Certain one-relator products of groups: Freiheitssatz and non-triviality

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Abstract

Combinatorial group theory is an aspect of group theory that deals with groups given by presentations. Many techniques both geometric and algebraic are used in tackling problems in this area. In this work, we mostly adopt the geometric approach. In particular we use pictures and clique-pictures to describe various properties of certain one-relator product of groups. These groups are our object of study and they have presentations of the form

$$G = \frac{G_\Lambda}{N(w)},$$

where G_Λ is a free product of groups $G_{\lambda\in\Lambda}$ and w is a (cyclically reduced) word in the free product G_Λ with length at least two.

It is known that most results about G are obtained by putting some conditions on the G_λ's, on w, or both. In the case where $\Lambda = 2$, say $\Lambda = \{1, 2\}$, there are two forms of conditions we put on w.

The first condition is that $w = r^n$, where r is cyclically reduced word of length at most eight and $n > 0$. More precisely suppose A and B non-trivial groups and $G = (A * B)/N(r^n)$. Suppose that any of the following conditions holds:

1. $\ell(r) = 8$ and $n \geq 3$; or
2. $2 \leq \ell(r) < 8$ and $n \geq 2$; or
3. $n = 1$ and $\ell(r) = 4$ say $r = abcd$, with $\langle a, c\rangle$ and $\langle b, d\rangle$ isomorphic to 2-generator subgroups of $PSL_2(\mathbb{C})$.

We show that at least one of the following holds:

i. A and B embed in G naturally and r has order n in G.

ii. $n = 2$ and up to conjugation r has the form $r = axbx^{-1}cz$ with $z^2 = 1$, where $a, b, c \in A$ and $x, z \in B$.

We then deduce that r has order n in G.

The other condition is that $w = r^n$ as before but we require also that r is a word in $\langle a\rangle * \langle UbU^{-1}\rangle$, where $a, b \in G_1 \cup G_2$ and U is a word in $G_1 * G_2$. We prove that a minimal clique-picture over G satisfies the small-cancellation condition $C(6)$ under the condition that r has no element of order two or r has length at least two in the generators $\{a, UbU^{-1}\}$ and whenever both a and b belong to same factor, say G_1, then either the subgroup of G_1 generated by $\{a, b\}$ is cyclic or $\langle a\rangle \cap \langle b\rangle = 1$. From this we deduce the following results.

1. Let $\mathcal{H}$ be the quotient of $\langle a\rangle * \langle UbU^{-1}\rangle$ by the normal closure of R^n. Then G_1, G_2 and $\mathcal{H}$ all embed in G via the natural maps.

2. If the word problems are soluble for $\mathcal{H}$, G_1 and G_2, then it is soluble for G.

3. No proper cyclic sub-word of w is trivial in G. In particular r has order n in G.

We also consider the case where $\Lambda = 3$, say $\Lambda = \{1, 2, 3\}$, and prove the following results.

1. Suppose G_1, G_2 and G_3 are all finite cyclic groups, then each embed in G via the natural maps provided w has non-zero exponent sum in each of the generators.

2. If w has length at most eight, then G is non-trivial.

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Introduction

For some indexing set Λ, let G_Λ be the free product of groups G_λ, where $\lambda \in \Lambda$. The object of study in this thesis falls under the class of groups of the form

$$G = \frac{G_\Lambda}{N(w)},$$

otherwise known as a one-relator product of groups G_λ, with relator w.

Such groups have been widely studied for various reasons. One of such reasons follows from the fact that G generalises one-relator groups. Hence one can try to see how much the rich theory of one-relator groups can be generalised.

On the other hand many problems in combinatorial group theory and low dimensional topology are indeed problems about one-relator products. For us the motivation to study one-relator products comes from a simple construction in topology known as Dehn surgery. As the name implies, this construction is due to Max Dehn in 1910, motivated by the works of French mathematician Henri Poincaré.

Poincaré at the end of a long article [80] asked the following seemingly innocuous question:

> Is every closed 3-manifold with trivial fundamental group topologically equivalent to the 3-dimensional sphere S^3?

An affirmative answer to the above question was known as Poincaré conjecture, now a theorem of Perelman (see [11]). For us the interesting bit is that this was not the first formulation of the problem. The earlier version was in terms of homology rather than fundamental group, of which Poincaré found a counter-example himself. Dehn [17] then invented the Dehn surgery construction to show that there are in fact an infinite family of homology spheres which are all counter-examples to the earlier version of this problem.

Dehn surgery on a knot K in S^3 is the following. Remove a regular neighbourhood $N(K)$ of K from S^3 to get the manifold $S^3(K) \cong S^3 - N(K)$; glue in a solid torus using some boundary homeomorphism to get the surgered manifold $S^3_s(K) \cong S^3(K) \cup_s (S^1 \times D^2)$, where $s \in \mathbb{Q} \cup \{\infty\}$, also known as surgery *slope*, is the isotopy class of essential simple closed curves on the boundary of $N(K)$. The surgered manifold $S^3_s(K)$ determined up to homeomorphism by s. The knot K can be replaced by a link, in which case the surgery is done on each component. In this setting, an important result due to Lickorish and Wallace [[68], [94]] states that every closed, orientable, connected 3-manifold is obtained by Dehn surgery. There have been hundreds of publications on this subject alone. To learn more about history and scope, we refer the reader to [35].

If we restrict to the case where K is a knot, a basic question to ask is what the possibilities for $S^3_s(K)$ can be. Even more basic is the question of when $S^3_s(K)$ is *reducible* (i.e. decomposes as non-trivial connected sum or homeomorphic to $S^1 \times S^2$ (in the orientable case)). The first non-trivial examples of this phenomenon were provided by Moser, who showed in [78] that

$$S^3_{pq}(T_{p,q}) \cong L(p,q)\#L(q,p),$$

where $T_{p,q}$ is a (p,q)-torus knot and $L(p,q), L(q,p)$ are lens spaces. Gabai showed in [30] that $S^1 \times S^2$ occurs only when K is the unknot U and $s = 0$. It follows that any non-trivial reducible $S^3_s(K)$ decomposes as a connected sum. It is also known that $S^3_s(K)$ can be reducible if K is a cabled knot [36]. If $C_{p,q}$ with $p > 1$, is a (p,q)-cable of a knot C, then

$$S^3_{pq}(C_{p,q}) \cong L(p,q)\#S^3_{q/p}(C).$$

These are the only known examples where the phenomenon occurs. Of course one can think of $T_{p,q}$ as a (p,q)-cable of U. This motivates the following conjecture due to Gonzalez-Acuna and Short [34].

Conjecture 0.0.1. *(Cabling Conjecture) If K is a non-trivial knot in S^3 and $S^3_s(K)$ is reducible, then $K = C_{p,q}$ and $s = pq$.*

A lot is known in support of an affirmative answer to this conjecture. For example it known that the conjecture is true for symmetric knots [[47], [46], [25]], knots with low bridge-number [[42], [48], [85], [59], [97]], alternating knots [77], arborescent knots [96], satellite knots [86] and strongly invertible knots [37].

Also it is known that if $S^3_s(K)$ is reducible, then s must be an integer [[38], [7], see also [34]], and $S^3_s(K)$ contains a lens space summand. Moreover when $S^3_s(K)$ is the connected sum of two lens spaces, then K is either a torus knot or a cable of a torus knot [40]. Other results by Howie [58] and Valdez Sánchez [93] imply that three is the upper bound for the number of summands, and if three summands occur, two must be lens spaces and the third is an integer homology sphere.

Conjecture 0.0.2. *(Two-summand Conjecture) If K is a non-trivial knot in S^3 and $S^3_s(K)$ is reducible, then $S^3_s(K) \cong M_1\#M_2$, with M_1 and M_2 irreducible.*

Clearly, the Cabling conjecture implies the Two-summand conjecture. The latter is known to be true for slice knots [76], knots with bridge number at most five and knots with positive braid closures [97].

In particular both conjectures lead to questions about one-relator products of groups with two or three factors. Since knot groups have single weight, a question of when such groups are trivial arises. This thesis studies these groups under various conditions on the factors, or the relator, or both.

0.1 Layout

There are seven chapters: Chapters 1, 2 and 7 are the three minor chapters while Chapters 3, 4, 5 and 6 are the major chapters.

In Chapter 1, we mention a few basic concepts which will be used throughout this thesis. In particular we define pictures which is our main geometric tool for proving various results.

Chapter 2 contains a brief review of groups with one-relator presentations. We mention some important existing results about them.

In Chapter 3, we consider the one-relator group $G = (G_1 * G_2)/N(R^n)$, where R is a word in the free product $\langle a \rangle * \langle UbU^{-1} \rangle$ for some letters $a, b \in G_1 \cup G_2$ and word $U \in G_1 * G_2$. We show that if $n > 1$ and R has no letter of order 2, or $\ell(R) > 2$ as a cyclically reduced word in the free product $\langle a \rangle * \langle UbU^{-1} \rangle$ and (a, b) is admissible, then the following holds.

1. The natural maps $G_1 \to G$, $G_2 \to G$ and $\mathcal{H} \to G$ are all injective, where $\mathcal{H}$ is the quotient of $\langle a \rangle * \langle UbU^{-1} \rangle$ by $N(R^n)$.
2. If the word problems are soluble for $\mathcal{H}$, G_1 and G_2, then it is soluble for G.
3. No proper cyclic subword of R^n is trivial in G. In particular R has order n in G.

We do this using (clique-)pictures which we shall also describe in this chapter.

In Chapter 4, we consider the one-relator product $G = (A * B)/N(r^n)$ under the following conditions:

1. $\ell(r) = 8$ and $n \geq 3$; or
2. $2 \leq \ell(r) < 8$ and $n \geq 2$; or
3. $n = 1$ and $\ell(r) = 4$ say $r = abcd$, with $\langle a, c \rangle$ and $\langle b, d \rangle$ isomorphic to 2-generator subgroups of $PSL_2(\mathbb{C})$.

We show that either:

i. A and B embed in G naturally and r has order n in G, or

ii. $n = 2$ and up to conjugation r has the form $r = axbx^{-1}cz$ with $z^2 = 1$, where $a, b, c \in A$ and $x, z \in B$.

We then deduce that r has order n in G.

In Chapter 5, we consider a one-relator product G of three finite cyclic groups G_a, G_b and G_c with generators a, b, and c respectively, and relator w. We show that if the exponent sum of each of the three generators in w is non-zero (modulo their respective orders), then the natural maps $G_a \to G$, $G_b \to G$ and $G_c \to G$ are all injective.

In Chapter 6, we apply various techniques and results from previous chapters to show that a one-relator product of three non-trivial groups is non-trivial under the condition that the relator has free product length at most 8.

Chapter 7 is the concluding chapter. It contains the summary of this work as well various open problems related or arising from this work.

Chapter 1

Preliminaries

1.1 Preamble

In this Chapter we define some basic concepts that are relevant in this thesis. More specific definitions shall be made in the Chapters where they are first used. We begin with the concept of free groups and group presentations.

1.2 Free groups and Group presentations

Combinatorial group theory is mostly about group presentations. We mention a few things about free groups and group presentations. Our account is based on [65].

Definition 1.2.1. A group $F(X)$ is said to be *free* on $X \subseteq F(X)$ if, given any group G and any map $\phi : X \to G$, there is a unique homomorphism $\phi' : F(X) \to G$ extending ϕ. We call X the *basis* and $|X|$ the *rank* of $F(X)$.

Details about existence and properties of $F(X)$ can be found in [65]. By the above definition, we know that given any group G and a subset X of G such that every element in G is a word in $X \cup X^{-1}$, the inclusion map ϕ of X into G extends to a homomorphism ϕ' from $F(X)$ onto G. Hence it follows from the *First isomorphism theorem* that

$$G \simeq \frac{F(X)}{K}, \tag{1.1}$$

where K is the kernel of ϕ'. This tells us that every group is the quotient of some free group. It also leads to the idea of group presentation.

Definition 1.2.2. Let X be a set and R, a subset of $F(X)$. A (free) presentation of a group G is a pair $\wp = \langle X \mid R \rangle$, such that

$$G = \frac{F(X)}{N(R)}, \tag{1.2}$$

where $N(R)$ is the normal closure of R in $F(X)$. Henceforth, we shall use G instead of $\wp$.

In Definition 1.2.1, the elements of X and R are called the generators and relators of G respectively. We say that G is *finitely generated* if $|X| < \infty$. If in addition $|R| < \infty$, we say G is *finitely presented*.

The following proposition tells us when a mapping between two groups can be extended to a group homomorphism.

Proposition 1.2.3. *Given groups $G = \langle X \mid R \rangle$ and $H = \langle Y \mid S \rangle$, a mapping $\phi : X \longrightarrow H$ extends to a homomorphism $\phi' : G \longrightarrow H$ if for all $x \in X$ and $r \in R$, the result of substituting $\phi(x)$ for x in r yields a word in $N(S)$.*

1.3 Algorithmic problems

Although presentations are nice, sometimes it is hard to tell some basic properties of a group by presentation. In 1912 Max Dehn listed three problems about finitely presented groups. Let G be a group given by a finite presentation

$$G = \langle X \mid R \rangle. \tag{1.3}$$

1. (Word problem) Is there an algorithm which decides whether or not any given word in G is the identity?

2. (Conjugacy problem) Is there an algorithm which decides whether or not any pair of words in G are conjugate?

3. (Isomorphism problem) Is there an algorithm which decides whether or not any pair of finite presentations define isomorphic groups?

For this thesis, we shall focus only on the word problem. The works of Novikov, 1952, 1955, and, independently, Boone, and Britton, both in 1958, proved that the word problem has no general solution. In particular they produced a finitely presented group in which there is no effective way of showing that a word in the given generators is trivial or not. However, the word problem is known to have a solution for certain classes of groups. These includes free groups [[73], Corollary 1.2.2], one-relator groups [[73], Theorem 4.14], automatic groups [[24], Theorem 2.3.10], small cancellation groups [[39],[70],[92]], residually finite groups and word hyperbolic groups. In particular for word hyperbolic groups, the algorithm that solves the word problem is extremely efficient.

1.3.1 Tietze transformations

It easy to see that any group has infinitely many different presentations. In 1906, Tietze gave a way to move from one presentation to another of the same group. Let G be a group given by a finite presentation

$$G = \langle X \mid R \rangle. \tag{1.4}$$

(T1) Addition of a relation: If $r = 1$ in G, then replace R with $R \cup \{r\}$.

(T2) Removal of a relation: If $r \in R$ and $N(R - \{r\}) = N(R)$, then replace R with $R - \{r\}$.

(T3) Addition of a new generator: Replace X with $X \cup \{t\}$ and R by $R \cup \{tw^{-1}\}$ for some $t \notin X$ and word w in $F(X)$.

(T4) Removal of a generator: Replace X with $X - \{x\}$ and R by $R - \{r\}$ where $r = xw^{-1}$ if w and all other words in R do not contain $x^{\pm 1}$.

The transformations T1-T4 are called *Tietze transformations.* Tietze showed that any two presentations of the same group are connected by repeated application of Tietze transformations.

1.3.2 Nielsen Equivalence

Let F be a free group on X and U a finite subset of F. We think of U as an n-tuple $(u_1, u_2, \ldots, u_n)$ of elements of F.

Definition 1.3.1. An elementary Nielsen transformation of U is one of the following three types: for some i, $1 \leq i \leq n$,

(T0) delete u_i if $u_i = 1$,

(T1) replace u_i by u_i^{-1},

(T2) replace u_i by $u_i u_j$, $j \neq i$

and leave u_k fixed for all $k \neq i$. A Nielson transformation is a finite sequence of elementary Nielsen transformations.

Definition 1.3.2. Two subsets U and V of F are called Nielsen equivalent if U is carried into V by some Nielsen transformation τ. In such a case we write $V = U_\tau$.

Lemma 1.3.3. *The subgroups of F generated by U and U_τ are equal, for any Nielsen transformation τ.*

A generating set $U = \{u_1, u_2, \ldots, u_n\}$ is called *Nielsen reduced* if the following conditions are satisfied for every u_i, u_j and u_k in U.

1. $1 \notin \{u_i, u_j, u_k\}$
2. $u_i u_j \neq 1$ implies $\ell(u_i u_j) \geq \ell(u_i), \ell(u_j)$
3. $u_i u_j \neq 1$ and $u_j u_k \neq 1$ implies $\ell(u_i u_j u_k) > \ell(u_i) - \ell(u_j) + \ell(u_k)$

The point here is that Nielsen reduced sets are free generators for the subgroup they generate.

1.4 Free Product, Free Product with Amalgamation and HNN extension

In this section we study the definitions, some properties and theorems of free products, free product with amalgamation and HNN extensions. The reference for this section is [70]. We begin with free products.

1.4.1 Free Products

Definition 1.4.1. Let $A_1, A_2, \ldots, A_n$ be non-trivial groups for some $n \in \mathbb{N}$. Suppose that $A_\lambda = \langle X_i \mid R_i \rangle$ and where $X_i \cap X_j = \emptyset$ for $i \neq j$. The *free product*, $*A_i = A_1 * A_2 * \ldots * A_n$, of groups A_i is the group

$$*A_i = \langle \cup_{i=1}^n X_i \mid \cup_{i=1}^n R_i \rangle. \tag{1.5}$$

The groups A_i are called the *factors* of $*A_i$. It is known that free product is independent of the choice of presentations for A_i.

Definition 1.4.2. (Normal Form) Let G be a group with trivial pairwise intersecting subgroups $A_{\lambda \in \Lambda}$, for an indexing set Λ. A *reduced sequence* (or *normal form*) is a sequence $g_1, g_2, \ldots, g_k$, with $k \geq 0$, of elements of $\cup_{\lambda \in \Lambda} A_\lambda$ such that each g_i is in one of A_λ, and if $i > 1$ and $g_i \in A_\lambda$ then $g_{i-1}, g_{i+1} \notin A_\lambda$.

Definition 1.4.3. (Reduced Word) A *reduced word* in a group G is a word of the form $g_1 g_2 \ldots g_k$, for some reduced sequence $g_1, g_2, \ldots, g_k$ in G. A word is said to be *cyclically reduced*

Note that it follows from Definition 1.4.2 that in a reduced word $w = g_1 g_2 \ldots g_k$ with $k > 1$, no g_i is trivial.

Definition 1.4.4. (Cyclically Reduced Word) A word $w = g_1 g_2 \ldots g_k$ a group G said to be *cyclically reduced* if it is reduced and g_1, g_k do not belong to same A_λ (using the notation in Definition 1.4.2).

Theorem 1.4.5. *(Normal Form Theorem for Free Products) In a free product $A*B$, the following statements are equivalent.*

1. *If $w = g_1g_2\dots g_k$ where $k > 0$ and $g_1, g_2, \dots, g_k$ is a reduced sequence, then $w \neq 1$ in $A * B$.*

2. *Each $w \in A * B$ can be uniquely expressed as a product $w = g_1g_2\dots g_k$ where $g_1, g_2, \dots, g_k$ is a reduced sequence.*

The Normal Form Theorem allows us to have a notion of length for elements in a free product.

It also follows from the Normal Form Theorem that the word and conjugacy problems are solvable for $A * B$ if the factors are finitely generated and these problems are solvable in them. Also elements of finite order in $A * B$ are conjugates of finite order elements in the factors.

Definition 1.4.6. (Length) A word w in has *length* k if its normal form is of the form $g_1g_2\dots g_k$. We denote the length of w by $\ell(w)$.

Lemma 1.4.7. *Let A and B be subgroups of a group G such that $A \cup B$ generates G, $A \cap B = \{1\}$, and no reduced word in G is equal to 1. Then $G = A * B$.*

Next we state two of the most important theorems about free products. The first is due to Grushko (1940) and Neumann (1943), while the second is due to Kurosh (1934).

Theorem 1.4.8. *Let F be a free group, and let $\phi : F \to *A_i$ be a homomorphism of F onto $*A_i$. Then there is a factorisation of F as a free product, $*F_i$, such that $\phi(F_i) = A_i$*

However most applications of Theorem 1.4.8 uses the following corollary of it.

Corollary 1.4.9. *If $G = A_1 * A_2 * \dots * A_k$ and the minimum number of generators of A_i is n_i, then the minimum number of generators for G is $n_1 + n_2 + \dots + n_k$.*

Theorem 1.4.10. *Let $G = *A_i$, and let H be a subgroup of G. Then H is a free product, $H = F * (*H_j)$ where F is a free group and each H_j is the intersection of H with a conjugate of some factor A_i of G.*

1.4.2 Free Product with Amalgamation and HNN extensions

Free product with amalgamation was introduced by Schreier (1926), while HNN extensions was a joint work G, Higman, B. H. Neumann and N. Neumann in 1949.

These two constructions have the same basic idea. Both involves two subgroups and an isomorphism between them. To describe these constructions, let $G = \langle X \mid R\rangle$ and $H = \langle Y \mid S\rangle$ be groups with subgroups $A \subseteq G$ and $B \subseteq H$, such that there exists an isomorphism $\phi : A \to B$.

Definition 1.4.11. The *free product of G and H, amalgamating the subgroups A and B* is the group

$$G *_{A=B} H = \langle X,\ Y \mid R,\ S,\ a = \phi(a),\ a \in A\rangle. \tag{1.6}$$

When A and B are the trivial groups, we get the free product of G and H. Another way to think of $G *_{A=B} H$ is as the quotient of $G * H$ by $N(\{a\phi^{-1}(a)\ :\ a \in A\})$. When the isomorphism between A and B is known, the shorter notation $G *_A H$ is preferred to $G *_{A=B} H$.

Definition 1.4.12. A sequence $c_1, c_2, \ldots, c_k$, where $k \geq 0$, of elements of $G * H$ will be called *reduced* if:

1. Each c_i is in one of the factors G or H.
2. Successive c_i, c_{i+1} come from different factors.
3. If $k > 1$, then no c_i is in A or B.
4. If $k = 1$, then $c_1 \neq 1$.

Theorem 1.4.13. *(Normal Form Theorem for Free Products with Amalgamation) If $c_1, c_2, \ldots, c_k$, where $k \geq 1$ is a reduced sequence, then $c_1c_2 \ldots c_k \neq 1$ in $G *_A H$. In particular, G and H are embedded in $G *_A H$ by the maps $g \longrightarrow g$ and $h \longrightarrow h$.*

Theorem 1.4.14. *(Torsion Theorem for Free Products with Amalgamation) Every element of finite order in $G *_A H$ is a conjugate of a finite order element in G or H.*

Theorem 1.4.15. *(Conjugacy Theorem for Free Products with Amalgamation) Let $k > 1$ and $u = c_1c_2 \ldots c_k$ a cyclically reduced word. Then every cyclically reduced conjugate of u is a cyclic permutation of u conjugated by an element of A*

We move on to describe HNN extensions. In this case we want A and B above to be subgroups of G. Throughout g with or without subscript refers to an element of G. Also ϵ with or with subscript is either 1 or -1.

Definition 1.4.16. The *HNN extension of G relative to A, B and ϕ* is the group

$$G^* = \langle X \cup \{t\} \mid R,\ t^{-1}at = \phi(a),\ a \in A\rangle. \tag{1.7}$$

We call G the *base* of G^*, t is the *stable letter*, and A and B are the *associated subgroups*.

Definition 1.4.17. A sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \ldots t^{\epsilon_k}, g_k$, where $k \geq 0$, is said to be *reduced* if there is no consecutive subsequence t^{-1}, g_i, t with $g_i \in A$ or t, g_j, t^{-1} with $g_j \in B$.

Lemma 1.4.18. *(Britton's Lemma) If a sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \ldots, g_k, t^{\epsilon_k}$ is reduced and $k \geq 1$, then $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots t^{\epsilon_k} g_k \neq 1$ in G^*.*

Definition 1.4.19. A *normal form* is a sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \ldots, g_k$, where $k \geq 0$, such that:

1. g_0 is an arbitrary element of G,
2. if $\epsilon_i = -1$, then g_i is a representative of a coset of A in G,
3. if $\epsilon_i = 1$, then g_i is a representative of a coset of B in G, and
4. there is no consecutive subsequence $t^{\epsilon}, 1, t^{-\epsilon}$.

Theorem 1.4.20. *(Normal Form Theorem for HNN extensions) Every element w of G^* has a unique representation $w = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_k$, where $w = g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \ldots, g_k$ is a normal form. In particular G embeds in G^* via the map $g \mapsto g$.*

1.5 Small cancellation

In this section we recall some basic results about general *maps*. Almost every piece of information in this section is gotten from [70].

Let S be a subset of the Euclidean plane $\mathbb{E}^2$. We use ∂S to mean the boundary of S, $\bar{S}$ denotes the topological closure of S, and $-S$ will denote $\mathbb{E}^2 - S$. A *vertex* is a point in $\mathbb{E}^2$. An *edge* is a subset of $\mathbb{E}^2$ homeomorphic to the open unit interval. A *region* is a bounded subset of $\mathbb{E}^2$ homeomorphic to the open unit disk.

Definition 1.5.1. A *map* M is a finite collection of vertices, edges, and regions which are pairwise disjoint and satisfy:

1. If e is an edge of M, there are vertices u and v (possibly the same one) in M such that $\bar{e} = e \cup u \cup v$.
2. The boundary, ∂D, of each region D of M is connected and there is a set of edges $e_1, \ldots, e_n$ in M such that $\partial D = \bar{e}_1 \cup \ldots \cup \bar{e}_n$.

1.5.1 Basic formulas

In [69], Lyndon obtained results on maps which generalize the properties of regular tessellations of the plane. Previously Blanc [5] had used similar results but in a different context. The basic idea is the following.

Let p and q be positive real numbers satisfying $1/p + 1/q = 1/2$. The only integral solutions are the tuples $(3, 6)$, $(4, 4)$ and $(6, 3)$. These pairs correspond to the tilings of the plane by regular triangles, squares, and hexagons respectively.

Some definitions

Let M be a map. A *boundary vertex* or *boundary edge* of M is a vertex or edge in ∂M. A *boundary region* is a region D of M such that $\partial D \cap \partial M \neq \emptyset$. Note that by definition, if D is a boundary region of M, $\partial D \cap \partial M$ need not contain an edge. A vertex, edge or region which is not a boundary vertex, edge, or region is called *interior*. The *degree* of a vertex v, denoted $d(v)$, is the number of edges incident on it. The *degree* of a region D, denoted $d(D)$, is the number of edges on ∂D. The symbol $i(D)$ will denote the number of *interior* edges of ∂D, with an edge counted twice if it appears twice in the boundary cycle of D.

Definition 1.5.2. A non-empty map is called a $[p, q]$-map if each interior vertex has degree at least p and all regions have degree at least q.

Definition 1.5.3. A non-empty map is called a (p, q)-map if each interior vertex has degree at least p and each <u>interior</u> region has degree at least q.

Small cancellation conditions

Let R be a subset of a free group F. We call R *symmetrized* if each $r \in R$ is cyclically reduced and all cyclically reduced conjugates of r and r^{-1} are in R.

Definition 1.5.4. A word w is called a *piece* (relative to a symmetrized set R) if it is identically equal to initial segments of two distinct elements in R.

Let R be a symmetrized set, λ some positive real number and p a natural number. The following are the hypothesis of small cancellation.

1. $C'(\lambda)$: If $r \in R$, $r = ws$ where w is a piece, then $\ell(w) < \lambda\ell(r)$.

2. $C(p)$: No element of r is a product of fewer than p pieces. In particular we observe that $C'(\lambda)$ implies $C(p)$ for $\lambda \leq 1/(p-1)$.

3. $T(p)$: Let $3 \leq \mu < p$. Suppose $r_1, \dots, r_\mu$ are the elements of R with no successive elements $r_i r_{i+1} = 1$. Then at least one in $r_1r_2, r_2r_3, \dots r_{\mu-1}r_\mu, r_\mu r_1$ is reduced without cancellation.

Area and Curvature Theorems

Here we mention two theorems which we shall use later. For some notations, $\sum_M$ will denote summation over vertices or regions in the map M. $\sum_M^{\bullet}$ will denote summation over boundary vertices or boundary regions in the map M.

Theorem 1.5.5 (Curvature). *[70] Let M be a simply-connected (q,p)-map which contains more than one region. Then*

$$\sum_M^{\bullet}\left[\frac{p}{q}+2-i(D)\right]\geq p. \tag{1.8}$$

Theorem 1.5.6 (Area). *[70] Let M be a simply-connected $[p,q]$-map. Then*

$$V_M\leq\frac{q}{p^2}\left[\sum_M^{\bullet}p-d(v)\right]^2, \tag{1.9}$$

where V_M is the total number of vertices in M.

1.5.2 Pictures

Pictures are one of the most powerful tools available in combinatorial group theory. Essentially, pictures are the duals of van Kampen diagrams [66]. We will describe pictures briefly as it relates to groups with presentations of the form $G=\langle X_1,X_2 \mid R_1,R_2,R\rangle$, where $G_1=\langle X_1\mid R_1\rangle$, $G_1=\langle X_2\mid R_2\rangle$, and R is a word with free product length at least two.

Groups of the form G above are called *one-relator products of groups G_1 and G_2*. In the next section we shall discuss such groups in more details. Pictures were first introduced by Rourke [84] and adapted to work for such groups (as G) by Short [89]. Since then they have been used extensively and successfully by various authors in a variety of different ways (see [[18], [19], [21], [34], [56], [57], [63]]). We describe below the basic idea, following closely the account in [62]. A more detailed description can be found in [55] and also [[14], [6], [64], [27], [82]].

Let G be as above, a picture Γ over G on an oriented surface $\mathcal{S}$ (usually D^2) consists of the following:

1. A collection of disjoint closed discs in the interior of $\mathcal{S}$ called vertices;
2. A finite number of disjoint arcs, each of which is either:
   (a) a simple closed curve in the interior of $\mathcal{S}$ that meets no vertex,
   (b) an arc joining two vertices (or one vertex to itself),

(c) an arc joining a vertex to the boundary $\partial\mathcal{S}$ of $\mathcal{S}$, or

(d) an arc joining $\partial\mathcal{S}$ to $\partial\mathcal{S}$;

3. A collection of labels, one at each corner of each region of $\mathcal{S}$ (i.e. connected component of the complement in $\mathcal{S}$ of the arcs and vertices) at a vertex, and one along each component of the intersection of the region with $\partial\mathcal{S}$. The label at each corner is an element of G_1 or G_2. Reading the labels round a vertex in the clockwise direction yields R^n (up to cyclic permutation), as a cyclically reduced word in $G_1 * G_2$.

A region is a boundary region if it meets $\partial\mathcal{S}$, and an interior region otherwise. If $\mathcal{S} \cong S^2$ or if $\mathcal{S} \cong D^2$ and no arcs of meet D^2, then Γ is called *spherical*. In the latter case ∂D^2 is one of the boundary components of a non-simply connected region (provided, of course, that Γ contains at least one vertex or arc), which is called the *exceptional region*. All other regions are interior. The labels of any region $\triangle$ of Γ are required all to belong to either G_1 or G_2. Hence we can refer to regions as G_1-regions and G_2-regions accordingly. Similarly each arc is required to separate a G_1-region from a G_2-region. Observe that this is compatible with the alignment of regions around a vertex, where the labels spell a cyclically reduced word, so must come alternately from G_1 and G_2. A region bounded by arcs that are closed curves will have no labels; nevertheless the above convention requires that it be designated a G_1- or G_2-region.

An important rule for pictures on D^2 or S^2 is that the labels within any interior G_1-region (respectively G_2-region) represents the identity element in G_1 (respectively G_2). The label around any given boundary component of the region are formed into a single word read anti-clockwise.

Two distinct vertices of a picture are said to *cancel* along an arc e if they are joined by e and if their labels, read from the endpoints of e, are mutually inverse words in $G_1 * G_2$. Such vertices can be removed from a picture via a sequence of *bridge moves* (see Figure 1.1 and [21] for more details), followed by deletion of a *dipole* without changing the boundary label.

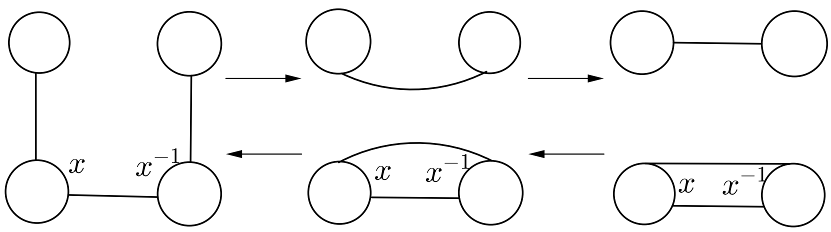

Figure 1.1: Diagram showing bridge-move.

A *dipole* is a connected spherical picture containing precisely two vertices, does not meet $\partial\mathcal{S}$, and none of its interior regions contain other components of Γ. This gives an alternative picture with the same boundary label and two fewer vertices.

Let Γ be a picture over G on some surface $\mathcal{S}$. We say that Γ is *reduced* if it cannot be altered by bridge moves to a picture with a pair of cancelling vertices. If $\mathcal{W}$ is a set of words, then Γ is $\mathcal{W}$-minimal if it is non-empty and has the minimum number of vertices amongst all pictures over G with boundary label in $\mathcal{W}$. Any cyclically reduced word in $G_1 * G_2$ representing the identity element of G occurs as the boundary label of some reduced picture on D^2. A picture is *connected* if the union of its vertices and arcs is connected. In particular, no arc of a connected picture is a closed arc or joins two points of $\partial\mathcal{S}$, unless the picture consists only of that arc.

Dehn function and Isoperimetric Inequality

In differential geometry, isoperimetric functions are classical. However its use in group theory was originally due to the work of Gromov. In his seminal article [41], Gromov characterised (word) hyperbolic groups as groups with linear isoperimetric functions. We describe these functions and explain the relationship with the solution of the word problem. Throughout this section, we fix a finite presentation

$$\wp = \langle X \mid R \rangle \tag{1.10}$$

for G.

A word $w \in F(X)$ represents the identity element in G if and only if it can be expressed as a product of the form

$$w = \prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1}, \tag{1.11}$$

where $u_i \in F(X)$ and $r_i \in R$.

We define the area of such w, $\mathrm{Area}(w)$, to be the smallest $n \geq 0$ such that w can be expressed as a product of the form 1.11.

Definition 1.5.7. A function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(k) = \max\{\mathrm{Area}(w) \mid w \in F(X), w = 1 \text{ in } G \text{ with } \ell(w) \leq k\},$$

is called the *Dehn function* or *isoperimetric function* for the presentation $\wp$.

Lemma 1.5.8. *Let $\wp_1$ and $\wp_2$ be two finite presentations for a group G with corresponding Dehn functions f_1 and f_2. Then for each $k \in \mathbb{N}$, there exist constants $\mathcal{A}$,*

$\mathcal{B}$, $\mathcal{C}$, $\mathcal{D} \in \mathbb{N}$ *such that*

$$f_1(k) \leq \mathcal{A} f_2(\mathcal{B}k + \mathcal{C}) + \mathcal{D}. \tag{1.12}$$

In particular, if f_2 is bounded above by a function that is linear (or quadratic, or polynomial, or exponential, ...) in k, then the same is true for f_1. These properties are thus invariants of the group G.

Definition 1.5.9. A finitely presented group G has a linear (or quadratic, or polynomial, or exponential, ...) isoperimetric inequality if for some (and hence for any) finite presentation with Dehn function f, there is a linear (or quadratic, or polynomial, or exponential, ...) function g such that $f(k) \leq g(k)$ for each $k \in \mathbb{N}$.

As mentioned earlier, finitely presented groups with linear Dehn functions are called *(word) hyperbolic groups.*

The following theorem shows the connection with solution of the word problem.

Theorem 1.5.10. *[31] For a finite presentation $\wp$ of a group G, the following statements are equivalent.*

1. *G has a recursive isoperimetric function.*
2. *The word problem is soluble in G.*

1.5.3 Combinatorial curvature

For any compact orientable surface (with or without boundary) $\mathcal{S}$ with a triangulation, we assign real numbers β to the corners of the faces in $\mathcal{S}$. We will think of these numbers as interior angles. A vertex which is on $\partial\mathcal{S}$ is called a boundary vertex, and otherwise interior. The *curvature* of an interior vertex v in $\mathcal{S}$ is defined as

$$\kappa(v) = \left[2 - \sum_i \beta(v)_i\right] \pi, \tag{1.13}$$

where the $\beta(v)_i$ range over the angles at v. If v is a boundary vertex then we define

$$\kappa(v) = \left[1 - \sum_i \beta(v)_i\right] \pi. \tag{1.14}$$

The curvature of a face Δ is defined as

$$\kappa(\Delta) = \left[2 - d(\Delta) + \sum_i \beta(\Delta)_i\right] \pi \tag{1.15}$$

where $\beta(\Delta)_i$ are the interior angles of Δ. The combinatorial version of the Gauss-Bonnet Theorem states that the total curvature is the multiple of Euler characteristic of the surface by 2π:

$$\kappa(\mathcal{S}) = \left[\sum_v \kappa(v) + \sum_\Delta \kappa(\Delta)\right]\pi = 2\pi\chi(\mathcal{S}). \tag{1.16}$$

We use curvature to prove results by showing that this value cannot be realised. We assign to each corner of a region of degree k an angle $(k-2)/k$. This will mean that regions are flat in the sense that they have zero curvature (alternatively we can make vertices flat instead). This will be the standard assignment for this work. In other words, wherever curvature is mentioned with no specified assignments, it is implicitly assumed that we are using the one described above.

In some cases, it may be needful to redistribute curvature (see [23]). This involves locating positively-curved vertices (or regions), and using its excess curvature to compensate its negatively curved neighbours. Hence the total curvature is preserved. We shall describe how to do this in Chapter 6.

1.6 Bass-Serre Theory

Suppose a group G *acts* on a set X. One can ask if it is possible to reconstruct G (and possibly X and the action) in a way that shows a decomposition theorem for G. For symmetric groups this turns out to be a natural question. It turns out that there is a very good answer when X or its quotient by the action is a tree. In this section we recall some of the ideas about the fundamental theory of Bass and Serre on groups acting freely and without inversion on trees. More details can be found in [[88], [3]]. We begin with the concept of group actions.

1.6.1 Group actions

Definition 1.6.1. Let G be a group and X a non-empty set. A left *action* of G on X is a map

$$f : G \times X \to X$$

such that the following holds for each $x \in X$ and $g, h \in G$.

1. $f(g, f(h, x)) = f(gh, x)$.
2. $f(1, x) = x$.

In such case, we call X a G-set or a G-space (if we are interested in other properties of X). For simplicity, we shall henceforth write gx instead of $f(g, x)$. For each $x \in X$, we associate to it two sets $G_x \subseteq G$ and $Gx \subseteq X$.

Definition 1.6.2. Let X be a G-set. The *stabilizer* of $x \in X$ is the subset

$$G_x = \{g \in G \mid gx = x\}.$$

Definition 1.6.3. Let X be a G-set. The *orbit* of $x \in X$ is the subset

$$Gx = \{y \in X \mid gx = y,\ g \in G\ \}.$$

Definition 1.6.4. Let X be a G-set. The quotient space (of G action on X) is the set

$$G/X = \{Gx \mid x \in X\}.$$

Definition 1.6.5. If X is a G-set and $G_x = 1$ for all $x \in G$, then we say the action is *free.*

Lemma 1.6.6. *If $y = gx$, then $G_x = (g^{-1}Gg)_y$*

1.6.2 Group actions on graphs

We recall some notion of graphs.

Definition 1.6.7. A *graph* X is a pair of sets, $V = V(X) \neq \emptyset$ and $E = E(X)$, termed the *vertices* and *edges* of X, equipped with three maps

$$o : E \to V, \qquad t : E \to V, \qquad ^{-} : E \to E,$$

satisfying the following conditions: if $e \in E$, then

1. $e \neq \bar{e}$ and $\bar{\bar{e}} = e$;
2. $o(e) = t(\bar{e})$.

We call $o(e)$, $t(e)$ and $\bar{e}$ the *origin*, *terminus* and *inverse* of e respectively. If $o(e) = t(e)$, we call e a *loop.* Together, we call $o(e)$ and $t(e)$ the *extremities* of e. Two vertices are adjacent if they are the extremities of some edge.

