Hidden classical symmetry in quantum spaces at roots of unity : From q-sphere to fuzzy sphere

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Brown Het-1229

Hidden classical symmetry in quantum spaces at roots of unity : From q-sphere to fuzzy sphere.

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We study relations between different kinds of non-commutative spheres which have appeared in the context of ADS/CFT correspondences recently, emphasizing the connections between spaces that have manifest quantum group symmetry and spaces that have manifest classical symmetry. In particular we consider the quotient $SU_q(2)/U(1)$ at roots of unity, and find its relations with the fuzzy sphere with manifest classical $SU(2)$ symmetry. Deformation maps between classical and quantum symmetry, the $U_q(SU(2))$ module structure of quantum spheres and the structure of indecomposable representations of $U_q(SU(2))$ at roots of unity conspire in an interesting way to allow the relation between manifestly $U_q(SU(2)$ symmetric spheres and manifestly $U(SU(2))$ symmetric spheres. The relation suggests that a subset of field theory actions on the q-sphere are equivalent to actions on the fuzzy sphere. The results here are compatible with the proposal that quantum spheres at roots of unity appear as effective geometries which account for finite N effects in the ADS/CFT correspondence.

July 2000

1. Introduction

Non-commutative spacetimes which are deformations $ADS \times S$ backgrounds have been studied as space-time explanation of the stringy exclusion principle [1], beginning in [2] and further in [3][4][5][6][7], Another mechanism emphasizing non-commutativity uses the fuzzy sphere world-volumes of fat gravitons moving as dipoles on a transverse non-commutative space was given in [8], and further explored recently in [9][10][11]. The fuzzy sphere worldvolumes appear by the polarization mechanism of Myers [12]. Some qualitative properties of finite N $ADS \times S$ correlators [4][13] are reproduced by overlaps of spherical harmonics on the fuzzy sphere [14].

For odd-dimensional spheres quantum groups give the natural non-commutative candidates. For even spheres the candidates discussed so far keep the classical symmetries manifest and are generalizations of the fuzzy sphere [15]. One motivation of this paper is to begin a study of the relation between the non-commutative spaces based on quantum groups and those based on fuzzy sphere and its generalizations. The goal in this direction is to study the relations, from the point of view of quantum space-time, which are expected to exist in string theory [16]. Another motivation is to better understand the relations between the candidate non-commutative spheres appearing as part of a non-commutative space-time and the non-commutative structures appearing from choosing a splitting of the spheres into non-commutative world-volume directions of a fat graviton and non-commutative transverse directions as in [8]. Another purely mathematical motivation is to ask if there is a generalization to the world of non-commutative spheres of relations of the kind $S^2 = S^3/U(1) = SU(2)/U(1)$.

The important feature that has to emerge from any convincing step in this direction is to clarify in what sense having a quantum group symmetry is compatible with the classical symmetries. Uncovering quantum group symmetry in physical systems with manifest classical symmetry has been undertaken in the context of WZW-quantum group correspondences [17]. Some mathematical work in the direction of uncovering classical symmetry in quantum groups has also been done, see for example [18][19][20][21], and at appropriate points in this paper we will use some of these results.

In this paper we study the connection between $SU_q(2)/U(1)$ for $q = e^{\frac{i\pi}{M}}$ and fuzzy sphere generated by $SU(2)$ generators satisfying $\sum_i S_i^2 = J(J+1)$. The fuzzy sphere algebra decomposes under representations of $SU(2)$ acting iby commutators as irreducible representations of spin s ranging from 0 to $2J$. $U_qSU(2)$ has reps. which are cutoff at

$2s \leq M-2$ for $q = e^{\frac{i\pi}{M}}$. This suggests a relation for $M = 2(2J+1)$ between the q-sphere and the fuzzy sphere.

Section 2 reviews some properties of the fuzzy sphere $\mathcal{A}_f$. Section 3 reviews properties of the q-sphere $\mathcal{A}_q$ and its $U_q(SU(2))$ symmetry. Section 4 obtains the decomposition under $U_q(SU(2))$ of $\mathcal{A}_q$. The decomposition consists entirely of indecomposable representations We observe that this spectrum of indecomposable representations contains a sub-module which is the direct sum of standard representations of the kind appearing in the decomposition of the fuzzy sphere. This allows us in section 5, to develop, using deformation maps [18], the precise relations between $\mathcal{A}_q$ and $\mathcal{A}_f$.

