Quasi-Fuchsian correspondences.

Samarasinghe, Manjula L

QUASI-FUCHSIAN CORRESPONDENCES

Manjula L. Samarasinghe

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I declare that the content of this thesis is my own work carried out while a student at University of London unless otherwise stated.

M. L. Samarasinghe.

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ABSTRACT

We consider the action of holomorphic correspondences or equally algebraic functions acting on the Riemann sphere $\overline{\mathbb{C}}$ and their limit sets: a holomorphic correspondence is a polynomial relation, $P(z,w)=0$ in z and w. A holomorphic correspondence $P(z,w)=0$ is said to be an $(n:m)$ correspondence if the degrees of z and w in P are n and m respectively.

We identify a class of $(2:2)$ holomorphic correspondences whose limit set is a topological circle where on a component of the complement of the limit set, the action of the correspondence is conjugate to the action of the Modular group $PSL(2,\mathbb{Z})$ on the upper half plane.

Further, we generalise these results to a class of $(3:3)$ holomorphic correspondences with analogous properties.

CONTENTS

1. INTRODUCTION AND PRELIMINARIES

1.1 An overview of holomorphic correspondences, and conventions

A holomorphic correspondence (a correspondence for brevity), or an algebraic function, is a polynomial relation $P(z, w) = 0$ where
$P(z, w) = z^n A_n(w) + z^{n-1} A_{n-1}(w) \ldots + A_0(w)$, $n \in \mathbb{N}$ and for each integer i with $0 \le i \le n$, A_i is a polynomial in w with coefficients in $\mathbb{C}$.

By expressing P in homogeneous coordinates, that is by letting $z = \frac{a}{b}$ and $w = \frac{c}{d}$ one can make sense of the expressions such as $P(\infty, w) = 0$ or $P(z, \infty) = 0$ as follows:
$P(\frac{a}{b}, \frac{c}{d})$ is of the form $\frac{T(a,b,c,d)}{b^n d^m}$ where T is a polynomial in a, b, c and d, and n and m are the degrees of z and w (in $P(z, w)$) respectively. We now define $P(\infty, w) = 0 \iff T(a, 0, c, d) = 0$, and $P(z, \infty) = 0 \iff T(a, b, c, 0) = 0$. Likewise, we set that $P(\infty, \infty) = 0$ if and only if $T(a, 0, c, 0) = 0$. In this regard, we implicitly assume that a correspondence is expressed in homogeneous coordinates whenever we deal with the point at infinity.

Moreover, in this thesis, we assume that all our correspondences satisfy the following notions and conventions:

(1) all correspondences are assumed to have no repeated factors.

(2) if $z_0, w_0 \in \mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}}$, and $P(z, w) = 0$ is a correspondence then the cardinalities of $\{w \in \overline{\mathbb{C}} : P(z_0, w) = 0\}$ and $\{z \in \overline{\mathbb{C}} : P(z, w_0) = 0\}$ are finite. This ensures in particular that $P(z, w)$ has no factors such as $sz + t$ or $qw + r$ where $s, q \in (\mathbb{C} - \{0\})$, and $t, r \in \mathbb{C}$.

(3) we regard two correspondences $P(z, w) = 0$ and $Q(z, w) = 0$ being the same if they have the same graph; that is,

$\{(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} : P(z, w) = 0\} = \{(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} : Q(z, w) = 0\}$.

(4) if $P(z, w) = 0$ is a correspondence, and n and m are the degrees of z and w in $P(z, w)$ respectively then we say that $P(z, w) = 0$ is an $(n : m)$ correspondence. Hence, for instance, if $w_0 \in \overline{\mathbb{C}}$ then $z \mapsto P(z, w_0)$ has n zeros, counting multiplicities.

(5) let $P(z, w) = 0$ be an $(n : m)$ correspondence, and g and h be polynomials with no common zeros. By replacing z in $P(z, w)$ with $\frac{g(z)}{h(z)}$ the resulting expression $P(\frac{g(z)}{h(z)}, w)$ can be written as $\frac{S(z,w)}{h^n(z)}$ for some polynomial S in z and w. Now, we define $P(\frac{g(z)}{h(z)}, w) = 0$ if and only if $S(z, w) = 0$: in this sense, for brevity we shall continue to refer to $P(\frac{g(z)}{h(z)}, w) = 0$ as a correspondence even though strictly speaking it is $S(z, w) = 0$ that we study. Note that, $S(z, w) = 0$ is a $(kn : m)$ correspondence where k is the degree of $\frac{g}{h}$. The

same discussion applies if we replace w by $\frac{g(w)}{h(w)}$: in this case we obtain an $(n : km)$ correspondence.

In this thesis we primarily consider classes of $(2 : 2)$ correspondences that are generated by degree three rational maps, and show that limit sets of such correspondences are topological circles. We further conjecture that these limit sets are indeed quasicircles, and outline a method of proving this. The complement (in the Riemann sphere, $\overline{\mathbb{C}}$) of the limit set of a correspondence is called the regular set. If the said rational map has a critical point of order two then we show that the action of the correspondence on a component of the regular set is conjugate (in an appropriate sense) to the action of the modular group $PSL(2, \mathbb{Z})$ on the upper half plane of $\mathbb{C}$. Analogous statements are shown for degree four rational maps, in Chapter 4.

The study of correspondences can be thought of as a generalisation of the study of finitely generated Kleinian groups, which are discrete subgroups of $PSL_2(\mathbb{C})$, and the study of iterated rational functions on $\overline{\mathbb{C}}$, as described below.

For, let G be a Kleinian group with generators $\frac{\psi_1}{\varphi_1}, \frac{\psi_2}{\varphi_2}, \ldots$ and $\frac{\psi_n}{\varphi_n}$ where for each integer i with $1 \leq i \leq n, \psi_i(z) = a_i z + b_i$ and $\varphi_i(z) = A_i z + B_i$ are linear functions of $z \in \overline{\mathbb{C}}$ for some constants A_i, B_i, a_i and b_i.

Then the relation

$$P(z,w)=0 \iff \prod_{i=1}^{n}(w\varphi_i(z)-\psi_i(z))=0 \tag{1.1}$$

is an $(n:n)$ correspondence: in this case, we say that P represents G as a correspondence.

We remark that representing G as a correspondence is not unique: for instance, if we multiply (1.1) by $(w-\frac{\psi_1}{\varphi_1}\circ\frac{\psi_2}{\varphi_2}(z))$ then we obtain an $(n+1,n+1)$ correspondence, Q say, which is different from (1.1). However, P and Q have the same grand orbits (cf. Sec. 1.2, Definition 1), and these grand orbits coincide with the grand orbits of G. So, as far as grand orbits are concerned, P and Q represent the same group G. Hence, whenever we represent a Kleinian group as a correspondence our choice of representation is dictated by purpose and context.

Finally, if $\frac{f}{g}$ is a rational map of degree $d>1$, where f and g are polynomials with no common zeros, then the relation

$$P(z,w)=0 \iff wg(z)-f(z)=0 \tag{1.2}$$

gives rise to a $(d:1)$ correspondence. Thus, the iterative behaviour of $\frac{f}{g}$ can be studied as a correspondence.

If R is a rational map of degree at least two then the dynamics (the iterative behaviour) of R partitions $\overline{\mathbb{C}}$ into two completely invariant sets called the Julia set (closed) and its complement, the Fatou set (open) [1].

Here, the Fatou set is defined to be the largest (in the set theoretic sense) open set where the iterates of R form a normal family or equivalently, where the iterates are equicontinuous [1]. It turns out that Julia set is the smallest (in the set theoretic sense) completely invariant closed set of cardinality at least three; this follows from Montel's theorem on normal families and is referred to as Montel's criterion.

The action of a Kleinian group G also exhibits a similar behaviour where the action partitions $\overline{\mathbb{C}}$ into two completely invariant disjoint sets, namely the limit set which is closed and its complement the regular set or the discontinuity set [8].

The limit set of G consists of all points $z \in \overline{\mathbb{C}}$ such that there exists $w \in \overline{\mathbb{C}}$ and a sequence of distinct elements $\{g_n\}$ of G for which $g_n(w) \to z$ as $n \to \infty$.

If G is not an elementary group, that is the limit set of G has at least three points, then as in the case of rational maps the limit set coincides with the smallest G invariant closed set which has cardinality at least three. In fact, in this instance G forms a normal family[1] on its regular set.

Thus, if R is a rational map of degree at least two, or G a non-elementary Kleinian group, then we can define the Julia set of R or the limit set of G by Montel's criterion. Ideally, for a holomorphic correspondence we would

[1] Every Kleinian group is countable ([8], P.8).

like to define a Montel-like criterion which would enable us to recover the corresponding Julia set or the limit set when the correspondence is obtained from a rational map or a finitely generated Kleinian group respectively.

If this were possible then the study of the action of a holomorphic correspondence would fully generalise the theory of iteration of rational maps and the action of finitely generated Kleinian groups.

However, it seems that there is no such unifying criterion for the action of correspondences [3]. In the absence of such a criterion we proceed to generalise the definition of the regular set of a Kleinian group to that of the regular set of a correspondence, as given in [3]. Thus, with this particular choice of definition of the regular set, the study of correspondences will include that of finitely generated Kleinian groups. Even so, correspondences should be treated in their own right as another type of complex dynamical system which sometimes behaves like a Kleinian group, sometimes like a rational map and sometimes something "in between"; we will later see examples of correspondences whose action exhibits both a group-like behaviour and non-group-like behaviour on their regular set.

As expected, the regular set of a correspondence is open and as far as the grand orbits are concerned we may regard the regular set as being analogous to the Fatou set of a rational map. In some sense [3] the regular set of a correspondence is the closest generalisation of the Fatou set of a rational map of degree at least two. We shall later give an example to show why the definition of the regular set is not satisfactory when it comes to describing the iteration of a rational map Q of degree at least two when the relation $Q(z) = w$ is considered as a correspondence.

1.2 The regular set and the limit set of a correspondence

1.2.1 Forward and backward branches of a correspondence

Let $P(z, w) = 0$ be an $(n : m)$ correspondence. Then, we define its forward branch $F_P : \overline{\mathbb{C}} \to 2^{\overline{\mathbb{C}}}$ and backward branch $F_P^{-1} : \overline{\mathbb{C}} \to 2^{\overline{\mathbb{C}}}$, where $2^{\overline{\mathbb{C}}}$ is the set of all subsets of $\overline{\mathbb{C}}$, as follows:

$$F_P(x) = \{w \in \overline{\mathbb{C}} : P(x, w) = 0\} \text{ and } F_P^{-1}(x) = \{z \in \overline{\mathbb{C}} : P(z, x) = 0\}. \tag{1.3}$$

If $U \subseteq \overline{\mathbb{C}}$ then its images under the forward branch and the backward branch are defined as

$$F_P(U) = \bigcup_{x \in U} F_P(x) \text{ and } F_P^{-1}(U) = \bigcup_{x \in U} F_P^{-1}(x). \tag{1.4}$$

For convenience we shall write F_P (or F_P^{-1}) for the graph of F_P (or F_P^{-1}). So, $(z, w) \in F_P$ (or F_P^{-1}) if and only if $w \in F_P(z)$ (or $z \in F_P^{-1}(w)$). Whenever P is clear from the context we shall just write F and F^{-1} for F_P and F_P^{-1}

respectively.

1.2.2 Iterates and grand orbits of a correspondence

Let $\mathcal{X} = \{(s_1, s_2, \ldots, s_n) : n \in \mathbb{N}, s_i \in \{1, -1\} \text{ where } 1 \leq i \leq n\}$ and P be a correspondence with forward and backward branches F and F^{-1} respectively. If $s = (s_1, s_2, \ldots, s_n) \in \mathcal{X}$ then we define

$$F^s = F^{s_1}(F^{s_2}(F^{s_3} \ldots (F^{s_n}))) \tag{1.5}$$

with the convention that F^1 stands for F, and $\mathcal{G}_P = \{F^s : s \in \mathcal{X}\}$.

Now, if $x \in \overline{\mathbb{C}}$ then we let

$$F^s(x) = F^{s_1}(F^{s_2}(F^{s_3} \ldots (F^{s_n}(x)))) \tag{1.6}$$

where the right hand side of the expression is derived via (1.3). Likewise, for $U \subset \overline{\mathbb{C}}$ we let

$$F^s(U) = F^{s_1}(F^{s_2}(F^{s_3} \ldots (F^{s_n}(U)))) \tag{1.7}$$

which can be derived using (1.4).

For utility, we write F^s (or $F^{s_1}(F^{s_2}(F^{s_3} \ldots (F^{s_n}))))$ to represent its graph which is a subset of $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

Definition 1: (The grand orbit of a point)

If P is a correspondence and $z \in \overline{\mathbb{C}}$ then we call $\{g(z) : g \in \mathcal{G}_P\}$ the grand orbit of z under the correspondence P. When the correspondence P is clear from the context we shall omit the phrase " under the correspondence P " and simply refer to $\{g(z) : g \in \mathcal{G}_P\}$ as the grand orbit of z.

Note 1: In this thesis we study a particular class of correspondences called covering correspondences for which the forward branch and backward branch are the same. Thus, the grand orbit of z is given by $\{F^n(z) : n \in \mathbb{N}\}$.

1.2.3 The regular and limit set of a correspondence

We now define the main objects which we wish to investigate for a given correspondence P, namely the regular set and limit set of P.

Definition 2: [3] (The regular set and limit set of a correspondence)

Let P be a correspondence and F and F^{-1} be as above. We say that the correspondence P acts discontinuously at $z \in \overline{\mathbb{C}}$ if there is an open neighbourhood U of z and $N \in \mathbb{N}$ so that

$$F^s \cap (U \times U) \subseteq \bigcup_{\substack{|m| \leq N \\ m \in \mathcal{X}}} F^m \qquad \forall \ s \in \mathcal{X}. \tag{1.8}$$

The set of all points where P acts discontinuously is said to be the regular set $\Omega(P)$ of P while $\Lambda(P) = \overline{\mathbb{C}} - \Omega(P)$ is said to be the limit set of P.

Remark 1: (i) Loosely speaking (1.8) states that there is a finite number of words of F and F^{-1} under which the images of z return to U.

(ii) It is clear from the definition that the regular set is open while the limit set is closed, and together they form a (disjoint) partition of $\overline{\mathbb{C}}$. Furthermore, it is shown in [3] that $\Omega(P)$ and $\Lambda(P)$ are completely invariant under F and F^{-1}.

Definition 3: (Quasi-fuchsian correspondence)

A correspondence is said to be a quasi-fuchsian correspondence if its limit set is a topological circle, and each component of the complement of the limit set is invariant under forward and backward branches of the correspondence.

1.3 Correspondences and subgroups of $PSL(2, \mathbb{C})$

The regular set in Definition 2 has a close connection with the classical definition of the regular set of a subgroup G of $PSL(2, \mathbb{C})$ in the following sense [8]. We say that G acts discontinuously at $z \in \overline{\mathbb{C}}$ if there is an open neighbourhood $U \neq \emptyset$ of z such that for $g \in G, g(U) \cap U = \emptyset$ for all but a finite number of elements of G. The set of all such points is called the regular set $\Omega(G)$ of G and $\Lambda(G) = \overline{\mathbb{C}} - \Omega(G)$ its limit set. In fact, the regular set is the largest open set where G is a normal family. It can be shown (see [8]) that the limit set can be described as the set of accumulation points of $\{g(z) : g \in G\}$ for any $z \in \overline{\mathbb{C}}$.

If G is generated by $\{\phi_1, \phi_2 \ldots, \phi_n\}$ then by considering the action of G

as the correspondence P (cf.(1.1)) one checks that $\Omega(G) = \Omega(P)$. Thus, with the choice of Definition 2, the study of correspondences fully generalises that of finitely generated subgroups of $PSL(2, \mathbb{C})$.

We note that if $z \in \overline{\mathbb{C}}$, then the $G-$ orbit of z, $\{g(z) : g \in G\}$ coincides with the grand orbit of z under the correspondence P : in other words, $\{g(z) : g \in \mathcal{G}_P\}$ and $\{g(z) : g \in G\}$ are the same. This is still the case if we multiply P by $(w - f(z))$ where $f \in G$.

Another interesting analogy between the action of a correspondence and the action of a subgroup of $PSL(2, \mathbb{C})$ (not necessarily finitely generated) that follows as a consequence of Definition 2 is the following theorem.

Theorem 1: ([3]) Let P be a correspondence with regular set $\Omega(P)$. Consider the equivalence relation $\sim$ on $\Omega(P)$ given by $z \sim w \iff z \in g(w)$ for some $g \in \mathcal{G}_P$, and denote the set of equivalence classes by $\Omega(P)/\mathcal{G}_P$. Then, $\Omega(P)/\mathcal{G}_P$ is Hausdorff with respect to the usual quotient topology.

The analogous result for subgroups of $PSL(2, \mathbb{C})$ is:

Theorem 2: ([8], p16) If G is a subgroup of $PSL(2, \mathbb{C})$ with regular set $\Omega(G)$ then $\Omega(G)/G$ is a Hausdorff space with respect to the quotient topology, where $\Omega(G)/G$ is the space of equivalence classes with respect to the equivalence relation $\sim$ on $\Omega(G)$ given by $z \sim w \iff z = g(w)$ for some $g \in G$.

1.4 Correspondences and iteration of rational maps

In the classical theory of iteration of a rational map R (or in general a meromorphic map on a sub-domain of $\mathbb{C}$) one is interested in the domain where $\{R^n : n \in \mathbb{N}\}$ forms a normal family. In contrast, if we consider R as a correspondence P say, where $P(z,w) = 0 \iff R(z) = w$ then we are concerned with the regular set $\Omega(P)$ of P which involves looking at the family $\mathcal{G}_P = \{R^n : n \in \mathbb{Z}\}$.

One of the disadvantages of the Definition 2 is that we are unable to fully recover the Fatou set of R as $\Omega(P)$. In fact, $\Omega(P)$ is a subset of the Fatou set of R.

For assume that there exists z_0 in $\Omega(P) \cap J(R)$ where $J(R)$ is the Julia set of R. Now, from a standard result of the iteration of a rational map (see [1]) we have that $J(R)$ is the closure of $\cup_{n=0}^{\infty} R^{-n}(z_0)$. Furthermore, since $J(R)$ is uncountable with no isolated points we conclude that there exists $w_0 \in J(R) \cap \Omega(P)$ such that $w_0 \notin \left(\cup_{n=0}^{\infty} R^{-n}(z_0)\right) \cup \left(\cup_{n=1}^{\infty} R^{n}(z_0)\right)$, in other words, w_0 is a limit point of $\cup_{n=0}^{\infty} R^{-n}(z_0)$. Now, it follows from

Theorem 1 that there exist disjoint open sets U and V containing

$\{g(z_0) : g \in \mathcal{G}\}$ and $\{g(w_0) : g \in \mathcal{G}\}$ respectively; but then the inverse images of U and V under the projection map $\Pi : \Omega \to \Omega(P)/\mathcal{G}$ are disjoint open sets in $\overline{\mathbb{C}}$, contradicting that w_0 is a limit point of $\cup_{n=0}^{\infty} R^{-n}(z_0)$. This proves that $\Omega(P)$ is a subset of the Fatou set of R.

In fact, $\Omega(P)$ consists of parabolic basins [2] and attractive basins[3] of R from which we remove the grand orbits of the attractive points under the correspondence P. The following example illustrates why the basins of super-attracting fixed points are disjoint from $\Omega(P)$.

Example 1: For $R(z) = z^2$, $z = 0$ is a super-attracting fixed point. Now, the grand orbit of 0 under R, $\{R^{-n}(0) : n \in \mathbb{Z}\}$ is just $\{0\}$. Thus, writing $P(z, w)$ for the polynomial $w - z^2$, the grand orbit of 0 under the correspondence $P(z, w) = 0$ is also $\{0\}$. If U is any open neighbourhood of 0 then U intersects every grand orbit of points of $\{z : |z| < 1\}$ under R and P since every point of the open unit disc converges to 0 under the iteration of R. This together with Theorem 1 shows that $0 \notin \Omega(P)$.

[2] see [1] for definition.

[3] Fatou components containing points z such that $R^n(z) = z$ and $0 < |(R^n)'(z)| < 1$ for some integer $n \geq 1$. Such points are called attracting periodic points of R.