In the plane, graphs are represented by diagrams, with points as vertices and line segments joining its extremities as edges. Usually, an arrow is affixed on such a line segment, whose direction begins from the origin of the edge and directed towards the terminus. When we use diagrams, it is customary to omit $\bar{e}$.

Definition 1.6.8. Let X_1 and X_2 be graphs with vertex sets $V(X_1)$ and $V(X_2)$ respectively. A map $\phi : V(X_1) \to V(X_2)$ is called a *morphism* if it is edge-preserving. In other words, two vertices in $V(X_2)$ are adjacent if their pre-images in $V(X_1)$ are adjacent. A morphism is is called an *isomorphism* if it is a bijection. An isomorphism between $V(X_1)$ and itself is called an *automorphism*

Definition 1.6.9. A group G acts on a graph X *without inversion* if $g(e) \neq \bar{e}$ for each $g \in G$.

Definition 1.6.10. An *orientation* of a graph X is a decomposition $E(X) = E_+ \cup E_-$ of $E(X)$ into two disjoint sets E_+ and E_- such that $\bar{E_+} = E_-$ and $\bar{E_-} = E_+$.

Since $^-$ has order two and is fixed point free, every graph has such an orientation. A graph with a prescribed orientation is called an *oriented* graph.

1.6.3 Theory of covering spaces

We present briefly some notions in covering space theory, taking for granted many basic notions such as *topological space*, *arc-wise connectedness* and the *fundamental group* $\pi_1(X, *)$ of a space X bases at a point $*$. These concepts can be found in most algebraic topology texts such as Massey [74] and Hatcher [45]. For the rest of this section we assume all spaces are *Hausdorff* i.e., distinct points have disjoint neighbourhoods, and *locally arc-wise connected*, i.e., if V is an open set containing a point x, there exists an open subset U contained in V and containing x such that any pair of points in U have a path in U joining them.

Definition 1.6.11. Let X and $\tilde{X}$ be two arc-wise connected, locally arc-wise connected spaces, $p : \tilde{X} \to X$ a continuous map. We call $(\tilde{X}, p)$ a covering space of X if

1. p is onto;
2. each $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets homeomorphic via p to U.

Definition 1.6.12. Let $\tilde{X}$ be two arc-wise connected, locally arc-wise connected space. Then a group G acts *properly discontinuously* on $\tilde{X}$ if there is an action of G on $\tilde{X}$ such that every point $\tilde{x} \in \tilde{X}$ is contained in an open neighbourhood V satisfying

$$V \cap gV = \emptyset \quad (g \in G,\ g \neq 1.$$

It is known that if a group G acts properly discontinuously on a suitable space, then G can be recaptured from the quotient space of the action. This idea is the motivation for what we do for the rest of this chapter.

1.6.4 Graph of groups

Suppose a group G acts without inversion on a tree X. We form the quotient graph G/X, keeping track of the stabilisers of some of the vertices and edges in X under the action of G. This information is captured in the so-called *graph of groups.*

Definition 1.6.13. A *graph of groups* $(\mathcal{G}, X)$ consists of the following:

1. a connected graph X;
2. a mapping $\mathcal{G}$ from $V(X) \cup E(X)$ into the class of all groups;
3. the image of $v \in V(X)$ under $\mathcal{G}$ is denoted by G_v and is called the *vertex group* at v;
4. the image of $e \in E(X)$ under $\mathcal{G}$ is denoted by G_e and is called the *edge group* at e;
5. $G_e = G_{\bar{e}}$ for every $e \in E(X)$;
6. each edge group G_e comes equipped with a monomorphism

$$G_e \to G_{t(e)} \quad \text{denoted by} \quad a \mapsto a_e.$$

Given a graph of groups $(\mathcal{G}, Y)$ we associate to it a group which is analogous to the fundamental group of a topological space, called its *fundamental group*.

1.6.5 The fundamental group of a graph of groups

Let $(\mathcal{G}, X)$ be a graph of groups, and T a maximal subtree of X. The following is a definition of the fundamental group $\pi_1(\mathcal{G}, X)$ of $(\mathcal{G}, X)$.

Definition 1.6.14. [3] Suppose $(\mathcal{G}, X)$ and T are as above. Suppose that $e \in E(T)$. Then we have two monomorphisms $G_e \to G_{t(e)}$ and $G_e \to G_{o(e)}$ sending a to a_e and a to $a_{\bar{e}}$ respectively. Define G_T to be the group generated by $G_v,\ v \in V(T)$, with the two images of the edge group G_e in $G_{o(e)}$ and $G_{t(e)}$ identified according to the prescription $a_e = a_{\bar{e}},\ a \in G_e$, where e ranges over the edges in T. In other words G_T is simply obtained from the vertex groups, by repeatedly forming amalgamated free products, where the graph of groups $(\mathcal{G}, X)$ determines the subgroups to be amalgamated. The group G_T can be described in more precise terms as follows. Choose a vertex $v_0 \in V(T)$ and define

$$L_0(T) = \{v_0\}.$$

We call $L_0(T)$ the set of vertices at level 0. We then define $L_n(T)$ to consist of those vertices of T which are at a geodesic distance n from v_0. We note that since T is a tree, none of the vertices in $L_n(T)$ are the extremities of an edge in T. Also since T is connected,

$$V(T) = \bigcup_{n=0}^{\infty} L_n(T).$$

Define

$$G(0) = G_{v_0}.$$

We then define $G(n+1)$ inductively, assuming that $G(n)$ has already been defined in such a way that it is generated by all the vertex groups occurring at levels at most n. For each $v \in L_{n+1}(T)$ there is a unique edge $e \in E(T)$ such that $o(e) \in L_n(T)$ with $t(e) = v$. We define $G(v)$ to be the generalised free product of $G(n)$ and G_v with G_e amalgamated according to the monomorphisms of G_e into the vertex groups at its extremities, as dictated by the graph of groups $(\mathcal{G}, X)$. $G(n+1)$ is then defined to be the generalised free product of all the groups $G(v)$ with $G(n)$ amalgamated. Finally we define

$$G_T = \bigcup_{n=0}^{\infty} G(n).$$

It follows that G_T contains an isomorphic copy of each of the vertex groups G_v ($v \in V(T)$ and that the subgroup of G_T generated by $G_{o(e)}$ and $G_{t(e)}$ is the free product of $G_{o(e)}$ and $G_{t(e)}$ amalgamated over G_e for every edge $e \in E(T)$. By inspection on the obvious presentation of G_T it is clear that G_T does not depend on the choice of v_0. The fundamental group $\pi_1(\mathcal{G}, X)$ (with respect to T) of $(\mathcal{G}, X)$, written $\pi_1(\mathcal{G}, X, T)$ is defined to be the HNN extension with possibly infinitely many stable letters. The base group is G_T. The choice of the stable latters depends on an orientation, say

$$E(X) = E_+ \cup E_-$$

of X. Given such an orientation, for each edge $e \in E_+$, $e \notin E(T)$ we choose a stable latter t_e and define

$$\pi_1(\mathcal{G}, X, T) = \langle G_T, \{t_e \mid e \in E_+ - E(T)\} \mid t_e a_e t_e^{-1} = a_{\bar{e}} \ (a \in G_e, e \in E_+ - E(T))\rangle.$$

The definition of $\pi_1(\mathcal{G}, X)$ given above seems to depend on the choice of T. However it can be shown that this is not the case.

1.6.6 Group action on trees

In this section we state the fundamental theorem of Bass and Serre. We also give examples of groups that act on trees.

Theorem 1.6.15. *(Bass-Serre) Let G be a group acting without inversions on a connected tree T. Denote by X the quotient graph G/T. Then there is a graph of groups $(\mathcal{G}, X))$ such that for any maximal subtree T_1 of X, G is isomorphic to $\pi_1(\mathcal{G}, X, T_1)$. Conversely if $(\mathcal{G}, X)$ is a graph of groups, then $\pi_1(\mathcal{G}, X, T_1)$ acts on a tree in such a way that the resulting graph of groups is isomorphic to $(\mathcal{G}, X)$, for* *some choice of maximal tree T_1.*

We now mention examples of some groups that act on trees. Note that every group acts freely and transitively on its Cayley graph by left multiplication. However, the Cayley graph is not a tree in general.

Example 1.6.16. Let G a free group on two generators x and y. Then the Cayley graph X of G is a tree. The vertex and edge stabilizers of G action on X are trivial. Hence the quotient graph is a graph with one vertex and two edges as shown in Figure 1.2.

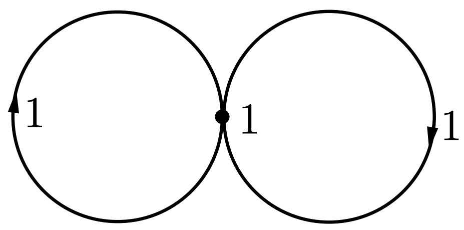

Figure 1.2: A graph with two edges and one vertex.

Theorem 1.6.17. *A group is free if and only if there exists a tree on which it acts freely.*

Example 1.6.18. Let G be the free product $G = A * B$. Then G acts by left translation on a tree T with $E(T) = G$ and

$$V(T) = \{gA \mid g \in G\} \sqcup \{gB \mid g \in G\}.$$

Each edge $g = (gA, gB)$. Clearly the action is free on the edges, but not on the vertices. In particular there is exactly one edge orbit and two vertex orbits $GA = \{gA \mid g \in G\}$ and $GB = \{gB \mid g \in G\}$ respectively. Also $G_{gA} = gAg^{-1} \simeq A$, $G_{gB} = gBg^{-1} \simeq B$, and as mentioned before, $G_g = 1$. Hence the quotient graph is the tree with one edge and two vertices as shown in Figure 1.3.

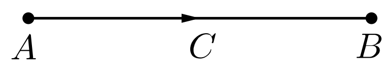

Figure 1.3: Graph of groups for free product of A and B.

Example 1.6.19. Let G be the free product with amalgamation $G = A *_C B$. Then G acts by left translation on a tree T with

$$V(T) = \{gA \mid g \in G\} \sqcup \{gB \mid g \in G\}.$$

There is an edge $e = (gA, hB)$ between gA and gB whenever $hB = gaB$ for some $a \in A$ (alternatively $gA = hbA$ for some $b \in B$). In such case we use gC instead of e. There are two vertex orbits; GA and GB, and one edge orbit. Also $G_{gA} = gAg^{-1} \simeq A$, $G_{gB} = gBg^{-1} \simeq B$, and for $gC = (gA, gaB)$, $G_{gC} = gaCa^{-1}g^{-1} \simeq C$. Hence the quotient graph is the tree with one edge and two vertices as shown in Figure 1.4.

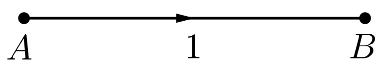

Figure 1.4: Graph of groups for free product of A and B amalgamated over C.

Example 1.6.20. Let G be the HNN extension of group H with stable letter t and amalgamated subgroup C. Then G acts by left translation on a tree T with $V(T) = \{gH \mid g \in G\}$ and $E(T) = \{gC \mid g \in G\}$, where $gC = (gH, gtH)$. All the vertices form a single orbit as well as all the edges. Also $G_{gH} = tHt^{-1} \simeq H$ and $G_{gC} = gCg^{-1} \simeq C$. Hence the quotient graph is the tree with one edge and two vertices as shown in Figure 1.5.

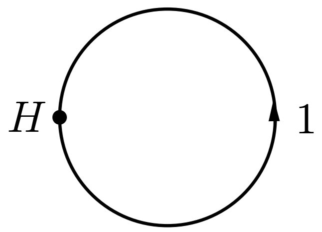

Figure 1.5: Graph of groups for HNN extension.

Chapter 2

Short review of one-relator product of groups

2.1 Preamble

One can think of Dehn's most influential contribution to combinatorial group theory to be his series of papers in which he posed the algorithmic problems. These problems arose naturally in his study of *surface groups* or the fundamental groups of two dimensional manifolds. These groups happen to fall into a larger class of groups which are presented with a single defining relator, otherwise known as one-relator groups. The account on one-relator products given here is mostly based on [1].

In this chapter we study these class of groups, their generalisations, known as one-relator products, and some known results about them. We begin with one-relator groups.

2.2 One relator groups

Definition 2.2.1. A *one-relator group* is a group G whose presentation contains a single defining relator. In other words, G has the form

$$G = \langle X \mid w \rangle.$$

Some 20 years after Dehn proposed the algorithmic problems, Magnus [71] proved his famous Freiheitssatz.

Theorem 2.2.2. *(Freiheitssatz). Let $G = \langle X \mid w \rangle$ where w is cyclically reduced. If L is a subset of X which omits a generator occurring in w, then the subgroup of G generated by L is free.*

The Freiheitssatz was so popular that many versions of its proof were produced. These include Lyndon [75] and Schupp[87]. In particular, it led to the first major break-through on the word problem.

Theorem 2.2.3. *[72] The word problem for one-relator groups has a positive solution.*

Theorem 2.2.4. *[75] Let $G = \langle X \mid w \rangle$ where w is cyclically reduced. The group G is torsion free if w is not a proper power. If $w = R^n, n > 1$, where R itself is not a proper power, then R has order n in G and all elements of finite order of G are conjugates of powers of R.*

Next we mention another fundamental result known as the Spelling Theorem.

Theorem 2.2.5. *Let $G = \langle X \mid R^n \rangle$ where R is cyclically reduced and $n > 1$. Any non-empty word which represents the identity element must contain a (cyclic) subword of R^n or R^{-n} longer than R^{n-1}.*

The Spelling Theorem was first due to Newman [79], and subsequently extended by Gurevich [43], Schupp [87] and Pride [81] (see also [61]).

2.3 One relator product of groups

For an indexing set Λ, let G_Λ be the free product $*_{\gamma\in\Lambda} G_\gamma$ of groups G_γ. Let G be the quotient of G_Λ by the normal closure of a word $w \in G_\Lambda$, assumed to be cyclically reduced with length at least two. We write

$$G = G_\Lambda / N(w). \tag{2.1}$$

Definition 2.3.1. A group of the form 2.1 is called a *one-relator product*

Each G_γ is called a *factor* of G and w is the *relator*. We say w *involves* the factor G_γ if it contains a non-trivial element of G_γ. If $w = R^n$ for some cyclically reduced word R and $n > 1$, then we call w a *proper power*, just like in the case of one-relator groups, with *root* R.

In the case where each factor is isomorphic to the infinite cyclic group, we get a one-relator group. So the question of whether we can prove analogues of results about one-relator groups seems natural. We shall be mostly interested in generalising the Freiheitssatz. For a complete discussion beyond the Freiheitssatz, see the survey article [54] by Howie. We will also mention other results along the line.

Definition 2.3.2. Let G be as in 2.1. We say that G satisfies the Freiheitssatz if each G_γ embeds in G by the natural maps. We call G_γ a *Freiheitssatz factor* if it embeds in G.

However, things are not so nice in this setting as there is no hope to prove Freiheitssatz in general. For example, consider $G = (A * B)/N(ab)$, where a and b are elements in A and B respectively with co-prime orders. In this case Freiheitssatz fails since $a = b = 1$ in G. So to get things to work, more conditions must be imposed. These conditions can be on the factors, the relator or both. We begin with the case of putting condition on the factors.

Theorem 2.3.3. *Let G be as in 2.1 with each G_γ locally indicable. Then G satisfies the Freiheitssatz.*

Recall that a group is said to be *locally indicable* if every non-trivial finitely-generated subgroup surjects onto the infinite cyclic group. Theorem 2.3.3 was originally due to Brodskii [[8],[9]]. Later on Howie [51] gave an independent proof. Short [89] gave another proof using pictures. An entirely algebraic proof was given by Baumslag [2].
We mention a few other results about one-relator product of locally indicable groups.

Theorem 2.3.4. *[51] Let $G = (A * B)/N(w)$, where A and B are locally indicable groups and $w = r^m$ for integer $m \geq 1$. No proper subword of w is trivial in G.*

Theorem 2.3.5. *[61] Let $G = (A * B)/N(w)$, where A and B are locally indicable groups and $w = r^m$ for integer $m \geq 1$. For every non-empty word $s \in N(w)$, either:*

1. *s is conjugate to $w^{\pm 1}$, or*
2. *s contains two almost-disjoint cyclic subwords, each of which is a cyclic subword of $w^{\pm 1}$ longer than r^{m-1}.*

As a corollary to Theorem 2.3.5, we have that the solubility of the word problems for A and B implies that of G.

It seemed at the time that torsion-freeness was the nice property we require of the factor groups to make things work. In particular we have the following conjecture.

Conjecture 2.3.6. *The Freiheitssatz holds for one-relator products with torsion-free factors.*

This conjecture is still widely open. However there are some results supporting it. We mention one next and another when we discuss equation over groups.

Theorem 2.3.7. *[10] Let $G = (A * B)/N(w)$ where A and B are torsion-free groups and $w = a_1 b_1 \ldots a_k b_k$ is a cyclically reduced word in $A * B$ such that some a_i is isolated in $(a_1, \ldots, a_k)$ and some b_j is isolated in $(b_1, \ldots, b_k)$. Then G satisfies the Freiheitssatz.*

We say that x_i is *isolated* in $(x_1, \dots, x_k)$ if $x_j \notin \langle x_i \rangle$ for $j \neq i$.

Next consider the case where a condition is put on the relator instead. Predominant among these is that the relator w is a proper power i.e. $G = (A * B)/N(w)$ where $w = R^n$ for some $n > 1$. It turns out that the higher n is, the easier it is to handle. For instance if $n > 6$, then Freiheitssatz for G follows from small-cancellation results of Lyndon [39] (see also [[70],Chapter V]), since the symmetrized closure of R^n satisfies the metric small-cancellation condition $C'(1/6)$. More generally the following result holds.

Theorem 2.3.8. *Let $G = (A * B)/N(w)$, where $w = R^n$. Then G satisfies the Freiheitssatz if $n \geq 4$, or $n \geq 3$ and R has no element of order 2.*

The case where $n = 6$ is due to Gonzalez-Acuna and Short [34], using pictures. It is also a consequence of [[15] Theorem 5.1] due to Collins and Perraud. The cases of $n = 4, 5$ are due to Howie [[55], [56]], again by use of pictures. The remaining case is due to Howie and Duncan [20].

Conjecture 2.3.9. *[53] Let $G = (A * B)/N(w)$, where $w = r^n$ and $n > 1$. Then G satisfies the Freiheitssatz.*

Most of the results proved above are either geometric or combinatorial. We move on to a special class of one-relator products where results are proved using representation theory.

2.4 One relator product of cyclics

In the next chapter we will be studying one-relator product induced from generalised triangle groups. Hence it is important to recall what generalized triangle groups are. But first we describe the larger family of one-relator products of cyclics. These are groups of the form

$$G = \langle x_1, x_2, \dots, x_n \mid x_1^{e_1} = x_2^{e_2} = \dots = x_n^{e_n} = w(x_1, x_2, \dots, x_n) = 1 \rangle, \tag{2.2}$$

where $w = w(x_1, x_2, \dots, x_n)$ is cyclically reduced and $e_i > 1$ for each $i = 1, 2, \dots, n$. In other words G is a one-relator product in which each factor is finite cyclic. We shall only be interested in the case where w is a proper power - say $w = R^m$ with $m > 1$, so henceforth we assume that we are in this situation.

One-relator product of cyclics provides a natural generalization of discrete subgroups of $PSL_2(\mathbb{R})$, otherwise known as *Fuchsian groups*. A very important special case is when $n = 2$, i.e.

$$G = \langle x_1, x_2 \mid x_1^{e_1} = x_1^{e_2} = R(x_1, x_2)^m = 1\rangle \tag{2.3}$$

A group of the form 2.3 is called a *generalised triangle group.* When $R(x_1, x_2) = x_1x_2$ (up to permutation and inversion) we get the ordinary *triangle group.*

These groups have been much studied and a lot is known about them. For example Baumslag, Morgan and Shalen used a technique based on the idea of Ree and Mendelsohn [83] to prove the following.

Theorem 2.4.1. *[4] A generalised triangle group admits an essential representation into $PSL_2(\mathbb{C})$.*

Recall that the generalised triangle group in 2.3 is said to admit an essential representation $\rho :\ G \to H$ if in H, $\rho(x_i)$ has order e_i and $\rho(R)$ has order m. So in particular, this shows that not only is the group non-trivial, the factor groups all embed. Theorem 2.4.1 was independently proved by Boyer [7] who produced an essential representation in $SU(2)$. A variant of Boyer's method was utilized by Howie [58] to prove the non-triviality of one-relator product with three cyclic factors. However much more was possible using representations in $PSL_2(\mathbb{C})$. Indeed many problems about generalised triangle groups tackled using this technique. One of such problems is the *Rosenberger conjecture* which states that generalised triangle groups either contains a non-abelian free group or a solvable subgroup of finite index. In other words, generalised triangle groups satisfy the *Tits Alternative.* So it makes sense to recap the basic idea.

Suppose X_1 and X_2 are matrices in $SL_2(\mathbb{C})$ with traces $\mathrm{Tr}(X_1)$ and $\mathrm{Tr}(X_2)$ respectively, and $W = W(X_1, X_2)$ is a word in X_1 and X_2. Then the trace $\mathrm{Tr}(W)$ of W is a polynomial in the variables $\mathrm{Tr}(X_1)$, $\mathrm{Tr}(X_2)$ and $\mathrm{Tr}(X_1X_2)$ (the trace of X_1X_2) [49]. Using this an essential representation of the generalised triangle group G in 2.3 can be constructed.

It is known that an element $X \in PSL_2(\mathbb{C})$ has order $e > 1$ if and only if $\mathrm{Tr}(X) = 2\cos(k\pi/e)$ for some integer k co-prime to e. Hence the images of x_1 and x_2 can be forced to have orders e_1 and e_2 by mapping them to matrices X_1 and X_2 with traces $2\cos(\pi/e_1)$ and $2\cos(\pi/e_2)$ respectively. Then the trace $\mathrm{Tr}(W)$ of W (the image of $R^m(x_1, x_2)$) becomes a one-variable polynomial in $\mathrm{Tr}(X_1X_2)$ of degree $m\ell(R(x_1, x_2))$. An essential representation of G is obtained by choosing $\mathrm{Tr}(X_1X_2)$ to be the root of $\mathrm{Tr}(W)$.

A refinement of the technique of Baumslag, Morgan and Shalen shows that it is sufficient for the factor groups to have faithful representation in $PSL_2(\mathbb{C})$.

Theorem 2.4.2. *[28] Let A and B be groups with faithful representations in $PSL_2(\mathbb{C})$, and suppose $R \in A{*}B$ is cyclically reduced with $\ell(R) \geq 2$. Then $G = (A{*}B)/N(R^m)$,* *where $m > 1$, has a representation $\rho : G \to PSL_2(\mathbb{C})$ such that ρ is faithful on each factor and $\rho(R)$ has order m. In particular the Freiheitssatz holds for G*

In chapter 4 we shall be proving various results similar to Theorem 2.4.2. Another result about generalised triangle groups which will be very useful in the analysis of the so-called *clique-pictures* is the following.

Theorem 2.4.3. *[60](Spelling Theorem for generalised triangle groups) For a generalised triangle group $G = \langle\, x,\ y \mid x^p,\ y^q,\ W(x,y)^r \,\rangle$, where $W(x,y) = \prod_{i=1}^{k} x^{\alpha_i} y^{\beta_i}$, $(k > 0, 0 < \alpha_i < p, 0 < \beta_i < q)$. If $V(x,y) = \prod_{i=1}^{l} x^{\gamma_i} y^{\delta_i}$, $(l > 0, 0 < \gamma_i < p, 0 < \delta_i < q)$ is a trivial word in G, then $l \geq kr$.*

Suppose we have two subgroups G_1 and G_2 of G, with the natural inclusion maps $G_1 \to G$ and $G_2 \to G$. We get a unique map $f : G_1 * G_2 \to G$. The *Gersten-Stallings angle* $(G; G_1, G_2)$ [91] is defined to be:

1. 0, if f is injective, or
2. π/n, where $2n$ is the minimal length of a non-trivial element in the kernel of f.

In general, it is not easy to calculate these angles. However, the Spelling Theorem gives a lower bound for generalised triangle groups.

Finally before we end this section, we mention the following result of Fine, Howie and Rosenberger [28] about one-relator product of cyclics.

Theorem 2.4.4. *Let G be as in 2.2. Then G admits an essential representation $\rho : G \to PSL_2(\mathbb{C})$ such that ρ restricted to any subgroup generated by any proper subset of $\{x_1, x_2, \ldots, x_n\}$ is faithful. In particular Freiheitssatz holds for G and any proper subset of $\{x_1, x_2, \ldots, x_n\}$ generates the obvious free product.*

2.5 Relative presentations

Let H be a an arbitrary group, and F a free group. A *relative presentation* is a one-relator product of the form

$$H = \frac{(G * F)}{N(w)}, \tag{2.4}$$

where w is a word in $G * F$.
We will only discuss the case where F is a free group of rank 1 with generator t.

Conjecture 2.5.1. *(Kervaire) H is non-trivial if G is non-trivial.*

This conjecture has been proved for a large class of groups, for example when G is locally residually finite [32] or locally indicable [[8], [50], [89]]. It is still open in general (even in the case where F is an arbitrary group) whether or not H is trivial.

We think of $w = 1$ as an equation with coefficients in G and variable t. We say that the equation $w = 1$ has a solution in G if the identity map on G extends to a homomorphism from $H * \langle t \rangle$ to G with w in its kernel. We say that $w = 1$ has a solution over H if H can be embedded in a group $\tilde{G}$ such that $w = 1$ has a solution in $\tilde{G}$. This property is equivalent to the canonical map from G to H being injective.

Problem 2.5.2. *(Adjunction) Under what conditions does $w = 1$ have a solution over G.*

In general it is not possible to find a solution to an arbitrary equation $w = 1$ over an arbitrary group G. As in the case of Freiheitssatz, some sort of restriction has to be placed on the group, on the equation, or both. For example the equation $xtyt^{-1} = 1$ has no solution in any group if x and y have co-prime orders. Also there is no chance if w is an element of G. So a possible restriction on the equation is that the w is non-singular, in other words the exponent sum of w in t is non-zero. Even with this restriction on the w, it is still a widely open problem whether a solution over G always exists. There is a positive answer when w is non-singular and G is locally residually finite or locally indicable. Klyachko gave a positive answer in the case where G is torsion-free and w has exponent sum ± 1 in t, thereby providing further evidence in favour of Conjecture 2.3.6. Levin [67] showed that a solution exists if t occurs in w only with positive exponents. Some other works show that the result holds when the sum of exponent sums of t in w is small [see [22], [52], [26]].

Chapter 3

One-relator product induced from generalised triangle group

3.1 Preamble

In this chapter we consider one-relator product of two non-trivial groups G_1 and G_2 of the form

$$G = \frac{(G_1 * G_2)}{N(R^n)}, \tag{3.1}$$

where R is contained in a subgroup $A*B$ of G_1*G_2 with A and B conjugate to cyclic subgroups of G_1 and/or G_2, and natural number $n \geq 2$. We can of course assume that A is generated by some element a and B is generated by UbU^{-1} for some word U and letter b in $G_1 * G_2$. Hence R is a word in the subgroup $\langle a \rangle * \langle UbU^{-1} \rangle$ of $G_1 * G_2$ which we can assume to be cyclically reduced. As in [62], we also require in some cases the technical condition that (a, b) be *admissible*: whenever both a and b belong to same factor, say G_1, then either the subgroup of G_1 generated by $\{a, b\}$ is cyclic or $\langle a \rangle \cap \langle b \rangle = 1$.

If a one-relator product is in the form of G described above, then we say G is induced by a generalised triangle group. In such a case G can be realised as a push-out of groups as shown in Figure 3.1,

$$\begin{tikzcd} A * B \arrow[r] \arrow[d] & H \arrow[d] \\ G_1 * G_2 \arrow[r] & G \end{tikzcd}$$

Figure 3.1: Push-out diagram.

where H is the corresponding generalised triangle group

$$H = \langle x,\ y \mid x^p,\ y^q,\ R'(x,y)^n \ \rangle,$$

such that $x \mapsto a$, $y \mapsto UbU^{-1}$ and $R'(x,y) \mapsto R$.

One-relator product induced by a generalised triangle group crops up in different forms when proving results. One frequent place is in the analysis of *zones* (to be defined later) of size $\ell(W)/2$ or $\ell(W)$, connecting vertices of same sign, when proving results about groups with one-relator presentation (with relator W) using pictures. Some of the references where such groups have cropped up include [[21], [44], [55], [56], and [57]]. However, the group only earned its name in [62], where a number of results were proved under the hypotheses that $n \geq 3$ and that the pair (a, b) is admissible. Here we prove similar results under hypotheses that are in general weaker than those assumed in [62].

Hypothesis A. $n \geq 2$, R has length at least 4 as a word in the free product $\langle a\rangle * \langle UbU^{-1}\rangle$, and the pair (a, b) is admissible.

Hypothesis B. $n \geq 2$ and no letter of R (as a word in $G_1 * G_2$) has order 2 .

Under either of the above hypotheses, we prove the following.

Theorem 3.1.1. *The natural maps $G_1 \to G$, $G_2 \to G$ and $\mathcal{H} \to G$ are all injective, where $\mathcal{H}$ is the quotient of $\langle a\rangle * \langle UbU^{-1}\rangle$ by $N(R^n)$.*

Theorem 3.1.2. *If the word problems are soluble for $\mathcal{H}$, G_1 and G_2, then it is soluble for G.*

Theorem 3.1.3. *If some cyclic permutation of R^n has the form W_1W_2 with $0 < \ell(W_1), \ell(W_2) < \ell(R^n)$ as words in G, then $W_1 \neq 1 \neq W_2$ as words in G. In particular R has order n in G.*

The first part of Theorem 3.1.1 is a generalisation of Magnus' Freiheitssatz for one-relator groups [71]. There are many generalisations of the Freiheitssatz to one-relator products of a special nature. Theorem 3.1.3 is a version of a result of Weinbaum [95] for one-relator groups. All three of these results were proved in [62] under the hypotheses that $n \geq 3$ and that the pair (a, b) is admissible.

The above theorems will be proven using pictures and *clique-picture* (see Section 3.4). Throughout this Chapter $\mathcal{W}$ denotes the set of non-trivial elements in $G_1 \cup G_2 \cup \langle a\rangle * \langle UbU^{-1}\rangle$.

3.2 Periodicity

Definition 3.2.1. A word w of length n has a period γ if $\gamma \leq n$ and $w_i = w_{i+\gamma}$ for all $i \leq n - \gamma$.

Example 3.2.2. Suppose that $w = x_1x_2x_3x_1x_2$ is a group G. Then w has period 3.

Definition 3.2.3. A word w is said to be bordered by u and v if u and v are proper initial and terminal segments of w respectively. Furthermore we say that w is *bordered* by u if $u \equiv v$.

Remark 3.2.4. It follows immediately that a w bordered by u has period $\gamma = \ell(w) - \ell(u)$.

Theorem 3.2.5. *[29] Let w be a word having periods γ and ρ with $\rho \leq \gamma$. If $\ell(w) \geq \gamma + \rho - gcd(\gamma, \rho)$, then w has period $gcd(\gamma, \rho)$.*

Even though Theorem 3.2.5 was originally due to Fine and Wilf, there has been other proofs. In particular we shall include a proof as given in [[55] Proposition 1]. We find this version useful as many of the arguments presented later not only uses this result, but are constructed in a similar way.

Proof of Theorem 3.2.5. Let $\gamma = s\nu$ and $\rho = t\nu$ where $\nu = \gcd(\gamma, \rho)$. If $\nu \in \{\gamma, \rho\}$, then there is nothing to show. Hence we assume without loss of generality that $s > t > 1$.

Let V denote the initial segment of w of length γ. We express V as

$$V = V_1V_2 \dots V_s \tag{3.2}$$

with each V_i of length ν. Hence to show that w has period ν, it suffices to show that

$$V_1 = V_2 = \dots = V_s. \tag{3.3}$$

By hypothesis, w has an initial segment of the form

$$V_1V_2 \dots V_sV_1V_2 \dots V_{t-1}. \tag{3.4}$$

By period $\rho = t\nu$ of w, we have that $V_i = V_{i+t}$ for $1 \leq i \leq s-1$ (all indices modulo s). This gives $s-1$ equations which involves each V_i for $i = 1, 2, \dots, s$. For suppose that $i \equiv i + mt$ modulo s for $0 < m < s$, then $m \equiv 0$ modulo s (since s and t are co-prime). But this implies that $m = 0$, which is impossible. $\square$

Corollary 3.2.6. *Let w be a word having initial segment w_1 with period γ and terminal segment w_2 with period ρ. If w_1 and w_2 intersect in a segment u with $\ell(u) \geq \gamma + \rho - gcd(\gamma, \rho)$, then w has period $gcd(\gamma, \rho)$.*

Lemma 3.2.7. *Suppose that W is a word in the free monoid S^* equipped with an involution, $s \mapsto s^{-1}$, possibly with fixed points.*
Suppose that W is a word in S^* the form

$$W = x_1 V_1 y_1 V_1^{-1} = z_0 z_1 \cdots z_{2k-1},$$

for some letters x_1, y_1 and some word V_1, where $\ell(W) = 2k$. Suppose also that W has a cyclic permutation of the form

$$z_j z_{j+1} \cdots z_{2k-1} z_0 \cdots z_{j-1} = x_2 V_2 y_2 V_2^{-1},$$

for some letters x_2, y_2 and some word V_2, where $j \not\equiv 0$ modulo k. Then one of the following holds:

1. *$\{x_1, y_1\} = \{x_2, y_2\}$ and*

$$W \equiv \prod_{j=1}^{s} [x_1^{\alpha(j)} V_3 y_1^{\beta(j)} V_3^{-1}], \tag{3.5}$$

for some odd integer $s > 1$ and some word V_3, with $\alpha(j), \beta(j) = \pm 1$ for each j.

2. *$y_i = x_i^{-1}$ for $i = 1, 2$, and*

$$W \equiv \prod_{j=1}^{s} [x_1^{\alpha(j)} V_3 x_2^{\beta(j)} V_3^{-1}], \tag{3.6}$$

for some even integer $s > 0$ and some word V_3, with $\alpha(j), \beta(j) = \pm 1$ for each j.

Proof. In what follows all subscripts are counted modulo $2k$. By hypothesis we have $x_1 V_1 y_1 V_1^{-1} = z_0 z_1 \cdots z_{2k-1}$ and so $z_i = z_{2k-i}^{-1}$ unless $i \equiv 0$ modulo k. Similarly $z_i = z_{2j-i}^{-1}$ unless $i \equiv j$ modulo k.

Let $V_3 = z_1 z_2 \cdots z_{m-1}$ where $m = \gcd(j, k)$. For $i \not\equiv 0$ modulo m, we have $z_i = z_{2j-i}^{-1} = z_{i-2j}$ and also $z_i = z_{i+2k}$, and so $z_i = z_{i+2m}$. It follows that

$$x_1 V_1 y_1 V_1^{-1} = \prod_{t=1}^{s} [\xi_t V_3 \eta_t V_3^{-1}] \tag{3.7}$$

for some letters ξ_t, η_t, where $s = k/m$. By hypothesis $j \not\equiv 0$ modulo k, and so $s > 1$.

Suppose first that s is odd. Replacing j by $k + j$ if necessary, we may assume that

j/m is also odd. This gives a chain of equalities

$$z_0 = z_{2j}^{-1} = z_{2k-2j} = \ldots, \tag{3.8}$$

which continues until it reaches $z_d^{\pm 1}$ for some subscript $d \in \{j, k, j+k\}$. Since the equalities link letters with subscripts of the same parity, we must have $d = j + k$. Moreover, every z_e with $e \equiv 0$ modulo $2m$ appears in the chain. There are precisely s such letters, so an even number of equalities, and hence $\xi_1 = x_1 = z_0 = z_{j+k} = y_2$ and $\xi_t = x_1^{\pm 1} = y_2^{\pm 1}$ for each t. By a similar argument $\eta_t = y_1^{\pm 1} = x_2^{\pm 1}$ for each t.

Now suppose that s is even. Then j/m is odd. Arguing as above, we have a chain of $s-1$ equalities

$$z_0 = z_{2j}^{-1} = \ldots, \tag{3.9}$$

which must end with z_k^{-1}, and a similar chain of equalities equating z_j with z_{j+k}^{-1}. Hence in this case $\xi_t = x_1^{\pm 1} = y_1^{\mp 1}$ for each t, and $\eta_t = x_2^{\pm 1} = y_2^{\mp 1}$ for each t. □

3.3 Generalised triangle group description and refinements

Let R be a cyclically reduced word of length at least 2 in the free product $G_1 * G_2$. As mentioned in the introduction, we are interested in the case where R is contained in the subgroup $A * B$, where A and B are cyclic subgroups of conjugates of G_1 or G_2. Let $N(R^n)$ denote the normal closure of R^n in $G_1 * G_2$ with $n \geq 2$. Then the group of interest is the following:

$$G = \frac{(G_1 * G_2)}{N(R^n)}$$

If S and T are the generators of A and B respectively, then we can construct a generalised triangle group $H = \langle x,\ y \mid x^p,\ y^q,\ R'(x,y)^n \rangle$, such that $x \mapsto S$, $y \mapsto T$ and $R'(x,y) \mapsto R'(S,T)$ (freely equal to R in $G_1 * G_2$). The group G can then be realised the push-out in Figure 3.1. We will call the set $\{S, T\}$ or Figure 3.1 the choice of *generalised triangle group description* for G

The subgroup $A * B$ of $G_1 * G_2$ may not be unique amongst all two-generator subgroups containing R. If $A' * B'$ is another two-generator subgroup of $G_1 * G_2$ with suitable generating set $\{S', T'\}$ and $A * B \subseteq A' * B'$, then we can write S, T and $R'(S,T)$ as words in this new generating set. In general we have that

$$\ell(S') + \ell(T') \leq \ell(S) + \ell(T) \tag{3.10}$$

and $\ell(R') \leq \ell(R'')$ where $R \equiv R'(S,T) \equiv R''(S',T')$. If any of the two inequalities is strict, we say that the generalised triangle group description given by R'' is a *refinement* of the one given by R'.

Let p' and q' be the orders of S' and T' respectively. Then we have that the group $H' = \langle\, x',\ y' \mid x^{p'},\ y^{q'},\ R''(x,y)^n \,\rangle$ and the refinement gives a commutative diagram as in Figure 3.2, in which both squares are push-outs.

$$\begin{tikzcd}
C_{p'} * C_{q'} \arrow[r] \arrow[d] & C_p * C_q \arrow[r] \arrow[d] & G_1 * G_2 \arrow[d] \\
H' \arrow[r] & H \arrow[r] & G
\end{tikzcd}$$

Figure 3.2: Double push-out diagram.

A generalised triangle group description for G is said to be *maximal* if no refinement is possible. In other words, $A * B$ is maximal (with respect to inclusion) amongst all two-generator subgroups of $G_1 * G_2$ containing R. It follows from Equation 3.10 that maximal refinements always exist (but are not necessarily unique). From now on we will assume that we are working with maximal refinements.

We next note a useful consequence in this context of Lemma 3.2.7.

Lemma 3.3.1. *Suppose that, in the above, the generators S and T of A and B respectively have the forms $S = a$, $T = UbU^{-1}$ for some letters a, b and some word U. Suppose that (a, b) is an admissible pair. If there are integers $\alpha, \beta, \gamma, \delta$ such that $a^\alpha U b^\beta U^{-1}$ and $a^\gamma U b^\delta U^{-1}$ are proper cyclic conjugates in $G_1 * G_2$, then a refinement is possible.*

Proof. We may apply Lemma 3.2.7 to $a^\alpha U b^\beta U^{-1}$ and $a^\gamma U b^\delta U^{-1}$ except in the situation where $a^\alpha U b^\beta U^{-1} \equiv b^\delta U^{-1} a^\gamma U$. Let us first consider this exceptional situation. Then in particular $U \equiv U^{-1}$ and so U has the form VxV^{-1} for some word V and some letter x of order 2. But we also have $a^\alpha = b^\delta$, and so by definition of admissibility a, b have a common root c say. But then $A * B$ is a proper subgroup of $C * D$, where C and D are the cyclic subgroups of $G_1 * G_2$ generated by c, VxV^{-1} respectively. This is a refinement, as required.