2. Fuzzy sphere

The fuzzy sphere is defined as the algebra generated by the three generators S_3, S_+, S_- obeying the relations of the $SU(2)$ Lie algebra :

$$\begin{aligned}
&[S_+, S_-] = 2S_3 \\
&[S_3, S_+] = S_+ \\
&[S_3, S_-] = -S_-.
\end{aligned} \tag{2.1}$$

together with a constraint on the Casimir :

$$S_3^2 + \frac{1}{2}(S_+S_- + S_-S_+) = J(J+1) \tag{2.2}$$

This algebra is infinite dimensional. For example S_-^l for any l are independent elements. It admits however a finite dimensional quotient which is isomorphic to the algebra of $N \times N$ matrices where $N = 2J+1$. We will call this finite dimensional truncation $\mathcal{A}_f(N)$.

It admits a right action of the universal enveloping algebra of $SU(2)$, by taking commutators from the right. We could also work with a left action instead but choose to work with the right action for convenience. Under this action of $U(SU(2))$, the $\mathcal{A}_f(N)$ decomposes as a direct sum of representations of integer spin s with unit multiplicity with s ranging over integers s from 1 to $2J = N-1$.

$$\mathcal{A}_f = \oplus_{s=0}^{2J} V_s \tag{2.3}$$

3. The q-sphere

We start with the q-deformed algebra of functions $SU(2)$ which we call $Fun_q(SU(2))$, generated by $\alpha, \beta, \gamma, \delta$ which obey

$$\begin{aligned}
&\alpha\beta = q\beta\alpha \quad \alpha\gamma = q\gamma\alpha \\
&\beta\gamma = \gamma\beta \quad \beta\delta = q\delta\beta \\
&\alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma \\
&\alpha\delta - q\beta\gamma = 1
\end{aligned} \tag{3.1}$$

The choice of $SU_q(2)$ real form is the choice of the involution

$$\begin{aligned}
&\alpha^* = \delta \\
&\beta^* = \gamma
\end{aligned} \tag{3.2}$$

In the last line we have set to 1 the central element.

The algebra (3.1) has a left and a right action of $Fun_q(SU(2))$. Under the left $U(1)$, the generators α, γ have charge 1, and the generators β, δ have charge -1. The $U(1)$ invariant sub-algebra is generated by $\alpha\beta$, $\alpha\delta$ and $\gamma\beta$. For a more complete discussion of the quantum geometry of these q-spheres see for example [22].

Defining the combinations

$$\begin{aligned}
&e_0 = 1 + (q + q^{-1})\beta\gamma, \\
&e_+ = q^{-1}(q + q^{-1})^{1/2}\alpha\beta, \\
&e_- = -(q + q^{-1})^{1/2}\gamma\delta
\end{aligned} \tag{3.3}$$

one finds that we have an algebra belonging to the family of Podles quantum 2-spheres [23]

$$\begin{aligned}
&e_+e_- - e_-e_+ + \lambda e_0^2 = \mu e_0 \\
&qe_0e_+ - q^{-1}e_+e_0 = \mu e_+ \\
&qe_-e_0 - q^{-1}e_0e_- = \mu e_- \\
&e_0^2 + qe_-e_+ + q^{-1}e_+e_- = 1,
\end{aligned} \tag{3.4}$$

where $\lambda = (q - q^{-1})$, with $\mu = (q - q^{-1})$.

This algebra has infinite dimensional representations for generic q. For roots of unity $q = e^{\frac{i\pi}{M}}$ it displays some special properties. It is easy to prove for example that e_-^M and e_+^M are central elements. We expect that there will be finite dimensional representations of the q-sphere algebra for any μ when q is a root of unity, which can be constructed by the method of highest weights much the same way we construct representations of $U(SU(2))$ or of $U_qSU(2)$. In the following we will focus on the case of the quotient 2-sphere. Finite dimensional representations will lead to finite dimensional quotients of the q-sphere algebra, much the way they do for the fuzzy sphere as discussed in section 2. We will denote these finite dimensional q-sphere algebras as $\mathcal{A}_q$.

