Permutation Centralizer Algebras and Multi-Matrix Invariants

Mattioli, P; Ramgoolam, S

PHYSICAL REVIEW D **93,** 065040 (2016)

Permutation centralizer algebras and multimatrix invariants

Paolo Mattioli* and Sanjaye Ramgoolam†

Centre for Research in String Theory, School of Physics and Astronomy, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom

(Received 7 February 2016; published 21 March 2016)

We introduce a class of permutation centralizer algebras which underly the combinatorics of multimatrix gauge-invariant observables. One family of such noncommutative algebras is parametrized by two integers. Its Wedderburn-Artin decomposition explains the counting of restricted Schur operators, which were introduced in the physics literature to describe open strings attached to giant gravitons and were subsequently used to diagonalize the Gaussian inner product for gauge invariants of two-matrix models. The structure of the algebra, notably its dimension, its center and its maximally commuting subalgebra, is related to Littlewood-Richardson numbers for composing Young diagrams. It gives a precise characterization of the minimal set of charges needed to distinguish arbitrary matrix gauge invariants, which are related to enhanced symmetries in gauge theory. The algebra also gives a star product for matrix invariants. The center of the algebra allows efficient computation of a sector of multimatrix correlators. These generate the counting of a certain class of bicoloured ribbon graphs with arbitrary genus.

DOI: 10.1103/PhysRevD.93.065040

I. INTRODUCTION

A number of questions on gauge-invariant functions and correlators of multiple matrices have been studied in the context of $\mathcal{N}=4$ super Yang-Mills (SYM) theory. The impetus for these developments in physics has come from the AdS/CFT correspondence [1–3], notably the duality between the $\mathcal{N}=4$ SYM theory with $U(N)$ gauge group and $\mathrm{AdS}_5\times S^5$. Local composite operators are $U(N)$ gauge invariants. CFT gives extra motivation because of the operator-state correspondence. Quantum states correspond to local operators, which are composite fields. These can be matrix-valued fields which are space-time scalars, fermions, field strengths or covariant derivatives of these. A generic problem is to understand $U(N)$ invariants constructed from a number n of such fields

$$\mathcal{F}^{j_1}_{1,i_1}\cdots\mathcal{F}^{j_n}_{n,i_n}. \tag{1.1}$$

This is subsequently used to understand their correlation functions. The n upper indices each transform in the fundamental of $U(N)$ while the lower indices transform in the antifundamental. Hence, an important ingredient is the nature of the invariants in

$$V^{\otimes n}\otimes\bar{V}^{\otimes n}. \tag{1.2}$$

The number of linearly independent invariants is $n!$. They are obtained by multiplying (1.1) with a product of n Kronecker delta functions, contracted with a permutation $\sigma\in S_n$. As n varies, we are interested in all possible values of n, so the properties of

$$\mathbb{C}[S_\infty]=\bigoplus_{n=0}^{\infty}\mathbb{C}[S_n]$$

become important. If all the n operators are the same e.g. a complex matrix $X=\phi_1+i\phi_2$ where ϕ_1, ϕ_2 are two of the six Hermitian matrices transforming in the vector of $SO(6)$, then the invariants are multitraces, of which there are $p(n)$, the number of partitions of n. In terms of the permutations, the composite operators are

$$\mathcal{O}_\sigma(X)=X^{i_1}_{i_{\sigma(1)}}\ldots X^{i_n}_{i_{\sigma(n)}}. \tag{1.3}$$

Distinct σ related by conjugation, i.e. σ and $\gamma\sigma\gamma^{-1}$ for some $\gamma\in S_n$ give the same operator

$$\mathcal{O}_\sigma(X)=\mathcal{O}_{\gamma\sigma\gamma^{-1}}(X). \tag{1.4}$$

When we consider invariants built from two types of matrices, say m copies of X and n copies of Y, then we encounter equivalence classes

$$\sigma\sim\gamma\sigma\gamma^{-1} \tag{1.5}$$

where $\sigma\in S_{m+n}$ and $\gamma\in S_m\times S_n$.

The fact that the enumeration of gauge-invariant operators can be effectively done by using a formulation in terms of equivalence classes of permutations has driven significant progress in the construction of operators and computation of correlators for the half-BPS sector, the perturbations of the half-BPS operators as well as quarter BPS operators. Two key facts have been used. One is that, by using the Fourier transformation which relates

*p.mattioli@qmul.ac.uk
†s.ramgoolam@qmul.ac.uk

functions on a group to matrix elements of irreducible representations, nice orthogonal bases of functions on these equivalence classes can be found. In mathematics, in the context of compact groups this is known as the Peter-Weyl theorem. In the context of finite groups, this follows from the Schur orthogonality relations. This leads to the construction of operators in the half-BPS sector parametrized by Young diagrams [4,5]. For the two-matrix sector, one application of this thinking leads to *restricted Schur operators*. These are labeled by three young diagrams and a pair of multiplicity labels: a Young diagram R_1 with m boxes, a Young diagram R_2 with n boxes and a third diagram R with $m+n$ boxes. The two multiplicity labels each run over a space of dimension equal to $g(R_1, R_2; R)$, which is equal to the Littlewood-Richardson (LR) coefficient for the number of times R appears in the tensor product of $R_1 \otimes R_2$ [6–10]. LR coefficients will be reviewed as needed in this paper (see Appendix B).

One reason for the efficacy of permutation groups in enumeration of gauge-invariant operators is Schur-Weyl duality. This states that the tensor product of n copies of the fundamental of $U(N)$ decomposes into a direct sum of irreps of $S_n \times U(N)$

$$V_N^{\otimes n} = \bigoplus_{\substack{R \vdash N \\ c_1(R) \leq N}} V_R^{S_n} \otimes V_R^{U(N)}. \tag{1.6}$$

Each summand is labeled by a Young diagram, and the Young diagrams are constrained to have no more than N rows, equivalently the first column $c_1(R)$ is no greater than N. This uses the fact that Young diagrams are used to classify representations of S_n as well as representations of $U(N)$. This is useful in the permutation approach to gauge-invariant operators, because it says that once we have organized operators according to representation data for S_n, it is easy to implement finite N constraints. In the one-matrix problem, the single Young diagram label R is cut off at N, $c_1(R) \leq N$. This leads directly to the connection between the stringy exclusion principle for giant gravitons and Young diagrams [4,11–13]. In the two-matrix problem, the Young diagram R is cut off at $c_1(R) \leq N$, which implies cutoffs for R_1, R_2. The two-matrix problem can also be approached using the walled Brauer algebra $B_N(m, n)$ and its representation theory [14]. A third way to enumerate two-matrix invariants, also based on permutations but involving Clebsch-Gordan multiplicities of S_n, keeps the $U(2)$ global symmetry manifest [15,16].

Aside from enumerating gauge-invariant operators, the permutation structures have been used to compute correlators. Correlators in free field theory are obtained by sums over Wick contractions. These sums are themselves parametrized by permutations. Correlators of gauge-invariant operators are thus given in terms of these Wick permutations and the permutations which enumerate the operators. Hence there are elegant formulas for the correlation functions in terms of permutations. It can be shown that the two-point functions of gauge-invariant operators in the two-matrix sector are diagonalized by operators constructed using representation bases. This was done with the Brauer basis in [14], with the $U(2)$ covariant basis in [15,16] and with the restricted Schur basis in [17,18]. The restricted Schur and covariant basis results have been extended beyond $\mathcal{N} = 4$ SYM to the sector of holomorphic operators in general quiver gauge theories [19–24] which have been shown to include sectors related to generalized oscillators [25]. Aspects involving Frobenius algebras have been studied in [26]. Within $\mathcal{N} = 4$ SYM, perturbations of half-BPS giant graviton operators have been studied and integrability at one-loop [24,27,27–30] and beyond has been established.

As a way to understand the existence of the different bases in the multimatrix problems, the authors of [31] conducted a detailed study of enhanced symmetries in the free limit of Yang Mills theories. The authors showed that Casimir-like elements constructed from Noether charges of these enhanced symmetries can be used to understand these different bases. Different sets of these Casimir-like charges each consist of mutually commuting simultaneously diagonalizable operators, which associate the labels of the basis with eigenvalues of Casimir-like charges. Thus there is a set of Casimir-like elements for the restricted Schur basis, another set for the covariant basis and yet another set for the Brauer basis. The enhanced symmetries themselves take the form of products of unitary groups, but the action of these Casimirs on gauge-invariant operators can be related, through applications of Schur-Weyl duality, to the algebraic structure of certain algebras constructed from the equivalence classes of permutations or of Brauer algebra elements discussed above. The discussion of charges which identify matrix invariants for general classical groups has been given using a different approach in [32]. While a uniform treatment of the Young diagram labels has been achieved, a treatment of the multiplicity labels running over Littlewood-Richardson coefficients in that approach remains an interesting open problem.

This paper was motivated by the goal of obtaining a systematic understanding of the algebraic structures involved in the construction of charges in [31]. To be more precise, we will define the notion of *permutation centralizer algebras*. A particular class of these, denoted as $\mathcal{A}(m, n)$, will be our main focus. Many of the important formulas we will use have already appeared in the physics literature. Nevertheless the $\mathcal{A}(m, n)$, as associative algebras with nondegenerate pairing, have not been made fully explicit. This paper proposes that these algebras are interesting to study intrinsically, disentangled from the contingencies of being embedded in a bigger symmetric group algebra, their simplicity hidden among the application to matrix correlators for matrices of size N. Here we define the algebras $\mathcal{A}(m, n)$, study their structure, and subsequently describe how they are relevant to matrix theory invariants. We expect that a deeper study of this algebraic structure has the potential to give a lot of information about correlators in free Yang-Mills theory, in the loop corrected theory, at all orders in the $1/N$ expansion. This paper is a step in this direction. Much as it is valuable to abstract Riemannian geometry from the study of submanifolds of Euclidean spaces, abstracting a family of algebras intrinsic to permutations hidden in the mathematics of matrix theory should be fruitful.

We describe the organization of the paper. In Sec. II we introduce the definition of permutation centralizer algebras. We consider four key examples of these algebras, which are useful in the context of gauge-invariant operators. In Sec. III, we focus on the algebras $\mathcal{A}(m,n)$ formed by equivalence classes of permutations in S_{m+n}, with equivalence generated by conjugation with permutations in $S_m \times S_n$. The dimension of this algebra is

$$|\mathcal{A}(m,n)| = \sum_{\substack{R_1\vdash m, R_2\vdash n\\ R\vdash m+n}} g(R_1,R_2;R)^2 \qquad (1.7)$$

where $g(R_1,R_2;R)$ is the LR coefficient for the triplet of Young diagram (R_1,R_2,R) made with $(m,n,m+n)$ boxes respectively. We will show that this is an associative algebra with a nondegenerate pairing. As a result, we know from the Wedderburn-Artin theorem that it is isomorphic to a direct sum of matrix algebras $\mathcal{M}at$ [33,34]:

$$\mathcal{A}(m,n) = \bigoplus_a \mathcal{M}at_a. \qquad (1.8)$$

In Eq. (3.5) we give a more precise version of this formula, where the index a is identified with triplets (R_1,R_2,R) with nonvanishing LR coefficient $g(R_1,R_2;R)$. The construction of restricted Schur operators in gauge theory is used to give the Wedderburn-Artin decomposition of $\mathcal{A}(m,n)$. Two subalgebras will be of interest. The center of the algebra $\mathcal{Z}(m,n)$ is the subspace of the algebra which commutes with any element of $\mathcal{A}(m,n)$. The dimension of this center is equal to the number of triples (R_1,R_2,R) of Young diagrams, with numbers of boxes equal to $(m,n,m+n)$, for which the LR coefficient is nonzero. It is useful to develop some formulas for the nondegenerate pairing on the center, using characters of S_{m+n}, S_m, S_n. The Wedderburn-Artin decomposition also highlights the importance of a maximally commuting subalgebra $\mathcal{M}(m,n)$. The dimension of this subalgebra is the sum of Littlewood-Richardson coefficients $g(R_1,R_2;R)$. Appendix A gives a multivariable generating function for this sum of LR coefficients. We explain the relevance of the this subalgebra to the enhanced symmetry charges studied in [35]. In particular we give a precise algebraic characterization (4.45) for the *minimal number of charges needed to identify all two-matrix gauge-invariant operators*. The evaluation of this number is an open problem for the future.

In Sec. IV, we explain some further physical implications of the permutation centralizer algebras. The simplest of these algebras is the algebra of class sums of permutations. Given the one-to-one correspondence between matrix operators and conjugacy classes of permutations given in (1.3), this means that there is a corresponding product on half-BPS operators. This is not the usual product obtained by multiplying the gauge-invariant operator built from X under which the dimension of the operator adds. The product on the class sums rather gives a product for the BPS operators of fixed dimension, a product which is associative and admits a nondegenerate pairing. We will refer to this as a *star product for half-BPS operators*. We explain the relevance of this star product for the computation of correlators. Similarly the product on the algebra $\mathcal{A}(m,n)$ gives a *star product for gauge-invariant polynomials in two matrices*, with degree m in the X's and degree n in the Y's. In the physics application, there is a closed associative star product on the space of quarter-BPS operators at zero Yang-Mills coupling. Conversely the usual product of gauge invariants gives a product on $\mathcal{A}(\infty,\infty)$

$$\mathcal{A}(\infty,\infty) = \bigoplus_{m,n=0}^{\infty} \mathcal{A}(m,n) \qquad (1.9)$$

which is the direct sum over all m, n. Thus $\mathcal{A}(\infty,\infty)$ has two products one of which closes at fixed m, n. This generalizes a structure seen in the study of symmetric polynomials.

In Sec. V, we show that the study of the structure of the algebra $\mathcal{A}(m,n)$ we developed in Sec. III is useful for the computation of correlators of two-matrix gauge invariants. In particular, we identify an *efficiently computable sector of central gauge-invariant operators* whose correlators can be computed using the knowledge of characters of S_{m+n}, S_m, S_n. It does not require the knowledge of more detailed data such as matrix elements $D^R_{ij}(\sigma)$ or branching coefficients for $S_{m+n} \to S_m \times S_n$. To illustrate the simplicity of this central sector, we compute the two-point function

$$\langle \mathrm{Tr}(X^m Y^n)\mathrm{Tr}((X^\dagger)^m (Y^\dagger)^n)\rangle \qquad (1.10)$$

at finite N. The computation requires a calculation of Littlewood-Richardson coefficients $g(R_1,R_2;R)$ where R_1, R_2 are hook-shaped Young diagrams. This computation is given in Appendix B. Further technical aspects of the computation are given in Appendix C. The computation agrees with the one in [36] which was done with explicit Young-Yamanouchi symbols which can be used to construct states in irreps R and describe their reduction to R_1, R_2.

In Sec. VI, we outline some future research directions related to the present results.

II. DEFINITIONS AND KEY EXAMPLES

When studying the representation theory of a group G, it is useful to introduce the algebra $\mathbb{C}[G]$ which consists of formal linear combinations of group elements, equipped with the multiplication inherited from the group. In the group algebra $\mathbb{C}[G]$, for each conjugacy class, we can form a sum over all the elements in the conjugacy class of G. Such class sums commute with any element of G and form the central subalgebra of $\mathbb{C}[G]$, i.e. the subalgebra which commutes with all $\mathbb{C}[G]$. We will refer to $\mathcal{Z}[\mathbb{C}[G]]$ as the center of $\mathbb{C}[G]$. Conjugacy classes are in 1-1 correspondence with irreducible representations and there is a basis of the center consisting of projectors of the form

$$P_R = \frac{d_R}{|G|}\sum_{g\in G}\chi_R(g)g^{-1}. \tag{2.1}$$

Of primary interest to us is the group algebra of $\mathbb{C}[S_n]$ and its center $\mathcal{Z}[\mathbb{C}[S_n]]$. The elements in $\mathcal{Z}[\mathbb{C}[S_n]]$ are sums over conjugacy classes t of S_n

$$T = \sum_{\sigma\in t}\sigma. \tag{2.2}$$

Given any $\sigma \in S_n$, we can generate an element of this subalgebra by summing over $\gamma \in H$,

$$\sum_{\gamma\in S_n}\gamma\sigma\gamma^{-1}. \tag{2.3}$$

Some properties of group algebras and their center can be found in [33,37]. In the context of AdS/CFT, group algebras $\mathbb{C}[S_n]$ and associated representation theory play a role in the half-BPS sector of $\mathcal{N} = 4$ SYM in 4D [4,5] and also in the symmetric orbifolds in AdS3/CFT2 [11,38]. Motivated by developments in AdS/CFT we will introduce a generalization of this construction.

Definition.—Consider an associative algebra $\mathbb{A}$ containing a subalgebra $\mathbb{B} = \mathbb{C}[H]$, the group algebra of a finite group H. Now define the subspace of $\mathbb{A}$ of elements which are invariant under conjugation by H. This subspace will contain group averages of the form

$$\sum_{\gamma\in H}\gamma\sigma\gamma^{-1}, \qquad \sigma\in\mathbb{A} \tag{2.4}$$

which commute with elements of $\mathbb{B}$. It is easy to verify that these subspaces are subalgebras. We have

$$\left(\sum_{\gamma_1\in H}\gamma_1\sigma\gamma_1^{-1}\right)\left(\sum_{\gamma_2\in H}\gamma_2\sigma\gamma_2^{-1}\right) = \sum_{\gamma_1\in H}\gamma_1\left(\sum_{\gamma_3\in H}\sigma_1\gamma_3\sigma_2\gamma_3^{-1}\right)\gamma_1^{-1} \tag{2.5}$$

where we set $\gamma_3 = \gamma_1^{-1}\gamma_2$. This shows that the product of two group averages is still a group average. This subalgebra of $\mathbb{A}$ commuting with $\mathbb{B}$, in cases where H is a permutation group, will be called a *permutation centralizer algebra*.