Hence we are in a situation where we can apply Lemma 3.2.7 with $x_1 = a^\alpha$, $y_1 = b^\beta$, $x_2 = a^\gamma$, $y_2 = b^\delta$, and $V_1 = V_2 = U$. There are two possibilities to consider depending on the parity of s.

If s is odd in the conclusion of Lemma 3.2.7. In this case, $A*B$ is a proper subgroup of $A*B'$, where B' is the cyclic subgroup generated by $V_3bV_3^{-1}$, so again we have a refinement.

Otherwise if s is even in Lemma 3.2.7. Then $a^{\alpha} = b^{-\beta}$, so by admissibility a, b have a common root c. Then $A*B$ is a proper subgroup of $C*D$, where C and D are the cyclic subgroups generated by c and $V_3cV_3^{-1}$ respectively. As before this gives a refinement.

□

3.4 Clique-pictures

Clique-pictures appeared in [62] and are modelled on generalised triangle groups. For the rest of this Thesis,

$$G = \frac{(G_1 * G_2)}{N(R^n)}$$

is a one-relator product induced by the generalised triangle group

$$H = \langle x, y \mid x^p, y^q, R'(x,y)^n \rangle.$$

In other words R is a word in $\{a, UbU^{-1}\}$ for some word $U \in G_1 * G_2$ and letters a and b in $G_1 \cup G_2$ with orders p and q respectively.

If u and v are two vertices in a picture over G that are joined by an arc e, then we may use the endpoints of e as the starting points for reading the labels L_u and L_v of u and v respectively. In each case the label is a cyclic permutation of $R'(a, UbU^{-1})^{\pm n}$. We may assume, without loss of generality, that the word $R'(x,y)$ begins with the letter x. Choose a cyclic permutation $R^*(x,y)$ of $R'(x,y)^{-1}$ that also starts with $x^{\pm 1}$.

Each of L_u and L_v^{-1} is a cyclic conjugate of $R'(a, UbU^{-1})^n$ or $R^*(a, UbU^{-1})^n$, say $L_u = YZ$, where $ZY = R'(a, UbU^{-1})^n$ or $ZY = R^*(a, UbU^{-1})^n$ and $L_v^{-1} = Y'Z'$, where $Z'Y' = R'(a, UbU^{-1})^n$ or $Z'Y' = R^*(a, UbU^{-1})^n$.

We u and v *equivalent* written $u \sim v$, if and only if $\ell(Y') \equiv \ell(Y)$ modulo $l = \ell(aUbU^{-1})$. It follows immediately from Lemma 3.3.1 that $\ell(Y')$ and $\ell(Y)$ are unique modulo l, and so the relation $\sim$ is well-defined. The point of the relation $\sim$ is that, when $u \sim v$, then the 2-vertex sub-picture consisting of u and v, joined by e and any arcs *parallel* (see Remark 3.4.4 for definition) to e, has boundary label a word in $\{a, UbU^{-1}\}$, after cyclic reduction and cyclic permutation. Indeed, the cyclic reduction of the label can be achieved by performing bridge moves to make the number of edges parallel to e be a multiple of $l/2$ (see Figures [3.3, 3.4, 3.5]). Now let $\approx$ denote the transitive, reflexive closure of $\sim$. Then $\approx$ is an equivalence relation on vertices. After a sequence of bridge moves, we may assume that arcs joining equivalent vertices do so in parallel classes each containing a multiple of $l/2$ arcs (see Figures [3.6, 3.14]).

As an illustration, suppose $U = u_1 u_2 \ldots u_{l/2-1}$ and u and v are vertices of a picture with labels

$$\text{label}(u) = \prod_{i=1}^{k} a^{\alpha_i} U b^{\beta_i} U^{-1} \tag{3.11}$$

and

$$\text{label}(v) = \prod_{j=1}^{k} a^{\gamma_j} U b^{\delta_j} U^{-1} \tag{3.12}$$

respectively. Suppose also that u and v are as shown in Figure 3.3 below.

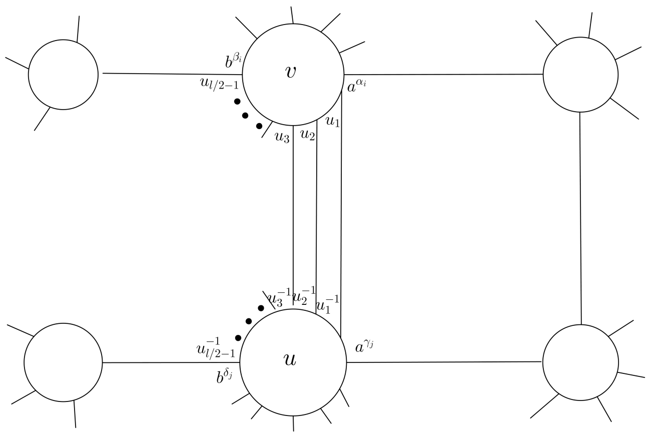

Figure 3.3: Diagram showing an example of $u \sim v$.

Then by definition $u \sim v$. After doing bridge-moves u and v are joined by $l/2$ parallel class of arcs as shown in Figure 3.4.

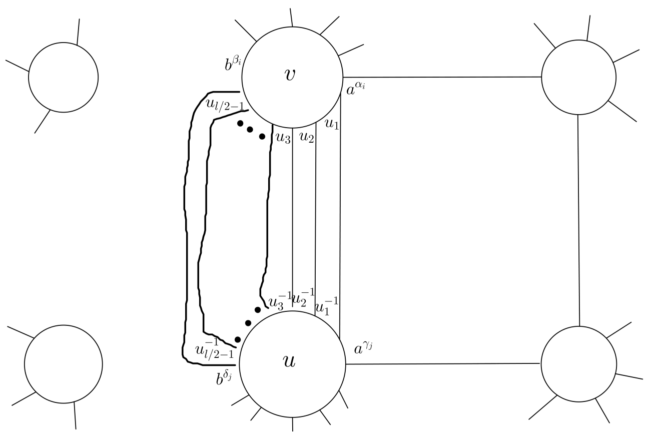

Figure 3.4: Diagram showing u and v joined by $l/2$ arcs after doing bridge-moves to Figure 3.3.

Note that this $l/2$ parallel class of arcs is special in the sense that the label on u contained in it is U and that of v is U^{-1}. We replace such class by a shaded rectangle as shown in Figure 3.5.

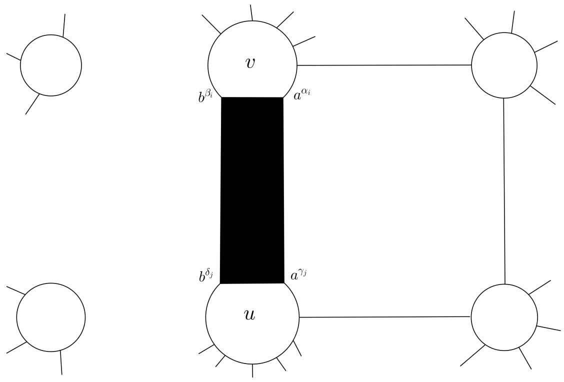

Figure 3.5: Diagram showing the $l/2$-zone in Figure 3.4 replaced by a single shaded rectangle.

Similarly if u and v are two equivalent vertices joined by a multiple of $l/2$ parallel class (possibly after doing bridge-moves) as shown in Figure 3.6, we replace such class by a shaded rectangle as shown in Figure 3.14.

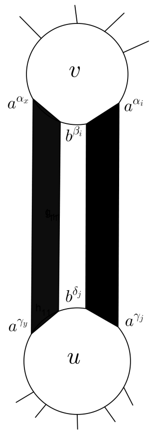

Figure 3.6: Diagram showing two vertices $u \sim v$ joined by l-zone.

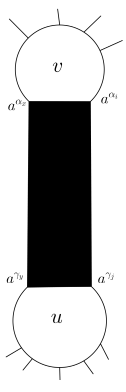

Figure 3.7: Diagram showing the l-zone in Figure 3.6 replaced by a single shaded rectangle.

Definition 3.4.1. A *clique* is a sub-picture consisting of any $\approx$-equivalence class of vertices, together with all arcs between vertices in that $\approx$-class (assumed to occur in parallel classes of multiples of $l/2$ arcs), and all regions that are enclosed entirely by such arcs.

In other words vertices that can be joined by shaded rectangles make up the vertices in a single clique.

Definition 3.4.2. A clique is said to be simply-connected if these collection of shaded rectangles in it forms a tree or a simple closed curve (i.e. a curve with no self-intersections).

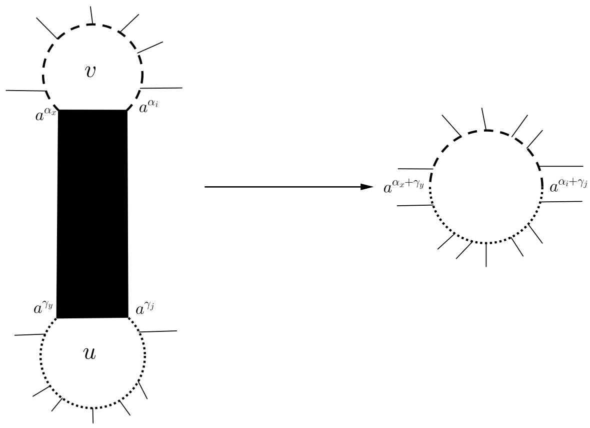

Figure 3.8: Diagram showing a simply-connected clique consisting of only two vertices and how the two vertices are combined to form one single vertex.

The point here is that vertices in a simply-connected clique can be combined along the shaded rectangle (see Figure 3.8) to form a single vertex, which we shall also call a clique. Note that we can have cliques which are not simply-connected. For example a clique containing the vertices $v_1, v_2, \ldots, v_5$ as shown in Figure 3.9 below is not simply-connected.

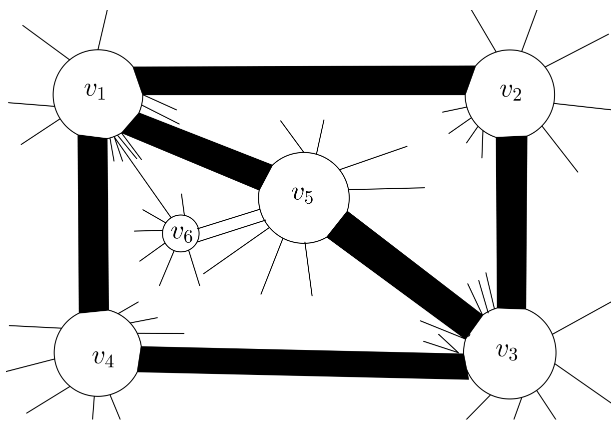

Figure 3.9: Diagram showing a non-simply connected clique.

This leads us to the notion of *Clique-picture*

Definition 3.4.3. Let G be a one-relator product induced from a generalised triangle group as above, and let P be a picture on a surface Σ, such that every clique of P is simply-connected. Then the *clique-quotient* of P is the picture formed from P by contracting each clique to a point, and regarding it as a vertex. A *clique-picture* $\mathbf{P}$ over G is the clique-quotient of some (reduced) picture over G. The label of a vertex in a clique-picture is called a *clique-label.*

The process of joining two vertices of P or two cliques of $\mathbf{P}$ to form a single clique is called *amalgamation.* (Here we also include the possibility of amalgamating a clique with itself. By this we mean adding arcs from v to v and/or regions to an existing clique v, which could alter some properties of the clique such as simple-connectivity.) If it is possible to amalgamate two cliques in $\mathbf{P}$(possibly after doing bridge-moves), we say that $\mathbf{P}$ is not *reduced*, and *reduced* otherwise. A picture whose vertices are cliques is said to be $\mathcal{W}$-minimal if it is non-empty and has the minimum number of cliques amongst all pictures over G with boundary label in the set $\mathcal{W}$.

Remark 3.4.4. Let Γ be P or $\mathbf{P}$. Two arcs of Γ are said to be *parallel* if they are the only two arcs in the boundary of some simply-connected region $\triangle$ of Γ. We will also use the term *parallel* to denote the equivalence relation generated by this relation, and refer to any of the corresponding equivalence classes as a *class of* ω *parallel arcs* or ω*-zone* (see Figure 3.10).

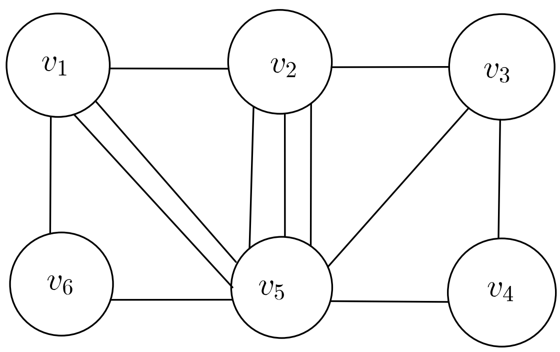

Figure 3.10: Diagram showing vertices v_1 and v_5 joined by a 2-zone and vertices v_2 and v_5 joined by a 3-zone.

Given a ω*-zone* joining vertices u and v of Γ, consider the $\omega - 1$ two-sided regions separating these arcs. Each such region has a corner label x_u at u and a corner label x_v at v, and the picture axioms imply that $x_u x_v = 1$ in G_1 or G_2. The $\omega - 1$ corner labels at v spell a cyclic subword s of length $\omega - 1$ of the label of v. Similarly the corner labels at u spell out a cyclic subword t of length $\omega - 1$. Moreover, $s = t^{-1}$. If we assume that Γ is reduced, then u and v do not cancel. Hence the cyclic permutations of the labels at v and u of which s and t are initial segments respectively are not equal. Hence t and s are *pieces*.

As in graphs, the *degree* of a vertex in Γ is the number of *zones* incident on it. For a region, the *degree* is the number corners it has. We say that a vertex v of Γ satisfies the local $C(m)$ condition if it is joined to at least m *zones*. We say that Γ satisfies $C(m)$ if every interior vertex satisfies local $C(m)$.

3.5 Clique-labels and virtual periodicity

In this section we obtain some preliminary results about clique-pictures. One advantage clique-pictures has over ordinary pictures is that some cyclic permutation of the inverse of any clique-label can also be interpreted as a clique-label. Thus we may regard any clique as having either possible orientation, as convenient. We make the convention that all our cliques have the same (positive or clockwise) orientation. Throughout this Chapter we shall assume that our clique-picture is $\mathcal{W}$-minimal. Note that up to cyclic permutation the clique-label of a clique u has the form

$$\text{label}(u) = \prod_{i=1}^{k} a^{\alpha_i} U b^{\beta_i} U^{-1} \tag{3.13}$$

for $0 < \alpha_i < p$ and $0 < \beta_i < q$. Denote our clique-picture by Γ and let v be a clique of Γ. Take a cyclic permutation $c(k)$ of the label of v of the form (3.13) and express it as

$$c(k) = z_0 z_1 \cdots z_{kl-1} \tag{3.14}$$

where $l = \ell(a^{\alpha_i} U b^{\beta_i} U^{-1})$.

We call a letter z_j of a clique-label

$$\text{label}(u) = \prod_{i=1}^{k} a^{\alpha_i} U b^{\beta_i} U^{-1} = z_0 z_1 \cdots z_{kl-1} \tag{3.15}$$

special if $j \equiv 0$ modulo $l/2$. Note that every special letter is equal to a power of a or of b.

Let $\Omega := (G_1 \cup G_2) \setminus \{1\}$. Define $\sim$ to be the smallest equivalence relation on Ω with the property that $a^\alpha \sim a$ for all α such that $a^\alpha \neq 1$ and $b^\beta \sim b$ for all β such that $b^\beta \neq 1$. Note that the natural involution $x \mapsto x^{-1}$ on Ω descends to an involution on $\Omega/\sim$, under which the $\sim$-classes of a and b are fixed points.

We will sometimes work in the free monoid $(\Omega/\sim)^*$. In particular clique-labels are periodic in $(\Omega/\sim)^*$ with period l.

Let v be a clique of degree k. This means that there are k zones incident at v, say $\mathrm{Z}_1, \mathrm{Z}_2, \ldots, \mathrm{Z}_k$ labelled consecutively in clockwise order around v as shown in Figure 3.11.

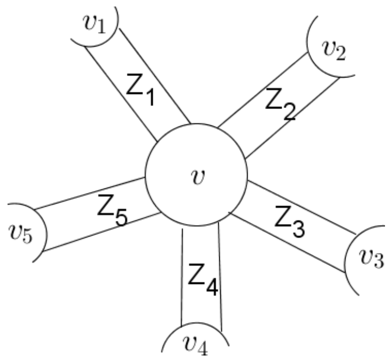

Figure 3.11: Diagram showing a vertex v of degree five.

Recall that each zone Z_i is a class of parallel arcs. The number of arcs in Z_i is denoted by ω_i. If Z_i connects cliques u and v (possibly $u = v$), then Z_i determines cyclic subwords s_i and t_i of length $\omega_i - 1$ of the clique-labels of u and v respectively, such that $s_i \equiv t_i^{-1}$ in $G_1 * G_2$

Definition 3.5.1. A zone Z_i is said to be *large* if $\omega_i > l/2$ and *small* otherwise.

We will use the following generalisation of the concept of periodic word, as applied to cyclic subwords of

$$\text{label}(v) = z_0 z_1 \cdots z_{nl-1} = \prod_{k=1}^{n} [a^{\alpha(k)} U b^{\beta(k)} U^{-1}]. \tag{3.16}$$

Definition 3.5.2. We say that a cyclic subword $W = z_j \cdots z_k$ (subscripts modulo nl) of label(v) is *virtually periodic* with *virtual period* μ if, for each $i \in \{j, j+1, \ldots, k-\mu\}$, one of the following happens:

1. $z_i = z_{i+\mu}$;

2. a special letter $z_d = a^\psi$ belongs to W, for some $d \equiv 0$ modulo l, $i \equiv d$ modulo μ, and each of z_i, $z_{i+\mu}$ is equal to a power of a;

3. a special letter $z_d = b^\psi$ belongs to W, for some $d \equiv l/2$ modulo l, $i \equiv d$ modulo μ, and each of z_i, $z_{i+\mu}$ is equal to a power of b;

4. a and b have a common root c in G_1 or G_2, a special letter $z_d = c^\psi$ belongs to W, for some $d \equiv 0$ modulo $l/2$, $i \equiv d$ modulo μ, and each of z_i, $z_{i+\mu}$ is equal to a power of c.

Recall that the pair (a, b) is assumed to be admissible. If a and b have a common power in G_1 and G_2, then they have a common root, and in that case the second and third possibilities in the above definition are subsumed in the fourth. Otherwise the fourth possibility cannot occur.
By definition, the clique-label label(v) itself is virtually periodic of virtual period l. Other examples of virtually periodic words arise from zones incident at v.

Lemma 3.5.3. *Suppose that Z_i is a zone incident at v. Then there is a positive integer $\mu \leq l/2$ and a cyclic subword s_i^+ of* label(v) *of length $\omega_i + \mu - 1$ and virtual period μ, such that s_i is either an initial or a terminal segment of s_i^+.*

Proof. Let s_i be the cyclic subword $z_j z_{j+1} \cdots z_k$ of

$$\text{label}(v) = z_0 z_1 \cdots z_{nl-1} = \prod_{k=1}^{n} [a^{\alpha(k)} U b^{\beta(k)} U^{-1}]. \tag{3.17}$$

The zone Z_i links v to an adjacent clique u and identifies s_i with t_i^{-1} for some cyclic subword t_i of label(u). Thus t_i^{-1} is a cyclic subword of label$(u)^{-1}$. Write $t_i^{-1} = y_{j'} y_{j'+1} \cdots y_{k'}$ where

$$\text{label}(u)^{-1} = y_0 y_1 \cdots y_{ml-1} = \prod_{k=1}^{m} [a^{\gamma(k)} U b^{\delta(k)} U^{-1}]. \tag{3.18}$$

Since $s_i \equiv t_i^{-1}$, then in particular $k' - j' \equiv \ell(t_i) - 1 = \ell(s_i) - 1 \equiv k - j$ modulo l. If $j \equiv j'$ modulo l, then we may amalgamate the cliques u and v, contrary to hypothesis. Hence there are integers q and μ_i such that $0 \leq q \leq m$, $0 < \mu_i \leq l/2$ such that $j' = j + ql \pm \mu_i$. Define

$$s_i^+ = \begin{cases} z_{j-\mu_i} \cdots z_k & \text{if} \quad j' = j + ql - \mu_i \\ z_j \cdots z_{k+\mu_i} & \text{if} \quad j' = j + ql + \mu_i \end{cases}$$

In the first case, s_i is a terminal segment of s_i^+, while the initial segment of the same length agrees with $t_i^{-1} \equiv s_i$, except possibly at special letters z_d ($d \equiv 0$ modulo $l/2$) which may be a different power of a (or of b) than the corresponding letter of s_i. It follows that s_i^+ is virtually periodic of virtual period μ_i, as claimed.
The second case is entirely analogous, except that s_i is an initial rather than a terminal segment of s_i^+. □

We need to analyse the interaction of virtually periodic subwords of label(v) obtained by applying Lemma 3.5.3 to two adjacent large zones at v. To do this we will use the following analogue of Corollary 3.2.6.

Lemma 3.5.4. *Suppose that the cyclic subword $W = z_j \cdots z_k$ (subscripts modulo nl) of*

$$\mathrm{label}(v) = z_0 z_1 \cdots z_{nl-1} = \prod_{k=1}^{n} [a^{\alpha(k)} U b^{\beta(k)} U^{-1}]. \tag{3.19}$$

is the union of a virtually periodic segment W_1 of virtual period μ and a virtually periodic segment W_2 of virtual period ν. Let $\gamma = \gcd(\mu, \nu)$. If the intersection of these segments has length at least $\mu + \nu - \gamma$, then W is virtually periodic of virtual period γ.

Proof. Let i, m be such that z_i and $z_{i+m\gamma}$ are letters of W. Then we claim there is a finite chain of subscripts $i(0), i(1), \ldots, i(N)$ with $i(0) = i$ and $i(N) = i + m\gamma$ such that, for each t either

$$|i(t) - i(t+1)| = \mu \tag{3.20}$$

and $z_{i(t)}$ and $z_{i(t+1)}$ are letters of W_1, or

$$|i(t) - i(t+1)| = \nu \tag{3.21}$$

and $z_{i(t)}$ and $z_{i(t+1)}$ are letters of W_2.

Certainly each letter in W_1 (respectively W_2) is linked to some letter in $W_0 := W_1 \cap W_2$ by such a chain, since $\ell(W_0) \geq \max(\mu, \nu)$, so it suffices to prove the claim when $z_i, z_{i+m\gamma}$ are letters of W_0. Write $m\gamma = \alpha\mu + \beta\nu$ where $\alpha, \beta \in \mathbb{Z}$, and argue by induction on $|\alpha| + |\beta|$. Without loss of generality, assume that $\alpha > 0$. If $z_{i+\mu}$ is a letter of W_0, then the result follows by applying the inductive hypothesis to $z_{i+\mu}, z_{i+m\gamma}$. Otherwise, $\beta < 0$ and $z_{i-\nu}$ is a letter of W_0, so we may apply the inductive hypothesis to $z_{i-\nu}, z_{i+m\gamma}$. This proves the claim.

Now we take $m = 1$ in the above, and prove that at least one of the alternative conditions for virtual periodicity holds.
If $z_{i(t)} = z_{i(t+1)}$ for all n then $z_i = z_{i+\gamma}$. Suppose next that $z_{i(t)} \neq z_{i(t+1)}$ for at least one value of t, and a, b have a common root c. Then there is a special letter z_d in W, $i \equiv i(t) \equiv d$ modulo μ or modulo ν (and hence in either case modulo γ). Moreover for each t either $z_{i(t)} = z_{i(t+1)}$ or each of $z_{i(t)}, z_{i(t+1)}$ is a power of c. Since $z_{i(t)} \neq z_{i(t+1)}$ for at least one value of t, it follows that each $z_{i(t)}$ is a power of c. In particular z_i and $z_{i+\gamma}$ are both powers of c.
Finally, suppose that $z_{i(t)} \neq z_{i(t+1)}$ for at least one value of t, and a, b have no common root. By admissibility, a, b also have no common non-trivial power. As above, W contains a special letter $z_d = a^{\psi}$ or $z_d = b^{\psi}$, and $i \equiv d$ modulo γ. Consider the least t for which $z_{i(t)} \neq z_{i(t+1)}$. Then either $z_{i(t)}$ and $z_{i(t+1)}$ are both powers of a or both powers of b. Assume the former. Then $z_i = z_{i(0)} = z_{i(1)} = \cdots = z_{i(t)}$ are also

powers of a. We claim that $z_{i(t+1)}, \ldots z_{i(N)} = z_{i+\gamma}$ are also powers of a, which will complete the proof. Suppose by way of contradiction that this is not true. Then there is an m for which $z_{i(m)}$ is a power of a and $z_{i(m+1)}$ is not a power of a. By the definition of virtual periodicity, it follows that both $z_{i(m)}$ and $z_{i(m+1)}$ must be powers of b. But then $z_{i(m)}$ is simultaneously a power of a and of b, contrary to the admissibility hypothesis. □

Corollary 3.5.5. *If a clique-label has virtual period $\mu < l$, then a refinement is possible.*

Proof. By Lemma 3.5.4 the clique-label

$$\mathrm{label}(v) = \prod_{j=1}^{n} [a^{\alpha(j)} U b^{\beta(j)} U^{-1}] \tag{3.22}$$

has virtual period $\gcd(\mu, l)|l$, so without loss of generality $\mu|l$. Let $V = z_1 \cdots z_{\mu/2}$ and let $s = l/\mu$. Then by definition of virtual periodicity and by the admissibility hypothesis one of the following is true:

1. s is odd and
$$\mathrm{label}(v) = \prod_{j=1}^{sn} [a^{\gamma(j)} V b^{\delta(j)} V^{-1}], \tag{3.23}$$
for some $\gamma(j), \delta(j)$.

2. s is even, a, b have a common root c, and
$$\mathrm{label}(v) = \prod_{j=1}^{sn} [c^{\gamma(j)} V x V^{-1}], \tag{3.24}$$
for some letter x of order 2 and some $\gamma(j)$.

In either case, we have a refinement. □

Corollary 3.5.6. *Suppose that v is a clique in a $\mathcal{W}$-minimal clique-picture over G satisfying Hypothesis A. Suppose also that the generalised triangle group description of G has no refinement. Then the length of any zone incident at v is strictly less than l.*

Proof. The i'th zone Z_i contains ω_i arcs. Assume that $\omega_i \geq l > l/2$. Then by Lemma 3.5.3, the word s_i^+ has length $\omega_i + \mu_i - 1$, which is strictly greater than $l + \mu_i - \gcd(l, \mu_i)$ since $\gcd(l, \mu_i)$ is even. Moreover, s_i^+ is virtually periodic of virtual period μ_i. But s_i^+ is a cyclic subword of the clique-label label(v) which is virtually periodic with virtual period l.

By Lemma 3.5.4 it follows that label(v) has virtual period $\gcd(l, \mu_i) \leq l/2$. But by Corollary 3.5.5 this leads to a refinement of our generalised triangle group description of G, contrary to hypothesis.

□

3.6 The relator has free product length at least 4 and the pair (a, b) is admissible.

Assuming Hypothesis A, we have an $\mathcal{W}$-minimal clique-picture over a one-relator product

$$G = \frac{(G_1 * G_2)}{N(R)^n},$$

with $n \geq 2$ and $\ell(R) \geq 4$ as a word in $\langle a \rangle * \langle UbU^{-1} \rangle$. Any clique-label has the form

$$\text{label}(u) = \prod_{j=1}^{k} [a^{\alpha(j)} U b^{\beta(j)} U^{-1}] \tag{3.25}$$

By the Spelling Theorem 2.4.3 we must have $k \geq 2nl$ where $l = \ell(aUbU^{-1})$. But by Corollary 3.5.6 each zone has fewer than l arcs, so all cliques have degree at least $2n + 1$.

Theorem 3.6.1. *Assuming Hypothesis A, any $\mathcal{W}$-minimal clique-picture over G on D^2 satisfies $C(6)$.*

To prove the Theorem 3.6.1, we assume that $n = 2$ and that v is a clique of degree 5, with zones $Z_1, \ldots Z_5$ of sizes $\omega_1, \ldots \omega_5$ respectively, in cyclic order around v, and aim to derive a contradiction.

The key tool in the proof of Theorem 3.6.1 is the following.

Lemma 3.6.2. *For each $i = 1, \ldots, 5$, one of the following holds:*

1. $\omega_i + \omega_{i-1} < 3l/2$;

2. $\omega_i + \omega_{i+1} < 3l/2$.

Proof. By Lemma 3.5.3, s_i is either an initial segment or a terminal segment of a subword s_i^+ of label(v) of length $\omega_i + \mu_i - 1$ and virtual period μ_i, where $0 < \mu_i \leq l/2$. We will assume that s_i is an initial segment of s_i^+ and show that $\omega_i + \omega_{i+1} < 3l/2$. (An entirely analogous argument shows that, if s_i is a terminal segment of s_i^+, then $\omega_{i-1} + \omega_i < 3l/2$.)

Now Lemma 3.5.3 to s_{i+1}: s_{i+1} is either an initial segment or a terminal segment of a cyclic subword s_{i+1}^+ of label(v) of length $\omega_{i+1}+\mu_{i+1}$ and virtual period μ_{i+1}, where $0<\mu_{i+1}\le l/2$. Our argument splits into two cases, depending on whether s_{i+1} is an initial or terminal segment of s_{i+1}^+.

Case 1. s_{i+1} is a terminal segment of s_{i+1}^+.
Consider the cyclic subword $W := s_i z_{s(i+1)} s_{i+1}$ of label(v). Since $\mu_i \le l/2 < \omega_{i+1}$, the virtually periodic subword s_i^+ is an initial segment of W. Similarly, s_{i+1}^+ is a terminal segment of W. These segments intersect in a segment of length

$$\mu_i+\mu_{i+1}-1>\mu_i+\mu_{i+1}-\lambda, \tag{3.26}$$

where $\lambda=\gcd(\mu_i,\mu_{i+1})$, so by Lemma 3.5.4 W is virtually periodic with period λ. Now recall that W is a cyclic subword of label(v), which is virtually periodic with virtual period l. If W has length greater than $l+\lambda-2$, then label(v) has virtual period $\gcd(l,\lambda)<l$, by another application of Lemma 3.5.4.
But by Corollary 3.5.5 this leads to a refinement of our generalised triangle group description of G, contrary to hypothesis. Thus

$$\omega_i+\omega_{i+1}=1+\ell(W)\le l+\lambda-1<3l/2. \tag{3.27}$$

Case 2. s_{i+1} is an initial segment of s_{i+1}^+.
Let $\bar{s}_i$, $\bar{s}_i^+$ denote the cyclic subwords of label(v) that begin with the letter exactly l places after the first letter of s_i. By the virtual periodicity of label(v) it follows that $\bar{s}_i^+$ is also virtually periodic, of virtual period μ_i. Moreover, $\bar{s}_i^+$ has length $\omega_i+\mu_i-1$ and has $\bar{s}_i$ as an initial segment.
By construction, the union of the subwords s_{i+1} and $\bar{s}_i$ of label(v) has length $l-1$. Let W be the union of the subwords s_{i+1}^+ and $\bar{s}_i^+$ of label(v). Then W has length at least $l+\mu_i-1$. Arguing as in Case 1, we obtain a refinement of the generalised triangle group description of G, contrary to hypothesis, if s_{i+1}^+ and $\bar{s}_i^+$ intersect in a segment of length $\mu_i+\mu_{i+1}-\gcd(\mu_i,\mu_{i+1})$ or greater. So we may assume that this does not happen.
In particular, $\mu_{i+1}<l-\omega_{i+1}+\mu_i$, for otherwise $\bar{s}_i^+$ is a subword of s_{i+1}^+, of length

$$\omega_i+\mu_i-1>l/2+\mu_i-1>\mu_{i+1}+\mu_i-\gcd(\mu_i,\mu_{i+1}). \tag{3.28}$$

Hence $\bar{s}_i^+$ is a terminal segment of W, and the intersection of s_{i+1}^+ and $\bar{s}_i^+$ has length precisely $\omega_{i+1}+\mu_{i+1}+\omega_i-l-1$.
Thus

$$\omega_i+\omega_{i+1}<\mu_i+\mu_{i+1}-\gcd(\mu_i,\mu_{i+1})+l+1-\mu_{i+1}<l+\mu_i\le 3l/2 \tag{3.29}$$

as claimed.

□

Using Lemma 3.6.2, we complete the proof of Theorem 3.6.1 as follows. Renumbering the zones if necessary, we may assume by Lemma 3.6.2 that $\omega_1+\omega_2 < 3l/2$. Applying Lemma 3.6.2 again, with $i = 4$, either $\omega_3+\omega_4 < 3l/2$ or $\omega_4+\omega_5 < 3l/2$. In the first case $\omega_5 > l$; in the second $\omega_3 > l$. Either of these is a contradiction.

3.7 The relator has no letter of order 2

In this section our main aim is to prove the following.

Theorem 3.7.1. *Assuming Hypothesis B, any $\mathcal{W}$-minimal clique-picture over G on D^2 satisfies $C(6)$.*

In order to do so we will need a number of lemmas that are particular to the situation of Theorem 3.7.1, which we also collect together in this section.

Recall that

$$G = \frac{(G_1 * G_2)}{N(R^n)},$$

where $n \geq 2$ and the relation $R = R(a, UbU^{-1})$ contains no letters of order 2. We assume that v is a clique of degree less than 6 in a clique picture, joined to neighbouring cliques u_i by zones Z_i. The first step in our proof is designed to further restrict the form of R (and hence also of v).

Lemma 3.7.2. *For any zone Z_i, if $s(i)+j \equiv t(i)+m$ modulo l where $1 \leq j, m < \omega_i$, then s_i has an element of order 2 in G_1 or G_2.*

Proof. Suppose by way of contradiction that $s(i) + j \equiv t(i) + m$ modulo l where $1 \leq j, m < \omega_i$. Recall that s_i is a cyclic subword of

$$\text{label}(v) = z_0 z_1 \cdots z_{nl-1} = \prod_{k=1}^{n} [a^{\alpha(k)} U b^{\beta(k)} U^{-1}]. \tag{3.30}$$

Write $s_i = z_{s(i)+1} \cdots z_{s(i+1)-1}$. Similarly, t_i is a cyclic subword of

$$\text{label}(u_i) = y_0 y_1 \cdots y_{ql-1} = \prod_{k=1}^{q} [a^{\gamma(k)} U b^{\delta(k)} U^{-1}]. \tag{3.31}$$

Write $t_i = y_{t(i)+1} \cdots y_{t(i+1)-1}$. By hypothesis, $y_{t(i)+m} = z^{-1}_{s(i)+m'}$ for some m' with $1 \leq m' < \omega_i$. In particular $s(i) + j \equiv t(i) + m \equiv s(i) + m'$ modulo 2, since $y_{t(i)+m}$ and $z_{s(i)+m'}$ belong to the same free factor. Thus $j + m'$ is even. Moreover,

$z_{s(i)+(j+m')/2} = y^{-1}_{t(i)+(j+m')/2}$. If $z_{s(i)+(j+m')/2}$ is a special letter, then so is $y^{-1}_{t(i)+(j+m')/2}$. But in this case an amalgamation is possible, contrary to hypothesis. Otherwise $z_{s(i)+(j+m')/2} = y_{t(i)+(j+m')/2}$, so $z_{s(i)+(j+m')/2}$ has order 2 in G_1 or G_2. □

Lemma 3.7.3. *If no letter of $R(a, UbU^{-1})$ has order 2, then there is no zone Z_i with $\omega_i > l/2$.*

Proof. Suppose that $\omega_i > l/2$ for some zone Z_i. Write $s_i = z_{s(i)+1} \cdots z_{s(i+1)-1}$ and $t_i = y_{t(i)+1} \cdots y_{t(i+1)-1}$. By Lemma 3.7.2 we may assume that $s(i) + j \not\equiv t(i) + m$ modulo l for all $j, m \in \{1, 2, \ldots, \omega_i - 1\}$, so in particular $\omega_i \le l/2 + 1$. Moreover, if $\omega_i = l/2 + 1$ then we must have $t(i) + 1 \equiv s(i+1)$ modulo l. But since $z_{s(i+1)-1} = y^{-1}_{t(i)+1}$ belongs to the same free factor as $z_{t(i)+1} = z_{s(i+1)}$, this is a contradiction. Hence $\omega_i \le l/2$. □

In what follows next, we will need the following result about generalised triangle groups.

Proposition 3.7.4. *Let $H = \langle\, x,\ y \mid x^p,\ y^q,\ (xy)^2 \,\rangle$ be a triangle group. If $v(x, y) = x^\alpha y^\beta x^\gamma y^\delta$ is trivial in H, with $\alpha, \gamma \in \{1, 2, \ldots, p-1\}$ and $\beta, \delta \in \{1, 2, \ldots, q-1\}$, then one of the following holds:*

1. $2 \in \{p, q\}$;
2. $\alpha = \beta = \gamma = \delta = 1$;
3. $\alpha = \gamma = p - 1$ *and* $\beta = \delta = q - 1$.

Proof. Assume that $p \ne 2 \ne q$ and consider the elements $X = \begin{pmatrix} e^{\frac{\pi i}{p}} & 0 \\ 1 & e^{\frac{-\pi i}{p}} \end{pmatrix}$ and $Y = \begin{pmatrix} e^{\frac{\pi i}{q}} & t \\ 0 & e^{\frac{-\pi i}{q}} \end{pmatrix}$ in $PSL_2(\mathbb{C})$. It follows that X and Y have orders p and q respectively in $PSL_2(\mathbb{C})$. If we take $t = -2\cos\left(\frac{\pi}{p} + \frac{\pi}{q}\right)$, then $Tr(XY) = 0$ and hence the map $x \mapsto X$, $y \mapsto Y$ extends to a faithful representation of H in $PSL_2(\mathbb{C})$. Suppose that $X^\alpha Y^\beta = Y^{-\delta} X^{-\gamma}$. By comparing the left lower entries of both sides of the equation we have $\alpha \equiv \pm\gamma$ modulo p and

$$\sin\frac{\alpha\pi}{p} e^{\frac{\beta\pi i}{q}} \mp \sin\frac{\gamma\pi}{p} e^{\frac{\delta\pi i}{q}} = 0. \tag{3.32}$$

By expanding and solving component-wise, we have that

$$\sin\frac{\gamma\pi}{p} \sin\left(\frac{\beta - \delta}{q}\right)\pi = 0. \tag{3.33}$$

In particular, $\beta = \delta$. Similarly we have $\alpha = \gamma$.

Hence $v = (X^\alpha Y^\beta)^2 = \pm I$. By comparing off-diagonal entries, we conclude that $X^\alpha \neq \pm Y^{-\beta}$, so $v \neq +I$. Hence $v = -I$, and so $Tr(X^\alpha Y^\beta) = 0$. In other words

$$2\cos\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)\pi + t\frac{\sin\frac{\alpha\pi}{p}\sin\frac{\beta\pi}{q}}{\sin\frac{\pi}{p}\sin\frac{\pi}{q}} = 0. \tag{3.34}$$

Hence we deduce that

$$\tan\frac{\alpha\pi}{p}\tan\frac{\beta\pi}{q} = \tan\frac{\pi}{p}\tan\frac{\pi}{q}. \tag{3.35}$$

Since $p, q > 2$, the last equality holds if and only if either $\alpha = \beta = 1$ or $\alpha = p-1$ and $\beta = q-1$. $\square$

The clique v fails to satisfy the $C(6)$ property, so by Lemma 3.7.3 its label has length at most $5l/2$. But this length is a multiple of l, so at most $2l$. By Theorem 2.4.3 and Proposition 3.7.4 we can assume that $R = aUbU^{-1}$ (up to cyclic permutation) and that the label of v is label$(v) = (aUbU^{-1})^{\pm 2}$ (up to cyclic permutation). Without loss of generality we will assume throughout that this label is label$(v) = (aUbU^{-1})^2$ and the letters a, b will mean the corresponding special letters.
This remark enables us to strengthen Lemma 3.7.3 as follows.