1.5 Basic definitions, and the statement of the main theorem for $(2:2)$ correspondences

Scheme

In this section we define the covering correspondence $H_R(z,w)=0$ of a rational map R and state the basic properties of H_R. We investigate $(2:2)$ Hecke correspondences $H_Q(z,\phi(w))=0$ where Q is a rational map of degree three and ϕ is a Möbius involution. We show that if ϕ is chosen in a certain way then $H_Q(z,\phi(w))=0$ is a quasi-fuchsian correspondence. The proof of this requires several key steps as follows.

First, due to a property of covering correspondences it is sufficient to consider Q in a certain form referred to as a normal form. The next piece of machinery is to identify a class of Möbius involutions so that the correspondence $H_Q(z,\phi(w))=0$ is of Hecke type. Finally, using the notion of a $3-$chain of discs which we introduce in Chapter 2, we show that $H_Q(z,\phi(w))=0$ is quasi-fuchsian.

1.5.1 The covering correspondence H_Q and its forward branch F_{H_Q}

If Q is a rational map of degree $d>1$ then the covering correspondence of Q, $H_Q(z,w)=0$ is the correspondence obtained as "$\frac{Q(z)-Q(w)}{z-w}=0$." More precisely, Q can be written as $\frac{f}{g}$ for some polynomials f and g with no common zeros. For $z, w \in \mathbb{C}$, the expression $f(z)g(w) - f(w)g(z)$ can be written as $(z-w)H_Q(z,w)$ for some $(d-1 : d-1)$ correspondence H_Q : we refer to H_Q as the covering correspondence of Q. We note that H_Q is symmetric with respect to z and w; so, the forward branch and the backward branch (cf. sec. 1.2) of H_Q are the same, and we denote this common branch by F_{H_Q}. In fact, if $z \in \overline{\mathbb{C}}$ then $F_{H_Q}(z)$ can be expressed as

$$z \mapsto F_{H_Q}(z) = \begin{cases} Q^{-1}\{Q(z)\} - \{z\} & \text{if } z \text{ is not a critical point of } Q \\ Q^{-1}\{Q(z)\} & \text{otherwise.} \end{cases} \tag{1.9}$$

We shall see later that on certain domains of $\overline{\mathbb{C}}$, $z \mapsto F_{H_Q}(z)$ consists of $d-1$ single valued analytic branches.

Remark 2: (i) Each grand orbit of H_Q contains at most d points, and in particular if $F_{H_Q}(z) = U$ then the grand orbit of z is $U \cup \{z\}$. For example, consider $Q(z) = z^3$. Then, $H_Q(z,w) = 0$ if and only if $z^2 + w^2 + wz = 0$; so, $F_{H_Q}(z) = \{w_1 = ze^{\frac{2\pi i}{3}}, w_2 = ze^{\frac{4\pi i}{3}}\}$ and $F_{H_Q}(F_{H_Q}(z)) = \{w_2, w_1, z\}$.
Thus, in a dynamical point of view, $z \mapsto F_{H_Q}(z)$ is of little interest.

(ii) If $z \xrightarrow{Q} \frac{az^2+bz+c}{pz^2+qz+r}$ is a rational map of degree two then $z \mapsto F_{H_Q}(z)$ is a Möbius involution. Indeed,

$$F_{H_Q}(z) = \frac{z(cp - ar) + (cq - br)}{z(aq - bp) + (ar - cp)} \tag{1.10}$$

and if α and β are the critical points of Q (ie the solutions of $z^2(aq - pb) + 2z(ar - pc) + br - cq = 0$) then (1.10) can be written as

$$F_{H_Q}(z) = \frac{z(\alpha + \beta) - 2\alpha\beta}{2z - (\alpha + \beta)}. \tag{1.11}$$

1.5.2 Fundamental sets, analytic branches of F_{H_Q} and the contact condition

Definition 4: (Fundamental sets) Let Q be a rational map of degree $d > 1$, and $U \subset \overline{\mathbb{C}}$. Then, U is a fundamental set for Q if $Q : U \to \overline{\mathbb{C}}$ is a bijection.

It follows that for any such fundamental set U :

(i) $U \cup F_{H_Q}(U) = \overline{\mathbb{C}}$ and

(ii) if $z \in U$ and $w \in \overline{\mathbb{C}}$ such that $H_Q(z, w) = 0$ then either $w = z$ or $w \notin U$: in fact, the case $w = z$ occurs only when z is a critical point of Q.

A fundamental set whose boundary consists of a union of Jordan arcs can be obtained as follows. Let D be a simply connected domain obtained from $\overline{\mathbb{C}}$ by removing a finite number of Jordan arcs that connect the critical values of Q. Then, the components of $Q^{-1}(D)$ are simply connected domains with Jordan arcs as their boundaries. In fact, there are d number of such components say, $D_1, D_2, \dots, D_d$, and $Q : D_i \to D$ is univalent. So, each D_i can be made into a fundamental set for Q by adjoining some of D_i's boundary arcs to itself.

Example 2: (fundamental sets)

Let $f(z) = z^3 - 3z$. The set of critical points of f is $\{1, -1, \infty\}$ while the set of critical values is $\{2, -2, \infty\}$. Now, the complement of $f^{-1}\{\{x \in \mathbb{R} : x \leq 2\} \cup \{\infty\}\}$ is a union of three disjoint Jordan domains say, D_1, D_2 and D_3. Let D_1 be the domain that contains $(2, \infty)$; so, the boundary of D_1 consists of the Jordan arcs $[1, 2]$,
$B^+ = \{z \in f^{-1}\{(-\infty, -2)\} : Im(z) \geq 0\} \cup \{\infty\}$ and
$B^- = \{z \in f^{-1}\{(-\infty, -2)\} : Im(z) \leq 0\} \cup \{\infty\}$. It now follows that
$D_1 \cup B^+ \cup [1, 2]$ is a fundamental set for f.

Analytic branches of F_{H_Q}

Since Q (using the same set up as in the discussion of fundamental sets) is injective on each D_i, if $z \in D_i$ then $F_{H_Q}(z)$ consists of $d-1$ distinct points, $z_1, z_2, \dots, z_{d-1}$ such that each D_j where $j \neq i$ contains one and only one of z_k where $1 \leq k \leq d-1$. Thus, if $i \neq j$ then, the map $F_{H_Q} : D_i \to D_j$ given by $z \mapsto Q^{-1} \circ Q(z)$ is univalent, and surjective.

Hence, for instance, considering these univalent branches (each denoted by

F_{H_Q}) the composition

$$D_1 \xrightarrow{F_{H_Q}} D_2 \xrightarrow{F_{H_Q}} D_3 \xrightarrow{F_{H_Q}} D_4 \ldots \xrightarrow{F_{H_Q}} D_{d-1} \xrightarrow{F_{H_Q}} D_d \xrightarrow{F_{H_Q}} D_1$$

is the identity map of order d on D_1. In particular, for any $z \in D_i$, those univalent branches of F_{H_Q} permute $Q^{-1}(Q(z))$.

Hecke type correspondences

For an integer $n > 2$, an $(n-1 : n-1)$ correspondence is of Hecke type if it satisfies certain aspects of the Hecke group $H(n)$ (that is, the group generated by $z \xrightarrow{\delta} \frac{-1}{z}$ and $z \xrightarrow{\rho} \frac{-1}{z+2\cos(\frac{\pi}{n})}$) when the latter is considered as an $(n-1 : n-1)$ correspondence (cf. Proposition 3 (Chapter 4)). One of those aspects is that $\delta\rho$ and $\delta\rho^{-1}$ have parabolic fixed points at ∞ and 0 respectively; these fixed points are interchanged by the involution δ while $\rho(\infty) = 0$ and $\rho^{-1}(0) = \infty$. So, when we represent[4] $H(n)$ (or a conjugate of it) as a correspondence of the form $H_{z\mapsto z^n}(z, \phi(w)) = 0$ where ϕ is an involution we require that $H_{z\mapsto z^n}(z, \phi(z)) = 0$ be in the form $S(z)(z-p)^2(z-q)^2 = 0$ for some non zero polynomial S and distinct points p and q : indeed, p and q are the fixed points of the correspondence $H_{z\mapsto z^n}(z, \phi(w)) = 0$, and each of p and q is a parabolic fixed point of $H_{z\mapsto z^n}(z, \phi(z)) = 0$. The correspondences that we wish to study in this thesis bear this requirement of having two

[4] See Proposition 3.

parabolic fixed points, and those fixed point are the only fixed points of the correspondence. However, this requirement on its own in general, does not guarantee that the limit set of the correspondence is a topological circle (cf. Comment 1 and 2, p.71 and p.85). So, as a way forward we formulate the following definition: let Q be a rational map of degree $d > 1$ and consider the $(d-1 : d-1)$ correspondence $H_Q(z, \phi(w)) = 0$ where ϕ is the Möbius involution F_{H_R} for some degree two rational map R (cf. Remark 2(ii)).

Definition 5: (Hecke type correspondences)

We say that $H_Q(z, \phi(w)) = 0$ is of Hecke type if there exist fundamental sets T_Q and T_ϕ of Q and R respectively having Jordan boundaries such that $\overline{T_\phi} \subset \overline{T_Q}$, $\partial T_Q \cap \partial T_\phi \subseteq \{z \in \overline{\mathbb{C}} : H_Q(z, \phi(z)) = 0\}$[5], and $\partial T_Q \cap \partial T_\phi$ has cardinality two. In this case, the two points of $\partial T_Q \cap \partial T_\phi$ are called the contact points of $H_Q(z, \phi(w)) = 0$.

A note concerning Definition 5:

(i) The requirement for fundamental sets to have a Jordan boundary is not essential however, we find them easy to work with so, there is no harm in assuming so.

(ii) the points p and q are interchanged by ϕ while a single valued branch of the multi-valued map $z \mapsto F_{H_Q}(z)$ maps one to the other, as reminiscent of

[5] Note that $H_Q(z, \phi(z)) = 0$ is a polynomial of degree $2(d-1)$.

$H(n)$.

(iii) The property $\overline{T_\phi} \subset \overline{T_Q}$ implies that for any $z \in \overline{\mathbb{C}}$, its grand orbit under $H_Q(z, \phi(w)) = 0$ intersects $\overline{T_Q}$. Thus, when dealing with grand orbits of $H_Q(z, \phi(w)) = 0$ we need to consider only the grand orbits of points of $\overline{T_Q}$ instead of the whole of $\overline{\mathbb{C}}$.

1.5.3 Forward and backward branches of $H_Q(z, \phi(w)) = 0$

Let $H_Q(z, \phi(w)) = 0$ be denoted by $P(z, w) = 0$, and F_P and F_P^{-1} be the forward and backward branches of P respectively. Now, if F_{H_Q} is the forward branch of the covering correspondence $H_Q(z, w) = 0$ then, for any $z \in \overline{\mathbb{C}}$ we have that $F_P(z) = \phi(F_{H_Q}(z))$ and $F_P^{-1}(z) = F_{H_Q}(\phi(z))$: we view these expressions as compositions of ϕ and F_{H_Q}, and denote the set all such arbitrary finite compositions of $\{\phi, F_{H_Q}\}$ by $\mathcal{C}[\phi, F_{H_Q}]$.

Then $\{h(z) : h \in \mathcal{G}_P\} = \{h(z) : h \in \mathcal{C}[\phi, F_{H_Q}]\}$ (see sec. (1.2.2) for $\mathcal{G}_P$).

Thus, when studying the properties of P (such as the limit set) there is no loss of generality if we replace each of ϕ and F_{H_Q} by $M \circ \phi \circ M^{-1}$ and $M(F_{H_Q}(M^{-1}))$ respectively where M is a Möbius map: this is the reason that we restrict our attention to rational maps in normal forms as stated below.

1.5.4 Normal forms of rational maps

Consider the equivalence relation $\sim$, defined on the class of rational maps by $S \sim R$ if and only if there exists Möbius maps K and M such that $S = M \circ R \circ K$ where S and R are rational maps. If $S \sim R$ then

(i) S and R have the same number of critical points with their multiplicities preserved, and

(ii) S and R have the same number of critical values (the images of the critical points).

However, these two properties are not sufficient to conclude that two rational maps belong to the same equivalence class. For instance, suppose that S and R satisfy the properties (i) and (ii), and that the critical points of S lie on a line in $\overline{\mathbb{C}}$ while the critical points of R do not lie on a line in $\overline{\mathbb{C}}$. Then, S and R do not belong to the same equivalence class.

It turns out, as given in Lemma 1 that when studying correspondences of the form $H_Q(z, \phi(w)) = 0$[6] it suffices to consider Q from each equivalence class of $\sim$: to this end, we select our Q conveniently so as to make the computations simpler and a so chosen form of a map is called a normal form. We remark that it is possible for infinitely many equivalence classes to have a map in the form of a particular normal form. For instance, in the case of degree three

[6] See also Appendix A3.

rational maps there are infinitely many equivalence classes but there are only three normal forms as follows:

(i) $z \mapsto z^3$ determines the equivalence class of maps with two distinct critical points, each with order two [7]

(ii) $z \mapsto z^3 - 3z$ determines the equivalence class of maps having one critical point of order two, and two distinct critical points (each of which with order one)

(iii) any map with four distinct critical points (so, each of which has order one) is equivalent to some $z \mapsto \frac{z^2(z+b)}{z+c}$ where $b, c \neq 0,\ 9c^2 + b^2 - 10bc \neq 0$ and $b + 3c + 2(1 + bc) = 0$: the later condition implies that the critical points of $z \mapsto \frac{z^2(z+b)}{z+c}$ are $0, 1, bc$ and ∞. Note that the images of critical points (critical values) under $z \mapsto \frac{z^2(z+b)}{z+c}$ are distinct: so, there are four critical values.

So, every equivalence class of degree three rational maps with four distinct critical points contains a map of the form (iii). For if $0, 1, \xi$ and ∞ are distinct points then a generic form of a map having those points as its critical points is $z \xrightarrow{Q} \frac{a(abz^3+b^2z^2+bcz+ac\xi)}{Aabz^3+Ab^2z^2+Babz+\xi a^2 B}$ where $A, a, B, b, c \in \mathbb{C}$ and $-2ab(1 + \xi) = b^2 + 3a^2\xi$. Now, if $z \xrightarrow{M} \frac{-B(z-\frac{c}{B})}{A(z-\frac{a}{A})}$ then $M \circ Q(z) = \frac{z^2(z+\frac{b}{a})}{z+\frac{\xi a}{b}}$ as required.

Lemma 1: Let Q be a rational map of degree $n > 1$, and K be a Möbius map. Then,

[7] A map Q has a critical point of order n at w if $\frac{\mathrm{d}^m Q(z)}{\mathrm{d}z^m}|_{z=w} = 0$ for $m \leq n$, and $\frac{\mathrm{d}^{n+1} Q(z)}{\mathrm{d}z^{n+1}}|_{z=w} \neq 0$.

(i) $H_{K\circ Q}(z,w) = 0 \iff H_Q(z,w) = 0$. So, for all $z \in \overline{\mathbb{C}}$ we have that $F_Q(z) = F_{K\circ Q}(z)$ where F_Q and $F_{K\circ Q}$ are the forward branches of H_Q and $H_{K\circ Q}$ respectively.

(ii) For all $z \in \overline{\mathbb{C}}$, $F_Q(K(z)) = K(F_{Q\circ K}(z))$ or equivalently,

$F_{Q\circ K}(z) = K^{-1}(F_Q(K(z)))$.

Proof: Let $Q(z) = \frac{S(z)}{T(z)}$ where $S(z) = \sum_{i=0}^{n} s_i z^i$ and $T(z) = \sum_{i=0}^{n} t_i z^i$ are polynomials (with $s_i, t_i \in \mathbb{C}$) with no common zeros, and $K(z) = \frac{az+b}{cz+d}$ be a Möbius map.

(i) As $K \circ Q = \frac{aS+bT}{cS+dT}$ we have for some $\lambda \neq 0$ that

$\lambda(z-w)H_{K\circ Q}(z,w) = (aS+bT)(z)(cS+dT)(w) - (aS+bT)(w)(cS+dT)(z) = (ad-bc)[S(z)T(w) - S(w)T(z)]$. Since, $S(z)T(w) - S(w)T(z)$ is of the form $\mu(z-w)H_Q(z,w)$ for some $\mu \neq 0$, the claim follows.

(ii) Write $f(z) = az+b$ and $g(z) = cz+d$, so, $K = \frac{f}{g}$.

Now, since $Q \circ K = \frac{\sum_{i=0}^{n} s_i f^i g^{n-i}}{\sum_{i=0}^{n} t_i f^i g^{n-i}}$ it follows that

$(\sum_{i=0}^{n} s_i f^i g^{n-i}(z))(\sum_{i=0}^{n} t_i f^i g^{n-i}(w)) - (\sum_{i=0}^{n} s_i f^i g^{n-i}(w))(\sum_{i=0}^{n} t_i f^i g^{n-i}(z))$ is in the form $\lambda(z-w)H_{Q\circ K}(z,w)$ for some $\lambda \neq 0$.

Likewise, $(\sum_{i=0}^{n} s_i z^i)(\sum_{i=0}^{n} t_i w^i) - (\sum_{i=0}^{n} s_i w^i)(\sum_{i=0}^{n} t_i w^i)$ can be written as $\mu(z-w)H_Q(z,w)$ for some $\mu \neq 0$. By letting $z = K(z)$ and $w = K(w)$ in $H_Q(z,w) = 0$ we find that $H_Q(K(z),K(w)) = 0 \iff H_{Q\circ K}(z,w) = 0$. Hence, $w \in F_Q(z) \iff K^{-1}(w) \in F_{Q\circ K}(K^{-1}(z))$, that is,

$F_Q(z) = K(F_{Q\circ K}(K^{-1}(z)))$ as required.

Note 2: Let ϕ be a Möbius involution, and R and Q be rational maps where $R = K \circ Q \circ M$ for some Möbius maps K and M. Then, from Lemma 1 it follows that $H_R(z, \phi(w)) = 0 \iff H_Q(M(z), M(\phi(w))) = 0$. So, if Ω and Λ are the regular and limit sets of $H_Q(z, \phi(w)) = 0$ respectively then the regular and limit sets of $H_R(z, \phi(w)) = 0$ are $M^{-1}(\Omega)$ and $M^{-1}(\Lambda)$ respectively: this justifies restricting our consideration to normal forms in studying the properties of $H_Q(z, \phi(w)) = 0$.

1.5.5 The statement and an overview of the main results

We now state the background to the main result of the thesis, viz.
Theorem 4 where we consider correspondences of the form $H_Q(z, \phi(w)) = 0$. The motivation to study this particular class of correspondences is due to Theorem 3: we begin with introducing some terminology.

Let G be a Kleinian group, freely generated by Möbius maps σ and ρ which have orders two and $n > 2$ respectively; so G is isomorphic to the cyclic group $C_2 * C_n$. We may associate to G the $(n-1 : n-1)$ correspondence $G(z, w) = 0$ given by $(w - \sigma\rho(z))(w - \sigma\rho^2(z)) \ldots (w - \sigma\rho^{n-1}(z)) = 0$. It follows that $F_G^{-1} = \sigma \circ F_G \circ \sigma$, where F_G and F_G^{-1} are the forward and backward branches of $G(z, w) = 0$ respectively.

Now, suppose that $P(z,w) = 0$ is any $(n-1 : n-1)$ correspondence equipped with a Möbius involution J such that $F_P^{-1} = J \circ F_P \circ J$, where F_P and F_P^{-1} are the forward and backward branches of $P(z,w) = 0$ respectively.

Definition 6: (Hecke conjugacy for correspondences)
Let $U \subset \overline{\mathbb{C}}$ be invariant under F_P and J, and let $V \subset \overline{\mathbb{C}}$ be invariant under F_G (here $G(z,w) = 0$ is an $(n-1 : n-1)$ correspondence as above) and σ (as above). An analytic bijection $\psi : U \to V$ is called a conjugacy from F_P to G, compatible with J and σ, if $F_G = \psi \circ F_P \circ \psi^{-1}$ and $\sigma = \psi \circ J \circ \psi^{-1}$. If such a conjugacy exists then we say that the action of $P(z,w) = 0$ on U is Hecke conjugate to the action of G on V.

Theorem 3: ([4], Thm. 4.1) Suppose that $P(z,w) = 0$ is an $(n-1 : n-1)$ correspondence equipped with a Möbius involution J such that $F_P^{-1} = J \circ F_P \circ J$. Let G be a Kleinian group which is freely generated by σ and ρ of orders two and n respectively. Now, let U and V be non empty open sub sets of $\overline{\mathbb{C}}$ and $\psi : U \to V$ be as in Definition 6. Then, there exists a rational map Q of degree n such that $H_Q(z, J(w)) = 0$ defines the same correspondence as $P(z,w) = 0$ on $\overline{\mathbb{C}}$.