Three cases of primary interest will be

(i) Example 1.—The algebra $\mathbb{A} = \mathbb{C}[S_n]$. The algebra $\mathbb{B} = \mathbb{C}[S_n]$. The centralizer of $\mathbb{B}$ is $\mathcal{Z}[\mathbb{C}[S_n]]$.
(ii) Example 2.—$\mathbb{A} = \mathbb{C}[S_{m+n}]$; $\mathbb{B} = \mathbb{C}[S_m \times S_n]$. We will call this algebra $\mathcal{A}(m,n)$.
(iii) Example 3.—$\mathbb{A} = B_N(m,n)$—the walled Brauer algebra; $\mathbb{B} = \mathbb{C}[S_m \times S_n]$. This algebra is called $\mathcal{B}_N(m,n)$.
(iv) Example 4.—$\mathbb{A} = \mathbb{C}[S_n \times S_n]$; $\mathbb{B} = \mathbb{C}[S_n]$ where the latter is the S_n diagonally embedded in the product group. This should be called $\mathcal{K}(n)$.

The case where $\mathbb{A}$ is itself a group algebra has been studied in mathematics, for example, in [39].

Our primary interest in this paper will be in $\mathcal{A}(m,n)$ of example 2. $\mathcal{Z}[\mathbb{C}[S_n]]$ of example 1 will be a useful guide and a source of analogies in our investigations. Fourier transformation on $\mathcal{A}(m,n)$ will be related to restricted Schur operators studied in AdS/CFT. These are parametrized by representation theory data (R, R_1, R_2, i, j) consisting of Young diagrams R_1, R_2, R with m, n, $m+n$ boxes as well as multiplicity indices i, j. The latter take values $1 \leq i, j \leq g(R_1, R_2; R)$ where $g(R_1, R_2; R)$ is the LR multiplicity for the triple of Young diagrams computed with the LR combinatoric rule (see for example [40]). Unlike $\mathcal{Z}[\mathbb{C}[S_n]]$, the algebra $\mathcal{A}(m,n)$ is not commutative. The central subalgebra $\mathcal{Z}(m,n)$, consisting of the subspace $\mathcal{Z}(m,n) \subset \mathcal{A}(m,n)$ which commutes with all of $\mathcal{A}(m,n)$ will play a predominant role. Likewise the algebras $\mathcal{B}_N(m,n)$ and $\mathcal{K}(n)$ in examples 3 and 4 are noncommutative.

III. STRUCTURE OF THE $\mathcal{A}(m,n)$ ALGEBRA

The algebra $\mathcal{A}(m,n)$ is constructed by taking all the elements in $\mathbb{C}[S_{m+n}]$ which are invariant under $\mathbb{C}[S_m \times S_n]$. Any element of $\sigma \in \mathbb{C}[S_{m+n}]$ can be mapped to a $\bar{\sigma} \in \mathcal{A}(m,n)$ by the group averaging

$$\bar{\sigma} = \sum_{\gamma\in S_m\times S_n}\gamma^{-1}\sigma\gamma. \tag{3.1}$$

The $\bar{\sigma}$ are formal sums of permutations τ lying in the same orbit of σ under the $S_m \times S_n$ action. Each τ has a stabilizer group, given by those $\gamma \in S_m \times S_n$ for which

$$\gamma^{-1}\tau\gamma = \tau. \tag{3.2}$$

The stabilizers of two permutations τ_1, τ_2 in the same orbit are generally different (they are conjugate to each other), but they have the same dimension. By the orbit-stabiliser theorem, $\bar{\sigma}$ is then a sum of permutations weighted by the same coefficient:

$$\bar{\sigma} = |\mathrm{Aut}_{S_m\times S_n}(\sigma)| \sum_{\tau\in\mathrm{Orbit}(\sigma,S_m\times S_n)} \tau. \tag{3.3}$$

$\mathcal{A}(m,n)$ is a finite-dimensional associative algebra (the associativity follows from the associativity of $\mathbb{C}[S_{m+n}]$), which we can equip with the nondegenerate symmetric bilinear form

$$\langle\bar{\sigma}_1,\bar{\sigma}_2\rangle = \delta(\bar{\sigma}_1\bar{\sigma}_2),\quad \bar{\sigma}_{1,2}\in\mathcal{A}(m,n). \tag{3.4}$$

Here the delta function on the group algebra $\mathbb{C}[S_{m+n}]$ is a linear function which obeys $\delta(\sigma)=1$ for $\sigma=1$ and $\delta(\sigma)=0$ otherwise.

The nondegeneracy of the bilinear form (3.4) implies that $\mathcal{A}(m,n)$ is semisimple. According to the Wedderburn-Artin theorem, it can then be decomposed into a direct sum of matrix algebras:

$$\mathcal{A}(m,n) = \bigoplus_{\substack{R\vdash m+n\\ R_1\vdash m, R_2\vdash n}} \mathrm{Span}\{Q^R_{R_1,R_2,i,j}; i,j\}. \tag{3.5}$$

In this equation R, R_1 and R_2 are representations of S_{m+n}, S_m and S_n respectively. The integers i, j run over the multiplicity $g(R_1,R_2;R)$ of the branching $R\to R_1\otimes R_2$: $0\le i,j\le g(R_1,R_2;R)$. An explicit expression for $Q^R_{R_1,R_2,i,j}$ is given in terms of the restricted Schur characters [17,22,31], defined as

$$\chi^R_{R_1,R_2,i,j}(\sigma) = D^R_{m,m'}(\sigma)B^{R\to R_1,R_2;i}_{m'\to l_1,l_2}\quad B^{R\to R_1,R_2;j}_{m\to l_1,l_2}. \tag{3.6}$$

Here $D^R_{m,m'}(\sigma)$ are the matrix elements of σ in the irreducible representation R. $B^{R\to R_1,R_2;j}_{m\to l_1,l_2}$ is the branching coefficient for the representation branching $R\to R_1\otimes R_2$, in the jth copy of $R_1\otimes R_2\subset R$. $l_{1,2}$ are states in $R_{1,2}$. The restricted Schur characters $\chi^R_{R_1,R_2,i,j}(\sigma)$ are invariant under conjugation by $\mathbb{C}[S_m\times S_n]$ elements. With these definitions we can write

$$Q^R_{R_1,R_2,i,j} = \sum_\sigma \chi^R_{R_1,R_2,i,j}(\sigma)\sigma \tag{3.7}$$

which is manifestly invariant under the action of $\mathbb{C}[S_m\times S_n]$. It follows that

$$Q^R_{R_1,R_2,i,j}Q^S_{S_1,S_2,k,l} = \delta^{R,S}\delta_{R_1,S_1}\delta_{R_2,S_2}(\delta_{jk}Q^R_{R_1,R_2,i,l}). \tag{3.8}$$

This is in accordance with the decomposition (3.5). Consequently it is useful to write $Q^R_{R_1,R_2,i,j}$ as

$$Q^R_{R_1,R_2,i,j} = \sum_{m_1,m_2}|R\to R_1,R_2,m_1,m_2,i\rangle \times\langle R\to R_1,R_2,m_1,m_2,j|. \tag{3.9}$$

Moreover, the basis $\{Q^R_{R_1,R_2,i,j}\}$ is complete as we now explain. The number of distinct $Q^R_{R_1,R_2,i,j}$'s is equal to the number of restricted Schur characters, which is in turn equal to $\sum_{R_1,R_2,R}g(R_1,R_2;R)^2$. On the other hand the dimension of $\mathcal{A}(m,n)$ is by definition equal to the number of elements of $\mathbb{C}[S_{m+n}]$ invariant under the $\mathbb{C}[S_m\times S_n]$ action. Using the Burnside lemma, it is possible to show that this dimension $|\mathcal{A}(m,n)|$ is given as

$$|\mathcal{A}(m,n)| = \sum_{\substack{R_1\vdash m,R_2\vdash n\\ R\vdash m+n}} g(R_1,R_2,R)^2. \tag{3.10}$$

In each of the blocks in (3.5) there is a projector of the form $P^R_{R_1,R_2}=\sum_i Q^R_{R_1,R_2,i,i}$. Let now P_R, P_{R_1} and P_{R_2} be the projectors onto the irreps R, R_1 and R_2 of S_{m+n}, S_m and S_n respectively. Since

$$\begin{aligned}\langle R\to R_1,R_2,m_1,m_2,i|P_RP_{R_1}P_{R_2}|R\to R_1,R_2,m_1',m_2',j\rangle\\ =\langle R\to R_1,R_2,m_1,m_2,i|P^R_{R_1,R_2}|R\to R_1,R_2,m_1',m_2',j\rangle\\ =\delta_{m_1,m_1'}\delta_{m_2,m_2'}\delta_{i,j}\end{aligned} \tag{3.11}$$

for all triplets R, R_1, R_2, we can write

$$P^R_{R_1,R_2} = P_RP_{R_1}P_{R_2} \tag{3.12}$$

so that the projectors $P^R_{R_1,R_2}$ are just products of ordinary S_{m+n}, S_m and S_n projectors. The set $\{P^R_{R_1,R_2}\}$ forms a basis for the center of $\mathcal{A}(m,n)$, which we call $\mathcal{Z}(m,n)$. Its dimension is then given by the number of nonvanishing LR coefficients $g(R_1,R_2;R)$, or

$$|\mathcal{Z}(m,n)| = \sum_{\substack{R_1\vdash m,R_2\vdash n\\ R\vdash m+n}}(1-\delta(g(R_1,R_2;R))). \tag{3.13}$$

Here $\delta(g(R_1,R_2;R))=1$ if $g(R_1,R_2;R)=0$ and $\delta(g(R_1,R_2;R))=0$ otherwise. The generating function for the dimension of the center is [41]

$$\mathcal{Z}(x,y) = \prod_i \frac{1}{(1-x^i-y^i)}. \tag{3.14}$$

We will now argue that the collection of the generators of the centers of $\mathbb{C}[S_{m+n}]$, $\mathbb{C}[S_m]$ and $\mathbb{C}[S_n]$, that we denote as $\{T^{(m+n)}_p\}$, $\{T^{(m)}_{q_1}\}$ and $\{T^{(n)}_{q_2}\}$ respectively, is a set of generators for $\mathcal{Z}(m,n)$. Here p, q_1 and q_2 are integer partitions of $m+n$, m and n respectively. For example, for the partition $p = (p_1, p_2, \ldots)$ of $m+n$, the operator $T_p^{(m+n)}$ consists of a sum over permutations belonging to the conjugacy class $p = (p_1, p_2, \ldots)$:

$$T_p^{(m+n)} = \sum_{i_1,\ldots,i_{p_1+p_2+\ldots} \in [m+n]} (i_1 i_2 \cdots i_{p_1}) \times (i_{p_1+1} i_{p_1+2} \cdots i_{p_1+p_2}) \cdots . \tag{3.15}$$

$T_p^{(m+n)}$ are sums of conjugates by elements of S_{m+n}, whereas $T_{q_1}^{(m)}$ and $T_{q_2}^{(n)}$ are sums over $S_m \subset S_{m+n}$ and $S_n \subset S_{m+n}$ respectively. To show that $\{T_p^{(m+n)}, T_{q_1}^{(m)}, T_{q_2}^{(n)}\}$ generate the whole center $\mathcal{Z}(m,n)$ we can use the following argument. Using the Wedderburn-Artin decomposition (3.5), we see that the center of $\mathcal{A}(m,n)$ is the direct sum of the centers of the matrix algebras $\mathrm{Span}\{Q^R_{R_1,R_2,i,j}; i,j\}$. For each of these matrix blocks, that is for any fixed representations R, R_1, R_2 for which $g(R_1,R_2;R) \neq 0$, the center is one-dimensional, and is spanned by

$$P^R_{R_1,R_2} = \sum_{i=1} Q^R_{R_1,R_2,i,i}. \tag{3.16}$$

Using Eq. (3.8), it is immediate to check that

$$[P^R_{R_1,R_2}, Q^R_{R_1,R_2,i,j}] = 0, \quad \forall\ i,j. \tag{3.17}$$

We know that $P^R_{R_1,R_2} = P_R P_{R_1} P_{R_2}$, with P_R, P_{R_1} and P_{R_2} projectors on the representations R, R_1 and R_2. Therefore every central element of $\mathcal{A}(m,n)$ can be generated with the collection of projectors $\{P_R, P_{R_1}, P_{R_2}\}$. For an R irrep of S_n, the projector is

$$P_R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma = \frac{1}{n!} \sum_{p \in \mathrm{Partitions}(n)} \chi_R(\sigma_p) T_p^{(n)} \tag{3.18}$$

where σ_p is a representative permutation belonging to the conjugacy class $p \vdash n$. This means that every projector P_R can be written as a linear combination of the central elements $\{T_p^{(n)}\}$. We can then write the set $\{P_R, P_{R_1}, P_{R_2}\}$ in terms of the central elements $\{T_p^{(m+n)}, T_{q_1}^{(m)}, T_{q_2}^{(n)}\}$. Since we know that the former generates the whole $\mathcal{Z}(m,n)$, we can now conclude that the latter is a complete set of generators for the center $\mathcal{Z}(m,n)$ as well. The basis thus obtained will be useful in the following sections. However, it is important to point out that such a basis is overcomplete. An easy way to see it is to note that, given (3.12), $P_R P_{R_1} P_{R_2} = 0$ if $g(R_1,R_2,R) = 0$. Therefore, taking a triplet (R_1,R_2,R) for which $g(R_1,R_2,R) = 0$ we have, using (3.18),

$$\frac{1}{(m+n)!m!n!} \sum_{\substack{p \vdash (m+n) \\ q_1 \vdash m, q_2 \vdash n}} \chi_R(\sigma_p)\chi_{R_1}(\sigma_{q_1})\chi_{R_2}(\sigma_{q_2}) \times T_p^{(m+n)} T_{q_1}^{(m)} T_{q_2}^{(n)} = 0. \tag{3.19}$$

This shows that $\{T_p^{(m+n)}, T_{q_1}^{(m)}, T_{q_2}^{(n)}\}$ is indeed an overcomplete basis.

We can also argue that $\{T_p^{(m+n)}, T_{q_1}^{(m)}, T_{q_2}^{(n)}\}$ generate $\mathcal{Z}(m,n)$ just by using the Schur-Weyl duality as in [31]. The $T^{(m)}$ elements are Schur-Weyl dual to $U(N)$ Casimirs of acting on the upper m indices of X-type matrices. This action is generated by

$$(E_x)^i_j = (D_x)^i_j = X^i_l \frac{\partial}{\partial X^j_l} \tag{3.20}$$

The $T^{(n)}$ elements are Schur-Weyl dual to $U(N)$ Casimirs acting on the upper n indices of Y-type matrices. We have

$$(E_y)^i_j = (D_y)^i_j = Y^i_l \frac{\partial}{\partial Y^j_l}. \tag{3.21}$$

Finally, the $T^{(m+n)}$ elements are Schur-Weyl dual to $U(N)$ Casimirs acting on the upper n and m indices of both X- and Y-type matrices, and the generator is

$$E^i_j = (E_x)^i_j + (E_y)^i_j. \tag{3.22}$$

We then have three distinct types of Casimirs:

$$\begin{aligned} C_k^{(m+n)} &= E^{i_1}_{i_2} E^{i_2}_{i_3} \cdots E^{i_k}_{i_1}, \\ C_k^{(m)} &= (E_x)^{i_1}_{i_2} (E_x)^{i_2}_{i_3} \cdots (E_x)^{i_k}_{i_1}, \\ C_k^{(n)} &= (E_y)^{i_1}_{i_2} (E_y)^{i_2}_{i_3} \cdots (E_y)^{i_k}_{i_1}. \end{aligned} \tag{3.23}$$

But the $C_k^{(m+n)}$, the $C_k^{(m)}$ and the $C_k^{(n)}$ operators measure respectively the R, R_1 and R_2 labels of the restricted Schurs $\chi^R_{R_1,R_2,i,j}$. Therefore they can be used to isolate every subspace $R_1 \otimes R_2 \subseteq R$, and to build all the correspondent projectors $P^R_{R_1,R_2}$. Since we know that each of these projectors is in a 1-1 correspondence with an element of $\mathcal{Z}(m,n)$, the whole center $\mathcal{Z}(m,n)$ is obtained.

On the other hand, noncentral elements are needed to measure the multiplicity labels i, j. This observation will be developed in Sec. IV.

A. Symmetric group characters and the pairing on the center $\mathcal{Z}(m,n)$

A central element $Z_a \in \mathcal{Z}(m,n)$ can be expanded in terms of the projectors $P^R_{R_1,R_2}$ as

$$Z_a = \sum_{R,R_1,R_2} Z_a^{R,R_1,R_2} P^R_{R_1,R_2}. \tag{3.24}$$

We can then define

$$\begin{aligned}\chi^R_{R_1,R_2;i,j}(Z_a) &= \sum_{m_1,m_2} \langle R \to R_1, R_2, m_1, m_2, i | Z_a | R \to R_1, R_2, m_1, m_2, j\rangle \\ &= \delta_{ij} \sum_{S,S_1,S_2} \sum_{m_1,m_2} Z_a^{S,S_1,S_2} \langle R \to R_1, R_2, m_1, m_2, i | P^S_{S_1,S_2} | R \to R_1, R_2, m_1, m_2, j\rangle \\ &= \delta_{ij} Z_a^{R,R_1,R_2} d_{R_1} d_{R_2}\end{aligned} \tag{3.25}$$

and

$$\chi^R_{R_1,R_2}(Z_a) = \sum_i \chi^R_{R_1,R_2;i,i}(Z_a) = Z_a^{R,R_1,R_2} g(R_1,R_2,R) d_{R_1} d_{R_2}. \tag{3.26}$$

From these equations it also follows that for any central element Z_a

$$\chi^R_{R_1,R_2,i,j}(Z_a) = \frac{\delta_{i,j}}{g(R_1,R_2;R)} \chi^R_{R_1,R_2}(Z_a). \tag{3.27}$$

Another useful expansion is in terms of $\{T_p^{(m+n)}\}$, $\{T_{q_1}^{(m)}\}$ and $\{T_{q_2}^{(n)}\}$. Since these elements generate the center, we can write

$$Z_a = Z_a^{p,q_1,q_2} T_p^{(m,n)} T_{q_1}^{(m)} T_{q_2}^{(n)} \tag{3.28}$$

for some Z_a^{p,q_1,q_2} coefficients. However, since the basis generated by $\{T_p^{(m+n)}, T_{q_1}^{(m)}, T_{q_2}^{(n)}\}$ is overcomplete, such coefficients are not unique. Using the expansion (3.28), we can write

$$\chi^R_{R_1,R_2,i,j}(Z_a) = \delta_{ij} Z_a^{p,q_1,q_2} \frac{\chi_R(T_p^{(m+n)})}{d_R} \chi_{R_1}(T_{q_1}^{(m)}) \chi_{R_2}(T_{q_2}^{(n)}) \tag{3.29}$$

and

$$\chi^R_{R_1,R_2}(Z_a) = \sum_i \chi^R_{R_1,R_2,i,i}(Z_a) = Z_a^{p,q_1,q_2} g(R_1,R_2,R) \frac{\chi_R(T_p^{(m+n)})}{d_R} \chi_{R_1}(T_{q_1}^{(m)}) \chi_{R_2}(T_{q_2}^{(n)}). \tag{3.30}$$

From these equation we see that all the restricted characters of central elements are determined by characters of S_{m+n}, S_m, S_n. Just as the center of S_n is generated by class sums, which are dual to irreducible characters of S_n, the center $\mathcal{Z}(m,n)$ of $\mathcal{A}(m,n)$ is dual to the characters $\chi^R_{R_1,R_2}$ which are nothing but products of characters. Therefore, to compute restricted characters of elements in $\mathcal{Z}(m,n)$ we only need the ordinary symmetric group character theory.

