

Separably injective C^* -algebras

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Abstract. We show that a C^* -algebra is a 1-separably injective Banach space if, and only if, it is linearly isometric to the Banach space $C_0(\Omega)$ of complex continuous functions vanishing at infinity on a substonean locally compact Hausdorff space Ω .

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1. Introduction

It is well-known that there are few examples of 1-injective Banach spaces. These are Banach spaces V for which every continuous linear map $T : Y \rightarrow V$ on a Banach space Y admits a norm preserving extension to a super space $Z \supset Y$, equivalently, contractive linear maps $T : Y \rightarrow V$ extend to contractive ones on Z . Indeed, a Banach space V is 1-injective if and only if it is linearly isometric to the continuous function space $C(\Omega)$ on some stonean space Ω [5, 9]. If the 1-injectivity condition is relaxed to requiring that each continuous linear map $T : Y \rightarrow V$ extends to a continuous one on $Z \supset Y$, then it is unclear what constitutes the larger class of Banach spaces, called λ -injective or \mathcal{P}_λ spaces, satisfying this condition. However, one can consider the class of λ -separably injective Banach spaces to which only continuous linear maps on separable spaces are extendable to separable super spaces. Of particular interest is the subclass of 1-separably injective Banach spaces to which contractive linear maps on separable spaces admit contractive extension on separable super spaces. While c_0 is the only λ -separably injective space among infinite dimensional separable Banach spaces [14], it has been shown recently in [1] that among nonseparable real Banach spaces, there are indeed many interesting examples of λ -separably injective spaces. In particular, the Banach space $C(\Omega, \mathbb{R})$ of real continuous functions on a compact Hausdorff space Ω is 1-separably injective if, and only if, Ω is an F-space. It is natural

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to ask if this result also holds for the space $C_0(S)$ of continuous functions vanishing at infinity on a locally compact space S . This case has not been discussed in [1] and in fact, the example of c_0 provides a negative partial answer since \mathbb{N} is an F-space, but c_0 is not 1-separably injective although it is 2-separably injective.

In this paper, we give a complete answer to the above question and prove, more generally, that a C*-algebra is 1-separably injective if, and only if, it is linearly isometric to the Banach space $C_0(S)$ of complex continuous functions vanishing at infinity on a *substonean* locally compact Hausdorff space S . Particularly, abelian monotone sequentially complete C*-algebras are 1-separably injective. This example may be of interest as the class of monotone complete C*-algebras is closely related to generic dynamics [13].

2. Separably injective Banach spaces

The concept of a separably injective *real* Banach space, considered in [1], can be extended naturally to that for a complex Banach space.

Definition 2.1. A complex (resp. real) Banach space V is said to be *1-separably injective* if for every complex (resp. real) *separable* Banach space Z and every closed subspace $Y \subset Z$, every bounded linear operator $T : Y \rightarrow V$ extends to a bounded linear operator $\tilde{T} : Z \rightarrow V$ with $\|\tilde{T}\| = \|T\|$.

Given a locally compact Hausdorff space Ω , we will denote by $C_0(\Omega)$ the abelian C*-algebra of complex continuous functions on Ω vanishing at infinity. If Ω is compact, then we omit the subscript 0 and denote by $C(\Omega, \mathbb{R})$ the Banach space of real continuous functions on Ω .

Definition 2.2. Let Ω be a locally compact Hausdorff space. It is called an *F-space* if for each real continuous function f on Ω , there is a real continuous function k on Ω such that $f = k|f|$ (cf. [3, 14.25]). Following [4], we call Ω *substonean* if any two disjoint open σ -compact subsets of Ω have disjoint compact closures.

The compact substonean spaces are exactly the compact *F-spaces*. However, infinite discrete spaces are F-spaces without being substonean. We refer to [8, Example 5] for an example of a substonean space which is not an F-space.