3.1. Finite dimensional truncations of the quotient sphere

There are known representations of the $Fun_q(SU(2))$ algebra which we will use to obtain representations of its $U(1)$ quotient. By specializing to roots of unity we can obtain finite dimensional truncations of these algebras. The $Fun_qSU(2)$ algebra has a family of reps. labelled by t [24]:

$$\begin{aligned} &\alpha|k>=(1-q^{2k})^{1/2}|k-1> \\ &\beta|k>=-q^{k+1}t^{-1}|k> \\ &\gamma|k>=q^{k}t|k> \\ &\delta|k>=(1-q^{2k+2})^{1/2}|k+1> \end{aligned} \tag{3.5}$$

Using the expressions (3.3) we get a representation of the S_q^2 algebra.

$$\begin{aligned} &e_0|k>=(1-q^{2k}(1+q^2))|k> \\ &e_+|k>=-q^kt^{-1}\sqrt{q+q^{-1}}(1-q^{2k})^{1/2}|k-1> \\ &e_-|k>=-t\sqrt{q+q^{-1}}(1-q^{2k+2})^{1/2}q^{k+1}|k+1> \end{aligned} \tag{3.6}$$

The parameter t will not affect the form of the finite dimensional quotient algebra $\mathcal{A}_q$, as we will see in explicit examples in later sub-sections.

Specializing to roots of unity, we find finite dimensional reps. with k extending from 0 to $M-1$, since e_+ annihilates $|M>$ and e_- annihilates $|M-1>$. In these finite M^2 dimensional reps. e_0 van be expressed as a sum of

$$e_0=\sum_{l=0}^{M-1} C_l e_-^l e_+^l \tag{3.7}$$

The coefficients can be determined recursively by acting successively on $|0>$ (which determines C_0 directly), and then $|1>$ (which determines C_1 in terms of C_0), and so forth. We will write down the explicit expressions for the cases $M=3,4$ below.

The first technical result of this paper is to give the decomposition of this M^2 dimensional U_q module algebra in terms of representations of U_q. We have explicit proofs for the cases $M=3,4$, and we have several tests of the proposed decomposition in the general case.

3.2. Action of $U_qSl(2)$

The right action of the $U_qSl(2)$ on the q-sphere is given below. It can be obtained from [25] after some changes of variables.

$$
\begin{aligned}
&(e_-)K = q(e_-) \quad (e_0)K = e_0 \quad (e_+)K = q^{-1}e_+ \\
&(e_-)X_+ = -e_0 \quad (e_0)X_+ = \ e_+ \quad (e_+)X_+ = 0 \\
&(e_-)X_- = 0 \quad (e_0)X_- = -(q+q^{-1})e_- \quad (e_+)X_- = (q+q^{-1})e_0
\end{aligned}
\tag{3.8}
$$

We can check that these are indeed consistent with the standard relations of U_q, which in our conventions, are :

$$
\begin{aligned}
&KX_+K^{-1} = qX_+ \quad KX_-K^{-1} = q^{-1}X_- \\
&X_+X_- - X_-X_+ = \frac{(K^2 - K^{-2})}{(q-q^{-1})}
\end{aligned}
\tag{3.9}
$$

To obtain the standard form of classical $U(SU(2))$ algebra in the $q \to 1$ limit we write $K = q^H$ and get :

$$
\begin{aligned}
&[H, X_+] = X_+ \\
&[H, X_-] = -X_- \\
&[X_+, X_-] = 2H
\end{aligned}
\tag{3.10}
$$

The U_q algebra admits finite dimensional quotients at roots of unity, and acts as finite dimensional symmetry algebras on the q-sphere, and as we will show in section 4 on the fuzzy sphere.

We can also check that the relations of the q-sphere are invariant under the action of the U_q symmetry. In checking this we have to use the following action of U_q on products :

$$
(e_ie_j)X_+ = (e_i)X_+(e_j)K + (e_i)K^{-1}(e_j)X_+
\tag{3.11}
$$

This form of the action on products uses the co-product of the quantum group and the q-sphere is a module-algebra for the quantum group. For a general discussion of module algebras acted on by Hopf algebras we refer the reader to [24].

For real q there is a conjugation operation on the q-sphere where

$$
\begin{aligned}
&(e_+)^\dagger = -(q+q^{-1})e_- \\
&(e_-)^\dagger = -\frac{1}{(q+q^{-1})}e_+
\end{aligned}
\tag{3.12}
$$

Using this we can reconstruct the action of X_- from that of X_+.

