Instanton Soliton Loops in 5D super-Yang-Mills

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Instanton-soliton loops in 5D super-Yang-Mills

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ABSTRACT. Soliton contributions to perturbative processes in QFT are controlled by a form factor, which depends on the soliton size. We provide a demonstration of this fact in a class of scalar theories with generic moduli spaces. We then argue that for instanton-solitons in 5D super-Yang-Mills theory the analogous form factor does not lead to faster-than-any-power suppression in the perturbative coupling. We also discuss the implications of such contributions for the UV behavior of maximally supersymmetric Yang-Mills in 5D and its relation to the (2,0) CFT in 6D. This is a contribution to the proceedings of the "String Math 2013" conference and is a condensed version of results appearing in [**1, 2**].

1. 5D MSYM and the (2,0) SCFT in 6D

Over the last few years, there has been renewed interest in the relation between the maximally supersymmetric Yang-Mills theory (MSYM) in 5D and the nonabelian (2,0) tensor CFT in 6D. In [**3, 4**] it was conjectured that 5D MSYM with coupling g_{YM} is exactly the (2,0) theory on a circle of radius

$$(1.1) \qquad\qquad R = \frac{g_{\mathrm{YM}}^2}{4\pi} \ .$$

This relies on the observation that Kaluza-Klein momentum along the circle can be identified with instanton charge in the 5D theory; the latter is a topological charge carried by soliton configurations [**5, 6**]. Equivalently, in the strong coupling limit 5D MSYM should define the fully decompactified (2,0) theory, which in turn is expected to describe the low-energy dynamics of multiple M5-branes [**7, 8**].

At the same time, since gauge theories above four dimensions are power-counting nonrenormalizable, one would expect that 5D MSYM should be treated as an effective theory in the Wilsonian sense, that is only defined up to some cutoff scale. It is then reasonable to wonder what it means to consider such a theory at strong coupling (and hence high energies) and how that would give rise to the well-defined (2,0) CFT.

This tension naturally leads to revisiting the UV behavior of 5D MSYM in the context of perturbative renormalization. Despite $\mathcal{N} = 2$ supersymmetry being responsible for the absence of UV divergences at low orders [**9**], the first logarithmic divergence was explicitly seen at 6 loops in [**10**]. However, instead of immediately taking the cutoff to infinity and declaring the theory UV-divergent, one has to also investigate possible contributions associated with virtual soliton states. The

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simplest such contribution can be related, via the optical theorem, to the soliton-antisoliton pair production amplitude. The latter is conventionally believed to be "exponentially suppressed" and as a result soliton loops are usually ignored.

Here we will argue that instanton-soliton pair production in 5D MSYM does not fall faster than any power in the effective dimensionless coupling controlling a given perturbative process [1]. Motivated by [3, 4], one could then envision a mechanism through which soliton loops would lead to exact cancelations against the perturbative UV divergences and render the theory well defined. We stress that we are not treating 5D MSYM as an effective theory in the Wilsonian sense, and instead we are supposing that the Lagrangian provides a microscopic definition of the theory. This point of view can only make sense if the theory is finite.

2. Soliton pair production and form factors

According to standard QFT lore, soliton production is "exponentially suppressed" at small coupling and hence unimportant for perturbative physics. However, upon careful consideration one can formulate a more refined version of that statement: the exponential dependence should appear in the dimensionless ratio of the soliton size (R_S) over its Compton wavelength (R_C), e^{-R_S/R_C} [11, 1]. On the one hand, when the size is fixed to a value much larger than the Compton wavelength one recovers the expected suppression, as e.g. for 't Hooft–Polyakov monopoles in Yang–Mills–Higgs theory. On the other, for situations where the size is a modulus that ranges over values on the order of the Compton wavelength, one might expect that small solitons would not be suppressed compared to perturbative processes. As we will see, this is precisely the setup for instanton-solitons in five dimensions.

