda\cong\Lambda^\star$ via a quantum metric. The example of braided Fourier transform on the standard 3D noncommutative calculus on the algebra of functions on S_3 (the group of permutations on 3 elements) is in [19].

We can go further to the full Fourier transform $\Omega\to\Omega^\star$ as a super-Hopf algebra as in Theorem 5.2. This is a tensor product of Hodge duality as above and ordinary quantum Fourier transform on the bosonic degree 0 quantum group A. The physical role of this full Fourier-Hodge transform remains to be explored.

7. Self-duality v de Morgan duality

Another algebraic self-dual concept, this time at the 'birth of geometry', is in the notion of Boolean algebra. From the point of view of setting (1) in Section 5, if X is a set then the collection of subsets is a Boolean algebra, or equivalently a ring with product given by $\cap$ and addition by exclusive-or $\oplus$, and such that every element obeys $x^2 = x$. Conversely, from a Boolean algebra one can construct a certain type of highly disconnected space. Also in this context there is a 'de Morgan duality' given by complementation, or $x\mapsto\bar{x} = 1-x$ in the ring-theoretic language. This operation is an isomorphism if we also interchange $\cap,\cup$ and $0,1$ (where the latter are the empty set, and all of X respectively from the Boolean point of view). Although this de Morgan duality is not obviously the same as representation-theoretic duality, there are physical reasons to think that it is somewhat in the same spirit. These physical reasons are speculative but are as follows [15, 2].

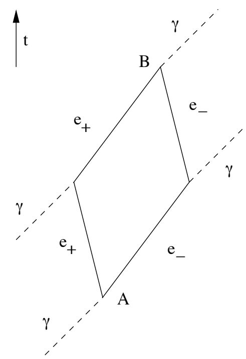

FIGURE 3. Unlikely creation and annihilation process of electron-positron pair could also be seen as a time-travel loop.

Firstly, dropping the axioms $x \cup \bar{x} = 1$ of a Boolean algebra leads us to a more general notion of Heyting algebra, often regarded as the correct (intuitionistic) logic for quantum theory. We refer to [9, 8] for recent work in this area. The de Morgan dual concept is that of a co-Heyting algebra where we drop the law that $x \cap \tilde{x} = 0$ and where now $\tilde{}$ denotes the complementation. In this case $\partial x := x \cap \tilde{x}$ behaves like the boundary of the set (and is a derivation in the co-Heyting algebra) so can be seen as the beginning of geometry[14]. So, roughly speaking, quantum theory and gravity are interchanged by the extension of de Morgan duality beyond Boolean algebra. Yet we have explained in Section 5 that they are also interchanged by 'quantum Born reciprocity' under a process of semidualisation where the position and momentum space quantum groups in certain bicrossoroduct models are interchanged[16, 2, 3, 24]. So it would appear to fit in with the representation-theoretic self-duality. Following [2], we can illustrate these ideas at the level of logic:

> Schrödinger's cat (which we assume the reader is familiar with) has a cat inside a sealed box in a mixed quantum state between dead and alive. We take the view that the cat is *neither dead nor alive* as an illustration of intuitionistic logic.
>
> Dual to this we introduce the *Co-Schrödinger's cat* thought experiment. A cat (in a spacesuit) is falling into a large black hole. From the cat's perspective it falls in and dies in a finite time when crushed near the initial singularity. But from our point of view as a distant observer, however, the cat never dies but is frozen forever hovering on the black hole event horizon due to time dilation effects. So the right logic here is that the cat is *both dead and alive.*