Lemma 3.7.5. *If no letter of $R(a, UbU^{-1})$ has order 2, then there is no zone Z_i with $\omega_i \geq l/2$.*

Proof. By Lemma 3.7.3 we are reduced to the case where $\omega_i = l/2$. By Lemma 3.7.2, we must have $t(i) \in \{s(i+1)-1, s(i+1), s(i+1)+1\}$ modulo l. The first possibility leads to a contradiction as in the proof of Lemma 3.7.3. The third possibility also leads to a contradiction for similar reasons, since it implies that $t(i+1) \equiv s(i)+1$ modulo l. Hence we may assume that $t(i) \equiv s(i+1)$ modulo l and hence that

$$s(i) \equiv s(i+1) + l/2 \equiv t(i) + l/2 \equiv t(i+1) \text{ modulo } l. \tag{3.36}$$

If $z_{s(i)}$ is special, then so is $y_{t(i+1)}$, and we may amalgamate cliques, contrary to hypothesis. Hence $z_{s(i)}$ is not special. In other words $s(i) \not\equiv 0$ modulo $l/2$. Thus s_i contains precisely one special letter – say a without loss of generality. Hence also t contains precisely one special letter, which is necessarily a power of b. Thus $aUb^\psi U^{-1}$ is a proper cyclic permutation of $z_{s(i)}s_i z_{s(i+1)}t_i \equiv z_{s(i)}s_i z_{s(i+1)}s_i^{-1}$ for some ψ. Applying Lemma 3.2.7, we see that

$$aUb^\psi U^{-1} = \prod_{k=1}^{s}[a^{\alpha(k)}Vd^{\beta(k)}V^{-1}] \tag{3.37}$$

for some word V, some $\alpha(k), \beta(k) = \pm 1$ and some $d \in \{b^\psi, z_{s(i)}, z_{s(i+1)}\}$, where $s > 1$.

Moreover, from the proof of Lemma 3.2.7 we see that we may take $d = b^{\psi}$ if s is odd, while if s is even then $a = b^{\psi}$. In either case, $\langle a \rangle * \langle UbU^{-1} \rangle$ is a proper subgroup of $\langle a \rangle * \langle VbV^{-1} \rangle$ or $\langle a \rangle * \langle VdV^{-1} \rangle$, giving a refinement of the generalised triangle group description of G. This contradiction completes the proof. □

Remark 3.7.6. From Lemma 3.7.5, our interior clique v which fails the $C(6)$ condition must have exactly five zones. Let these zones be Z_1, Z_2, …, Z_5 listed consecutively in clockwise order.

Proposition 3.7.7. *There are exactly three or four of the zones Z_1, Z_2, …, Z_5 containing a special letter.*

Proof. By assumption

$$\text{label}(v) = (aUbU^{-1})^2 = z_0 z_1 \cdots z_{2l-1} \tag{3.38}$$

so there are precisely four special letters in label(v), namely $z_0 = z_l = a$ and $z_{l/2} = z_{3l/2} = b$. By Lemma 3.7.3 no zone can contain more than one special letter, so it suffices to show that at most one of the special letters is not contained in a zone. Suppose by way of contradiction that z_0 and z_l are not contained in zones. By Lemma 3.7.5, each of the subwords $z_1 \cdots z_{l-1}$ and $z_{l+1} \cdots z_{2l-1}$ must contain at least three zones, contradicting our assumption that there are only five zones in total. A similar contradiction arises if z_0 and $z_{l/2}$ are not contained in zones: $z_1 \cdots z_{l/2-1}$ and $z_{l/2+1} \cdots z_{2l-1}$ must contain at least two and four zones respectively. By symmetry, any other combination of two special letters not contained in zones also contradicts our underlying hypotheses, hence the result. □

Therefore we have exactly two possible configurations as depicted in the Figures 3.12 and 3.13 below.

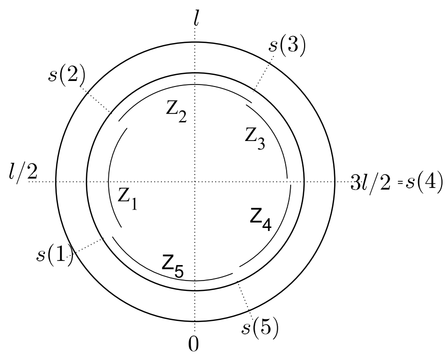

Figure 3.12: Diagram showing the case where only the special letters z_0, $z_{l/2}$ and z_l are contained in zones

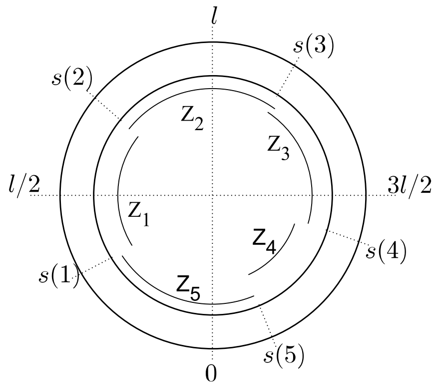

Figure 3.13: Diagram showing the case where all four special letters z_0, $z_{l/2}$, z_l and $z_{3l/2}$ are contained in zones

Remark 3.7.8. For each $z_{s(i)}$, we have $s(i) \neq s(j)$ modulo $l/2$ for $i \neq j$ since no $\omega_i \geq l/2$.

Remark 3.7.8 gives the inequalities

$$0 < s(3) < s(1) < l/2 = s(4) < s(2) < s(5) \leq l - 1 \tag{3.39}$$

and

$$0 < s(3) < s(1) < l/2 < s(4) < s(2) < s(5) \leq l - 1 \tag{3.40}$$

corresponding to Figure 3.12 and Figure 3.13 respectively. (Note that $s(i)$ in the inequalities is the modulo l equivalent. The actual value can be read from the Figures 3.12 and 3.13. For example the actual value for $s(4)$ in Figure 3.12 is $3l/2$.) The corresponding subwords are:

$$\begin{aligned}
s_1 &= z_{s(1)+1} \cdots z_{s(2)-1}, \\
s_2 &= z_{s(2)+1} \cdots z_{s(3)-1}, \\
s_3 &= z_{s(3)+1} \cdots z_{s(4)-1}, \\
s_4 &= z_{s(4)+1} \cdots z_{s(5)-1}, \\
s_5 &= z_{s(5)+1} \cdots z_{s(1)-1}.
\end{aligned}$$

Lemma 3.7.9. *Let W be a cyclically reduced word of length $2m$ in a free product. Let X be a reduced word of length m such that both X and X^{-1} appear as cyclic subwords of W. Then X contains a letter of order 2.*

Proof. The subwords X and X^{-1} of W must intersect, for otherwise W is a cyclic conjugate of XX^{-1}, contradicting the fact that it is cyclically reduced. Hence there is an initial segment Y of X or X^{-1} that coincides with a terminal segment of X^{-1} or X respectively. Thus $Y \equiv Y^{-1}$ and so Y has an odd length and its middle letter has order 2. $\square$

More notations

Think of $aUbU^{-1} = z_0 \cdots z_{l-1}$ as a cyclic word satisfying a partial reflectional symmetry using the special letters as mirrors. Thus U has *mirror image* U^{-1}. More generally the *mirror image* of $z_j \ldots z_k$ is $z_{l+2-j} \ldots z_{l+2-k}$ (subscripts modulo l). Unless stated otherwise, X (with or without subscript) denotes an initial or terminal segment of U. Similarly Y (with or without subscript) will denote an initial or terminal segment of U^{-1}. Also X^{-1} and Y^{-1} are mirror images of X and Y respectively. Using this notation, we can express the subwords in the zones as:

$$\begin{aligned} s_1 &= X_1 x_1 Y_1, \\ s_2 &= Y_2 y_2 X_2, \\ s_3 &= X_3 x_3 Y_3, \\ s_5 &= Y_5 y_5 X_5. \end{aligned}$$

where $x_i = b$ and $y_i = a$ are the corresponding special letters. In other words if X_i is a terminal segment of s_i (as in the case of s_2), then it is an initial segment of U. In which case Y_i is a terminal segment of U^{-1}. If $s(4) = l/2$, then $s_4 = Y_4$. Otherwise, s_4 is a subword of U^{-1} which is neither an initial nor terminal segment. Note that some of the x_i, X_i, y_i and Y_i are allowed to be empty as in the case of s_3 in Figure 3.13.

Let s'_i be the mirror image of t_i. Then s'_i is identically equal to s_i (modulo $\sim$). In particular if $s_i = X_i x_1 Y_i$, then $s'_i = X'_i x'_1 Y'_i$ where $X'_i \sim X_i$, $x'_1 \sim x_1$ and $Y'_i \sim Y_i$.

We use M to denote an initial segment of U or a terminal segment of U^{-1}. Similarly N denotes a terminal segment of U or an initial segment of U^{-1}. M^+_k is the initial segment of U or terminal a terminal segment of U^{-1} of length $\ell(M) + k$. If M is a subword of s_i, then M_i denotes the image of M under Z_i and M' is the mirror image of M_i. N^+_k, N_j and N' are defined similarly.

Remark 3.7.10. One thing to note is that if M is a subword of U, say, with $\ell(M) \geq l/4$, then M_i can not be a subword of U by Lemma 3.7.2. So that either M' is a subword of U or neither a subword of U nor U^{-1}. In the case where M and M' are both subwords of U or U^{-1}, then what we call M^+_k will be the union of the two. It follows that $\ell(M^+_k) > \ell(M)$ for otherwise $M' = M$ (i.e M' is also an initial segment of U), and hence M_i is a terminal segment of U^{-1}, and hence there is an amalgamation of cliques, contrary to hypothesis.

Lemma 3.7.11. *Let M, M' and $\ell(M)+k$ be as in Remark 3.7.10. If $\ell(M) \geq l/4$ and both M and M' are subwords of U or U^{-1}, then M_k^+ has period $\gamma = \ell(M_k^+) - \ell(M) \leq \ell(U) - \ell(M)$.*

The proof of Lemma 3.7.11 follows easily from Remarks 3.2.4 and 3.7.10.

Lemma 3.7.12. *Suppose that U has a period $\gamma < \ell(U)$ with X and Y as initial and terminal segments respectively both of length $\gamma/2$. Then no segment of s_i is of the form Xy_iX^{-1} or Yx_iY^{-1}.*

Proof. Suppose without loss of generality that s_i has Xy_iX^{-1} as a segment. Then it is identically equal to a subword of UxU^{-1}. Thus X is identically equal to a subword of U or U^{-1}. Take W to be any subword of U of length γ. The periodicity of U implies that each of X, X^{-1} is identically equal to a cyclic subword of W. By Lemma 3.7.9 it follows that X contains a letter of order 2 contrary to hypothesis. $\square$

Lemma 3.7.13. *Suppose that W has period $\gamma < \ell(W)$ and no element of order 2. Then W has no subword of the form $L = wrw^{-1}$ with $\ell(w) \geq \gamma/2$.*

Proof. Suppose by contradiction that W has a subword of the form $L = wrw^{-1}$ with $\ell(w) \geq \gamma/2$. Let r_1 and r_2 be the letters $\gamma/2$ places before and after r in L respectively, then $r_1^{-1} = r_2$. But by the periodicity of W, $r_1 = r_2$. Hence W has an element of order 2 contradicting hypothesis. $\square$

Lemma 3.7.14. *Let X be an initial (respectively terminal) segment of U of length $\ell(X) \geq l/4$. Suppose $s_i = YaX$ (respectively $s_i = XaY$) for some terminal (respectively initial) segment Y of U^{-1}. If s_i' is not contained entirely in U, then U has period $\gamma \leq 2(\ell(U) - \ell(X))$. Furthermore if $\gamma = 2(\ell(U) - \ell(X))$, then U has a terminal (respectively initial) segment of the form $xw^{-1}z_{s(i+1)}w$ (respectively $wz_{s(i)}w^{-1}x$) for some letter x and some word w with $\ell(w) = \gamma/2 - 1$.*

Proof. By symmetry it suffices to prove the first case, where X is an initial segment of U. Write $U \equiv Xz_{s(i+1)}V$ for some terminal segment V of U. Consider the mirror image s_i' of t_i. By Lemma 3.7.2, t_i is a subword of $Vb^{\psi}U^{-1}$ for some ψ. Hence s_i' is a subword of $Ub^{-\psi}V^{-1}$. Therefore s_i' has the form $w_1b^{-\psi}w_2^{-1}$ for some terminal segments w_1 and w_2 of U with $\ell(w_2) \leq \ell(V) = \ell(U) - (\ell(X)+1) < \frac{l}{4} - 1$. Denote by S the terminal segment of U of length

$$\begin{aligned}\ell(S) &= \ell(X) - \ell(w_2) - 1 \\ &= \ell(s_i') - \ell(w_2) - \ell(Y) - 2 \\ &= \ell(w_1) - \ell(Y) - 1. \end{aligned} \tag{3.41}$$

Then $w_1 \equiv YaS$ and $X \equiv Sb^{-\psi}w_2^{-1}$ and so S is identically equal to an initial segment of U. Thus by Remark 3.2.4, we have that U has period

$$\begin{aligned}\gamma &= \ell(U) - \ell(S) \\ &= \ell(U) - \ell(X) + \ell(w_2) + 1 \\ &\leq 2(\ell(U) - \ell(X)). \end{aligned} \tag{3.42}$$

Finally if $\gamma = 2(\ell(U) - \ell(X))$, then $1 + \ell(X) + \ell(w_2) = \ell(U)$. Thus $V \equiv w_2$ and so $U \equiv Xz_{s(i+1)}V \equiv Sb^{-\psi}w_2^{-1}z_{s(i+1)}w_2$. The result follows by talking $w = w_2$ and $x = b^{-\psi}$.

□

Lemma 3.7.15. *Let Z_i and Z_j be distinct. Suppose that the intersection of s_i and s_j contains a subword of the form $L = w\phi w^{-1}$ where ϕ is a special letter. Let L_i and L_j be the images of L under Z_i and Z_j respectively, and $[L_i], [L_j]$ their $\sim$-classes in $(\Omega/\sim)^*$. Then $[L_i]$ and $[L_j]$ do not intersect as cyclic subwords of the $\sim$-class of $aUbU^{-1}$ in $(\Omega/\sim)^*$.*

Proof. Let σ be the intersection of s_i and s_j. Since σ is non-empty, the zones Z_i and Z_j are not consecutive, so $i - j \equiv \pm 2$ modulo 5. Without loss of generality, suppose that $j \equiv i + 2$ modulo 5. Then σ is an initial segment of s_i and a terminal segment of s_j. So $\sigma = z_{s(i)+1} \dots z_{s(j+1)-1}$ where $s_j = z_{s(j)+1} \dots z_{s(j+1)-1}$ and $s_i = z_{s(i)+1} \dots z_{s(j-1)-1}$. Choose L with maximal length among all subwords of σ of that form. Let ϱ be the union of s_i and s_j. Then $\varrho = z_{s(j)+1} \dots z_{s(j-1)-1}$ and a cyclic permutation of the clique-label of v has the form $(\varrho z_{s(j-1)} s_{j-1} z_{s(j)})^2$.

Now define $t_j' = z_{t(j)+1} \dots z_{t(j+1)-1}$ and $t_i' = z_{t(i)+1} \dots z_{t(i+1)-1}$. Then t_i' is identical to the image t_i of s_i, with the possible exception of a special letter of t_i: the corresponding letter of t_i' is also special, and these two special letters may be different powers of a (or of b). Similarly, t_j' agrees with t_j except possibly at a special letter. Now define σ_i to be the terminal segment $\sigma_i = z_{d+1} \dots z_{t(i+1)-1}$ of t_i', where

$$d := t(i+1) + s(i) - s(j+1) \text{ modulo } l, \tag{3.43}$$

and σ_j to be the initial segment $\sigma_j = z_{t(j)+1} \dots z_{e-1}$ of t'_j, where

$$e := t(j) + s(j+1) - s(i) \text{ modulo } l. \tag{3.44}$$

Then σ_i and σ_j agree with the images L_i and L_j of σ under Z_i and Z_j respectively, again with the possible exception that they may differ at a special letter.

If $[L_i]$ and $[L_j]$ coincide as cyclic subwords of $[aUbU^{-1}]$, then σ_i and σ_j coincide as cyclic subwords of $aUbU^{-1}$. In this case we can form the union $\sigma* = t'_i \cup t'_j = z_{t(i)+1} \dots z_{t(m)-1}$: a cyclic subword of $aUbU^{-1}$ that is disjoint from σ and hence contains at most one special letter. But σ^* is identically equal to ϱ^{-1}, with the possible exception of a special letter of σ^*. It follows from Corollary 3.7.5 that $\ell(\varrho) > l/2$. Thus $\varrho \cap \sigma* \neq \emptyset$ and so without loss of generality some initial segment τ of ϱ coincides with a terminal segment of σ^*. Since the only special letter of ϱ is ϕ which does not appear in σ^*, τ does not contain a special letter. It follows that $\tau^{-1} \equiv \tau$, whence τ contains a letter of order 2, contrary to hypothesis.

Suppose then that $[L_i]$ and $[L_j]$ intersect but do not coincide. Consider the subword $\varrho* = \sigma_i \cup \sigma_j$ of $t'_i \cup t'_j$. As above, ϱ^* is a cyclic subword of $aUbU^{-1}$ which is disjoint from σ and hence contains at most one special letter. Write $\varrho* = x_1 x_2 \dots x_r$ for some r. Note that r is odd, since by definition σ begins and ends with letters in the same free factor, and hence the same holds for ϱ.
Assume without loss of generality that σ_i, σ_j are the initial and terminal segments respectively of ϱ^*. Then

$$\sigma_i = x_1 x_2 \dots x_{\ell(L)} \tag{3.45}$$

and

$$\sigma_j = x_{r+1-\ell(L)} x_{r+2-\ell(L)} \dots x_r. \tag{3.46}$$

Since each of σ_i, σ_j agrees with σ^{-1} except possibly at a special letter of ϱ^*, it follows that, for $1 \le \mu \le \ell(L)$,

$$x_\mu = x_{r+\mu-\ell(L)} \tag{3.47}$$

unless one of $x_\mu, x_{r+\mu-\ell(L)}$ is a special letter of ϱ^*.
Also, since σ^{-1} agrees with σ except for the middle letter $\phi^{\pm 1}$, it follows that for $1 \le \mu \le \ell(L)$,

$$x_\mu = x^{-1}_{\ell(L)+1-\mu} \tag{3.48}$$

unless either $\mu = (\ell(L)+1)/2$ or one of $x_\mu = x^{-1}_{\ell(L)+1-\mu}$ is a special letter of ϱ.
Similarly, for $r + 1 - \ell(L) \le \mu \le r$,

$$x_\mu = x^{-1}_{2r-\ell(L)-\mu} \tag{3.49}$$

unless either $\mu = r - \ell(L)/2$ or one of $x_\mu = x^{-1}_{2r-\ell(L)-\mu}$ is a special letter of ϱ^*.

Now consider the three letters $x_{\ell(L)+1-(r+1)/2}$, $x_{(r+1)/2}$ and $x_{3(r+1)/2-\ell(L)-1}$ of ϱ^*. We know at most one of these three letters can be special.

Suppose first that neither $x_{\ell(L)+1-(r+1)/2}$ nor $x_{(r+1)/2}$ is special. Then

$$\begin{aligned} x_{\frac{r+1}{2}} &= x^{-1}_{\ell(L)+1-\frac{r+1}{2}} && \text{(by Equation 3.48)} \\ &= x^{-1}_{r+\ell(L)+1-\frac{r+1}{2}-\ell(L)} && \text{(by Equation 3.47)} \\ &= x^{-1}_{\frac{r+1}{2}} && (3.50) \end{aligned}$$

It follows that $x_{(r+1)/2}$ has order 2, contrary to hypothesis.

Similarly, if neither $x_{(r+1)/2}$ nor $x_{3(r+1)/2-\ell(L)-1}$ is special, then we may deduce that $x_{(r+1)/2}$ using Equations 3.47 and 3.49.

Finally, suppose that $x_{(r+1)/2}$ is a special letter of ϱ^*. Then $x_\mu = x^{-1}_{r+1-\mu}$ for each $\mu = 1, 2, \ldots, (r-1)/2$. In particular

$$x_{\frac{\ell(L)+1}{2}} = x^{-1}_{r+1-\frac{\ell(L)+1}{2}}. \tag{3.51}$$

But by Equation 3.47,

$$x_{\frac{\ell(L)+1}{2}} = x_{r+1-\frac{\ell(L)+1}{2}}. \tag{3.52}$$

Hence $x_{(\ell(L)+1)/2}$ has order 2, again contrary to hypothesis. This completes the proof. □

Remark 3.7.16. We remark that the first part of the proof of Lemma 3.7.15 does not assume any form for the intersection.

3.7.1 Proof of Theorem 3.7.1

We are now ready to complete the proof of Theorem 3.7.1. Our proof is by contradiction. Suppose that some interior vertex v of Γ fails to satisfy $C(6)$, then we know from Remark 3.7.6 that v has exactly five incident zones $\mathrm{Z}_1, \ldots, \mathrm{Z}_5$. We also assume by Corollary 3.7.5 and Remark 3.7.6 that $\omega_i < l/2$ for $i = 1, \ldots, 5$. The proof is divided into two cases.

Case 1. $s(4) = l/2$.

Case 2. $s(4) > l/2$.

Proof of Case 1. For Case 1 (alternatively case 3.39), take N to be the longer of X_3 and Y_4 and M to be the longer of X_5 and Y_2. In each case M is either an initial segment of U or a terminal segment of U^{-1}. Similarly N is either a terminal segment of U or an initial segment of U^{-1}. Also $\ell(N), \ell(M) \geq 1/4$ for otherwise $\ell(s_i) \geq l/2$ for some $i \in \{1, 2, 5\}$, contradicting hypothesis.

By Lemma 3.7.14, U has an initial segment of period $\rho \leq 2(\ell(U) - \ell(M))$ and a terminal segment of period $\gamma \leq 2(\ell(U) - \ell(N))$. If $\gamma < 2(\ell(U) - \ell(N))$, then $L = z_{l-\gamma/2} z_{l+1-\gamma/2} \dots z_{l+\gamma/2}$ is a proper subword of $s_2 \cap s_5$. It follows from Lemma 3.7.15 that at least one of the images of L under Z_2 and Z_5 is identically equal to a periodic subword of U (with period γ). This can not happen by Lemma 3.7.13.

Otherwise U has period $\gamma = 2(\ell(U) - \ell(N))$. It is easy to see that

$$\ell(X_1), \ell(Y_1) \geq \ell(U) - \ell(N) \tag{3.53}$$

as otherwise either $\ell(s_2) \geq l/2$ or $\ell(s_5) \geq l/2$, thereby contradicting hypothesis. Hence the result follows by applying Lemma 3.7.12 to s_1.

□

For Case 2 (alternatively case 3.40), take N to be the longer of X_3 and Y_1 and M to be the longer of X_5 and Y_2. In each case M is either an initial segment of U or a terminal segment of U^{-1}. Similarly N is either a terminal segment of U or an initial segment of U^{-1}. Also $\ell(N), \ell(M) \geq 1/4$. The proof is subdivided into three sub-cases namely:

Case 2a. Each of M' or N' is identically equal to a subwords of U^{-1} or U.

Case 2b. Exactly one of M' or N' is identically equal to a subword of U or U^{-1}.

Case 2c. Both M' and N' are not identically equal to a subword of U^{-1} or U.

Proof of Case 2a. Using Lemma 3.7.11, we conclude that U has a periodic initial segment M^+_γ of period $\gamma \leq \ell(U) - \ell(M)$ and a periodic terminal segment N^+_ρ of period $\rho \leq \ell(U) - \ell(N)$. Moreover by Remark 3.7.6, the two segments intersect in a segment S with

$$\ell(S) = \ell(M^+_\gamma) + \ell(N^+_\rho) - \ell(U) \geq \rho + \gamma + 1 > \rho + \gamma.$$

Hence U is periodic with period $\nu = gcd(\rho, \gamma) \leq \min\{\rho, \gamma\} \leq \min\{\ell(Y_1), \ell(X_1)\}$ by Corollary 3.2.6. It follows that s_1 contains a subword the form $w x_1 w^{-1}$ with $\ell(w) = \nu/2$. The result will then follow from Lemma 3.7.12. □

Proof of Case 2*b*. Suppose N' is identically equal to a subword of U or U^{-1}. By Lemma 3.7.14, U is periodic with period $\nu \leq 2(\ell(U)-\ell(M))$. If $\nu < 2(\ell(U)-\ell(M))$, the result follows by applying Lemma 3.7.12 to s_1 as in Case 2a.

Hence we may assume that $\nu = 2(\ell(U) - \ell(M))$. Then by Lemma 3.7.14 U a has a terminal segment of the form $u_1 w u_2 w^{-1}$ for some letters u_1, u_2 and some word w with $\ell(w) = \nu/2 - 1$.
Also by Lemma 3.7.11, U has a periodic terminal segment N_ρ^+ of length $\ell(N_\rho^+) = \ell(N) + \rho$ and period $\rho \leq \ell(U) - \ell(N)$. If $\rho + 2 \leq \nu$, then N_ρ^+ has a subword of the form $\hat{w} u_2 \hat{w}^{-1}$ where $\hat{w}$ is the terminal segment of w of length $\rho/2$. This contradicts Lemma 3.7.13.
Finally if $\rho + 2 > \nu$, then $\rho + \nu \leq 2\rho + 1 \leq \ell(N) + \rho$. It follows from Corollary 3.2.6 that U has period $\lambda = \gcd(\nu, \rho)$. Hence we can apply Lemma 3.7.12 to s_2 since

$$\frac{\lambda}{2} \leq \lambda - 1 \leq \rho - 1 \leq \ell(U) - \max\{\ell(X_3), \ell(Y_1)\} - 1 \leq \min\{X_2, Y_2\}. \tag{3.54}$$

The proof for the case where M' is identically equal to a subword of U or U^{-1} is similar. $\square$

Proof of Case 2*c*. Then U is periodic with periods $\rho \leq 2(\ell(U) - \ell(M))$ and $\gamma \leq 2(\ell(U) - \ell(N))$. If $\rho < 2(\ell(U) - \ell(M))$ or $\gamma < \rho$, then the result follows by applying Lemma 3.7.12 to s_2. Similarly if $\gamma < 2(\ell(U) - \ell(N))$ or $\rho < \gamma$, then the result follows by applying Lemma 3.7.12 to s_1.

Hence suppose $\rho = \gamma = 2(\ell(U) - \ell(M)) = 2(\ell(U) - \ell(N))$. Then we have $\ell(X_5) \leq \ell(M) = \ell(U) - \gamma/2$, whence $\ell(X_1) = \ell(U) - 1 - \ell(X_5) \geq \gamma/2 - 1$ and so

$$\ell(Y_1) = \ell(s_1) - 1 - \ell(X_1) < \ell(U) - 1 - \ell(X_1) \leq \ell(U) - \gamma/2 = \ell(N). \tag{3.55}$$

It follows that $N = X_3$. By a similar argument, $M = X_5$. It follows from Lemma 3.7.14 that U has an initial segment of the form $w_1 z_{s(3)} w_1^{-1} u_1$ and a terminal segment of the form $u_2 w_2 z_{s(1)}^{-1} w_2^{-1}$ for some letters $u_1 = z_\rho, u_2 = z_{l/2-\rho}$, and words w_1, w_2 satisfying $\ell(w_1) = \ell(w_2) = \rho/2 - 1$.

We also have from Lemma 3.7.14 that $t(3) + \omega_3 \equiv s(3) \equiv \rho/2$ modulo l. The $(l-\rho)/2$-th letter of s_3 is $x_3 = z_{l/2} = b$. The corresponding letter of t_3 is therefore $z_{s(3)-(l-\rho)/2} = z_{l/2+\rho}$, which is not a special letter of t_3. Hence $z_{l/2+\rho} = z_{l/2}^{-1} = b^{-1}$. Therefore

$$u_2 = z_{\frac{l}{2}-\rho} = z_{\frac{l}{2}+\rho}^{-1} = b. \tag{3.56}$$

A similar argument shows that

$$u_1 = z_\rho = z_{l-\rho}^{-1} = y_5 = a. \tag{3.57}$$

Moreover by the periodicity of U

$$w_1 z_{s(3)} w_1^{-1} a = z_1 \cdots z_\rho \tag{3.58}$$

is a cyclic conjugate of

$$b w_2 z_{s(1)}^{-1} w_2^{-1} = z_{\frac{l}{2}-\rho} \cdots z_{\frac{l}{2}-1}. \tag{3.59}$$

Now apply Lemma 3.2.7 to this pair of cyclically conjugate words to obtain

$$b w_2 z_{s(1)}^{-1} w_2^{-1} = \prod_{t=1}^{s} b^{\beta(t)} V x^{\alpha(t)} V^{-1} \tag{3.60}$$

for some letter x, some word V and some integer s and some $\alpha(t), \beta(t) \in \{\pm 1\}$. The integer s in the statement of Lemma 3.2.7 is defined to be k/m, where $k = \rho/2$ and $m = \gcd(j,k)$. Here j in turn is defined as the number of places by which one has to cyclically permute $b w_2 z_{s(1)}^{-1} w_2^{-1}$ to obtain $w_1 z_{s(3)} w_1^{-1} a$ – in other words

$$z_1 \cdots z_\rho \equiv z_{\frac{l}{2}-\rho+j} \cdots z_{\frac{l}{2}-1} z_{\frac{l}{2}-\rho} \cdots z_{\frac{l}{2}-\rho+j-1}. \tag{3.61}$$

Suppose first that s is even. Then Lemma 3.2.7 gives that $b = z_{s(1)}$, $a = z_{s(3)}^{-1}$, and $x = a$ in the above expression. This U has a terminal segment

$$b w_2 z_{s(1)}^{-1} w_2^{-1} = \prod_{t=1}^{s} b^{\beta(t)} V a^{\alpha(t)} V^{-1} \tag{3.62}$$

Similarly U has an initial segment of the form

$$w_1 z_{s(3)} w_1^{-1} a = \prod_{t=1}^{s} V^{-1} b^{\delta(t)} V a^{\gamma(t)} \tag{3.63}$$

Putting these together, using the periodicity of U, we obtain

$$U = V^{-1} \prod_{t=1}^{p} b^{\zeta(t)} V a^{\eta(t)} V^{-1} \tag{3.64}$$

for some integer p and some $\zeta(t), \eta(t) \in \{\pm 1\}$.
Thus $UbU^{-1} \in \langle a, V^{-1}bV \rangle$ and we have a refinement of our generalised triangle group description of G, contrary to our underlying hypotheses.

Next suppose that s is odd in Lemma 3.2.7. Then $\{b, z_{s(1)}^{-1}\} = \{a, z_{s(3)}\}$, and we have an expression

$$b w_2 z_{s(1)}^{-1} w_2^{-1} = \prod_{t=1}^{s} b^{\beta(t)} V z_{s(1)}^{\alpha(t)} V^{-1} \tag{3.65}$$

and an analogous expression

$$w_1 z_{s(3)} w_1^{-1} a = \prod_{t=1}^{s} V^{-1} z_{s(3)}^{\delta(t)} V a^{\gamma(t)} \tag{3.66}$$

Again we can fit these together using the periodicity of U to get an expression for U.
If $b = z_{s(3)}$ and $a = z_{s(1)}^{-1}$ then again this has the form

$$U = V^{-1} \prod_{t=1}^{p} b^{\zeta(t)} V a^{\eta(t)} V^{-1} \tag{3.67}$$

which leads to a refinement, contrary to hypothesis.
If on the other hand $b = a$ and $z_{s(3)} = z_{s(1)}^{-1}$ then we obtain an expression of the form

$$U = V z_{s(3)}^{\eta(0)} V^{-1} \prod_{t=1}^{p} a^{\zeta(t)} V z_{s(3)}^{\eta(t)} V^{-1} \in \langle a, V z_{s(3)} V^{-1} \rangle \tag{3.68}$$

As before, this yields a refinement, contrary to hypothesis. This completes the proof. □

3.8 Proof of Theorems

Here we give the proofs for Theorems 3.1.1, 3.1.2 and 3.1.3. As before we suppose that the triangle group description for G is maximal. By Theorems 3.6.1 and 3.7.1, an $\mathcal{W}$-minimal clique-picture over G satisfies the $C(6)$ property.

Theorem 3.8.1 (Theorem 3.1.1)**.** *The maps $G_1 \to G$, $G_2 \to G$ and $\mathcal{H} \to G$ are all injective.*

Proof. Suppose that there is a non-trivial word w in $\mathcal{H}$, G_1 or G_2 that is trivial in G. Then we obtain a $\mathcal{W}$-minimal picture P over G on D^2 with boundary label w. We prove the theorem by induction on the number of cliques in P; the case of 0 cliques corresponds to the empty picture P, for which there is nothing to prove.
Suppose first that some clique v in P is not simply connected, and let C be one of the boundary components of the surface carrying the clique v. By an innermost argument, we may assume that C bounds a disc $D \subset D^2$ such that every clique in $P \cap D$ is simply-connected (see Figure.

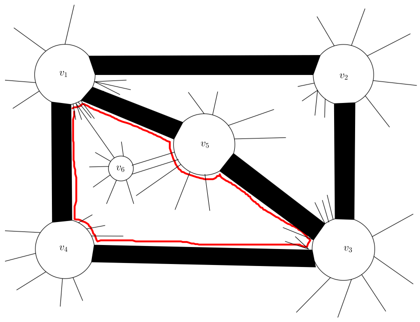

Figure 3.14: Diagram showing a clique containing vertices $v_1, v_2, \ldots, v_5$ which is not simply-connected. The region bounded by v_1, v_5, v_4, v_5 and containing v_6 (indicated by the red curve) contains only simply-connected cliques, and so is a good choice for C since v_6 is in $P \cap C$.

Now the label on C is a word in H which is the identity in G, and $P \cap D$ has at least one fewer clique than P, so by inductive hypothesis the label is trivial in $\mathcal{H}$. We then amend P by replacing $P \cap D$ by a picture over $\mathcal{H}$, all of the arcs, vertices and regions of which will belong to the same clique as v in the amended picture P'. Since $C \cap P$ was not empty (for otherwise D is contained in v), the new picture P' also has fewer cliques than P, and the result follows from the inductive hypothesis. Hence we are reduced to the situation where every clique in P is simply connected, and hence we may form from P a clique-picture Γ over G on D^2 with boundary label w. Without loss of generality, we may assume that Γ is $\mathcal{W}$-minimal. It follows that Γ satisfies $C(6)$.

If Γ is empty, then w is already trivial in $\mathcal{H}$, G_1 or G_2 and so we get a contradiction. On the other hand suppose that Γ is non-empty. If no arcs of Γ meet ∂D^2, then Γ is a spherical picture (i.e a picture on S^2) and the $C(6)$ property implies $\chi(S^2) \neq 2$. This contradiction implies $G_1 \to G$ and $G_2 \to G$ are both injective.

Suppose then that some arcs of Γ meet ∂D^2. Then $w \notin G_1, G_2$. Moreover if w is a word in $\{a, UbU^{-1}\}$, then the $C(6)$ condition combined with Theorem 1.5.5 implies that some boundary clique v_0 has at most degree three.

Under Hypothesis A, by Lemma 3.5.6, v_0 is connected to ∂D^2 by a zone Z_i with $\omega_i > l$. Either a refinement is possible by Corollary 3.3.1 or we can amalgamate v_0 with ∂D^2 to form a new clique-picture with fewer cliques whose boundary label also belongs to $\mathcal{H}$. The former possibility contradicts the maximality of the triangle group description for G while the latter contradicts the minimality of Γ.

Under Hypothesis B, it follows from Remark 3.7.6 that v_0 is connected to ∂D^2 by a zone Z_i with $\omega_i > l/2$. By Corollary 3.7.5, U has a letter of order 2 or a refinement is possible, contradicting hypothesis. Otherwise as before we can amalgamate v_0 with ∂D^2 to form a new clique-picture with fewer cliques. Either case is a contradiction. $\square$

Theorem 3.8.2 (Theorem 3.1.2)**.** *If the word problems are soluble for $\mathcal{H}$, G_1 and G_2, then it is soluble for G.*

Proof. Any clique-picture Γ over G satisfies $C(6)$, and hence a quadratic isoperimetric inequality by Theorem 1.5.6. In other words, there is a quadratic function f such that any word of length m representing the identity element of G is the boundary label of a clique-picture with at most $f(m)$ cliques. Also there is a bound (as a function of m) on the length of any clique-label of Γ. By Corollary 3.5.6 and Corollary 3.7.5, a clique with label of length ml has degree at least m. Moreover, there is a linear isoperimetric inequality of the form

$$\ell(\Gamma) \geq \sum_v [\deg(v) - 6] \tag{3.69}$$

where $\ell(\Gamma)$ is the number of regions of Γ and the sum is over all cliques v. Hence no clique can have degree greater than $\ell(\Gamma) + 6f(\ell(\Gamma))$. Since no zone has length greater than l, this gives an upper bound of $l[\ell(\Gamma) + 6f(\ell(\Gamma))]$ on the length of any clique-label. Since both the number of cliques and the length of any clique-label are bounded, there are only a finite number of connected graphs that could arise as clique-pictures for words of length less than or equal to that of a given word w. Moreover, any such graph can be labelled as a clique-picture only in a finite number of ways. For any such potential labelling, we may check whether or not the clique-labels are equal to the identity in $\mathcal{H}$, and whether or not the region-labels are equal to the identity in G_1 or G_2, using the solution to the word problem in H, G_1 and G_2 respectively. Hence we may obtain an effective list of all words of length less than or equal to $\ell(w)$ that appear as boundary labels of connected clique-pictures over G. In particular, we may check, for all cyclic subwords w_1 of w, whether or not $w_1 g$ belongs to this list for some letter $g \in G_1 \cup G_2$. (Note that this check also uses the solution to the word problem in G_1 and G_2, and that the letter g, if it exists, is unique by the Freiheitssatz). If so, then w is a cyclic conjugate of $w_1 w_2$ for some w_2 , so $w = 1$ in G if and only if $g = w_2$ in G, which we may assume inductively is decidable. Hence the word problem is soluble for G. □

Theorem 3.8.3 (Theorem 3.1.3)**.** *If a cyclic permutation of R^n has the form $W_1 W_2$ with $0 < \ell(W_1), \ell(W_2) < \ell(R^n)$ as words in G, then $W_1 \neq 1 \neq W_2$ as words in G. In particular R has order n in G.*

Proof. Suppose by way of contradiction that $W_1 = 1 = W_2$ in G. We can assume by Theorem 3.1.1 that $\ell(W_1) \neq 1 \neq \ell(W_2)$. We obtain a $\mathcal{W}$-minimal clique-picture Γ over G with boundary label W_1 or W_2. Suppose without loss of generality that Γ has boundary label W_1. Form a new clique picture $\tilde{\Gamma}$ with boundary label W_2^{-1} by adding a vertex labelled R^{-n}. $\tilde{\Gamma}$ has only one boundary vertex and is reduced since Γ is $\mathcal{W}$-minimal. It follows from Theorem 1.5.5 that $\tilde{\Gamma}$ has a single vertex or clique. Hence up to conjugacy W_2 (and hence W_1) is a word in H which is trivial and has length strictly less than the length of R^n. This contradicts the Spelling Theorem, hence the result. $\square$

Theorem 3.8.4. *[12] The pushout of groups in Figure 3.1 is* geometrically Mayer-Vietoris *in the sense of [23]. In particular it gives rise to Mayer-Vietoris sequences*

$$\cdots \to H_{k+1}(G, M) \to H_k(A * B, M) \to$$

$$H_k(G_1 * G_2, M) \oplus H_k(H, M) \to H_k(G, M) \to \cdots$$

and

$$\cdots \to H^k(G, M) \to H^k(G_1 * G_2, M) \oplus H^k(H, M)$$
$$\to H^k(A * B, M) \to H^{k+1}(G, M) \to \cdots$$

for any $\mathbb{Z}G$-module M.

The proof of Theorem 3.8.4 is done by constructing a push-out square of *aspherical CW- complexes* and embeddings which realises Figure 3.1 on fundamental groups. These ideas are not covered in this thesis, so the reader can see the given reference for a complete proof. Also the proof uses the $C(6)$ property of clique-pictures and Theorem 3.1.1, which justifies its inclusion here.