To that end, it is instructive to revisit the derivation of the soliton form factor for the simple case of scalar theories [1]. Consider the following class of Lagrangians

$$(2.1) \qquad L = \frac{1}{g^2} \int d\mathbf{x} \left\{ \frac{1}{2} \dot{\Phi} \cdot \dot{\Phi} - \frac{1}{2} \partial_\mathbf{x} \Phi \cdot \partial_\mathbf{x} \Phi - V(\Phi) \right\} .$$

We denote by $\mathbf{x}$ a $(D-1)$-dimensional position vector, while $d\mathbf{x}$ is shorthand for $d^{D-1}x$. The field Φ is $\mathbb{R}^n$-valued and $\cdot$ denotes the Euclidean dot product. Here we assume that the potential has a dimensionless parameter g controlling the perturbative expansion. Then, in terms of canonically normalized fields $\tilde{\Phi} = g^{-1}\Phi$, we have $\tilde{V}(\tilde{\Phi}; g) = g^{-2}\tilde{V}(g\tilde{\Phi}; 1)$, while we have also set $V(\Phi) = \tilde{V}(g\tilde{\Phi}; 1)$ [12].

We are interested in soliton solutions, classically described by localized, finite-energy field configurations and denoted by ϕ. Such classical solutions for a fixed topological sector[1] usually come in a smooth family parameterized by a collection of moduli, U^M, where $M = i, m$. A subset of these moduli always consist of the center-of-mass positions, $U^i = \mathbf{X}$; we will call their conjugate momenta $\mathbf{P}$. U^m then parameterize all remaining "centered" moduli. We denote the moduli space of soliton solutions for a given fixed topological charge as $\mathcal{M}$; it represents a local minimum of the energy functional.

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[1]The sectors are labeled by homotopy equivalence classes $\pi_{D-2}(M_{\text{vac}})$, where $M_{\text{vac}} := \{\Phi \mid V(\Phi) = 0\}$. See e.g. [13]. Note that although Derrick's theorem [14, 13] precludes the existence of soliton solutions for $D > 2$ in the class of linear sigma models considered here, it is no more difficult to leave D arbitrary.

In the presence of a soliton a new sector of the quantum theory opens up, which is orthogonal to the vacuum sector since solitons carry a conserved topological charge [**12**]. Single particle states in the soliton sector form a subspace of the total single-particle Hilbert space and one can study processes involving both perturbative particles and solitons as asymptotic states. Such soliton states can be chosen to be momentum eigenstates, $|\mathbf{P}\rangle$. Note that, apart from the soliton's actual energy and momenta, these states can carry extra labels corresponding to eigenvalues of additional operators that commute with the Hamiltonian. These depend on the particulars of the theory and will be left implicit for the rest of our discussion.

Let us now study the soliton pair-production amplitude, involving a perturbative incoming excitation of (off-shell) momentum k and a soliton-antisoliton outgoing pair of (on-shell) momenta P_f and $-P_i$ respectively. Although it is unclear how one should proceed directly—since there exists no known associated analytic classical solution and hence no semiclassical expansion scheme—crossing symmetry relates soliton pair-production to a process where the soliton (baryon) absorbs the perturbative excitation (meson)

$$(2.2) \qquad \mathcal{A}(k \to P_f, -\bar{P}_i) = \mathcal{A}(P_i, k \to P_f) \,.$$

Note that this is an equality between amplitudes in distinct topological sectors. The RHS is related to the form factor

$$i(2\pi)^D \delta^{(D)}(k + P_i - P_f)\mathcal{A}(P_i, k \to P_f) =$$
$$(2.3) \qquad\qquad = \int \mathrm{d}^D x \, e^{-ik \cdot x} \langle \mathbf{P}_f | \mathrm{T}\{\Phi(x) \, e^{-i \int \mathrm{d}t' H_I(t')}\} |\mathbf{P}_i\rangle \,,$$

where H_I denotes the interaction Hamiltonian.

The Hamiltonian obtained from (2.1) is simply

$$(2.4) \qquad H = \int \mathrm{d}\mathbf{x} \left\{ \frac{g^2}{2}\Pi \cdot \Pi + \frac{1}{g^2}\left(\frac{1}{2}\partial_\mathbf{x}\Phi \cdot \partial_\mathbf{x}\Phi + V(\Phi)\right)\right\} \,.$$