We will now use some of the known equations for the character of symmetric group and use them to compute restricted characters in $\mathcal{Z}(m,n)$. Our aim will be to compute the dual pairing (3.4) for central elements. Equation (B.12) in [22] reads

$$\frac{(m+n)!}{m!n!} \sum_{\gamma \in S_m \times S_n} \delta(\sigma\gamma\tau\gamma^{-1}) = \sum_{R,R_1,R_2,i,j} \frac{d_R}{d_{R_1} d_{R_2}} \chi^R_{R_1,R_2,i,j}(\sigma) \chi^R_{R_1,R_2,i,j}(\tau). \tag{3.31}$$

By setting $\tau = 1$ this equation simplifies to

$$(m+n)!\delta(\sigma) = \sum_R d_R \chi^R_{R_1,R_2,i,i}(\sigma) \tag{3.32}$$

where we used

$$\chi^R_{R_1,R_2,i,j}(1) = \delta_{ij} d_{R_1} d_{R_2}. \tag{3.33}$$

We can immediately use this result to show that $\delta(Q^R_{R_1,R_2,i,j}) = \delta_{ij} d_{R_1} d_{R_2}$. This is because, using (3.7)

$$\delta(Q^R_{R_1,R_2,i,j}) = \sum_\sigma \chi^R_{R_1,R_2,i,j}\delta(\sigma) = \chi^R_{R_1,R_2,i,j}(1) = \delta_{ij} d_{R_1} d_{R_2}. \tag{3.34}$$

It is also worthwhile to notice that, for $\mathcal{O} \in \mathcal{A}(m,n)$, $\mathrm{Tr}(\mathcal{O}) = \delta(\mathcal{O})$. Therefore we could have obtained the same result by considering

$$\begin{aligned}\mathrm{Tr}(Q^R_{R_1,R_2,i,j}) &= \sum_{S,S_1,S_2} \sum_{\substack{m_1,m_2\\ m_1',m_2'}} \sum_k \langle S \to S_1, S_2, m_1', m_2', k | R \to R_1, R_2, m_1, m_2, i\rangle \langle R \to R_1, R_2, m_1, m_2, j | S \to S_1, S_2, m_1', m_2', k\rangle \\ &= \delta_{ij} d_{R_1} d_{R_2}\end{aligned} \tag{3.35}$$

where we used the definition (3.9).

Let us now go back to Eq. (3.32). If we replace σ by a central element Z_a, using the expansion (3.28) and Eq. (3.30), we find

$$(m+n)!\delta(Z_a) = \sum_{R,R_1,R_2} Z_a^{p,q_1,q_2} g(R_1,R_2,R)\chi_R(T_p^{(m+n)})\chi_{R_1}(T_{q_1}^{(m)})\chi_{R_2}(T_{q_2}^{(n)}). \tag{3.36}$$

By further replacing $\sigma \to Z_a$, $\tau \to Z_b$ in (3.31) we get, in a similar fashion

$$\begin{aligned}(m+n)!\delta(Z_a Z_b) &= \sum_{R,R_1,R_2,i,j} \frac{d_R}{d_{R_1} d_{R_2}} \chi^R_{R_1,R_2,i,j}(Z_a)\chi^R_{R_1,R_2,i,j}(Z_b) \\ &= Z_a^{p,q_1,q_2} Z_b^{p',q_1',q_2'} \sum_{R,R_1,R_2} \frac{g(R_1,R_2,R)}{d_R d_{R_1} d_{R_2}} \chi_R(T_p^{(m+n)})\chi_{R_1}(T_{q_1}^{(m)})\chi_{R_2}(T_{q_2}^{(n)})\chi_R(T_{p'}^{(m+n)})\chi_{R_1}(T_{q_1'}^{(m)})\chi_{R_2}(T_{q_2'}^{(n)}).\end{aligned} \tag{3.37}$$

Comparing the left-hand side above with Eq. (3.4) we find that for central elements Z_a, Z_b

$$\langle Z_a, Z_b\rangle = Z_a^{p,q_1,q_2} Z_b^{p',q_1',q_2'} \frac{1}{(m+n)!} \sum_{R,R_1,R_2} \frac{g(R_1,R_2,R)}{d_R d_{R_1} d_{R_2}} \times \chi_R(T_p^{(m+n)})\chi_{R_1}(T_{q_1}^{(m)})\chi_{R_2}(T_{q_2}^{(n)})\chi_R(T_{p'}^{(m+n)})\chi_{R_1}(T_{q_1'}^{(m)})\chi_{R_2}(T_{q_2'}^{(n)}). \tag{3.38}$$

Thus we have an explicit way of computing the dual paring on the center $\mathcal{Z}(m,n)$ in terms of ordinary S_n characters.

Similarly, there is a character expansion for $\delta(Z_a Z_b Z_c)$. We begin by writing

$$\begin{aligned}(m+n)!\delta(Z_a Z_b Z_c) &= \sum_{R,R_1,R_2,i,j} \frac{d_R}{d_{R_1} d_{R_2}} \chi^R_{R_1,R_2,i,j}(Z_a Z_b)\chi^R_{R_1,R_2,i,j}(Z_c) \\ &= \sum_{R,R_1,R_2} \frac{d_R}{d_{R_1} d_{R_2} g(R_1,R_2;R)} \chi^R_{R_1,R_2}(Z_a Z_b)\chi^R_{R_1,R_2}(Z_c).\end{aligned} \tag{3.39}$$

Since Z_a is central, $Z_a = (c_a)^R_{R_1,R_2} 1$, where $(c_a)^R_{R_1,R_2}$ is a constant. This constant can be obtained by considering

$$\chi^R_{R_1,R_2,i,j}(Z_a) = (c_a)^R_{R_1,R_2}\chi^R_{R_1,R_2}(1) = (c_a)^R_{R_1,R_2} d_{R_1} d_{R_2} g(R_1,R_2;R). \tag{3.40}$$

We therefore have that

$$\chi^R_{R_1,R_2}(Z_a Z_b) = \frac{\chi^R_{R_1,R_2}(Z_a)\chi^R_{R_1,R_2}(Z_b)}{d_{R_1} d_{R_2} g(R_1,R_2;R)}. \tag{3.41}$$

Using (3.41) in (3.39), and then exploiting (3.30), we obtain

$$\begin{aligned}(m+n)!\delta(Z_aZ_bZ_c) &= \sum_{R,R_1,R_2}\frac{d_R}{d_{R_1}^2 d_{R_2}^2 g(R_1,R_2;R)^2}\chi^R_{R_1,R_2}(Z_a)\chi^R_{R_1,R_2}(Z_b)\chi^R_{R_1,R_2}(Z_c)\\ &= Z_a^{p,q_1,q_2}Z_b^{p',q_1',q_2'}Z_c^{p'',q_1'',q_2''}\sum_{R,R_1,R_2}\frac{g(R_1,R_2;R)}{d_R^2 d_{R_1}^2 d_{R_2}^2}\chi_R(T_p^{(m+n)})\chi_{R_1}(T_{q_1}^{(m)})\chi_{R_2}(T_{q_2}^{(n)})\chi_R(T_{p'}^{(m+n)})\\ &\quad\times\chi_{R_1}(T_{q_1'}^{(m)})\chi_{R_2}(T_{q_2'}^{(n)})\chi_R(T_{p''}^{(m+n)})\chi_{R_1}(T_{q_1''}^{(m)})\chi_{R_2}(T_{q_2''}^{(n)}).\end{aligned} \tag{3.42}$$

More generally, we can use (3.41) to compute the identity coefficient of arbitrary large products of central elements, $\delta(Z_aZ_b\cdots Z_k)$, just by using ordinary symmetric group characters.

B. Maximal commuting subalgebra

In this section we describe the Maximal commuting subalgebra $\mathcal{M}(m,n)$ of $\mathcal{A}(m,n)$:

$$\mathcal{Z}(m,n)\subseteq\mathcal{M}(m,n)\subseteq\mathcal{A}(m,n). \tag{3.43}$$

We often refer to $\mathcal{M}(m,n)$ as the Cartan subalgebra of $\mathcal{A}(m,n)$. $\mathcal{M}(m,n)$ is spanned by elements of the form $Q^R_{R_1,R_2,i,i}$ (no sum over i). For fixed R_1, R_2 and R, the total number of basis elements is $g(R_1,R_2;R)$, so that its dimension is

$$|\mathcal{M}(m,n)| = \sum_{\substack{R_1\vdash m,R_2\vdash n\\ R\vdash m+n}} g(R_1,R_2;R). \tag{3.44}$$

In Appendix A we derived the dimension formula

$$|\mathcal{M}(m,n)| = \sum_{p\vdash m}\sum_{q\vdash n}\mathcal{F}_p\mathcal{F}_q\mathcal{F}_{p+q}Sym(p+q) \tag{3.45}$$

where p, q are partitions of m and n, $\mathcal{F}_p$, $\mathcal{F}_q$, $\mathcal{F}_{p+q}$ are combinatorial quantities dependent only on the partitions p, q and $p+q$ respectively, and $\mathrm{Sym}(p+q) = \prod_i i^{p_i+q_i}(p_i+q_i)!$ is a symmetry factor.

We now turn to the problem of constructing a basis for $\mathcal{M}(m,n)$. According to the definition (3.9), to write the basis elements $Q^R_{R_1,R_2,i,i}$ we first need to compute the branching coefficients for the branching $R\to R_1\otimes R_2$. These quantities are in general computationally difficult to obtain[1] and require a choice of a basis in S_{m+n} representations adapted to $S_m\times S_n$. However, using the correspondence with matrix algebras given by the Wedderburn-Artin decomposition, we can construct the Cartan by solving, in each block, the following equations for $(g(R_1,R_2;R)-1)$ linearly independent elements $Q^R_{R_1,R_2,a}\in\mathcal{A}(m,n)$:

$$P^R_{R_1,R_2}Q^R_{R_1,R_2,a} = Q^R_{R_1,R_2,a}, \tag{3.46a}$$

$$\langle P^R_{R_1,R_2}, Q^R_{R_1,R_2,a}\rangle = 0, \tag{3.46b}$$

$$[Q^R_{R_1,R_2,a}, Q^R_{R_1,R_2,b}] = 0. \tag{3.46c}$$

In the second equation, we are using the pairing defined in (3.4).

IV. STAR PRODUCT FOR COMPOSITE OPERATORS

In the previous sections we discussed the algebra $\mathcal{A}(m,n)$ and its center $\mathcal{Z}(m,n)$. We noted that central elements are special, as all their properties only depend on ordinary symmetric group character theory. An example of this is Eq. (3.42). In this section we will take advantage of this fact to compute physically relevant quantities, in particular two and three point functions of BPS operators in $\mathcal{N}=4$ SYM. To do so, we will first start by discussing the one matrix sector in $\mathcal{N}=4$ SYM, reviewing the permutation description of $U(N)$ matrix invariants which are gauge-invariant operators (GIOs) in the conformal field theory. We will stress that for this case there is an underlying $\mathcal{Z}[\mathbb{C}[S_n]]$ algebra. The one matrix problem will be used as a guide to extend to the two-matrix problem, which we treat in Sec. IV B. Here the underlying algebra will be $\mathcal{A}(m,n)$.

A. One matrix problem

Let us consider a matrix invariant constructed with n copies of the same matrix Z. Any such invariant can be written in terms of a contraction

$$\mathcal{O}_\sigma(Z) = \mathrm{tr}(Z^{\otimes n}\sigma), \qquad \sigma\in S_n \tag{4.1}$$

subject to the equivalence relation

[1]See for example a discussion of the difficulty and the simplifications in a "distant corners approximation" in [30].

$$\mathcal{O}_{\sigma}(Z) = \mathcal{O}_{\gamma^{-1}\sigma\gamma}(Z), \qquad \gamma \in S_n. \tag{4.2}$$

Polynomials in Z like the one in (4.1) can be multiplied together. Set $\sigma_1 \in S_{n_1}$, $\sigma_2 \in S_{n_2}$. By multiplying together $\mathcal{O}_{\sigma_1}(Z)$ and $\mathcal{O}_{\sigma_2}(Z)$ we get

$$\mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z) = \mathcal{O}_{\sigma_1\circ\sigma_2}(Z) \tag{4.3}$$

where $\sigma_1\circ\sigma_2 \in S_{n_1} \times S_{n_2} \subset S_{n_1+n_2}$. Therefore for the usual product of matrix invariants, $\sigma_1\circ\sigma_2$ lives in the symmetric group of degree $n_1 + n_2$. We can define

$$\mathbb{C}[S_\infty] = \bigoplus_n \mathbb{C}[S_n] \tag{4.4}$$

which is closed under the circle product

$$\circ\colon\ \mathbb{C}[S_\infty] \otimes \mathbb{C}[S_\infty] \to \mathbb{C}[S_\infty]. \tag{4.5}$$

However, we can define another associative product, which we call *star product*, which closes on the operators of fixed degree:

$$\mathcal{O}_{\sigma_1}(Z) * \mathcal{O}_{\sigma_2}(Z) = \mathcal{O}_{\sigma_1\sigma_2}(Z), \qquad \sigma_{1,2} \in S_n. \tag{4.6}$$

It is immediate to see how this product is different from the ordinary GIO multiplication product (4.3): σ_1, $\sigma_2$and $\sigma_1\sigma_2$ are all permutations of n elements, and the star product is generally noncommutative. Let $[\sigma]$ be the conjugacy class of σ. We now define a map from the multitrace GIOs to the class algebra

$$\mathcal{O}_{\sigma}(Z) \to \frac{1}{\text{size of } [\sigma]} \sum_{\tau\in[\sigma]} \tau \equiv \frac{T_\sigma}{|T_\sigma|}. \tag{4.7}$$

This map is 1-1 at large N. Let us focus on this case. We can expand the product of T_i, $T_j \in \mathcal{Z}[\mathbb{C}[S_n]]$as

$$T_i T_j = C_{ij}^k T_k. \tag{4.8}$$

Here the C_{ij}^k are the class algebra structure constants. By multiplying both sides above by T_l and taking the coefficient of the identity we get

$$\delta(T_i T_j T_l) = C_{ij}^k \delta(T_k T_l) = \delta_{k,l} C_{ij}^k |T_l| = C_{ij}^k |T_k|. \tag{4.9}$$

Now we expand the star product $\mathcal{O}_{\sigma_1}(Z) * \mathcal{O}_{\sigma_2}(Z)$ as

$$\begin{aligned}\mathcal{O}_{\sigma_1}(Z) * \mathcal{O}_{\sigma_2}(Z) &= \sum_p \frac{|T_{\sigma_p}|}{|T_{\sigma_1}||T_{\sigma_2}|} C_{[\sigma_1][\sigma_2]}^p \mathcal{O}_{\sigma_p}(Z) \\ &= \sum_p \frac{\delta(T_{\sigma_1}T_{\sigma_2}T_{\sigma_p})}{|T_{\sigma_1}||T_{\sigma_2}|} \mathcal{O}_{\sigma_p}(Z)\end{aligned} \tag{4.10}$$

where the sum is over the conjugacy classes p of S_n. σ_p is a representative element of the conjugacy class p. This equation will lead to a new expression for the two point functions of GIOs built from Z, $Z^\dagger$ in $\mathcal{N} = 4$ SYM. First observe that setting Z to the identity $N \times N$ matrix

$$\mathcal{O}_\sigma(Z = 1_N) = N^{C_\sigma} \tag{4.11}$$

where C_σ is the number of cycles in the permutation σ. Now consider taking the star product of $\mathcal{O}_{\sigma_1}(Z)$, $\mathcal{O}_{\sigma_2}(Z)$ and then setting $Z = 1_N$. We have, according to (4.10),

$$\begin{aligned}\mathcal{O}_{\sigma_1}(Z) * \mathcal{O}_{\sigma_2}(Z)|_{Z=1_N} &= \frac{1}{|T_{\sigma_1}||T_{\sigma_2}|} \sum_p \delta(T_{\sigma_1}T_{\sigma_2}T_{\sigma_p}) \mathcal{O}_{\sigma_p}(1_N) \\ &= \frac{1}{|T_{\sigma_1}||T_{\sigma_2}|} \sum_p \delta(T_{\sigma_1}T_{\sigma_2}T_{\sigma_p}) N^{C_{\sigma_p}} \\ &= \frac{1}{n!|T_{\sigma_1}||T_{\sigma_2}|} \sum_{\gamma\in S_n} \delta(\gamma T_{\sigma_1}\gamma^{-1}T_{\sigma_2}\Omega)\end{aligned} \tag{4.12}$$

where we set $\Omega = \sum_p T_{\sigma_p} N^{C_{\sigma_p}}$. On the other hand the free field correlator is known to be [4]

$$\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}^\dagger(Z)\rangle = \frac{1}{|T_{\sigma_1}||T_{\sigma_2}|} \sum_{\gamma\in S_n} \delta(\gamma T_{\sigma_1}\gamma^{-1}T_{\sigma_2}\Omega) \tag{4.13}$$

so that

$$\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}^\dagger(Z)\rangle = n! \mathcal{O}_{\sigma_1}(Z) * \mathcal{O}_{\sigma_2}(Z)|_{Z=1_N}. \tag{4.14}$$

The two point function $\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}^\dagger(Z)\rangle$ is therefore proportional to the star product $\mathcal{O}_{\sigma_1}(Z) * \mathcal{O}_{\sigma_2}(Z)$ followed by the evaluation $Z \to 1_N$.