Another bit of intuition in support of our hypothesis is that if something moves along a worldline then its absence moves backwards along the worldline, so a kind of time-reversal. This is also what happens with random walks based on Hopf algebras[1] basically because if we dualise a sequence of compositions then its adjoint is a sequence of compositions of maps between the dual vector spaces but *in reverse order.* How these observations translate into physics is not clear particularly as it is normally assumed that antiparticles experience gravity in the same way as particles. However, the arrow of time, according to relative realism, is a prime example of a reality that it created by hidden assumptions in the way we talk and handle probability in much the same way as, within a country, it is normal for people to agree to drive on a fixed side of the road. Anyone that tried to go on the wrong side would tend to crash and in the same way the equations of physics are time-reversal (or CPT) invariant, but anyone who insists on interpreting everything with the reversed arrow of macroscopic time will not be able to interact consistently with the rest of us. This freedom can, however, be seen at the subatomic level where, for example, the process in Figure 3, can be read one way from bottom to top as a photon splitting into an electron and anti-electron then these happen to recombine producing a photon. Alternatively we could take the view that an electron appeared out of nowhere (absorbing a photon), travelled up the page where it disappeared producing a photon and travelled back in time to become the electron we started with. This kind of time-travel loop is possible at the subatomic level if we chose to interpret it that way but with very low probability. The reasons for the arrow of macroscopic time being non-reversible then become, as for thermodynamics, the nature of entropy or probability and the key thing here is that probability and associated concepts are a prime example of relative realism as they depend on subjective choices.

Another speculative but physical idea related to de Morgan duality in [15] refers to the nature of vacuum energy as dual to a black hole in the sense that space as empty of matter as quantum fluctuations allow means as full of not-matter (the real objects from the de Morgan dual side) as the not-gravity (gravity from the de Morgan dual side) will allow due to the formation of a not-black hole (a black-hole from the de Morgan dual side). Or simply put, we have already argued that cramming space full of matter is the dual statement to space being as empty as possible, i.e. black holes are dual to vacuum energy in some sense. We do not pretend to have a theory here but we do note that this very issue of the right way to think of vacuum energy is widely accepted as a likely test of quantum gravity in that naive ideas for its origin from the zero point energy of each frequency mode gives 10^{122} times the observed and otherwise unexplained vacuum density of $10^{-29}\mathrm{g/cm}^3$. In this regard let us note that we have proposed in [21] that vacuum energy arises as a noncommutative geometry or Planck-scale correction; it is in some sense zero when computed in noncommutative geometry but arises when this is seen as a correction to classical geometry, which would explain why it is so much smaller than what would arise from a quantum correction.

In spite of this circumstantial evidence it remains to relate this type of duality to representation-theoretic duality more mathematically. Here we want to make a few small comments. From a mathematical point of view a Heyting algebra is a (bounded) distributive lattice with an additional binary operation $x \Rightarrow y$ where $\bar{x} = (x \Rightarrow 0)$. A bounded lattice here means a type of poset where any two elements have a supremum denoted $x \cup y$ and an infimum denoted $x \cap y$ with respect to the partial ordering $\leq$, and in the bounded case we have associated $0, 1$. The $\cap, \cup$ are compatible by certain absorption axioms. In the distributive case the latter extend to the distributivity of $\cap, \cup$ over each other (as for Boolean algebras). The extended duality notion $x \Rightarrow y$ for a Heyting algebra is the largest z such that $z \cap x \leq y$. It is also known that $x \leq \bar{\bar{x}}$ and $x \cap \bar{x} = 0$. The subset of elements where double-duality does hold form a Boolean subalgebra.

(1) Our first observation is that as a poset a Heyting or co-Heyting algebra A is itself a discrete differential geometry in that there is an algebra of 1-forms $\omega_{x<y}$ spanned by the edges $x < y$ (where we do not include the self-edges). The differential calculus Ω^1 is defined on the algebra of functions $C(A)$ with $\mathrm{d}f = \sum_{x<y}(f(y) - f(x))\omega_{x<y}$ which makes sense at least in the finite case (one would need some analysis in the infinite case). From the point of view above, there is also a topological space X of which a Heyting algebra elements can be seen as the collection of open sets, or a co-Heyting algebra as the closed sets. Thus we are in some sense doing discrete geometry on the set of open or closed sets on a topological space.

(2) Next we can also think of A as a category with at most one morphism $\leq$ between any two objects and $\cap$ as product. In this context one can view the Heyting algebra map $x \Rightarrow y$ as an 'exponential element' y^x or internal hom $\underline{\mathrm{Hom}}(x, y)$ depending on notation. The 'evaluation map' is the morphism $\mathrm{ev} : \underline{\mathrm{Hom}}(x, y) \cap x \leq y$ where in our category the morphisms are $\leq$ and the product is $\cap$. So in this context $\bar{x} = \underline{\mathrm{Hom}}(x, 0)$ which starts to look like a representation-theoretic dual space of all maps to a trivial object, but with respect to this internal hom structure. Also $x \leq \bar{\bar{x}}$ is still close to double-duality in the sense of a morphism from x to $\bar{\bar{x}}$. Similarly for a co-heyting algebra with respect to $\cup$ and a dual interpretation.

