NONASSOCIATIVE DEFORMATIONS OF NON-GEOMETRIC FLUX BACKGROUNDS AND FIELD THEORY

by

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Abstract

In this thesis we describe the nonassociative geometry probed by closed strings in flat non-geometric R-flux backgrounds, and develop suitable quantization techniques. For this, we propose a Courant sigma-model on an open membrane with target space M, which we regard as a topological sector of closed string dynamics on R-space. We then reduce it to a twisted Poisson sigma-model on the boundary of the membrane with target space the cotangent bundle T^*M. The pertinent twisted Poisson structure is provided by a $U(1)$ gerbe in momentum space, which geometrizes R-space.

From the membrane perspective, the path integral over multivalued closed string fields in Q-space (i.e. the T-fold endowed with a non-geometric Q-flux which is T-dual to the R-flux), is equivalent to integrating over open strings in R-space. The corresponding boundary correlation functions reproduce Kontsevich's global deformation quantization formula for the twisted Poisson manifolds, which we take as our proposal for quantization. We calculate the corresponding nonassociative star product and its associator, and derive closed formulas for the case of a constant R-flux. We then develop various versions of the Seiberg–Witten map, which relate our nonassociative star products to associative ones and add fluctuations to the R-flux background.

We also propose a second quantization method based on quantizing the dual of a Lie 2-algebra via convolution in an integrating Lie 2-group. This formalism provides a categorification of Weyl's quantization map, and leads to a consistent quantization of Nambu–Poisson 3-brackets. We show that the convolution product coincides with the star product obtained by Kontsevich's formula, and clarify its relation with the twisted convolution products for topological nonassociative torus bundles.

As a first step towards formulating quantum gravity on non-geometric spaces, we develop a third quantization method to study nonassociative deformations of geometry in R-space, which is analogous to noncommutative deformations of geometry (i.e. noncommutative gravity). We find that the symmetries underlying these nonassociative deformations generate the non-abelian Lie algebra of translations and Bopp shifts in phase space. Using a suitable cochain twist, we construct the quasi-Hopf algebra of symmetries that deforms the algebra of functions, and the exterior differential calculus in R-space. We define a suitable integration on these nonassociative spaces, and find that the usual cyclicity of associative noncommutative deformations is replaced by weaker notions of 2-cyclicity and 3-cyclicity. In this setting, we consider extensions to non-constant R-flux backgrounds as well as more generic twisted Poisson structures emerging from non-parabolic monodromies of closed strings.

As a first application of our nonassociative star product quantization, we develop nonassociative quantum mechanics based on phase space state functions, wherein 3-cyclicity is instrumental for proving consistency of the formalism. We calculate the expectation values of area and volume operators, and find coarse-graining of the string background due to the R-flux. For a second application, we construct nonassociative deformations of fields, and study perturbative nonassociative scalar field theories on R-space. We find that nonassociativity induces modifications to the usual classification of Feynman diagrams into planar and non-planar graphs, which are controlled by 3-cyclicity. The example of φ^4 theory is studied in detail and the one-loop contributions to the two-point function are calculated.

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Contents

Ἓν οἶδα ὅτι οὐδὲν οἶδα.

(The one thing I know is that I know nothing.)

– Socrates –

Chapter 1

Introduction

String theory vacua consist of higher dimensional spaces that require compactification of the extra dimensions in order to relate them to observable phenomenology and cosmology. In the presence of p-form field fluxes along the compact dimensions, these compactifications are referred to as *flux compactifications*, and they have been intensively studied in recent years because of their ability to cure some of the problems suffered by the more conventional Calabi–Yau compactifications (see e.g. [55, 46, 26] for reviews). Flux compactifications lead to generalized non-geometric structures wherein open neighbourhoods are patched together by T-duality [116], and in this sense they arise as consistent string theory solutions [43].

Non-geometric flux backgrounds exhibit noncommutative and even nonassociative structures. To understand how these structures are probed by strings, let us briefly recall how noncommutative spaces emerge when we take *open strings* ending on D-branes in a constant B-field background. Canonical quantization of the open string sigma-model results in commutation relations for the string endpoints given by [38]

$$\big[X^i(\tau,\sigma)\,,\,X^j(\tau,\sigma'\,)\big]\Big|_{\sigma=\sigma'=0,2\pi} = \,\mathrm{i}\,\theta^{ij}\;, \tag{1.1}$$

where $\theta = -2\pi\,\alpha'\,\mathcal{F}\,(1+\mathcal{F}^2)^{-1}$ and $\mathcal{F} = B-F$, with F the gauge field strength two-form on the D-brane. By (1.1) it is evident that the D-brane worldvolume becomes noncommutative. In the double-scaling limit $\alpha'\to 0\,, B\to\infty$ the open string modes decouple from the closed string modes, i.e. from gravity, and the open string two-point function becomes a well-defined target space entity that is independent of the worldsheet coordinates [115]. The deformation of the commutator (1.1) is then proportional to B^{-1} and yields a star product of fields on D-brane worldvolumes, i.e. a noncommutative gauge theory, which can be related to ordinary gauge theories via Seiberg–Witten maps [115].