Chapter 4

One-relator product of two non-trivial groups with short relator

4.1 Preamble

Let A and B be non-trivial groups. Consider the one-relator product

$$G = \frac{(A * B)}{N(r^n)}, \tag{4.1}$$

where n is a natural number and r is a word in $A * B$ which is not a proper power. We will be mostly interested in the case where $n \geq 2$ and $\ell(r) \leq 8$. Our aim is to show that G satisfies the Freiheitssatz and r has order n in G. In very few cases it might not be possible to prove both results. However, either of the two results will be enough for later applications. More precisely using pictures and clique pictures, we prove the following theorem.

Theorem 4.1.1. *Let A and B non-trivial groups and $G = (A * B)/N(r^n)$. Suppose that any of the following conditions holds:*

1. *$\ell(r) = 8$ and $n \geq 3$; or*

2. *$2 \leq \ell(r) < 8$ and $n \geq 2$; or*

3. *$n = 1$ and $\ell(r) = 4$ say $r = abcd$, with $\langle a, c\rangle$ and $\langle b, d\rangle$ isomorphic to 2-generator subgroups of $PSL_2(\mathbb{C})$.*

Either:

i. A and B embed in G naturally and r has order n in G, or *ii.* $n = 2$ *and up to conjugation* r *has the form* $r = axbx^{-1}cz$ *with* $z^2 = 1$*, where* $a, b, c \in A$ *and* $x, z \in B$*.*

Corollary 4.1.2. *Let* G *be as in Theorem 4.1.1. Then* r *has order* n *in* G*.*

Proof. Suppose that $n = 2$ and the relations $xy = z^2 = 1$ holds in B, then one can show easily that B embeds in G. More precisely the identity map from B to itself factors through the projection map from G to B. So if r fails to have order 2, then $r = 1$ in G and so the image of r (under this projection map) which is z is trivial in B. But this is impossible. This completes the proof. □

The rest of this Chapter is divided into five sections. In each section we consider a specific $\ell(r) = l$. But first we make some general assumptions as well as prove general results which apply to arbitrary l.

General assumptions

Most of the proofs we present in this chapter dwells on analysis of pictures. These pictures have some properties which we shall establish. We show by rigorous analysis that these properties will give us the desired result. If the group G in 4.1 fails to satisfy the Freiheitssatz or the word r has order m strictly less than n, then we can construct a non-trivial picture M over G on D^2. In the first case M has an exceptional region Δ_o whose label is a non-trivial word in $A \cup B$ which is trivial in G. In the second case M has an exceptional vertex v_o whose label is a non-trivial word $r^{\pm m}$ in $A * B$ which is trivial in G. In both cases M can in fact be thought of as a spherical picture. Also we will always assume that M is $\mathcal{W}$-minimal, where $\mathcal{W}$ is the union of the set on non-trivial elements in $A \cup B$ and the symmetrized closure of the set $\{r^k \mid k < n\}$.

If both A and B have faithful representations in $PSL_2(\mathbb{C})$ and $n > 1$, then G satisfies the Freiheitssatz and r has order n in G by Theorem 2.4.2. Hence we assume that at least one of A or B does not have any faithful representation in $PSL_2(\mathbb{C})$. In particular if A – say, is cyclic or dihedral, then B is neither cyclic nor dihedral.

Another assumption is that the root r of the relator r^n is a cyclically reduced word in $A * B$, which is not a proper power. Hence without loss of generality, it has the form

$$r = a_1 b_1 \dots a_{\frac{l}{2}} b_{\frac{l}{2}}, \tag{4.2}$$

where $\{a_i \mid 1 \leq i \leq l/2\} = X \subseteq A$, $\{b_i \mid 1 \leq i \leq l/2\} = Y \subseteq B$ and $X \cup Y$ does not contain the the identity element of A or B.

Finally we assume that X generates A and Y generates B. This is a pretty standard assumption, for suppose Freiheitssatz holds for $G_o = (A_o * B_o)/N(r^n)$ where A_o is the subgroup of A generated by X and B_o is the subgroup of B generated by Y, then same is true for $G = A *_{A_o} G_o *_{B_o} B$. Also if r has order n in G_o, then it has order n in G.

4.2 Preliminary results

Most of the arguments we present involve analysis of zones incident on an interior vertex of M. We show that $(l-1)$-zones do not occur, and if an $(l-2)$-zone occurs, then there is only one possibility for its label. Later on l will either be 4, 6, or 8. Unless stated otherwise, we assume $n \geq 2$. In what follows all subscripts are modulo $l/2$.

Lemma 4.2.1. *Suppose an $(l-1)$-zone Z is incident on some vertex, and none of the regions in Z is exceptional. Then one of the following holds.*

1. *r is a proper power.*

2. *$n = 2$ and there exist letters $x, y \in A \cup B$ with $y^2 = 1$, and a $W \in A * B$ with $\ell(W) = l/2 - 1$, such that r is conjugate to a word of the form $xWyW^{-1}$.*

Proof. Suppose that Z joins vertices of opposite signs. By Lemma 3.5.3, a cyclic subword of r^n of length $l + \mu - 2 \geq l + \mu - \gcd(l, \mu)$ has period $\mu \leq l/2$. Hence by Theorem 3.2.5, r has period $\gcd(l, \mu)$ since $\gcd(l, \mu) \geq 2$. This implies that r is a proper power.

On the other hand suppose that Z joins vertices of same sign. Up to cyclic relabelling of r, we may assume that the zone Z identifies $b_1 a_2 b_2 \ldots a_{l/2}$ with the subword $(a_\lambda b_\lambda \ldots b_{l/2+\lambda-2})^{-1}$ of r^n for some $1 \leq \lambda \leq l/2$. In particular $a_j = a_{l/2+\lambda-j}^{-1}$ for $2 \leq j \leq l/2$ and $b_j = b_{l/2+\lambda-1-j}^{-1}$ for $1 \leq j \leq l/2 - 1$.

If $\lambda > 2$, then $a_{l/2+\lambda-1} = a_{\lambda-1} = a_{l/2+\lambda+1-\lambda}^{-1} = a_1$. Similarly $b_{l/2} = b_{\lambda-1}^{-1}$. So we can get an $(l+1)$-zone by bridge-moves. If $\lambda = 1$, then $a_{l/2} = a_1^{-1}$. Similarly if $\lambda = 2$, then $b_2 = b_{l/2-1}^{-1}$. In either case we can do a bridge-move, changing Z to get an l-zone. Hence r has the form $r = xWyW^{-1}$, with (x, y) admissible since $y^2 = 1$. If $n > 2$, we can apply Theorems 1 and 3 in [62] to show that Freiheitssatz holds for G and r has order n in G. Since this contradicts our assumption, we must have that $n = 2$. This completes the proof. □

Corollary 4.2.2. *If an $(l-1)$-zone Z whose regions are all interior joins vertices of same sign, then either the size of Z can be increased by bridge-moves or r has the form $r = xWyW^{-1}$.*

As a consequence, we have the following result which is a special case to a more general result proved in [[55],[56]]. However we include the proof here for completeness.

Theorem 4.2.3. *Suppose $n > 3$ and $\ell(r) \leq 10$, then Freiheitssatz holds for G. Also r has order n in G.*

Proof. By Lemma 4.2.1, the number of edges incident on a vertex v is at most $5(l-2) \leq 4l \leq nl$. If any of the inequalities is strict, then there is nothing to prove. Hence we suppose that $l = 10$ and $n = 4$. In which case all the zones must have size $l-2$. It easy to see that we can apply bridge-moves to show that some letter is trivial. This contradiction completes the proof. □

Lemma 4.2.4. *Let Z be an $(l-2)$-zone joining two interior vertices. Suppose that Z can not be changed to $(l-1)$-zone by a bridge-move. Then up to cyclic relabelling of r, we may assume that the zone Z identifies the subword $b_1a_2b_2\dots b_{l/2-1}$ of r^n with the subword $(b_\lambda a_{\lambda+1}b_{\lambda+1}\dots b_{l/2+\lambda-2})^{\pm 1}$ of $r^{\pm n}$, for some $\lambda \leq l/2$, and one of the following holds:*

1. *If the two vertices have opposite signs, then B is cyclic. Furthermore neither of the two regions on both sides of Z is an m-gon for $m < 4$.*
2. *If the two vertices have same sign, then $\lambda - 2 \equiv 1$ mod $l/2$. In particular, the middle 2-zone of Z has both corners labelled by same letter.*

Proof. First suppose that the two vertices have opposite signs. Then $b_1a_2b_2\dots b_{l/2-1}$ is identified with the subword $b_\lambda a_{\lambda+1}b_{\lambda+1}\dots b_{l/2+\lambda-2}$. By Lemma 3.5.3, a cyclic subword W of r^n of length $\ell(W) = l+\mu-3$, with $b_1a_2b_2\dots b_{l/2-1}$ as an initial or terminal segment, has period $\mu = 2(\lambda-1) \geq 2$. Without loss of generality, we assume that $W = b_1a_2b_2\dots b_{l/2+\mu-2}$. If $\mu = 2$, then B is cyclic. Also $a_i = a_{i+1}$ for $1 < i < l/2$. If any of the two regions is an m-gon for $n < 4$, we get the relation $a_1a_{l/2}a_i^{\pm1} = 1$, for some i. This will imply that A is cyclic, contradicting assumption.

Suppose $\mu > 2$. If $\gcd(l,\mu) > 2$, then $l+\mu-3 > l+\mu-\gcd(l,\mu)$. And so by Theorem 3.2.5, r^n has period $\gcd(l,\mu) \leq \mu/2$. Hence r is a proper power. Otherwise suppose that $\gcd(l,\mu) = 2$. Then $\gcd(l/2,\lambda-1) = \gcd(l/2,\mu/2) = \gcd(l,\mu)/2 = 1$. But we have that $b_i = b_{i+\lambda-1}$ for $1 \leq i \leq l/2-1$ and

$$\frac{l}{2}+\lambda-2 = \frac{l}{2}+\lambda-1-\gcd\left(\frac{l}{2},\lambda-1\right). \tag{4.3}$$

So $b_1b_2\dots b_{l/2}$ has period 1. Hence B is cyclic.

Similarly, $a_i = a_{i+\lambda-1}$ for $2 \leq i \leq l/2-1$. So we get $l/2-2$ equalities amongst $l/2$ elements. Any circuit will imply that $m(\lambda-1) \equiv 0$ mod $l/2$ for some $m < l/2$.

This is not possible by assumption. Also either a_1 and a_λ or $a_{l/2}$ and $a_{l/2+\lambda-1}$ do not belong to same chain of equality. Hence without loss of generality $a_1 a_\lambda a_i = 1$ for some i. This implies A is cyclic.

Suppose the two vertices have same sign. In which case $b_1 a_2 b_2 \ldots b_{l/2-1}$ is identified with $(b_\lambda a_{\lambda+1} b_{\lambda+1} \ldots b_{l/2+\lambda-2})^{-1}$, where $\lambda \leq l/2$. It follows that for $2 \leq i \leq l/2-1$ and $1 \leq j \leq l/2-1$

$$a_i = a^{-1}_{l/2+\lambda-i} \tag{4.4}$$

and

$$b_j = a^{-1}_{l/2+\lambda-j-1} \tag{4.5}$$

respectively. There is nothing to prove when $\lambda = 1$. Also if $\lambda = l/2$, then by Equation 4.4

$$a_{\frac{l}{2}-1} = a_1^{-1}. \tag{4.6}$$

Finally if $2 \leq \lambda \leq l/2-1$, then

$$a_\lambda = a^{-1}_{l/2}. \tag{4.7}$$

In each case a bridge-move can be applied to increase the size of Z to $l-1$. This contradicts hypothesis, hence the proof is complete. □

One property we require of M is the existence of a positively-curved interior vertex, which we shall denote by ϑ. Throughout ϑ is assumed to have positive sign. This leads us to the following result.

Lemma 4.2.5. *Let M be as above and let t be the minimum number of interior regions bounded by a boundary vertex which are not 2-gonal. Then one of the following holds*

1. *M has a positively curved interior vertex*
2. *If the exceptional region Δ_o has degree k, then $k(3-t) \geq 6$. In particular $k \geq 4$ and there exist a boundary vertex of degree 3.*

Proof. Note that if M has an exceptional vertex v_o (with label a proper subword of r^n), then we can assume that all other vertices are interior. For each region Δ of M of degree $d(\Delta)$, assign an angle $(d(\Delta)-2)\pi/d(\Delta)$ to each of its $d(\Delta)$ corners. Using the curvature formulas in Section 1.5.3 we have that regions of M have zero curvature. Since v_o has curvature at most 2π, we must have a positively-curved interior vertex. Hence we assume that M has an exceptional region Δ_o. We prove this Lemma by considering the degree $d(\Delta_o)$ of Δ_o.

By assumption, there are no interior regions Δ with degree $d(\Delta) = 1$. Suppose that $d(\Delta_o) = 1$. Then there is only one boundary vertex namely the vertex – v say –

bounding Δ_o. All other vertices are interior. Since v has one region of degree 1 and $(ln-2)$ of degree at least 2, it follows that the curvature of v is at most 3π. But the total curvature of M is 4π, so there must be an interior vertex of positive curvature, as claimed.

Suppose then that $d(\Delta_o) = 2$. Then there are no regions of degree 1, so any vertex v has curvature at most 2π, with equality only if every region bounded by v has degree 2. But if every region bounded by v has degree 2, then v has only one neighbouring vertex v'. Since M is $\mathcal{W}$-minimal, it consists of only these two vertices. If this happens, then by the proof of Lemma 4.2.1, either r is a proper power or the label of Δ_o was already trivial in $A \cup B$. Either case we get a contradiction. Otherwise there are at most two vertices bounding Δ_o, each with curvature strictly less than 2π. Since the total curvature of M is 4π, there must be a positively curved interior vertex, as claimed.

Suppose then that Δ_o has degree $d(\Delta_o) = k \geq 3$. Then there are no regions of degree 1. For each vertex v bounding Δ_o, let $\tau(v)$ be number of regions it bounds outside Δ_o which are 2-gonal. It follows that any such vertex has curvature

$$\kappa(v) \leq \left[2 - \frac{\tau(v)}{3} - \frac{(k-2)}{k}\right]\pi. \tag{4.8}$$

If there are no positively-curved interior vertices, then

$$4\pi \leq \sum \kappa(v) \leq \left[2 - \frac{t}{3} - \frac{(k-2)}{k}\right]k\pi, \tag{4.9}$$

where the summation is over all boundary vertices. It follows that $6 \leq k(3-t)$. This can only happen if $t \leq 2$ and $k \geq 3$. Note that none of the zones occurring can have size greater than l by minimality. Hence in particular that $t \neq 0$. If $k = 3$, then there are no interior vertices and all three boundary vertices bound exactly two regions which are not 2-gonal. In other words M consists of the three boundary vertices and the l-zones joining them as shown in Figure 4.1.

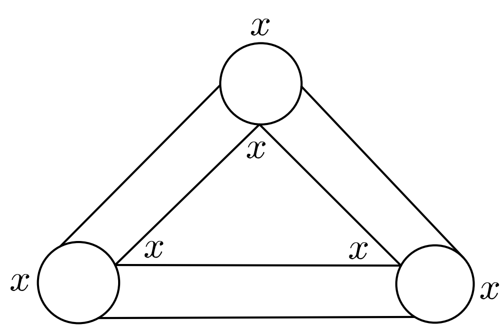

Figure 4.1: A picture consisting of only k boundary vertices joined to each other by l-zones.

However each of the two 3-gons (one of which is Δ_o) has label x^3. Since one is interior, it follows that $x^3 = 1$. In other words the label of Δ_o was already trivial in G. Hence $k \geq 4$ and there exist a boundary vertex of degree 3. This completes the proof. □

Remark 4.2.6. It is easy to see that the existence of a boundary vertex of degree 3 in Lemma 4.2.5 implies the existence of a positively-curved boundary vertex of degree 3.

Corollary 4.2.7. *Suppose an $(l-3)$-zone Z joins ϑ to a vertex u of the same sign, with $l \geq 6$. Then*

1. *either the size of Z can be increased by bridge-moves, or*
2. *$l = 6$ and without loss of generality, $r = a_1 b_1 a_2 b_2 a_2^{-1} b_1^{-1}$.*

Proof. As in Lemma 4.2.4, without loss of generality Z identifies $b_1 a_2 b_2 \dots b_{l/2-2} a_{l/2-1}$ with $a_\lambda b_\lambda \dots a_{l/2+\lambda-3} b_{l/2+\lambda-3}$, where $\lambda \leq l/2$. Hence for $2 \leq i \leq l/2-1$ and $1 \leq j \leq l/2-2$,

$$a_i = a^{-1}_{l/2+\lambda-i-1} \tag{4.10}$$

and

$$b_j = b^{-1}_{l/2+\lambda-2-j} \tag{4.11}$$

respectively. By Equation 4.10, $a_{l/2-1} = a_1^{-1}$ when $\lambda = 1$. Similarly by Equation 4.11, $b_1 = b^{-1}_{l/2-1}$ when $\lambda = 2$. In either case a bridge-move can be done on the left side or right side of Z, increasing its size to $l-2$. When $l = 6$, the only other possibility is $\lambda = 3$. In this case Equations 4.10 and 4.11 reduces to $a_i = a^{-1}_{5-i}$ and $b_j = b^{-1}_{4-j}$, where $2 \leq i \leq 3$ and $1 \leq j \leq 2$ respectively. In other words, $a_2 = a_3^{-1}$, $b_1 = b_3^{-1}$ and $b_2 = b_2^{-1}$. Hence $r = a_1 b_1 a_2 b_2 a_2^{-1} b_1^{-1}$.

Now suppose that $l > 6$ and $\lambda > 2$, then $b_{\lambda-1} = b^{-1}_{l/2-1}$ by Equation 4.11. As before, we can do a bridge-move to increase the size of Z. This completes the proof. □

We also include the following trivial results which shall be used later.

Proposition 4.2.8. *The group $H = \langle x, y \mid x^2 = 1, w(x,y) = 1 \rangle$ with $w(x,y) \in \{xy^{\pm 1}xy^2, xyxy\}$ is either dihedral or cyclic.*

This can be proved by applying Tietze transformations. In fact H is the cyclic group of order 6 when $w(x,y) = xyxy^2$, dihedral group of order 6 when $w(x,y) = xy^{-1}xy^2$ and infinite dihedral when $w(x,y) = xyxy$. We are also interested in the case where $w(x,y) = xyxy^{-1}$. In other words, H is abelian.

Lemma 4.2.9. *Let $G = (A * B)/N(r^2)$ where A is Abelian with factor groups A_i, and B has a faithful representation in $PSL_2(\mathbb{C})$. If $p_i(r)$ is neither in $\bigcup_i N(A_i)$ nor $N(B)$, where $p_i : A * B \to A_i$ is the canonical projection map, then G satisfies the Freihetssatz.*

Proof. For each i, both A_i and B have faithful representations in $PSL_2(\mathbb{C})$. Hence both embed in G since they embed in $p_i(G)$. Moreover

$$\left(\bigcap \mathrm{Ker}(p_i)\right) \cap A = 1,$$

so A also embeds in G. □

We now focus on positively-curved interior vertices ϑ. All the zones mentioned here are incident on ϑ. Recall that a vertex of degree m has m zones incident on it. Let the zones be Z_1,..., Z_m listed consecutively in clockwise order.

Definition 4.2.10. A *configuration* of a vertex of degree m is the m-tuple $(\omega_1, \ldots, \omega_m)$ or any of its cyclic permutations, where ω_i is the size of Z_i.

A bridge-move induces a map from one configuration of a vertex to another. Since we will be adopting a rigorous method which involves checking all configurations, we can try to that do that effectively. One way is to consider two configurations equal up to orientation and cyclic permutation. For example $(6, 5, 5, 4, 4)$, $(6, 4, 4, 5, 5)$ and $(4, 6, 5, 5, 4)$ are all equal.

4.3 Relator of length eight with power $n \geq 3$

In the section we consider the one-relator group

$$G = \frac{(A * B)}{N(r^n)}, \tag{4.12}$$

where $n \geq 3$ and $r = axbyczdw$ with $a, b, c, d \in A$ and $x, y, z, w \in B$. The aim is to prove the following theorem.

Theorem 4.3.1. *Let G be as in 4.12. Then the following holds for G.*

1. *G satisfies the Freiheitssatz.*

2. *The word r has order n in G.*

To prove Theorem 4.3.1, it is enough by Theorem 4.2.3 to restrict to the case of $n = 3$. Suppose by contradiction that the Theorem fails. Then we obtain a non-trivial $\mathcal{W}$-minimal spherical picture M over G (with exceptional vertex or region as the case

Type 1	Type 2	Type 3	Type 4
$(6,6,6,6)$	$(6,3,6,5,4)$	$(6,5,5,5,3)$	$(6,5,4,4,5)$
$(6,6,1,6,5)$	$(6,4,6,3,5)$	$(6,5,5,4,4)$	
$(6,6,2,6,4)$	$(6,4,6,4,4)$	$(6,5,5,3,5)$	
$(6,6,3,6,3)$	$(6,5,6,3,4)$	$(6,4,5,5,4)$	
$(6,6,3,5,4)$	$(6,5,6,2,5)$	$(6,5,5,6,2)$	
$(6,6,4,4,4)$		$(6,4,5,4,5)$	
$(6,6,5,2,5)$			
$(6,6,5,3,4)$			
$(6,6,5,4,3)$			
$(6,6,5,5,2)$			

Table 4.1: A table containing all the possible configurations of a positively-curved interior vertex.

may be). To each region Δ in M of $d(\Delta)$ we assign an angle of $(d(\Delta)-2)\pi/d(\Delta)$. By Lemma 4.2.5, we deduce that M contains a positively-curved interior vertex ϑ i.e. $d(\vartheta) \leq 5$. If $d(\vartheta) \leq 3$, then the result follows from Lemma 4.2.1. Hence $d(\vartheta) = 4$ or 5.

We list all the possible configurations of ϑ (see Table 4.1) and show that each of them leads to a contradiction to the general assumptions. More precisely, analysis of these configurations will show that each of A and B has faithful representation in $PSL_2(\mathbb{C})$.

Throughout this Section we shall use the letter t (with or without subscript) to refer to an element in $\{a,b,c,d\}^{\pm 1}$, and s (with or without subscript) to refer to an element in $\{x,y,z,w\}^{\pm 1}$. With the exception of $(6,6,6,6)$, all other configurations in Table 4.1 has degree five. Since ϑ has positive curvature, it follows that at least four of the five regions bounded by ϑ which are not 2-gons must be 3-gons. Hence by Lemma 4.2.4, 6-zones of a configuration of degree five joins vertices of same sign. Without loss of generality we assume that at least one the 6-zones in any configuration has corners at ϑ labelled x, b, y, c, z respectively in the clock-wise direction. So again by Lemma 4.2.4, we get the relations $xz = bc = y^2 = 1$. Note that the same assumptions hold even in the case of $(6,6,6,6)$ by restricting to the 6-zones bounding a 3-gon.

Lemma 4.3.2. *Configurations of Type 1 do not occur.*

Proof. The proof uses only the fact in each of the configurations in Type 1, there are two adjacent 6-zones and at least one of A-regions bounded by these zones is a 3-gon. In the case of $(6,6,6,6)$, we choose the two 6-zones bounding a 3-gon. By Lemma 4.2.4 these 6-zones connect vertices of same sign. So without loss of generality we can assume that the two adjacent 6-zones are labelled as shown in Figure 4.2 below.

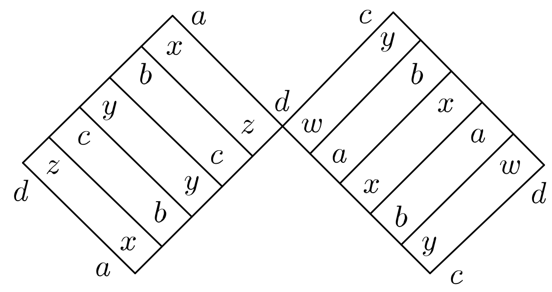

Figure 4.2: Diagram showing adjacent 6-zones of a configuration of Type 1

So in B the relations $xz = y^2 = yw = x^2 = 1$ hold. Hence B is a quotient of D_∞. In A we have the relations $bc = ab = 1$ and $tcd = 1$ with $t \in \{b^{\pm 1}, a, c, d\}$, since one of the regions with a corner labelled c must be a 3-gon. Hence A is cyclic, contradicting assumptions. $\square$

Lemma 4.3.3. *Configurations of Type* 2 *do not occur.*

Proof. Every configuration in Type 2 has the form $(6, 3, 6, *, *)$, $(6, 4, 6, *, *)$, or $(6, 5, 6, *, *)$, where $*$ is a place holder. We shall treat each of the three cases separately beginning with $(6, 3, 6, *, *)$.

1. Without loss of generality Figure 4.3 gives a labelling of ϑ in case of $(6, 3, 6, *, *)$.

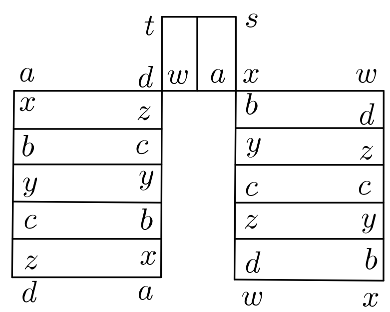

Figure 4.3: Diagram showing configuration $(6, 3, 6, *, *)$ in Type 1

We can assume that A-region and the B-region containing t and s respectively are 3-gons. In other words $tad = 1 = sxw$. The two 6-zones give the relations $xz = y^2 = yz = 1$ and $bc = bd = c^2 = 1$. Hence both A and B are cyclic which contradicts assumption.

2. In the case of $(6, 4, 6, *, *)$ we can assume without loss of generality that ϑ is labelled as shown in Figure 4.4.

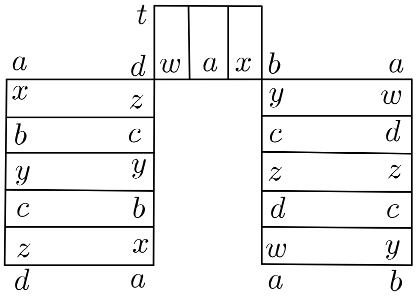

Figure 4.4: Diagram showing configuration $(6, 4, 6, *, *)$ in Type 2

The two 6-zones give the relations $bc = cd = 1$ and $xz = y^2 = yw = z^2 = 1$. Hence we conclude that B is a quotient of D_∞. Also we can assume that $tad = 1$. It follows that A is cyclic. This contradicts assumption.

3. Finally in the case of configuration $(6, 5, 6, *, *)$ we can assume without loss of generality that ϑ is labelled as shown in Figure 4.5.

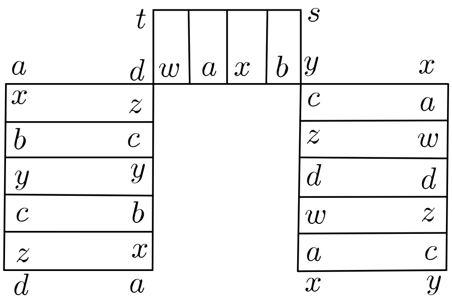

Figure 4.5: Diagram showing configuration $(6, 5, 6, *, *)$ in Type 2

Hence the two 6-zones give the relations $ac = bc = d^2 = 1$ and $xz = zw = y^2 = 1$. Also we can assume as before that $tad = 1 = sxy$. Hence both A and B are cyclic, a contradiction to assumption.

□

Lemma 4.3.4. *Configurations of Type 3 do not occur.*

Proof. Configurations in Type 3 can be grouped essentially in two: configurations of the form $(6, 4, 5, *, *)$ or $(6, 5, 5, *, *)$. We assume that the middle 5-zone can not be changed to a 6-zone via a bridge-move, as this will take us back to Type 2 for which the result already holds. Hence the middle 5-zones joins vertices of opposite signs by Lemma 4.2.7. We begin with case $(6, 4, 5, *, *)$.

1. Without loss of generality Figure 4.6 shows all the possible labellings of 6-zone and the middle 5-zone of configuration $(6, 4, 5, *, *)$.

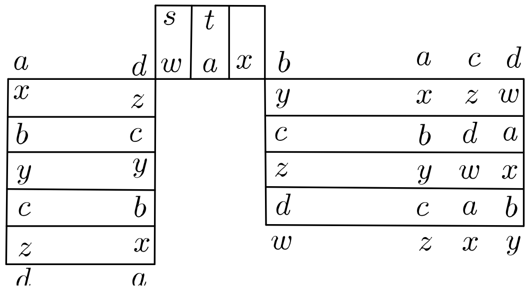

Figure 4.6: Diagram showing configuration $(6, 4, 5, *, *)$ in Type 3

The 6-zone gives the relations $xz = y^2 = 1$ and $bc = 1$. The 5-zone gives three possible set of relations.

(a) In the first case $x = y = z$ and $b = c = d$. Since $at = 1$, A is either cyclic or a quotient of D_∞. Similar conclusion holds for B since $ws = 1$.

(b) In the second case $y = z = w$ and $c = d = a$. Hence both A and B are cyclic.

(c) In the third case $y = w$, $x = z$ and $a = c$, $b = d$. Hence B is a quotient of D_∞ and A is cyclic.

In each case we get a contradiction to assumption.

2. Figure 4.7 shows all the possible labellings of 6-zone and the middle 5-zone of configuration $(6, 5, 5, *, *)$.

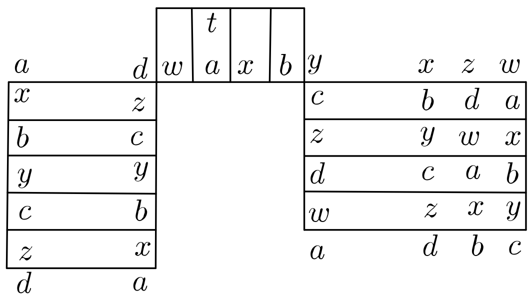

Figure 4.7: Diagram showing configuration $(6, 5, 5, *, *)$ in Type 3

The 6-zone gives the relations $xz = bc = y^2 = 1$. The 5-zone gives three possible set of relations.

(a) In the first case $y = z = w$ and $b = c = d$. So B is cyclic and A is either cyclic or a quotient of D_∞ (since $at = 1$).

(b) In the second case $x = z = w$ and $c = d = a$. Hence A is cyclic and B quotient of D_∞.

(c) In the third case $y = w$, $x = z$ and $a = c$, $b = d$. Hence B is a quotient of D_∞ and A is cyclic.

In each case we get a contradiction to assumption.

□

Lemma 4.3.5. *Configurations of Type 4 do not occur.*

Proof. There is only one configuration of Type 4 namely $(6, 5, 4, 4, 5)$. We assume that no bridge-moves can be done to change any of the two 5-zones to a 6-zone. For otherwise we get one of configurations $(6, 5, 5, 4, 4)$ or $(6, 6, 3, 4, 5)$ for which the result is already known. As before both 5-zones connect vertices of opposite signs. In Figure 4.8, we have shown all the possible labellings of ϑ with this configuration, where the 2-zone indicated is part of the second 5-zone.

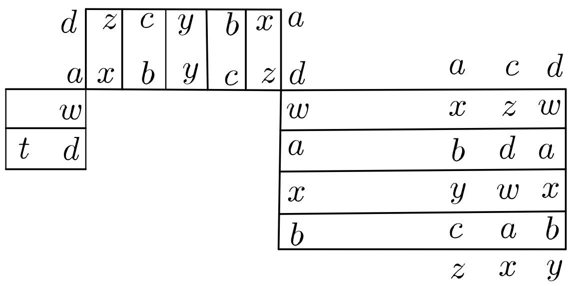

Figure 4.8: Diagram showing configuration $(6, 5, 4, 4, 5)$ in Type 4

The 6-zone gives the relations $xz = bc = y^2 = 1$. The 5-zone gives three possible set of relations. Also $td = 1$ where $t \in \{a^{-1}, b^{-1}, c^{-1}\}$.

1. In the first case $x = y = w$ and $a = b = c$. So both A and B are cyclic.

2. In the second case $y = w$, $x = z$ and $a = c$, $b = d$. Hence B is a quotient of D_∞ and A is cyclic.

3. In the third case $x = z = w$ and $a = b = d$. Hence A is cyclic and B quotient of D_∞.

In each case we get a contradiction to assumption that at least one of A or B does not have a faithful representation in $PSL_2(\mathbb{C})$. □

This shows that our assumption that Theorem 4.3.1 fails was false, hence the proof of the Theorem 4.3.1 is complete.

4.4 Relator of length six with power $n \geq 2$

In the section we consider the one-relator group

$$G = \frac{(A * B)}{N(r^n)}, \tag{4.13}$$

where $n \geq 2$ and $r = axbycz$ with $a, b, c \in A$ and $x, y, z \in B$.

As in the case of $l = 8$, we list all the possible configurations of a positively-curved interior vertex. However things are a little more complicated in this case. We call a configuration of a positively-curved interior vertex *good* if it is not of the form $(4, 2, 2, 2, 2)$, $(4, 2, 4, 1, 1)$ or $(4, 2, 4, 2)$, and *not good* (or *bad*) otherwise. We say that a picture M is *not good* (or *bad*) if the configuration of each positively-curved interior vertex of M is not good. The main result of this section is the following.

Theorem 4.4.1. *Let G be as in 4.13. Then either:*

1. *A and B have faithful representations in $PSL_2(\mathbb{C})$, or*
2. *$n = 2$ and up to conjugation r has the form $r = axbx^{-1}cz$ with $z^2 = 1$, where $a, b, c \in A$ and $x, z \in B$.*

We can restrict to the case where $n = 2$ for the following reasons. Suppose $n > 2$ and M does not have a positively-curved interior vertex ϑ. Then there is a boundary vertex v of degree 3 by Lemma 4.2.5. Hence v bounds an ω-zone with $\omega \geq 5$. Same conclusion holds if M has a positively-curved interior vertex ϑ of degree less than 5 or $n > 3$. In these cases the result follows from Lemma 4.2.1 and Lemma 4.4.2 below. Hence we suppose that $n = 3$ and ϑ has degree 5 and a configuration of the form $(4, 4, *, *, *)$. In other words, there are two adjacent 4-zones incident on ϑ. Without loss of generality and using Lemma 4.2.4, we assume the two adjacent 4-zones are labelled as shown in Figure 4.9.

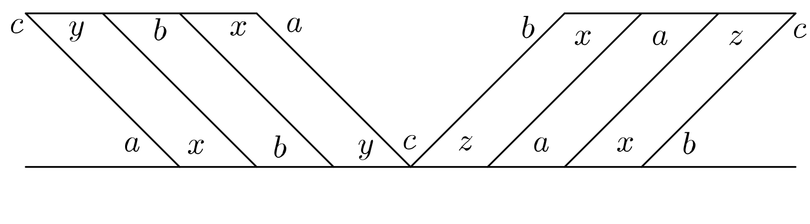

Figure 4.9: Configuration $(4, 4, *, *, *)$

We deduce from the two 4-zones that $xy = xz = 1$ and $a^2 = b^2 = 1$. In particular B is cyclic. Also at least one of $bac = 1$ or $tac = 1$, with $t \in \{a, c\}$ holds. So A is cyclic or a quotient of D_∞. Hence the Theorem holds. For the rest of this section we assume that $n = 2$.

We assume that Theorem 4.4.1 fails and proceed to derive a contradiction. We do this in a series of Lemmas. In particular, most of our conclusions will follow from Lemma 4.4.2 below.

Lemma 4.4.2. *If r has the form $r = axbyb^{-1}x^{-1}$, then $G = (A * B)/N(r^n)$ satisfies the Freiheitssatz and r has order n.*

Proof. Suppose $r = axbyb^{-1}x^{-1}$ and either G fails to satisfy the Freiheitssatz or r does not have order n in G. We construct a non-trivial $\mathcal{W}$-minimal spherical clique-picture M over G. A typical clique-label will have the form

$$a^{\alpha_1}(xb)y^{\beta_1}(b^{-1}x^{-1})\dots a^{\alpha_k}(xb)y^{\beta_k}(b^{-1}x^{-1}).$$

So G is induced by the generalised triangle group

$$T(p,q,2) = \langle a, y \mid a^p, y^q, (ay)^n \rangle,$$

where p and q are the orders of a and y respectively. Hence by the Spelling Theorem for generalised triangle groups, it follows that $k \geq n$. In any region of M, we may assume that no two consecutive corner labels are of the form a^α or y^β (for otherwise we could combine the two cliques into one, forming a smaller clique-picture).

In particular, if a 2-zone is labelled by a^α at one corner, the opposite corner has to be a $b^{\pm 1}$ and so A will be cyclic. Same argument applies to B. The results holds if both A and B are cyclic. Hence we can assume without loss of generality that A is not cyclic and proceed to derive a contradiction.

There are at least two corners labelled by a^α and atleast two corners labelled y^β (since $n \geq 2$). This implies by Lemma 4.2.5 that M has a positively-curved interior clique ϑ. Curvature argument tells us that in any positively-curved interior clique, there are at most five regions which are not 2-gons. Since A is not cyclic, no A-region with corner labelled a^α is a 2-zone.

Suppose a 2-zone has a corner labelled b (respectively b^{-1}), then the opposite corner has label b (respectively b^{-1}). If not, we get a contradiction to minimality or the fact that A is not cyclic. Consequently, there is no zone with label $(by^{\beta_i}b^{-1})^{\pm 1}$.

Hence the two A-regions with corner labelled by a^α and at most one in the sequence $x, b, y^{\beta_k}, b^{-1}, x^{-1}$ are contained in regions which are not 2-gons. This implies (wlog) that there is a 3-zone with corners at ϑ labelled x and b respectively. Hence $b^2 = 1$ in A, and B is cyclic since x must be a power of y. Also there are at least four regions which are not 2-gonal and any A-region with corner labelled a^α is at least a 4-gon. Since ϑ is assumed to be positively-curved, it must have degree four. Hence at least one of these regions which are not 2-gons must be a 3-gon. Since no zone has label $(by^{\beta_i}b^{-1})^{\pm 1}$, any such 3-gon must have two consecutive corners labelled x, x^{-1} or $b^{\pm 1}, a^{\alpha_i}$. Both cases gives a contradiction. Hence the proof. □

Using Lemma 4.2.1 and Lemma 4.4.2, we can assume that there are no ω-zone joining two vertices of same sign with $\omega > 4$, and all the 2-zones are interior. Hence we can easily deduce that M contains a positively-curved interior vertex. As before we shall use the letter t (with or without subscripts) to refer to an element in $\{a, b, c\}^{\pm 1}$, and s (with or without subscripts) to refer to an element $\{x, y, z\}^{\pm 1}$.

Lemma 4.4.3. *M has a positively-curved interior vertex ϑ.*

Proof. Suppose by contradiction that M has no positively-curved interior vertex. Since every vertex has degree at least 3, it follows from the proof of Lemma 4.2.5 that the exceptional region has degree at least 6. Also we have that M contains a positively-curved boundary vertex v of degree 3 which bounds at least one interior region of degree at most 4 (for otherwise curvature of M is less than 4π).

By the comments succeeding Lemma 4.4.2, we assume that all the zones have equal size which is 4, and that no bridge-moves can be applied to increase the size of any of the zones. We can arrange so that the middle 2-zones have corners labelled a, c and b consecutively in the clockwise-sense as shown in Figure 4.10. Hence the three regions which are not 2-gonal are all A-regions.

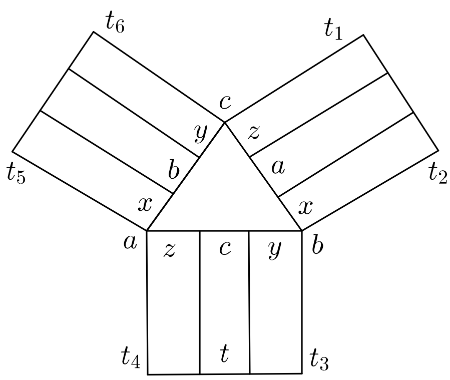

Figure 4.10: Diagram showing positively-curved boundary vertex of degree 3

If all three 4-zones connect vertices of opposite signs, then by Lemma 4.2.4, $xy = xz$ and $a^2 = b^2 = c^2 = 1$. In particular B is cyclic. Also we have a relation of length at most 5 involving a, b, and c. So A is a quotient of D_∞. Hence we suppose that at least one of the 4-zones joins vertices of opposite signs. It follows from Lemma 4.2.4 that B is cyclic.