Using (3.8) and (3.4) we can write down the action of X_+ on e_-^L as

$$((e_-)^L)X_+ = -\frac{q^{-L+3}}{1-q^4}(1-q^{2L})(1-q^{2(L-1)})e_-^{L-1} - q^{-L+1}\frac{(1-q^{4L})}{(1-q^4)}e_-^{L-1}e_0 \tag{3.13}$$

Using the conjugations symmetry at real q or directly (3.8) and (3.4), we can obtain :

$$((e_+)^L)X_- = \frac{q^{-L+2}}{(1-q^2)}(1-q^{2L})(1-q^{2(L-1)})e_+^{L-1} + q^{-1}\frac{(1-q^{4L})}{(1-q^4)}(1+q^4)e_0e_+^{L-1} \tag{3.14}$$

For $q = e^{\frac{i\pi}{M}}$, the above equations imply

$$(e_+^M)X_- = 0, \quad (e_-^M)X_+ = 0. \tag{3.15}$$

This means that the constraints $e_+^M = 0, e_-^M = 0$ are consistent with the action of the U_q symmetry. The finite dimensional quotient should be a U_q module algebra. As $q \to 1$, only the term $e_+^{L-1}e_0$ survives. When $L = M/2$, the coefficient of $e_+^{L-1}e_0$ vanishes. In this sense this power of e_+ shows very non-classical behaviour.

4. Reduction to Indecomposables

The q-sphere module algebra can be decomposed into reps. of U_q. We would like to know what kind of reps. appear. We first perform a reduction $e_+^M = 0, e_-^M = 0$ together with the polynomial expression of e_0 in terms of $\sum_{l=0}^{M-1} C_l e_-^l e_+^l$. This reduced algebra is spanned by $e_-^{l_1}e_+^{l_2}$, with $0 \leq l_1, l_2 \leq M-1$, and is therefore M^2 dimensional. It contains M highest weights $1, e_-, e_-^2, \cdots e_-^{M-1}$ annihilated by X_- (this may seem unusual but it is because we are using right action of U_q rather than left action) and M lowest weights $1, e_+, e_+^2, \cdots e_+^{M_1}$ annihilated by X_+. The following is a proposed reduction of the module algebra in terms of indecomposables which is consistent with the dimension M^2 and with the above set of highest and lowest weights. For even M we propose,

$$A_{M^2} = \oplus_{k=1}^{\frac{M}{2}} I_0^{2k}. \tag{4.1}$$

We are here using the notation of [26] for the indecomposables. Each of these indecomposables has dimension $2M$, so the above is consistent with the dimension count $M^2 = \frac{M}{2}(2M)$. For odd M we propose

$$A_{M^2} = \oplus_{k=1}^{\frac{(M-1)}{2}} I_0^{2k+1} \oplus I_0^1 \tag{4.2}$$

Since I_0^1 has dimension M the above is consistent with the dimension count $M^2 = \frac{(M-1)}{2}(2M) + M$. A similar discussion of quantum group structure of Matrix algebras has appeared in [27][28], where the the geometric objects considered were quantum planes rather than quantum spheres.

Counting the number of highest and lowest weights gives another check of (4.1) and (4.2). Each rep. of the form I_0^{2k} has two highest weight states and two lowest weight states. Each rep. of the form $I_0^{(1)}$ has one highest weight and one lowest weight. The decomposition in (4.1) has $2(M/2) = M$ highest weights. The decomposition in (4.2) has $2\frac{(M-1)}{2} + 1 = M$ highest weights.

Each representation of the type I_0^{2k} contains as a submodule an ordinary representation of dimension $M - p + 1$. The dimensions of these reps add up to $(M/2)^2$ for even M, and to $(\frac{M-1}{2})^2$for odd M. So we can map them to the fuzzy sphere corresponding to the respective matrix algebras while keeping the same structure of $SU(2)$ representations as the ones of classical $SU(2)$.

The above proposal for the decomposition of the A_q algebra in terms of indecomposables has some implications which can be checked. In the case of even M the set of highest weights $e_-^{M/2}, e_-^{M/2+1}, e_-^{M/2+2}, \cdots e_-^{M-1}$ pair up, respectively, with $e_-^{M/2-1}, e_-^{M/2-2} \cdots 1$. By applying an appropriate number of powers of X_+ we get from the upper highest weights to the lower ones according to the structure of I_0^p described in [26]. We need to check that

$$(e_-^{\frac{M-2+p}{2}})X_+^{p-1} \sim e_-^{k_1-k_2} \tag{4.3}$$

for $p = 2, 4, \cdots M$. The equation (3.13) proves the desired result for $p = 2$, since it shows that in $(e_-^{M/2})X_+$ only the first term survives. Similarly from (3.14) $(e_+^{M/2})X_-$ is proportional to $e_-^{M/2-1}$. It will be an interesting exercise to give the explicit proof for other values of p. Rather than pursuing this direct route for general M and p we give a counting arguments which works in the general case, and we give explicit formulae for $q^3 = -1$ and $q^4 = -1$ in the following sections.