(3) Usually a bi-Heyting algebra[26] is defined as something that is both a Heyting and co-Heyting algebra at the same time (but generally with different complementations for the two cases). In this case we have both

$$\partial x := x \cap \tilde{x}, \quad \delta x := x \cup \bar{x}.$$

If we view ∂ as a homology version of differential as in Section 5 then we could view δ as a homology version of a codifferential. According to Section 5 we could start to think of this as the beginning of a 'Riemannian manifold' but let us stress that we are not claiming that we really have the structure in that section.

(4) Based on the 'self-dualisation' ideas in Section 5, a slightly different approach to a bi-Heyting algebra would our following proposal: a bounded distributive lattice with a single operation complementation obeying double duality and de Morgan duality interchanging $\cap, \cup$. In this case $\delta x = \overline{\partial \bar{x}}$ just as in Hodge theory, i.e. de Morgan duality could from this point of view be more in the spirit of Hodge duality. We again have ∂ a derivation with respect to $\cap$ and dually $\delta(x \cup y) = (\delta x \cup y) \cap (x \cup \delta y)$ as well as

$$\delta x = \delta \bar{x}, \quad \partial x = \partial \bar{x}, \quad x \cap \delta x = x \cup \partial x = x$$
$$\partial^2 = \partial, \quad \delta^2 = \delta, \quad \delta\partial = \delta, \quad \partial\delta = \partial$$

under our assumptions. One may also show

$$x \oplus y := (x \cap \bar{y}) \cup (\bar{x} \cap y)$$

is commutative with zero and associative up to ∂ terms. The latter is because, omitting the product $\cap$ and grouping it ahead of $\cup$, we have

$$(x \oplus y) \oplus z = (\partial x \cup \partial y)z \cup xyz \cup \bar{x}\bar{y}z \cup \bar{x}y\bar{z} \cup x\bar{y}\bar{z}$$

where the last group of terms is symmetric in permutations of x, y, z. We also have

$$xy \oplus xz = x(y \oplus z) \cup (\partial x)(y \cup z).$$

Thus the Boolean ring gets replaced by some kind of 'homotopy ring' meaning up to ∂. Instead of x being inverse to itself, and instead of $x \oplus \bar{x} = 1$ as in the Boolean ring case, we have

$$x \oplus x = \partial x, \quad x \oplus \bar{x} = \delta x, \quad \bar{x} = x \oplus 1, \quad \overline{x \oplus y} = (x \oplus \bar{y}) \cup \partial x \cup \partial y.$$

The realisation of this proposal in terms of proving the existence or not of such objects (beyond the Boolean case) remains to be seen.

8. Relative realism and the origin of Riemannian geometry

We now address what could be considered as the biggest problem in quantum gravity at the moment, namely how exactly does gravity and classical Riemannian geometry emerge. In the spirit of relative realism we are going to argue that Riemannian metrics and connections are forced by the choices already made in having a manifold and assuming that quantum gravity effects make things noncommutative. This does not answer the question of how does a manifold itself emerge, i.e. we assume some kind of differential structure encoded as in setting (2) of Section 5, as a differential graded algebra, but within that, i.e. assuming basically nothing but the Leibniz rule of differential calculus, we shall see that this assumption creates or forces on us Riemannian geometry. This is philosophically then the origin of Riemannian geometry, while at the practical level it also translates into a concrete mechanism, for the first time, for how Riemannian structures can emerge from quantum gravity *without* having assumed anything like a metric or Riemannian structure in advance.