Similar considerations lead to the following expression for the extremal three point function. In this case, we find that $\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z)\mathcal{O}_{\sigma_3}^\dagger(Z)\rangle$ is proportional to the usual product $\mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z)$, followed by the star product with $\mathcal{O}_{\sigma_3}(Z)$, followed by the evaluation $Z \to 1_N$. To see this, take $\sigma_1 \in S_{n_1}$, $\sigma_2 \in S_{n_2}$ and consider

$$\begin{aligned}&(\mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z)) * \mathcal{O}_{\sigma_3}(Z)|_{Z=1_N} \\ &\quad= \frac{1}{|T_{\sigma_1\circ\sigma_2}||T_{\sigma_3}|} \delta(T_{\sigma_1\circ\sigma_2}T_{\sigma_3}\Omega)\end{aligned} \tag{4.15}$$

where $T_{\sigma_1\circ\sigma_2} \in \mathcal{Z}[\mathbb{C}[S_{n_1+n_2}]]$, $T_{\sigma_3} \in \mathcal{Z}[\mathbb{C}[S_{n_1+n_2}]]$ and $\Omega = \sum_{\sigma\in S_{n_1+n_2}} \sigma N^{C_\sigma}$. On the other hand the correlator in $\mathcal{N} = 4$ SYM [4] is

$$\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z)\mathcal{O}^{\dagger}_{\sigma_3}(Z)\rangle = \sum_{\gamma\in S_{n_1+n_2}} \delta(\gamma(\sigma_1\circ\sigma_2)\gamma^{-1}\sigma_3^{-1}\Omega) = \frac{(n_1+n_2)!}{|T_{\sigma_1\circ\sigma_2}||T_{\sigma_3}|}\delta(T_{\sigma_1\circ\sigma_2}T_{\sigma_3}\Omega) \tag{4.16}$$

so that

$$\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z)\mathcal{O}^{\dagger}_{\sigma_3}(Z)\rangle = (n_1+n_2)!(\mathcal{O}_{\sigma_1\circ\sigma_2}(Z)) * \mathcal{O}_{\sigma_3}(Z)|_{Z=1_N}. \tag{4.17}$$

Given that these correlators are neatly expressed in terms of the star product, it would be interesting to give an interpretation of the latter in the dual $\mathrm{AdS}_5\times S_5$ side.

We will now write similar equations for the two-matrix problem.

B. Two-matrix problem

For the two-matrix problem, the GIOs are polynomials in the X, Y matrices. Formally, we can write them in terms of a permutation $\sigma\in S_{m+n}$ as

$$\mathcal{O}_\sigma(X,Y) = \mathrm{Tr}(X^{\otimes m}\otimes Y^{\otimes n}\sigma). \tag{4.18}$$

As in the one-matrix problem, there is an equivalence relation

$$\mathcal{O}_\sigma(X,Y) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(X,Y), \qquad \gamma\in S_m\times S_n. \tag{4.19}$$

To each of these GIO $\mathcal{O}_\sigma$ we can associate a specific element N_σ of $\mathcal{A}(m,n)$ that we call a necklace. We define a necklace N_σ as

$$N_\sigma = \frac{1}{|\mathrm{Aut}_{S_m\times S_n}(\sigma)|}\sum_{\gamma\in S_m\times S_n}\gamma\sigma\gamma^{-1} \tag{4.20}$$

or equivalently as

$$N_\sigma = \sum_{\tau\in \mathrm{Orbit}(\sigma,S_m\times S_n)}\tau \tag{4.21}$$

where the sum is restricted to the permutations τ in the group orbit of σ under $S_m\times S_n$. We can think of the necklaces as the normalized version of the $\bar\sigma$ elements defined in (3.3). The set of necklaces form a basis for $\mathcal{A}(m,n)$. We associate a GIO to a necklace simply by mapping

$$\mathcal{O}_\sigma(X,Y)\to\frac{1}{|N_\sigma|}N_\sigma. \tag{4.22}$$

For example, for the GIO corresponding to the permutation $\tilde\sigma = (1,2,4,5)(3,6)\in S_6$,

$$\mathcal{O}_{\tilde\sigma}(X,Y) = \mathrm{Tr}(X^2Y^2)\mathrm{Tr}(XY), \tag{4.23}$$

we associate, through the map (4.22), the $\mathcal{A}(3,3)$ element

$$N_{\tilde\sigma} = \sum_{S_3\times S_3}\gamma\tilde\sigma\gamma^{-1} = \sum_{\substack{a_1\neq a_2\neq a_3\in\{1,2,3\}\\ \bar b_1\neq\bar b_2\neq\bar b_3\in\{4,5,6\}}}(a_1,a_2,\bar b_1,\bar b_2)(a_3,\bar b_3). \tag{4.24}$$

Similarly, for the GIO specified by $\tilde\sigma = (1,2,3)\in S_6$

$$\mathcal{O}_{\tilde\sigma}(X,Y) = \mathrm{Tr}(X^2Y)\mathrm{Tr}(Y)^3 \tag{4.25}$$

we associate the $\mathcal{A}(2,4)$ necklace

$$N_{\tilde\sigma} = \sum_{\substack{a_1\neq a_2\in\{1,2\}\\ \bar b_1\in\{3,4,5,6\}}}(a_1,a_2,\bar b_1). \tag{4.26}$$

Notice that in the necklaces we do not explicitly write the single cycle permutations, but rather we leave them implicit. In the last example, these single cycle permutations would account for the multitrace $\mathrm{Tr}(Y)^3$ component of $\mathcal{O}_{\tilde\sigma} = \mathrm{Tr}(X^2Y)\mathrm{Tr}(Y)^3$.

From these examples it is clear how these necklaces are built by taking products of cyclic objects, which in turn are constructed using two different types of *beads*. Such cyclic objects are well studied in Polya theory. They can be related to the single cycle permutations in S_{m+n} with equivalences generated by $S_m\times S_n$. These equivalence classes form the algebra $\mathcal{A}(m,n)$. We can imagine having blue beads corresponding to integers $[1,2,..m]$ and red beads corresponding to integers $[m+1,m+2,\ldots,m+n]$. Therefore, we can pictorially depict the necklaces of examples (4.24) and (4.26) as in Fig. 1. The same structure is present in the GIO $\mathcal{O}_\sigma$ corresponding to the necklace N_σ. In this case the single-traces are the cyclic objects, and the role of the blue and red beads is played by the X and Y type fields respectively.

The map (4.22) is 1-1 at large N: as in the 1-matrix problem, we now focus on this case. There is a natural product on the space of two matrix GIOs coming from

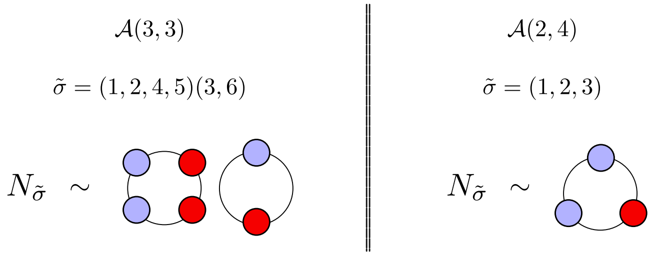

FIG. 1. Pictorial interpretations of the necklaces in the examples (4.24) and (4.26).

multiplying the multitraces. For such a product, the degrees of the permutations add

$$\mathcal{O}_{\sigma_1}(X,Y)\mathcal{O}_{\sigma_2}(X,Y) = \mathcal{O}_{\sigma_1\circ\sigma_2}(X,Y). \tag{4.27}$$

Here $\sigma_1 \in S_{m_1+n_1}$ is a representative of a class in $\mathcal{A}(m_1,n_1)$ and $\sigma_2 \in S_{m_2+n_2}$ represents a class in $\mathcal{A}(m_2,n_2)$, while $\sigma_1\circ\sigma_2 \in S_{m_1+n_1}\times S_{m_2+n_2} \subset S_{m_1+m_2+n_1+n_2}$ represents a class in $\mathcal{A}(m_1+m_2,n_1+n_2)$. Continuing the analogy with (4.4), we can define

$$\mathcal{A}(\infty,\infty) = \bigoplus_{m,n}\mathcal{A}(m,n) \tag{4.28}$$

and for $\bar{\sigma}_1 \in \mathcal{A}(m_1,n_1)$ and $\bar{\sigma}_2 \in \mathcal{A}(m_2,n_2)$ we have

$$\circ\colon\ \mathcal{A}(\infty,\infty)\otimes\mathcal{A}(\infty,\infty)\to\mathcal{A}(\infty,\infty). \tag{4.29}$$

As in the one-matrix case, there is however a second type of product of GIOs that we can construct. The product on $\mathcal{A}(m,n)$ can in fact be used to define a closed and associative star product on the space of the multitrace operators with fixed numbers (m,n) of (X,Y), in the same fashion as (4.6):

$$\mathcal{O}_{\bar{\sigma}_1}(X,Y)*\mathcal{O}_{\bar{\sigma}_2}(X,Y) = \mathcal{O}_{\bar{\sigma}_1\bar{\sigma}_2}(X,Y),$$
$$\bar{\sigma}_{1,2}\in\mathcal{A}(m,n). \tag{4.30}$$

Notice that here $\bar{\sigma}_1$, $\bar{\sigma}_2$ and $\bar{\sigma}_1\bar{\sigma}_2$ are all of the same degree, and that the star product is noncommutative. We will use this star product to express the two point function of GIOs built from X, Y.

Since the set of necklaces $\{N_a\}$ forms a basis for $\mathcal{A}(m,n)$, we can expand the product N_aN_b as

$$N_aN_b = C^c_{a,b}N_c \tag{4.31}$$

for some structure constants $C^c_{a,b}$. Moreover, the necklaces are orthogonal in the metric (3.4):

$$\langle N_a, N_b\rangle = \delta(N_aN_b) = \delta_{a,b}|N_b|. \tag{4.32}$$

Here $|N_a|$ is the number of permutations in the necklace N_a. We can write

$$\delta(N_aN_bN_c) = |N_c|C^c_{a,b}. \tag{4.33}$$

Now use the map (4.22) to map the two matrix invariants $\mathcal{O}_a(X,Y)$ and $\mathcal{O}_b(X,Y)$ to the necklaces N_a and N_b respectively. Then

$$\mathcal{O}_a(X,Y)*\mathcal{O}_b(X,Y) = \sum_c C^c_{a,b}\frac{|N_c|}{|N_a||N_b|}\mathcal{O}_c(X,Y)$$
$$= \sum_c \frac{1}{|N_a||N_b|}\delta(N_aN_bN_c)\mathcal{O}_c(X,Y). \tag{4.34}$$

As for the one-matrix problem case, by setting $X=Y=1_N$ we get

$$\mathcal{O}_a(X,Y)*\mathcal{O}_b(X,Y)|_{X=Y=1_N} = \frac{1}{|N_a||N_b|}\delta(N_aN_b\Omega) \tag{4.35}$$

where $\Omega = \sum_{\sigma\in S_{m+n}}\sigma N^{C_\sigma}$. On the other hand the free field correlator [5,17] is

$$\langle\mathcal{O}_a(X,Y)\mathcal{O}^\dagger_b(X,Y)\rangle = \sum_{\gamma\in S_m\times S_n}\delta(\gamma a\gamma^{-1}b^{-1}\Omega)$$
$$= \frac{1}{|N_a||N_b|}\sum_{\gamma\in S_m\times S_n}\delta(\gamma N_a\gamma^{-1}N_b\Omega)$$
$$= \frac{m!n!}{|N_a||N_b|}\delta(N_aN_b\Omega). \tag{4.36}$$

Therefore, in analogy with (4.14) and (4.17), we can write the two point function as

$$\langle\mathcal{O}_a(X,Y)\mathcal{O}^\dagger_b(X,Y)\rangle = m!n!\mathcal{O}_a(X,Y)*\mathcal{O}_b(X,Y)|_{X=Y=1_N} \tag{4.37}$$

and the extremal three point function as

$$\langle\mathcal{O}_a(X,Y)\mathcal{O}_b(X,Y)\mathcal{O}^\dagger_c(X,Y)\rangle$$
$$= (m_1+m_2)!(n_1+n_2)!\mathcal{O}_{a\circ b}(X,Y)*\mathcal{O}_c(X,Y)|_{X=Y=1_N} \tag{4.38}$$

where $a\in S_{m_1+n_1}$, $b\in S_{m_2+n_2}$ and $c\in S_{m_1+n_1+m_2+n_2}$. Finally, notice that the pairing (3.4) is proportional to the planar correlator [42–44] of BPS operators: given $\mathcal{O}_a(X,Y)$ and $\mathcal{O}_b(X,Y)$, we have

$$\langle\mathcal{O}_a(X,Y)\mathcal{O}^\dagger_b(X,Y)\rangle_{\text{planar}} = m!n!\langle a,b\rangle \tag{4.39}$$

where the pairing on the right-hand side is the one in Eq. (3.4).

Let us now focus on the center of $\mathcal{A}(m,n)$. In Sec. III we argued that the center is generated by $\{T_p^{(m+n)}, T_{q_1}^{(m)}, T_{q_2}^{(n)}\}$. We remind the reader that $\{T_p^{(m+n)}\}$, $\{T_{q_1}^{(m)}\}$ and $\{T_{q_2}^{(n)}\}$ are the generators of the centers of $\mathbb{C}[S_{m+n}]$, $\mathbb{C}[S_m]$ and $\mathbb{C}[S_n]$ respectively, and that p, q_1 and q_2 are integer partitions of $m+n$, m and n. A GIO $\mathcal{O}_{T_p^{(m+n)}}(X,Y)$ can be understood as a descendant of a single matrix 1/2 BPS state $\mathcal{O}_{T_p^{(m+n)}}(X)$ under the $U(2)$ internal symmetry that mixes the X and Y fields. In fact, given $(D^-)^i_j = Y^i_k \frac{\partial}{\partial X^j_k}$, we can write

$$\mathcal{O}_{T_p^{(m+n)}}(X,Y) \sim (D^-)^n \mathcal{O}_{T_p^{(m+n)}}(X). \tag{4.40}$$

This means that central elements (and their corresponding matrix gauge invariants), described in terms of the over-complete basis $\{T_p^{(m,n)} T_{q_1}^{(m)} T_{q_2}^{(n)}\}$, are formed from composites which employ both the usual product and the star product:

$$[\text{descendant operators}] * \{(X\text{-operators})(Y\text{-operators})\}. \tag{4.41}$$

The descendant GIOs are associated with $T_p^{(m+n)}$ elements, X- and Y- GIOs to $T_{q_1}^{(m)}$ and $T_{q_2}^{(n)}$ elements, respectively. In terms of the permutations we are taking the product in $\mathcal{A}(m,n)$ along with the circle product $\circ: \mathcal{A}(m,0) \otimes \mathcal{A}(0,n) \to \mathcal{A}(m,n)$.

Single-trace symmetrized traces are $U(2)$ descendants of single-trace operators built from a single matrix. In terms of the permutation language, they correspond to single-cycle permutations that are invariant under any reshuffling.[2] On the other hand, $U(2)$ descendants of multitrace operators built from one matrix form a subspace of the space spanned by products of symmetrized single-trace states. In other words, not all products of single-trace descendants are themselves descendants. One way to see this explicitly is the following. Let $ST_{m,n}$ be the space of symmetrized traces with m copies of X and n copies of Y matrices. The generating function for the dimension $\mathrm{Dim}(ST_{m,n})$ is

$$\prod_{i,j\in\Omega} \frac{1}{1-x^i y^j} = \sum_{m,n} \mathrm{Dim}(ST_{m,n}) x^m y^n \tag{4.42}$$

where $\Omega = \{0 \le i \le \infty\} \cup \{0 \le j \le \infty\} \backslash \{i = j = 0\}$. Let ST_{m+n} be the space of symmetrized traces with a total of $m+n$ matrices, with any number of X or Y. We have

$$\mathrm{Dim}(ST_{m+n}) = \sum_{i=0}^{m+n} \mathrm{Dim}(ST_{i,m+n-i}). \tag{4.43}$$

On the other hand, the total number of $U(2)$ descendants obtained from a multitrace operator with $m+n$ copies of X is

$$(m+n+1)p(m+n). \tag{4.44}$$

$p(m+n)$ is the number of partitions of $m+n$ (the number of highest weight states), while $m+n+1$ is the number of descendants for a fixed highest weight. It can now be checked that $\mathrm{Dim}(ST_{m+n}) > p(m+n)(m+n+1)$. This indeed proves our original claim.

C. Cartan subalgebra and the minimal set of charges

In [31], it was observed that, in the free limit, multimatrix gauge theories have enhanced symmetries including products of unitary groups. There are Noether charges for these enhanced symmetries. Casimirs constructed from these charges have eigenvalues that can distinguish all the labels R, R_1, R_2, i, j of restricted Schur operators. Because of Schur-Weyl duality, these charges are also expressible in terms of permutations. Given the definitions in this paper, this action of permutations amounts to the action of $\mathcal{A}(m,n)$ on itself by the left or right regular representation. We can now characterize more precisely what is a minimal set of charges that can measure all the labels. In Sec. III B we introduced the Cartan subalgebra $\mathcal{M}(m,n)$, and gave a prescription to build a basis for it. We need to find a subspace $C_{m,n}$ of $\mathcal{M}(m,n)$ such that polynomials in some basis elements $c_a \in C_{m,n}$ with coefficients taking values in the center $\mathcal{Z}(m,n)$ span $M(m,n)$. In other words $C_{m,n}$ contains a minimal set of generators for $\mathcal{M}(m,n)$ as a polynomial algebra over $\mathcal{Z}(m,n)$. A minimal set of generators for $\mathcal{Z}(m,n)$, along with the basis elements of the subspace $C_{m,n}$, provide a complete set of charges, which can measure all the labels of the $Q^R_{R_1,R_2,i,j}$ by left and right multiplication. Let $N^{\min}(\mathcal{Z}(m,n))$ be the minimal number of elements of $\mathcal{Z}(m,n)$ which generate $\mathcal{Z}(m,n)$ as a polynomial algebra. Also, let $N^{\min}_{\mathcal{Z}(m,n)}(\mathcal{M}(m,n))$ be the minimal number of elements of $\mathcal{M}(m,n)$ which generate $\mathcal{M}(m,n)$ as a polynomial algebra over $\mathcal{Z}(m,n)$. Left multiplication by these generators correspond to enhanced symmetry charges that measure the multiplicity index i of restricted Schur operators. Right multiplication by the same generators correspond to other enhanced symmetry charges that measure the multiplicity index j of restricted Schur operators. Hence the minimal number of charges is

$$N^{\min}(\mathcal{Z}(m,n)) + 2N^{\min}_{\mathcal{Z}(m,n)}(\mathcal{M}(m,n)). \tag{4.45}$$

An *important open problem* is to determine this function of (m,n) in general. This will tell us how many bits of

[2]Further details of symmetrized traces in terms of an operation on the permutations in the $\mathcal{O}_\sigma(X,Y)$ can be found in [44].

information completely specify all the operators in a multimatrix setup.