If exactly one 4-zone joins to a vertex of negative sign, then without loss of generality $a^2 = b^2 = 1$ and $t \in \{a^{-1}, b^{-1}\}$. Hence A is dihedral. If exactly two 4-zones joins to a vertex of negative sign, then without loss of generality we assume that $a^2 = 1$ and $b = a$ or $b = c$. Similarly $c = a$ or $c = b$. Any of the possibilities must imply that A is cyclic, except when $a^2 = bc^{-1} = 1$ are the only relations. In other words $t_1 = b$, $t_2 = a$, $t_3 = a^{-1}$, $t_4 = c^{-1}$, $t_5 = b^{-1}$ and $t_6 = a^{-1}$. If any of the three regions is interior and has degree 3, then A is cyclic. So we conclude that there is a relation of the form $ab^2p = 1$, $babp = 1$ or $bapa = 1$, for some $p \in \{a, b, c\}^{\pm 1}$. Hence A is either cyclic, dihedral or abelian. So G satisfies the Freiheitssatz. Clearly r has order 2 in G since B is isomorphic to $\mathbb{Z}_2$.

Finally, if all 4-zones connect to vertices of negative signs, then it is easy to see that $a = b = c$, and so A is cyclic. □

Type 1	Type 2	Type 3	Type 4	Type 5	Type 6
$(4,3,4,1)$	$(3,4,2,1,2)$	$(4,4,2,1,1)$	$(2,3,2,4,1)$	$(3,3,2,2,2)$	$(4,2,2,2,2)$
$(3,3,3,3)$	$(3,4,2,2,1)$	$(4,4,1,2,1)$	$(2,2,3,1,4)$	$(3,2,3,2,2)$	$(4,2,4,1,1)$
$(4,3,3,2)$	$(3,4,1,3,1)$	$(4,1,4,2,1)$	$(2,4,2,1,3)$	$(3,3,3,2,1)$	$(4,2,4,2)$
$(4,3,2,3)$	$(3,4,3,1,1)$	$(1,4,1,3,3)$		$(3,3,1,3,2)$	
$(4,4,2,2)$	$(3,4,1,1,3)$				
$(4,4,3,1)$	$(3,4,1,2,2)$				

Table 4.2: A table containing all the possible configurations of a positively-curved interior vertex.

Table 4.2 contains all the possible configurations of ϑ. We assume that at least one of the 4-zones in each configuration has corners at ϑ labelled x, b and y consecutively. So such a 4-zone gives the relations $xy = b^2 = 1$ (if it joins vertices of same sign) or B is cyclic and $b = a$ or c (if it joins vertices of opposite signs) by Lemma 4.2.4. The proof on Theorem 4.4.1 is divided in two parts depending on whether or not M is good.

Part 1

Here we consider the case where M is good and we prove that part 1 of Theorem 4.4.1 holds.

4.4.1 Configurations of an interior vertex of degree 4

In this section we prove that the existence of a positively-curved interior vertex of degree 4 implies Theorem 4.4.1. We do this in a series of Lemmas, each dealing with a specific configuration in column 1 of Table 4.2. We begin with configuration $(4, 4, 3, 1)$.

Lemma 4.4.4. *Theorem 4.4.1 holds in the case of configuration* $(4, 4, 3, 1)$

Proof. First suppose that each of the two 4-zones joins vertices of same sign. Then from the two 4-zones we get the relations $xy = xz = 1$ and $a^2 = b^2 = 1$. So in particular B is cyclic. We assume that $c \notin \langle a, b \rangle$, as otherwise A is a quotient of D_∞. In which case the result holds. It follows that the 3-zone must be labelled as shown in Figure 4.11.

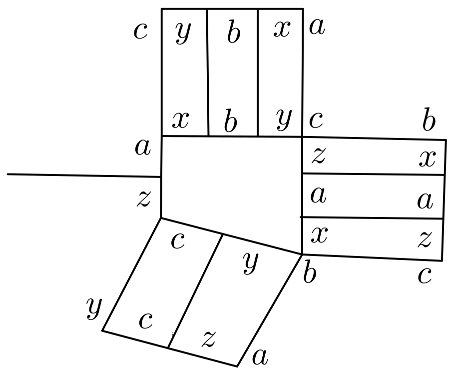

Figure 4.11: Configuration $(4, 4, 3, 1)$

Hence the B-region can not be a 3-gon, so at least one of the A-regions is a 3-gon. Hence we get a relation of length at most 3 involving c and at least one of a or b. It follows that A is cyclic or dihedral.

Suppose that one of the 4-zones joins vertices of opposite signs. Then without loss of generality B is cyclic and $b \in \langle a, c \rangle$. Since the other 4-zone and the 3-zone have corners labelled a and c respectively, A is either cyclic or a quotient of D_∞. This is a contradiction, hence the proof. □

Lemma 4.4.5. *Theorem 4.4.1 holds in the case of configuration* $(4, 3, 4, 1)$

Proof. Suppose that each of the two 4-zones joins vertices of same sign as shown in Figure 4.12. Then from the two 4-zones we get the relations $xy = y^2 = 1$ and $bc = b^2 = 1$. If the 3-zone joins vertices of same sign, then one of $xz = 1$, $ab = 1 = yz$, or $ac = 1$ holds. Also if exactly one of the two 4-zones joins vertices of same sign, then by symmetry we can assume that $x = y = z$, $b = a$ or $b = c$, and $bc = y^2 = 1$. In each case a cyclic conjugate of r satisfies the conditions of Lemma 4.4.2. The result follows.

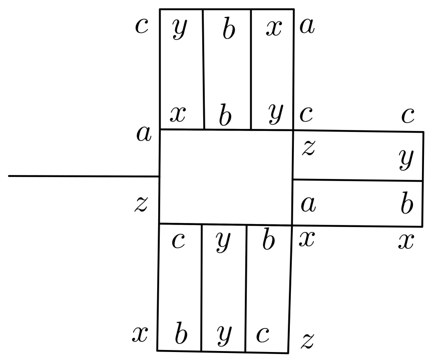

Figure 4.12: Configuration $(4, 3, 4, 1)$

If each of the two 4-zones joins vertices of opposite signs, then both A and B are cyclic. The result follows by contradiction. □

Lemma 4.4.6. *Theorem 4.4.1 holds in the case of configuration* $(4, 4, 2, 2)$

Proof. Suppose that each of the two 4-zones joins vertices of same sign. Then $xy = xz = 1$ and $a^2 = b^2 = 1$. Hence in particular B is cyclic.

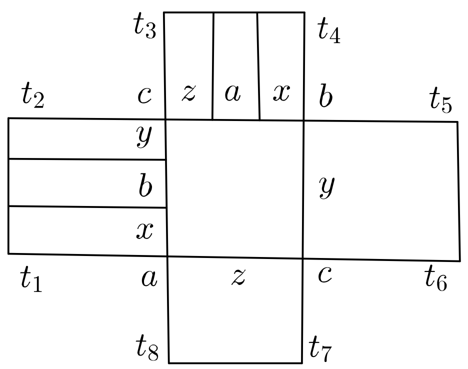

Figure 4.13: Configuration $(4, 4, 2, 2)$

If $c \in \langle a, b \rangle$, then we are done. Otherwise we can assume that the only triangular region is the one with corner labelled t_6 with $t_6 = t_7 = c$. Hence $c^3 = 1$. Without loss of generality we assume that $t_1 t t_8 a = ctba = 1$ for some $t \in \{a, b, c\}$. For each possibility A is a quotient of D_∞.

Suppose that each of the two 4-zones joins vertices of opposite signs. Then B is cyclic as well as A, unless $t_1 = c^{-1}, t_2 = b^{-1}, t_3 = a^{-1}$ and $t_4 = c^{-1}$. As before we can assume that the only triangular region is the one with corner labelled t_6 with $t_6 = t_7 = c$. Hence $c^3 = 1$. Without loss of generality we assume that $t_1 tba = 1$, where $t = c^{-1}$ or a (alternatively b). Either possibility implies A is cyclic.

Finally, we assume that only one of the two 4-zones joins vertices of opposite signs. Hence without loss of generality $t_1 = c^{-1}, t_2 = b^{-1}, t_3 = b$ and $t_4 = c$. Hence

$ab^{-1} = a^2 = 1$ and $B = \mathbb{Z}_2$. It follows that a cyclic conjugate of r has the form $x(by)c(za) = x(by)c(y^{-1}b^{-1})$. Hence the result follows from Lemma 4.4.2. $\square$

Lemma 4.4.7. *Theorem 4.4.1 holds in the case of configuration* $(4, 3, 3, 2)$

Proof. Suppose the 4-zone joins vertices of same sign, then $t_4 = a, t_3 = c$ and $xy = b^2 = 1$. By Lemma 4.4.2 we can assume that $ac \neq 1$.

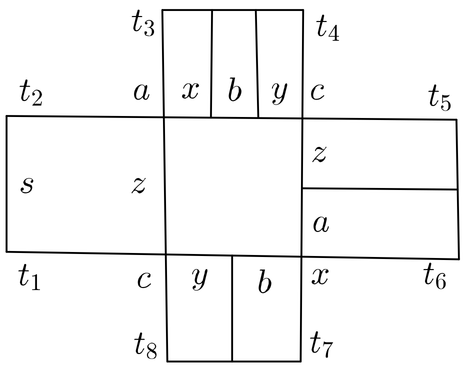

Figure 4.14: Configuration $(4, 3, 3, 2)$

1. If the 3-zone adjacent to the 4-zone joins vertices of same sign, then $t_5 = c, t_6 = x$ and $zy = ab = 1$. Note that if $t_5 \neq a$ by assumption and if $t_5 = b$, then $a^2 = xz = 1$ and so in particular B is cyclic. Also we can assume that the only triangular region is the one with corner labelled t_2. So $t_2ac = 1 = tt_4ct_5$. It follows that A is cyclic.

   Hence the result follows from Lemma 4.4.2.

2. Suppose then that the 3-zone adjacent to the 4-zone joins vertices of opposite signs. There are two cases to consider.

   (a) If $t_5 = b^{-1}$, $t_6 = z^{-1}$ and $zy^{-1} = ac^{-1} = 1$. Then in particular B is cyclic. So we assume that $b \notin \langle a, c \rangle$. This implies that the other 3-zone joins vertices of same sign with $t_7 = y$, $t_8 = a$ and so no new relations. However in such case the B-region can not be a 3-gon (as otherwise $z = 1$). Hence we get a relation $act = 1$, and so A is cyclic.

   (b) If $t_5 = a^{-1}$, $t_6 = y^{-1}$ and $zx^{-1} = ab^{-1} = 1$. Then the result follows from Lemma 4.4.2.

Suppose the 4-zone joins vertices of opposite signs, then B is cyclic and we can assume that none of the 3-zones joins vertices of same sign as that will imply that either A is also cyclic or $B = \mathbb{Z}_2$ and $ab^{-1} = b^2 = 1$ (alternatively $cb^{-1} = b^2 = 1$). The result follows from Lemma 4.4.2.

1. If $t_3 = b^{-1}$, $t_4 = a^{-1}$ and hence $b = c$, then we conclude that $a \in \langle b, c\rangle$ using the 3-zone with corner labelled a.

2. If $t_3 = c^{-1}$, $t_4 = b^{-1}$ and hence $b = a$, we can assume that $t_5 = a^{-1}$, $t_6 = y^{-1}$, $t_7 = z^{-1}$, $t_8 = b^{-1}$. It follows that the B-region can not be a 3-gon (since $x = y = z$). Hence one of the A-regions is a 3-gon and since each has corners labelled $c^{\pm 1}$ and $a^{\pm 1}$ or $b^{\pm 1}$, A is cyclic.

The result follows by contradiction. □

Lemma 4.4.8. *Theorem 4.4.1 holds in the case of configuration* $(4, 3, 2, 3)$

Proof. Suppose the 4-zone joins vertices of same sign, then $xy = b^2 = 1$.

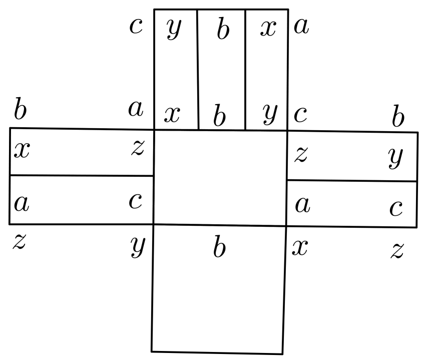

Figure 4.15: Configuration $(4, 3, 2, 3)$

Suppose the 3-zone on the right joins vertices of same sign. Then we can apply Lemma 4.4.2 except in the case where $a^2 = xz = 1$. Similarly we can apply Lemma 4.4.2 if the 3-zone on the right joins vertices of same sign except when $c^2 = yz = 1$. So if both 3-zones connect vertices of same sign, then $B = \mathbb{Z}_2$. Hence $abc = 1$ and so A is a quotient of D_∞. Same conclusion holds if just one of the 3-zones joins vertices of opposite signs.

Hence we assume that both 3-zones connect vertices of opposite signs. If the two 3-zones both give the relation $ac^{-1} = xz^{-1} = yz^{-1} = 1$ as shown in Figure 4.15, then $B = \mathbb{Z}_2$. Hence neither of the two B-regions can be triangular. So A is cyclic as $abc = 1$. Otherwise B is still cyclic and A is cyclic or a quotient of D_∞.

Finally, if the 4-zone joins vertices of opposite signs. Then by symmetry we can assume from the 4-zone that $x = y = z$ and $a = b$. Since the two 3-zones have corners labelled a and c, A is either cyclic or a quotient of D_∞ unless the two 3-zones give the relations $yz = c^2 = 1$ and $xz^{-1} = ab^{-1} = 1$. But in this case $B = \mathbb{Z}_2$

and so for similar reason as before, we conclude that $capa = 1$, where $p = 1$ or $p \in \{a, b, c\}^{\pm 1}$. Hence A is cyclic or a quotient of D_∞. □

Lemma 4.4.9. *Theorem 4.4.1 holds in the case of configuration* $(3, 3, 3, 3)$

Proof. Suppose all the zones connect vertices of opposite signs.

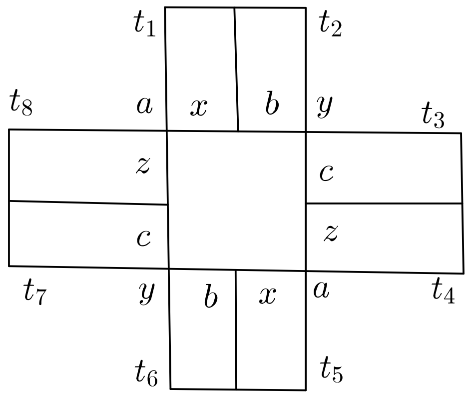

Figure 4.16: Configuration $(3, 3, 3, 3)$

If $t_1 = c^{-1}$, $t_2 = x^{-1}$, then $x = z$ and $b = a$. We can conclude that A is cyclic (since c is a corner label of some 3-zone). If $t_3 = x^{-1}$, then $z = y^{-1}$ and so B is cyclic. Hence we suppose that $t_3 = z^{-1}$ and so $t_4 = b^{-1}$. It follows that none of the two A-regions can be a 3-gon, and each B-region has a corner labelled y and x^{-1} or z^{-1}. Hence B is cyclic. The second case is similar. Hence in this case the result follows by contradiction.

So we are reduced to the case where at least one of the zones joins vertices of same sign. By symmetry we can take this zone to be the 'northern' 3-zone. Suppose A is neither cyclic nor dihedral. Then by Lemma 4.4.2, we can assume that $ac = 1$ and $xy = 1$ (alternatively $ab = 1$ and $yz = 1$) do not hold simultaneously. In particular this implies that either $t_1 = b$ or $t_1 = c$.

If $t_1 = b$, then $t_2 = t_3 = t_7 = z$. It follows that none of the four regions bounded by ϑ can be a 3-gon. This contradicts the assumption that ϑ is positively-curved. Hence $t_1 = c$. It follows that either

1. $t_2 = t_6 = x$, $t_3 = t_7 = z$, $t_4 = t_8 = b$, and $t_5 = c$, or

2. $t_2 = t_6 = x$, $t_3 = t_7 = z^{-1}$, $t_4 = t_8 = b^{-1}$, and $t_5 = c$.

In both cases the two B-regions can not be 3-gons and so at least one of the two A-regions is a 3-gon. So $bac = 1$. This contradicts the assumption that A is neither cyclic nor dihedral. By symmetry, B is also cyclic or dihedral. This completes the proof. □

4.4.2 Configurations of an interior vertex of degree 5

In this section we consider positively-curved interior vertices ϑ of degree 5. It follows from Corollary 4.4.2 that 4-zones connect vertices of same sign.

Lemma 4.4.10. *Configurations of Type 2 do not occur.*

Proof. In each case, a 4-zone is adjacent to a 3-zone. The 4-zone joins vertices of same sign and is labelled as shown in Figure 4.17. Hence we get the relations $xy = b^2 = 1$. So we assume by Lemma 4.4.2 that $ac \neq 1$.

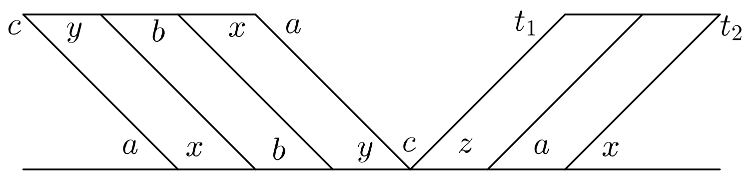

Figure 4.17: Configuration of Type 2

1. When $t_1 = a^{-1}$ or $t_1 = c$, we have $ab = yz = 1$. Hence the result follows from Lemma 4.4.2.

2. When $t_1 = b$ or $t_1 = b^{-1}$, the B-region with corner labelled t_2 can not be a 3-gon. Hence in the first case $a^2 = xz = acb = 1$. In the second we have $yz^{-1} = ac^{-1} = acb^{-1} = 1$. So A is either cyclic or a quotient of D_∞.

□

Lemma 4.4.11. *Configurations of Type 3 do not occur.*

Proof. The proof is divided into three cases: the case with adjacent 4-zones, the case with non-adjacent 4-zones, and the case with one 4-zone.

1. Suppose that ϑ has two adjacent 4-zones (as in $(4,4,2,1,1)$ and $(4,4,1,2,1)$). Then from Figure 4.18, B is cyclic and $b^2 = a^2 = 1$.

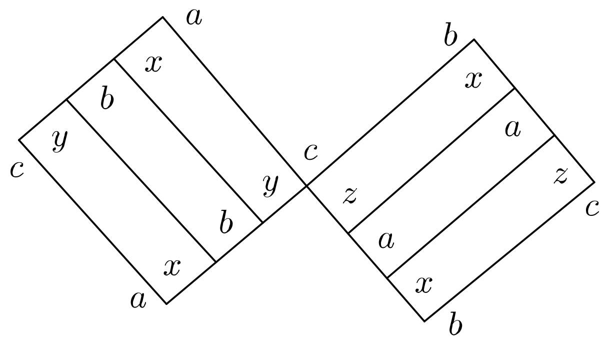

Figure 4.18: Adjacent 4-zones

Also $tca = 1$ where $t \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$. Hence A is cyclic or dihedral.

2. Suppose that ϑ has configuration $(4, 1, 4, 2, 1)$. Then from Figure 4.19, the 4-zones give the relations $ab = b^2 = 1$ and $x^2 = xy = 1$.

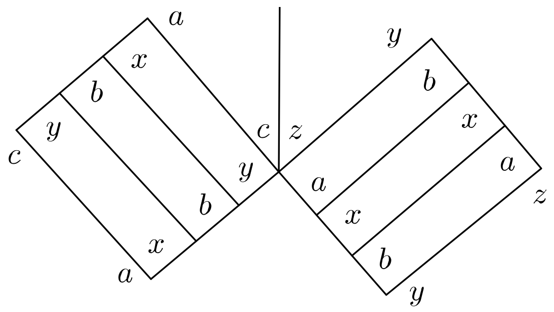

Figure 4.19: Two 4-zones separated by a 1-zone

Also we have relations $szx^{\pm 1} = 1$ and $tac = 1$. So both A and B are cyclic, hence the result.

3. Finally, suppose that ϑ has configuration $(4, 1, 3, 3, 1)$. Then from Figure 4.20, the 4-zones give the relations $xy = b^2 = 1$.

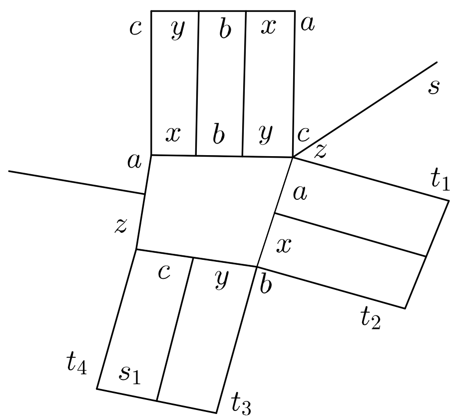

Figure 4.20: Configuration $(4, 1, 3, 3, 1)$

(a) When $t_1 = x$, we have the relations $a^2 = xz = 1$ and $tac = 1$. Hence both A and B are cyclic.

(b) When $t_1 = y$, then the A-region containing t_2 can not be a 3-gon. Hence $szy = 1$ and $tac = 1$. So each of A and B is either cyclic or dihedral. Similar conclusion holds when $t_1 = x^{-1}$ (since $ab^{-1} = xy^{-1} = 1$).

(c) When $t_1 = z$, then we have $ac = xy = 1$. The result for this case follows from Lemma 4.4.2.

(d) When $t_1 = y^{-1}$, we have $ac^{-1} = xz^{-1} = 1$, and so both A and B are cyclic.

$\square$

Lemma 4.4.12. *Configurations of Type 4 do not occur.*

We prove this Lemma in a series of propositions one for each configuration in Type 4.

Proposition 4.4.13. *Configuration* $(4, 1, 2, 3, 2)$ *does not occur.*

Proof. Figure 4.21 shows the labelling of ϑ with configuration $(4, 1, 2, 3, 2)$. From the 4-zone we get the relation $xy = b^2 = 1$. Also $at = 1 = sz$ and at least one of $t_1ac = 1$ or $t_2ac = 1$ holds.

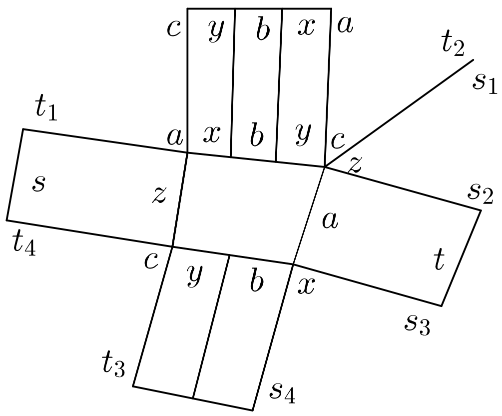

Figure 4.21: Configuration $(4, 1, 2, 3, 2)$

We consider the 3-zone.

1. When $s_4 = z^{-1}$, then $ab^{-1} = xy^{-1} = 1$. So A is cyclic and B is either cyclic or dihedral.

2. When $s_4 = y^{-1}$, we have $bc^{-1} = yz^{-1} = 1$. Hence B is cyclic and A is cyclic or dihedral. Similar conclusion holds when $s_4 = z$ (since $y^2 = bc = 1$).

3. When $s_4 = x$, we have $yz = bc = 1$. In this case the result follows from Lemma 4.4.2.

4. When $s_4 = y$, we get no new relations from the 3-zone. However we conclude that the B-region containing s_4 is not a 3-gon. So if $s \neq z$, then $t_1 = b^{\pm 1}$ or $t_2 = b^{\pm 1}$, and hence B is cyclic and A is either cyclic or dihedral. Hence we suppose that $s = z$, $t_1 = c$, $t_2 = a$, so in particular $z^2 = 1$ and $a = c$. We conclude from $zs_2s_1 = 1$ that B is cyclic (since $s_1, s_2 \in \{x, y\}^{\pm 1}$). Also we can

assume that $t = c^{-1}$, $s_2 = y^{-1}$ and so $s_1 = x$ and $t_2 = b$. Hence A is also cyclic (since $t_2ac = 1$).

□

Proposition 4.4.14. *Configuration* $(4, 1, 3, 2, 2)$ *does not occur.*

Proof. Figure 4.22 shows the labelling of ϑ with configuration $(4, 1, 3, 2, 2)$. From the 4-zone we get the relations $xy = b^2 = 1$. Also $bt = 1 = sz$, and at least one of $t_1ac = 1$ or $t_2ac = 1$ holds. As before we assume $ac \neq 1$ by Lemma 4.4.2, so in particular $s_2 \neq z$.

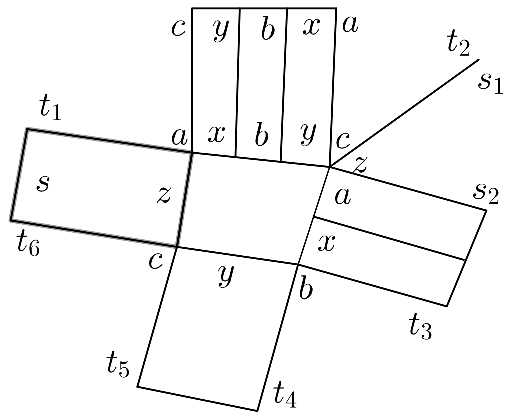

Figure 4.22: Configuration $(4, 1, 3, 2, 2)$

We consider the 3-zone.

1. When $s_2 = x$ and so $t_3 = c$, then $xz = a^2 = 1$, and so in particular B is cyclic. It follows that the B-region containing s_2 is not a 3-gon. So A is cyclic (since $bt_4t_3 = 1$).

2. When $s_2 = x^{-1}$ and so $t_3 = c^{-1}$, then $ab^{-1} = xy^{-1} = 1$. Hence either A is cyclic or dihedral. Same conclusion holds for B too. Also similar conclusion holds when $s_2 = y$ (since $x^2 = ab = 1$).

3. When $s_2 = y^{-1}$ and so $t_3 = a^{-1}$, then $ac^{-1} = xz^{-1} = 1$. In particular B is cyclic. If $s \neq z$, then $t_1 = b^{\pm 1}$ or $t_6 = b^{\pm 1}$, and A is either cyclic or dihedral (given also that the region containing $t_3 (= c^{-1})$ contains a corner labelled b). So we suppose that $s = z$ and so $B = \mathbb{Z}_2$. Hence no B-region is a 3-gon. So again A is cyclic or dihedral.

□

Proposition 4.4.15. *Configuration* $(4, 2, 3, 1, 2)$ *does not occur.*

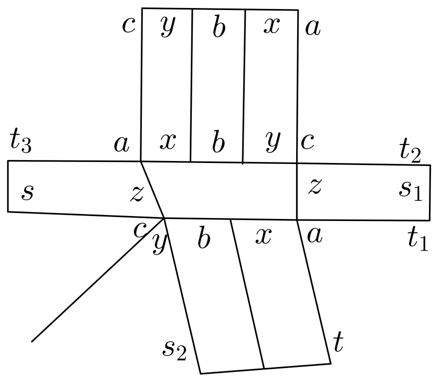

Figure 4.23: Configuration $(4, 2, 3, 1, 2)$

Proof. From the 4-zone we get the relations $xy = b^2 = 1$. Also $s_1 z = 1$ and $t_i ac = 1$, where $i \in \{2, 3\}$. If the 3-zone joins vertices of opposite signs, then either $x = y$, $b = c$ or $x = z$, $b = a$. In the first case A is cyclic and B is cyclic or a quotient of D_∞. In the second case both A and B are cyclic.

Suppose the 3-zone joins vertices of same sign. If $xz = 1 = bc$, then both A and B are cyclic. If $x^2 = 1 = ab$, then A cyclic and B is either cyclic or a quotient of D_∞. Finally if the 3-zone gives the relations $xy = b^2 = 1$, then we can assume that the B-region is not a 3-gon. This forces $s_1, s_2 \neq z$, and hence both A and B are cyclic (since $t_2 = b$ or $t_3 = b$). Otherwise $z^2 = 1$ and $a = c^{-1}$ (since $t_2 = a$ and $t_3 = c$). Then the result follows from Lemma 4.4.2. $\square$

Lemma 4.4.16. *Configurations of Type 5 do not occur.*

The proof of Lemma 4.4.16 is divided into four propositions, one for each configuration in Type 5. In each case we can assume by minimality and Lemma 4.4.2 that $t_1 \in \{b^{\pm 1}, c^{\pm 1}\}$. In particular the following holds:

1. If $t_1 = b$, then $x^2 = ab = 1$, $s_2 = z$ and the region containing t_1 is not a 3-gon. Also by Lemma 4.4.2 we assume $yz \neq 1$.
2. If $t_1 = b^{-1}$, then $xy^{-1} = bc = 1$ and $s_1 = z^{-1}$.
3. If $t_1 = c$, then then $xy = b^2 = 1$, $s_1 = x$ and the region containing s_1 is not a 3-gon. Also by Lemma 4.4.2 we assume $ac \neq 1$.
4. If $t_1 = c^{-1}$, then $xz^{-1} = ab^{-1} = 1$ and $s_1 = x^{-1}$.

Proposition 4.4.17. *Configuration* $(3, 3, 2, 2, 2)$ *does not occur.*

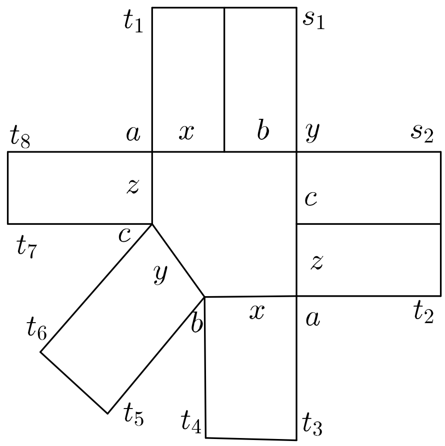

Figure 4.24: Configuration $(3, 3, 2, 2, 2)$

Proof. 1. Suppose that A is neither cyclic nor dihedral.

(a) If $t_1 = b$, then we can assume that $s_2 = z$, so that $c^2 = 1$. Either A is cyclic or $s_2 = z$ and so $yz = 1$. In either case we get a contradiction to assumption.

(b) If $t_1 = b^{-1}$, then we can assume that $t_2 = b$ and $c^2 = 1$ or $t_2 = c^{-1}$. Since at least one of $t_3t_2a = 1$ or $t_1t_8a = 1$ holds, then A is cyclic.

(c) If $t_1 = c$, then it follows that A is cyclic or dihedral if $c^2 = 1$ using $t_7t_6b = 1$. Otherwise if $c = a^{\pm 1}$, we use $t_5t_4b = 1$ to show A is cyclic.

(d) If $t_1 = c^{-1}$, then we can assume that $t_2 = b$ and so $c^2 = 1$. Since at least one of $t_7t_6c = 1$ or $t_8at_1 = 1$ holds, we conclude that A is cyclic.

2. Suppose that B is neither cyclic nor dihedral.

(a) If $t_1 = b$, then we can assume that $s_2 = z$, so that $c^2 = 1$. By Lemma 4.4.2 and minimality we can assume $t_2 \in \{c^{\pm 1}, b^{-1}\}$. If $t_2 = b^{-1}$, then $xz^{-1} = 1$ and so $1 = s_1ys_2 = zyz^{-1}$. This contradicts the assumption that $y \neq 1$. If $t_2 = c$, then $z^2 = 1$ and $s_2 = x$. Similarly f $t_2 = c^{-1}$, then $yz^{-1} = 1$ and so $s_2 = x^{-1}$. In both cases it follows that B is cyclic or dihedral since $zys_2 = 1$.

(b) If $t_1 = b^{-1}$, then we can assume that $t_2 = c$ and so $z^2 = ac = 1$, $s_2 = x$. Since $t_3t_2a \neq 1$, it follows that B is cyclic as $1 = s_1ys_2 = z^{-1}yx$.

(c) If $t_1 = c$, we conclude from the other 3-zone that B is cyclic or $ac = z^2 = 1$. In the latter, the result follows from Lemma 4.4.2.

(d) If $t_1 = c^{-1}$, then we can assume that $t_6 = b$ and so $y^2 = 1$. Also we can assume that $t_2 = b^{-1}$, so $xz^{-1} = ac^{-1} = 1$ and $s_2 = z^{-1}$. Hence B is cyclic since $s_2ys_2 = 1$ (as $t_3t_2a \neq 1$).

□

Proposition 4.4.18. *Configuration* $(3, 2, 3, 2, 2)$ *does not occur.*

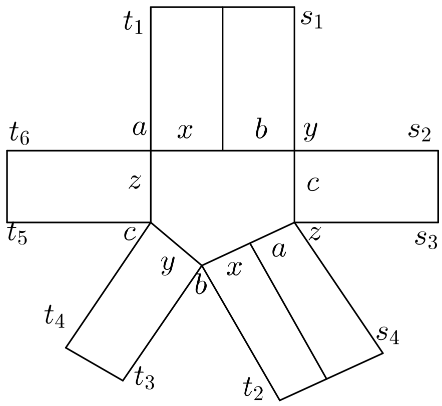

Figure 4.25: Configuration $(3, 2, 3, 2, 2)$

Proof. 1. Suppose that A is neither cyclic nor dihedral.

(a) If $t_1 = b$, we assume that $s_3 = y$, so that $c^2 = 1$. We then conclude from $t_3t_4c = 1$ that A is cyclic.

(b) Suppose $t_1 = b^{-1}$. We can assume that $t_2 = c$, so that $a^2 = xz = 1$ and $s_4 = x$. It follows that $s_4s_3z \neq 1$. Hence $1 = t_1t_6a = b^{-1}t_6a$, and so A is cyclic.

(c) If $t_1 = c$, we conclude from the 2-zone with corner labelled c that either $c^2 = 1$ or $c = a^{\pm 1}$. It follows from $1 = t_1t_6a = ct_6a$ that A is cyclic or dihedral.

(d) If $t_1 = c^{-1}$, then we can assume that $s_4 = x$ so that $t_2 = c$, $xz = a^2 = 1$ and $s_4s_3z \neq 1$. It follows from $1 = t_1t_6a = c^{-1}t_6a$ that A is cyclic.

2. Suppose that B is neither cyclic nor dihedral.

(a) If $t_1 = b$, then we can assume that $x = y^{\pm 1}$ (alternatively $x = z^{\pm 1}$) since $t_2 \neq a$. It follows that B is cyclic or dihedral (using the 2-zone containing y (alternatively z).

(b) If $t_1 = b^{-1}$, then we can assume that $t_6 = c$ and so $z^2 = 1$. Since at least one of $s_1ys_2 = 1$ or $s_4s_3z = 1$ holds, B is cyclic.

(c) If $t_1 = c$, then we can assume that $t_6 = c$ and so $z^2 = 1$. It follows that $s_4s_3z = 1$ (since $s_1ys_2 \neq 1$). Hence B is cyclic.

(d) If $t_1 = c^{-1}$, we can assume that $y^2 = 1$ (using the 2-zone with corner labelled y). Also we can assume that $t_5 = a^{-1}$, $t_6 = b^{-1}$ and so $t_6at_1 \neq 1$. It follows from $1 = s_1ys_2 = x^{-1}ys_2$ that B is cyclic.

□

Proposition 4.4.19. *Configuration* $(3,3,3,2,1)$ *does not occur.*

Proof. By Lemma 4.4.2 and minimality we can assume $t_2 \neq a^{\pm 1}$.

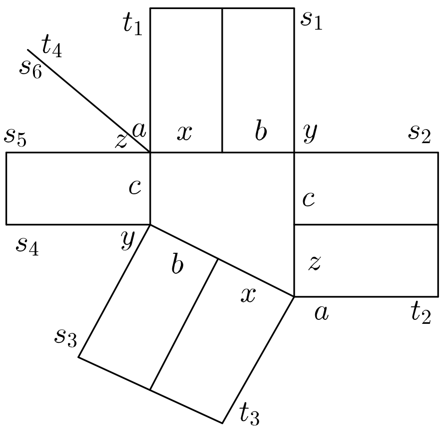

Figure 4.26: Configuration $(3,3,3,2,1)$

1. Suppose that A is neither cyclic nor dihedral.

   (a) If $t_1 = b$, we can assume that $t_2 = b$, so that $yz = c^2 = 1$. The rest of the result follows from Lemma 4.4.2.

   (b) Suppose $t_1 = b^{-1}$, then A is cyclic since at least one of $t_1 t_4 a = 1$ or $t_2 a t_3 = 1$ holds.

   (c) If $t_1 = c$, then using the 2-zone with corner labelled c we conclude that $c^2 = 1$ or $c = a^{\pm 1}$ or $c = b^{\pm 1}$. Since $t_4 a t_1 = t_3 t_2 a = 1$, we have that A is cyclic or dihedral.

   (d) If $t_1 = c^{-1}$, we assume that $t_2 = b$, so that $c^2 = zy = 1$. Hence $s_1 y s_2 \neq 1$. So $t_4 a c^{-1} = 1$ and hence A is cyclic.

2. Suppose that B is neither cyclic nor dihedral.

   (a) If $t_1 = b$, then by minimality and Lemma 4.4.2 $s_2 \in \{y^{\pm 1}, z^{\pm 1}\}$. Hence $z = y^{-1}$ or $z^2 = 1$ and $1 = s_1 y s_2 = zyx^{\pm 1}$. So B is cyclic or dihedral.

   (b) If $t_1 = b^{-1}$, we can assume that $t_2 = c$ and so $z^2 = 1$, $t_2 a t_3 \neq 1$. It follows that $s_1 y s_2 = 1$ and so B is cyclic.

   (c) If $t_1 = c$, we can assume that $t_2 = c$. Hence the result follows from Lemma 4.4.2 since $ac = 1$.

   (d) Suppose that $t_1 = c^{-1}$. Since at least one of $s_1 y s_2 = 1$ or $s_4 s_3 y = 1$ holds, we conclude that B is cyclic unless $s_1 y s_2 \neq 1$ and $s_4 = s_3 = y$. But in this case $bc = xz = 1$ and we can apply Lemma 4.4.2.

□

Proposition 4.4.20. *Configuration* $(3,3,1,3,2)$ *does not occur.*

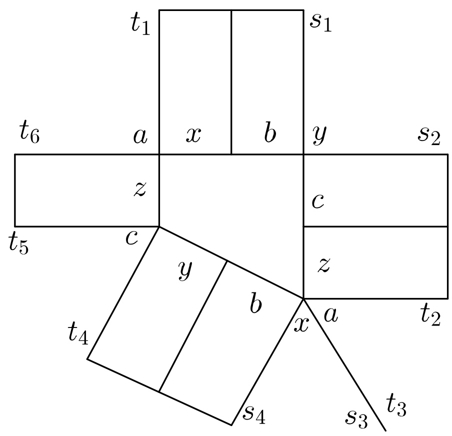

Figure 4.27: Configuration $(3,3,1,3,2)$

Proof. 1. Suppose that A is neither cyclic nor dihedral.

(a) If $t_1 = b$, then using $ct = 1$ (from the other 3-zone) and $t_5t_6c = 1$ we conclude that A is cyclic.

(b) Suppose $t_1 = b^{-1}$. If $c^2 = 1$ (alternatively $b^2 = 1$), we conclude that A is cyclic since at least one of $t_1t_6a = 1$ or $t_4ct_5 = 1$ holds. Hence we assume that $s_2 = x^{\pm 1}$, and so $t_2 = c^{-1}$. But also at least one of $t_1t_6a = 1$ or $t_2at_3 = 1$ holds. Hence A is again cyclic.

(c) If $t_1 = c$, then we can assume that $t_2 = c$ and so the result follows from Lemma 4.4.2.

(d) If $t_1 = c^{-1}$, then we can assume that $s_4 = c$. Since $x^{-1}yx = s_1ys_2 \neq 1$, we have $1 = at_3t_2 = at_3c$, and so A is cyclic.