Through a canonical transformation [**2**] the original conjugate pair of variables (Φ, Π) can be related to the new pairs (U^M, p_N), (χ, π) capturing the collective coordinate and massive oscillator dynamics respectively. We have

$$\Phi(x) = \phi(\mathbf{x}; U) + g \, \chi(x; U)$$
$$(2.5) \qquad \Pi(x) = \frac{1}{2}\left(a^M \partial_M \phi(\mathbf{x}; U) + \partial_M \phi(\mathbf{x}; U)\bar{a}^M\right) + \frac{1}{g} \, \pi(x; U) \,,$$

subject to the constraints

$$(2.6) \qquad F_{1,M} := \int \mathrm{d}\mathbf{x} \, \chi \cdot \partial_M \phi = 0 \,, \qquad F_{2,M} := \int \mathrm{d}\mathbf{x} \, \pi \cdot \partial_M \phi = 0 \,,$$

which ensure that the fluctuations χ, π are orthogonal to the zero-modes $\partial_M \phi$. Here we have inserted factors of g so that the fluctuation fields are canonically normalized. The functionals $a^M, \bar{a}^M$ are given by

$$(2.7) \; a^N = \frac{1}{g^2}\left(p_M - \int \pi \cdot \partial_M \chi\right) C^{MN} \,, \quad \bar{a}^M = \frac{1}{g^2}C^{MN}\left(p_M - \int \partial_M \chi \cdot \pi\right) \,,$$

where $C = (G - g\Xi)^{-1}$ with

$$(2.8) \qquad \Xi_{MN} = \frac{1}{g^2}\int \chi \cdot \partial_M \partial_N \phi \,, \qquad G_{MN} = \frac{1}{g^2}\int \partial_M \phi \cdot \partial_N \phi \,.$$

Here G_{MN} is the metric on moduli space, induced from the flat metric on field configuration space.

In terms of these new variables the Hamiltonian can be written as

$$H = \frac{g^4}{2} a^M G_{MN} a^N + \mathrm{v}(U^m) + \int \left[\frac{1}{2} \pi \cdot \pi + g \, s \cdot \chi + \frac{1}{2} \chi \cdot \Delta \chi + V_I(\chi) \right] +$$

(2.9)
$$+ \mathcal{O}(g^2) \ ,$$

with $V_I(\chi)$ denoting cubic and higher-order interaction terms in the fluctuations χ coming from the original potential. In writing the above, we have ignored operator-ordering ambiguities, such that $a^M = \bar{a}^M + \mathcal{O}(g^2)$. These corrections correspond to two-loop effects that will not be important for the rest of our calculation.

We have also defined

$$s(\mathbf{x}; U^m) := \frac{1}{g^2} \left(- \partial_{\mathbf{x}}^2 \phi + \frac{\partial V}{\partial \Phi} \bigg|_{\Phi = \phi} \right) \ , \qquad \Delta := -\delta_{ab} \partial_{\mathbf{x}}^2 + \frac{\delta^2 V}{\delta \Phi \delta \Phi} \bigg|_{\Phi = \phi} \ ,$$

(2.10) $$\mathrm{v}(U^m) := \frac{1}{g^2} \int \mathrm{d}\mathbf{x} \left(\frac{1}{2} \partial_{\mathbf{x}} \phi \cdot \partial_{\mathbf{x}} \phi + V(\phi) \right) = M_{\mathrm{cl}} + \delta\mathrm{v}(U^m) \ .$$

If ϕ is an exact solution to the time-independent equations of motion then $s(\mathbf{x}; U^m) = 0$ and $\delta\mathrm{v}(U^m) = 0$. However, in theories with centered moduli it is sometimes convenient to expand around a configuration that is only an approximate solution.

The regime needed to extract information about the pair-creation process through crossing symmetry requires large velocity exchange and hence momentum transfer of the order of the soliton mass, $\mathbf{P} \sim \frac{m}{g^2}$ with m the meson mass. Therefore, the conventional small-velocity (Manton) approximation usually implemented in the literature is not sufficient for our purposes.

3. Relativistic scalar form factor

In the case of the two-dimensional kink in Φ^4 theory, seminal work by Gervais, Jevicki and Sakita [15] showed how velocity corrections can be systematically accounted for, to recover the covariant expression for the soliton energy, $M_{\mathrm{cl}} \to \sqrt{\mathbf{P}^2 + M_{\mathrm{cl}}^2}$. This answer is to be expected, since the starting point is a Lorentz-invariant theory. We will now show how the same techniques can be applied in the more general class of Lorentz-invariant theories considered here. We will be interested in evaluating the form factor (2.3) rather than the soliton energy. Fortunately, the techniques of [15] have been adapted to this context by [16], the methodology of which we will be following closely.

The two qualitative differences between the general case and the kink in Φ^4 theory are: a) lack of an explicit classical soliton solution to work with and b) the possible presence of centered moduli. Both can be taken into account and their discussion can be appropriately modified, provided we continue to make the simplifying assumptions of the Manton (small-velocity and small moduli-space-potential) approximation for the dynamics of the centered moduli. Specifically, we will impose $p_m/m \sim \mathcal{O}(1/g)$ and $s(\mathbf{x}; U^m) \sim \mathcal{O}(1)$, but we will assume $p_i/m = \mathbf{P}/m \sim \mathcal{O}(1/g^2)$.