Note that, there are examples of quasi-fuchsian correspondences of the form $H_Q(z, \phi(w)) = 0$ where $H_Q(z, \phi(w)) = 0$ is Hecke conjugate on both components of the complement of the limit set (cf.Proposition 3 in Chapter 4).

The statement of the main theorem, whose proof is given in Chapter 3 is as follows:

Theorem 4: (1)(a). Let $Q(z) = z^3 - 3z$. Then, there is a non-empty class $\mathcal{M}$ of Möbius involutions so that for each $\phi \in \mathcal{M}$, the $(2:2)$ correspondence $H_Q(z, \phi(w)) = 0$ which we denote by P_ϕ is quasi-fuchsian, say with the limit set Λ_ϕ.

(b). If Ω_ϕ is the component of $\overline{\mathbb{C}} - \Lambda_\phi$ containing the point at infinity then the action of P_ϕ on Ω_ϕ is Hecke conjugate to the action of $PSL(2, \mathbb{Z})$[8] on the upper half plane $\{z \in \mathbb{C} : Im(z) > 0\}$.

(2) If $R(z) = \frac{z^2(z+b)}{(z+c)}$ with all its critical points lie on a line in $\overline{\mathbb{C}}$ then, there exists a non empty class of Möbius involutions $\mathcal{E}$ so that for each $\psi \in \mathcal{E}$ the correspondence $H_R(z, \psi(w)) = 0$ is quasi-fuchsian.

In this case, the action of $H_R(z, \psi(w)) = 0$ on any component of the complement of the limit set is not Hecke conjugate to a Kleinian group.

The case $Q(z) = z^3$, that is the normal form corresponding to degree three maps having exactly two critical points, is treated in Proposition 3 in Chapter 4.

[8] The group generated by $z \mapsto \frac{-1}{z}$ and $z \mapsto \frac{-1}{z+1}$.

2. 3-CHAINS OF DISCS

2.1 The modular group and 3-chains of discs

In this chapter we introduce the notion of a $3-$chain of discs and show that its limit set is a topological circle. The notion of a $3-$chain of discs is based on observing how the action of the modular group $PSL(2,\mathbb{Z})$ on a fundamental domain tessellates $\mathbb{C}-\mathbb{R}$: we formalise this process as a $3-$chain of discs.

Notations and terminology

(1) From now on, we shall use the symbol ∂ to denote the boundary of a set; so for instance, ∂S or $\partial(S)$ stands for the boundary of the set S.

(2) An open disc is the image of the open unit disc under a univalent map. A Jordan domain is an open disc whose boundary is a closed Jordan curve.

(3) If n is a positive integer, and h is a function then h^n stands for the $n-$th iterate of h, viz. $h \circ h \circ \ldots \circ h$.

(4) If f and h are maps then we shall write fh for $f \circ h$ whenever it is convenient to do so. Likewise, if $U \subset \overline{\mathbb{C}}$ or $z \in \overline{\mathbb{C}}$ then we write from time to

time, $fg(U)$ and $fg(z)$ for $f(g(U))$ and $f(g(z))$ respectively.

(5) The closure of a set S is denoted by $\overline{S}$.

2.1.1 The model for a 3−chain of discs

Informally, a 3−chain of discs or in general an n−chain of discs where n is an integer greater than two, can be visualised as a necklace of n pearls, where each pearl is glued to exactly two pearls, thus creating a chain of pearls without any gaps. This situation can be observed by considering the modular group[1] $PSL(2,\mathbb{Z})$, that is the group generated by $\psi(z) = \frac{-1}{z}$ and $g(z) = \frac{-1}{z+1}$, with its action on a particular fundamental domain.

For, by considering the standard fundamental domain

$D = \{z \in \overline{\mathbb{C}} : \frac{-1}{2} \leq Re(z) \leq \frac{1}{2} \text{ and } |z| \geq 1\}$ of $PSL(2,\mathbb{Z})$ for its action on the upper half plane, we see that

$$U = \{z \in \overline{\mathbb{C}} : Re(z) > \frac{1}{2} \text{ or } |z-1| < 1\} \tag{2.1}$$

contains a copy of the fundamental domain for the action of $PSL(2,\mathbb{Z})$ on $\overline{\mathbb{C}}$ in the sense that U intersects every grand orbit of the modular group: the limit set of this action on U is $\overline{\mathbb{R}}$. Now, we can think of $U, g(U)$ and $g^2(U)$ as three pearls forming a chain of three pearls which we refer to as

[1] Or more generally the Hecke group $H(n)$, the group generated by $\psi(z) = \frac{-1}{z}$ and $g(z) = \frac{-1}{z+2cos\frac{\pi}{n}}$.

a $3-$chain of discs: pearl U is attached to the pearls $g(U)$ and $g^2(U)$ at 0 and ∞ respectively, while $g(U)$ and $g^2(U)$ are attached to each other at -1. We now let $PSL(2,\mathbb{Z})$ act on these three pearls and observe as to how our necklace changes, as follows.

First, note that $g(\overline{U}) \cup g^2(\overline{U}) - \{0,\infty\} \subset \psi(U)$ and $\partial(\psi(U)) \cap \partial U = \{0,\infty\}$. Now, the images of $g(U)$ and $g^2(U)$ under ψ lie in U in such a way that we obtain a smaller necklace with four pearls, $g(U), g^2(U), \psi(g(U))$ and $\psi(g^2(U))$. Taking the images of $\psi(g(U))$ and $\psi(g^2(U))$ under g and g^2 we obtain another necklace with six pearls namely, $\psi(g(U)), \psi(g^2(U)), g(\psi(g(U))), g(\psi(g^2(U))), g^2(\psi(g(U)))$ and $g^2(\psi(g^2(U)))$. Next, we take the ψ images of $g(\psi(g(U))), g(\psi(g^2(U))), g^2(\psi(g(U)))$ and $g^2(\psi(g^2(U)))$, and obtain a necklace with eight pearls and so on. As the number of pearls increases, they become smaller and smaller. By repeating the process ad infinitum we end up with a necklace whose pearls are nothing more than points; in other words, we are left with the limit set of the modular group.

We formalise this notion of a necklace with three pearls as a $3-$chain of discs, and define its limit set, which will be shown to be a topological circle. Once this formalism is done, the modular group can be considered as an example of a $3-$chain of discs.

Before we define a $3-$chain of discs, we highlight a certain aspect of pearls shrinking to a point.

First, we note that ψg and ψg^2 have parabolic fixed points at ∞ and 0 respectively, and for $n \in \mathbb{N}$ we have that $(\psi g)^{n+1}(\overline{U}) \subset (\psi g)^n(\overline{U}) \subset \overline{U}$ and $(\psi g^2)^{n+1}(\overline{U}) \subset (\psi g^2)^n(\overline{U}) \subset \overline{U}$.

Moreover, we note that, $\bigcap_{n=1}^{\infty}(\psi g)^n(\overline{U}) = \{\infty\}$ and $\bigcap_{n=1}^{\infty}(\psi g^2)^n(\overline{U}) = \{0\}$: we refer to this property as the shrinking condition.

In general, there is no certainty that a pair of maps satisfying all of the above properties except the shrinking condition would automatically satisfy the shrinking condition as well: to see this, consider the following counter example which is due to A. Eremenko:

Let $f(z) = z - 1$, and D be the compact, simply connected domain above and including the graph $y = tan^{-1}(x)$. So, the boundary of D consists of the graph $y = tan^{-1}(x)$ together with the point at infinity. Now, for any $n \in \mathbb{N}$, $f^{n+1}(D) \subset f^n(D)$; but f does not satisfy the shrinking condition at infinity since $\bigcap_{n=1}^{\infty} f^n(D)$ is the compact half plane $\{x + \imath y : y \geq \frac{\pi}{2}\}$.

Thus, it is necessary to include the shrinking condition in our set up of a $3-$chain of discs, and we state this formally as follows:

Definition 7: (shrinking condition)

Let U be a Jordan domain, and $F : \overline{U} \to \overline{U}$ be univalent[2] (so F is analytic

[2] F is analytic and injective.

in a domain that contains $\overline{U}$). Let $a \in \partial U$ be the only fixed point of F in $\overline{U}$. Then, we say that F satisfies the shrinking property (at a) if

$$F(\overline{U} - \{a\}) \subset U \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} F^n(\overline{U}) = \{a\}.$$

Later, in Lemma 4 we give some geometric conditions under which the shrinking condition is satisfied.

2.1.2 3-chains of discs

Let V be an open disc, and ϕ and f be univalent maps on $\overline{V}$ with the following properties:

(1) $\phi : \phi(\overline{V}) \to \overline{V}$ is univalent and $\phi^2(z) = z$ for all z in $\overline{V} \cup \phi(\overline{V})$

(2) $f : f(\overline{V}) \to f^2(\overline{V})$ and $f : f^2(\overline{V}) \to \overline{V}$ are univalent and $f^3(z) = z$ for all z in $\overline{V} \cup f(\overline{V}) \cup f^2(\overline{V})$.

Definition 8: (3−chain of discs)

We say that V, ϕ and f give rise to a 3−chain of discs if there exist $p, q \in \partial(V)$ satisfying the following conditions:

(3) $\phi(\overline{V}) \cap \overline{V} = \{p, q\}$ where $\phi(p) = q$

(4) $f(\overline{V}) \cup f^2(\overline{V}) - \{p, q\} \subset \phi(V)$

(5) $f(\overline{V}) \cap \overline{V} = \{q\}, f^2(\overline{V}) \cap \overline{V} = \{p\}$ and $f^2(\overline{V}) \cap f(\overline{V}) = \{r\}$ where $f(p) = q$ and $f(q) = r$

(6) $\phi f : \overline{V} \to \overline{V}$ and $\phi f^2 : \overline{V} \to \overline{V}$ satisfy the shrinking condition at p and q respectively.

We shall denote this 3−chain of discs by $\left\langle \phi, f, V, p, q, r \right\rangle$.

Remark 3: Since we have modelled our 3−chain of discs on the action of the modular group with the fundamental set $\overline{U}$ (as given in (2.1)) it follows that $\mathcal{M} = \left\langle z \overset{\psi}{\to} \frac{-1}{z}, z \overset{g}{\to} \frac{-1}{z+1}, U, \infty, 0, -1 \right\rangle$ is a 3−chain of discs: we will show later in Proposition 2 (in section 2.2.1) that $\mathcal{M}$ satisfies the shrinking condition at 0 and ∞.

2.2 The limit set of a 3−chain of discs

Our next task is to define and show that the limit set of a 3−chain of discs is a topological circle. We shall define the limit set of a 3−chain of discs to be the closure of the the grand orbit of p under all compositions of ϕ and f : we refer to the grand orbit of p as the set of "finite points".

This notion of limit set of a 3−chain of discs is analogous to that of the modular group where the limit set $\overline{\mathbb{R}}$ is the closure of the grand orbit of 0 : the grand orbit of 0 is in fact the set of rational numbers. Thus, for a 3−chain of discs, we need to know what constitutes the counterpart of irrational numbers of the limit set of modular group: we define (for a 3−chain of discs) those points that correspond to irrational numbers as "infinite points": we refer to them as infinite point since each such point is the intersection of a nested sequence of compact sets, as set out in Lemma 2. Once, we prove Lemma 2, it remains to show that the limit set is indeed equal to the union of infinite points and finite points, and that the limit set is a topological circle.

In the Appendix A1, we give a direct proof that the limit set is a topological circle: this proof is based on identifying each infinite point with an irrational point, and each finite point with a rational number.

Finite words and finite points of a $3-$chain of discs

We first introduce the notations that we use for the remainder of this chapter.

For $m \in \mathbb{N} \cup \{\infty\}$ we denote by $(n_i)_{i=1}^m$ the ordered $m-$tuple $(n_1, n_2, \ldots, n_k, \ldots)$ whose entries are positive integers.

Now, let $S(m) = \{(n_i)_{i=1}^m : n_i \in \mathbb{N}\}$ be the set of all $m-$tuples. Likewise, let $T(m) = \{(n_i)_{i=1}^m : n_i \neq n_{i+1},\ \text{each } n_i \text{ is either } 1 \text{ or } 2\}$: so, an element of $T(m)$ has one of the forms $(1, 2, 1, 2, 1, 2, \ldots)$ or $(2, 1, 2, 1, 2, 1 \ldots)$. Note that $T(1) = \{(1), (2)\}$.

Finite words of g_1 and g_2

Let g_1 and g_2 be a pair of distinct maps, and m be a positive integer. For $t = (t_i)_{i=1}^m \in T(m)$ and $s = (n_i)_{i=1}^m \in S(m)$ we define the finite word associated with the pair (t, s) with respect to the maps g_1 and g_2 as

$$W(t, s) = W((t_i)_{i=1}^m, (n_i)_{i=1}^m) = g_{t_1}^{n_1} \circ g_{t_2}^{n_2} \circ g_{t_3}^{n_3} \ldots \circ g_{t_{m-1}}^{n_{m-1}} \circ g_{t_m}^{n_m}. \tag{2.2}$$

For brevity, whenever it is clear from the context, we shall omit the reference to the maps (in this case g_1 and g_2) when we talk about finite words. The same convention will be used later when we consider infinite words.

The set of all such finite words is denoted by $\mathcal{C}[g_1, g_2]$, that is

$$\mathcal{C}[g_1, g_2] = \{W(t,s) : m \in \mathbb{N},\ (t,s) \in T(m) \times S(m)\}. \tag{2.3}$$

Definition 9: (Finite words and finite points of a 3$-$chain of discs)

Let $\mathcal{H} = \left\langle \phi, f, V, p, q, r \right\rangle$ be a 3$-$chain of discs, $g_1 = \phi f$ and $g_2 = \phi f^2$. Then, the set of finite words of $\mathcal{H}$ is defined as

$$\mathcal{F}_{Words} = \mathcal{C}[g_1, g_2] \cup \{\phi h : h \in \mathcal{C}[g_1, g_2]\}. \tag{2.4}$$

Likewise, the set of finite points $\mathcal{F}_{Points}$ of $\mathcal{H}$ is defined to be the grand orbit of p under $\mathcal{F}_{Words}$, that is,

$$\mathcal{F}_{Points} = \{g(p) : g \in \mathcal{F}_{Words}\}.^3 \tag{2.5}$$

Note 3: (i) $\mathcal{F}_{Words}$ is a proper subset of $\mathcal{C}[f, \phi]$, the set of all compositions of ϕ and f : e.g., $\nu = (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ (\phi f)^{n_3} \ldots (\phi f^2)^{n_k} \in \mathcal{F}_{Words}$ while

$$\begin{aligned} \nu^{-1} &= (f\phi)^{n_k} \circ (f^2\phi)^{n_{k-1}} \ldots (f\phi)^{n_2} \circ (f^2\phi)^{n_1} \\ &= f \circ \Big((\phi f)^{n_k - 1} \circ (\phi f^2)^{n_{k-1}} \ldots (\phi f)^{n_2} \circ (\phi f^2)^{n_1} \Big) \circ \phi \ \notin \mathcal{F}_{Words}. \end{aligned}$$

[3] Or equally $\mathcal{F}_{Points} = \{g(q) : g \in \mathcal{F}_{Words}\}$ since $\phi f^2(p) = \phi f(q)$.

However, $\mathcal{F}_{Points} = \{g(p) : g \in \mathcal{C}[f, \phi]\}$. For example,

$$\begin{aligned}
\nu^{-1}(p) &= f \circ \Big((\phi f)^{n_k - 1} \circ (\phi f^2)^{n_{k-1}} \dots (\phi f)^{n_2}\Big)(q) \\
&= f \circ \Big((\phi f)^{n_k - 1} \circ (\phi f^2)^{n_{k-1}} \dots (\phi f)^{n_2}\Big) f(p) \\
&= f \circ \Big((\phi f)^{n_k - 1} \circ (\phi f^2)^{n_{k-1}} \dots (\phi f)^{n_2 - 1}\Big)(\phi f^2(p)) \\
&= \phi \circ \Big((\phi f)^{n_k} \circ (\phi f^2)^{n_{k-1}} \circ (\phi f)^{n_2 - 1} \circ (\phi f^2)\Big)(p) \in \mathcal{F}_{Points}.
\end{aligned}$$

Likewise, if $\theta = \phi \circ \big((\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ (\phi f)^{n_3} \dots (\phi f^2)^{n_k}\big)$ then θ^{-1} is given by $f \circ \Big((\phi f)^{n_k - 1} \circ (\phi f^2)^{n_{k-1}} \dots (\phi f)^{n_2} \circ (\phi f^2)^{n_1}\Big)$ thus
$\theta^{-1}(p) = \phi \circ \Big((\phi f)^{n_k} \circ (\phi f^2)^{n_{k-1}} \dots (\phi f)^{n_2} \circ (\phi f^2)^{n_1}\Big)(p) \in \mathcal{F}_{Points}$.

(ii) If $w_1, w_2 \in \mathcal{F}_{Words}$ then $w_1^{-1} \circ w_2(p), w_1 \circ w_2^{-1}(p) \in \mathcal{F}_{Points}$.

For example, if $w_1 = (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ (\phi f)^{n_3} \dots (\phi f^2)^{n_k}$ and

$w_2 = (\phi f)^{m_1} \circ (\phi f^2)^{m_2} \circ (\phi f)^{m_3} \dots (\phi f^2)^{m_l}$ then $w_1^{-1} \circ w_2$ is given by

$\Big((f\phi)^{n_k} \circ (f^2\phi)^{n_{k-1}} \dots (f\phi)^{n_2} \circ (f^2\phi)^{n_1}\Big) \circ \Big((\phi f)^{m_1} \circ (\phi f^2)^{m_2} \circ (\phi f)^{m_3} \dots (\phi f^2)^{m_l}\Big)$.

Now, without loss of generality assume that $n_1 > m_1 \geq 0$.[4] Then,

[4] Here, we have relaxed our original assumption that all m_i's are positive integers by allowing m_1 to be 0 : this deviation causes no ambiguity in this context.

$$\begin{aligned}
w_1^{-1} \circ w_2 &= \Big((f\phi)^{n_k} \circ (f^2\phi)^{n_{k-1}} \dots (f\phi)^{n_2} \circ (f^2\phi)^{n_1-m_1}\Big)\circ \\
&\qquad \Big((\phi f^2)^{m_2} \circ (\phi f)^{m_3} \dots (\phi f^2)^{m_l}\Big) \\
&= \Big((f\phi)^{n_k} \circ (f^2\phi)^{n_{k-1}} \dots (f\phi)^{n_2} \circ (f^2\phi)^{n_1-m_1-1}\Big)\circ \\
&\qquad f \circ \Big((\phi f^2)^{m_2-1} \circ (\phi f)^{m_3} \dots (\phi f^2)^{m_l}\Big) \\
&= \phi \circ \Big((\phi f)^{n_k} \circ (\phi f^2)^{n_{k-1}} \dots (\phi f)^{n_2} \circ (\phi f^2)^{n_1-m_1-1} \circ (\phi f)\Big)\circ \\
&\qquad \Big((\phi f^2)^{m_2-1} \circ (\phi f)^{m_3} \dots (\phi f^2)^{m_l}\Big) \in \mathcal{F}_{Words}.
\end{aligned}$$

So, $w_1^{-1} \circ w_2(p) \in \mathcal{F}_{Points}$ as required.

Similarly, if $w_1 = (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ (\phi f)^{n_3} \dots (\phi f^2)^{n_k}$ and $w_2 = \phi \circ \Big((\phi f)^{m_1} \circ (\phi f^2)^{m_2} \circ (\phi f)^{m_3} \dots (\phi f^2)^{m_l}\Big)$ then $w_1^{-1} \circ w_2 \in \mathcal{F}_{Words}$. For

$$\begin{aligned}
w_1^{-1} \circ w_2 &= f \circ \Big((\phi f)^{n_k-1} \circ (\phi f^2)^{n_{k-1}} \dots (\phi f)^{n_2} \circ (\phi f^2)^{n_1}\Big)\circ \\
&\qquad \Big((\phi f)^{m_1} \circ (\phi f^2)^{m_2} \circ (\phi f)^{m_3} \dots (\phi f^2)^{m_l}\Big) \\
&= \phi \circ \Big((\phi f)^{n_k} \circ (\phi f^2)^{n_{k-1}} \dots (\phi f)^{n_2} \circ (\phi f^2)^{n_1}\Big)\circ \\
&\qquad \Big((\phi f)^{m_1} \circ (\phi f^2)^{m_2} \circ (\phi f)^{m_3} \dots (\phi f^2)^{m_l}\Big) \in \mathcal{F}_{Words} \text{ as required.}
\end{aligned}$$

We now turn our attention to the limit points of $\mathcal{F}_{Points}$: these limit points consist of infinite points as in the following lemma.