The discovery of noncommutative geometry in string theory has led to a flurry of investigation into the structures and properties of these noncommutative field theories; see e.g. [47, 117] for early pedagogical reviews on the subject. At the level of the worldsheet theory, it was soon realised that θ defines a Poisson bivector which is dual to the two-form B-field. The associated Poisson sigma-model describes the dynamics of the topological sector of the open string theory, and it can be quantized using Kontsevich's deformation quantization [32, 34]. On the other hand, twist deformation techniques were employed in [8] to systematically deform differential geometry and general relativity on noncommutative spacetime, and thus formulate a noncommutative theory of gravity.

However, it is more natural to seek noncommutative gravity structures emerging in the *closed string* sector. This question had remained somewhat unclear until the recent works [27, 90, 21]. In closed string theory, the two-point function on a sphere depends explicitly on the worldsheet coordinates in the low-energy limit, and so is not a well-defined target space quantity. The three-point function does not exhibit the same pathology; it depends only on the relative orientation of the three insertion points on the sphere, and hence the following tenary bracket emerges on target space

$$[X^i, X^j, X^k] := \lim_{\sigma_i \to \sigma} \big[[X^i(\tau,\sigma_1), X^j(\tau,\sigma_2)], X^k(\tau,\sigma_3)\big] + \text{cyclic} \ . \tag{1.2}$$

The 3-bracket (1.2) is naturally identified as a measure of nonassociativity of the closed string coordinates. For a linearized conformal field theory in flat space with constant Neveu–Schwarz *H-flux* $H = \mathrm{d}B$, it yields the non-trivial result [27, 21]

$$[X^i, X^j, X^k] = \mathrm{i}\,\kappa\,\theta^{ijk} \ , \tag{1.3}$$

where θ^{ijk} is proportional to the background flux, and $\kappa = 0$ for the H-flux background while $\kappa = 1$ after an odd number of T-duality transformations. The deformation is now provided by a trivector induced by a non-geometric R-flux which is T-dual to the constant three-form H-flux. In chapters 3, 4 and 5 we will describe techniques for the proper quantization of this nonassociative deformation of geometry, which is anticipated to lead to closed string nonassociative gravity.

The same type of nonassociativity emerges in the context of closed string flux compactifications. The prototypical example is provided by the flat three-torus $\mathbb{T}^3$ endowed with a constant three-form H-flux. Applying the Büscher rules in this background gives rise to geometric and non-geometric fluxes via the T-duality chain [73, 116]

$$H_{ijk} \xrightarrow{T_i} f^i{}_{jk} \xrightarrow{T_j} Q^{ij}{}_k \xrightarrow{T_k} R^{ijk} \ . \tag{1.4}$$

Here T_i, $i = 1, 2, 3$ denotes a T-duality transformation along the i-th cycle of $\mathbb{T}^3$, which in each step maps the flux to a new flux with a raised index. Geometrically this means that a given differential form component is dualised to a vector field component. Let us now examine more closely the geometric and non-geometric interpretations of each duality frame in (1.4).

The first member of the T-duality chain is the flat three-torus $\mathbb{T}^3$ endowed with a constant Neveu–Schwarz *H-flux* $H = \mathrm{d}B$, which can be regarded as the identity fibration $\mathbb{T}^3 \to \mathbb{T}^3$ with zero-dimensional fibres. Since abelian fluxes in string theory obey analogues of the Dirac quantization condition, the three-form determines a cohomology class $[H] \in \mathrm{H}^3(\mathbb{T}^3; \mathbb{Z}) = \mathbb{Z}$ which is the characteristic class of a gerbe.

The H-flux frame is mapped under T-duality to a circle bundle over $\mathbb{T}^2$ of degree equal to $[H] \in \mathrm{H}^3(\mathbb{T}^3; \mathbb{Z}) = \mathbb{Z}$. In this frame the B-field vanishes; however, a *metric flux* f appears which determines a torsion through the Maurer–Cartan equation. Therefore, the geometry is that of a twisted torus or Heisenberg nilmanifold, which is still a geometric frame.