The above discussion is complete for the case where $m+n<N$, which is adequate for a treatment of the physics at all orders in the $1/N$ expansion. For finite N effects, where we consider $m+n>N$, the charges given by the above still determine all the multimatrix invariants, but they are not a minimal set any more. The discussion can be easily adapted to this case. Define

$$\mathcal{A}_N^{\mathrm{null}}(m,n)=\bigoplus_{R\vdash m+n:c_1(R)>N}\bigoplus_{R_1\vdash m,R_2\vdash n}\mathrm{Span}\{Q^R_{R_1,R_2,i,j};i,j\}. \tag{4.46}$$

The quotient

$$\mathcal{A}_N(m,n)=\mathcal{A}(m,n)/\mathcal{A}_N^{\mathrm{null}}(m,n) \tag{4.47}$$

is a closed subalgebra of blocks surviving the finite N cut. It has a center $\mathcal{Z}_N(m,n)$ and a Cartan $\mathcal{M}_N(m,n)$ that are simply related to $\mathcal{Z}(m,n)$ and $\mathcal{M}(m,n)$ by quotienting out the parts belonging to $\mathcal{A}_N^{\mathrm{null}}(m,n)$. Let $N^{\min}(\mathcal{Z}_N(m,n))$ be the number of generators in a minimal generating set for $\mathcal{Z}_N(m,n)$ as a polynomial algebra. Let $N^{\min}_{\mathcal{Z}_N(m,n)}(\mathcal{M}_N(m,n))$ be the number of generators in a minimal generating set for $\mathcal{M}_N(m,n)$ as a polynomial algebra over $\mathcal{Z}_N(m,n)$. The minimal number of charges needed is

$$N^{\min}(\mathcal{Z}_N(m,n))+2N^{\min}_{\mathcal{Z}_N(m,n)}(\mathcal{M}_N(m,n)). \tag{4.48}$$

We expect (4.45), (4.48) will have implications for information theoretic discussions of AdS/CFT such as [45,46].

V. COMPUTATION OF THE FINITE N CORRELATOR

In this section we will derive a finite N generating function for the two point function of operators of the form

$$\mathcal{O}=\mathrm{Tr}(X^mY^n) \tag{5.1}$$

in the free field metric. Operators like the one in (5.1) correspond to $\mathcal{A}(m,n)$ elements

$$\frac{1}{m!n!}T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]} \tag{5.2}$$

where $T_{\bar{1},1}=T_2^{(X,Y)}-T_2^{(X)}-T_2^{(Y)}$. Here $T_2^{(X,Y)}$, $T_2^{(X)}$ and $T_2^{(Y)}$ are the sum of transpositions in S_{m+n}, S_m and S_n respectively. $T_{\bar{1},1}$ can be understood as a joining operator, merging the $(1\ldots m)$ type cycles with the $(m+1\ldots m+n)$ type cycles.

The two point function (4.36) therefore reads, with $\mathcal{O}=\mathrm{Tr}(X^mY^n)$

$$\begin{aligned}\langle\mathcal{O}\mathcal{O}^\dagger\rangle&=\frac{1}{m!^2n!^2}\sum_{\gamma\in S_m\times S_n}\sum_{\sigma\in S_{m+n}}\delta(\gamma T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}\gamma^{-1}T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}\sigma)N^{C_\sigma}\\&=\frac{1}{m!n!}\delta(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}\Omega)\end{aligned} \tag{5.3}$$

where we set $\Omega=\sum_{\sigma\in S_{m+n}}\sigma N^{C_\sigma}$. This quantity can be computed using only ordinary character theory. Using Eq. (3.42) and using the shorthand notation $g=g(R_1,R_2;R)$ we write

$$\langle\mathcal{O}\mathcal{O}^\dagger\rangle=\frac{1}{(m+n)!m!n!}\sum_{R_1\vdash m,R_2\vdash n}\sum_{R\vdash m+n}\frac{d_R}{d_{R_1}^2d_{R_2}^2g^2}(\chi^R_{R_1,R_2}(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}))^2\chi^R_{R_1,R_2}(\Omega). \tag{5.4}$$

We now expand $T_{\bar{1},1}=T_2^{(X,Y)}-T_2^{(X)}-T_2^{(Y)}$ so that

$$T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}=T_2^{(X,Y)}T^{(X)}_{[m]}T^{(Y)}_{[n]}-T_2^{(X)}T^{(X)}_{[m]}T^{(Y)}_{[n]}-T^{(X)}_{[m]}T_2^{(Y)}T^{(Y)}_{[n]}. \tag{5.5}$$

We also have (see e.g. [22])

$$\chi^R_{R_1,R_2}(\Omega)=\chi^R_{R_1,R_2}\left(\sum_{\sigma\in S_{m+n}}\sigma N^{C_\sigma}\right)=\frac{gd_{R_1}d_{R_2}}{d_R}(n+m)!\mathrm{Dim}_N(R). \tag{5.6}$$

Equation (5.4) simplifies then to

$$\langle \mathcal{O}\mathcal{O}^\dagger \rangle = \frac{1}{m!n!}\sum_{\substack{R_1\vdash m \\ R_2\vdash n}}\sum_{R\vdash m+n}\frac{1}{d_{R_1}d_{R_2}g}\mathrm{Dim}_N(R)(\chi^R_{R_1,R_2}(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}))^2. \tag{5.7}$$

On the other hand, as shown in Appendix C

$$\chi^R_{R_1,R_2}(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}) = \begin{cases} (-1)^{c_{R_1}+c_{R_2}}g(m-1)!(n-1)!\left[\frac{\chi_R(T_2^{(X,Y)})}{d_R} - \frac{\chi_{R_1}(T_2^{(X)})}{d_{R_1}} - \frac{\chi_{R_2}(T_2^{(Y)})}{d_{R_2}}\right]; & R_1, R_2 \text{hooks} \\ 0 \quad \text{otherwise.} \end{cases}$$

Here c_{R_i} is the number of boxes in the first column of the Young diagram associated with the representation R_i. This expression restricts the sums over representations $R_1\vdash m$, $R_2\vdash n$ in (5.7) to a sum over hook representations $h_1\vdash m, h_2\vdash n$.

We now need an equation for $g(h_1, h_2; R)$, with h_1 and h_2 hook representations of S_m and S_n respectively. We specify any representation R by the sequence of pairs of integers $R = ((a_1, b_1), (a_2, b_2), \ldots (a_d, b_d))$. In a Young diagram interpretation, a_j $(1 \le j \le d)$ is the number of boxes to the right of the jth diagonal box, and b_j is the number of boxes below the jth diagonal box. We refer to d as the "depth" of the representation R. Let us write $h_1 = (k_1, l_1)$, $h_2 = (k_2, l_2)$ and $R = ((a_1, b_1), (a_2, b_2))$. In Appendix B we show that

$$\begin{aligned} g(h_1, h_2; R) = {} & \delta_{k_1+k_2,a_1}\delta_{l_1+l_2+1,b_1}\delta_{-1,a_2}\delta_{0,b_2} + \delta_{k_1+k_2+1,a_1}\delta_{l_1+l_2,b_1}\delta_{0,a_2}\delta_{-1,b_2} \\ & + \sum_{\epsilon_1,\epsilon_2=0}^{1}\sum_{i=\epsilon_1\bar{\epsilon}_2}^{\min(k_1-\bar{\epsilon}_1\bar{\epsilon}_2,k_2-\epsilon_1\epsilon_2)}\sum_{j=\bar{\epsilon}_1\epsilon_2}^{\min(l_1-\bar{\epsilon}_1\bar{\epsilon}_2,l_2-\epsilon_1\epsilon_2)}\delta_{k_1+k_2-i+\bar{\epsilon}_1\epsilon_2,a_1}\delta_{l_1+l_2-j+\epsilon_1\bar{\epsilon}_2,b_1}\delta_{i-\epsilon_1\bar{\epsilon}_2,a_2}\delta_{j-\bar{\epsilon}_1\epsilon_2,b_2} \end{aligned} \tag{5.8}$$

where $\bar{\epsilon}_{1,2} = 1 - \epsilon_{1,2}$. Using this identity, in Appendix C we derive the formula

$$\langle \mathrm{Tr}(X^mY^n)\mathrm{Tr}(X^mY^n)^\dagger \rangle = \sum_{k_1,l_1=0}^{m}\sum_{k_2,l_2=0}^{n}\sum_{\substack{a_1,b_1=0 \\ a_2,b_2=0}}^{n+m} g\delta(k_1+l_1-m)\delta(k_2+l_2-n)F(a_1,b_1,a_2,b_2,k_1,l_1,k_2,l_2) \tag{5.9}$$

where we defined the function

$$\begin{aligned} F(a_1,b_1,a_2,b_2,k_1,l_1,k_2,l_2) = {} & \frac{k_1!k_2!l_1!l_2!(a_1-a_2)(b_1-b_2)}{4(a_1+b_2+1)(a_2+b_1+1)(k_1+l_1+1)(k_2+l_2+1)} \\ & \times \binom{a_1+b_1}{b_1}\binom{a_2+b_2}{b_2}\binom{N+a_1}{a_1+b_1+1}\binom{N+a_2}{a_2+b_2+1} \\ & \times ((a_1+b_1+1)(a_1-b_1) + (a_2+b_2+1)(a_2-b_2)+ \\ & - (k_1+l_1+1)(k_1-l_1) - (k_2+l_2+1)(k_2-l_2))^2. \end{aligned} \tag{5.10}$$

In [36] a closed form for the two point function has been given by using a different approach based on Young-Yamanouchi symbols. We have checked agreement of (5.9) with that closed form for up to $n = m = 10$. It is an interesting exercise to simplify (5.9) into the closed form obtained in [36]. It will also be interesting to apply the present franework to obtain formulas analogous to (5.9) for more general GIOs corresponding to central elements of $\mathcal{A}(m,n)$.

In this section we have shown how to calculate a particular two point function of a central operator, without explicitly constructing projectors. The result rather follows from knowing how central operators of interest are generated via the star product of pure X gauge invariants, pure Y gauge invariants and descendants of half-BPS operators.

A. Coloured ribbon graphs

The correlator computations above can be expressed in terms of ribbon graphs, equivalently the usual double-line graphs of large N expansions, but with edges coming in two colors, as explained for example in [47]. The graphs can be organized by the minimum genus of the surface they can be embedded in and these graphs of a given genus contribute to a fixed power of N. For small m, n, we have checked with a Groups, Algorithms, Programming (GAP) code that directly computing the permutation sums for a given genus agrees with the analytic result (5.9) we have derived.

VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we initiated a systematic study of permutation centralizer algebras, in connection with gauge-invariant operators. We focused our attention on the algebras $\mathcal{A}(m,n)$ which are related to restricted Schur operators studied in the context of giant gravitons in AdS/CFT. Other closely related algebras are related to the Brauer basis for multimatrix invariants, the covariant basis and to tensor models.

While many of the key formulas we have used were already understood in the literature on giant gravitons, we have emphasized the intrinsic structure of $\mathcal{A}(m,n)$ as an associative algebra with a nondegenerate pairing. This means that it has a Wedderburn-Artin decomposition, which gives a basis for the algebra in terms of matrixlike linear combinations. The construction of these matrix units in terms of representation theory data from S_{m+n}, S_m, S_n has already been extensively used in the context of giant gravitons, although the link to the Wedderburn-Artin decomposition has not been made explicit before. In addition to explicating this link, the new emphasis in this paper has been on the structure of the center $\mathcal{Z}(m,n)$ and the maximally commuting subalgebra $\mathcal{M}(m,n)$.

We have used the structure of $\mathcal{M}(m,n)$ as a polynomial algebra over $\mathcal{Z}(m,n)$ to characterize the minimal number of charges needed to identify any two-matrix gauge invariant (Sec. IV C). It will be interesting to generalize this discussion to gauge invariants for more general gauge groups.

Two key structural facts about $\mathcal{A}(m,n)$ have played a role in the computation of correlators in Sec. V. The first is that $(x^m)*(y^n) = (x^m y^n)$ and the second is that $(x^m y^n)$ is part of $\mathcal{Z}(m,n)$. The nondegenerate pairing on $\mathcal{A}(m,n)$, when restricted to elements in the center, can be expressed in terms of characters of S_n, S_m, S_{n+m} without requiring more detailed representation theory data such as matrix elements and branching coefficients. These are in general computationally difficult to calculate, although there has been progress in the context of "perturbations of half-BPS giants." This makes it very interesting to understand the structure of the center $\mathcal{A}(m,n)$. A special case is $\mathcal{Z}[\mathbb{C}[S_n]]$, which is the algebra of class sums in S_n.

A. Structure of the center

A number of questions about $\mathcal{A}(n)$, $\mathcal{A}(m,n)$ and the center $\mathcal{A}(m,n)$ can be explored experimentally, with the help of group theory software, notably GAP. In particular, since $\mathcal{Z}(m,n)$ is generated by the center of S_m, the center of S_n and that of S_{n+m} it is a useful first step to know about these centers.

Since S_n is generated by transpositions, one might naively expect that the sum of permutations T_2 will generate $\mathcal{A}(n)$. This is actually not true. We know that T_2 obeys a relation of degree $p(n)$

$$\prod_{R\vdash n}\left(T_2 - \frac{\chi_R(T_2)}{d_R}\right) = 0. \tag{6.1}$$

If this is the only relation, then we know that T_2 alone generates $\mathcal{Z}[\mathbb{C}[S_n]]$. However simpler relations occur when there are coincidences in the normalized characters, e.g. two different irreps have the same normalized character. In fact the failure of T_2 to generate center is always correctly predicted by the degeneracies of the normalized characters. If we take

$$\prod\nolimits_R{}'\left(T_2 - \frac{\chi_R(T_2)}{d_R}\right) = 0 \tag{6.2}$$

where the product is taken over a maximal set of irreps with distinct normalized characters, we are getting an element in $\mathbb{C}[S_n]$ which vanishes in all irreps. It is a central element, so the matrix elements in any irrep are proportional to the identity. We conclude that the above element vanishes. Given that the Peter-Weyl theorem gives an isomorphism between $\mathbb{C}[S_n]$ and matrix elements of irreps, it follows that something which has vanishing matrix elements in all irreps should be identically zero.

Even for large n, it is possible to check that the center of $\mathbb{C}[S_n]$ is generated by a small number of T_p's. Using GAP we tested that $T_{[2,1^{n-2}]}$ and $T_{[3,1^{n-3}]}$ are enough to generate the center for $\mathbb{C}[S_n]$ up to $n = 14$. The procedure we used to perform these checks is the following. We know that the set of projectors $\{P_R\}$, with R integer partition of n, generate the center of S_n. We can compute the overlap of P_R with the kth power of T_p, that we simply write as T_p^k:

$$\begin{aligned}\langle T_p^k, P_R\rangle &= \delta(T_p^a P_R)\\ &= \frac{1}{n!}\sum_{S\vdash n}\chi_S(T_p^k)\chi_S(P_R)\\ &= \frac{1}{n!}\sum_{S\vdash n} d_S\left(\frac{\chi_S(T_p)}{d_S}\right)^k\chi_S(P_R)\\ &= d_R\left(\frac{\chi_R(T_p)}{d_R}\right)^k.\end{aligned} \tag{6.3}$$

Similarly, we can derive

$$\langle T_p^k T_q^l, P_R\rangle = d_R\left(\frac{\chi_R(T_p)}{d_R}\right)^k\left(\frac{\chi_R(T_q)}{d_R}\right)^l. \tag{6.4}$$

Now we construct the $AB \times p(n)$ matrix $M(A,B)$, whose matrix elements are the overlaps (6.4)

$$M(A,B)|_{(k,l),R} = d_R\left(\frac{\chi_R(T_p)}{d_R}\right)^k\left(\frac{\chi_R(T_q)}{d_R}\right)^l \tag{6.5}$$

with $0 \le k < A$ and $0 \le l < B$. By computing the rank of this matrix we obtain the number of independent central elements in $\mathbb{C}[S_n]$ that are obtained by taking at most $A-1$ powers of T_p and $B-1$ powers of T_q. This method can be easily generalized to obtain the number of central elements generated by the string of operators $T_{p_1}^{k_1} T_{p_2}^{k_2} \cdots T_{p_N}^{k_N}$.

These studies on the center of $\mathbb{C}[S_n]$ inspire a similar analysis for center of $\mathcal{A}(m,n)$. The task is to find a minimal set of generators for $\mathcal{Z}(m,n)$ as a polynomial algebra. The importance of this problem is discussed in Sec. IV C. Concretely, we would like to determine $N^{\min}(\mathcal{Z}(m,n))$. There are many approaches one can take in this case, which would be interesting to investigate in the future. For example, using GAP we checked that low powers of the sum of two- and three-cycles permutations, $T_2^{(m+n)}$ and $T_3^{(m+n)}$, together with the generators of the centers of $\mathbb{C}[S_m]$ and $\mathbb{C}[S_n]$, generate the whole center $\mathcal{Z}(m,n)$. We leave a more systematic discussion of this problem for future work.