Lemma 2: Let $\mathcal{H} = \Big\langle \phi, f, V, p, q, r \Big\rangle$ be a 3$-$chain of discs, and $g_1 = \phi f$ and $g_2 = \phi f^2$ be as in (2.2). Then whenever $t = (t_i)_{i=1}^{\infty} \in T(\infty)$ and $s = (n_i)_{i=1}^{\infty} \in S(\infty)$,

$$K(t,s) = \bigcap_{k=1}^{\infty} W\Big((t_i)_{i=1}^{k}, (n_i)_{i=1}^{k}\Big)(\overline{V}) \tag{2.6}$$

is a point. We refer to $K(t,s)$ as the infinite point associated with (t,s).

Definition 10: (Infinite points of a 3−chain of discs)

We define the set of infinite points $\mathcal{I}_{Points}$ of $\mathcal{H}$ (using the same notation as above) to be

$$\mathcal{I}_{Points} = \{K(t,s) : (t,s) \in T(\infty) \times S(\infty)\} \cup \{\phi(K(t,s)) : (t,s) \in T(\infty) \times S(\infty)\}. \tag{2.7}$$

Note 4: Each $K(t,s) \in V$, and if $h \in \mathcal{F}_{Words}$ then it follows that $h(K(t,s)) = \bigcap_{k=1}^{\infty} h \circ W\Big((t_i)_{i=1}^{k}, (n_i)_{i=1}^{k}\Big)(\overline{V})$ is also a point. In fact, $h(K(t,s))$ is equal to either $\phi(K(\tilde{t},\tilde{s}))$ or $K(\tilde{t},\tilde{s})$ for some $(\tilde{t},\tilde{s}) \in T(\infty) \times S(\infty)$.

For the proof of Lemma 2 we need the following theorem concerning a sequence of annuli:

An annulus (a doubly connected domain) is a univalent image of a standard annulus $\{z : r < |z| < R\}$ for some $0 \leq r$ and $0 < R \leq \infty$. For such an annulus, the quantity $\ln(\frac{R}{r})$ is said to be its modulus, which is invariant under univalent maps. When $r = 0$ or $R = \infty$ the modulus of the annulus is defined to be ∞.

Theorem 5 ([11]): Let A be an annulus and $\{A_n\}_{n=1}^{\infty}$ be a sequence of disjoint annuli in A. Let C_1 and C_2 be the components of $\overline{\mathbb{C}} - A$. If each A_i separates C_1 and C_2, then

$$\sum_{n=1}^{\infty} mod(A_n) \leqslant mod(A)$$

where *mod* stands for the modulus of the annulus concerned.

Proof of Lemma 2. We first note that since $\left\{W\left((t_i)_{i=1}^{2k}, (n_i)_{i=1}^{2k}\right)(\overline{V})\right\}_{k=1}^{\infty}$ is a nested sequence of compact and connected sets, $K(t,s)$ is non empty and connected. Furthermore, $K(t,s)$ lies in the interior of V.

Now, assume without loss of generality that $t_1 = 1$, and as before let $g_1 = \phi f$ and $g_2 = \phi f^2$. Let Γ be a closed Jordan curve in V that separates ∂V and $g_1 g_2(\overline{V}) \cup K(t,s)$, and A be the annulus surrounded by ∂V and Γ. Further, let $\widehat{A}$ be the component of $\overline{\mathbb{C}} - A$ which lies in V.

We show that there is a sequence $\{A_n\}_{n=1}^{\infty}$ of annuli having the same modulus lying in $\overline{V} - K(t,s)$ as described in Theorem 5: hence, $\sum_{n=1}^{\infty} mod(A_n) = \infty$ and so $mod(\overline{V} - K(t,s))$ is infinite. This implies that $K(t,s)$ is a point.

There are four cases to consider: in all cases the construction of the desired sequence of annuli is essentially the same as in Case 1 below.

We introduce the following terminology: If Q is an annulus with finite modulus then we say that a set R is surrounded by Q if $R \cap Q = \emptyset$ and R

lies in the bounded component of $\overline{\mathbb{C}} - Q$.

Case 1. None of the n_{2i+1} equal to 1.

Let $F_k = g_1 \circ g_2^{n_{2k}} \circ g_1^{n_{2k+1}-1}$ where $k = 1, 2, \ldots$. Further, let $G_1 = F_1$ and $G_{k+1} = G_k \circ F_{k+1}$ for each positive integer k. So $K(t,s)$ can be written as $g_1^{n_1-1}\big(\bigcap_{k=1}^{\infty} G_k(\overline{V})\big)$.

Now, by construction we have the following:

(i) for each $k, F_k(A) \subset \widehat{A}$ and $F_k(A)$ is surrounded by A. So, for any positive integers l and k, $F_k \circ F_l(A)$ is surrounded by $F_k(A)$

(ii) for each $k, G_k(V)$ is surrounded by A. So, for each $k, G_{k+1}(A)$ is surrounded by $G_k(A)$. For, as $G_{k+1} = G_k \circ F_{k+1}$ and $F_{k+1}(A)$ is surrounded by A it follows that $G_k \circ F_{k+1}(A)$ is surrounded by $F_{k+1}(A)$.

Thus, the sequence of annuli $\Big\{g_1 \circ G_k(A)\Big\}_{k=1}^{\infty}$ with each annulus lying in $\overline{V} - K(t,s)$ satisfies the Theorem 5. So, we conclude that $K(t,s)$ is a point.

Case 2. A non-zero finite number of the $n_{2i+1} = 1$ equal to 1.

In this case, $K(t,s) = W\Big((t_i)_{i=1}^{2j}, (n_i)_{i=1}^{2j}\Big)\Big(\bigcap_{k=1}^{\infty} W\Big((t_i)_{i=1}^{k}, (n_{2j+i})_{i=1}^{k}\Big)(\overline{V})\Big)$ where j is chosen so that $n_{2j+i} \neq 1$ for all i. So, using the Case 1 above we see that $\bigcap_{k=1}^{\infty} W\Big((t_i)_{i=1}^{k}, (n_{2j+i})_{i=1}^{k}\Big)(\overline{V})$ is a point and this in turn proves the claim.

Case 3. All of the n_{2k+1} equal to 1.

In this case $K(t,s) = \bigcap_{j=1}^{\infty} W\Big((t_i)_{i=1}^{2j}, (n_i)_{i=1}^{2j}\Big)(\overline{V})$.

Here, for each positive integer l, $g_1 g_2^{l}(A)$ is surrounded by A.

So, for each positive integer $k, W\Big((t_i)_{i=1}^{2(k+1)},(n_i)_{i=1}^{2(k+1)}\Big)(A)$ is surrounded by $W\Big((t_i)_{i=1}^{2k},(n_i)_{i=1}^{2k}\Big)(A)$.

Thus, the sequence of annuli $\Big\{W\Big((t_i)_{i=1}^{2j},(n_i)_{i=1}^{2j}\Big)(A)\Big\}_{j=1}^{\infty}$ where each annulus lies in $\overline{V}-K(t,s)$ satisfies Theorem 5; so it follows that $mod(\overline{V}-K(t,s))$ is infinite, as desired.

Case 4 Infinitely many but not all of the $n_{2i+1}=1$. So, assume that $\{n_{2l_k+1}\}$ is the proper infinite subsequence of $\{n_{2i+1}\}$ where $n_{2l_k+1}=1$ for all k.

Let $F_k = W((t_i)_{i=1}^{2l_k-2l_1},(n_i)_{i=n_{2l_1+1}}^{2l_k})$ for $k=2,3,\ldots$.

So, $K(t,s)=W((t_i)_{i=1}^{n_{2l_1}},(n_i)_{i=1}^{2l_1})\Big(\bigcap_{k=2}^{\infty}F_k(\overline{V})\Big)$.

Now, the sequence $\Big\{F_k(A)\Big\}_{k=1}^{\infty}$ of annuli with each annulus lying in $\overline{V}-K(t,s)$ satisfies Theorem 5; hence, $mod(\overline{V}-K(t,s))=\infty$ as required.

Definition 11: (Limit set of a 3−chain of discs)

Let $\mathcal{H}=\Big\langle\phi,f,V,p,q,r\Big\rangle$ be a 3−chain of discs. Then, the limit set Λ of $\mathcal{H}$ is the closure of the grand orbit of p : that is $\Lambda=\overline{\mathcal{F}_{Points}}$.

Corollary 1: If $\mathcal{H}=\Big\langle\phi,f,V,p,q,r\Big\rangle$ is a 3−chain of discs then,

(i) $\overline{\mathcal{F}_{Points}}=\mathcal{F}_{Points}\cup\mathcal{I}_{Points}$.

(ii) Λ is invariant under ϕ and f. In particular, Λ is invariant under $\mathcal{F}_{Words}$.

Before proving Corollary 1 we make the following observation regarding a 3−chain of discs:

Let w_1 and w_2 be distinct maps in $\mathcal{F}_{Words}$. Then one and only one of the following holds:

(i) $w_1(\overline{V}) \cap w_2(\overline{V}) = \emptyset$

(ii) $w_1(\overline{V}) \cap w_2(\overline{V}) \in \mathcal{F}_{Points}$ and $w_1(V) \cap w_2(V) = \emptyset$

(iii) $w_1(\overline{V})$ is contained in the interior of $w_2(\overline{V})$ or vise versa

(iv) $w_1(\overline{V}) \subset w_2(\overline{V})$ (or $w_2(\overline{V}) \subset w_1(\overline{V})$) and $\partial(w_1(\overline{V})) \cap \partial(w_2(\overline{V})) \in \mathcal{F}_{Points}$.

Proof:

(i) We first prove that $\overline{\mathcal{F}_{Points}} \supset \mathcal{F}_{Points} \cup \mathcal{I}_{Points}$. With regard to Note 5 it is enough to show that for any $(t,s) \in T(\infty) \times S(\infty)$, $K(t,s)$ (cf. 2.6) is a limit point of $\mathcal{F}_{Points}$: for, let U be an open neighbourhood of $K(t,s) = \bigcap_{k=1}^{\infty} W\Big((t_i)_{i=1}^{k}, (n_i)_{i=1}^{k}\Big)(\overline{V})$. Then, as a consequence of Lemma 2, there is a positive integer m such that U contains the sequence of (distinct) nested compact sets $\Big\{W\Big((t_i)_{i=1}^{j}, (n_i)_{i=1}^{j}\Big)(\overline{V})\Big\}_{j=m}^{\infty}$. Hence, $\Big\{W\Big((t_i)_{i=1}^{j}, (n_i)_{i=1}^{j}\Big)(p)\Big\}_{j=m}^{\infty}$ is a sequence of distinct points (all lying in U) of $\mathcal{F}_{Points}$ converging to $K(t,s)$ as $m \to \infty$, as required.

For the converse assume that $z \in (\overline{\mathcal{F}_{Points}} - \mathcal{F}_{Points})$. Again, it is enough to consider those z in V (cf. Note 4) and we show that $z = K(t,s)$ for some $(t,s) \in T(\infty) \times S(\infty)$.

Let $\{z_m\}$ be a sequence of finite points converging to z : so each z_m is of the form $W((t_i(m))_{i=1}^{k}, (n_i(m))_{i=1}^{k})(p)$ for some $k \in \mathbb{N}$. Without loss of generality assume that $t_1(m) = 1$ for all positive integers m.

We claim that there is an infinite subsequence of $\{z_m\}$ such that

(a) $k > 1$ for all m

(b) there is a positive integer say n_1 such that $n_1 = n_1(m)$ for all m.

Proof:

(a) If $k = 1$ for infinitely many m then $z_m = p$ for infinitely many m which is a contradiction since $p \neq z$.

(b) If not each positive integer is assumed by at most a finite number of $n_1(m)$: hence, there exists an (increasing) infinite sequence $\{l_i\}$ of positive integers such that $n_1(l_i) \to \infty$ as $i \to \infty$. This implies that $z_{l_i} \to p$ as $i \to \infty$ since $g_1^{n_1(l_i)}(\overline{V}) \to p$ as $n_1(l_i) \to \infty$. This contradiction establishes the claim.

In fact we can say more: since $g_1^x \circ g_2(V) \cap g_1^y \circ g_2(V) = \emptyset$ for all positive integers $x \neq y$, and z is not a finite point, $n_1(m)$ is a constant say n_1 for all but finitely many m.

Now, let $\{Z_l(1)\}_{l\in\mathbb{N}}$ be a subsequence of $\{z_m\}$ satisfying the properties (a) and (b) above, and for a positive integer $N > 1$ let $S(N)$ be the statement that there is an infinite subsequence $\{Z_m(N)\}_{m\in\mathbb{N}}$ of $\{Z_l(N-1)\}_{l\in\mathbb{N}}$ such that

(i) $k > N$ for all m

(ii) there exists t_i, n_i where $i = 1, 2, \dots N$ such that each $Z_m(N)$ has the form $W((t_i)_{i=1}^N, (n_i)_{i=1}^N) \circ W_m(p)$ for some finite word W_m.

We prove by induction that $S(N)$ is true for all positive integers greater than one:

Proof:

Base case : $N = 2$.

(i) For a contradiction assume that $k = 2$ for infinitely many $\{Z_l(1)\}$. In this case we have two cases to consider: either $A = \{n_2(l) : l \in \mathbb{N}\}$ is unbounded or there is a maximum element of A. If A is unbounded then $z = g_1^{n_1}(q)$ which is a contradiction as z is not a finite point. On the other hand if A has a maximum element then a subsequence of $\{Z_l(1)\}$ converges to a finite point (since a subsequence of $\{Z_l(1)\}$ consists of a particular finite point) again a contradiction. Thus, we have established (i) when $N = 2$.

(ii) Now we pass to a subsequence of $\{Z_l(1)\}$ where $k > 2$: each point of this subsequence has the form $g_1^{n_1} \circ W((t_i(l))_{i=2}^{2+i}, (n_i(l))_{i=2}^{2+i})(p)$ for some positive integer i.

Now, using the same argument used in the proof of (b) above but this time with the sequence of points $\left\{W((t_i(l))_{i=2}^{2+i}, (n_i(l))_{i=2}^{2+i})(p)\right\}_{l \in \mathbb{N}}$ we find a positive integer say n_2 such that $n_2 = n_2(l)$ for infinitely many l. Thus, we obtain a subsequence $\{Z_m(2)\}$ of $\{Z_l(1)\}$ with the properties stated.

Induction step: assume that $S(N)$ is true for some integer $N > 1$. We show that $S(N + 1)$ is true. The proof is essentially analogous to the base case.

We first show that $k > N+1$ for infinitely many points of $\{Z_m(N)\}$. For a contradiction assume that $k = N+1$ for infinitely many points of $\{Z_m(N)\}$, and let $B = \{n_{N+1}(m) : m \in \mathbb{N}\}$. If B is unbounded then a subsequence of $\{Z_m(N)\}$ converges to a finite point which is a contradiction

If B has a maximum element then infinitely many points of $\{Z_m(N)\}$ equal to a single finite point which is a contradiction. So, we have shown that there is a subsequence of $\{Z_m(N)\}$ having the property (i).

Now consider a subsequence of $\{Z_m(N)\}$ having the property (i).

Each point of this subsequence has the form

$W((t_i)_{i=1}^N, (n_i)_{i=1}^N) \circ W((t_i(m))_{i=N+1}^{N+1+k}, (n_i(m))_{i=N+1}^{N+1+k})(p)$ for some positive integer k (which may depend on m) and for some finite word $W((t_i)_{i=1}^N, (n_i)_{i=1}^N)$ which is fixed for each m by definition of $\{Z_m(N)\}$.

We now apply the base case to the sequence

$\left\{W((t_i(m))_{i=N+1}^{N+1+k}, (n_i(m))_{i=N+1}^{N+1+k})(p)\right\}_{m\in\mathbb{N}}$ to obtain $\{Z_m(N+1)\}$ having the property (ii): note that this sequence already satisfies the property (i) by construction. This completes the proof of our claim.

Now, it follows from Lemma 2 that $\bigcap_{k=1}^{\infty} Z_k(k)(\overline{V})$ is a point: in fact, this point is z since, $Z_k(k)(p)$ converges to z as $k \to \infty$, as required.

(ii) Since, $\mathcal{F}_{Points}$ is invariant under ϕ and f it follows that Λ is also invariant under ϕ and f.

Note 5: The reasoning in the proof of Corollary 1(i) shows that if $z_0 \notin \Lambda$

then there exist an open neighbourhood O of z_0, and $n_0 \in \mathbb{N}$ so that for all $k \geq n_0$ we have that $O \cap W\Big((t_i)_{i=1}^k, (n_i)_{i=1}^k\Big)(\overline{V}) = \emptyset$ and $O \cap \phi \circ W\Big((t_i)_{i=1}^k, (n_i)_{i=1}^k\Big)(\overline{V}) = \emptyset$. In fact, we can find such an O and $n_0 \in \mathbb{N}$ so that if $m \geq n_0$ and $(t,s) \in T(m) \times S(m)$ then, $O \cap W\big(t,s\big)(\overline{V}) = \emptyset$ and $O \cap \phi \circ W\big(t,s\big)(\overline{V}) = \emptyset$

Proposition 1: Let $P(z,w) = 0$ be a correspondence, and $\mathcal{H} = \Big\langle \phi, f, V, p, q, r \Big\rangle$ be a 3$-$chain of discs with the limit set Λ. If

(i) each grand orbit of P intersects $\overline{V}$, and

(ii) the forward branch F_P of P restricted to $\overline{V}$ is the pair of maps $\{\phi f, \phi f^2\}$ while the backward branch F_P^{-1} of P restricted to $\phi(\overline{V})$ is the pair of maps $\{f\phi, f^2\phi\}$ then, Λ is the limit set of $P(z,w) = 0$ in the sense of Definition 2.

Proof:

It follows from (i) that, each grand orbit of P intersects $\phi f(\overline{V}) \cup \phi f^2(\overline{V})$; thus, each grand orbit of P intersects $V \cup \{p,q\}$. Now, let $z_0 \in V \cap \overline{\mathbb{C}} - \Lambda$. From Note 5, we can find $n_0 \in \mathbb{N}$ and an open neighbourhood O of z_0 so that whenever $m \geq n_0$, we have that $F_P^m(\overline{V}) \cap O = \emptyset$. Hence, the relation (1.8) is satisfied with $N = 2 + n_0$. This implies that the limit set of P is contained in Λ. In order to show that Λ is indeed the limit set of P it suffices to show that p is not in the regular set of P. For assume that p is in the regular set of P.

So, there is an open neighbourhood U of p, and a positive integer N so that the relation (1.8) holds. Now, let z be any point of $U \cap V$. Since, ϕf satisfies the shrinking condition at p there is a positive integer m such that if $n \geq m$ then $(\phi f)^n(\overline{V}) \subset U \cup \{p\}$. Now, considering the left hand side of (1.8), we have that $(z, (\phi f)^n(z)) \in F_P^n \cap (U \times U)$ for all $n \geq m$. But, $(z, (\phi f)^n(z))$ is not in the right hand side of (1.8) if $n \neq N$. This contradiction establishes that p is a limit point of P, and so is the grand orbit of p. This in turn shows that the closure of the grand orbit of p is the limit set of P : in other words, Λ is the limit set of P.

Lemma 3: If $\mathcal{H} = \left\langle \phi, f, V, p, q, r \right\rangle$ is a $3-$chain of discs then its limit set Λ is a topological circle.

We give two proofs of Lemma 3: one utilising a continued fractions argument (see Appendix A1), and the other using the following result.

Proposition 2: The modular group $\mathcal{M} = \left\langle z \stackrel{\psi}{\to} \frac{-1}{z}, z \stackrel{g}{\to} \frac{-1}{z+1}, U, \infty, 0, -1 \right\rangle$ is a $3-$chain of discs where $U = \{z \in \overline{\mathbb{C}} : Re(z) > \frac{1}{2} \text{ or } |z-1| < 1\}$.

Proof of Proposition 2: It is easy to see that $\mathcal{M}$ satisfies all the requirements of a $3-$chain of discs: the fact that ψg and ψg^2 satisfy the shrinking condition at ∞ and 0 respectively is done in Appendix (cf. A2).