The transition amplitude from an initial state i described by the functional $\Psi_i(U^M(-T); \chi)$ to a final state f described by the functional $\Psi_f(U^M(T); \chi)$ is

$$S_{fi} = \int [DU Dp D\chi D\pi] \delta(F_1) \delta(F_2) e^{i \int_{-T}^{T} \mathrm{d}t L} \Psi_f^* \Psi_i \ , \quad \text{with}$$

(3.1) $$L = p_M \dot{U}^M + \int \mathrm{d}\mathbf{x} \, \pi \dot{\chi} - H \ .$$

An incoming soliton state of momentum $\mathbf{P}_i$ is defined by taking $\Psi_i = e^{i\mathbf{P}_i \cdot \mathbf{X}_i} \tilde{\Psi}_i(U^m)$, where $\mathbf{X}_i = \mathbf{X}(-T)$, and similarly for outgoing soliton states. The $\tilde{\Psi}_{i,f}$ are wavefunctions on the centered moduli space. In general we will denote quantities associated with the centered part of the moduli space with a tilde. We can consider time-ordered correlators of the meson field between soliton states by inserting appropriate factors of $\Phi(x_1) \cdots \Phi(x_n)$ under the path integral, and using the relation $\Phi(x) = \phi(\mathbf{x} - \mathbf{X}(t); U^m) + g\chi(t, \mathbf{x} - \mathbf{X}(t); U^m)$.

We are interested in the particular case of the 1-point function and hence in

$$\langle \mathbf{P}_f, T | \Phi(x) | \mathbf{P}_i, -T \rangle = \int [D\mathbf{X} D\mathbf{P}] e^{i(\mathbf{X}_i \cdot \mathbf{P}_i - \mathbf{X}_f \cdot \mathbf{P}_f)} \int [DU^m Dp_n] \tilde{\Psi}_f^* \tilde{\Psi}_i \times$$

(3.2) $$\times \int [D\chi D\pi] \delta(F_1) \delta(F_2) e^{i \int_{-T}^{T} \mathrm{d}t L} \Phi[U, p; \chi](x) \ .$$

Let us focus first on the internal path integral over χ and π for which we will proceed to compute the leading contribution at small g. This was done in [15] for the case of the 0-point function by evaluating the action on the saddle-point solution for χ, π corresponding to the moving soliton. One can argue [16, 1] that the same saddle point solution gives the leading contribution to the 1-point function, even though one should now be solving the equations of motion with source. This is a special feature of working with the 1-point function and would not be true for higher-point functions. We denote this saddle point $(\chi_{\mathrm{cl}}, \pi_{\mathrm{cl}})$ and expand the fields as $\chi = \chi_{\mathrm{cl}} + \delta\chi$, $\pi = \pi_{\mathrm{cl}} + \delta\pi$.

Starting with the Hamiltonian (2.9) one can find a saddle-point solution to the χ, π equations of motion perturbatively in g by making use of the above-mentioned scaling assumptions for the coordinate momenta [1]. One finds

(3.3) $$\chi_{\mathrm{cl}} = g^{-1}\phi\left(\Lambda(\mathbf{x} - \mathbf{X}); U^m\right) - g^{-1}\phi\left((\mathbf{x} - \mathbf{X}); U^m\right) + \mathcal{O}(1) \ ,$$

where

(3.4) $$\Lambda^i{}_j = \delta^i{}_j + \left(\sqrt{1 + \frac{\mathbf{P}^2}{M_{\mathrm{cl}}^2}} - 1\right) \frac{p^i p_j}{\mathbf{P}^2}$$

is a Lorentz contraction factor. The insertion can then be expressed as

(3.5) $$\Phi = \phi\left(\Lambda(\mathbf{x} - \mathbf{X}); U^m\right) + \mathcal{O}(g) \equiv \Phi_{\mathrm{cl}} + \mathcal{O}(g) \ .$$

With this solution in hand, we want to evaluate (3.2) in the presence of centered moduli. For this, we also need the Lagrangian evaluated on the solution:

(3.6) $$L = \mathbf{P} \cdot \dot{\mathbf{X}} - \sqrt{\mathbf{P}^2 + M_{\mathrm{cl}}^2} + L^{(0)}[U^m, p_m; \delta\chi, \delta\pi; \mathbf{P}] + L_{\mathrm{int}} \ ,$$

where L_{int} starts at $\mathcal{O}(g)$ and

(3.7) $$L^{(0)} = p_m \dot{U}^m - \tilde{H}_{\mathrm{eff}}[U^m, p_m; \mathbf{P}]$$

is an $\mathcal{O}(1)$ contribution describing the dynamics of the centered moduli, whose precise form we will not require. $\tilde{H}_{\text{eff}}$ includes the 1-loop potential from integrating out the fluctuation fields $(\delta\chi, \delta\pi)$. The leading contribution to (3.2) then takes the form

$$\langle \mathbf{P}_f | \Phi(x) | \mathbf{P}_i \rangle = \int [D\mathbf{X} D\mathbf{P}] e^{i(\mathbf{X}_i \cdot \mathbf{P}_i - \mathbf{X}_f \cdot \mathbf{P}_f)} e^{i \int dt(\mathbf{P}\cdot\dot{\mathbf{X}} - \sqrt{\mathbf{P}^2 + M_{\text{cl}}^2})} \times$$

$$(3.8) \qquad \times \int_{\tilde{\mathcal{M}}} dU \sqrt{\tilde{G}} \, \tilde{\Psi}_f^*(U^m; \mathbf{P}) \Phi_{\text{cl}}[\mathbf{X}, \mathbf{P}; U^m] \tilde{\Psi}_i(U^m; \mathbf{P}) (1 + \mathcal{O}(g)) \ .$$

In the above we have expressed the centered moduli space path integral as a position-basis matrix element in the quantum mechanics on the centered moduli space with Hamiltonian $\tilde{H}_{\text{eff}}$. Note that the $(\mathbf{X}, \mathbf{P})$ path integral is a functional integral representation of the quantum mechanics for a relativistic particle. From the point of view of the translational moduli space dynamics, U^m are merely parameters, so we can carry out the functional integration over $\mathbf{X}$ and $\mathbf{P}$ first and then integrate over the centered moduli space.

This was carried out in [1] employing the techniques of [16]. Using that result, which is specific to two dimensions, we find that the amplitude (2.3) takes the following form to leading order:

$$(3.9) \qquad \mathcal{A}(P_i, k \to P_f) \sim \int_{\tilde{\mathcal{M}}} dU \sqrt{\tilde{G}} \, \tilde{\Psi}_f^* \mathcal{F}[\phi] \left(\frac{2R_S(U^m)}{R_C} \zeta(P_f, P_i) \right) \tilde{\Psi}_i \ ,$$

where $\mathcal{F}[\phi](u) = \int dv \, e^{-iuv} \phi(v)$ is the Fourier transform of the classical soliton profile, $\tilde{\Psi}_{i,f} = \tilde{\Psi}_{i,f}(U^m; P_{i,f})$ and

$$(3.10) \qquad \zeta(P_f, P_i) := \frac{2\epsilon_{\mu\nu} P_f^\mu P_i^\nu}{(P_f + P_i)^2} \ .$$

The quantity $R_S(U^m)$, inserted on dimensional grounds, characterizes the size of the soliton. For example, in Φ^4 theory $R_S = 1/m$, with m the meson mass. As we previously indicated, in the general class of theories considered here it can in principle be a function of the centered moduli. $R_C = 1/M_{\text{cl}}$ is the soliton Compton wavelength.

Now, given that the classical soliton profile ϕ is a smooth (C^∞) function of $\mathbf{x} - \mathbf{X}$, we can draw a rather strong conclusion about the asymptotic behavior of the Fourier transform in (3.9). For any values of momenta such that ζ is not $O(g^2)$ or smaller, it is the $2R_S/R_C$ factor that controls the parametric size of the argument of the Fourier transform. Given this, and as long as the soliton size is bounded away from zero, $R_S^{\text{min}} > 0$, we will have that $(2R_S/R_C)|\zeta| \to \infty$ in the semiclassical limit. The Riemann–Lebesgue lemma then implies that[2]

$$(3.11) \qquad \lim_{g \to 0} \mathcal{F}[\phi] \left(\frac{2R_S(U^m)}{R_C} \zeta \right) \sim e^{-\frac{2R_S(U^m)}{R_C}|\zeta|} \ .$$

Let us emphasize that the exponential on the RHS is a typical function exhibiting a faster-than-any-power falloff that we use for concreteness, but the exact expression will depend on the details of the theory under consideration. In any case, the important property for our purposes is the faster-than-any-power falloff.