Specifically, we interpret recent results in [23] now in the context of relative realism. The starting assumption is that whatever quantum gravity is, it must in some limit recover classical Riemannian geometry and classical gravity in the vanishing limit of some parameter, assumed to be the Planck scale λ. If so then at non-zero values of the parameter we should see corrections or deformation to classical Riemannian geometry. As these are quantum gravity effects one could assume – and this is now widely accepted as an expectation for quantum gravity – that these corrections take the form of noncommutative geometry where spacetime coordinates become noncommuting operators or noncommuting algebra variables. This is the reason, proposed in [16], for interest in noncommutative geometry as the foundation of current generation quantum gravity models.

Now if this assumption of noncommutative geometry as a better approximation of what emerges from quantum gravity holds, how much reality is created or forced by that? The following is not the whole story but let's suppose for the sake of analysis that we are in only the mildest form of noncommutative geometry where the coordinate algebra A remains classical and the noncommutativity enters only between 1-forms and functions, i.e. only the differential calculus is being quantised. This is so as to focus on this issue alone. Now, due to previously identified mathematical constraints it turns out that in many cases one has to have one or more extra cotangent direction in order to keep associativity[5] so we allow for this possibility too. Thus we assume:

(1) There is in the classical limit a classical manifold X and let $\Omega(X)$ be its associated differential exterior algebra

(2) For general parameter values we assume the same algebra $A = C^\infty(X)$ but allow all possible differential 'manifold' structures in the form of some differential graded algebra E over A but *not necessarily graded-commutative* and not necessarily generated by A, d in that we allow possibly an extra dimension.

(3) We assume an extension of associative differential graded algebras of the form

$$\Omega_{\theta'} \to E \to \Omega(X)$$

as maps of differential algebras, where $\Omega_{\theta'}$ is the differential graded algebra with a single 1-form θ' with $\theta'^2 = 0$ and $\mathrm{d}\theta' = 0$. There is a technical cleftness condition[23] similar to assumptions made in Section 5 in those other contexts.

(3) We assume that the extension is graded-central in that θ' anticommutes with 1-forms.

In short we are looking at all possible 1-dimensional central extensions of the classical differential 'manifold' structure on X within noncommutative geometry while still close to classical. We are *not* assuming any kind of metric or Riemannian structure. We use the notation

$$L_B(\omega, \eta) := B(\omega\eta) - (B\omega)\eta - (-1)^{b|\omega|}\omega B\eta, \quad \forall \omega, \eta \in \Omega$$

for the 'Leibnizator' measuring failure of any operator B of degree b to obey the Leibniz rule.

Theorem 8.1. [23] *All possible central extensions E as above of a fixed classical exterior algebra are given by '2-cocycle' data $(\Delta, [[\ ,\ ]])$ where*

$$\Delta : \Omega \to \Omega, \quad [[\ ,\ ]] : \Omega \otimes \Omega \to \Omega$$

of degrees 0, -1 respectively, obey

$$[\Delta, \mathrm{d}] = 0$$

$$[[\omega\eta, \zeta]] + [[\omega, \eta]]\zeta = [[\omega, \eta\zeta]] + (-1)^{|\omega|}\omega[[\eta, \zeta]]$$

$$L_\Delta(\omega, \eta) = \mathrm{d}[[\omega, \eta]] + [[\mathrm{d}\omega, \eta]] + (-1)^{|\omega|}[[\omega, \mathrm{d}\eta]]$$

for all $\omega, \eta, \zeta \in \Omega$. Moreover, given such an extension there is an associated possibly-degenerate metric and form-covariant derivative

$$(\omega, \mathrm{d}a) = \frac{1}{2}[[\omega, a]], \quad \nabla_\omega \eta = \frac{1}{2}[[\omega, \eta]], \quad \forall a \in C^\infty(X),\ \omega, \eta \in \Omega^1$$

which are compatible in the sense

$$(\omega, \mathrm{d}(\eta, \zeta)) - (\nabla_\omega \eta, \zeta) - (\eta, \nabla_\omega \zeta) = T(\omega, \eta)(\zeta) + T(\omega, \zeta)(\eta), \quad \forall \omega, \eta, \zeta \in \Omega^1$$

where T is the torsion tensor.