2. Suppose that B is neither cyclic nor dihedral.

(a) If $t_1 = b$, we conclude from one of the other 3-zones that $z^2 = 1$ or $z = y^{\pm 1}$. Either way it follows from $1 = s_1ys_2 = s_4s_3x$ that B is cyclic or dihedral.

(b) If $t_1 = b^{-1}$, then we can assume that $t_2 = c$, so $z^2 = 1$ and $s_2 = x$. Since $t_3t_2a \neq 1$, we have that $s_1ys_2 = 1$ and so B is cyclic.

(c) If $t_1 = c$, we can assume that $t_2 = c$ and so the result follows from Lemma 4.4.2.

(d) If $t_1 = c^{-1}$, we can assume that $t_4 = b$ and so $y^2 = 1$. Since $z^2 \neq 1$ by assumption, $s_2 = z^{-1}$ and so $ac^{-1} = 1$. It follows that $s_1 y s_2 = 1$ (since $t_3 t_2 a \neq 1$). Hence B is cyclic.

□

Part 2

Here we consider the bad cases and show that part 2 of Theorem 4.4.1 holds. We do this in number of Lemmas, one for each of the three configurations.

Lemma 4.4.21. *Configuration* $(4, 2, 2, 2, 2)$ *does not occur.*

Proof. Configuration $(4, 2, 2, 2, 2)$ is depicted in Figure 4.28. From the 4-zone we get the relations $xy = b^2 = 1$. Clearly, this implies B is cyclic whenever $s_1, s_2 \neq z$. We can assume without loss of generality that the two regions with corners labelled t_1 and t_2 are 3-gons.

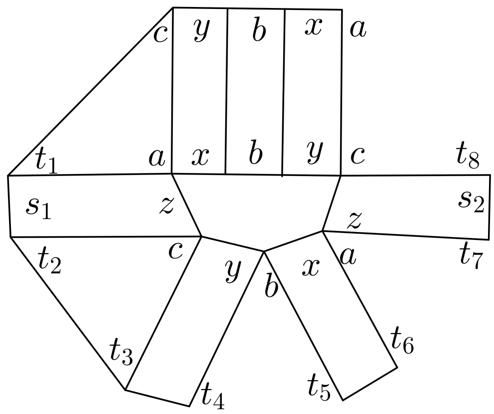

Figure 4.28: Configuration $(4, 2, 2, 2, 2)$

In other words $t_1 ac = s_1 z = t_2 t_3 c = 1$. There are four cases to consider depending on the value of s_1. Note that $s_1 \neq y^{-1}$ (as that will imply that $t_1 = c^{-1}$ and so $a = 1$).

1. Suppose $s_1 = x$ and hence $t_1 = a$ and $t_2 = b$. Then $c = a^{-2}$ and $t_3 \in \{a^{\pm 1}, b, c\}$. In each case A is cyclic.

2. Suppose $s_1 = x^{-1}$, and hence $t_1 = b^{-1}$ and $t_2 = a^{-1}$. Then $b = ac$ and $a = t_3 c$. If $t_3 \neq b$, then A is cyclic. Otherwise if $t_3 = b$, then $b = c^{-1}a$. It follows that $a^2 = (ac)(c^{-1}a) = b^2 = 1$. So A is a quotient of D_∞.

3. Suppose $s_1 = y$ and hence $t_1 = b$ and $t_2 = c$. Then $bac = 1 = t_3 c^2$. If $t_3 \neq c$, then A is cyclic. So we suppose that $t_3 = c$, hence $c^3 = 1$ and $t_4 = a$. At least

one of $t_5ba = 1$ or $act_8 = 1$ holds. In the former, we can assume that $t_8 = b$ (as otherwise A is cyclic). It follows that $s_2 \in \{x, y^{-1}\}$, and so $z^2 = 1$. Similarly in the latter, we can assume that $t_5 = c$, and so from the 2-zone we have that $xz = 1$. Again $z^2 = 1$. This completes the proof.

□

Lemma 4.4.22. *Configuration* $(4, 2, 4, 1, 1)$ *does not occur.*

Proof. Without loss of generality, Figure 4.29 shows ϑ with configuration $(4, 2, 4, 1, 1)$. The 4-zone gives the relations $xy = b^2 = 1$.

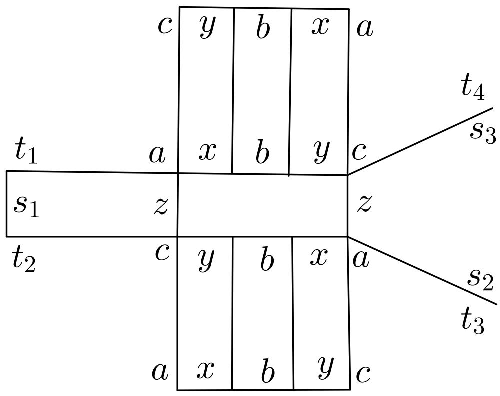

Figure 4.29: Configuration $(4, 2, 4, 1, 1)$

Suppose $s_1 = x$, and so $t_1 = a$ and $t_2 = b$. If the region containing s_2 is not a 3-gon, then all other regions which are not 2-gonal are 3-gons. In particular $t_2ac = bac = 1 = t_1ac = a^2c$. So A is cyclic. Hence we suppose the region containing s_2 is a 3-gon, so $s_2s_3z = 1$. Note that $t_4 \neq b$ (as otherwise $s_3 \in \{x, y^{-1}\}$ and so $s_2 = 1$). So $act_4 = 1 = bac$, where $t_4 \in \{a, c\}$. Again A is cyclic.

Suppose $s_1 = x^{-1}$, and so $t_1 = b^{-1}$ and $t_2 = a^{-1}$. In particular the region containing t_2 is not a 3-gon. Hence all other regions which are not 2-gonal are 3-gons. So $t_1ac = bac = 1 = s_2s_3z$. Note that $t_3, t_4 \neq b$ (since $z = x = y^{-1}$). It follows that $t_3ac = 1$ with $t_3 \in \{a, c\}$. Hence A is cyclic.

By symmetry, similar argument takes care of the cases where $s_1 = y^{\pm 1}$. Hence we must have that $s_1 = z$. This completes the proof. □

Lemma 4.4.23. *Configuration* $(4, 2, 4, 2)$ *does not occur.*

Proof. Without loss of generality, Figure 4.30 shows ϑ with configuration $(4, 2, 4, 2)$. If any of the two 4-zones joins vertices of opposite signs, then in particular B is cyclic. Also either $b = a$ (alternatively $b = c$). We can use the four A-regions to show that A is cyclic or dihedral, hence the result.

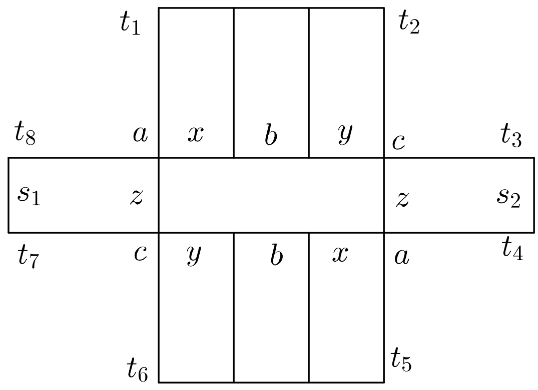

Figure 4.30: Configuration $(4, 2, 4, 2)$

So we assume that $t_2 = t_6 = a$, $t_1 = t_5 = c$. Hence $xy = b^2 = 1$. If $s_1 = z$ or $s_2 = z$ or more generally B is $\mathbb{Z}_2$, then we are done. Suppose not, then $\{s_1, s_2\} \in \{\{x, y^{-1}\}, \{x^{-1}, y\}, \{y\}, \{x\}\}$. At this point we amend the picture M to get $\hat{M}$ in which the number of 2-gonal A-regions is maximal. We can assume that the only positively-curved interior vertex in $\hat{M}$ has configuration $(4, 2, 4, 2)$; and proceed as before. Since there are two A-regions with consecutive corners labelled $b^{\pm 1}, a, c$, we have that either $bac = 1$, or $tbac = 1$ and there is a relation of length at most 3 involving only a and c. In the latter A is cyclic. In the former r has order 2 for otherwise its image under the projection map onto A is abc. Hence $abc = acb = 1$, hence A is abelian. In both case G satisfies the Freiheitssatz. This completes the proof. $\square$

This shows that our assumption that Theorem 4.4.1 fails was false, hence the proof of the Theorem 4.3.1 is complete.

4.5 Relator of length four with power $n \geq 2$

In the section we consider the one-relator group

$$G = \frac{(A * B)}{N(r^n)}, \tag{4.14}$$

where $n \geq 2$ and $r = abcd$ with $a, c \in A$ and $b, d \in B$.

Theorem 4.5.1. *Let G be as in 4.13. Then the following holds for G.*

1. *Freiheitssatz holds for G.*

2. *The word r has order n in G.*

Lemma 4.5.2. *If $b = d^{-1}$ or $a = c^{-1}$, then Theorem 4.14 holds.*

Proof. Without loss of generality, we assume that $b = d^{-1}$. If $n \geq 3$, then the result holds by [62]. Hence we assume that $n = 2$. By assumption, A is not cyclic.

Suppose Freiheitssatz fails. We construct a $\mathcal{W}$-minimal spherical picture M over G on D^2. The label of each vertex contains two of letter a and two of letter b. Hence by Lemma 4.2.5, at least one of the interior A-regions must have degree 2 or 3. Since A is not cyclic, any relation of length 2 or 3 between a and c must be a power of a or a power of c. Suppose that a has order $p \in \{2, 3\}$ and c has order $q \geq 2$. Then in particular $\langle a \rangle \cap \langle c \rangle = \{1\}$ in A.

The normal closure N of A in G is the quotient of m copies $A_i := b^i A b^{-i}$ of A (where m is the order of b in B) by the relators $(a_i c_{i+1})^n$ (subscripts mod m). Let $H_i := \langle A_i,\ a_{i-1} \mid (a_{i-1}c_i)^n = 1\rangle$, the free product of A_i with a triangle group $T_i := \langle a_{i-1}, c_i \mid a_{i-1}^p = c_i^q = (a_{i-1}c_i)^n = 1\rangle$, amalgamated over the cyclic subgroup $\langle c_i \rangle$.

If $m = \infty$ then G is the HNN extension of H_0 with associated subgroups $\langle a_0 \rangle$ and $\langle a_{-1} \rangle$ and stable letter b, and the result clearly holds in this case. Suppose then that $m < \infty$. By [[88], Theorem 7], H_i acts on a tree X with fundamental domain a segment T (see Figure 4.31) with A_i, T_i and $\langle c_i \rangle$ the stabilizers of the vertices u, v

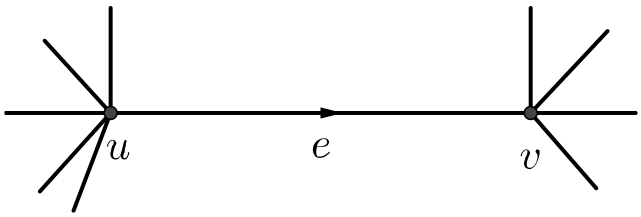

Figure 4.31: A segment T of X

and the edge e respectively. The edge e partitions X into two disjoint components: X_u (the component containing the vertex u) and X_v (the component containing the vertex v). Since $\langle a_{i-1}\rangle \cap \langle c_i \rangle = \{1\}$ in T_i and $\langle a_i \rangle \cap \langle c_i \rangle = \{1\}$ in A_i , $a_i^j X_u \subseteq X_v$ and $a_{i-1}^j X_v \subseteq X_u$ for $j \not\cong 0$ mod p (i.e no non-trivial element of $\langle a_{i-1}\rangle$ or $\langle a_i \rangle$ fixes the e). It follows from the Ping-Pong Lemma that the subgroup $F_i < H_i$ generated by a_{i-1} and a_i is the free product $\langle a_{i-1}\rangle * \langle a_i \rangle$. Let $H' := b *_{\langle a_1 \rangle} \cdots *_{\langle a_{m-2}\rangle} H_{m-1}$. Then the subgroup of H' generated by a_{m-1} and a_0 is isomorphic to F_0, and $N = H_0 *_{F_0} H'$. Hence $A = A_0 < H_0 < N < G$. Trivially $B < G$.

Suppose that r does not have order n in G. The proof is similar to that in Lemma 4.4.2. We construct a $\mathcal{W}$-minimal spherical clique-picture M over G. A typical clique-label will have the form

$$a^{\alpha_1} b c^{\beta_1} b^{-1} \ldots a^{\alpha_k} b y^{\beta_k} b^{-1}.$$

G is induced by the generalised triangle group

$$T(p,q,n) = \langle a, c \mid a^p, c^q, (ac)^2 \rangle. \tag{4.15}$$

Hence by the Spelling theorem for generalised triangle groups, it follows that for an interior clique-label $k \geq 2$. Also we assume that A is not cyclic. In any region of M, we may assume that no two consecutive corner labels are of the form a^α or c^β (for otherwise we could amalgamate the two cliques into one, forming a smaller clique-picture).

In particular, if a 2-zone is labelled by a^α at one corner, the opposite corner has to be a $c^{\pm 1}$. But also this can not happen since from the proof of Freiheitssatz, $p = 2$ or 3. Hence any A-region has degree at least four. It follows that M has at most one clique of positive curvature, namely the clique corresponding to the exceptional vertex. Hence the total curvature of M is less than 2π. This contradiction implies that r has order n.
This completes the proof. □

Corollary 4.5.3. *M has a positively-curved interior vertex.*

Lemma 4.5.4. *If a* 3*-zone is incident at ϑ, then Theorem 4.14 holds.*

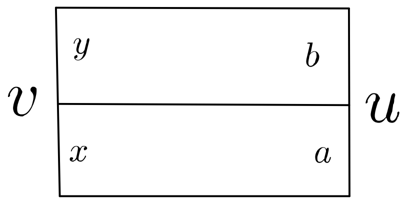

Figure 4.32

Proof. Suppose that two vertices u and v of M are joined by the 3-zone. Without loss of generality we can assume that the corner labels at u are a and b. Suppose x and y are the corresponding labels at v i.e $ax = 1 = by$. Then $(x,y) \in \{(a,d),(c,b),(c^{-1},d^{-1})\}$. In each case, the result holds by assumption and Lemma 4.5.2. □

Proof of Theorem 4.5.1. By Lemma 4.5.2 we can assume that neither $b = d^{-1}$ nor $a = c^{-1}$ holds.

Suppose that Theorem 4.5.1 fails. Then we obtain a obtain a spherical diagram M over G with boundary label $w \in A \cup B$. Each vertex has $4n$ ($n \geq 2$) edges incident on it and with label $r^{\pm n}$. By Lemma 4.2.5, M has a positively curved interior vertex

ϑ. It follows from Lemma 4.5.4 that $4n - 5 \leq \dfrac{4n}{2}$ i.e $n = 2$. This agrees with the already known result that Freiheitssatz holds for $n \geq 4$ and proves it also for $n = 3$.

When $n = 2$, it follows from Lemma 4.5.4 that it is enough to consider the cases where the number of regions of v of degree 2 is three or four. If there are four regions of degree 2, then without loss of generality there are two A-regions –say Δ_1, Δ_2 each of degree 2 separated by a B-region of degree 3.

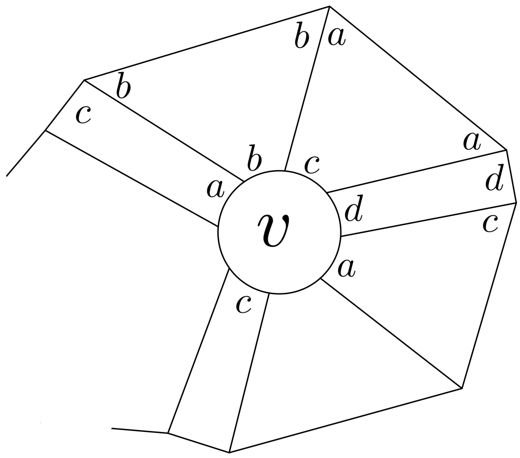

Figure 4.33: A positively-curved interior vertex

Without loss of generality, the possible cases for the labels of the two A-vertices are:

1. Δ_1 has label a^2 and Δ_2 has label c^2;
2. Δ_1 has label a^2 and Δ_2 has label ca^{-1};
3. Δ_1 has label ac^{-1} and Δ_2 has label ca^{-1}.

In all cases, A has faithful representation in $PSL_2(\mathbb{C})$. Also B is cyclic as the label of the B-region contains both the elements b and d. This contradicts the starting hypothesis and hence the proof.

If on the other hand there are three 2-zones, then v has degree 5. Hence v has at least four regions of degree 3. If there is a region of degree 3 separating two 2-zones, the the result follows as before. Without loss of generality, we can assume that the three 2-zones have labels ac, ca and b^2 respectively as shown in Figure 4.33. Hence $a = c = 1$, contradicting hypothesis. $\square$

4.6 Special case of $n = 1$ and relator has length four.

In the section we consider the one-relator group

$$G = \frac{(A * B)}{N(r)}, \tag{4.16}$$

where $r = abcd$ with $\langle a, c\rangle = A_0 \leq A$ and $\langle b, d\rangle = B_0 \leq B$. By putting restrictions on the relator and on the order of the factors, Shwartz [90] has some classification on when Freiheitsatz holds. For us the only requirement is that each of A_0 and B_0 is a two-generator groups with faithful representation in $PSL_2(\mathbb{C})$.

Theorem 4.6.1. *Let G be as in 4.16. Under above assumption, Freiheitssatz holds for G.*

We make the following observation about representation of groups in $SL_2(\mathbb{C})$. For more details see ([16],[33],[3]). Let $F_2 = \langle x, y\rangle$ be a free group of rank two. Then the irreducible representations $\rho : F_2 \longrightarrow SL_2(\mathbb{C})$ are parametrised (up to conjugacy) by the three independent parameters namely $\mathrm{Tr}\rho(x)$, $\mathrm{Tr}\rho(y)$ and $\mathrm{Tr}\rho(xy)$. In other words, the character variety of conjugacy classes of irreducible representations ρ is an affine algebraic set contained in $\mathbb{C}^3$. Recall that ρ is reducible if and only if $\mathrm{Tr}[\rho(x), \rho(y)] = 2$. This criterion translates to

$$\mathrm{Tr}(\rho(x))^2 + \mathrm{Tr}(\rho(y))^2 + \mathrm{Tr}(\rho(xy) - \mathrm{Tr}(\rho(x)\mathrm{Tr}(\rho(y)\mathrm{Tr}(\rho(xy) = 4. \tag{4.17}$$

So the irreducible character variety is the complement of a cubic surface in $\mathbb{C}^3$.

If we rather consider the free group in three variables $F_3 = \langle x, y, z\rangle$, then by [33] a conjugacy class of irreducible representations $\rho : F_3 \longrightarrow SL_2(\mathbb{C})$ is determined by seven parameters: $\mathrm{Tr}\rho(x)$, $\mathrm{Tr}\rho(y)$, $\mathrm{Tr}\rho(z)$, $\mathrm{Tr}\rho(xy)$, $\mathrm{Tr}\rho(xz)$, $\mathrm{Tr}\rho(yz)$ and $\mathrm{Tr}\rho(xyz)$ which are no longer independent but satisfy the following polynomial equation.

$$\begin{aligned}
&\mathrm{Tr}\rho(x)^2 + \mathrm{Tr}\rho(y)^2 + \mathrm{Tr}\rho(z)^2 + \mathrm{Tr}\rho(xy)^2 + \mathrm{Tr}\rho(xz)^2 + \mathrm{Tr}\rho(yz)^2 + \mathrm{Tr}\rho(xyz)^2 \\
&+\mathrm{Tr}\rho(xy)\mathrm{Tr}\rho(xz)\mathrm{Tr}\rho(yz) - \mathrm{Tr}\rho(x)\mathrm{Tr}\rho(y)\mathrm{Tr}\rho(xy) - \mathrm{Tr}\rho(x)\mathrm{Tr}\rho(z)\mathrm{Tr}\rho(xz) \\
&-\mathrm{Tr}\rho(y)\mathrm{Tr}\rho(z)\mathrm{Tr}\rho(yz) + \mathrm{Tr}\rho(xyz)\mathrm{Tr}\rho(x)\mathrm{Tr}\rho(y)\mathrm{Tr}\rho(z) - \mathrm{Tr}\rho(xyz)\mathrm{Tr}\rho(x)\mathrm{Tr}\rho(yz) \\
&-\mathrm{Tr}\rho(xyz)\mathrm{Tr}\rho(y)\mathrm{Tr}\rho(xz) - \mathrm{Tr}\rho(xyz)\mathrm{Tr}\rho(z)\mathrm{Tr}\rho(xy) = 4
\end{aligned} \tag{4.18}$$

The irreducible character variety is 6-dimensional, and therefore comprised of a hyper-surface in $\mathbb{C}^7$ defined by Equation 4.18, minus the subset corresponding to reducible representations, which is the intersection the three cubic curves arising from the commutators $\mathrm{Tr}[\rho(x), \rho(y)]$, $\mathrm{Tr}[\rho(x), \rho(z)]$ and $\mathrm{Tr}[\rho(y), \rho(z)]$, where $[x, y] = xyx^{-1}y^{-1}$.

Using above comments we give a proof of Theorem 4.6.1.

Proof of Theorem 4.6.1. Let X, Y and Z be variable matrices in $SL_2(\mathbb{C})$. The aim of the proof is to show that one can choose X, Y and Z such that $a \mapsto X$, $c \mapsto Z$ gives a faithful representation $A \mapsto PSL_2(\mathbb{C})$ and $b \mapsto Y$, $d \mapsto (XYZ)^{-1}$ gives a faithful representation $B \mapsto PSL_2(\mathbb{C})$.
Such a triple of matrices is a representation of the free group F_3 of rank 3. Recall that the character variety of representations $F_3 \to PSL(2, \mathbb{C})$ is given by the seven parameters $\mathrm{Tr}(X)$, $\mathrm{Tr}(Z)$, $\mathrm{Tr}(Y)$, $\mathrm{Tr}(XY)$, $\mathrm{Tr}(XZ)$, $\mathrm{Tr}(YZ)$ and $\mathrm{Tr}(XYZ)$ subject to a single polynomial equation

$$\begin{aligned}
&\mathrm{Tr}(X)^2 + \mathrm{Tr}(Y)^2 + \mathrm{Tr}(Z)^2 + \mathrm{Tr}(XY)^2 + \mathrm{Tr}(XZ)^2 + \mathrm{Tr}(YZ)^2 \\
&+ \mathrm{Tr}(XYZ)^2 + \mathrm{Tr}(XY)\mathrm{Tr}(XZ)\mathrm{Tr}(YZ) \\
&- \mathrm{Tr}(X)\mathrm{Tr}(Y)\mathrm{Tr}(XY) - \mathrm{Tr}(X)\mathrm{Tr}(Z)\mathrm{Tr}(XZ) - \mathrm{Tr}(Y)\mathrm{Tr}(Z)\mathrm{Tr}(YZ) \qquad (4.19) \\
&+ \mathrm{Tr}(X)\mathrm{Tr}(Y)\mathrm{Tr}(Z)\mathrm{Tr}(XYZ) - \mathrm{Tr}(X)\mathrm{Tr}(YZ)\mathrm{Tr}(XYZ) \\
&- \mathrm{Tr}(Y)\mathrm{Tr}(XZ)\mathrm{Tr}(XYZ) - \mathrm{Tr}(Z)\mathrm{Tr}(XY)\mathrm{Tr}(XYZ) = 4
\end{aligned}$$

By hypothesis, faithful representations of A and B in $PSL_2(\mathbb{C})$ exist. Moreover they are parametrised by fixing suitable values for $\mathrm{Tr}(X)$, $\mathrm{Tr}(Z)$, $\mathrm{Tr}(XZ)$, $\mathrm{Tr}(Y)$, $\mathrm{Tr}(XYZ)$ and $\mathrm{Tr}(XYZY^{-1})$. By the repeated application of trace relation

$$\mathrm{Tr}(MN) = \mathrm{Tr}(M)\mathrm{Tr}(N) - \mathrm{Tr}(MN^{-1}) \qquad (4.20)$$

for arbitrary matrices M and N can write:

$$\mathrm{Tr}(XYZY^{-1}) = \mathrm{Tr}(Y)\mathrm{Tr}(XYZ) - \mathrm{Tr}(XY)\mathrm{Tr}(YZ) + \mathrm{Tr}(X)\mathrm{Tr}(Z) - \mathrm{Tr}(XZ). \qquad (4.21)$$

Hence if we fix suitable values for $\mathrm{Tr}(X)$, $\mathrm{Tr}(Y)$, $\mathrm{Tr}(Z)$, $\mathrm{Tr}(XZ)$ and $\mathrm{Tr}(XYZ)$, we have two free variables $\alpha := \mathrm{Tr}(XY)$ and $\beta := \mathrm{Tr}(YZ)$ which are required to satisfy the quadratic equation which fixes the value of

$$\mathrm{Tr}(XYZY^{-1}) = \mathrm{Tr}(Y)\mathrm{Tr}(XYZ) - \alpha\beta + \mathrm{Tr}(X)\mathrm{Tr}(Z) - \mathrm{Tr}(XZ). \qquad (4.22)$$

Combining this with Equation (4.19), and fixing $\mathrm{Tr}(X)$, $\mathrm{Tr}(Z)$, $\mathrm{Tr}(XZ)$, $\mathrm{Tr}(Y)$, $\mathrm{Tr}(XYZ)$, we have a pair of quadratic equations in α, β of the form

$$\alpha\beta = c_1 \tag{4.23}$$

and

$$\alpha^2 + \beta^2 + c_2\alpha\beta + c_3\alpha + c_4\beta = c_5 \tag{4.24}$$

for suitable constants $c_1, \ldots, c_5$. It is routine to check that any such pair of equations can be solved in $\mathbb{C}$. Any solution gives a representation $\langle a, b, c, d\rangle \longrightarrow SL_2(\mathbb{C})$ that induces the given faithful representations of A and B in $PSL_2(\mathbb{C})$ up to conjugacy, mapping the word *abcd* to the identity element. This completes the proof. $\square$

Chapter 5

One-relator product of three non-trivial finite cyclic groups

5.1 Preamble

We prove a very important result about one relator free product of cyclic groups. Let
$$G = \frac{(G_a * G_b * G_c)}{N(w)}$$
where $G_a = \langle a \mid a^p \rangle$, $G_b = \langle b \mid b^q \rangle$, and $G_c = \langle c \mid c^r \rangle$. We show that each of G_a, G_b and G_c embeds in G in the case where w has non-zero exponent sum in each of the generators (modulo their respective orders). In [58], it was shown that such a group in non-trivial. Indeed the proof there shows that each of G_a, G_a and G_a embeds in G in the case where p, q and r are distinct primes. We will adapt the methods in [58] to show that a stronger conclusion holds for arbitrary non-zero triple p, q, r. More precise we show that in a one-relator product of three non-trivial cyclic groups, at least one of the factors is a Freiheitssatz factor. Before that we present some preliminary results most of which are based from [58].

5.2 S^1-equivariant homotopy and degree of maps

Definition 5.2.1. A *homotopy* between two continuous maps $f_1, f_2 : X \longrightarrow Y$ is a continuous map $H : X \times [0,1] \longrightarrow Y$ such that, if $x \in X$ then $H(x,0) = f_1(x)$ and $H(x,1) = f_2(x)$. If such H exist, we say that f_1 and f_2 are *homotopic.*

Definition 5.2.2. Let X and Y be G-spaces for some group G. Then a continuous map $f : X \longrightarrow Y$ is said to be *G-equivariant* if $f(gx) = gf(x)$ for each $x \in X$ and $g \in G$.

Definition 5.2.3. A *G-equivariant homotopy* between two G-equivariant maps

$f_1, f_2 : X \longrightarrow Y$ is a G-equivariant map $H : X \times [0,1] \longrightarrow Y$, where G acts on $X \times [0,1]$ by $g(x,t) = (gx,t)$, so that $H(x,0) = f_1(x)$ and $H(x,1) = f_2(x)$. If such H exist, we say that f_1 and f_2 are *G-equivariantly homotopic.*

Definition 5.2.4. A G-equivariant map $f_1 : X \longrightarrow Y$ is a *G-equivariant homotopy equivalence* if there is a G-equivariant map $f_2 : Y \longrightarrow X$ so that f_1f_2 and f_2f_1 are G-equivariant homotopic to the respective identity maps. In which case we say that X and Y are *G-equivariant homotopy equivalent.*

In what follows, G will be S^1 and the G-spaces will be S^2 and S^3. Later on we shall make precise what the S^1-action is. Recall that for some real number r and integer n, an n-sphere of radius r is the set

$$S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = r^2\}. \tag{5.1}$$

We shall be mostly interested in the unit n-sphere i.e $r = 1$. Identify S^3 with the group of unit quaternions

$$S^3 = \{x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k} \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}, \tag{5.2}$$

S^2 with the subset

$$S^2 = \{x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k} \mid x_2^2 + x_3^2 + x_4^2 = 1\}, \tag{5.3}$$

and S^1 with the subgroup

$$S^1 = \{x_1 + x_2\boldsymbol{i} \mid x_1^2 + x_2 = 1\}. \tag{5.4}$$

We shall use S^n and $SU(n)$ interchangeably. For example, by the trace of

$$L = x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k} \in S^3, \tag{5.5}$$

we mean the trace of the corresponding matrix in $SU(2)$. That is the trace of

$$L = \begin{pmatrix} x_1 + x_2\boldsymbol{i} & x_3 + x_4\boldsymbol{i} \\ -x_3 + x_4\boldsymbol{i} & x_1 - x_2\boldsymbol{i} \end{pmatrix}. \tag{5.6}$$

Definition 5.2.5. The degree of a continuous map $f : S^n \longrightarrow S^n$ is $\alpha \in \mathbb{Z}$ if the induced map $f_* : H_n(S^n) \longrightarrow H_n(S^n)$ is a multiplication by α, where $H_n(.)$ stands for the n-th homology group of S^n.

Lemma 5.2.6 ([58] Lemma 2.1)**.** *Let X be a simply-connected space equipped with an S^1-action, and let $f, g : S^2 \longrightarrow X$ be S^1-equivariant maps. Then f is equivariantly* *homotopic to g if and only if there are paths in the fixed subspace joining $f(+\boldsymbol{i})$ to $g(+\boldsymbol{i})$, and $f(-\boldsymbol{i})$ to $g(-\boldsymbol{i})$.*

Corollary 5.2.7 below gives us a way of computing the degree of S^1-equivariant maps from S^2 to itself.

Corollary 5.2.7 ([58] Corollary 2.2)**.** *The degree of any S^1-equivariant map $f : S^2 \longrightarrow S^2$ is one of the following:*

1. 0 *if* $f(\boldsymbol{i}) = f(-\boldsymbol{i}) \in \{\pm\boldsymbol{i}\}$*;*
2. $+1$ *if* $f(\boldsymbol{i}) = \boldsymbol{i}$ *and* $f(-\boldsymbol{i}) = -\boldsymbol{i}$*; or*
3. -1 *if* $f(\boldsymbol{i}) = -\boldsymbol{i}$ *and* $f(-\boldsymbol{i}) = \boldsymbol{i}$*.*

Lemma 5.2.8. *Two elements in S^3 are conjugates if and only if they have same eigenvalues.*

Proof. The forward direction is clear. For the reverse, let $T \in S^3$. It follows from the spectral theorem that $STS^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$ for some $S \in U(2)$ and λ an eigenvalue of T. The final step is obtained by replacing S with $\bar{s}S$ where $s = \sqrt{\det(S)}$. Since $\bar{s}S \in S^3$ and eigenvalues are completely determined by the trace, the result follows. $\square$

Lemma 5.2.9. *The conjugacy classes of S^3 are the sets $C(\theta) = \{\cos\theta + \sin\theta\boldsymbol{v} \mid \boldsymbol{v} \in S^2\}$ where $0 \leq \theta \leq \pi$.*

Proof. We begin by showing that any element $L = x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}$ in S^3 can be expressed in the form $\cos\theta + \sin\theta\boldsymbol{v}$ for some $0 \leq \theta \leq \pi$ and $\boldsymbol{v} \in S^2$. Since $-1 \leq x_1 \leq 1$, take $\theta = \cos^{-1}(x_1)$ and $\boldsymbol{v} = (x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k})/\sin\theta$. Note that L has trace $2x_1$.

The eigenvalues of L are $x_1 \pm \sqrt{x_1^2 - 1}$, so depends just on x_1 as does θ. The result follows from Lemma 5.2.8. $\square$

Now we are in a position to describe S^1 action on S^3 and S^2. S^3 is a group and so acts on itself by conjugation. The fixed set of the S^3-action consists of precisely $C(\theta)$ for $0 \leq \theta \leq \pi$, each of which is a topological 2-sphere with the exception of $C(0) = \{1\}$ and $C(\pi) = \{-1\}$. By restricting to the subgroup S^1, we get an action of S^1 on S^3 and on S^2. The fixed set of S^1-action on S^3 is itself, the fixed set of S^1 action on S^2 is $S^1 \cap S^2 = \{\pm\boldsymbol{i}\}$.

Lemma 5.2.10. *The spaces S^2 and $S^3 - \{\pm 1\}$ are S^3-equivariant homotopy equivalent.*

Proof. The inclusion map $\iota : S^2 \to S^3 - \{\pm 1\}$ is an S^3-equivariant map. It has an inverse $\psi : S^3 - \{\pm 1\} \to S^2$, sending $\cos\theta + \sin\theta \boldsymbol{v}$ to $\boldsymbol{v}$, which is also S^3-equivariant map. □

What we really need are the maps ι and ψ.

5.3 Main result

In this section, we prove the main result as a corollary to the following Lemma.

Lemma 5.3.1. *Let $G_a = \langle a \mid a^p \rangle$, $G_b = \langle b \mid b^q \rangle$ and $G_c = \langle c \mid c^r \rangle$ be cyclic groups, with p, q, r are prime powers.. Let $w \in G_a * G_b * G_c$ be an element such that the exponent sums of a, b, c in w are not divisible by p, q, r respectively (equivalently, w is not contained in the normal closure of any two of G_a, G_b, G_c). Then each of G_a, G_b, G_c embeds via the natural map into*

$$G = \frac{(G_a * G_b * G_c)}{N(w)}.$$

Proof. We know the result holds when p, q, r are primes by [[58], Theorem 4.1]. We suppose that atleast on of p, q, r is a prime power but not prime. Let $F(a, b, c)$ denote the free group of rank 3 with generating set $\{a, b, c\}$. The aim is to produce a representation of $F(a, b, c)$ in $\mathbb{H}$ (where $\mathbb{H}$ denotes the quaternions) such that a, b, c have orders p, q, r respectively and $w \in F(a, b, c)$ is sent to the identity.

Suppose that n is the exponent sum of a in w. The assumption that n is not divisible by p implies that $n = tp + s$ with $0 < s < p$. By replacing w with $wa^{-tp} \in F(a, b, c)$ which leaves G unchanged, we can always assume that $n < p$. If m is co-prime to p then $a \mapsto a^m$ induces an automorphism of G_a. Thus, replacing a by a^m in w gives a new word $w' \in G_a * G_b * G_c$ such that the resulting group

$$G' = \frac{(G_a * G_b * G_c)}{N(w')}$$

is isomorphic to G (and such that G_a embeds in G' if and only if it embeds in G). Moreover, the exponent sum of a in w' is mn. By Bezout's Lemma we may choose m such that $mn \equiv \gcd(n, p) \bmod p$. Thus without loss of generality we may assume that n divides p (and similarly the exponent sums of b, c in w divide q, r respectively).

Now suppose that p and n (the exponent sum of a in w) are powers of a prime τ – say $p = \tau^t$ and $n = \tau^s$ where $0 \leq s < t$. If τ is an odd prime, define

$$\theta_p = \frac{(\tau^{t-s} - 1)\pi}{2\tau^t}. \tag{5.7}$$

If $\tau = 2$, define

$$\theta_p = \frac{(2^{t-s-1} - 1)\pi}{2^t} \tag{5.8}$$

unless $s = t - 1$, in which case define $\theta_p = \frac{\pi}{2}$. Recall that an element $\cos(\theta_p) + \boldsymbol{v}\sin(\theta_p) \in SO(3)$ has order p if and only if θ_p is a multiple of π/p but not π/τ^{t-1}, for any vector $\boldsymbol{v} \in S^2$. Hence for any vector $\boldsymbol{v} \in S^2$, the map

$$\alpha_{\boldsymbol{v}} : G_a \to \mathbb{H}, \ a \mapsto \cos(\theta_p) + \boldsymbol{v}\sin(\theta_p),$$

induces a faithful representation $G_a \to S^3/\{\pm 1\} \cong SO(3)$. Moreover, the real part of $\alpha_v(a^n)$ is $\cos(\psi_p)$ where $\psi_p := n\theta_p$ and $\frac{\pi}{4} \leq \psi_p \leq \frac{\pi}{2}$.

Similarly, we can define maps $\beta_{\boldsymbol{v}} : G_b \to \mathbb{H}$ and $\gamma_{\boldsymbol{v}} : G_c \to \mathbb{H}$ that induce faithful representations $G_b, G_c \to SO(3)$, and such that, if e, f denote the exponent-sums of b, c in w, then the real part of $\beta_{\boldsymbol{v}}(b^e)$ is $\cos(\psi_q)$ and the real part of $\gamma_{\boldsymbol{v}}(c^f)$ is $\cos(\psi_r)$ where $\psi_q, \psi_r \in [\frac{\pi}{4}, \frac{\pi}{2}]$.

The numbers ψ_p, ψ_q, ψ_r satisfy a triangle inequality: none is greater than the sum of the other two. Hence, for example, the triple $(\alpha_{\boldsymbol{i}}, \beta_{\boldsymbol{i}}, \gamma_{-\boldsymbol{i}})$ induces a homomorphism

$$\delta : G_a * G_b * G_c \to S^3$$

that sends w to $\cos(\theta) + \boldsymbol{i}\sin(\theta)$ with $0 \leq \theta \leq \frac{3\pi}{4}$. If $\theta = 0$ then δ induces a representation $G \to SO(3)$ that is faithful on each of G_a, G_b, G_c and we are done. So assume that $\theta > 0$. In other words the imaginary part of $\delta(w)$ is greater than 0. Similar remarks apply to the triples $(\alpha_{\boldsymbol{i}}, \beta_{-\boldsymbol{i}}, \gamma_{\boldsymbol{i}})$ and $(\alpha_{-\boldsymbol{i}}, \beta_{\boldsymbol{i}}, \gamma_{\boldsymbol{i}})$. Hence also the triple $(\alpha_{\boldsymbol{i}}, \beta_{-\boldsymbol{i}}, \gamma_{-\boldsymbol{i}})$ induces a representation δ with the imaginary part of $\delta(w)$ less than 0. The map $S^2 \to S^3$, $\boldsymbol{v} \mapsto (\alpha_{\boldsymbol{i}}, \beta_{-\boldsymbol{i}}, \gamma_{\boldsymbol{v}})(w)$ is an S^1-equivariant map under the conjugation action. It follows that it either sends some $\boldsymbol{v}$ to $\pm 1 \in S^3$ (in which case $(\alpha_{\boldsymbol{i}}, \beta_{-\boldsymbol{i}}, \gamma_{\boldsymbol{v}})$ gives a representation $G \to SO(3)$ that is faithful on each of G_a, G_b, G_c), or represents $+1$ in $H_2(S^3 \setminus \{\pm 1\})$ (isomorphic to $\mathbb{Z}$), the second homology group of $S^3 \setminus \{\pm 1\}$ by Lemma 5.2.6.