4.1. Counting for even M.

Let us check that the number of states in the algebra is indeed consistent with the above decomposition. At $H = L$, for positive L, the polynomials are spanned by

$$e_-^k, e_-^{k+1}e_+, e_-^{k+2}e_+^2, \cdots e_-^{M-1}e_+^{M-1-k} \tag{4.4}$$

i.e a total of $M-k$ states.

For $L<M/2$, the proposed decomposition has representations which contribute states with multiplicity 1 and representations which contribute states with multiplicity 2. The representations with $p=2,4,\cdots(M-2k)$ contribute two states each, giving a total of $M-2L$. The representations with $p=M,M-2,\cdots(M-2L+2)$ contribute one state each giving a total of L states. Adding these up we get $M-L$ states in agreement with the explicit counting of polynomials.

For $H>L/2$ we have representations I_0^{2l} contributing one state each for $l=L-M/2+1, L-M/2+2,\cdots M-L$, giving us exactly $M-H$ states, in agreement with the counting of polynomials in e_+, e_-.

4.2. Counting for odd M

In this case, we propose a decomposition into $\frac{(M-1)}{2}$ repsresentations of dimension $2M$ and one representation of dimension M. This spectrum is the set of representations I_0^p with p ranging over the set $p=1,3,\cdots M$. The highest weight $e_-^{\frac{M-1}{2}}$ belongs to $p=1$. The remaining highest weights pair up as $(e_-^{\frac{M+p-2}{2}}, e_-^{\frac{M-p}{2}})$ in the reps. I_0^p, for p ranging over $3,5\cdots M$.

Consider eigenvalues of H which are equal to $L\leq\frac{(M-1)}{2}-1$. There are two states with such eigenvalue from all reps I with $(M-p)\geq 2L$, i.e for $p=3,5,\cdots M-2L$. From these we get a total of $M-1-2L$ states. I-reps with $(M-p)<2L$ give one state each. These values of p are $M-2L+2, M-2L+4,\cdots M$. The representation with $p=1$ also contributes one state, giving a total of $H+1$ states coming from representations which contribute one each. Adding up the states from reps which contribute 2 each we get

$$(M-2L-1)+(L+1)=M-L \tag{4.5}$$

This agrees with the count of independent polynomials with $H=L$.

For eigenvalues $H=L$ which obey $L\geq\frac{(M-1)}{2}$ we have I-reps contributing one state if $M+p-2\geq 2L$. This allows $M-L$ different values of p, again agreeing with the count of independent expressions of the form $e_-^{l_1}e_+^{l_2}$.

4.3. An explicit example : $q = e^{\frac{i\pi}{3}}$

The expression for e_0 in this case is :

$$e_0 = -q^2 - q^2 e_- e_+ + q e_-^2 e_+^2 \tag{4.6}$$

The highest eigenvalue of H in a rep. I_0^p is j which is given by $2j = M + p - 2$.

The structure of the rep. I_0^1 is given by

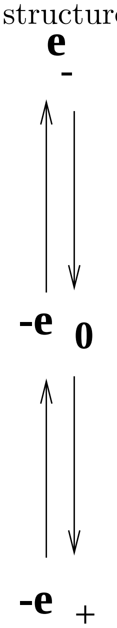

fig. 1

The structure of the rep. I_0^3 is given below.

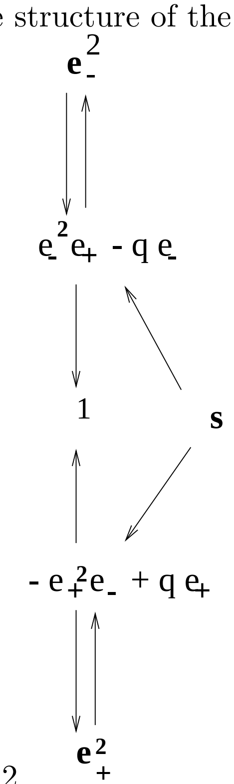

fig. 2

The element denoted as s can be determined up to an arbitrary constant to be :

$$s = A_1 + e_- e_+ + q(1+q) e_-^2 e_+^2 \tag{4.7}$$

4.4. An explicit example : $q = e^{\frac{i\pi}{4}}$

$$e_0 = -q^2 - q^3 e_- e_+ + \frac{(q^2-1)}{2} e_-^2 e_+^2 - \frac{q(1+q^2)}{2} e_-^3 e_+^3 \tag{4.8}$$

The rep. I_0^2 is shown below.