B. Construction of quarter-BPS operators beyond zero coupling and the structure constants of $\mathcal{A}(m,n)$

The center of $\mathbb{C}[S_{m+n}]$ is denoted by $\mathcal{Z}[\mathbb{C}[S_{m+n}]]$. $\mathcal{Z}[\mathbb{C}[S_{m+n}]]$ is a commutative subalgebra of $\mathcal{A}(m,n)$. The $\mathcal{A}(m,n)$ algebra is a module over $\mathcal{Z}[\mathbb{C}[S_{m+n}]]$. We can write

$$\mathcal{T}_p N_i = \tilde{C}_{pj}^k N_k \tag{6.6}$$

for some coefficients $\tilde{C}$. The $\mathcal{T}_p$ are themselves linear combinations of necklaces:

$$\mathcal{T}_p = T_p^i N_i. \tag{6.7}$$

Hence

$$\mathcal{T}_p N_i = T_p^j N_j N_i = T_p^j C_{jk}^l N_l. \tag{6.8}$$

Another subspace in $\mathcal{A}(m,n)$ is the subspace of symmetrized traces. A symmetrized trace S_v can be parametrized by a *vector partition* v of (m,n). We can expand S_v on the basis of necklaces $\{N_k\}$ as

$$S_v = S_v^k N_k. \tag{6.9}$$

Symmetrized traces and their products are quarter-BPS at weak coupling in the large N limit. One can get the complete set of $1/N$ corrected BPS states at large N by acting on S_v with Ω^{-1} which belongs to $\mathcal{Z}[\mathbb{C}[S_{m+n}]] \otimes \mathbb{C}(1/N)$ [15,16,22,43]. The coefficients of T_p are easily computable. The expansion of T_p in terms of necklaces is also easily computable. The nontrivial part of the calculation is the C_{ij}^k of the necklace algebra $\mathcal{A}(m,n)$. For any symmetrized trace S_v, the corrected operator is

$$\Omega_k^{-1} S_v = \Omega_p^{-1} T_p S_v^j N_j = \Omega_p^{-1} S_v^j \tilde{C}_{pj}^k N_k = \Omega_p^{-1} S_v^j T_p^l C_{lj}^k N_k. \tag{6.10}$$

1. Central quarter BPS sector

A subspace of symmetrized trace elements is central. The symmetrized trace elements give a subspace of $\mathcal{A}(m,n)$ and the central elements form another subspace. The intersection is the space of central symmetrized traces. The dimension of this subspace can be computed for small m,n using GAP. Suppose S^C is an element in this subspace. Then elements $\Omega^{-1} S^C$ in $\mathcal{A}(m,n)$ are very interesting. They are quarter-BPS beyond zero coupling and they are central, so computations of their correlators have the simplicity of the center. The computations can be done using knowledge of the characters of S_m, S_n, S_{m+n}, without knowing branching coefficients. From AdS/CFT this central quarter BPS sector should have a dual in the space-time theory, e.g. some subclass of states in the tensor product of supergraviton states. An interesting question is to compute their correlators in space-time and verify the matching with the gauge theory computations.

C. Noncommutative geometry and topological field theory

Studies in noncommutative geometry in string theory suggest that open strings can be associated to noncommutative algebras and the center is related to closed strings [48]. If we apply this thinking to $\mathcal{A}(m,n)$ and $\mathcal{Z}(m,n)$, how do we interpret these emergent open and closed strings? The traditional view is that Yang-Mills theory is the open-string picture in AdS/CFT with the closed string picture given by the AdS description, so this is an intriguing question. Noncommutative algebras and their center have also been discussed in noncommutative geometry in [49]. The study of the pair $\{\mathcal{A}(m,n), \mathcal{Z}(m,n)\}$ should form an interesting example of this discussion. Additionally we have the Cartan $\mathcal{M}(m,n)$ here, with physical relevance in distinguishing the multiplicity labels. So a more complete picture of strings and noncommutative geometry for the triple $\{\mathcal{A}(m,n), \mathcal{M}(m,n), \mathcal{Z}(m,n)\}$ looks desirable. Given that the infinite direct sum $\mathcal{A}(\infty,\infty)$ comes up in connection with matrix invariants, it would also be interesting to study the triple $\{\mathcal{A}(\infty,\infty), \mathcal{M}(\infty,\infty), \mathcal{Z}(\infty,\infty)\}$ from this point of view. Some relevant work in this direction is in [26] (see also [50]).

D. Other examples of permutation centralizer algebras and correlators

Based on our study of $\mathcal{A}(m,n)$, we outline some properties of the other examples of permutation centralizer algebras given in Sec. II and sketch the connection to correlators. We leave a more detailed development for the future.

Consider $\mathcal{B}_N(m,n)$, which is the subspace of the Brauer algebra $B_N(m,n)$ invariant under $\mathbb{C}[S_m \times S_n]$. This is example 3 in Sec. II. Brauer algebras were used to construct gauge-invariant operators in [14] from tensor products of a complex matrix and its conjugate. For an element b in the walled Brauer algebra $B_N(m,n)$, we use

$$tr_{m,n}(Z^{\otimes m}\otimes \bar{Z}^{\otimes n} b) \tag{6.11}$$

where the trace is taken in $V^{\otimes m}\otimes \bar{V}^{\otimes n}$, a tensor product of fundamentals and antifundamentals of $U(N)$. We focus here on the case $m+n\leq N$. The number of gauge-invariant operators is

$$\sum_{\gamma,\alpha,\beta}(M^{\gamma}_{\alpha,\beta})^2 \tag{6.12}$$

where γ labels an irrep of $B_N(m,n)$, while α,β are irreps of S_m and S_n respectively. $M^{\gamma}_{\alpha,\beta}$ is a multiplicity with which (α,β) appears in the reduction of γ from $B_N(m,n)$ to its $\mathbb{C}[S_m\times S_n]$ subalgebra. The sum of squared dimensions in (6.12) is the dimension of the algebra $\mathcal{B}_N(m,n)$. This is a noncommutative algebra. The dimension of its center is the number of triples (γ,α,β) for which $M^{\gamma}_{\alpha,\beta}$ is nonvanishing. There is a maximally commuting subalgebra of dimension equal to the sum

$$\sum_{\gamma,\alpha,\beta}M^{\gamma}_{\alpha,\beta}. \tag{6.13}$$

This follows since the (γ,α,β) give a Wedderburn-Artin decomposition of $\mathcal{B}_N(m,n)$. A tractable sector of correlators should be given by the center of $\mathcal{B}_N(m,n)$ and more detailed study of the structure of this center will be useful.

The next algebra of interest is the subalgebra $\mathcal{K}(n)$ of $\mathbb{C}[S_n]\times\mathbb{C}[S_n]$ which is invariant under conjugation by $\mathrm{Diag}(\mathbb{C}[S_n])$. Let us denote this as $\mathcal{A}_{\mathrm{diag}}(n,n)$. We can generate elements in this algebra by summing over the elements of the subgroup

$$\sigma_1\otimes\sigma_2\to\sum_{\gamma\in S_n}\gamma\sigma_1\gamma^{-1}\otimes\gamma\sigma_2\gamma^{-1}. \tag{6.14}$$

The dimension of this algebra is

$$\sum_{R,S,T}C(R,S,T)^2 \tag{6.15}$$

where $C(R,S,T)$ is the Kronecker coefficient, i.e. the number of times the irrep T of S_n appears in the tensor product $R\otimes S$. The dimension of the center is the number of triples (R,S,T) for which the $C(R,S,T)$ is nonzero. A maximal commuting subalgebra has dimension

$$\sum_{R,S,T}C(R,S,T). \tag{6.16}$$

These properties follow from the fact that Wedderburn-Artin decomposition of the algebra $\mathcal{K}(n)$ has blocks labeled by triples (R,S,T) with nonvanishing $C(R,S,T)$. An explicit formula for this decomposition is

$$Q^{R,S,T}_{\tau_1,\tau_2}=\sum_{\sigma_1,\sigma_2}\sum_{i_1,i_2,i_3,j_1,j_2}S^{R,S,T,\tau_1}_{i_1,i_2,i_3}S^{R,S,T,\tau_2}_{j_1,j_2,i_3}D^R_{i_1j_1}(\sigma_1)\times D^S_{i_2j_2}(\sigma_2)\sigma_1\otimes\sigma_2. \tag{6.17}$$

The D's are representation matrices for S_n irreps. The S's are Clebsch-Gordan coefficients. One verifies, using equivariance properties of the Clebsch's that these are invariant under conjugation by the diagonal S_n.

There is another definition of $\mathcal{K}(n)$ which is more symmetric in (R,S,T). $C(R,S,T)$ is also the multiplicity of invariants of the diagonal S_n acting on $R\otimes S\otimes T$. $\mathcal{K}(n)$ can be defined as the subalgebra of $\mathbb{C}[S_n]\otimes\mathbb{C}[S_n]\otimes\mathbb{C}[S_n]$ which is invariant under left action by the diagonal $\mathbb{C}[S_n]$ and right action by the diagonal $\mathbb{C}[S_n]$. These invariant elements can again be constructed by averaging

$$\sum_{\gamma_1,\gamma_2}(\gamma_1\sigma_1\gamma_2,\gamma_1\sigma_2\gamma_2,\gamma_1\sigma_3\gamma_2). \tag{6.18}$$

A representation basis is given by

$$\sigma_1\otimes\sigma_2\otimes\sigma_3 D^R_{i_1,j_1}(\sigma_1)D^S_{i_2,j_2}(\sigma_2)D^T_{i_3,j_3}(\sigma_3)\times S^{R,S,T,\tau_1}_{i_1,i_2,i_3}S^{R,S,T,\tau_2}_{j_1,j_2,j_3} \tag{6.19}$$

labeled by R, S, T, τ_1, τ_2.

These triples of permutations $(\sigma_1,\sigma_2,\sigma_3)$, with equivalences given by left and right diagonal action have appeared in the enumeration invariants for tensor models built from 3-index tensors [51]. The simplification from a description in terms of permutation triples to one in terms of permutation pairs was also described there, which lead to a connection between 3-index tensor invariants and Belyi maps. By analogy with the discussion in this paper, we expect that the center of $\mathcal{K}(n)$ will lead to a class of simpler correlators in tensor models. The discussion of $\mathcal{A}(\infty,\infty)$ will analogously lead to

$$\mathcal{K}(\infty)=\bigoplus_{n=0}^{\infty}\mathcal{K}(n). \tag{6.20}$$

This space will have two products: one related to the algebra structure of $\mathcal{K}(n)$ and one related to the multiplication of tensor invariants. Somewhat related algebraic structures appear in [52] and it would be useful to better understand these relations. As a last remark, consider the Kronecker multiplicities $C(R,R,T)$, i.e., in the special case where $R=S$. These have also appeared in the construction of gauge-invariant multimatrix operators in a basis which is covariant under the global symmetries [15,16]. The structure of $\mathcal{K}(n)$ can thus also be expected to have implications for multimatrix correlators in the covariant basis.

ACKNOWLEDGMENTS

We thank David Berenstein, Robert de Mello Koch and Edward Hughes for useful discussions, and Robert de Mello Koch for comments on an earlier draft. S. R. is supported by STFC consolidated Grant No. ST/L000415/1 "String Theory, Gauge Theory & Duality." S. R. thanks the Simons Summer workshop 2015 for hospitality while part of this work was done. P. M. is supported by a Queen Mary University of London studentship.

APPENDIX A: ANALYTIC FORMULA FOR THE DIMENSION OF $\mathcal{M}(m,n)$

In this section we derive a formula for the dimension of $\mathcal{M}(m,n)$. This dimension is equal to the sum of Littlewood-Richardson coefficients

$$\mathrm{Dim}\,\mathcal{M}(m,n) = \sum_{R_1\vdash m, R_2\vdash n}\sum_{R\vdash m+n} g(R_1,R_2,R). \tag{A1}$$

The sum of squares of the Littlewood-Richardson coefficients is the dimension of $\mathcal{A}(m,n)$ and has a simple 2-variable generating function. It is natural to ask if we can write a nice generating function for the dimension of $\mathcal{M}(m,n)$. While we have not been able to derive something of comparable simplicity, we will derive two interesting expressions (A11) and (A28) in terms of multivariable polynomials.

Let T_p denote a conjugacy class of permutations with cycle structure determined by a vector $(p_1,p_2,\ldots)$, i.e. permutations with p_i cycles of length i. Let now σ_p be an element in T_p. For $\sigma_p\in T_p$, it is known that [53]

$$\sum_R \chi_R(\sigma_p) = \prod_i \mathrm{Coeff}\left(f_i(t_i), \frac{t_i^{p_i}}{p_i!}\right) \tag{A2}$$

where

$$f_i(t_i) = e^{\frac{(1-(-1)^i)}{2}t_i + \frac{it_i^2}{2}}. \tag{A3}$$

We can define

$$F(t_1,t_2,\ldots) = \prod_i f_i(t_i) \tag{A4}$$

and write

$$\sum_R \chi_R(\sigma_p) = \mathrm{Coeff}\left(F(t_1,t_2,\ldots), \prod_i \frac{t_i^{p_i}}{p_i!}\right). \tag{A5}$$

It is also useful to define

$$\begin{aligned}\tilde f_i(t_i) &= f_i\left(\frac{t_i}{i}\right)\\ \tilde F(t_1,t_2,\ldots) &= F\left(t_1,\frac{t_2}{2},\frac{t_3}{3}\cdots\right) = F\left(\left\{\frac{t_i}{i}\right\}\right)\\ &= \prod_{i:\mathrm{odd}} e^{\frac{t_i}{i}}\prod_{i=1}^{\infty} e^{\frac{it_i^2}{2i}}.\end{aligned} \tag{A6}$$

We can write the LR coefficients in terms of T_p's as

$$\begin{aligned} g(R_1,R_2,R) &= \frac{1}{m!n!}\sum_{\sigma_1\in S_m}\sum_{\sigma_2\in S_n}\chi_{R_1}(\sigma_1)\chi_{R_2}(\sigma_2)\chi_R(\sigma_1\circ\sigma_2)\\ &= \sum_{p\vdash m}\sum_{q\vdash n}\chi_{R_1}(T_p)\chi_{R_2}(T_q)\chi_R(T_p\circ T_q)\prod_i\frac{1}{i^{p_i+q_i}p_i!q_i!}. \end{aligned} \tag{A7}$$

This uses the fact that the number of permutations in the class T_p is $n!/\prod_i i^{p_i}p_i!$. Now use the above formula for $\sum_R\chi_R(T_p)$ to obtain

$$\begin{aligned}\sum_{R_1,R_2,R} g(R_1,R_2,R) &= \sum_{p\vdash m}\sum_{q\vdash n}\prod_i \mathrm{Coeff}\,(\tilde f_i(s_i), t_i^{p_i})\mathrm{Coeff}\,(\tilde f_i(t_i), t_i^{q_i})\mathrm{Coeff}\,(f_i(u_i), u_i^{p_i+q_i})(p_i+q_i)!\\ &= \sum_{p\vdash m}\sum_{q\vdash n}\mathrm{Coeff}\left(\tilde F(\vec s)\tilde F(\vec t)F(\vec u), \prod_i s_i^{p_i}t_i^{q_i}u_i^{p_i+q_i}(p_i+q_i)!^{-1}\right)\\ &= \sum_{p\vdash m}\sum_{q\vdash n}\mathrm{Coeff}\left(\tilde F(\vec s)\tilde F(\vec t)\tilde F(\vec u), \prod_i s_i^{p_i}t_i^{q_i}u_i^{p_i+q_i}\right)i^{p_i+q_i}(p_i+q_i)!\end{aligned} \tag{A8}$$

It is useful to make the substitutions $s_i \to s^i z_i$, $t_i \to t^i z_i$, $u_i \to \bar{z}_i$ and to introduce a pairing[3]

$$\langle z_j^k, \bar{z}_i^l \rangle = \delta_{ij}\delta_{kl} k! i^k. \tag{A9}$$

With these substitutions define

$$\mathcal{F}(z_i, s) = \tilde{F}(t_i \to s^i z_i). \tag{A10}$$

Then we can write

$$\mathrm{Dim}(\mathcal{M}(m,n)) = \langle \mathrm{Coeff}(\mathcal{F}(z_i,s)\mathcal{F}(z_i,t)\mathcal{F}(z_i,u=1), s^m t^n)\rangle. \tag{A11}$$

This has been checked for very simple cases, e.g. up to $(m,n) = (3,3)$.