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[2]As stated by the Riemann-Lebesgue lemma, the Fourier transform $\mathcal{F}[f](p)$ of an L^1-function $f(x)$ goes to zero as $|p| \to \infty$. Accordingly, if $f(x)$ is C^∞, $\mathcal{F}[f^{(n)}](p) = (ip)^n \mathcal{F}[f](p)$ should also go to zero as $p \to \infty$; i.e. $\mathcal{F}[f](p)$ goes to zero faster than any power.

This leads to the expression

$$(3.12) \qquad \mathcal{A}(P_i, k \to P_f) \sim \int_{\tilde{\mathcal{M}}} \mathrm{d}U \sqrt{\tilde{G}} \, \tilde{\Psi}_f^* e^{-\frac{2R_S(U^m)}{R_C}|\zeta|} \tilde{\Psi}_i$$

for the leading contribution to the form factor as $g \to 0$. Note that the centered moduli space represents the internal degrees of freedom of the single-particle state. A field theory interpretation requires a single-particle state to have a finite number of internal degrees of freedom. The eigenvalues labeling these internal degrees of freedom should be discrete eigenvalues of the centered-moduli-space Hamiltonian $\tilde{H}_{\text{eff}}$. Hence the wavefunctions on the centered moduli space $\tilde{\Psi}$ should be L^2; this is automatically the case if $\tilde{\mathcal{M}}$ is compact. Then we have the inequalities

$$\int_{\tilde{\mathcal{M}}} \mathrm{d}U \sqrt{\tilde{G}} \, \tilde{\Psi}_f^* e^{-\frac{2R_S(U^m)}{R_C}|\zeta|} \tilde{\Psi}_i \leq \int_{\tilde{\mathcal{M}}} \mathrm{d}U \sqrt{\tilde{G}} \, |\tilde{\Psi}_f^* \tilde{\Psi}_i| e^{-\frac{2R_S(U^m)}{R_C}|\zeta|}$$

$$\leq e^{-\frac{2R_S^{\min}}{R_C}|\zeta|} ||\tilde{\Psi}_f^* \tilde{\Psi}_i||_{L^1}$$

$$\leq e^{-\frac{2R_S^{\min}}{R_C}|\zeta|} ||\tilde{\Psi}_f||_{L^2} ||\tilde{\Psi}_i||_{L^2}$$

$$(3.13) \qquad = e^{-\frac{2R_S^{\min}}{R_C}|\zeta|} \, ,$$

where in the second-last step we used Hölder's inequality. Hence we have reached the result

$$(3.14) \qquad \mathcal{A}(P_i, k \to P_f) \lesssim e^{-\frac{2R_S^{\min}}{R_C}|\zeta|} \, .$$

In order to use this result to obtain the pair-production amplitude using crossing symmetry, we first re-write ζ in terms of $k^2 = (P_f - P_i)^2$. Making use of the fact that $P_{i,f}$ are on shell, one can show

$$(3.15) \qquad \zeta = \sqrt{\frac{k^2}{k^2 - 4M_{\text{cl}}^2}} \, .$$

The above result is consistent with expectations. First, on physical grounds the form factor should be a function of the momentum transfer only; $\zeta = \zeta(k^2)$. Second, as $k^2 \to \infty$ we expect $\zeta(k^2) \to O(1)$; otherwise, one would obtain an amplitude with exponential behavior for large k^2, in contradiction with the large-momentum behavior of asymptotically free theories. Finally, it agrees exactly with the prescription proposed in [17] in the context of the Skyrme model, where it was also observed that exponential falloff is inconsistent with asymptotic freedom.

This quantity can be analytically continued from spacelike to timelike k^2 and goes to the same value as $k^2 \to \infty$ in any direction on the complex plane. Thus, via crossing symmetry we arrive at

$$(3.16) \qquad \mathcal{A}(k \to -P_i, P_f) \lesssim e^{-\frac{2R_S^{\min}}{R_C}\zeta} \, .$$

This is in agreement with the original expectation from dimensional analysis. Note that if $R_S^{\min}$ is of order R_C this does not lead to suppression.