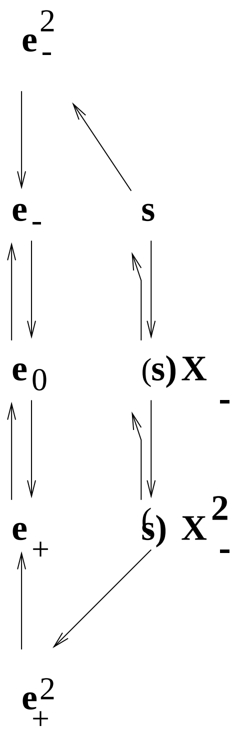

fig. 3

Using the formulae for the right action of U_q we find that $(e_-^2)X_+ = (q^3 - q)e_-$. Note that there is no $e_-^2 e_+$ appearing in $(e_-^2)X_+$. This means that $(e_-^2)X_+ X_- = 0$ as indicated in Fig.3 by the absence of a upward arrow emerging from the state at the second row. To get the form of the element s of the A_q, which sits at the right end of the second row of Fig. 3 we solve the equation $(s)X_+ = e_-^2$. This allows solutions

$$s = a e_- - e_-^3 e_+^2 - q(1-q^2) e_-^2 e_+, \tag{4.9}$$

where a is an arbitrary constant. Explicit expressions for the other states on the right leg of Fig. 3 can be obtained by acting with X_+ on the element s.

The rep. I_0^4 is shown below.

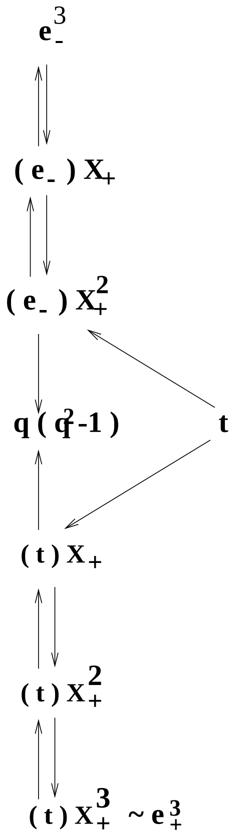

fig. 4

Similarly we find that $(e_3)X_+^2$ is proportional to the identity, with no combination of $e_-e_+, e_-^2e_+^2, e_-^3e_+^3$. And we can solve for the element t up to an arbitrary constant b by requiring that $(t)X_-$ gives $(e_-^3)X_+^2$. This gives an expression

$$t = b - qe_-e_+ + \frac{1}{2}e_-^2e_+^2 + \frac{q(1+q^2)}{2}e_-^3e_+^3. \tag{4.10}$$

The steps, described above for $q = e^{\frac{i\pi}{3}}$ and $q = e^{\frac{i\pi}{4}}$ make it clear how to obtain explicit expressions form for polynomials which fill out the appropriate set of I-reps in the case of general M.

4.5. Relation to fuzzy sphere

The decomposition (4.1) contains as a sub-module the direct sum of standard representations of U_q. These are representations which have the same structure as representations of ordinary $SU(2)$. For U_q at roots of unity there is a finite set of these, with spins $2s \leq M-2$. This set of representations forms a closed fusion ring and is used as a model for fusion rules of WZW models [17]. Integer spin representations within this range appear in the decomposition of the fuzzy sphere (2.3).

The representation I_0^p contain as a sub-module the representation V_s with spin given by $2s = M - p$. This means that included in (4.1) is a direct sum of the standard representations with integer spins ranging up to $M-2$. This will allow us to exhibit some interesting properties of a map $\rho : \mathcal{A}_q \to \mathcal{A}_f(\frac{M}{2})$ in the next section. The same property holds for the case of odd M, i.e we have a sub-module in (4.2) which is the direct sum of standard reps. with integer spins ranging up to $(M-3)/2$. In this case we have a map $\rho : \mathcal{A}_q \to \mathcal{A}_f(\frac{M-1}{2})$.

5. Deformation Map, deformed product and deformed co-product

The decomposition of A_q contains as a submodule the $\mathcal{H} = \oplus V_s$, where V_s is the standard representation of spin s. While $\mathcal{H}$ is a sub-module it is not a subalgebra of A_q. It turns out that we can define a new product on $\mathcal{H}$, which we call $\mu_q^* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, and which is a natural additional product to consider on any U_q module algebra given the existence of twistings of the standard co-product of U_q. This new product will allow the sub-module to be, in addition, a sub-algebra and will in fact be the same as the fuzzy sphere product. We will begin by elaborating the properties of the map $\rho : \mathcal{A}_q \to \mathcal{A}_f$ in connection with deformation maps, and then show the relation between the new product and twisted co-products.