1. Multivariable polynomials

It is useful to isolate the multivariable polynomials in the z_i variables at each order in the s, t variables. Let us introduce the quantities

$$\mathcal{A}(\vec{z}, s) = \prod_i \exp\left[\frac{s^{2i} z_i^2}{2i}\right]$$
$$\mathcal{B}(\vec{z}, s) = \prod_{i=1,3,..} \exp\left[\frac{s^i z_i}{i}\right]. \tag{A12}$$

It follows from previous formulas (A6) and (A10) that

$$\mathcal{F}(\vec{z}, s) = \mathcal{A}(\vec{z}, s)\mathcal{B}(\vec{z}, s). \tag{A13}$$

Introducing polynomials $\mathcal{F}_m(\vec{z})$ for each order in s we can rewrite the latter quantity as

$$\mathcal{F}(\vec{z}, s) = \sum_{m=0} \mathcal{F}_m(\vec{z}) s^m. \tag{A14}$$

We will now write formulas for the coefficients of s^m in $\mathcal{A}$ and $\mathcal{B}$. For $\mathcal{A}(\vec{z}, s)$ we derive

$$\mathcal{A}(\vec{z}, s) = \sum_{m=0}^{\infty} \mathcal{A}_{2m}(\vec{z}) s^{2m} = \sum_{p_1,p_2,\ldots=0}^{\infty} \prod_{i=1}^{\infty} \frac{s^{2ip_i} z_i^{2ip_i}}{(2i)^{p_i} p_i!} \tag{A15}$$

so that

$$\mathcal{A}_{2m}(\vec{z}) = \sum_{p\vdash m} \frac{z_i^{2ip_i}}{(2i)^{p_i} p_i!}. \tag{A16}$$

We can also define $\mathcal{A}_m(\vec{z})$ to be zero for odd m and equal to the above for the even values. It is useful to define the coefficients of $z_1^{2p_1} z_2^{4p_2} \ldots z_i^{2ip_i}$ in the $\mathcal{A}(\vec{z}, s=1)$ as

$$\mathcal{A}_{[p]} = \mathcal{A}_{[p_1,p_2\cdots]} = \prod_i \frac{1}{p_i!(2i)^{p_i}} \tag{A17}$$

so that we may write

$$\mathcal{A}_{2m} = \sum_{p\vdash m} \mathcal{A}_{[p]} \prod_{i=1}^{\infty} z_i^{2ip_i}. \tag{A18}$$

Similarly, for $\mathcal{B}(\vec{z}, s)$ we obtain

$$\mathcal{B}(\vec{z}, s) = \prod_{i=0}^{\infty} \exp\left[\frac{s^{(2i+1)} z_{2i+1}}{(2i+1)}\right] \tag{A19}$$

and

$$\mathcal{B}_m(\vec{z}) = \sum_{\{p_1,p_3\cdots\}\vdash m} \prod_{i\ \mathrm{odd}} \frac{z_i^{ip_i}}{(i)^{p_i} p_i!}. \tag{A20}$$

Therefore it is natural to define

$$\mathcal{B}_{[p_1,p_3,\ldots]} = \prod_i \frac{1}{i^{p_i} p_i!}$$
$$\mathcal{B}_m(\vec{z}) = \sum_{p\vdash m} \mathcal{B}_{[p_1,p_3,\ldots]} \prod_{i\ \mathrm{odd}} z_i^{ip_i}. \tag{A21}$$

Going back to (A14) we get, using the formulas just derived,

$$\mathcal{F}_m(\vec{z}) = \sum_{k=0}^{m} \mathcal{A}_k(\vec{z})\mathcal{B}_{m-k}(\vec{z}) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{A}_{2k}(\vec{z})\mathcal{B}_{m-2k}(\vec{z})$$
$$= \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{r\vdash k} \sum_{\substack{q\vdash m-2k \\ q\ \mathrm{odd}}} \mathcal{A}_{[r]}\mathcal{B}_{[q]} \prod_i z_i^{i(2r_i+q_i)}. \tag{A22}$$

Grouping terms with the same power of z_i we obtain

$$\mathcal{F}(\vec{z}, s=1) = \sum_{[p_1,p_2\ldots]} \mathcal{F}_{[p_1,p_2\ldots]} \prod_i z_i^{ip_i} \tag{A23}$$

with

[3]Alternatively we can think about expectation values in a Fock space with $z_i \to a_i$, $\bar{z}_i \to a_i^\dagger$. This would allow us to write the subsequent formulas in terms of quantities in a 2D field theory. This perspective could be fruitful, but we will leave its exploration for the future.

$$\mathcal{F}_{[p]} = \sum_{[r_1,r_2,\ldots]}\sum_{[q_1,q_2\cdots]}\mathcal{A}_{[r_1,r_2\cdots]}\mathcal{B}_{[q_1,q_2,\ldots]} \times \prod_{i\,\text{even}}\delta(p_i,2r_i)\prod_{i\,\text{odd}}\delta(p_i,2r_i+q_i). \quad (A24)$$

Note that the function $\mathcal{F}(\vec{z},s)$ is closely related to the generating function for the cycle indices of S_n which is

$$\mathcal{Z}(\vec{z},t) = \exp\left[\sum_{i=1}^{\infty}\frac{t^i z_i}{i}\right],$$
$$\tilde{\mathcal{A}}(\vec{z},s) = (\mathcal{Z}(z_i\to z_i^2, s\to s^2))^{1/2},$$
$$\tilde{\mathcal{B}}(\vec{z},s) = (\mathcal{Z}(z_{2i+1}\to z_{2i+1}, z_{2i}\to 0))^{1/2}. \quad (A25)$$

We can work with the same function if we change the pairing. With the pairing

$$\langle z_i^{k_i}, z_j^{k_j}\rangle = \delta_{i,j}\delta_{k_i,k_j}k_i!\,i^{k_i} \quad (A26)$$

we can write the above formulas as

$$\mathrm{Dim}(\mathcal{M}(m,n)) = \langle\mathcal{F}_m(\vec{z})\mathcal{F}_n(\vec{z}),\mathcal{F}_{m+n}(\vec{z})\rangle \quad (A27)$$

or, equivalently,

$$\mathrm{Dim}(\mathcal{M}(m,n)) = \sum_{p\vdash m}\sum_{q\vdash n}\mathcal{F}_{p_1,p_2\ldots}\mathcal{F}_{q_1,q_2,\ldots}\mathcal{F}_{p_1+p_2,q_1+q_2,\ldots} \times\prod_i i^{p_i+q_i}(p_i+q_i)! = \sum_{p\vdash m}\sum_{q\vdash n}\mathcal{F}_p\mathcal{F}_q\mathcal{F}_{p+q}\mathrm{Sym}(p+q). \quad (A28)$$

This is Eq. (3.45).

APPENDIX B: LR RULE FOR HOOK REPRESENTATIONS

Here we derive the LR decomposition rule for the tensor product of two hook representations. Let us consider three representations R, R_1 and R_2 of S_{m+n}, S_m and S_n respectively. The LR coefficient $g(R_1,R_2;R)$ gives the multiplicity with which the representation $R_1\otimes R_2$ appear in the representation R upon its restriction to $S_m\times S_n$. There is a systematic procedure to obtain such coefficients [40], that we now briefly review. We take the Young diagrams corresponding to R_1 and R_2, and we start by decorating the latter as follows. We write "1" in all the boxes of the first row, "2" in all the boxes of the second row and so on in a similar fashion until the last row. Then we proceed to move all the 1 boxes from R_2 to R_1, ensuring that we produce legal Young diagrams and no two copies of 1 appear in the same column. We then move the 2 boxes following the same rules, and so on. In doing so, we also require a *reading condition*. At any step, reading from right lo left along the first row and then subsequent rows, the number of 1 boxes must be greater or equal to the number of 2 boxes. Similarly, the number of 2 boxes must be greater or equal to the number of "3" boxes, and so on.

At the end of this procedure we are left with a collection of Young diagrams, made with $m+n$ boxes. If two or more of the resulting diagrams are identical (that is, they not only match in shape but also in the numbering of their boxes), we only retain one of them. Otherwise, if k diagrams R appear with the same shape but different numbering, we can say that $g(R_1,R_2;R)=k$. These will be the prescriptions that we will follow to derive our LR formula.

We specify any representation R by the sequence of pairs of integers $R=((a_1,b_1),(a_2,b_2),\ldots(a_d,b_d))$. In a Young diagram interpretation, a_j $(1\le j\le d)$ is the number of boxes to the right of the jth diagonal box, and b_j is the number of boxes below the jth diagonal box. We refer to d as the "depth" of the representation R. Hooks therefore are representations of depth 1. Schematically, in this appendix we will obtain the right-hand side of

$$(k_1,l_1)\otimes(k_2,l_2) = \bigoplus((a_1,b_1),(a_2,b_2)). \quad (B1)$$

In our derivation we imagine to keep the first hook fixed, and to add to it boxes coming from the second diagram. In doing so we are careful to follow the LR prescription. The boxes of the second diagram are decorated by a 1 or a v, depending whether they come from the first row of the diagram or not. The tensor product $(k_1,l_1)\otimes(k_2,l_2)$ will decompose into a direct sum of a varying number of depth 2 representation and precisely two hooks (regardless of the actual value of $k_{1,2}$, $l_{1,2}$). These hooks are

$$\text{Hook 1}:\ (k_1+k_2+1, l_1+l_2),$$
$$\text{Hook 2}:\ (k_1+k_2, l_1+l_2+1). \quad (B2)$$

Notice that we can rewrite them using the notation we use for the depth two diagram as

$$\text{Hook 1}:\ ((k_1+k_2+1,l_1+l_2),(0,-1)), \quad (B3)$$

$$\text{Hook 2}:\ ((k_1+k_2,l_1+l_2+1),(-1,0)). \quad (B4)$$

This notation will be helpful at a later stage.

We now turn to the depth two representations. We proceed systematically, grouping them into four categories according to the two yes/no questions:

(1) Is there a $\boxed{1}$ in the first column of the resulting diagram?
(2) Is there a $\boxed{v}$ in the first row of the inner hook of the resulting diagram?

We now analyze these four possibilities.

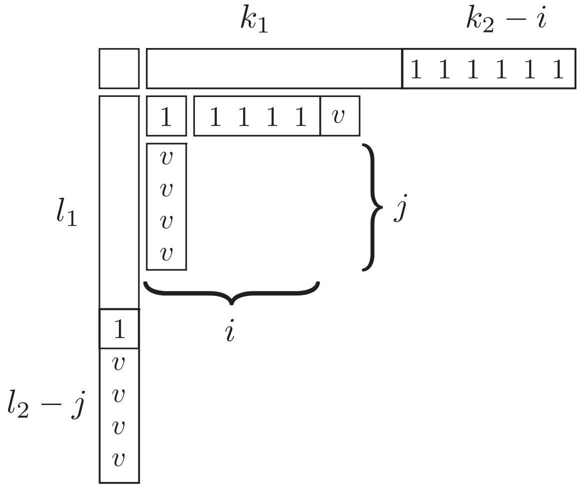

FIG. 2. (Y,Y) case.

1. (Y,Y) case

The diagrams in this class are of the form depicted in Fig. 2.

They can be described by the expression

$$(Y,Y)\colon\ ((k_1 + k_2 - i, l_1 + l_2 - j), (i, j)) \tag{B5}$$

where i and j are constrained by the boundaries

$$\begin{aligned} &0 \le i \le \min(k_1, k_2 - 1), \\ &0 \le j \le \min(l_1, l_2 - 1). \end{aligned} \tag{B6}$$

The upper bound on i is $\min(k_1, k_2 - 1)$ because, if $k_1 \ge k_2$, we cannot remove all the k_2 $\boxed{1}$ type boxes from the first row. This has to be avoided since by construction the rightmost box in the second row has to be a $\boxed{v}$ type box. A diagram with no $\boxed{1}$ type boxes on the first row and a $\boxed{v}$ type box at the end of the second row would violate the LR reading condition.

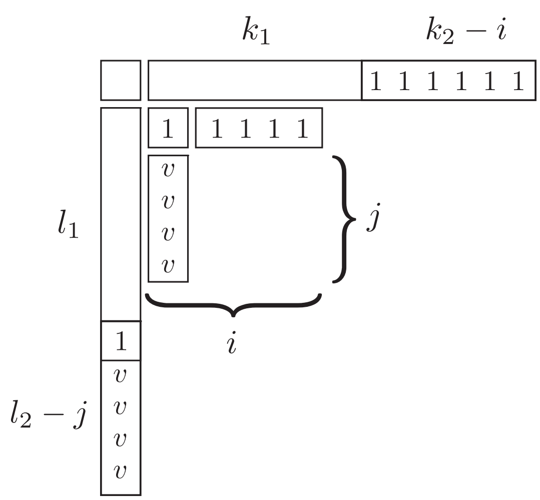

FIG. 3. (Y,N) case.

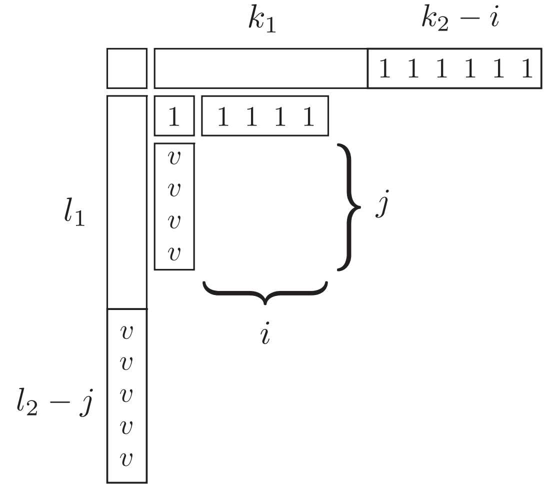

FIG. 4. (N,N) case.

2. (Y,N) case

The diagrams in this class are of the form depicted in Fig. 3.

They can be described by the expression

$$(Y,N)\colon\ ((k_1 + k_2 - i, l_1 + l_2 - j + 1), (i - 1, j)) \tag{B7}$$

with the boundaries

$$\begin{aligned} &1 \le i \le \min(k_1, k_2), \\ &0 \le j \le \min(l_1, l_2). \end{aligned} \tag{B8}$$

3. (N,N) case

The depth two diagrams in this class are of the form depicted in Fig. 4.

They can be described by the expression

$$(N,N)\colon\ ((k_1 + k_2 - i, l_1 + l_2 - j), (i, j)) \tag{B9}$$

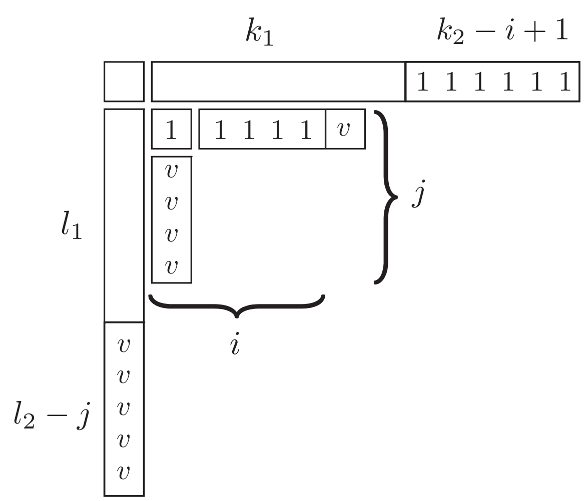

FIG. 5. (N,Y) case.

with the boundaries

$$0 \le i \le \min(k_1 - 1, k_2),$$
$$0 \le j \le \min(l_1 - 1, l_2). \quad \text{(B10)}$$

4. (N,Y) case

The diagrams in this class are of the form depicted in Fig. 5.

These can be described by the equation

$$(N,Y)\colon\ ((k_1 + k_2 - i + 1, l_1 + l_2 - j), (i, j - 1)). \quad \text{(B11)}$$

The boundary for i is

$$0 \le i \le \min(k_1, k_2). \quad \text{(B12)}$$

The upper bound is k_2 and not $k_2 + 1$ because we cannot remove all the $\boxed{1}$ from the first row, as the rightmost box in the second row has to be a $\boxed{v}$ type box. In this way, we are enforcing the LR reading condition. On the other hand, the boundary for j is

$$1 \le j \le \min(l_1, l_2). \quad \text{(B13)}$$

The lower bound is a 1 as by construction there has to be a $\boxed{v}$ box in the first row of the inner hook.

5. A summary

These four cases comprise all possible valid depth two diagrams. Summarizing our result, we have

(i) (Y,Y) case: $((k_1 + k_2 - i, l_1 + l_2 - j), (i, j))$,

$$0 \le i \le \min(k_1, k_2 - 1),$$
$$0 \le j \le \min(l_1, l_2 - 1). \quad \text{(B14)}$$

(ii) (Y,N) case: $((k_1 + k_2 - i, l_1 + l_2 - j + 1), (i - 1, j))$,

$$1 \le i \le \min(k_1, k_2),$$
$$0 \le j \le \min(l_1, l_2). \quad \text{(B15)}$$

(iii) (N,N) case: $((k_1 + k_2 - i, l_1 + l_2 - j), (i, j))$,

$$0 \le i \le \min(k_1 - 1, k_2),$$
$$0 \le j \le \min(l_1 - 1, l_2). \quad \text{(B16)}$$

(iv) (N,Y) case: $((k_1 + k_2 - i + 1, l_1 + l_2 - j), (i, j - 1))$,

$$0 \le i \le \min(k_1, k_2),$$
$$1 \le j \le \min(l_1, l_2). \quad \text{(B17)}$$

We now introduce the boolean parameters

$$\epsilon_1 = \begin{cases} 0 & \text{If the answer to the first question is } \textbf{no} \\ 1 & \text{If the answer to the first question is } \textbf{yes} \end{cases} \quad \text{(B18)}$$

and

$$\epsilon_2 = \begin{cases} 0 & \text{If the answer to the second question is } \textbf{no} \\ 1 & \text{If the answer to the second question is } \textbf{yes} \end{cases} \quad \text{(B19)}$$

With this notation we can compactly rewrite (B14)–(B17) as

$$((k_1 + k_2 - i + \bar{\epsilon}_1\epsilon_2, l_1 + l_2 - j + \epsilon_1\bar{\epsilon}_2), (i - \epsilon_1\bar{\epsilon}_2, j - \bar{\epsilon}_1\epsilon_2)) \quad \text{(B20)}$$

where the sign $\bar{}$ denotes the logical negation of a boolean variable, so that $\bar{\epsilon}_{1,2} = 1 - \epsilon_{1,2}$. In this notation, i and j have the boundaries

$$\epsilon_1\bar{\epsilon}_2 \le i \le \min(k_1 - \bar{\epsilon}_1\bar{\epsilon}_2, k_2 - \epsilon_1\epsilon_2),$$
$$\bar{\epsilon}_1\epsilon_2 \le j \le \min(l_1 - \bar{\epsilon}_1\bar{\epsilon}_2, l_2 - \epsilon_1\epsilon_2). \quad \text{(B21)}$$

By denoting $h_1 = (k_1, l_1)$ and $h_2 = (k_2, l_2)$, together with $R = ((a_1, b_1), (a_2, b_2))$ we can then write

$$g(h_1, h_2; R) = \delta_{k_1+k_2,a_1}\delta_{l_1+l_2+1,b_1}\delta_{-1,a_2}\delta_{0,b_2} + \delta_{k_1+k_2+1,a_1}\delta_{l_1+l_2,b_1}\delta_{0,a_2}\delta_{-1,b_2}$$
$$+ \sum_{\epsilon_1,\epsilon_2=0}^{1} \sum_{i=\epsilon_1\bar{\epsilon}_2}^{\min(k_1-\bar{\epsilon}_1\bar{\epsilon}_2,k_2-\epsilon_1\epsilon_2)} \sum_{j=\bar{\epsilon}_1\epsilon_2}^{\min(l_1-\bar{\epsilon}_1\bar{\epsilon}_2,l_2-\epsilon_1\epsilon_2)} \delta_{k_1+k_2-i+\bar{\epsilon}_1\epsilon_2,a_1}\delta_{l_1+l_2-j+\epsilon_1\bar{\epsilon}_2,b_1}\delta_{i-\epsilon_1\bar{\epsilon}_2,a_2}\delta_{j-\bar{\epsilon}_1\epsilon_2,b_2} \quad \text{(B22)}$$

where we also added the two hooks in the depth two notation, (B3) and (B4). Explicitly, summing over the $\epsilon_{1,2}$ parameters, we get the lengthier expression

$$g(h_1,h_2;R) = \delta_{k_1+k_2,a_1}\delta_{l_1+l_2+1,b_1}\delta_{-1,a_2}\delta_{0,b_2} + \sum_{i=1}^{\min(k_1,k_2)}\sum_{j=0}^{\min(l_1,l_2)}\delta_{k_1+k_2-i,a_1}\delta_{l_1+l_2-j+1,b_1}\delta_{i-1,a_2}\delta_{j,b_2}$$
$$+\delta_{k_1+k_2+1,a_1}\delta_{l_1+l_2,b_1}\delta_{0,a_2}\delta_{-1,b_2} + \sum_{i=0}^{\min(k_1,k_2)}\sum_{j=1}^{\min(l_1,l_2)}\delta_{k_1+k_2-i+1,a_1}\delta_{l_1+l_2-j,b_1}\delta_{i,a_2}\delta_{j-1,b_2}$$
$$+\left(\sum_{i=0}^{\min(k_1,k_2-1)}\sum_{j=0}^{\min(l_1,l_2-1)} + \sum_{i=0}^{\min(k_1-1,k_2)}\sum_{j=0}^{\min(l_1-1,l_2)}\right)\delta_{k_1+k_2-i,a_1}\delta_{l_1+l_2-j,b_1}\delta_{i,a_2}\delta_{j,b_2}. \quad \text{(B23)}$$

From this equation it is clear that $g(h_1,h_2;R)$ can be either 0, 1 or 2. In particular, $g(h_1,h_2;R)=2$ only if $R=((k_1+k_2-i,l_1+l_2-j),(i,j))$ and $0\leq i<\min(k_1,k_2)$, $0\leq j<\min(l_1,l_2)$.