4. Instanton-soliton loops in 5D MSYM

Let us now apply the above result to the case of interest, i.e. instanton-solitons in 5D MSYM. Instanton-solitons are finite-energy $\frac{1}{2}$-BPS configurations, obtained by solving the selfduality equation for the gauge field strength in the four spatial

directions, $F = \star_4 F$, and have mass $M_{\text{cl}} \propto 1/g_{\text{YM}}^2$. As such, they are described by standard 4D instanton solutions, which for topological charge $c_2(F) = 1$ and SU(2) gauge group, correspond to classical gauge fields given by

$$(4.1) \qquad A_i = U(\vec{\theta})^{-1} \Big(\frac{\eta_{ij}^a (\mathbf{x} - \mathbf{X})^j}{((\mathbf{x} - \mathbf{X})^2 + \rho^2)} T^a \Big) U(\vec{\theta}) \,, \qquad A_0 = 0 \,,$$

with $a = 1, 2, 3$, $i = 1, ..., 4$ and η_{ij}^a the 't Hooft symbols. This solution has eight moduli: four center-of-mass collective coordinates $\mathbf{X}$, a size modulus ρ and three Euler angles $\vec{\theta}$ parameterizing global gauge transformations. The associated moduli space is a hyperkähler manifold

$$(4.2) \qquad \mathcal{M} = \mathbb{R}^4 \times \mathbb{R}_+ \times S^3/\mathbb{Z}_2 \,,$$

with metric

$$(4.3) \qquad \mathrm{d}s^2 = \frac{4\pi^2}{g_{\text{YM}}^2} \Big[\delta_{ij} \, \mathrm{d}\mathbf{X}^i \, \mathrm{d}\mathbf{X}^j + 2(\, \mathrm{d}\rho^2 + \rho^2 \tilde{G}_{\alpha\beta} \, \mathrm{d}\theta^\alpha \, \mathrm{d}\theta^\beta) \Big] \,,$$

where $\tilde{G}_{\alpha\beta}$ is the metric on $\mathrm{SO}(3) \cong S^3/\mathbb{Z}_2$, the group of effective global gauge transformations.

The existence of the noncompact size modulus ρ means that we can have arbitrarily small or large soliton sizes. However, it is also responsible for the absence of L^2-normalizable wavefunctions on the centered moduli space $\tilde{\mathcal{M}}$. This renders the interpretation of instanton-solitons as asymptotic states confusing, since they would correspond to particles with a continuous infinity of internal degrees of freedom.

Moreover, the semiclassical expansion parameter in this theory is in fact $g^2 = g_{\text{YM}}^2/\rho$, which coincides with R_C/R_S. In particular, note that $g(\rho)$ is moduli dependent. In the context of finding the saddle-point solution (3.3) we can imagine a fixed ρ, such that $g(\rho)$ is small. However, when evaluating amplitudes, where one must integrate over all sizes, the semiclassical approximation breaks down. Consequently, the small-sized instanton-solitons invalidate our argument for exponential suppression.

One can attempt to circumvent this conclusion by turning on a scalar VEV, $\langle \Phi \rangle \neq 0$, and going out onto the Coulomb branch.[3] It is known that in this case finding instanton-soliton solutions requires turning on an electric field, which stabilizes the *classical* size [**18, 19**]. From the point of view of the quantum theory, turning on an electric field generates a potential on the centered moduli space,

$$(4.4) \qquad \delta\mathrm{v}(U^m) = \frac{2\pi^2}{g_{\text{YM}}^2} \langle \Phi \rangle^2 \rho^2 \,,$$

and lifts the flat direction associated with the instanton-soliton size. Although ρ is no longer a true modulus, the VEV provides an additional dimensionless parameter, $\epsilon := g_{\text{YM}}^2 \langle \Phi \rangle$, that can be adjusted so that we remain in the small-potential approximation, where it is still appropriate to represent states as L^2-wavefunctions on $\tilde{\mathcal{M}}$. In order to determine the precise form of the resulting L^2-wavefunctions, one would need to compute the centered-moduli-space Hamiltonian $\tilde{H}_{\text{eff}}$, appearing in (3.6) and (3.7)

Our formalism has been general enough to accommodate such potentials on moduli space. Thus, despite the classical stabilization, one must still integrate

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[3]Here Φ is one of the five adjoint scalars of 5D SYM and should not be confused with the scalar fields for the linear sigma models considered in the previous sections.

over all of moduli space, which includes arbitrarily small sizes. However, as we have already discussed, this means treating the solitons semiclassically when $\rho \sim O(g_{\mathrm{YM}}^2)$, which is not valid because quantum corrections that have been neglected become important. Hence, turning on the potential (4.4) does not enable one to salvage an argument for faster-than-any-power suppression.