We refer to [23] for the proof and merely note here that if we are given $(\Delta, [[\ ,\ ]])$ data as above, the product and exterior derivative of E are

$$\omega \wedge_E \eta = \omega \wedge \eta - \frac{\lambda}{2}\theta'[[\omega, \eta]], \quad \mathrm{d}_E\omega = \mathrm{d}\omega - \frac{\lambda}{2}\theta'\Delta\omega$$

for all $\omega, \eta \in \Omega$. The parameter λ here is included on dimensional grounds (and allows us to think of the classical limit as $\lambda \to 0$) but any non-zero value can be absorbed in the normalisation.

What Theorem 8.1 says is that the different possible metric geometries on a given manifold correspond to the different possible noncommutative differential structures that centrally extend the classical one. So if quantum gravity involves such a noncommutative differential algebra then classical Riemannian geometry will be induced when we write its structure in terms of the classical manifold. The result also says that the kind of geometry that emerges is not quite Levi-Civita unless the torsion is zero, and predicts a specific relationship between the two as stated.

Standard Riemannian geometry with zero torsion corresponds to a further assumption. Note that there is an obvious notion of equivalence of different extensions given by cocycles that differ by a kind of coboundary. We can sometimes use this equivalence to set $\Delta = 0$:

(4) We say a cleft extension as above is *flat* if it is equivalent to one where $\Delta = 0$, i.e. where d is undeformed.

Proposition 8.2. [23] *An extension as above is flat iff there exists a map δ of degree -1 such that $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$. If in this case $\delta^2 = 0$ and the associated bilinear is symmetric then the associated form-covariant derivative is torsion-free and given by our formula in Theorem 5.1.*

So this shows how the Levi-Civita connection emerges as a flat cleft extension of the classical manifold. Nondegeneracy in this context does not have a particular interpretation i.e. degenerate metrics could also arise. Aside from this, we see how Riemannian geometry emerges from nothing much but the requirements of a differential calculus and the assumption that the Leibniz rule also holds in the quantum case.

These methods also apply when the original calculus that we are extending is already noncommutative. So let Ω be a general standard differential calculus but this time Ω and its degree zero part A could be noncommutative in the first place. We consider possible cleft central extensions

$$\Omega_{\theta'} \to E \to \Omega$$

These are still classified by the same kind of '2-cocycle' data and this time the same formulae as above induce a quantum metric and quantum (bimodule[10]) connection on Ω, see [23]. What this means is that our above point of view as to what Riemannian geometry *is* also applies when the algebra A is noncommutative, i.e. our analysis and philosophy are not limited to the classical case that we focussed on. It also means that we can use this mechanism to generate examples of noncommutative Riemannian geometry. An example and some applications can be found in [23].

Next, we remark on a striking parallel with a different problem in which instead of centrally extending the differential calculus $\Omega(X)$, we deform it keeping dimensions classical but allowing it possibly to be nonassociative and also allowing the coordinate algebra $C^\infty(X)$ to be quantized. This was recently analysed in [7] as an extension of [5]. It is well-known that the data for the quantisation of the coordinate algebra is a Poisson tensor π, say, if we work to low order in the deformation parameter λ. The further data at this level for the noncommutativity of the differential structure is a connection ∇ on Ω^1 which typically has torsion and has,