APPENDIX C: DERIVING THE TWO POINT CORRELATOR

In this appendix we will derive Eq. (5.9) from Eq. (5.7). Let us start by considering the quantity

$$\chi^R_{R_1,R_2}(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}) \quad \text{(C1)}$$

where we remind the reader that R_1, R_2 and R are irreps of S_m, S_n and S_{m+n} respectively. Let us define $T_2^{(X,Y)}$, $T_2^{(X)}$ and $T_2^{(Y)}$ as the sum of transpositions in S_{m+n}, S_m and S_n respectively. We can expand (C1) as

$$\chi^R_{R_1,R_2}(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}) = \chi^R_{R_1,R_2}(T_2^{(X,Y)}T^{(X)}_{[m]}T^{(Y)}_{[n]}) - \chi^R_{R_1,R_2}(T_2^{(X)}T^{(X)}_{[m]}T^{(Y)}_{[n]}) - \chi^R_{R_1,R_2}(T^{(X)}_{[m]}T_2^{(Y)}T^{(Y)}_{[n]})$$
$$= g\frac{\chi_R(T_2^{(X,Y)})}{d_R}\chi_{R_1}(T^{(X)}_{[m]})\chi_{R_2}(T^{(Y)}_{[n]}) - \frac{1}{gd_{R_1}d_{R_2}}\chi^R_{R_1,R_2}(T_2^{(X)})\chi^R_{R_1,R_2}(T^{(X)}_{[m]}T^{(Y)}_{[n]}) +$$
$$-\frac{1}{gd_{R_1}d_{R_2}}\chi^R_{R_1,R_2}(T_2^{(Y)})\chi^R_{R_1,R_2}(T^{(X)}_{[m]}T^{(Y)}_{[n]})$$
$$= g\frac{\chi_R(T_2^{(X,Y)})}{d_R}\chi_{R_1}(T^{(X)}_{[m]})\chi_{R_2}(T^{(Y)}_{[n]}) - \frac{\chi_{R_1}(T_2^{(X)})}{d_{R_1}}\chi^R_{R_1,R_2}(T^{(X)}_{[m]}T^{(Y)}_{[n]}) + -\frac{\chi_{R_2}(T_2^{(Y)})}{d_{R_2}}\chi^R_{R_1,R_2}(T^{(X)}_{[m]}T^{(Y)}_{[n]})$$
$$= g\chi_{R_1}(T^{(X)}_{[m]})\chi_{R_2}(T^{(Y)}_{[n]})\left[\frac{\chi_R(T_2^{(X,Y)})}{d_R} - \frac{\chi_{R_1}(T_2^{(X)})}{d_{R_1}} - \frac{\chi_{R_2}(T_2^{(Y)})}{d_{R_2}}\right]. \quad \text{(C2)}$$

But now

$$\chi_{R_1}(T_{[m]}) = \begin{cases} (-1)^{c_{R_1}+1}(m-1)! & \text{if } R_1 \text{ is a hook representation} \\ 0 & \text{otherwise} \end{cases} \quad \text{(C3)}$$

where c_{R_1} is the number of boxes in the firs column of the Young diagram associated with the representation R_1. A similar equation holds for $\chi_{R_2}(T_{[n]})$. We then have

$$\chi^R_{R_1,R_2}(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}) = \begin{cases} (-1)^{c_{R_1}+c_{R_2}}g(m-1)!(n-1)!\left[\frac{\chi_R(T_2^{(X,Y)})}{d_R} - \frac{\chi_{R_1}(T_2^{(X)})}{d_{R_1}} - \frac{\chi_{R_2}(T_2^{(Y)})}{d_{R_2}}\right]; & R_1, R_2 \text{ hooks} \\ 0 \quad \text{otherwise} \end{cases} \quad \text{(C4)}$$

This is Eq. (5.8). Let us now restrict to the case in which both R_1, R_2 are hooks representations. We will denote there representations as $h_1=R_1=(k_1,l_1)$ and $h_2=R_2=(k_2,l_2)$. This also forces the representation R to be at most of depth two, as we derived in Appendix B. We now consider such a representation. With the notation given at the beginning of this section, $R=((a_1,b_1),(a_2,b_2))$, it is immediate to write an equation for the normalized character $\frac{\chi_R(T_2)}{d_R}$

$$\frac{\chi_R(T_2)}{d_R}=\frac{1}{2}\sum_i r_i(r_i-2i+1)=a_1(a_1+1)+(a_2+2)(a_2-1)+2\sum_{i=3}^{b_2+2}(3-2i)+2\sum_{i=b_2+3}^{b_1+1}(1-i) \tag{C5}$$

$$=\frac{1}{2}(a_1^2+a_2^2+a_1+a_2)-\frac{1}{2}(b_1^2+b_2^2+b_1+b_2)=\frac{1}{2}(a_1+b_1+1)(a_1-b_1)+\frac{1}{2}(a_2+b_2+1)(a_2-b_2). \tag{C6}$$

We now need the equivalent of this formula for the depth one representations h_1 and h_2, i.e. the hooks. Such an equation can be directly obtained by setting $(a_2,b_2)=(-1,0)$ or $(a_2,b_2)=(0,-1)$ in (C5). We can then write (C4) as

$$\chi^R_{h_1,h_2}(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]})=\frac{(-1)^{c_{h_1}+c_{h_2}}}{2}g(m-1)!(n-1)!\times[(a_1+b_1+1)(a_1-b_1)+(a_2+b_2+1)(a_2-b_2)+ \\ -(k_1+l_1+1)(k_1-l_1)-(k_2+l_2+1)(k_2-l_2)] \tag{C7}$$

where $R=((a_1,b_1),(a_2,b_2))$ and $h_1=(k_1,l_1)$, $h_2=(k_2,l_2)$.

The last piece we need is an equation for the $U(N)$ dimension of a depth two representation $R=((a_1,b_1),(a_2,b_2))$. It is straightforward to write

$$\mathrm{Dim}_N(R)=\frac{(a_1-a_2)(b_1-b_2)}{(a_1+b_2+1)(a_2+b_1+1)}\binom{a_1+b_1}{b_1}\binom{a_2+b_2}{b_2}\binom{N+a_1}{a_1+b_1+1}\binom{N+a_2}{a_2+b_2+1}. \tag{C8}$$

This equation reduces to its depth 1 equivalent by imposing $(a_2,b_2)=(-1,0)$ or $(a_2,b_2)=(0,-1)$. It is also helpful to recall the dimension formula for a S_{l+k+1} hook representation (k,l):

$$d_R=\binom{k+l}{k} \tag{C9}$$

Let us now consider Eq. (5.7):

$$\langle\mathcal{O}\mathcal{O}^\dagger\rangle=\frac{1}{m!n!}\sum_{\substack{R_1\vdash m\\ R_1\vdash n}}\sum_{R\vdash m+n}\frac{1}{d_{R_1}d_{R_2}g}\mathrm{Dim}_N(R)(\chi^R_{R_1,R_2}(T_{\bar{1},1}T^{(X)}_{[m]}T^{(Y)}_{[n]}))^2. \tag{C10}$$

Inserting Eq. (C7), (C8) and (C9) into the above equation gives

$$\langle\mathrm{Tr}(X^mY^n)\mathrm{Tr}(X^mY^n)^\dagger\rangle=\sum_{k_1,l_1=0}^{m}\sum_{k_2,l_2=0}^{n}\sum_{\substack{a_1,b_1=0\\ a_2,b_2=0}}^{n+m}g\,\delta(k_1+l_1-m)\delta(k_2+l_2-n)\,F(a_1,b_1,a_2,b_2,k_1,l_1,k_2,l_2) \tag{C11}$$

where we defined the function

$$F(a_1,b_1,a_2,b_2,k_1,l_1,k_2,l_2)=\frac{k_1!k_2!l_1!l_2!(a_1-a_2)(b_1-b_2)}{4(a_1+b_2+1)(a_2+b_1+1)(k_1+l_1+1)(k_2+l_2+1)} \\ \times\binom{a_1+b_1}{b_1}\binom{a_2+b_2}{b_2}\binom{N+a_1}{a_1+b_1+1}\binom{N+a_2}{a_2+b_2+1} \\ \times((a_1+b_1+1)(a_1-b_1)+(a_2+b_2+1)(a_2-b_2)+ \\ -(k_1+l_1+1)(k_1-l_1)-(k_2+l_2+1)(k_2-l_2))^2. \tag{C12}$$

[1] J. M. Maldacena, The Large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. **38**, 1113 (1999); Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).
[3] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).
[4] S. Corley, A. Jevicki, and S. Ramgoolam, Exact correlators of giant gravitons from dual $N = 4$ SYM theory, Adv. Theor. Math. Phys. **5**, 809 (2002).
[5] S. Corley and S. Ramgoolam, Finite factorization equations and sum rules for BPS correlators in $N = 4$ SYM theory, Nucl. Phys. **B641**, 131 (2002).
[6] V. Balasubramanian, M.-x. Huang, T. S. Levi, and A. Naqvi, Open strings from $N = 4$ super Yang-Mills, J. High Energy Phys. 08 (2002) 037.
[7] V. Balasubramanian, D. Berenstein, B. Feng, and M.-x. Huang, D-branes in Yang-Mills theory and emergent gauge symmetry, J. High Energy Phys. 03 (2005) 006.
[8] R. de Mello Koch, J. Smolic, and M. Smolic, Giant gravitons—with strings attached (I), J. High Energy Phys. 06 (2007) 074.
[9] R. de Mello Koch, J. Smolic, and M. Smolic, Giant gravitons—with strings attached (II), J. High Energy Phys. 09 (2007) 049.
[10] D. Bekker, R. de Mello Koch, and M. Stephanou, Giant gravitons—with strings attached. III., J. High Energy Phys. 02 (2008) 029.
[11] J. M. Maldacena and A. Strominger, AdS(3) black holes and a stringy exclusion principle, J. High Energy Phys. 12 (1998) 005.
[12] J. McGreevy, L. Susskind, and N. Toumbas, Invasion of the giant gravitons from anti-de Sitter space, J. High Energy Phys. 06 (2000) 008.
[13] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler, Giant gravitons in conformal field theory, J. High Energy Phys. 04 (2002) 034.
[14] Y. Kimura and S. Ramgoolam, Branes, anti-branes and brauer algebras in gauge-gravity duality, J. High Energy Phys. 11 (2007) 078.
[15] T. W. Brown, P. Heslop, and S. Ramgoolam, Diagonal multi-matrix correlators and BPS operators in $N = 4$ SYM, J. High Energy Phys. 02 (2008) 030.
[16] T. W. Brown, P. Heslop, and S. Ramgoolam, Diagonal free field matrix correlators, global symmetries and giant gravitons, J. High Energy Phys. 04 (2009) 089.
[17] R. Bhattacharyya, S. Collins, and R. d. M. Koch, Exact multi-matrix correlators, J. High Energy Phys. 03 (2008) 044.
[18] S. Collins, Restricted Schur Polynomials and Finite N Counting, Phys. Rev. D **79**, 026002 (2009).
[19] P. Caputa, R. d. M. Koch, and P. Diaz, Operators, Correlators and free fermions for SO(N) and Sp(N), J. High Energy Phys. 06 (2013) 018.
[20] R. de Mello Koch, B. A. E. Mohammed, J. Murugan, and A. Prinsloo, Beyond the planar limit in ABJM, J. High Energy Phys. 05 (2012) 037.
[21] J. Pasukonis and S. Ramgoolam, From counting to construction of BPS states in $N = 4$ SYM, J. High Energy Phys. 02 (2011) 078.
[22] J. Pasukonis and S. Ramgoolam, Quivers as calculators: Counting, correlators and Riemann surfaces, J. High Energy Phys. 04 (2013) 094
[23] P. Mattioli and S. Ramgoolam, Quivers, Words and fundamentals, J. High Energy Phys. 03 (2015) 105.
[24] R. de Mello Koch, R. Kreyfelt, and N. Nokwara, Finite N quiver gauge theory, Phys. Rev. D **89**, 126004 (2014).
[25] D. Berenstein, Extremal chiral ring states in the AdS/CFT correspondence are described by free fermions for a generalized oscillator algebra, Phys. Rev. D **92**, 046006 (2015).
[26] Y. Kimura, Multi-matrix models and Noncommutative Frobenius algebras obtained from symmetric groups and Brauer algebras, Commun. Math. Phys. **337**, 1 (2015).
[27] R. de Mello Koch and S. Ramgoolam, A double coset ansatz for integrability in AdS/CFT, J. High Energy Phys. 06 (2012) 083.
[28] W. Carlson, R. d. M. Koch, and H. Lin, Nonplanar Integrability, J. High Energy Phys. 03 (2011) 105.
[29] R. d. M. Koch, B. A. E. Mohammed, and S. Smith, Nonplanar integrability: Beyond the SU(2) sector, Int. J. Geom. Methods Mod. Phys. **26**, 4553 (2011).
[30] R. d. M. Koch, M. Dessein, D. Giataganas, and C. Mathwin, Giant graviton oscillators, J. High Energy Phys. 10 (2011) 009.
[31] Y. Kimura and S. Ramgoolam, Enhanced symmetries of gauge theory and resolving the spectrum of local operators, Phys. Rev. D **78**, 126003 (2008).
[32] P. Diaz, Novel charges in CFT‘s, J. High Energy Phys. 09 (2014).
[33] R. Goodman and N. Wallach, *Representations and Invariants of the Classical Groups*, Encyclopedia of Mathematics and Its Applications (Cambridge University Press, Cambridge, England, 2000).
[34] A. Ram, Ph.D. in Mathematics, University of California, San Diego, Representation theory; dissertation, chapter 1. http://www.ms.unimelb.edu.au/ram/Preprints/dissertationChapt1.pdf, 1990.
[35] Y. Kimura and S. Ramgoolam, Enhanced symmetries of gauge theory and resolving the spectrum of local operators, Phys. Rev. D **78**, 126003 (2008).
[36] R. Bhattacharyya, R. de Mello Koch, and M. Stephanou, Exact multi-restricted Schur polynomial correlators, J. High Energy Phys. 06 (2008) 101.
[37] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Dover Publications, New York, 1989).
[38] F. Larsen and E. J. Martinec, U(1) charges and moduli in the D1—D5 system, J. High Energy Phys. 06 (1999) 019.
[39] S. Danz, J. Ellers, and J. Murray, The centralizer of a subgroup in a group algebra, Proc. Edinburgh Math. Soc. **56**, 49 (2013).
[40] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics/Readings in Mathematics (Springer New York, 1991).
[41] M. Bianchi, F. A. Dolan, P. J. Heslop, and H. Osborn, $N = 4$ superconformal characters and partition functions, Nucl. Phys. **B767**, 163 (2007).
[42] D. Vaman and H. L. Verlinde, Bit strings from $N = 4$ gauge theory, J. High Energy Phys. 11 (2003) 041.
[43] T. Brown, Cut-and-join operators and $N = 4$ super Yang-Mills, J. High Energy Phys. 05 (2010) 058.

[44] J. Pasukonis and S. Ramgoolam, From counting to construction of BPS states in $N = 4$ SYM, J. High Energy Phys. 02 (2011) 078.
[45] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, The Library of Babel: On the origin of gravitational thermodynamics, J. High Energy Phys. 12 (2005) 006.
[46] V. Balasubramanian, B. Czech, K. Larjo, and J. Simon, Integrability versus information loss: A simple example, J. High Energy Phys. 11 (2006) 001.
[47] R. d. M. Koch and S. Ramgoolam, From matrix models and quantum fields to Hurwitz space and the absolute Galois Group, arXiv:1002.1634.
[48] G. W. Moore and G. Segal, D-branes and K-theory in 2D topological field theory, arXiv:hep-th/0609042.
[49] D. Berenstein and R. G. Leigh, Resolution of stringy singularities by noncommutative algebras, J. High Energy Phys. 06 (2001) 030.
[50] Y. Kimura, Noncommutative frobenius algebras and open strings, Talk at Queen Mary University of London, 2014 (unpublished).
[51] J. Ben Geloun and S. Ramgoolam, Counting tensor model observables and branched covers of the 2-sphere, arXiv:1307.6490.
[52] A. Mironov, A. Morozov, and S. Natanzon, A Hurwitz theory avatar of open-closed strings, Eur. Phys. J. C **73**, 2324 (2013).
[53] I. MacDonald, *The Theory of Groups* (Clarendon Press, Oxford, England, 1968).