Example 2.3. Let Ω be a compact Hausdorff space. Using the results in [6, 11], it has been shown in [1, Proposition 4.2] that the real continuous function space $C(\Omega, \mathbb{R})$ is 1-separably injective if, and only if, Ω is an F-space. This result remains true if we replace $C(\Omega, \mathbb{R})$ by the complex continuous function space $C(\Omega)$. Indeed, if $C(\Omega, \mathbb{R})$ is 1-separably injective and given a contractive complex linear operator $T : Y \rightarrow C(\Omega)$, where Y is a closed subspace of a separable complex Banach space Z , the real part $\text{Re } T : y \in Y_r \mapsto \text{Re } T(y) \in$

$C(\Omega, \mathbb{R})$ extends to a real linear contraction $T_r : Z_r \rightarrow C(\Omega, \mathbb{R})$ which, as in the proof of [5, Theorem 2], gives a complex linear contraction

$$z \in Z \mapsto T_r(z) - iT_r(iz) \in C(\Omega)$$

extending T since $T(y) = \operatorname{Re} T(y) - i\operatorname{Re} T(iy)$ for $y \in Y$. Hence $C(\Omega)$ is 1-separably injective. Conversely, if $C(\Omega)$ is 1-separably injective, it will follow from Theorem 3.5 that $C(\Omega, \mathbb{R})$ is 1-separably injective.

However, as noted earlier, the above result is not valid for the space $C_0(S)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space S . Separable injectivity of $C_0(S)$ has not been considered in [1]. A topological criterion for 1-separable injectivity of $C_0(S)$ follows from Theorem 3.5.

Let V be a complex 1-separably injective Banach space and let $T : Y \rightarrow V$ be a bounded linear operator on a closed subspace Y of a complex Banach space X . The arguments in the proof of [1, Proposition 3.5 (a)] for *real* 1-separably injective spaces can be extended to the complex case and one can show that T has a norm preserving extension $\tilde{T} : X \rightarrow V^{**}$. A further application of the arguments in the proof of [10, Theorem 2.1, (9) \Rightarrow (1)], which are also valid for complex spaces, gives the following result. The result for real 1-separably injective spaces has been shown in [1].

Lemma 2.4. *Let V be a 1-separably injective complex Banach space. Then the bidual V^{**} is 1-injective.*

3. Separably injective C^* -algebras

We characterize 1-separably injective C^* -algebras in this section. Let us begin with a simple lemma.

Lemma 3.1. *A 1-separably injective C^* -algebra is abelian.*

Proof. Let A be a 1-separably injective C^* -algebra. By Lemma 2.4, the bidual A^{**} is 1-injective and hence linearly isometric to a continuous function space $C(\Omega)$ on some stonian space Ω [5]. The linear isometry between the C^* -algebras A^{**} and $C(\Omega)$ preserves the Jordan triple product

$$\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a) \quad (a, b, c \in A^{**})$$

by a well-known result of Kadison [7] (see also [2, Theorem 3.1.7]). Since $C(\Omega)$ is an abelian algebra, we must have, via the isometry between $C(\Omega)$ and A^{**} ,

$$\{a, b, \{c, \mathbf{1}, \mathbf{1}\}\} = \{a, \{b, c, \mathbf{1}\}, \mathbf{1}\}$$

for $a, b, c \in A^{**}$, where $\mathbf{1}$ denotes the identity in A^{**} . Let p be a projection in A^{**} and $a \in A^{**}$. A simple computation gives

$$\begin{aligned} \frac{1}{2}(pa + ap) &= \{p, p, \{a, \mathbf{1}, \mathbf{1}\}\} = \{p, \{p, a, \mathbf{1}\}, \mathbf{1}\} \\ &= \frac{1}{4}(pa + ap + 2pap) \end{aligned}$$

and $pa + ap = 2pap$, which implies $pa = ap$. Hence A^{**} is abelian since it is generated by projections. In particular, A itself is abelian. \square

We have the following result readily.

Proposition 3.2. *Let A be a von Neumann algebra. The following conditions are equivalent.*

- (i) A is 1-separably injective.
- (ii) A is 1-injective.

Proof. (i) \Rightarrow (ii). By the above observation, the unital algebra A is abelian and hence linearly isometric to a continuous function space $C(\Omega)$ on some compact Hausdorff space Ω . Since A has a predual, Ω must be hyperstonean and therefore $C(\Omega)$ is 1-injective by [5]. \square

Remark 3.3. The above proposition is false for unital C^* -algebras. Indeed, let $\beta\mathbb{N}$ be the Stone-Ćech compactification of \mathbb{N} . Then $\beta\mathbb{N}\setminus\mathbb{N}$ is a compact F-space by [3, p. 210]. Hence the C^* -algebra $C(\beta\mathbb{N}\setminus\mathbb{N})$ is 1-separably injective (cf. Example 2.3), but not 1-injective since $\beta\mathbb{N}\setminus\mathbb{N}$ is not stonian [3, p. 98].