We have a vector space isomorphism between $\mathcal{H}$ and $\mathcal{A}_f$. Let us call this map $\rho : \mathcal{A}_f \to \mathcal{A}_q$. There is deforming map :

$$D : U \to U_q \tag{5.1}$$

Let the map $\lambda_f : A_f \otimes U \to A_f$ denote the right action of the universal enveloping algebra of $SU(2)$ on the fuzzy sphere. Let the map $\lambda_f : A_q \otimes U_q \to A_q$ denote the right action of U_q on the q-sphere. The deforming map satisfies the property

$$\rho \circ \lambda_f = \lambda_q \circ (\rho \otimes D) \tag{5.2}$$

This is illustrated diagrammatically in the figure below.

$$
\begin{tikzcd}
A_f \arrow[d, "\rho"'] & \mathbf{x} & U \arrow[d, "D"] \arrow[r, "\lambda_f"] & A_f \arrow[d, "\rho"] \\
A_q & \mathbf{x} & U_q \arrow[r, "\lambda_q"'] & A_q
\end{tikzcd}
$$

fig. 5

To be more explicit we can choose the isomorphism ρ and the deformation map as follows. A highest weight of U_q with H eigenvalue l is given by e_-^l. A highest weight of U with eigenvalue of H equal to l is given by S_-^l. We define $Y_{l,m}^{(q)} = \lambda_q(e_-^l, X_+^{(l-m)})$. We also define $Y_{l,m}^{(f)} = \lambda_f(S_-^l, X_+^{(l-m)})$. With these definitions ρ takes a simple form $\rho(Y^{(f)}) = Y^{(q)}$. And the deformation map is :

$$
\begin{aligned}
&K = q^H \\
&X_+^{(q)} = X_+ \\
&X_-^{(q)} = \frac{(l-H)(l+H+1)}{(l-H)_q(l+H+1)_q} X_-
\end{aligned}
\tag{5.3}
$$

Here l is understood to be expressed in terms of the generators of the algebra using $l(l+1) = H^2 + \frac{1}{2}(X_+X_- + X_-X_+)$. If we use this formula in a space where l takes the value $M/2$, the denominator can vanish with the numerator finite. In the application of interest the eigenvalues of l extend from 0 to $M/2-1$ (for M even) and to $(M-1)/2-1$ (for M odd).

The content of (5.2) can now be expressed more simply as

$$
\lambda_f(Y^{(f)}, X) = \lambda_q(Y^{(q)}, D(X)) \tag{5.4}
$$

The meaning of (5.2) and (5.4) is that using the action of U_q on the vector-subspace $\mathcal{H}$ of $\mathcal{A}_q$ we can reconstruct the action of the U on $\mathcal{A}_f$. So far we have only discussed the module structure of $\mathcal{H}$. We now turn to the product structure $\mathcal{H}$.

We can define a new product on the $Y^{(q)}$ by first mapping with ρ and then multiplying. The modified q-product is acted on by the q-symmetry through the co-product :

$$
\Delta_q^* = (D \otimes D) \circ \Delta \circ D^{-1} \tag{5.5}
$$

This may be seen as follows.

$$
\begin{aligned}
\lambda_q(Y^{(q)} * Y^{(q)}, X^{(q)}) &= (\lambda_f \otimes \lambda_f)(Y^{(q)} \otimes Y^{(q)}, \Delta \circ D^{-1}(X^{(q)})) \\
&= (\lambda_q \otimes \lambda_q)(Y^{(f)} \otimes Y^{(f)}, (D \otimes D) \circ \Delta \circ D(X^{(q)}))
\end{aligned}
\tag{5.6}
$$

This new co-product for the q-algebra can in fact be written as a conjugation of the standard co-product by an element F of $U_q \otimes U_q$. The existence of such an element for large M, follows from work of Drinfeld [21]. The work of [18] shows how to construct it from Clebsch-Gordan coefficients of the q-symmetry and the classical symmetry. The twist element is related to the R matrix but unlike the R matrix it is not very explicitly known [29]. Drinfeld twists have recently appeared in discussions of brane world-volume non-commutativity recently [30][31]. Some of their abstract properties are discussed in generality in [32].