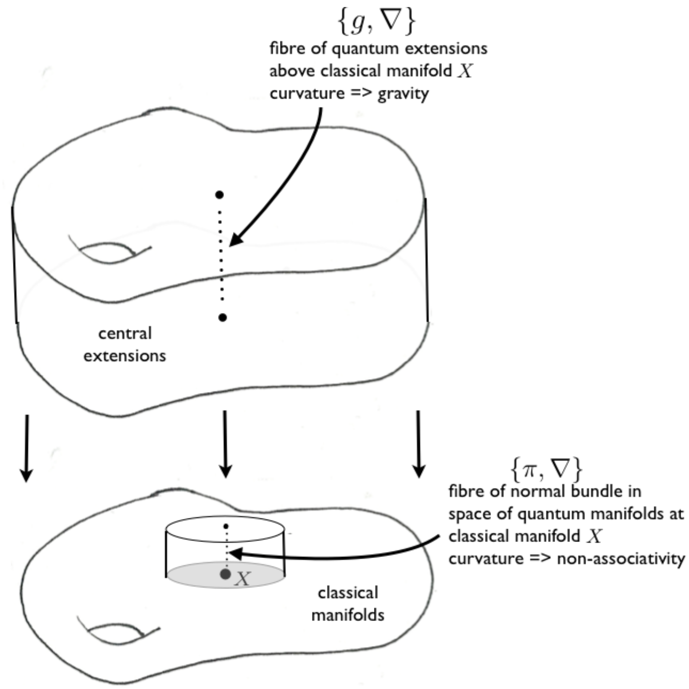

FIGURE 4. The moduli of Riemannian structures on a given manifold X can be viewed intuitively as the fibre of a 'bundle' of central extensions over different classical manifolds. It has parallels to the 'normal bundle' of deformations of classical manifolds within the space of possibly nonassociative quantum differential algebras with fibre at X the moduli of Poisson structures and compatible connections.

in the associative case, zero curvature. What we require is that this is Poisson compatible[12, 5], which we have written in [7] as

$$\pi^{\mu\nu}{}_{;\rho} = \pi^{\eta\mu}T^{\nu}_{\eta\rho} + \pi^{\nu\eta}T^{\mu}_{\eta\rho}$$

up to some changes of notation. Here the semicolon denotes covariant derivative by the Poisson-compatible connection and T denotes its torsion. Now compare the above Poisson-compatibility condition with the generalised metric-compatibility condition in Theorem 8.1, which in tensor notation we can write as

$$g^{\mu\nu}{}_{;\rho} = g^{\eta\mu}T^{\nu}_{\eta\rho} + g^{\nu\eta}T^{\mu}_{\eta\rho}$$

where now the semicolon and torsion refer to the metric connection. Instead of the antisymmetric Poisson tensor we have the symmetric metric tensor. Instead of being relevant to an extension by a 1-form θ' with $\theta'^2 = 0$ as in the second case, we can view the first case as relevant (one can say) to extending by a central bosonic variable λ with $\lambda^2 = 0$ since we work only at the semiclassical level. Thus the two compatibility conditions are parallel and would seem to be odd and even aspects of a single concept. For example, we can write the two together for a single hermitian tensor $g + \imath\pi$. We do not even need π strictly to be a Poisson tensor

– any antisymmetric bivector will do though in nice cases we do want it to be nondegenerate, thus even more analogous to the Riemannian case.

We have seen that metrics and generalised metric-compatible connections (as in Theorem 8.1) on a manifold X describe the moduli of its noncommutative central extensions and its curvature is related to gravity. By contrast, this parallel theory of Poisson tensors and Poisson-compatible connections on X describes the 'fibre of the normal bundle' at X in the moduli of all noncommutative deformations with differential structure including nonassociative ones, with curvature leading to the nonassociativity. This is depicted in Figure 4.

Nevertheless, these two parallel constructions are often related as follows: many noncommutative algebra deformations do not admit any associative differential calculi of classical dimensions, which is to say at semiclassical level for such π, all Poisson-compatible connections have curvature. This is the no-go theorem in [5] and is a kind of anomaly for differentiation. However, often with anomalies we can trade the curvature obstruction for an extra dimension which can be used to absorb the anomaly, i.e. the anomaly translates into an associative central extension. This then by Theorem 8.1 is governed by a metric and metric connection, typically with curvature. This suggests the possibility of an interchange between these two types of structure. It would mean that gravity originates in an anomaly for the associative implementation of the Leibniz rule when working with noncommutative algebras.

Finally, many noncommutative differential structures admit only a much reduced moduli of quantum metrics. Again this is part of the greater rigidity of noncommutative geometry. At the semiclassical level it means that many choices of π and Poisson-compatible connection admit only a much reduced moduli of classical metric g that are compatible with the connection. What that means is that if a classical geometry is to be a limit of a quantum one then the metrics that may appear are highly constrained and in particular are often forced to have curvature. This could ultimately hint at the origin of gravity. The analysis is in [7] and an example in 2 dimensions appeared in [6]. The curvature in a similar 4D model can be such that the Einstein tensor corresponds to a perfect fluid, so in this case one could say that the matter distribution is also in some sense forced by the metric quantizability assumption[6]. This is thus another example of how physical phenomena can be forced out of the assumed mathematical structure in the spirit of relative realism.