Proof of Lemma 3: Our proof is based on homeomorphically identifying the limit points of $\mathcal{H}$ with the corresponding limit points (that is $\mathbb{R} \cup \{\infty\}$)

of $\mathcal{M} = \left\langle z \overset{\psi}{\to} \frac{-1}{z}, z \overset{g}{\to} \frac{-1}{z+1}, U, \infty, 0, -1 \right\rangle$ (cf. Proposition 2). Indeed, it suffices to show that there is a bijective mapping $h : \Lambda \to \mathbb{R} \cup \{\infty\}$ such that h^{-1} is continuous. First, since we are dealing here with limit sets of two 3−chain of discs we introduce the following notation: if $t = (t_i)_{i=1}^m \in T(m)$ and $s = (n_i)_{i=1}^m \in S(m)$ then we shall denote the composition (2.2) associated with $\mathcal{M}$ by $W_{\mathcal{M}}(t, s)$ where $g_1 = \psi g$ and $g_2 = \psi g^2$ while retaining the original notation of (2.2) for those pertaining to $\mathcal{H}$.

Now, for $k \in \mathbb{N}$, and $((t_i)_{i=1}^k, (n_i)_{i=1}^k) \in T(k) \times S(k)$ let h be the bijection given by

$$h(x) = \begin{cases} \bigcap_{k=1}^{\infty} W_{\mathcal{M}}\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{U}) & \text{if} \quad x = \bigcap_{k=1}^{\infty} W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{V}) \\ \infty & \text{if} \quad x = p \\ 0 & \text{if} \quad x = q \\ W_{\mathcal{M}}\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\infty) & \text{if} \quad x = W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(p) \\ \psi(\bigcap_{k=1}^{\infty} W_{\mathcal{M}}\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{U})) & \text{if} \quad x = \phi(\bigcap_{k=1}^{\infty} W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{V})) \\ \psi(W_{\mathcal{M}}\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\infty)) & \text{if} \quad x = \phi(W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(p)). \end{cases}$$

Note that $b = W_{\mathcal{M}}\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\infty)$ and $\psi(b)$ are rational numbers while $a = \bigcap_{k=1}^{\infty} W_{\mathcal{M}}\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{U})$ and $\psi(a)$ are irrational numbers.

We now proceed to show that $h^{-1} : \mathbb{R} \cup \{\infty\} \to \Lambda$ is continuous where there are two cases to consider:

Case 1: h^{-1} is continuous at points of the form $a = \bigcap_{k=1}^{\infty} W_{\mathcal{M}}\big((t_i)_{i=1}^{k}, (n_i)_{i=1}^{k}\big)(\overline{U})$.

Case 2: Continuity of h^{-1} at points having the form $b = W_{\mathcal{M}}\big((t_i)_{i=1}^{k}, (n_i)_{i=1}^{k}\big)(\infty)$.

Proof of Case 1:

Let O be an open neighbourhood of $h^{-1}(a)$. Then, (as seen in the proof of Lemma 2) there is $m \in \mathbb{N}$ such that $W\big((t_i)_{i=1}^{k}, (n_i)_{i=1}^{k}\big)(\overline{V}) \subset O$ for all $k \geq m$. Now, $\widehat{O}$, the interior of $W_{\mathcal{M}}\big((t_i)_{i=1}^{m+2}, (n_i)_{i=1}^{m+2}\big)(\overline{U})$ is an open neighbourhood of a and $h^{-1}(\widehat{O} \cap \overline{\mathbb{R}}) \subset O$ as required.

Proof of Case 2:

We first show that h^{-1} is continuous at ∞. For, let $\{z_n\}$ be a sequence of reals such that $z_n \to \infty$ as $n \to \infty$, and O_1 be an open neighbourhood of p. So, there is $m_1 \in \mathbb{N}$ such that whenever $k \geq m_1$ we have that $(\phi f)^k(\overline{V}) \cup \phi \circ (\phi f^2)^k(\overline{V}) \subset O_1$. Now, since $I = \overline{\mathbb{R}} \cap \Big((\psi g)^{m_1}(\overline{U}) \cup \psi \circ (\psi g^2)^{m_1}(\overline{U})\Big)$ is a closed interval containing the point at infinity, there is $m_2 \in \mathbb{N}$ such that if $n \geq m_2$ then $z_n \in I$: that is, each z_n lies in either of $(\psi g)^{m_1}(\overline{U})$ or $\psi \circ (\psi g^2)^{m_1}(\overline{U})$. Thus, if $n \geq m_2$ then $h^{-1}(z_n) \in (\phi f)^{m_1}(\overline{V}) \cup \phi \circ (\phi f^2)^{m_1}(\overline{V}) \subset O_1$ as required. Now, the continuity of h^{-1} at a point of the form $b = W_{\mathcal{M}}\big((t_i)_{i=1}^{k}, (n_i)_{i=1}^{k}\big)(\infty)$ is proved as follows: let $\{z_n\}$ be a sequence of reals converging to b as $n \to \infty$. Since, $\zeta = W_{\mathcal{M}}\big((t_i)_{i=1}^{k}, (n_i)_{i=1}^{k}\big)$ is a continuous map in $\overline{\mathbb{C}}$, $\{\zeta^{-1}(z_n)\}$ is a sequence converging to ∞ as $n \to \infty$. Now, as detailed in Note 3, each $\zeta^{-1}(z_n)$ is either a finite point or an infinite point of $\mathcal{M}$, so, $\{\zeta^{-1}(z_n)\}$ is a sequence of reals. Thus, using the continuity of h^{-1} at ∞, and the continuity of

$W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)$ at p we conclude that h^{-1} is continuous at b.

2.2.1 Sufficient conditions for the shrinking property

In our applications of 3−chains of discs (or n−chains of discs in general) we are concerned with maps of the form $z \mapsto a + (z-a) + b(z-a)^2 + \ldots$ for some constants a and b with $b \neq 0$. Such maps have just one attracting petal at a and the following two theorems set out the behaviour of the map on an attracting petal.

Theorem 6 ([1], P.122): Let $f(z) = z + az^{n+1} + bz^{n+2}\ldots$ be analytic in a neighbourhood of zero, and $a \neq 0$. Then in a neighbourhood of zero, f is conjugate to an analytic function of the form $g(z) = z - z^{n+1} + \mathbf{O}(z^{2n+1})$. That is, near 0, $g = \varphi f \varphi^{-1}$ for some univalent map φ with $\varphi(0) = 0$.

Theorem 7 ([1], P.116): Let $g(z) = z - z^{n+1} + \mathbf{O}(z^{2n+1})$ be analytic near zero. Then,

(a) for all sufficiently small $t > 0$, g maps each petal

$P(j,t) = \{re^{i\beta} : r^n < t(1+\cos(n\beta)),\ |\beta - \frac{2j\pi}{n}| < \frac{\pi}{n},\ r > 0\}$ to itself where $j \in \{0, 1, 2, \ldots, n-1\}$.

(b) $g^k(z) \to 0$ uniformly on each petal as $k \to \infty$.

(c) If $z \in P(j,t)$ then $arg(g^k(z)) \to \frac{2\pi j}{n}$ locally uniformly as $k \to \infty$ where $j \in \{0, 1, 2, \ldots, n-1\}$.

Remark 4: (1) For each petal $P(j,t)$, the ray with argument $\frac{2\pi j}{n}$ is called the axis of $P(j,t)$ or the attracting direction in $P(j,t)$.

It follows that the attracting directions of f in Theorem 6 are the rays with arguments $\theta_k = -\frac{arg(a)}{n} + \frac{\pi(1+2k)}{n}$ where $k = 0, 1, \ldots n-1$ ([5], P.40).

(2) Since each $P(j,t)$ is invariant under g it follows that $g(P(j,t))$ is also a petal for g at 0 with the same attracting direction as $P(j,t)$. Thus, by a petal of g (at 0) we mean any open domain satisfying the conclusion of Theorem 7.

Lemma 4: (Sufficient condition for shrinking property)

Let U be a Jordan domain, $a \in \partial U$ and $f : \overline{U} \to U \cup \{a\}$ be analytic on $\overline{U}$ with $f(a) = a$. Suppose further that f has no fixed points in U, and near a, f has the Taylor series $f(z) = z + b(z-a)^{N+1} + \mathbf{O}(z-a)$ where $N \in \mathbb{N}$, and $b \neq 0$. So, f has a parabolic fixed point at a, viz, $f'(a) = 1$.

If there exist a positive integer q, an open petal P of f at a and an open neighbourhood D of a such that $D \cap f^q(\overline{U}) \subset P \cup \{a\}$ then $\bigcap_{n\in\mathbb{N}} f^n(\overline{U}) = \{a\}$.

For the proof of Lemma 4 we need the following result:

Theorem 8 ([2]): Let G be a domain in $\overline{\mathbb{C}}$ and $f : G \to G$ be meromorphic. If the iterates of f form a normal family in G then one and only one of the following holds:

(a) There exists $w \in G$ such that $f(w) = w$ and $f^n(z) \to w$ locally uniformly as $n \to \infty$ for all $z \in G$. If $w \neq \infty$ then $| f'(w) | < 1$.

(b) $dist(\ f^n(z), \partial G\ ) \ \to \ 0$ locally uniformly in G as $n \to \infty$. That is, if K is a compact subset of G then there exists $l \in \mathbb{N}$ such that for all $n \geq l, n \in \mathbb{N}$ we have that $f^n(K) \cap K = \emptyset$. Here, $dist(.)$ is the usual spherical distance between two sets in $\overline{\mathbb{C}}$, and ∂G is the boundary of G.

(c) $f : G \to G$ is univalent and surjective in G.

Proof of Lemma 4:

We first note that $f : U \to U$ satisfies Theorem 8(b). Now, suppose that D, q and an open petal P exist so that $D \cap f^q(\overline{U}) \subset P \cup \{a\}$. Let K be the compact set $(\overline{\mathbb{C}} - D) \cap f^q(\overline{U})$. So, by Theorem 8 there is $n_0 \in \mathbb{N}$ such that whenever $n > n_0$ we have that $f^n(K) \cap K = \emptyset$: thus, $f^n(K) \subset D \cap P \cup \{a\}$. Now, using the uniform convergence of $\{f^k : k \in \mathbb{N}\}$ on P, we deduce that for all sufficiently large integers k, $f^k(\overline{U}) \subset D \cap (P \cup \{a\})$. This in turn shows that if $\widehat{D}$ is any open neighbourhood of a that is contained in D then for all large k, $f^k(\overline{U}) \subset \widehat{D} \cap (P \cup \{a\})$ as required.

Note 6: In the simplest situations the condition in Lemma 4 can be satisfied by establishing that near a, the boundary curve of U lies in $P \cup \{a\}$ for some petal P. In particular, if f has just one attracting petal and the boundary curve of U near a is smooth then a sufficient requirement for the shrinking

Fig. 2.1:

condition is that the tangent line to U at a is not parallel to the petal axis of f at a. In this regard we note that the following scenario (see fig. 2.2) is not a possibility even though the tangent line to U and the petal axis are not parallel: suppose that P is a petal of f at a where a is an interior point of $P \cup U$. Then, $P \cup U$ is a domain which is invariant under f. Hence, by Theorem 8(a) we conclude that a is an attracting fixed point of f, that is $|f'(a)| < 1$, which is a contradiction.

3. APPLICATION OF 3−CHAINS OF DISCS TO HOLOMORPHIC (2:2) CORRESPONDENCES

We now proceed to prove Theorem 4 using the notion of a 3−chains of discs. We first recall the statement of Theorem 4:

Theorem 4 (1)(a). Let $Q(z) = z^3 - 3z$. Then, there is a non-empty class $\mathcal{M}$ of Möbius involutions so that for each $\phi \in \mathcal{M}$, the $(2:2)$ correspondence $H_Q(z, \phi(w)) = 0$ which we denote by P_ϕ is quasi-fuchsian, say with the limit set Λ_ϕ.

(b). If Ω_ϕ is the component of $\overline{\mathbb{C}} - \Lambda_\phi$ containing the point at infinity then the action of P_ϕ on Ω_ϕ is Hecke conjugate[1] to the action of $PSL(2, \mathbb{Z})^2$ on the upper half plane $\{z \in \mathbb{C} : Im(z) > 0\}$.

(2) If $R(z) = \frac{z^2(z+b)}{(z+c)}$ with all its critical points lie on a line in $\overline{\mathbb{C}}$ then, there exists a non empty class of Möbius involutions $\mathcal{E}$ so that for each $\psi \in \mathcal{E}$ the correspondence $H_R(z, \psi(w)) = 0$ is quasi-fuchsian.

In this case, the action of $H_R(z, \psi(w)) = 0$ on any component of the

[1] cf. Definition 6.

[2] The group generated by $z \mapsto \frac{-1}{z}$ and $z \mapsto \frac{-1}{z+1}$.

complement of the limit set is not Hecke conjugate to a Kleinian group.

The scheme of the proof of Theorem 4(1)[3] is as follows:

Step 1.

Identify $\mathcal{M}$ so that for each $\phi \in \mathcal{M}$, $P_\phi(z,w) = 0$ satisfies the contact condition for some fundamental sets T_ϕ and T_Q (such that $\overline{T_\phi} \subset \overline{T_\phi}$) with contact points say, p and q. Recall that for such T_ϕ and T_Q we have:

(i) each grand orbit of $P_\phi(z,w) = 0$ intersects $\overline{T_\phi}$

(ii) $F_{P_\phi}(\overline{T_\phi}) \subset \overline{T_\phi}$ and $F^{-1}_{P_\phi}(\overline{T_\phi}) = \overline{\mathbb{C}}$ where $F_{P_\phi} = \phi \circ F_{H_Q}$ and $F^{-1}_{P_\phi} = F_{H_Q} \circ\ \phi$.

Step 2.

For each $\phi \in \mathcal{M}$ we show that there exists a Jordan domain V in $T_\phi \setminus [-2,2]$ such that:

(i) $\partial V \cap \partial T_\phi$ consists of the contact points $\{p,q\}$ of P_ϕ, and every grand orbit of P_ϕ intersects $\overline{V}$

(ii) The boundaries of V and T_ϕ are tangent to each other at p and q

(iii) $\Big\langle \phi, f, V, p, q, f(q) \Big\rangle$[4] is a 3−chain of discs

Before carrying out these steps we describe the consequences of the above construction.

[3] We shall implement a similar argument for the proof of Theorem 4(2).

[4] See Appendix A3 for how the limit set behaves when we replace of Q by an equivalent map in normal form.

Note that on such $\overline{V}$, $z \to F_{H_Q}(z)$ is a pair of univalent maps, $g(z) = \frac{-z+\sqrt{12-3z^2}}{2}$ and $f(z) = \frac{-z-\sqrt{12-3z^2}}{2}$: in fact, $z \mapsto f(z) = w$ and $z \mapsto g(z) = w$ are the solution of $H_Q(z,w) = 0 \iff z^2 + w^2 + zw - 3 = 0$ satisfying $f(f(z)) = g(z)$ and $f^3(z) = z$ for all $z \in \overline{V}$. We distinguish these solutions by letting $f(p) = q$ and $g(q) = p$. With this identification, we have that $f(\overline{V}) \cap g(\overline{V}) = \{f(q)\}, f(\overline{V}) \cap \overline{V} = \{q\}$ and $g(\overline{V}) \cap \overline{V} = \{p\}$.

Finally, using Proposition 1 (cf. Step 2) we conclude that the limit set of $P_\phi(z,w) = 0$ coincides with the limit set of $\Big\langle \phi, f, V, p, q, f(q) \Big\rangle$.

We remark that the forward branch F_{P_ϕ} of $P_\phi(z,w) = 0$ consists of the pair of maps $\{\phi f, \phi g\}$ while the backward branch $F^{-1}_{P_\phi}$ of $P_\phi(z,w) = 0$ is given by the pair of maps $\{f\phi, g\phi\}$.

Proof of Theorem 4(1):

Part (a).

Let the class of involutions be

$$\mathcal{M} = \left\{ z \mapsto \frac{(a+b)z - 2ab}{2z - (a+b)} : a^2+b^2-14ab+48 = 0, a \in (\mathbb{R}-[-1,1]), ab > 7 \right\}. \tag{3.1}$$

Note that if $\phi \in \mathcal{M}$ with $\phi(z) = \frac{(a+b)z-2ab}{2z-(a+b)}$ then ϕ has fixed points at a and b and for $a \in (\mathbb{R} - [-1,1])$ the other fixed point b for which $ab > 7$ is obtained by solving $a^2 + b^2 - 14ab + 48 = 0$ as follows:

$$b = \begin{cases} 7a + 4\sqrt{3}\sqrt{a^2-1} & \text{if} \quad 1 < a \leq 7 \\ 7a \pm 4\sqrt{3}\sqrt{a^2-1} & \text{if} \quad a > 7 \\ 7a - 4\sqrt{3}\sqrt{a^2-1} & \text{if} \quad -7 \leq a < -1 \\ 7a \pm 4\sqrt{3}\sqrt{a^2-1} & \text{if} \quad a < -7. \end{cases} \tag{3.2}$$

The condition $a^2+b^2-14ab+48=0$ imposes that the degree four polynomial $H_Q(\phi(z),z)=0$ has only two solutions say, p and q where each of which has multiplicity two. In fact, p and q are the contact points of $P_\phi(z,w)=0$ and specifically we let $p := (\frac{a+b}{8}, \frac{\sqrt{3(ab-7)}}{2})$ and $q := (\frac{a+b}{8}, -\frac{\sqrt{3(ab-7)}}{2})$.

We now construct the fundamental sets T_Q and T_ϕ for which $P_\phi(z,w)=0$ satisfies the contact condition with contact points p and q as follows.

Firstly, it is easy to see that

$$T_\phi = \left\{ z \in \mathbb{C} : |z - \frac{a+b}{2}| < |\frac{a-b}{2}| \text{ or } |z - \frac{a+b}{2}| = |\frac{a-b}{2}| \text{ where } Im(z) \geq 0 \right\} \tag{3.3}$$

is a fundamental set for ϕ and the boundary ∂T_ϕ of T_ϕ is given by $\{(x+iy) \in \mathbb{C} : x^2+y^2+ab-x(a+b)=0\}$.

As for T_Q, considering the three-to-one covering map

$$Q : \mathbb{C} - Q^{-1}(\{x : x \leq 2\} \cup \{\infty\}) \to \{x : x \leq 2\} \cup \{\infty\}$$

we find that $\mathbb{C} - Q^{-1}(\{x : x \leq 2\} \cup \{\infty\})$ is a union of three pairwise disjoint simply connected domains and one of which say D, has $[1,2]$ as one of its boundary curves.

It follows that $B := D \cup [1,2] \cup \{z \in Q^{-1}(\{x : x \leq -2\}) : Im(z) > 0\} \cup \{\infty\}$ is a fundamental set for Q and we let (with respect to the fixed point a)

$$T_Q = \begin{cases} B & \text{if } \; a > 1 \\ \{-z : z \in B\} & \text{if } \; a < -1. \end{cases} \tag{3.4}$$

The boundary of T_Q is as follows

$$\partial T_Q = \begin{cases} \{z = (x+iy) \in \mathbb{C} : 3x^2 - y^2 - 3 = 0, x \geq 1\} \cup \{\infty\} & \text{if } \; a > 1 \\ \{z = (x+iy) \in \mathbb{C} : 3x^2 - y^2 - 3 = 0, x \leq -1\} \cup \{\infty\} & \text{if } \; a < -1. \end{cases}$$

We remark that the boundaries of T_Q and T_ϕ are tangent to each other at p and q.

It now follows that $P_\phi(z,w) = 0$ satisfies **Step 1** with respect to the fundamental sets given in (3.3) and (3.4) and the set of contact points of $P_\phi(z,w) = 0$ is $\partial T_Q \cap \partial T_\phi = \{p, q\}$.

We now proceed to show that $P_\phi(z,w) = 0$ satisfies **Step 2**(ii) for the case where $a > 1$ and we omit the proof when $a < -1$ since the two proofs are analogous.

The construction of V is done under the following three cases arising as to whether 2 is outside, on the boundary or inside of T_ϕ.

Case 1. $a > 2$ with $b = 7a + 4\sqrt{3}\sqrt{a^2-1}$: so, $b > 2$.

Case 2. $a = 26$ and $b = 2$: this is the case if we let $b = 7a - 4\sqrt{3}\sqrt{a^2-1}$.