Similarly, the map $\boldsymbol{v} \mapsto (\alpha_{\boldsymbol{i}}, \beta_{\boldsymbol{i}}, \gamma_{\boldsymbol{v}})(w)$ either maps some $\boldsymbol{v} \in S^2$ to $\pm 1 \in S^3$ and so gives a representation $G \to SO(3)$ that is faithful on each of G_a, G_b, G_c, or represents one of $0, -1 \in H_2(S^3 \setminus \{\pm 1\}) \cong \mathbb{Z}$. Now any path $P : [0,1] \to S^2$ from $-\boldsymbol{i}$ to $\boldsymbol{i}$ gives rise to a homotopy

$$t \mapsto \big(\boldsymbol{v} \mapsto (\alpha_i, \beta_{P(t)}, \gamma_v)(w)\big)$$

between the above two maps. If $(\alpha_{\boldsymbol{i}}, \beta_{P(t)}, \gamma_{\boldsymbol{v}})(w) \neq \pm 1$ for all t and for all $\boldsymbol{v}$, then we can regard this as a homotopy of maps $S^2 \to S^3 \setminus \{\pm 1\}$. This is a contradiction since

the two maps belong to different homology classes in $H_2(S^3 \setminus \{\pm 1\})$. Hence for some t and some $\boldsymbol{v}$, the map $(\alpha_{\boldsymbol{i}}, \beta_{P(t)}, \gamma_{\boldsymbol{v}})$ sends w to ± 1 and so induces a representation $G \to SO(3)$ that is faithful on each of G_a, G_b and G_c.
It follows that each of the natural maps from G_a, G_b and G_c to G is injective, as required. □

The general case of the Freiheitssatz for G follows from the special case of prime powers together with the Chinese Remainder Theorem by an easy induction.

Theorem 5.3.2. *Let $G_a = \langle a \mid a^p \rangle$, $G_b = \langle b \mid b^q \rangle$ and $G_c = \langle c \mid c^r \rangle$ be cyclic groups. Let $w \in G_a * G_b * G_c$ be an element such that the exponent sums of a, b, c in w are not divisible by p, q, r respectively. Then each of G_a, G_b and G_c embeds via the natural map into*

$$G = \frac{(G_a * G_b * G_c)}{N(w)}.$$

Proof. For the inductive step, suppose that $p = mn$ with $\gcd(m, n) = 1$. Since the exponent sum of the generator a in w is non-zero modulo p, we can assume that it is non-zero modulo m. Now factor out a^m and apply the inductive hypothesis. This shows that the maps $G_b \to \mathbb{H}$ and $G_c \to \mathbb{H}$ are injective. It also shows that the kernel K of $G_a \to \mathbb{H}$ is contained in the subgroup $\langle m \rangle$.

If the exponent sum of a in w is also non-zero modulo n, then by interchanging the roles of m and n in the above we see that K is contained in $\langle n \rangle$. However, if the exponent-sum of a in w is divisible by n, then the same is automatically true: K is contained in $\langle n \rangle$. Finally, we know that K is contained in the intersection of $\langle m \rangle$ and $\langle n \rangle$. But this intersection is trivial by the Chinese Remainder Theorem, so we deduce that $G_a \to \mathbb{H}$ is injective. □

Remark 5.3.3. It is clears that if a occurs in w with zero exponent sum modulo p, then A is a Freiheitssatz factor of G.

Combining this Remark 5.3.3 with Theorem 5.3.2 we obtain:

Corollary 5.3.4. *In a one-relator product of three non-trivial cyclic groups, at least one of the factors is a Freiheitssatz factor.*

Definition 5.3.5. A group H is said to be *finitely annihiliated* if for every non-trivial element $h \in H$, there exist a finite index normal subgroup N of H such that h is trivial in H/N.

In [13], Chiodo used the result of Howie [58] to show that the free product G of three cyclic groups of distinct prime orders is finitely annihiliated. The proof uses nothing more than the fact that finitely generated subgroups of $SO(3)$ are residually finite. Hence our result extends this to the case where the cyclic groups are arbitrary.

Corollary 5.3.6. *Free product of three cyclic groups is finitely annihiliated.*

Chapter 6

One-relator product of three non-trivial groups with short relator

6.1 Preamble

In Chapter 5 we considered a one-relator product of three finite cyclic groups. Here we do the same only that this time we allow the factor groups to be arbitrary. It is conjectured in [58] that such a group is non-trivial. We prove this conjecture in a special case. In particular we prove the following theorem.

Theorem 6.1.1. *Suppose A, B and C are any three non-trivial groups. Then*

$$G = \frac{(A * B * C)}{N(w)}$$

is non-trivial, for any word $w \in A * B * C$ with $\ell(w) < 9$.

Theorem 6.1.1 holds trivially if w is in the normal closure of any of the factors. Hence we can assume by results in Chapter 4 that w contains at least two letters in each of the three factors. To see why this is true, suppose by way of contradiction that w contains one letter from A, say α. Then a cyclic conjugate of w has the form $w = \alpha W$, where $W \in B * C$ and $\ell(W) \leq 7$ (and so some cyclic conjugate of W has length at most 6). If α has order n in A, then we define the group

$$H = \frac{(B * C)}{N(W^n)}.$$

From Corollary 4.1.2 we know that W has order n in H. Hence

$$G = A_{\langle \alpha \rangle} *_{\langle W \rangle} H.$$

So G is non-trivial. In particular it is enough to consider the cases where the length of the relator is at least six and at most eight, as a cyclically reduced word in the free product $A * B * C$.

We now prove some technical results from which Theorem 6.1.1 will follow as a corollary.

6.2 Technical Lemmas

We can assume that up to cyclic permutation w has the form $w = c_1 U c_2 V$ where $c_1, c_2 \in C$ and $U, V \in A * B$ with $\ell(U) + \ell(V) \leq 6$. The group G is non-trivial if $c_1 c_2 = 1$ or $UV \in N(A) \cup N(B)$. Hence we assume that neither of the two conditions holds. If without loss of generality we assume $\ell(U) \leq \ell(V)$, then the possibilities for U and V as words in the free product $A * B$ are as follows:

1. U or V is in the normal closure of A or B;
2. $U = (\alpha\beta)^{\pm 1}$ and $V \in \{\alpha_1\beta_1, \alpha_1\beta_1\alpha_2, \alpha_1\beta_1\alpha_2\beta_2\}$;
3. $U = \alpha\beta\alpha_1$ and $V \in \{\beta_1\alpha_2\beta_2, \alpha_2\beta_1\alpha_3\}$ (by symmetry),

where α (with or without subscript) is an element of A and β (with or without subscript) is an element of B. We show that in each of the possibilities listed above G is non-trivial.

Lemma 6.2.1. *If U or V is in the normal closure of A or B, then G is non-trivial.*

Proof. Without loss of generality, we can assume that U is in the normal closure of A. In other words, there exist a $\gamma \in A * B$ such that $U = \gamma^{-1}\alpha\gamma$. Hence

$$w = c_1 \gamma^{-1} \alpha \gamma c_2 V.$$

Let $\tilde{C} = \gamma C \gamma^{-1}$ and $W = \tilde{c_1} \alpha \tilde{c_2} \tilde{V} = \gamma c_1 \gamma^{-1} \alpha \gamma c_2 \gamma^{-1} \gamma V \gamma^{-1}$. So G is trivial if and only if

$$G' = \frac{(A * B * \tilde{C})}{N(W)} \tag{6.1}$$

is trivial. Hence it is enough to consider the case where $U = \alpha$ (i.e $\gamma = 1$).

By assumption $c_1 \neq c_2^{-1}$. Hence $c_1 \alpha c_2$ has infinite order in $A * C$. It follows that the subgroup of $A * C$ generated by A and $c_1 \alpha c_2$ is isomorphic to the free product $A * \mathbb{Z}$. If A and V generates a subgroup of $A * B$ which is also isomorphic to $A * \mathbb{Z}$, then

$$G = (A * B)_{\langle A, V \rangle} *_{\langle A, c_1 \alpha c_2 \rangle} (A * C). \tag{6.2}$$

So in this case G is non-trivial.

Suppose then that $\langle A, V\rangle$ is not isomorphic to $A*\mathbb{Z}$. By the comments immediately after the statement of Theorem 6.1.1 we can assume that V at least one letter from A and at least two letters from B as a reduced word in $A*B$. If V contains exactly two letters from B, then from the assumption that $\langle A, V\rangle$ is not isomorphic to $A*\mathbb{Z}$ it follows that the two letters must be inverses of each other. In such case w belongs to the normal closure of $A*C$, so G is non-trivial. Hence V contains exactly three B letters. Hence V contains exactly two letters from A and exactly three letters from B since $\ell(V) \leq 5$. Again it follows from the assumption that $\langle A, V\rangle$ is not isomorphic to $A*\mathbb{Z}$ that V is conjugate in $A*B$ to a letter $\beta \in B$ with order $r < \infty$. Define H to be the group

$$H := \frac{A*C}{N((c_1\alpha c_2)^r)} \tag{6.3}$$

$$= A \underset{\langle\alpha\rangle}{*} T \underset{\langle\gamma\rangle=\langle c_2c_1\rangle}{*} C, \tag{6.4}$$

where $T = \langle \alpha, \gamma \mid \alpha^p, \gamma^q, (\alpha\gamma)^r\rangle$ and p, q are the orders of α and c_2c_1 respectively. Then $G = (A*B)_{\langle A,V\rangle} *_{\langle A, c_1\alpha c_2\rangle} H$, provided of course that $\langle A, c_1\alpha c_2\rangle$ embeds in H. We show below that this is in fact the case.

By Bass-Serre theory, H acts on a tree Γ. Let the vertex set of Γ be X. There exists an edge e divides Γ (see Figure 6.1) into two components Γ_1 and Γ_2 with vertex sets X_1 (containing vertices u_A, u_T) and and X_2 (containing vertices $u_C, u_{c_1(u_T)}$) respectively such that $X = X_1 \cup X_2$. The vertices u_A, u_T and u_C have stabilizers A, T and C respectively. Similarly, $\langle\alpha\rangle$ and $\langle\gamma\rangle$ are the stabilizers of e_1 and e respectively. Let $e_2 = c_1(e)$, then the stabilizer of e_2 is $c_1\langle\gamma\rangle c_1^{-1}$.

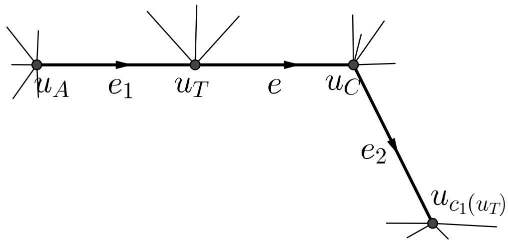

Figure 6.1: Diagram showing a section of the tree Γ on which H acts.

We aim to use the Ping-Pong Lemma to show that the subgroup of H generated by A and the element $c_1\alpha c_2$ is the free product $A * \langle c_1\alpha c_2\rangle$. Since

$$A \cap \langle\gamma\rangle = \langle\alpha\rangle \cap \langle\gamma\rangle, \tag{6.5}$$

it follows that for any non-trivial element $a \in A$, $a(X_2) \subset X_1$. Similarly since $\alpha\gamma$

stabilizes an edge e_3 incident to u_T other than e, $c_1\alpha c_2 = c_1\alpha\gamma c_1^{-1}$ stabilizes an edge $e_4 = c_1(e_3)$ incident to $u_{c_1(u_T)}$ other than $c_1(e) = e_2$. If we can show that $\langle\gamma\rangle$ and $\langle c_1\alpha c_2\rangle$ intersect trivially, then it will follow that $b(X_1) \subset X_2$ for every non-trivial element $b \in \langle c_1\alpha c_2\rangle$.

But $\langle\gamma\rangle \cap \langle c_1\alpha c_2\rangle$ stabilizes both e and e_4, and hence also e_2. So $c_1\langle\gamma\rangle c_1^{-1}$ is contains $\langle\gamma\rangle \cap \langle c_1\alpha c_2\rangle$. Hence $\langle\gamma\rangle \cap \langle c_1\alpha c_2\rangle = 1$ in T implies that in $c_1Tc_1^{-1}$,

$$c_1\langle\gamma\rangle c_1^{-1} \cap c_1\langle\alpha\gamma\rangle c_1^{-1} = 1 \tag{6.6}$$

Thus we have that $\langle\gamma\rangle \cap \langle c_1\alpha c_2\rangle = 1$ as required. Hence the Ping-Pong Lemma yields the result. $\square$

It follows in particular from Lemma 6.2.1 that $\ell(U) \geq 2$. The rest of the arguments we present rely heavily on Nielsen transformations. We transform $\{U, V\}$ into a more suitable Nielsen equivalent set depending on the subgroup they generate.

Lemma 6.2.2. *Suppose $\langle U, V\rangle$ is free, then G is non-trivial.*

Proof. First we suppose $\langle U, V\rangle$ is free of rank 1 say with generator t. Then w can be expressed in the form $w = c_1t^rc_2t^s$, where $V = t^s$ and $U = t^r$ for integers s, r. If $s + r = 0$, then $G \neq 1$ since $w \in N(C)$. Otherwise $w = 1$ is a non-singular equation over C. We assume that $t \notin N(A) \cup N(B)$ for otherwise $G \neq 1$ by Lemma 6.2.1. It follows that any cyclically reduced conjugate of t has length at least 2. So since $s, r \neq 0$ and $\ell(t^n) \geq 2n$ for any n,

$$|s| + |r| \leq 3. \tag{6.7}$$

Consider the group

$$H := \frac{(C * \langle t\rangle)}{N(w)}.$$

If $s, r \geq 1$, then C embeds in H by [67]. Otherwise without loss of generality $s = -1$ and $r = 2$. Again C embeds in H by [52]. If t is trivial in H, the $H = C$ and so $w \in N(A * B)$; again $G \neq 1$. If t has finite order $m > 1$ in H, then

$$G = \frac{(A * B)}{N(t^m)} *_{\langle t\rangle} H.$$

By the previous comment it follows that $\ell(t) = 2$–say $t = \alpha\beta$, so $(A * B)/N(t^m)$ is a triangle group and t has order m as required.

Finally if t has infinite order in H, then

$$G = (A * B) *_{\langle t\rangle} H.$$

Hence G is non-trivial.

Now suppose $\langle U, V\rangle$ is free of rank 2. Let $H := (C * \langle U, V\rangle)/N(w) = C * \langle U\rangle$. Note that the subgroup of H generated by $\{U, c_1Uc_2\}$ is a free group of rank 2. It follows that G is the free product of H and $A * B$ amalgamated over the subgroups $\langle U, c_1Uc_2\rangle$ and $\langle U, V\rangle$. It follows that G is non-trivial. $\square$

Lemma 6.2.3. *Suppose $\langle U, V\rangle$ is isomorphic to $C_p * C_q$ or $C_p * \mathbb{Z}$, where C_p and C_q are finite cyclic groups. Suppose further that $\ell(V) < 4$. Then G is non-trivial.*

By assumption $\langle U, V\rangle$ has an element of finite order. So Nielsen transformations can be applied to $\{U, V\}$ to get a new set $\{u, v\}$ with u or v having finite order. Note also that $V \neq \alpha_1\beta_1\alpha_2\beta_2$. Lemma 6.2.3 is a corollary to Propositions 6.2.4–6.2.7 below.

Proposition 6.2.4. *If $U = (\alpha\beta)^{\pm 1}$ and $V = \alpha_1\beta_1$, then G is non-trivial.*

Proof. First suppose $U = (\alpha\beta)^{-1}$. Since UV is not in the normal closure of A or B, neither $\alpha = \alpha_1$ nor $\beta = \beta_1$ holds. Hence $\langle U, V\rangle$ is free of rank 2. The result follows from Lemma 6.2.2.

Suppose then that $U = \alpha\beta$. If $\alpha \neq \alpha_1$ and $\beta \neq \beta_1$, then $\{U, V\}$ is Nielsen reduced, so $\langle U, V\rangle$ is free of rank 2 and the result follows from Lemma 6.2.2. Hence we may assume without loss of generality that $\alpha = \alpha_1$. If $\beta = \beta_1$, then $\langle U, V\rangle$ is isomorphic to $\mathbb{Z}$. In this case the result follows from Lemma 6.2.2.

If $c_1 = c_2 = c$ then we can replace w with $\hat{U}\beta\hat{U}\beta_1$, where $\hat{U} = c\alpha$. In this case $\langle\hat{U}\rangle$ is free of rank 1 and the result follows from Lemma 6.2.2.
Hence we may assume that $\beta \neq \beta_1$ and $c_1 \neq c_2$, so $\langle \beta c_2, \beta_1 c_1\rangle$ is free of rank 2. Again we apply Lemma 6.2.2 to show that G is non-trivial. $\square$

Proposition 6.2.5. *If $U = (\alpha\beta)^{\pm 1}$ and $V = \alpha_1\beta_1\alpha_2$, then G is non-trivial.*

Proof. Suppose $U = \alpha\beta$. We can assume that $\beta \neq \beta_1^{-1}$ as otherwise G maps onto B, hence non-trivial. Also $\alpha_1 \neq \alpha_2^{-1}$ by Lemma 6.2.1. Since $\alpha_1\alpha_2 \neq 1$, either $\alpha = \alpha_1$ and $\beta = \beta_1$ or $\alpha = \alpha_2^{-1}$ and $\beta = \beta_1^{-1}$, as otherwise $\langle U, V\rangle$ is free. Hence without loss of generality we assume $\beta = \beta_1$ and $\alpha = \alpha_1$. If $c_1 = c_2 = c$, then G surjects onto $(B * C)/N((\beta c)^2)$, so non-trivial. Otherwise take $U' = c_2\alpha$ and $V' = \alpha_2 c_1\alpha$, and so $\langle U', V'\rangle$ is free. Hence G is non-trivial by Lemma 6.2.2. The proof for the case where $U = (\alpha\beta)^{-1}$ is similar by symmetry. $\square$

Proposition 6.2.6. *If $U = \alpha\beta\alpha_1$ and $V = \alpha_2\beta_1\alpha_3$, then G is non-trivial.*

Proof. We may assume that $\beta \neq \beta_1$, as otherwise G surjects onto B. Without loss of generality, there are two possibilities to consider. Either $\alpha = \alpha_2$ and $\beta = \beta_1$ or

$\alpha = \alpha_2$ and $\alpha_1 = \alpha_3$. (Note that we can not have $\alpha = \alpha_3^{-1}$ and $\alpha_1 = \alpha_2^{-1}$, for otherwise w is contained in the normal closure of $B * C$).

In the first case, we take take $U' = \alpha_1 c_2 \alpha$ and $V' = \alpha_3 c_1 \alpha$. By Lemma 6.2.2, we can assume that $\langle U', V' \rangle$ is not free. Hence either $c_1 = c_2 = c$ or $\alpha_1 = \alpha_3$. In either case G maps onto

$$\frac{(B * C)}{N((c\beta)^2} \quad \text{or} \quad \frac{(A * B)}{N((\alpha\beta\alpha_1)^2)}$$

respectively. Hence G is non-trivial.

In the second case where $\alpha = \alpha_2$ and $\alpha_1 = \alpha_3$, we can replace B with its conjugate by α, and w by $W = c_1\beta\tilde{\alpha}c_2\beta_1\tilde{\alpha}$, where $\tilde{\alpha} = \alpha\alpha_1$. Since G is isomorphic to

$$G' = \frac{(A * \alpha^{-1}B\alpha * C)}{N(W)}, \tag{6.8}$$

the result follows from Lemma 6.2.4. □

Proposition 6.2.7. *If $U = \alpha\beta\alpha_1$ and $V = \beta_1\alpha_2\beta_2$, then G is non-trivial.*

Proof. In this case, the only possibility is $\alpha = \alpha_1^{-1}$ or $\beta_1 = \beta_2^{-1}$. Either case, the result follows from Lemma 6.2.1. □

Finally, we consider the case where $U = (\alpha\beta)^{\pm 1}$ and $V = \alpha_1\beta_1\alpha_2\beta_2$. Since $\langle U, V \rangle = C_p * K$, without loss of generality we have by Nielsen transformations that either

1. $U = \alpha\beta$ and $V \in \{\alpha\beta\alpha\beta_2, \alpha\beta\alpha_2\beta\}$ with $\beta \neq \beta_2$, or

2. $U = \beta^{-1}\alpha^{-1}$ and $V \in \{\alpha\beta\alpha\beta_2, \alpha\beta\alpha_2\beta\}$ with $\alpha \neq \alpha_2$.

Remark 6.2.8. In (1) and (2) above we gave two forms of V. If $V = \alpha\beta\alpha_2\beta$ we can replace U and V by $\beta U \beta^{-1}$ and $\beta V \beta^{-1}$ respectively (or equivalently replace C by $\beta^{-1}C\beta$) and interchange A and B to get the first $\alpha\beta\alpha\beta_2$.

In what follows we regard G as a one-relator product of $A*B$ and C. For convenience we let $U_1 = \alpha\beta$ and $U_2 = \beta^{-1}\alpha^{-1}$, so $U_2 = U_1^{-1}$. We use R to denote a relator in G which is a cyclically reduced word in $\{U, V\}$ and $\ell(R)$ denotes its length also as a word in $\{U, V\}$.

Definition 6.2.9. The *index* of R is the number of cyclic subwords of the form $(UU)^{\pm 1}$, $(VV)^{\pm 1}$, $(VU^{-1})^{\pm 1}$ or $(U^{-1}V)^{\pm 1}$.

Definition 6.2.9 generalizes the notion of sign-index. Recall that the sign-index of R is n (necessarily even) if a cyclic permutation of R has the form

$$W_1 W_2^{-1} W_3 \dots W_{n-1} W_n^{-1},$$

with each W_i a positive word in $\{U, V\}$. In particular the index of R is bounded below by its sign-index, and above by $\ell(R)$.

By Remark 6.2.8

$$\{U, V\} = \{(\alpha\beta)^{\pm 1}, \alpha\beta\alpha\beta_2\} \xrightarrow{\text{Nielsen transformation}} \{\alpha\beta, \beta^{-1}\beta_2\}.$$

There are two possibilities to consider. If $\{\alpha\beta, \beta^{-1}\beta_2\}$ is Nielsen reduced, then $\langle U, V\rangle$ is free (if $\beta^{-1}\beta_2$ has infinite order), or $\mathbb{Z} * \mathbb{Z}_m$ (if $\beta^{-1}\beta_2$ has order m). A second possibility is that $\{\alpha\beta, \beta^{-1}\beta_2\}$ is not Nielsen reduced. In which case β is a power of $\beta^{-1}\beta_2$, so B is cyclic (generated by $\beta^{-1}\beta_2$). In particular it follows that $\beta^{-1}\beta_2$ must have order at least 3 (since by assumption $\beta \neq \beta_2$).

Proposition 6.2.10. *Suppose that R is a cyclically reduced word in $\{U, V\}$ and that $(\beta^{-1}\beta_2)^2 \neq 1$. Then $\ell(R) \geq 2(6-k)$, where k is the index of R.*

Proof. It is easy to see that $(UV)^m \neq 1$. Hence since also $k \leq \ell(R)$, we can assume the $1 \leq k \leq 3$. In particular it is enough to consider the cases when R has sign-index 0 or 2.

First if $U = U_1$, then U and V generates a free sub-semigroup of $A * B$ of rank 2 (since $(\beta^{-1}\beta_2)^2 \neq 1$). In which case R is a positive word which is trivial in a free sub-semigroup or we get an equality between two positive words in a free sub-semigroup, depending on whether the sign-index is 0 or 2 respectively. None of these can happen.

Suppose then that $U = U_2$ and $\ell(R) < 2(6-k)$. Up to inversion and cyclic permutation the list of such R is contained in Table 6.1 below. It is easy to check that none of the words appearing in the table is trivial. □

Remark 6.2.11. We remark that if $(\beta^{-1}\beta_2)^2 = 1$, then B can not be cyclic since that will imply that $\beta = \beta_2$. In other words $\{U, \beta^{-1}\beta_2\}$ is Nielsen reduced.

Sign-index	Index	Length	List
0	1	< 10	U^2VUV $U^2(VU)^2V$ $U^2(VU)^3V$ V^2UVU $V^2(UV)^2U$ $V^2(UV)^3U$
0	2	< 8	U^3VUV U^2V^2 U^2V^2UV U^2VUV^2UV V^3UVU
0	3	< 6	U^3V^2 V^3U^2
2	2	< 8	$U^{-1}(VU)^2V$ $(VU)^{-1}(UV)^2$ $(UVU)^{-1}VUV$ $V^{-1}(UV)^2U$
2	3	< 6	$U^2(VUV)^{-1}$ $U^2V(VU)^{-1}$ $U^2VU^{-1}V$ $U^2V^{-1}UV$ $V^2(UVU)^{-1}$ $V^2U(UV)^{-1}$ V^2UVU^{-1} $V^2UV^{-1}U$ $V^2U^{-1}VU$ $V^2U^{-1}V^{-1}U$

Table 6.1: A table containing all the possible words R (up to cyclic permutation and inversion) in $\{U_2, V\}$ of index k, $1 \leq k \leq 3$ and $\ell(R) \geq 2(6-k)$.

Lemma 6.2.12. *If $U = (\alpha\beta)^{\pm 1}$, $V = \alpha_1\beta_1\alpha_2\beta_2$ and $\{U, \beta^{-1}\beta_2\}$ is not Nielsen reduced, then G is non-trivial.*

Proof. Suppose by contradiction that G is trivial, then we get a non-trivial $\mathcal{W}$-minimal spherical picture M over G with an exceptional region, where $\mathcal{W}$ is the set of non-trivial elements in $A * B \cup C$.

Note that both $A = \langle\alpha\rangle$ and $B = \langle\beta^{-1}\beta_2\rangle$ are cyclic. So by Theorem 4.6.1 we assume that C is neither cyclic nor dihedral. Hence the only relations between c_1, c_2 of length less than 4 are powers of c_1 or c_2. Without loss of generality, $c_1^2 = 1$. But then the only possible relator of length 4 involving c_2 is c_2^4, and so c_2 has order 3 or 4 in C.

By bridge-moves, we can assume that no two adjacent corners in a C-region are labelled by c_1 unless it is a 2-gon. Similarly any C-region has at most 2 or 3 consecutive c_2-corners depending on whether c_2 has order 3 or 4 respectively.

Every c_1-corner gets angle 0, every c_2-corner gets angle $\pi/3$, and every U- and V-corner gets angle $5\pi/6$. This ensures that vertices have curvature 0, and C-regions have non-positive curvature. However, $(A * B)$-regions can have positive curvature. We overcome this by redistributing any such positive curvature to neighbouring negatively curved C-regions, as follows.

Let R be the label of an interior region of Δ of M. If R has index n, we transfer $\pi/3$ of curvature across each of the edges of Δ joining a c_1-corner to c_2-corner of an adjacent C-region.

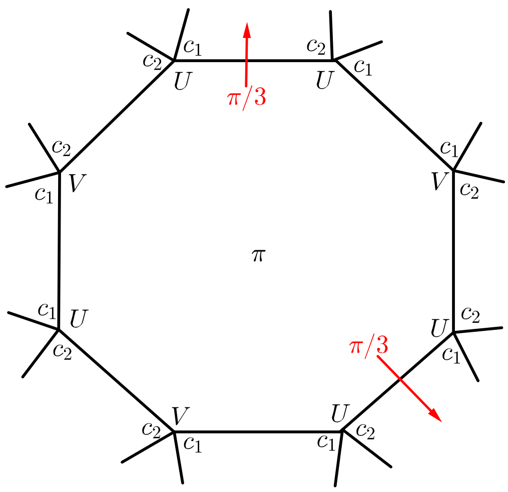

Figure 6.2: Diagram showing an $(A * B)$-region with label $(U^2V)^2UV$ of index 2. The head of the red arrows indicate the adjacent C-regions receiving $\pi/3$ curvature.

Now any interior $(A*B)$-region whose label is of the form $(U_2U_2V)^n$ (up to inversion) has index n and curvature at most $\pi/2$. However, $n > 2$ by Remark 6.2.11. Hence it has transferred at least π of curvature to neighbours, so becomes negatively curved. Similarly, it follows from Proposition 6.2.10 that any interior $(A * B)$-region whose label is of the form $(U_2 U_2 V)^n$ (up to inversion) has curvature at most π, and has become non-positively curved after transfer as well. It follows that interior $(A * B)$-regions non-positively curved after transfer.

A C-region Δ receives $\pi/3$ of positive curvature across each edge separating a c_1-corner from a c_2-corner. Suppose that in Δ there are p c_1-corners, q c_2-corners, and r edges separating a c_1-corner from a c_2-corner. The curvature of Δ after transfer is at most

$$2\pi - (p+q)\pi + \frac{q\pi}{3} + \frac{r\pi}{3} \leq \frac{(6-2p-q)\pi}{3}. \tag{6.9}$$

If $2p+q \geq 6$, then Δ still has non-positive curvature after transfer. Suppose Δ is an interior C-region and $2p+q \leq 5$. Then either $p=0$ or $q=0$ (since $p, q \neq 1$). Hence curvature after transfer is $(2-p)\pi$ or $(6-2q)\pi/3$, both of which are at most 0 since $p \geq 2$ and $q \geq 3$.

Since the curvature of the exceptional region is less than 4π, we get a contradiction that curvature of M is 4π. Hence G non-trivial. $\square$

Lemma 6.2.13. *If $U = (\alpha\beta)^{\pm 1}$, $V = \alpha_1\beta_1\alpha_2\beta_2$ and $\{U, \beta^{-1}\beta_2\}$ is Nielsen reduced, then G is non-trivial.*

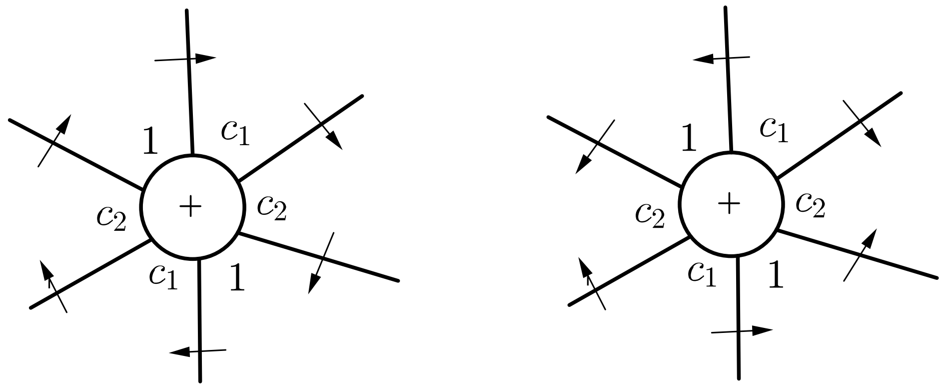

Figure 6.3: Positively oriented vertices of Γ when $n=2$. The figure on the left corresponds to a vertex of Γ when $r = (t^2c_1tc_2)^n$ and the other is when $r = (t^{-2}c_1tc_2)^n$.

Proof. By assumption $\langle U, V\rangle = \langle U\rangle * \langle \beta^{-1}\beta_2\rangle$, and is isomorphic to $\mathbb{Z} * \mathbb{Z}_n$, where $n > 1$ is the order of $\beta^{-1}\beta_2$. We can assume by Theorem 4.6.1 that $\langle c_1, c_2\rangle$ is not dihedral. Consider the relative presentation

$$H = \langle C, t \mid r = (t^{\pm 2}c_1tc_2)^n\rangle.$$

The aim is to show that G is the free product of H and $A * B$ amalgamated over the subgroups $\langle U, V\rangle$ and $\langle t, c_1tc_2\rangle$. To this end we must show that the latter is also isomorphic to $\mathbb{Z} * \mathbb{Z}_n$. In other words, any relation in H which is a word in $\{t, c_1tc_2\}$ is a consequence of $(t^{\pm 2}c_1tc_2)^n$.

If this is not so, we obtain a $\mathcal{W}$-minimal non-trivial picture Γ over H on D^2, where $\mathcal{W}$ is the set of non-trivial elements in the free product $\langle t\rangle * \langle c_1tc_2\rangle$. Figure 6.3 shows typical vertices with positive orientation in the case of $n = 2$. Note that there are no 2-zones with corners labelled by $1^{\pm 1}$, for in such case, either the two vertices cancel, or we can combine the vertex with the boundary. In both cases we get a smaller picture, thereby contradicting minimality of Γ.

Make regions of Γ *flat* by assigning angle $(d(\Delta)-2)\pi/d(\Delta)$ to each corner of a region Δ of Γ of degree $d(\Delta)$. We claim that interior vertices Γ are non-positively curved. The proof is in two stages depending on whether $r = (t^2c_1tc_2)^n$ or $r = (t^{-2}c_1tc_2)^n$.

Suppose that $r = (t^2c_1tc_2)^n$. Since r is a positive word, Γ is bipartite. More precisely, only vertices of opposite orientations are adjacent in Γ. In particular this implies that regions have even degrees. Every interior vertex v bounds at least two regions with a corner labelled 1. By minimality of Γ, every such region has degree at least 4. Also every 2-zone gives the relation $c_1 = c_2$, and so each of the two regions on both sides of the 2-zone has a corner labelled 1, hence is at least a 4-gon. Hence v bounds at least four regions of degree at least 4 and so is non-positively curved.

The case of $(t^{-2}c_1tc_2)^n$ is slightly different as regions can be odd. Note that any corner is either a *source* (the two arrows point outwards), *sink* (the two arrows point inwards), or *saddle* (one arrow points inwards and the other points outwards) depending on whether it is a c_1-, c_2- or 1-corner (see Figure 6.3). So in particular any 2-zone gives the relation $c_1^2 = 1$ or $c_2^2 = 1$. If an interior vertex v does not bound a 2-zone, then v satisfies $C(3n)$. Suppose it does. As before any region adjacent to a 2-zone has a 1-corner. It follows that such a region has degree at least 4. There are at least two regions with 1-corner at v. If v bounds only one 2-zone, then it has degree at least 5 and bounds at least three regions of degree 4. Otherwise v bounds at least four 4-gons. In all cases v has non-positive curvature.

It follows that there exists a boundary vertex of degree at most 3. This is clearly impossible if $n > 2$ (since we will get a 2-zone with corners labelled 1). So we assume that $n = 2$. An argument similar to the ones given above shows that such a vertex must connect to ∂D^2 by an ω-zone, with $\omega \geq 3$. It follows that either one of c_1 or c_2 is trivial or we can combine such a vertex with ∂D^2 to get a smaller picture. Both possibilities lead to a contradiction which completes the proof. $\square$

6.3 Main result

Let us recall that the statement of the theorem: if A, B and C are non-trivial groups, then the one-relator product $G = (A*B*C)/N(w)$ is non-trivial where $w \in A*B*C$ is a cyclically reduced word with $\ell(w) < 9$.

Proof of Theorem 6.1.1. By the comments immediately after the statement of Theorem 6.1.1, can assume $6 \leq \ell(w) < 9$ and w has the form $w = c_1Uc_2V$ (up to cyclic permutation) where $U, V \in A * B$ and $c_1, c_2 \in C$ with $c_1c_2 \neq 1$. It follows from Grushko's theorem that the subgroup of $A * B$ generated by U and V is isomorphic to one of the following:

1. Conjugate to subgroup of A (or B).
2. Free group of rank one.
3. Free group of rank two.
4. Free product of two finite cyclic groups.
5. Free product of finite and infinite cyclic groups.

In the case of part 1 the result follows from Lemma 6.2.1. For parts 2 and 3, the result follows from Lemma 6.2.2. And finally the result follows from Lemmas 6.2.3, 6.2.12 and 6.2.13 in the case of parts 4 and 5. This completes the proof. $\square$

Chapter 7

Conclusion and future work

In this work we proved a number of results about one-relator groups under some conditions. In particular we produced various lower bounds on the length of the relator of a one-relator product G for which G can be trivial (in the case of three factors) or fails to satisfy the Freiheitssatz (in the case of two factors). We now mention some future work.

7.0.1 Removing admissibility condition

In Chapter 3 we proved various results about one-relator products induced from generalised triangle groups. This was not without the requirement of admissibility (as in [62]) or that the relator had no element of order 2. Given also the conjecture of Freiheitssatz for any one-relator products in which the relator is a proper power, it seems possible that most, if not all these results are still true without these requirements. Hence we make the following conjecture.

Conjecture 7.0.1 (Conjecture)**.** *Let G be a one-relator product induced from a generalised triangle group H with factor groups G_1 and G_2. Let $\mathcal{H}$ be the quotient of $\langle a \rangle * \langle UbU^{-1} \rangle$ by $N(R^n)$. Then*

1. *The maps $G_1 \to G$, $G_2 \to G$ and $\mathcal{H} \to G$ are all injective.*

2. *If the word problems are soluble for $\mathcal{H}$, G_1 and G_2, then it is soluble for G.*

3. *If some cyclic permutation of R^n has the form W_1W_2 with $\ell(W_1) > 0$ and $\ell(W_2) > 0$ as words in G, then $W_1 \neq 1 \neq W_2$ as words in G. In particular R has order n in G.*

A special case of Conjecture 7.0.1 which surfaced in at least two places in this work is the case where the generalised triangle group H has the form

$$H = \langle x, y \mid x^2, y^p, w(x,y)^2 \rangle, \tag{7.1}$$

for some integer $p > 2$. Perhaps one can try to deal with this special case.

7.0.2 Solution of equations over free product of groups

For some non-trivial groups A and B, and the infinite cyclic group $\langle t\rangle$, consider the one-relator product of groups

$$G = \frac{(A * B)}{N(w)} \tag{7.2}$$

for some word $w \in A * B$ with $\ell(w) \geq 2$.

For the rest of this chapter $w(t) \in A * B * \langle t\rangle$ is a word obtained from $w \in A * B$ by inserting powers of t between the letters in w. For example suppose $w = a_1b_1 \ldots a_nb_n$ with $a_i \in A$ and $b_i \in B$. Then $w(t) = a_1t^{\alpha_1}b_1t^{\alpha_2} \ldots a_nt^{\alpha_{2n-1}}b_nt^{\alpha_{2n}}$ for integers α_i, $i = 1, \ldots, 2n$. We are interested in the relationship between Freiheitssatz for G and solution of the equation $w(t) = 1$ over $A * B$.

Theorem 7.0.2. *Suppose $w(t)$ is as above such that $|\alpha_i| = 1$ and $\alpha_i = -\alpha_{i+1}$. Then G satisfies the Freiheitssatz if and only if $w(t) = 1$ has a solution over $A * B$.*

Proof. By assumption and without loss of generality, we can write $w(t)$ in the form

$$w(t) = \prod_{i=1}^{k} a_itb_it^{-1},$$

where $w = \prod_{i=1}^{k} a_ib_i$. Let K be normal closure of $A * B$ in the universal solution group

$$H = \frac{A * B * \langle t\rangle}{N(w(t))}.$$

Then

$$K = \frac{*_{k\in\mathbb{Z}}(t^kAt^{-k} * t^kBt^{-k})}{N(\{t^kw(t)t^{-k}\}; k \in \mathbb{Z})}$$

However since $t^kw(t)t^{-k}$ is contained $t^kAt^{-k} * t^{(k+1)}Bt^{-(k+1)}$ for each k, it follows that $K = *_{k\in\mathbb{Z}}G_k$, where

$$G_k = \frac{(t^kAt^{-k} * t^{(k+1)}Bt^{-(k+1)})}{N(t^kw(t)t^{-k})}.$$

If G satisfies the Freiheitssatz, then A embeds in G_0 and B embeds in G_{-1}. This implies that $A * B$ embeds in H (via K). On the other hand suppose $w(t) = 1$ has a solution over $A * B$. Then $\alpha \in Ker(A \to G_0) \subseteq Ker(A \to K) \subseteq Ker(A * B \to H)$ implies $\alpha = 1$. Similarly if we replace A by B and G_0 by G_{-1}. This implies that Freiheitssatz holds for G. $\square$

Perhaps one can ask the following more general question.

Question 7.0.3. Is it true that Freihetssatz holds for $G = (A * B)/N(w)$ where $w = a_1 b_1 \dots a_n b_n$, $a_i \in A$ and $b_i \in B$, whenever the equation

$$w(t) = a_1 t^{\alpha_1} b_1 t^{\alpha_2} \dots a_n t^{\alpha_{2n-1}} b_n t^{\alpha_{2n}} = 1$$

has a solution over $A * B$?

As mentioned in Chapter 2, Question 7.0.3 has a positive answer when A and B are locally indicable groups, and is conjectured to be true in the more general setting where both are torsion-free. From Theorem 7.0.2 one might suspect that not only does Question 7.0.3 have a positive answer in general, but also the reverse direction also has a positive answer. In other words, Freihetssatz holds for G if and only if $w(t) = 1$ has a solution over $A * B$.

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