While none of these arguments definitively show that instanton-soliton contributions are *not* suppressed compared to perturbative processes, they at least allow for that possibility. Non-suppression of the pair-production amplitude would provide a mechanism via which the contribution of virtual soliton-antisoliton pairs to perturbative processes can compete with the contribution from loops of perturbative particles. Such a mechanism is precisely what is called for in order to avoid contradicting the assumption of finiteness: One would require that the soliton-antisoliton contribution be divergent, with exactly the right coefficient to cancel the divergence found in [**10**]. This is an intriguing possibility, the investigation of which would, however, require an alternative approach to the one used here.

References

[1] C. Papageorgakis and A. B. Royston, "Revisiting Soliton Contributions to Perturbative Amplitudes," `arXiv:1404.0016 [hep-th]`.

[2] C. Papageorgakis and A. B. Royston, "Scalar Soliton Quantization with Generic Moduli," *JHEP* **1406** (2014) 003, `arXiv:1403.5017 [hep-th]`.

[3] M. R. Douglas, "On D=5 super Yang-Mills theory and (2,0) theory," *JHEP* **1102** (2011) 011, `arXiv:1012.2880 [hep-th]`.

[4] N. Lambert, C. Papageorgakis, and M. Schmidt-Sommerfeld, "M5-Branes, D4-Branes and Quantum 5D super-Yang-Mills," *JHEP* **1101** (2011) 083, `arXiv:1012.2882 [hep-th]`.

[5] M. Rozali, "Matrix theory and U duality in seven-dimensions," *Phys.Lett.* **B400** (1997) 260–264, `arXiv:hep-th/9702136 [hep-th]`.

[6] M. Berkooz, M. Rozali, and N. Seiberg, "Matrix description of M theory on T**4 and T**5," *Phys.Lett.* **B408** (1997) 105–110, `arXiv:hep-th/9704089 [hep-th]`.

[7] E. Witten, "Some comments on string dynamics," `arXiv:hep-th/9507121 [hep-th]`.

[8] A. Strominger, "Open p-branes," *Phys.Lett.* **B383** (1996) 44–47, `arXiv:hep-th/9512059 [hep-th]`.

[9] P. Howe and K. Stelle, "Supersymmetry counterterms revisited," *Phys.Lett.* **B554** (2003) 190–196, `arXiv:hep-th/0211279 [hep-th]`.

[10] Z. Bern, J. J. Carrasco, L. J. Dixon, M. R. Douglas, M. von Hippel, *et al.*, "D = 5 maximally supersymmetric Yang-Mills theory diverges at six loops," *Phys.Rev.* **D87** (2013) 025018, `arXiv:1210.7709 [hep-th]`.

[11] T. Banks, "Arguments Against a Finite N=8 Supergravity," `arXiv:1205.5768 [hep-th]`.

[12] J. Goldstone and R. Jackiw, "Quantization of Nonlinear Waves," *Phys.Rev.* **D11** (1975) 1486–1498.

[13] N. Manton and P. Sutcliffe, *Topological solitons*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2004. `http://dx.doi.org/10.1017/CBO9780511617034`.

[14] G. Derrick, "Comments on nonlinear wave equations as models for elementary particles," *J.Math.Phys.* **5** (1964) 1252–1254.

[15] J.-L. Gervais, A. Jevicki, and B. Sakita, "Perturbation Expansion Around Extended Particle States in Quantum Field Theory. 1.," *Phys.Rev.* **D12** (1975) 1038.

[16] N. Dorey, M. P. Mattis, and J. Hughes, "Soliton quantization and internal symmetry," *Phys.Rev.* **D49** (1994) 3598–3611, `arXiv:hep-th/9309018 [hep-th]`.

[17] X.-D. Ji, "A Relativistic skyrmion and its form-factors," *Phys.Lett.* **B254** (1991) 456–461.

[18] N. D. Lambert and D. Tong, "Dyonic instantons in five-dimensional gauge theories," *Phys. Lett.* **B462** (1999) 89–94, `arXiv:hep-th/9907014`.

[19] K. Peeters and M. Zamaklar, "Motion on moduli spaces with potentials," *JHEP* **0112** (2001) 032, `arXiv:hep-th/0107164 [hep-th]`.

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