Case 3. (i). $1 < a < 2$ and $b = 7a + 4\sqrt{3}\sqrt{a^2 - 1}$. Here we have that $7 < b < 26$.

(ii). $7 < a < 26$ and $b = 7a - 4\sqrt{3}\sqrt{a^2 - 1}$, thus $1 < b < 2$.

In Case 1 and Case 2, we let $V = \phi(\overline{\mathbb{C}} - \overline{T_Q})$.

As for the Case 3, we consider only the part (i) and whose proof is analogous to that of part (ii).

We first note that for our choice of a and b as given in (3.2) we have that $\phi(-1) > 2$.

Now, there are three possibilities (A), (B) and (C) to consider:

(A). $\overline{\phi(\overline{\mathbb{C}} - T_Q)} \cap \{2\} = \emptyset$.

In this case, we let V be the interior of $\overline{\phi(\overline{\mathbb{C}} - T_Q)}$.

(B). 2 is on the boundary of $\overline{\phi(\overline{\mathbb{C}} - T_Q)}$.

Here, $\phi(F_{H_Q}(\overline{\phi(\overline{\mathbb{C}} - T_Q)}))$ is a union of two closed discs say, S_1 and S_1 (with Jordan curves as boundaries) whose intersection is $\{\phi(-1), \phi(f(q))\}$. Now, since $(S_1 \cup S_2) \subset \overline{T_\phi}$ and $2 \notin (S_1 \cup S_2)$ (2 is in the unbounded component of $\overline{\mathbb{C}} - (S_1 \cup S_2)$) we let V be any Jordan domain in $T_\phi \setminus [-2, 2]$ such that $(S_1 \cup S_2) - \{p, q\} \subset V, \partial V \cap \partial T_\phi = \{p, q\}$ and the boundary curves of V near p and q are tangent to the boundary curves of T_ϕ near p and q respectively.

(C). 2 is in the interior of $\overline{\phi(\overline{\mathbb{C}} - T_Q)}$: this occurs when $\phi(1) < 2$.

In this case, we claim that there is $n \in \mathbb{N}$ so that, $2 \notin (\phi \circ F_{H_Q})^n(\overline{\phi(\overline{\mathbb{C}} - T_Q)})$.

Note that if such an n exists then the claim is also true for any integer $m > n$.

For a contradiction assume that $2 \in (\phi \circ F_{H_Q})^n(\overline{\phi(\overline{\mathbb{C}} - T_Q)})$ for all $n \in \mathbb{N}$. This implies that for some single valued analytic branch say, β of $z \to F_{H_Q}(z)$ we have that $s_n = (\phi\beta)^n(\phi(1)) < 2$. Since, s_n is an increasing sequence of points it has a limit point say, $s \leq 2$. Now, if $s < 2$ then we have that $s = \phi(\beta(s))$: this implies that s is a solution of $H_Q(\phi(z), z) = 0$ which is a contradiction as the solutions of $H_Q(\phi(z), z) = 0$ are p and q.

On the other hand, if $s = 2$ then we consider $\phi(s_n) = \beta((\phi\beta)^n(\phi(1)))$ and obtain a contradiction as follows: since, $\phi(-1) > 2$, $\phi(s_n)$ tends to a limit $\mu > -1$ while $\beta((\phi\beta)^n(\phi(1)))$ tends to -1 which is a contradiction. Thus, there is an integer n for which $2 \notin (\phi \circ F_{H_Q})^n(\overline{\phi(\overline{\mathbb{C}} - T_Q)})$.

So, let V be any Jordan domain in $T_\phi \setminus [-2, 2]$ such that $(\phi \circ F_{H_Q})^n(\overline{\phi(\overline{\mathbb{C}} - T_Q)}) - \{p, q\} \subset V$, $\partial V \cap \partial T_\phi = \{p, q\}$ and the boundary curves of V near p and q are tangent to the boundary curves of T_ϕ near p and q respectively.

We now discuss **Step 2** (iii) of the proof where we show that ϕf and ϕg satisfy the shrinking condition (cf. Definition 7) at p and q respectively: this concludes the proof that $\left\langle \phi, f, V, p, q, r \right\rangle$ is a 3−chain of discs. Here, we consider only[5] the case of ϕf at p; it suffices to show (cf.Note 6) that the petal axis of ϕf at p is not parallel to the tangent to V at p.

First, by construction, the boundary of V at p and q is tangent to the

[5] The proof of the other case is analogous.

boundary of T_ϕ at p and q respectively. If χ is the angle (measured from the $X-$axis) of the tangent line to V at p then $\tan(\chi) = \frac{3(a+b)}{4\sqrt{3(ab-7)}} \neq 1, 0$.

We now show that if τ is the attracting direction (or the argument of the petal axis) of ϕf at p then $\tan(\tau) = \frac{-1}{\tan(\chi)}$: hence, the petal axis is perpendicular to the tangent to V at p. This, shows that ϕf satisfies the shrinking condition at p thereby completing the proof of **Step 2** (iii).

For consider the Taylor series of ϕf near p : that is,

$\phi f(z) = p + a_0(z-p) + a_1(z-p)^2 + \mathcal{O}(z-p)^3$.

Now, since p and q are the solutions (each of which is of order two) of $H_Q(\phi(z), z) = 0 \iff \phi(z)^2 + z^2 + z\phi(z) - 3 = 0$, it follows that $\phi f(z) - z$ has a zero of order two at p. Thus, $a_0 = 1$ and $a_1 \neq 0$, and by Remark 4(2) we have that $\tau = \pi - arg(a_1)$.

We now find an expression for $\tan(arg(a_1))$ as follows:

for $2a_1 = \phi''(q)(f'(p))^2 + f''(p)\phi'(q)$ and using the identities

$z^2 + f^2(z) + zf(z) - 3 = 0,\ \phi'(z) = \frac{(a+b)-\phi(z)}{2z-(a+b)},\ \phi''(z) = \frac{-4\phi'(z)}{2z-(a+b)},$

$f'(p) = \frac{-(f(p)+2p)}{2f(p)+p} = \frac{-(q+2p)}{2q+p}$ and $f''(p) = \frac{-2(1+f'(p)+(f'(p))^2)}{2f(p)+p} = \frac{-2(1+f'(p)+(f'(p))^2)}{2q+p}$

we obtain that $a_1 = \frac{-16^2 \times 8\mu^2(3(a+b)+i8\mu)}{(9(a+b)^2+64\mu^2)^2}$ where $\mu = \frac{\sqrt{3(ab-7)}}{2}$.

Thus, $\tan(\tau) = -\tan(arg(a_1)) = \frac{-8\mu}{3(a+b)} = \frac{-1}{\tan(\chi)}$ as required.

This completes the proof that $H_Q(z, \phi(w)) = 0$ is quasi-fuchsian.

Part (b):

First note that f and ϕ are univalent maps of Ω_ϕ onto itself (cf.Corollary 1(ii)) and let ψ be a Riemann map from Ω_ϕ to the upper half plane H_+. Thus, $\sigma = \psi \circ \phi \circ \psi^{-1}$ and $\rho = \psi \circ f \circ \psi^{-1}$ are Möbius maps of order two and three respectively, from H_+ onto itself. Now, consider the $(2:2)$ correspondence $G(z,w) = 0 \iff (w - \sigma\rho(z))(w - \sigma\rho^2(z)) = 0$. So, G represents $PSL(2,\mathbb{Z})$ as a correspondence, since, their grand orbits coincide.

It follows that ψ is a conjugacy from F_{P_ϕ} to $PSL(2,\mathbb{Z})$, compatible with ϕ and σ, in the sense of Definition 6, as required.

Proof of Theorem 4(2):

We highlight only the key points of the proof since the idea and the methodology are essentially the same as our previous case.

First, recall that $0, 1, \lambda = bc$ and ∞ are the critical points of $z \xrightarrow{R} \frac{z^2(z+b)}{z+c}$, where b and c satisfy the conditions $9c^2 + b^2 - 10bc \neq 0$ and $b + 3c + 2(1 + bc) = 0$. Now, if $\lambda \in \mathbb{C} - \{0, 1\}$ then there are two choices for c namely, $\frac{-(\lambda+1)+\sqrt{(\lambda^2-\lambda+1)}}{3}$ and $\frac{-(\lambda+1)-\sqrt{(\lambda^2-\lambda+1)}}{3}$ for which the critical points of R are $0, 1, \lambda$ and ∞. It is clear from the discussion on the normal forms, and Note 2 that it suffices to prove the theorem in the case where all the critical points of R are on $\overline{\mathbb{R}}$. In this case we have that $\lambda \in \mathbb{R} - \{0, 1\}$.
Out of two choices we have for c we consider $c = \frac{-(\lambda+1)+\sqrt{(\lambda^2-\lambda+1)}}{3}$, and the proof of the other case is analogous.
The scheme of the proof is as follows:

Step 1 (i) we find fundamental sets, $T_1(R), T_2(R)$ and $T_3(R)$ of R and corresponding classes of involutions, $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ so that if $\psi \in \mathcal{E}_i$ where $i = 1, 2, 3$ then $H_R(z, \psi(w)) = 0$ is of Hecke type correspondence with respect to the fundamental sets $T_i(R)$ and the fundamental set $T_i(\psi)$ of ψ.

Step 2

For each $H_R(z, \psi(w)) = 0$ as in step 1, if p and q are the contact points then we show that there is a domain V in $T_i(\psi)$ such that

(i) $\partial V \cap \partial T_i(\psi) = \{p, q\}$

(ii) each grand orbit of $H_R(z, \psi(w)) = 0$ intersects $\overline{V}$, and on $\overline{V}$, $z \mapsto F_{H_R}(z)$ is a pair of univalent maps say, f and g

(iii) if we set $f(p) = q$ and $g(q) = p$ then, $\left\langle \psi, f, V, p, q, f(q) \right\rangle$ is a 3−chain of discs.

Proof of Step 1(i)

Let $d = \sqrt{\lambda^2 - \lambda + 1}, \mu = d - (\lambda - 1)$ and $\delta = d + \lambda - 1$. Then, $R^{-1}(R(\lambda)) = \{\lambda, \mu\}$ and $R^{-1}(R(1)) = \{1, \delta\}$.

We construct T_R according to the values of λ viz, $\lambda < 0$, $0 < \lambda < 1$ and $\lambda > 1$.

Case 1: $\lambda < 0$

Here we have that $Q(1) < 0 < Q(\lambda)$ and $\lambda < -c < \delta < 0 < 1 < -b < \mu$. Now, $Q^{-1}(\overline{\mathbb{C}} - [-\infty, Q(\lambda)])$ consists of three simply connected domains, and let D_1 be the component of $Q^{-1}(\overline{\mathbb{C}} - [-\infty, Q(\lambda)])$ that intersects (μ, ∞). The boundary of D_1 consists of

$$\left\{(x,y)\in\mathbb{R}\times\mathbb{R} : y^2 = \frac{2x(x-1)(x-\lambda)}{-2x+c-b}, x>1\right\} \tag{3.5}$$

and $[1,\mu]$. Now, we let

$T_1(R) = D_1 \cup [1,\mu] \cup \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : y^2 = \frac{2x(x-1)(x-\lambda)}{-2x+c-b}, x>1, y>0\right\} \cup \{\infty\}.$

Case 2: $0<\lambda<1$

If $0<\lambda<1$ then $Q(1)<Q(\lambda)<0$ and $0<-c<\delta<\lambda<1<\mu<-b$. Now, if D_2 is the component of $Q^{-1}(\overline{\mathbb{C}}-[-\infty,0])$ that intersects $[-b,\infty]$ then let

$T_2(R) = D_2 \cup [1,-b] \cup \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : y^2 = \frac{2x(x-1)(x-\lambda)}{-2x+c-b}, x>1, y>0\right\} \cup \{\infty\}.$

Case 3: $1<\lambda$

When $1<\lambda$ we have that $0<-c<\mu<1<\lambda<\delta<-b$ and $Q(\lambda)<Q(1)<0$. Now, let D_3 be the component of $Q^{-1}(\overline{\mathbb{C}}-[-\infty,0])$ that intersects $[-b,\infty]$, and let

$T_3(R) = D_3 \cup [\lambda,-b] \cup \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : y^2 = \frac{2x(x-1)(x-\lambda)}{-2x+c-b}, x>\lambda, y>0\right\} \cup \{\infty\}.$

Finding the classes of involutions, $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$.

If $\psi(z) = \frac{(\alpha+\beta)z-2\alpha\beta}{2z-(\alpha+\beta)}$ then $H_R(z,\psi(z)) = 0$ is of the form $(z-p)^2(z-q)^2 = 0$ where $p \neq q$ if and only if α and β satisfy the relation $A(\beta)\alpha^2+B(\beta)\alpha+E(\beta)=0$ where $A(\beta) = \frac{3(2\beta+b-c)^2}{4}$, $E(\beta) = \frac{3(\beta(b-c)+2\lambda)^2}{4}$ and $B(\beta) = (-4d-2-2\lambda)\beta^2+\frac{2\beta}{3}(8d(1+\lambda)+(1+5\lambda+\lambda^2))-2\lambda(1+2d+\lambda)$. It is easy to see that $A(\beta)\alpha^2+B(\beta)\alpha+E(\beta)=0$ is symmetric with respect to α and β : so, if α_+ and α_-, where $\alpha_-<\alpha_+$ are the roots of $A(\beta)\alpha^2+B(\beta)\alpha+E(\beta)=0$

for some β then, β is a solution of $A(\alpha_+)\alpha^2 + B(\alpha_+)\alpha + E(\alpha_+) = 0$ and $A(\alpha_-)\alpha^2 + B(\alpha_-)\alpha + E(\alpha_-) = 0$.

We note that the discriminant $72\beta c(c^2-\lambda)(\beta-1)(\beta-\lambda)$ of $A(\beta)\alpha^2 + B(\beta)\alpha + E(\beta) = 0$ is positive for the following three cases:

(i) if $\lambda < 0$ and $\beta > 1$

(ii) if $0 < \lambda < 1$ and $\beta > 1$

(iii) if $\lambda > 1$ and $\beta > \lambda$.

Now, let

$\mathcal{E}_1 = \left\{ z \xrightarrow{\psi} \frac{(\alpha_+ + \beta)z - 2\alpha_+\beta}{2z-(\alpha_+ + \beta)} : A(\beta)\alpha_+^2 + B(\beta)\alpha_+ + E(\beta) = 0, 1 < \beta < \frac{c-b}{2} \right\}$,

$\mathcal{E}_2 = \left\{ z \xrightarrow{\psi} \frac{(\alpha_+ + \beta)z - 2\alpha_+\beta}{2z-(\alpha_+ + \beta)} : A(\beta)\alpha_+^2 + B(\beta)\alpha_+ + E(\beta) = 0, 1 < \beta < \frac{c-b}{2} \right\}$ and

$\mathcal{E}_3 = \left\{ z \xrightarrow{\psi} \frac{(\alpha_+ + \beta)z - 2\alpha_+\beta}{2z-(\alpha_+ + \beta)} : A(\beta)\alpha_+^2 + B(\beta)\alpha_+ + E(\beta) = 0, \lambda < \beta < \frac{c-b}{2} \right\}$.

Fundamental sets $T_1(\psi), T_2(\psi)$ and $T_3(\psi)$.

If $\psi(z) = \frac{(\alpha+\beta)z-2\alpha\beta}{2z-(\alpha+\beta)}$ is in $\mathcal{E}_i$ where $i = 1, 2, 3$ then let

$T_i(\psi) = \left\{ z \in \mathbb{C} : |z - \frac{\alpha+\beta}{2}| < |\frac{\alpha-\beta}{2}| \text{ or } |z - \frac{\alpha+\beta}{2}| = |\frac{\alpha-\beta}{2}|, Im(z) \geq 0 \right\}$.

Step 2

It follows that if $\psi \in \mathcal{E}_i$ then with respect to the fundamental sets $T_i(\psi)$ and $T_i(R)$, $H_R(z, \psi(w)) = 0$ is a Hecke type correspondence whose contact points are given by $p = \left(x, \sqrt{\frac{2x(x-1)(x-\lambda)}{-2x+c-b}}\right)$ and $q = \left(x, -\sqrt{\frac{2x(x-1)(x-\lambda)}{-2x+c-b}}\right)$ where $x = \frac{-(2\lambda-2\rho+\nu(b-c))}{4(2c+\nu)}$. Now, the construction of V is done in the same way as in the previous proof, and we now proceed to show that $\left\langle \psi, f, V, p, q, f(q) \right\rangle$ is a 3−chain of discs by showing that ψf satisfies the shrinking condition at

p. First, using the identities $f^2(z)(z+c)+f(z)(z+c)(z+b)+zc(z+b)=0$,

$\psi'(z)=\frac{\alpha+\beta-\psi(z)}{2z-(\alpha+\beta)}$, $\psi''(z)=\frac{-4\psi'(z)}{2z-(\alpha+\beta)}$ and $(\psi f)''(p)=\psi''(q)(f'(p))^2+\psi'(q)f''(p)$

we have $(\psi f)''(p)=\frac{-2}{(\alpha+\beta)^2-4\alpha\beta}\left\{2(\alpha+\beta-2(x-iy))-\frac{T}{(3x+b)(x+c)+y^2+\imath y(2x+b-c)}\right\}$

where $p=x+iy$ and

$T=c((\alpha^2+\beta^2+8(x^2-y^2)+6\alpha\beta)-(\alpha-\beta)^2(2x+b)-8(\alpha+\beta)(cx+y^2)$.

Now if θ_1 is the argument of the attracting direction of the petal of ψf at p

then $\theta=-arg((\psi f)''(p))$, and $\tan(\theta)=\frac{y\{4(t_1^2+t_2^2)+T(2x+b-c)\}}{Tt_1-2(\alpha+\beta-2x)(t_1^2+t_2^2)}$ where

$t_1=(3x+b)(x+c)+y^2$ and $t_2=y(2x+b-c)$.

Since the gradient to $\partial T_i(\psi)$ at p is $\xi=\frac{\alpha+\beta-2x}{2y}$ it follows that $\xi\tan(\theta)=-1$ which in turn shows that the petal axis is perpendicular to $\partial T_i(\psi)$ at p; so, ψf satisfies the shrinking condition at p. Likewise, one checks that ψg satisfies the shrinking at q, there by completing the proof that $\left\langle\psi, f, V, p, q, f(q)\right\rangle$ is a 3-chain of discs.

Comment 1: behaviour of $H_Q(z,\phi(w))=0$ when $ab<7$

The reason for the condition $ab>7$ in (3.1) is that if $ab<7$, then the "contact points" p and q are both real (provided that $a\in\mathbb{R}$ and $b\in\mathbb{R}$) since p and q are the solutions of $4z^2-z(a+b)+4(ab-6)=0$. In this case, we still have p and q as "parabolic fixed points" of the correspondence yet we do not obtain a 3−chain of discs, and the limit set of the correspondence is not a topological circle. This is why we have formulated the notion of Hecke correspondence (cf.Definition 5) in a rather restrictive manner.

We now discuss the behaviour of $H_Q(z, \phi(w)) = 0$ where the fixed points of ϕ are given by a and $b = 7a - 4\sqrt{3}\sqrt{a^2-1}$ [6] with $1 < a < 7$; so, $ab < 7$. First, when $a = 2$ we have that $b = 2$, which does not result in an involution, hence we have excluded this scenario. Let $p = \frac{(a+b)+4\sqrt{3}\sqrt{7-ab}}{8}$ and $q = \frac{(a+b)-4\sqrt{3}\sqrt{7-ab}}{8}$ be the solutions of $4z^2 - z(a+b) + 4(ab-6) = 0$: both p and q are real, and there is an analytic branch say f of F_{H_Q} that maps p to q. Likewise, let g be the other branch of F_{H_Q} and set $r = g(p)$.

One checks that $p \in (1, 2), r \in (-2, -1)$ and $q \in (-1, 1)$: so, q lies outside of the closure of T_Q (cf.(3.4) with $a > 1$) while p lies in the interior of T_ϕ (cf.(3.3) with $a > 1$). Now, let $D = \phi(F_{H_Q}(\overline{T_Q}))$. So, D is a closed Jordan disc whose boundary meets the positive real axis at $\phi(1)$ and $\frac{a+b}{2}$. Furthermore, D intersects every grand orbit of $H_Q(z, \phi(w)) = 0$, and by Definition 2 it follows that $\Lambda = \cap_{n=1}^{\infty}(\phi(F_{H_Q}(D)))^n(D) \cup \phi\Big(\cap_{n=1}^{\infty}(\phi(F_{H_Q}(D)))^n(D)\Big)$ is the limit set of $H_Q(z, \phi(w)) = 0$.