We now determine the class of 1-separably injective C^* -algebras. A useful fact noted in [4, Proposition 1.1] is that a locally compact Hausdorff space S is substonian if, and only if, the following condition holds: given f and g in $C_0(S)$ satisfying $fg = 0$, there are functions $f_1, g_1 \in C_0(S)$ such that $f_1g_1 = 0$, $f_1f = f$ and $g_1g = g$. We will need the following definition introduced in [8].

Definition 3.4. A nonempty subset S_0 of a topological space S is called a P -set if it is closed and any G_δ -set containing S_0 is a neighborhood of S_0 . A point $p \in S$ is called a P -point if $\{p\}$ is P-set in S .

It has been remarked in [8] that a nonempty subspace S_0 of S is a P-set if, and only if, each real continuous function f on S , vanishing on S_0 , must vanish on a neighborhood of S_0 . If S is a locally compact and noncompact space, then S is substonian if, and only if, the one-point compactification $S \cup \{\infty\}$ is an F-space and ∞ is a P-point in $S \cup \{\infty\}$ (cf. [8, Theorem 1]).

Theorem 3.5. *Let A be a C^* -algebra. The following conditions are equivalent.*

- (i) A is 1-separably injective.
- (ii) A is linearly isometric to the Banach space $C_0(S)$ of complex continuous functions vanishing at infinity on a substonian locally compact Hausdorff space S .

Proof. (i) \Rightarrow (ii). By Lemma 3.1, A is abelian and hence linearly isometric to the function space $C_0(S)$ on some locally compact Hausdorff space S . We show that S is substonian.

Given any function $f \in C_0(S)$, we define the *cozero set* of f to be the set $\text{coz}(f) = \{x \in S : f(x) \neq 0\}$. Let U and V be two disjoint open σ -compact sets in S . We show that they have disjoint compact closures. It has been

observed in [4, p.125] that one can find, via Urysohn's lemma, two functions $f, g \in C_0(S)$ with $0 \leq f, g \leq 1$ such that $U = \text{coz}(f)$ and $V = \text{coz}(g)$. We note that $fg = 0$ in this case.

Let $h = \chi_{\text{coz}(f)}$ be the characteristic function of $\text{coz}(f)$. Let Y be the closed linear span of $\{f^{1/n}, g^{1/n} : n = 1, 2, \dots\}$ in $C_0(S)$, and let $Z = Y + \mathbb{C}h \subset \ell^\infty(S)$. Since $C_0(S)$ is 1-separably injective, the identity map $\iota : Y \rightarrow C_0(S)$ admits a norm preserving extension $\tilde{\iota} : Z \rightarrow C_0(S)$. Write $k = \tilde{\iota}(h) \in C_0(S)$ and note that $\|k\| \leq 1$.

For each $n \in \mathbb{N}$, we have $\|h - 2f^{\frac{1}{n}}\| \leq 1$ and therefore $\|k - 2f^{\frac{1}{n}}\| = \|\tilde{\iota}(h) - 2\tilde{\iota}(f^{\frac{1}{n}})\| \leq 1$. In particular, $|k(x) - 2f^{\frac{1}{n}}(x)| \leq 1$ for all $n \in \mathbb{N}$ which, together with $|k(x)| \leq 1$, implies $k(x) = 1$ for $x \in \text{coz}(f)$. It follows that $kf = f$.

Since $hg = 0$, we have $\|h + e^{i\theta}g^{\frac{1}{n}}\| \leq 1$ for every $\theta \in [0, 2\pi)$ which gives $\|k + e^{i\theta}g^{\frac{1}{n}}\| = \|\tilde{\iota}(h) + \tilde{\iota}(e^{i\theta}g^{\frac{1}{n}})\| \leq 1$. Hence for each $x \in \text{coz}(g)$, one has $|k(x) + e^{i\theta}g^{\frac{1}{n}}(x)| \leq 1$ for all $n \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. This implies $k(x) = 0$ for $x \in \text{coz}(g)$ and therefore $kg = 0$.