References

[1] S. Majid, *Foundations of Quantum Group Theory*, Cambridge Univ. Press, 1996

[2] S. Majid, Algebraic approach to quantum gravity I: relative realism, to appear *Proceedings Road to Reality with Roger Penrose, Cracow 2010*, 50 pp.

[3] S. Majid, Algebraic approach to quantum gravity II: noncommutative spacetimes, in Approaches to Quantum Gravity, ed. D. Oriti. C.U.P. (2009) 466-492

[4] I.A. Batalin & G.A., Vilkovisky, Quantization of gauge theories with linearly dependent generators, Phys. Rev. D 28 (1983) 2567–2582

[5] E.J. Beggs and S. Majid, Semiclassical differential structures, Pac. J. Math. 224 (2006) 1–44

[6] E.J. Beggs & S. Majid, Gravity induced by quantum spacetime, Class. Quant. Gravity 31 (2014) 035020 (39pp)

[7] E.J. Beggs & S. Majid, Semiquantisation functor and Poisson Riemannian geometry, arXiv:1403.4231

[8] A. Döring, Topos-based logic for quantum systems and bi-Heyting algebras, arXiv:1202.2750

[9] A. Döring & C.J. Isham, What is a thing?: Topos theory in the foundations of physics, in *New Structures for Physics*, ed. B. Coecke, Lec. Notes Phys. 813 (2011) 753–937, Springer
[10] M. Dubois-Violette & P.W. Michor, Connections on central bimodules in noncommutative differential geometry, J. Geom. Phys. 20 (1996) 218 –232
[11] L. Freidel and S. Majid, Noncommutative harmonic analysis, sampling theory and the Duflo map in 2+1 quantum gravity, Class. Quant. Gravity 25 (2008) 045006 (37pp)
[12] E. Hawkins, Noncommutative rigidity, Comm. Math. Phys. 246 (2004) 211–235
[13] A. Kempf & S. Majid, Algebraic q-integration and Fourier theory on quantum and braided spaces, J. Math. Phys. 35 (1994) 6802-6837
[14] F. W. Lawvere, Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes, in *Proceedings, Category Theory, 1990*, eds. A. Carboni et al., Lec. Notes in Math. 1488 (1991) 279–281, Springer
[15] S. Majid, The principle of representation-theoretic self-duality, Phys. Essays. 4(3) (1991) 395-405
[16] S. Majid, Hopf algebras for physics at the Planck scale, Class. Quant. Gravity 5 (1988) 1587–1607
[17] S. Majid, Double bosonisation of braided groups and the construction of $U_q(g)$, Math. Proc. Camb. Phil. Soc.125 (1999) 151-192
[18] S. Majid, Classification of differentials on quantum doubles and finite noncommutative geometry, Lect. Notes Pure Appl. Maths 239 (2004) 167-188, Marcel Dekker
[19] S. Majid, Noncommutative differentials and Yang-Mills on permutation groups S_N, Lect. Notes Pure Appl. Maths 239 (2004) 189-214, Marcel Dekker
[20] S. Majid, Almost commutative Riemannian geometry: wave operators, Commun. Math. Phys. 310 (2012) 569–609
[21] S. Majid, Newtonian gravity on quantum spacetime, Euro Phys. J. Web of Conf, 70 (2014) 00082 (10pp)
[22] S. Majid, Emergence of Riemannian geometry and the massive graviton, to appear Euro Phys. J. Web of Conf, 14pp
[23] S. Majid, Reconstruction and quantisation of Riemannian structures, arXiv:1307.2778
[24] S. Majid & B. Schroers, q-Deformation and semidualisation in 3D quantum gravity, J. Phys A 42 (2009) 425402 (40pp)
[25] S. Majid & W. Tao, Duality for generalised differentials on quantum groups, arXiv:1207.7001
[26] G.E. Reyes & H. Zolfaghari, Bi-Heyting algebras, toposes and modalities. Journal of Philosophical Logic 25 (1996) 2543
[27] S.L. Woronowicz, Differential calculus on compact matrix pseudpgroups (quantum groups). Com. Math. Phys. 122 (1989) 125170

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