It is clear that Λ is compact, and $\Lambda \neq \emptyset$ since for instance the grand orbit of p is in Λ.

We show that $\Lambda \cap (\mathbb{R} \cup \{\infty\}) \subset (r, q) \cup (p, \phi(r))$.

We first note that $2 \in (\phi(F_{H_Q}(D)))^n(D)$ for each positive integer n. If not, there is $m \in \mathbb{N}$ so that 2 is not in $V = (\phi(F_{H_Q}(D)))^m(D)$. So, on V the branches of F_{H_Q}, f and g are univalent. So, $\phi \circ f$ maps V into the interior of

[6] The other case of b (cf. (3.2)) is similar.

V while fixing p. Thus, using Theorem 8 we obtain a contradiction that p is an attracting fixed point of $\phi \circ f$: by our construction $(\phi \circ f)^{'}(p) = 1$. Thus, we deduce that 2 is in every $(\phi(F_{H_Q}(D)))^n(D)$, and in particular $2 \in \Lambda$. Now, since $2 \in (\phi(F_{H_Q}(D)))^n(D)$ for each n, the boundary of $(\phi(F_{H_Q}(D)))^n(D)$ is a topological circle which cuts the positive real axis [7] at $(\phi f)^n(\phi(1))$ and $(\phi g)^n(\phi(1))$ where $(\phi f)^n(\phi(1)) < (\phi g)^n(\phi(1))$. As $p \in \Lambda$ we must have that $(\phi f)^n(\phi(1)) < p$. Now, consider the increasing sequence $z_n = (\phi f)^n(\phi(1))$ where $n \in \mathbb{N}$; since, $z_n = (\phi f)(z_{n-1})$, if z_n converges to a point $w \neq p$ then we have that $w = \phi f(w)$ which is a contradiction as p is the only fixed point of the correspondence that lies in T_ϕ. Thus, z_n converges to p, so, the interval $[1, p)$ is not in the limit set Λ. This implies that $[-2, r) \cup [1, q) \nsubseteq \Lambda$. A similar reasoning shows that $(\phi g)^n(\phi(1))$ converges to $\phi(r)$, thus the interval $(\phi(r), \frac{a+b}{2}]$ is outside of Λ.

Now, since $(\frac{a+b}{2}, \infty]$ is outside of D it follows that the interval $(\phi(r), \infty]$ does not intersect Λ. This in turn shows that $[-\infty, r)$ is outside of Λ, which concludes the proof.

We note that

(i) The above discussion shows that r, p, q and $\phi(r)$ are not interior points (if any) of Λ . Also, Λ contains non real points, for instance $f(\phi(r))$ and $g(\phi(r))$.

[7] Note that $1 < \phi(1) < 2$ and $\phi(-1) > 2$.

(ii) It is interesting to know whether Λ can have interior points, or indeed whether Λ is a cantor set.

Finally, we remark that the same situation occurs in Theorem 4(2) whenever the "contact points" are real with respect to the real fixed points: this occurs for instance by taking the root α_- in each of the cases we have studied.

4. GENERALISATION OF 3-CHAINS OF DISCS.

Here, we generalise the notion of a $3-$chain of discs to an $n-$chain of discs and show that its limit set is a topological circle. The limit points are defined in similar manner except there is somewhat a longer list of limit points arising from finite words; nonetheless, the picture one should keep in the mind is that of a $3-$chain of discs. Finally, we explore some examples of $(3:3)$ quasi-fuchsian correspondences.

4.0.2 $n-$chains of discs

Let V be an open disc, f and ϕ be univalent maps on $\overline{V}$, and $n \in \mathbb{N} - \{1, 2\}$ with the following properties:

(1) $\phi : \phi(\overline{V}) \to \overline{V}$ is univalent and $\phi^2(z) = z$ for all $z \in \overline{V} \cup \phi(\overline{V})$.

(2) for each $k \in \mathbb{N}$ such that $1 \leq k \leq n-1$, $f : f^k(\overline{V}) \to f^{k+1}(\overline{V})$ is univalent where $f^n(\overline{V}) = \overline{V}$ and $f^n(z) = z$ for all $z \in \bigcup_{i=1}^{n} f^i(\overline{V})$.

(3) there is $r_1 \in \partial V$, and if $k, l \in \mathbb{N}$ such that $1 \leq k, l \leq n-1$ then

$$\begin{cases} f^k(\overline{V}) \cap f^l(\overline{V}) = \emptyset & \text{if } |k-l| \neq 1 \\ \\ f(\overline{V}) \cap \overline{V} = \{r_1\} \\ \\ f^k(\overline{V}) \cap f^{k+1}(\overline{V}) = \{f^k(r_1)\} & \text{if } 1 \leq k \leq n-1. \end{cases}$$

We shall denote $f^k(r_1)$ by r_{k+1} where $1 \leq k \leq n-1$. Note that it follows from property (2) that $f(r_n) = r_1$.

Definition 12: We say that V, ϕ and f form an $n-$chain of discs if in addition to the above three properties the following properties are satisfied:

(4) $\phi(\overline{V}) \cap \overline{V} = \{r_1, r_n\}$ and $\phi(r_1) = r_n$. So, $\{r_1, r_n\} \subset \partial V$.

(5) $\left(\bigcup_{k=1}^{n-1} f^k(\overline{V})\right) - \{r_1, r_n\} \subset \phi(V)$.

(6) $\phi f : \overline{V} \to \overline{V}$ and $\phi f^{n-1} : \overline{V} \to \overline{V}$ satisfy the shrinking condition at r_n and r_1 respectively.

We denote this $n-$chain of discs by $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^n \right\rangle$.

We view an $n-$chain of discs being modelled on the Hecke group $H(n)$, that is, the group generated by $\psi(z) = \frac{-1}{z}$ and $g(z) = \frac{-1}{z+2cos\frac{\pi}{n}}$. For, if $U = \{z \in \overline{\mathbb{C}} : Re(z) > cos\frac{\pi}{n}\} \cup \ \{z \in \overline{\mathbb{C}} : |z - \frac{1}{2cos\frac{\pi}{n}}| < \frac{1}{2cos\frac{\pi}{n}}\}$,
$r_1 = 0$ and $r_{k+1} = g^k(0)$ where $1 \leq k \leq n-1$ then $\left\langle \psi, g, U, \left(r_i\right)_{i=1}^n \right\rangle$ is an $n-$chain of discs: it is easy to see that the properties from 1 to 5 are satisfied. We outline the proof of property 6 as follows: at $r_n = \infty$,
$\psi g(z) = z - 2z^2 cos\frac{\pi}{n} \ldots$; so, the petal axis is perpendicular to the imaginary

axis. Since, the boundary of U at ∞ is perpendicular the real axis we conclude that ψg satisfies the shrinking condition at ∞. The proof that ψg^{n-1} satisfies the shrinking condition at $r_1 = 0$ is similar.

Next, we define the limit set of an $n-$chain of discs in such a way that there is a choice of fundamental sets for the action of $H(n)$ on $\overline{\mathbb{C}}$ which provides an example of an $n-$chain of discs.

4.0.3 Limit set of an $n-$chain of discs

Finite words and finite points of an $n-$chain of discs

The concept of finite words and finite points of an $n-$chain of discs is reminiscent of what we have already seen in section (2.2), and before defining them we begin by introducing the notations as follows.

For $m \in \mathbb{N} \cup \{\infty\}$, let $(n_i)_{i=1}^m$ stand for the ordered $m-$tuple $(n_1, n_2, \ldots..)$. Now, let $S(m) = \{(n_i)_{i=1}^m : n_i \in \mathbb{N}\}$ and

$P(m) = \{(a_i)_{i=1}^m : a_i \in \mathbb{N}, a_i \neq a_{1+i} \text{ and } 1 \leq a_j \leq n-1 \quad \text{for } 1 \leq j \leq m\}$.

If $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^n \right\rangle$ is an $n-$chain of discs, $A_m = (a_i)_{i=1}^m \in P(m)$, $B_m = (n_i)_{i=1}^m \in S(m)$ and $g_i = \phi f^{a_i}$ where $1 \leq i \leq m$ then we define the finite word $W(A_m, B_m)$, associated with A_m and B_m by

$$W(A_m, B_m) = g_1^{n_1} \circ g_2^{n_2} \circ g_3^{n_3} \ldots \circ g_{m-1}^{n_{m-1}} \circ g_m^{n_m}. \tag{4.1}$$

Note that two such words $W(A_m, B_m)$ and $W(\hat{A}_m, \hat{B}_m)$ are equal if and only if $A_m = \hat{A}_m$ and $B_m = \hat{B}_m$.

Definition 13: (**Finite words and finite points**)
Suppose that $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^{n} \right\rangle$ is an $n-$chain of discs. We define its set of finite words $\mathcal{F}_{Words}$ by
$\{\phi(W(A_m, B_m)), W(A_m, B_m) : m \in \mathbb{N}, A_m \in P(m), \text{ and } B_m \in S(m)\}$ and the set of finite points $\mathcal{F}_{Points}$ by $\{h(r_1) : h \in \mathcal{F}_{Words}\}$.

We note that $\mathcal{F}_{Points}$ is the set of images of r_1 under all compositions of ϕ and f.

As in the case of a $3-$chain of discs we have the following Lemma whose proof we omit since it is essentially the same proof as of Lemma 2 .

Lemma 5: (**Infinite points**)

If $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^{n} \right\rangle$ is an $n-$chain of discs, $A_\infty = (a_i)_{i=1}^{\infty} \in P(\infty)$ and $B_\infty = (n_i)_{i=1}^{\infty} \in S(\infty)$ then

$$W(A_\infty, B_\infty) = \bigcap_{k=1}^{\infty} W\left((a_i)_{i=1}^{k}, \left(n_i\right)_{i=1}^{k}\right)(\overline{V}) \tag{4.2}$$

is a point, which we regard as an infinite point of $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^{n} \right\rangle$.

The set of infinite points $\mathcal{I}_{Points}$ of $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^{n} \right\rangle$ is defined as
$\{\phi(W(A_\infty, B_\infty)), W(A_\infty, B_\infty) : A_\infty \in P(\infty), B_\infty \in S(\infty)\}$.

We note that two infinite points $W(A_\infty, B_\infty)$ and $W(\hat{A}_\infty, \hat{B}_\infty)$ are equal if and only if $A_\infty = \hat{A}_\infty$ and $B_\infty = \hat{B}_\infty$.

Definition 14: (**Limit set**)

If $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^{n} \right\rangle$ is an $n-$chain of discs then its limit set Λ is defined to be $\overline{\mathcal{F}_{Points}}$, the closure of $\mathcal{F}_{Points}$.

As in the Corollary 1 the analogous statement for an $n-$chain of discs is:

Corollary 2: For an $n-$chain of discs the following holds:

(1) $\overline{\mathcal{F}_{Points}} = \mathcal{F}_{Points} \cup \mathcal{I}_{Points}$.

(2) Limit set is invariant under ϕ and f; in particular, the limit set is invariant under $\mathcal{F}_{Words}$.

Note 7: Analogous statements of Note 5 and Proposition 1 hold for an $n-$chain of discs; we omit these proofs as they are nothing more than re-tracing the same notions.

Likewise, the proof that the limit set of an $n-$chain of discs is a topological circle is done by constructing a homeomorphism h between $\Lambda = \overline{\mathcal{F}_{Points}}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ which is the limit set of the Hecke group,[1]
$H(n) = \left\langle z \xrightarrow{\psi} \frac{-1}{z}, z \xrightarrow{g} \frac{-1}{z+2cos\frac{\pi}{n}}, U, \left(q_i\right)_{i=1}^{n} \right\rangle$ (cf. section 4.0.2). Before

[1] We abuse the notation of Hecke group to denote both the Hecke group itself and the correspondence associated with Hecke group.

defining h, let us agree that, the expressions (4.1) and (4.2) that arise with respect to $H(n)$ be denoted by $W_H(A_m, B_m)$ and $W_H(A_\infty, B_\infty)$ respectively, while we retain the original notation for those expressions pertaining to $\left\langle \phi, f, V, (r_i)_{i=1}^n \right\rangle$.

Now, for $(A_m, B_m) \in \bigcup_{m\in\mathbb{N}} P(m) \times S(m)$, and $(A_\infty, B_\infty) \in P(\infty) \times S(\infty)$ define $h : \Lambda \to \overline{\mathbb{R}}$ as

$$h(x) = \begin{cases} W_H(A_\infty, B_\infty) & \text{if} \quad x = W(A_\infty, B_\infty) \\ q_n & \text{if} \quad x = r_n \\ q_1 & \text{if} \quad x = r_1 \\ W_H\big(A_m, B_m)(q_1) & \text{if} \quad x = W(A_m, B_m)(r_1) \\ \psi(W_H(A_\infty, B_\infty)) & \text{if} \quad x = \phi(W(A_\infty, B_\infty)) \\ \psi(W_H(A_m, B_m)(q_1)) & \text{if} \quad x = \phi(W(A_m, B_m)(r_1)). \end{cases}$$

By following the same argument of the proof of Lemma 3 we see that h is a homeomorphism and this in turn proves that:

Lemma 6: The limit set of an $n-$chain of discs is a topological circle.

4.1 Applications of n-chains of discs

As discussed in Note 2 (sec. 1.5.4), when studying correspondences of the form $H_Q(z, \phi(w)) = 0$ where Q is a rational maps of degree at least three, and ϕ is an involution, it suffices to consider a normal form of Q. Thus, the statements of the following results remain true for any rational map of the form $M \circ Q \circ K$ where M and K are Möbius maps.

The proofs of the following results follow a similar pattern to that of Theorem 4: so, we state claims without their proofs if those proofs are analogous in nature to what we have already done in the proof of Theorem 4.

Proposition 3: **(Hecke group case)**

(i) If n is an integer greater than or equal to three, and

$Q(z) = z^n$ then there is a class $\mathcal{I}$ of Möbius involutions so that for each $\phi \in \mathcal{I}$, $H_Q(z, \phi(w)) = 0$ is a quasi-fuchsian correspondence.

In fact, $H_Q(z, \phi(w)) = 0$ represents a conjugate of the Hecke group [2] $H(n)$, the group generated by $z \mapsto \frac{-1}{z}$ and $z \mapsto \frac{-1}{z+2cos\frac{\pi}{n}}$.

Proof: First note that,

$H_Q(z,w) = 0 \iff z^{n-1} + wz^{n-2} + w^2z^{n-3} \ldots + w^{n-2}z + w^{n-1} = 0$, and the forward branch $z \mapsto F_{H_Q}(z)$ of $H_Q(z,w) = 0$ is given by

$\{z\exp(\frac{2\pi ik}{n}) : k = 1, 2, \ldots, n-1\}$. So, if $f(z) = z\exp(\frac{2\pi i}{n})$ then $f^n(z) = z$ for all $z \in \mathbb{C}$. Thus, for a suitably chosen involution ϕ, the group generated by ϕ and f is conjugate to the Hecke group of order n. Indeed,

$\mathcal{I} = \left\{z \xrightarrow{\phi} \frac{(a+b)z-2ab}{2z-(a+b)} : a, b \in \mathbb{C}, (a+b)^2(1+\exp(\frac{2\pi i}{n}))^2 - 16ab\exp(\frac{2\pi i}{n}) = 0\right\}$

is such a collection of involutions:

[2] See (1.1) regarding representing $H(n)$ as a correspondence.

the condition $(a+b)^2(1+\exp(\frac{2\pi i}{n}))^2 - 16ab\exp(\frac{2\pi i}{n}) = 0$ imposes that ϕf (and ϕf^{n-1}) has just one fixed point, in fact, it is a parabolic fixed point. In particular, if a and b are positive then we can obtain an n-chains of discs as follows. We first choose fundamental sets T_ϕ and T_Q of $\phi \in \mathcal{I}$ and Q respectively so that $H_Q(z, \phi(w)) = 0$ satisfies the contact condition. For let $T_\phi = \left\{z \in \mathbb{C} : |z - \frac{a+b}{2}| < |\frac{a-b}{2}|\right\} \cup \left\{z \in \mathbb{C} : |z - \frac{a+b}{2}| = |\frac{a-b}{2}|, Im(z) \geq 0\right\}$ and $T_Q = \left\{z \in \mathbb{C} : -\frac{\pi}{n} < arg(z) \leq \frac{\pi}{n}\right\} \cup \infty$. So, the contact points of $H_Q(z, \phi(w)) = 0$ are $r_1 = \frac{a+b}{2\sec^2\frac{\pi}{n}} + i\frac{a+b}{2\sec^2\frac{\pi}{n}}\tan\frac{\pi}{n}$ and $r_n = \frac{a+b}{2\sec^2\frac{\pi}{n}} - i\frac{a+b}{2\sec^2\frac{\pi}{n}}\tan\frac{\pi}{n}$. These contact points are the fixed points of ϕf and ϕf^{n-1} respectively. It is easy to see that $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^{n} \right\rangle$ is an $n-$chain of discs where $V = \{z \in \mathbb{C} : |z - \frac{a+b}{2}| < |\frac{a-b}{2}|\}$, and $f^k(r_1) = r_{k+1}$ for $1 \leq k \leq n-1$: we sketch the proof that ϕf satisfies the shrinking condition at r_n. For, if β is the argument of the petal axis of ϕf at r_n then $\tan\beta = -\tan(\arg((\phi f)''(r_n))) = -\tan(\frac{4\pi}{n} - 3\arg(2r_1 - (a+b))) = \frac{1}{\tan\frac{\pi}{n}}$. So, the petal axis is perpendicular to the tangent line to V at r_n : this in turn proves the claim. In the same manner, one shows that ϕf^{n-1} satisfies the shrinking condition at r_1.

Theorem 9: (i) If $Q(z) = z^4 - 2a^2z^2$ where $a > 0$ then there exists a class of Möbius involutions $\mathcal{I}$, such that for each $\phi \in \mathcal{I}$, $H_Q(z, \phi(w)) = 0$ is a quasi-fuchsian correspondence.

(ii) If Λ_ϕ is the limit set of $H_Q(z,\phi(w)) = 0$, and Ω_ϕ is the component of $\overline{\mathbb{C}} - \Lambda_\phi$ containing the point at infinity then the action of $H_Q(z,\phi(w)) = 0$ on Ω_ϕ is Hecke conjugate[3] to the action of $H(4)$ on the upper half plane.

Note that Q has four critical points (at $\infty, 0, a$ and $-a$ where ∞ has order three while the other critical points have order one each) and three critical values at $0, -a^4$ and ∞.

We take Q as the normal form representing maps of the form $M \circ Q \circ N$ for Möbius maps M and N. Notice that $M \circ Q \circ N$ is not an even map in general, thus, $M \circ Q \circ N$ may assume four critical values.

Proof:(i) We first identify $\mathcal{I}$ and then for $z \xrightarrow{\phi} \frac{(\alpha+\beta)z-2\alpha\beta}{2z-(\alpha+\beta)}$ in $\mathcal{I}$ we construct a fundamental set T_ϕ of ϕ.

Now since $H_Q(z,w) = 0 \iff (z+w)(z^2+w^2-2a^2) = 0$, $z^2+\phi(z)^2-2a^2 = 0$ has exactly two roots, each with order two if and only if

$(\alpha+\beta)^2 - 8\alpha\beta + 8a^2 = 0$. So, we let

$$\mathcal{I} = \left\{ z \xrightarrow{\phi} \frac{(\alpha+\beta)z - 2\alpha\beta}{2z-(\alpha+\beta)} : (\alpha+\beta)^2 - 8\alpha\beta + 8a^2 = 0, \alpha \in (a,\infty) \cup (-\infty,-a), \alpha\beta > 3a^2 \right\} :$$

note that $\mathcal{I} \neq \emptyset$, for instance the involution with fixed points $\alpha > a$ and $\beta = 3\alpha + 2\sqrt{(2)}\sqrt{(\alpha^2-a^2)}$ is in $\mathcal{I}$. If $\phi \in \mathcal{I}$ then let its fundamental set be $T_\phi = \left\{ z : |z - \frac{\alpha+\beta}{2}| < |\frac{\alpha-\beta}{2}| \right\} \cup \left\{ z : |z - \frac{\alpha+\beta}{2}| = |\frac{\alpha-\beta}{2}|, Im(z) \geq 0 \right\}$.