The existence of this twist of the standard co-product of U_q allows the definition of a new product on the sub-module $\mathcal{H}$ of the A_q module algebra. This new product allows the sub-module to be, in addition, a sub-algebra. The action of U_q on the sub-module with multiplication on the sub-module defined by the star product recovers the fuzzy sphere and its $U(SU(2)$ symmetry. This transformation was possible because of the fact observed in section 4 that $\mathcal{A}_q$ contains a sub-module which transforms under U_q exactly the way $\mathcal{A}_f$ trasnforms under the classical symmetry. We expect that these observations will allow an understanding of how to relate field theories naturally written in terms of q-sphere variables to field theories written in terms of fuzzy sphere variables. These transformations may have analogies to the Seiberg-Witten map [33].

6. Summary and Outlook

We have elaborated on connections between different kinds of non-commutative spheres. We gave a strategy for recovering the fuzzy 2-sphere from the q-sphere at roots of unity. Our main result relates to the module structure of $\mathcal{A}_q$ and its relation via deformation maps to the module structure and the product structure of the fuzzy sphere. Further work is needed to understand the detailed implications for maps between field theories defined on q-sphere and field theories defined on fuzzy sphere. A preliminary remark is that the subset of theories defined on $\mathcal{A}_q$ which only use the sub-module made of V reps and use the deformation product, can be mapped to fuzzy sphere field theories. It remains to study in more detail issues of reality, invariant traces and field theoretic Feynman rules in the light of the deformation maps and Drinfeld twists. Some works that are likely to be useful in this direction are [30].

One intriguing fact we have uncovered about the fuzzy sphere in this investigation is that $\mathcal{A}_f(M/2)$ and $\mathcal{A}_f(M)$ (for M even) are both module algebras which have the same finite dimensional symmetry algebra (the finite U_q algebra for $q = e^{\frac{i\pi}{M}}$). $\mathcal{A}_f(M)$ is a module algebra with the standard q-co-product. $\mathcal{A}_f(M/2)$ is a module algebra with a twisted q-co-product.

We already discussed two physical motivations for this work in the introduction. Another direction where work along these lines can be useful is towards the formulation of a precise relation between the structure of the chiral ring of theories dual to string theory on $AdS \times S$ and quantum group symmetries. In some sense the chiral ring of the orbifold CFT dual to $AdS_3 \times S^3$ would be analogous to the q-sphere algebra. It would be a module algebra which admits a left and right action of $U_q(SU(2))$. We might also look for action of $U_q(SU(1,1))$ but even if such an action exists, it would be simpler to first focus of $SU(1,1)$ highest weights and identify the $U_q(SU(2))$ action. The existence of some relations between the fuzzy sphere and the chiral ring [14] and the results of the current paper relating fuzzy sphere to quantum groups give good reason to expect that a lot of information about the chiral ring might be encoded in the existence of a hidden quantum group symmetry.

There are other connections between fuzzy spheres and quantum group symmetric spaces explored in [34]. These q symmetric spaces are module algebras (with the standard q-co-product) which involve the V-representations of [26]. The finite dimensional quotient we studied has I-reps which in turn contain the V-reps sub-module. By considering conjugations of the standard co-product of U_q by the deformation map, we were lead to define a new product on the sub-module which makes it a sub-algebra as well. It will be interesting to see if there are examples of finite dimensional U_q module algebras where one has mixtures of I and V reps (when we use the standard q-coproduct) and which can nevertheless be related to fuzzy spheres after an appropriate modification of the product based on deformation maps.

It will be interesting to look for generalization of the connections between q-2sphere and fuzzy 2-sphere to the case of 4-spheres. q-4-spheres can be constructed along the lines of [35]. At roots of unity we expect finite dimensional truncations to exist. Relations between the q-sphere and its differential calculi with the fuzzy 4-sphere [36] would be a good test of usefulness of the non-commutative spheres in providing models of non-commutative space-time where similarities in the finite N physics of different backgrounds entering the ADS/CFT correspondence can be made manifest. The need for considering differential calculi would appear necessary from the observation of [37] that the deformed algebras relevant in the fuzzy 4-sphere case mix momenta and coordinates. Similar questions can be asked about the relation between classical and quantum spaces for the AdS part of space-time. Some relevant works on non-compact quantm groups are [38][39][40][41][42][43].

There have been other recent appearances of fuzzy spheres in the literature [44]. It will be very interesting to explore whether a picture of non-commutativity of space-time in string theory, e.g along the lines of [45][46], can coherently account for these diverse appearances of fuzzy structures.

Acknowledgements: We are especially grateful to Phillippe Roche for several illuminating discussions near the beginning of this project. We have also benefitted from useful communications with P.M. Ho, A. Matusis, J. Pawelczyk, H. Steinacker, L. Susskind. This research was supported by DOE grant DE-FG02/19ER40688-(Task A).

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