We have $\text{coz}(f) \subset \{x \in S : k(x) = 1\}$ and $\text{coz}(g) \subset \{x \in S : k(x) = 0\}$, where $\{x \in S : k(x) = 1\}$ is compact and contained in the open set $\{x \in S : |k(x)| > \frac{3}{4}\}$. Applying Urysohn's lemma to the compact set $\overline{\text{coz}(f)} \subset \{x \in S : k(x) = 1\}$, one can find a function $f_1 \in C_0(S)$ such that $0 \leq f_1 \leq 1$, $f_1 = 1$ on $\overline{\text{coz}(f)}$ and $f_1 = 0$ outside $\{x \in S : |k(x)| > \frac{3}{4}\}$.

Considering the function $h' = \chi_{\text{coz}(g)}$ with similar arguments, we can find another function $k' \in C_0(S)$ with $\|k'\| \leq 1$ such that $\text{coz}(g) \subset \{x \in S : k'(x) = 1\}$ and hence $\overline{\text{coz}(g)}$ is a compact subset of S . Since $\{x \in S : |k(x)| < \frac{1}{4}\}$ is an open subset of S containing $\overline{\text{coz}(g)}$, Urysohn's lemma again yields a function $g_1 \in C_0(S)$ such that $0 \leq g_1 \leq 1$, $g_1 = 1$ on $\overline{\text{coz}(g)}$ and $g_1 = 0$ outside $\{x \in S : |k(x)| < \frac{1}{4}\}$. It follows that $f_1g_1 = 0$ and $f_1f = f, g_1g = g$.

It follows that the closures $\overline{U} = \overline{\text{coz}(f)} \subset \{x \in S : f_1(x) \geq 1/2\}$ and $\overline{V} = \overline{\text{coz}(g)} \subset \{x \in S : g_1(x) \geq 1/2\}$ are compact and disjoint.

(ii) \Rightarrow (i). Let A be linearly isometric to $C_0(S)$ where S is a substonean locally compact Hausdorff space. We show that $C_0(S)$ is separably injective. This is true if S is compact, as shown in Example 2.3. Let S be noncompact. Since the one-point compactification $S \cup \{\infty\}$ is an F -space, $C(S \cup \{\infty\})$ is 1-separably injective, again by Example 2.3.

Let Y be a closed subspace of a separable Banach space Z and let $T : Y \rightarrow C_0(S)$ be a bounded linear operator. We identify $C_0(S)$ with the closed subspace $\{u \in C(S \cup \{\infty\}) : u(\infty) = 0\}$ of $C(S \cup \{\infty\})$. Let $\tilde{T} : Z \rightarrow C(S \cup \{\infty\})$ be a norm preserving extension of T .

Let $\{f_n : n \in \mathbb{N}\}$ be a countable dense subset of $T(Y) \subset C_0(S)$. Since S is substonean, ∞ is a P-point and there exists an open neighbourhood U_n of ∞ such that $f_n = 0$ on U_n .

Moreover, the G_δ -set $\bigcap_n U_n$ contains an open neighbourhood U of ∞ . Hence Urysohn's lemma again enables us to choose a function $e \in C_0(S)$ with

$\|e\| = 1$ such that $e = 1$ on $S \setminus U$ and $e(\infty) = 0$. This gives $f_n e = f_n$ for all $n \in \mathbb{N}$.

It can now be seen readily that the linear map $T_e : Z \rightarrow C_0(S)$ defined by

$$T_e(z) = \tilde{T}(z)e \quad (z \in Z).$$

is a norm preserving extension of T . □

We conclude by mentioning some interesting examples of substonean spaces. For any locally compact, σ -compact Hausdorff space S with Stone-Ćech compactification βS , the space $\beta S \setminus S$ is a compact F-space [3, 14.27]. Locally compact substonean spaces include the *Rickart spaces* which are exactly those locally compact spaces S for which $C_0(S)$ is monotone sequentially complete [4]. In particular, abelian monotone sequentially complete C^* -algebras are 1-separably injective.

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