We now construct T_Q a fundamental set for Q so that $H_Q(z,\phi(w)) = 0$ is

[3] cf. Definition 6.

of Hecke type if $z \xrightarrow{\phi} \frac{(\alpha+\beta)z-2\alpha\beta}{2z-(\alpha+\beta)}$ is in $\mathcal{I}$: for if D is the component of $Q^{-1}\{\mathbb{C}-(-\infty,0]\}$ that intersects $(a\sqrt{2},\infty)$, and

$E = D \cup \{\infty\} \cup [a, a\sqrt{2}] \cup \{x+iy : x^2-y^2 = a^2, x > a, y > 0\}$ then let

$$T_Q = \begin{cases} E & \text{if} \quad \alpha \in (a,\infty) \\ \{z : -z \in E\} & \text{if} \quad \alpha \in (-\infty,-a). \end{cases}$$

It follows that the contact points of $H_Q(z,\phi(w)) = 0$ are $r_4 = \frac{\alpha+\beta}{4} + i\frac{\sqrt{\alpha\beta-3a^2}}{\sqrt{2}}$ and $r_1 = \frac{\alpha+\beta}{4} - i\frac{\sqrt{\alpha\beta-3a^2}}{\sqrt{2}}$.

The next step is to show that (see the proof of Theorem 4) there is a Jordan domain V such that on $\overline{V}$, $z \mapsto F_{H_Q}(z)$ consists of three univalent maps f, g and h where $f(r_4) = r_1, g = f^2$ and $h = f^3$: hence, f^4 is the identity map on $\overline{V}$. For, if $\alpha \in [a\sqrt{2},\infty) \cup (-\infty,-a\sqrt{2}]$ then let V be the interior of T_ϕ. On the other hand, if $\alpha \in (a, a\sqrt{2})$ (or $\alpha \in (-a\sqrt{2},-a)$) then there is $n \in \mathbb{N}$ so that, $a\sqrt{2} \notin (\phi \circ F_{H_Q})^n(\overline{\phi(\mathbb{C}-E)})$ (or $-a\sqrt{2} \notin (\phi \circ F_{H_Q})^n(\overline{\phi(\mathbb{C}-\widehat{E})})$). For such V, it follows that $\left\langle \phi, f, V, \left(r_i\right)_{i=1}^n \right\rangle$ is an $n-$chain of discs where $r_2 = f(r_1)$ and $r_3 = f^2(r_1)$: among the properties concerning an $n-$chain of discs we sketch the proof that ϕf satisfies the shrinking condition at r_4. For, since $(\phi f)''(r_4) = \frac{2(a^2-r_1r_4)}{r_4r_1^2}$, if τ is the argument of the petal axis of ϕf at r_4 then $\tan\tau = \frac{-4\sqrt{\alpha\beta-3a^2}}{\sqrt{2}(\alpha+\beta)}$. Now, if θ is the angle made by the tangent line to V at r_4 then we have that $\tan\theta = \frac{-1}{\tan\tau}$; so, the petal axis of ϕf is not tangential to V at r_4, as required. One shows in a similar manner that ϕf^2 satisfies the shrinking condition at r_1. Thus, we have shown that $H_Q(z,\phi(w)) = 0$ is a quasi-fuchsian correspondence.

Finally, we omit the proof of part (ii) since it is analogous to the proof of Theorem 4.

Comment 2.

As in the case of Theorem 4 (cf. Comment 1), when $\alpha\beta < 3a^2$ the limit set of the resulting correspondence is not a topological circle, and its limit set is given by $\Big(\cap_{n=1}^{\infty}(\phi\circ F_{H_Q})^n(\overline{T_Q})\Big)\cup\phi\Big(\cap_{n=1}^{\infty}(\phi\circ F_{H_Q})^n(\overline{T_Q})\Big)$.

5. APPENDIX (INCLUDING COMMENTS AND CONJECTURES)

Here we pose some questions and conjectures which are of some interest. For some conjectures we propose a strategy together with the obstacles that needed to overcome in order to make our argument into a proof. In this regard, where necessary we will make certain definitions, and state relevant results for clarity and completeness.

- **Hecke type correspondences**

As we have seen, in some cases if Q is a rational map of degree at least three then there is a class of involutions $\mathcal{I}$ so that if $\phi \in \mathcal{I}$ then $H_Q(z, \phi(z))$ has a factor $(z-p)^2(z-q)^2$ where $p \neq q$. Note that this is certainly the case if Q is of degree three or Q is of degree four and an even map, that is $Q(z) = Q(-z)$ for all z. In general, it is not clear whether such a class of involutions exists for a given Q. If such an $\mathcal{I}$ exists then we were able to identify a subset $\mathcal{J}$[1] of $\mathcal{I}$ so that whenever $\phi \in \mathcal{J}$, $H_Q(z, \phi(w)) = 0$ is

[1] So, the cases mentioned in Comments 1 and 2 do not occur.

a Hecke type correspondence (cf.Definition 5), and indeed a quasi-fuchsian correspondence whose proof is based on the notion of an $n-$chain of discs. The drawback of our methodology of showing that a correspondence is quasi-fuchsian is twofold: first there is a geometric condition[2] where we require the existence of a pair of fundamental sets T_Q and T_ϕ for Q and ϕ respectively so that $\partial T_Q \cap \partial T_\phi$ consists of the two contact points, and the other being rather technical in nature concerning the shrinking condition. It seems that the existence of such fundamental sets is the hardest to check in practice. Recall from Comment 1 and 2 that the existence of $\mathcal{I}$ does not guarantee that the correspondence is quasi-fuchsian. So, it is of interest to know whether every Hecke type correspondence is a quasi-fuchsian correspondence: failing this means that either there is no open set V as mentioned in the proof of Theorem 4 or the correspondence fails to satisfy the shrinking condition.

- $\alpha-$**plane**

Keeping the same set up as above, let the $\alpha-$plane $A_\alpha \subset \mathbb{C}$ be defined by

$\beta \in A_\alpha$ if and only if there is $\phi_\beta \in \mathcal{I}$ such that $\phi_\beta(\beta) = \beta$, and

$H_Q(z, \phi_\beta(w)) = 0$ is quasi-fuchsian: we denote this correspondence by P_β, and its limit set by Λ_β. In the cases we have considered it was shown that A_α is a union of open intervals of the real line; so, the question is that if $\beta \in A_\alpha$ is given can one "generate" an open (in $\mathbb{C}$) neighbourhood U of β so that

[2] cf. Definition 5.

$U \subset A_\alpha$?

Conjecture 1: A_α is an open set in $\mathbb{C}$.

We outline two possible approaches for proving this conjecture:

Approach 1. For each $\beta \in A_\alpha$ there corresponds a Hecke type correspondence which is equipped with a pair of fundamental sets satisfying Definition 5. Since, each contact point is given by an analytic function of β, if one can perturb the original pair of fundamental sets so as to retain the conditions of Definition 5 then the resulting correspondence is of Hecke type. Now, using the fact that petal axis moves analytically on β we deduce that the so obtained correspondence is a quasi-fuchsian correspondence: this shows that A_α is open in $\mathbb{C}$.

Approach 2. If $\beta \in A_\alpha$ then there is an open set V (such as in the proof of Theorem 4 or 9) such that every grand orbit of $P_\beta(z, w) = 0 \iff H_Q(z, \phi_\beta(w)) = 0$ intersects $\overline{V}$. Now, put an ellipse field say μ on $\overline{V}$: the correspondence P_β then distributes μ to an ellipse field of $\overline{\mathbb{C}}$. Then, there is a quasi conformal map h_β on $\overline{\mathbb{C}}$ that straightens μ into a circle field. Let F_β and F_β^{-1} be the forward and backward branches of P_β, and consider $g_\beta = h_\beta \circ F_\beta \circ h_\beta^{-1}$ and $g_\beta^{-1} = h_\beta \circ F_\beta^{-1} \circ h_\beta^{-1}$. It follows that g_β and g_β^{-1} are forward and backward branches of a correspondence say, G_β. So, G_β and P_β are topologically conjugate to each other, and a branch of $g_\beta \circ g_\beta^{-1} \circ g_\beta$ is an involution say ψ_β. Now, at this stage one requires to show the following:

(i). $G_\beta(z,w) = 0 \iff H_Q(z, \psi_\beta(w)) = 0$. Note that it is important that we have the "same Q". Indeed, we have already noted[3] that if $H_Q(z, \phi(w)) = 0$ is quasi-fuchsian then so is $H_{M \circ Q \circ K}(z, K^{-1} \circ \phi \circ K(w)) = 0$ where K and M are Möbius maps.

(ii). assuming that (i) holds, as μ varies the set of corresponding β gives rise to an open (in $\mathbb{C}$) neighbourhood of β.

Here we have to ascertain that we find a class of "new" involutions other than what we have already obtained through our construction of Hecke type correspondences.

Conjecture 2: If $\beta \in A_\alpha$ then there is a neighbourhood D of β so that for each $\gamma \in D$, Λ_γ is a quasi conformal image of Λ_β.

The above assertion is true if A_α is open in $\mathbb{C}$. For if A_α is open and $\beta \in A_\alpha$ then there is a neighbourhood D_1 of β such that $D_1 \subset A_\alpha$. Now, if p_β is the contact point that corresponds to r_n (see Definition 12) in P_β then there is a neighbourhood $D \subset D_1$ of β, and an analytic map $h : D \to \overline{\mathbb{C}}$ such that $h(\beta) = p_\beta$. For each $\gamma \in A_\alpha$ let W_γ denote a generic expression in (4.1) and $\mathcal{F}_\gamma$ be the set of finite points of P_γ : so, each element of $\mathcal{F}_\gamma$ is of the form $W_\gamma(p_\beta)$ or $\phi_\gamma \circ W_\gamma(p_\beta)$, and the closure of $\mathcal{F}_\gamma$ is the limit set Λ_γ. Then the map

[3] cf. Appendix A3.

$$f : D \times \mathcal{F}_\beta \to \overline{\mathbb{C}} \text{ given by } (z,x) \mapsto \begin{cases} W_\gamma \circ h(z) & \text{if } x = W_\beta(p_\beta) \\ \\ \phi_\gamma \circ W_\gamma \circ h(z) & \text{if } x = \phi_\gamma \circ W_\beta(p_\beta) \end{cases}$$

is a holomorphic motion[4] of $\mathcal{F}_\beta$. So, by $\lambda-$ Lemma it follows that if $\gamma \in D$ then Λ_γ is a quasi conformal image of Λ_β.

The following conjecture is particularly interesting if the correspondence is Hecke conjugate as in Definition 6:

Conjecture 3: The limit set Λ_ϕ of $H_Q(z, \phi(w)) = 0$ is a quasi-circle: that is, Λ_ϕ is a quasi-conformal image of $\mathbb{R} \cup \{\infty\}$.

Conjecture 4: A result analogous to Theorem 4(2) holds if the critical points of R do not lie on a line in $\overline{\mathbb{C}}$.

Appendix

A1 (Another proof of Lemma 3): Let $[n_1; n_2, n_3, \ldots]$ stand for the continued fraction

$$n_1 + \cfrac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4} \cdots}}$$

where $n_1 \in \mathbb{N} \cup \{0\}$ and $n_i \in \mathbb{N}$ for all $i \geq 2$. An infinite continued fraction, that is when there are infinitely many n_i's, converges to an irrational number, and each irrational number can be expressed as a unique continued fraction [9]. On the other hand, a finite continued fraction $[n_1; n_2, n_3, \ldots, n_k]$ is a

[4] See the Appendix A4 for the definition and the $\lambda-$ Lemma.

rational number and vise versa. Moreover, each rational number has two different continued fraction expressions namely,

$[n_1; n_2, n_3, \ldots, n_k, 1] = [n_1; n_2, n_3, \ldots, n_k + 1]$.

Now, let $\mathcal{H} = \left\langle \phi, f, V, p, q, r \right\rangle$ be a 3−chain of discs with limit set Λ. Using the usual notation of (2.2) with $g_1 = \phi f$ and $g_2 = \phi f^2$, and that of Lemma 2, define $h : \mathbb{R} \cup \{\infty\} \to \Lambda$ as follows:

$$h(x) = \begin{cases} p & \text{if } x = \infty \\ \bigcap_{k=1}^{\infty} W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{V}) \ \text{ where } t_1 = 1 & \text{if } x = [n_1; n_2, \ldots] \\ & \ \text{where } n_1 \neq 0 \\ \bigcap_{k=1}^{\infty} W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{V}) \ \text{ where } t_1 = 2 & \text{if } x = [0; n_1, n_2, \ldots] \\ (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ \ldots \circ (\phi f)^{n_{2k-1}} \circ (\phi f^2)^{n_{2k}}(p) & \text{if } x = [n_1; n_2, \ldots, n_{2k}] \\ (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ \ldots \circ (\phi f^2)^{n_{2k}} \circ (\phi f)^{n_{2k+1}}(q) & \text{if } x = [n_1; n_2, \ldots, n_{2k+1}] \\ q & \text{if } x = 0 \\ \phi\Big(\bigcap_{k=1}^{\infty} W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{V})\Big) \ \text{ where } t_1 = 1 & \text{if } x = -[0; n_1, n_2, n_3, \ldots] \\ \phi\Big(\bigcap_{k=1}^{\infty} W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(\overline{V})\Big) \ \text{ where } t_1 = 2 & \text{if } x = -[n_1; n_2, n_3, \ldots] \\ & \ \text{where } n_1 \neq 0 \\ \phi\Big((\phi f^2)^{n_1} \circ (\phi f)^{n_2} \circ \ldots \circ (\phi f^2)^{n_{2k-1}} \circ (\phi f)^{n_{2k}}(q)\Big) & \text{if } x = -[n_1; n_2, \ldots, n_{2k}] \\ \phi\Big((\phi f^2)^{n_1} \circ (\phi f)^{n_2} \circ \ldots \circ (\phi f)^{n_{2k}} \circ (\phi f^2)^{n_{2k+1}}(p)\Big) & \text{if } x = -[n_1; n_2, \ldots, n_{2k+1}]. \end{cases}$$

It is clear that h is well defined at irrational numbers, and we show that h is well defined at rational numbers. For let $x = [n_1; n_2, \ldots, n_{2k}]$. If $n_{2k} \neq 1$

then x can also be written as $[n_1; n_2, \ldots, n_{2k} - 1, 1]$. So,

$$\begin{aligned} h([n_1; n_2, \ldots, n_{2k} - 1, 1]) &= (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ \ldots \circ (\phi f^2)^{n_{2k}-1} \circ (\phi f)(q) \\ &= (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ \ldots \circ (\phi f^2)^{n_{2k}-1} \circ (\phi f)(f(p)) \\ &= (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ \ldots \circ (\phi f^2)^{n_{2k}-1} \circ (\phi f^2)(p) \\ &= (\phi f)^{n_1} \circ (\phi f^2)^{n_2} \circ \ldots \circ (\phi f^2)^{n_{2k}}(p) \\ &= h([n_1; n_2, \ldots, n_{2k}]) \text{ as required.} \end{aligned}$$

The cases where $n_{2k} = 1$ or $x = [n_1; n_2, \ldots, n_{2k+1}]$ are dealt with analogously. Furthermore, it is clear that h is a bijective map. We now show that h is a continuous map as follows:

Case 1 continuity of h at irrational points.

Let $x = [n_1; n_2, n_3, \ldots,]$ and without loss of generality assume that $n_1 \neq 0$. We first note that for any $k \geq 1$, $[n_1; n_2, \ldots, n_{2k+1}] < x < [n_1; n_2, \ldots, n_{2k}]$. Moreover, $\{[n_1; n_2, \ldots, n_{2k+1}]\}_k^\infty$ is an increasing sequence converging to x while $\{[n_1; n_2, \ldots, n_{2k}]\}_k^\infty$ is a decreasing sequence converging to x (see [9] for example). Now, let U be an open neighbourhood of $h(x)$. Then (as seen in the proof of Lemma 2) there is some k_0 so that if $k \geq k_0$ then $W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(V)$ is an open neighbourhood of $h(x)$ where $W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(V) \subset U$. We show that there is an open neighbourhood (a, b) of x such that $h((a, b)) \subset U$. Indeed, let $a = [n_1; n_2, n_3, \ldots, n_{2k_0+1}]$ and $b = [n_1; n_2, n_3, \ldots, n_{2k_0}]$. Since, every $y \in (a, b)$ has a continued fraction expansion of the form $[n_1; n_2, n_3, \ldots, n_{2k_0}, \ldots]$ we conclude that $h(y) \in U$.

Case 2. continuity of h at rational points.

It suffices to show that h is continuous at ∞, since if x is a rational number then $h(x)$ is a finite point which is a continuous image of p. In fact, $h(x)$ is given by $W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(p)$ or $\phi(W\big((t_i)_{i=1}^k, (n_i)_{i=1}^k\big)(p))$.

Now, let U be an open neighbourhood of p. Then, by the shrinking condition at p and q, there is $k_0 \in \mathbb{N}$ so that whenever $n \geq k_0$, $(\phi f)^n(\overline{V}) \subset U$ and $\phi \circ (\phi f^2)^n(\overline{V}) \subset U$. Now, if $y \in (k_0, \infty)$ then y has a continued fraction of the form $[n; \ldots..]$ where $n \geq k_0$. So, $h(y) \in (\phi f)^n(\overline{V}) \subset U$. Likewise, if $y \in (-\infty, -k_0)$ then y has a continued fraction of the form $-[n; \ldots..]$ where $n \geq k_0$, and so $h(y) \in \phi \circ (\phi f^2)^n(\overline{V}) \subset U$. These two facts together show that h is continuous at ∞ as claimed.

Remark 5: To generalise the above proof for an $n-$chain of discs with $n > 3$, one would need to generalise the notion of continued fractions.

A2 (Shrinking condition for Proposition 2) We note that it is easy to see that the properties 1 to 5 of Definition 8 are satisfied by $\mathcal{M}$, and we show that the shrinking condition is satisfied. For at infinity, $\psi(g(z)) = z+1$ has the Taylor series $z - z^2 + z^3 + \ldots$ which is valid near 0. So, the argument of the petal axis of ψg near ∞ is $\pi - arg(-1) = 0$: hence, the petal axis is perpendicular to the imaginary axis which is tangent to U at ∞ and this proves that ψg satisfies the shrinking condition at ∞.

Similarly, one can show that ψg^2 satisfies the shrinking condition at 0.

A3

Let $\left\langle \phi, f, V, p, q, f(q) \right\rangle$ be the 3−chain of discs associated with the rational map Q (such as in the proof of Theorem 4).

Now, if $R = K \circ Q \circ M$ for some Möbius maps K and M then $\left\langle M^{-1} \circ \phi \circ M, M^{-1} \circ f \circ M, M^{-1}(V), M^{-1}(p), M^{-1}(q), M^{-1}(f(q)) \right\rangle$ is the 3−chain of discs associated with R. The situation is the same if Q is of degree $n > 3$, and in that case we would be dealing with an n−chain of discs. Thus, it suffices to consider a normal form in studying Hecke type correspondences.

A4 (Holomorphic motion)

Definition 15: [6, 7] Let $A \subseteq \overline{\mathbb{C}}$ be any non empty set and D be the open unit disc[5] in $\overline{\mathbb{C}}$. A holomorphic motion of A with respect to the base point 0 is a map $f : D \times A \to \overline{\mathbb{C}}$ such that

(i) for any fixed $a \in A, f(z, a)$ is analytic in z. That is $f : D \times \{a\} \to \overline{\mathbb{C}}$ is analytic.

(ii) for any fixed $z_0 \in D, f : \{z_0\} \times A \to \overline{\mathbb{C}}$ is injective. Thus,

$f(z_0, w_1) = f(z_0, w_2) \Rightarrow w_1 = w_2$.

(iii) $f : \{0\} \times A \to \overline{\mathbb{C}}$ is the identity map on A. That is $f(0, w) = w$ for all

[5] In practice we can replace D by any other domain.

$w \in A$.

Theorem 10 ([6, 10]): Let $f : D \times A \to \overline{\mathbb{C}}$ be a holomorphic motion of A. Then,

(i) f has a unique extension F in the sense that $F : D \times \overline{A} \to \overline{\mathbb{C}}$ is a holomorphic motion of $\overline{A}$ and the restriction of F to $D \times A$ is identical to f.

(ii) for each $z_0 \in D, F(z_0, w) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quasi-conformal with respect to w. In particular, for any $z_0 \in D, F(z_0, w) : \overline{A} \to \overline{\mathbb{C}}$ is a quasi-conformal map in w.

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