A further T-duality along a cycle of the base yields a non-geometric space with *Q-flux* that can be locally modelled as a $\mathbb{T}^2$ fibre bundle over a circle S^1. Contrary to the situation encountered above, the closed string sigma-model fields do not commute on Q-space. Via T-duality, this maps back to the original geometric space as a noncommutativity relation [90]

$$\left[X^i(\tau,\sigma)\,,\,\tilde{X}_j(\tau,\sigma)\right]\neq 0 \tag{1.5}$$

between string coordinates $X^i \in M$ and the dual coordinates $\tilde{X}_i \in M^*$. In particular, as the closed strings wrap around the base they fail to satisfy the periodicity condition in S^1 up to non-trivial $SL(2,\mathbb{Z})$ monodromies of the $\mathbb{T}^2$ fibres. For the *parabolic flux model*, which we treat throughout this thesis, the monodromies of $\mathbb{T}^2$ lie in a parabolic conjugacy class of $\mathrm{SL}(2,\mathbb{Z})$. The fibre directions then acquire a noncommutative deformation which is given by the commutation relation $[x^i,x^j] = \mathrm{i}\, Q^{ij}{}_k\, w^k$, where x^i are the local $\mathbb{T}^2$ coordinates and w^k is the winding number along the base direction [90, 6, 61]. As a result, the transition functions between local charts now involve T-duality transformations therefore the metric and B-field are well-defined locally but not globally. In this sense Q-space is locally geometric but globally non-geometric, i.e. it is an example of a *T-fold* [42, 73]. Non-geometric Q-flux backgrounds dual to three-spheres were also recently constructed in [106].

In the case of open strings, the non-geometry of the T-fold can be globalised at the topological level by a fibration of noncommutative two-tori $\mathbb{T}^2_\theta$ over S^1 [95, 57]. In particular, within a C^*-algebra framework the T-fold is described as a topological approximation to a $\mathbb{T}^2$-equivariant gerbe with 2-connection on $\mathbb{T}^3$ [31]. The noncommutativity parameters are parametrized by the local coordinates x^k of the base S^1 as $\theta^{ij}(x) = Q^{ij}{}_k\, x^k$, and thus they define a non-trivial Poisson bivector field. This Poisson structure is naturally dual to the B-field which is a potential for the original H-flux in the T-duality chain. In chapter 2 we shall see how to regard this identification in the context of closed strings which wind in the non-geometric background.

The final background of the T-duality chain (1.4) involving the non-geometric *R-flux* is not even locally geometric [116]. It comes about by taking a final T-duality of the base S^1; however, since this is not a Killing isometry, the standard Büscher rules cannot be applied to the Q-flux background. Nevertheless, a well-defined prescription exists at the level of worldsheet conformal field theory, and the final T-duality transformation is performed by flipping the sign of the corresponding right-moving closed string coordinate. The failure of the Büscher rules simply reflects the absence of local geometric structures (see e.g. [20] and references therein). In the context of open string theory the globalisation of this non-geometry is a (topological) nonassociative three-torus, regarded as a fibration over a point [28, 51].

In the parabolic flux model, the final T-duality along S^1 maps Q-flux to R-flux, and the closed string winding variables w^k to the momentum modes p_k, which are canonically conjugate to the local string position coordinates x^i. Together with the standard Heisenberg commutation relations, this yields the *nonassociative algebra* [90]

$$[x^i, x^j] = \mathrm{i}\,\hbar\, R^{ijk}\, p_k \qquad , \qquad [x^i, p_j] = \mathrm{i}\,\hbar\, \delta^i{}_j \qquad , \qquad [p_i, p_j] = 0 \ , \tag{1.6}$$

which exhibits the same type of nonassociativity that was observed in [27, 21]; in particular, it has a non-trivial Jacobiator given by

$$[x^i, x^j, x^k] := \big[[x^i, x^j], x^k\big] + \text{cyclic permutations} = 3\,\hbar^2\, R^{ijk} \ , \tag{1.7}$$

that reproduces (1.3) for an odd (three) number of T-dualities. The above relations arise from a twisted Poisson structure on phase space [91]; the nonassociative deformation is controlled by a three-vector which is T-dual to the original Neveu–Schwarz H-flux since its components are the R-flux components. In chapter 2 we will describe a membrane sigma-model which encompasses the nonassociative structure of the parabolic R-flux model; its boundary will be an R-twisted Poisson sigma-model on the cotangent bundle of the target space that will provide the precise geometric interpretation of R-space. Noncommutative and nonassociative phase space structures were also found on geometric twisted tori in [36] as solutions of matrix theory compactification conditions.

In the context of worldsheet conformal field theory, non-geometric backgrounds are regarded as left-right asymmetric orbifold theories with respect to the action of the $SL(2,\mathbb{Z})$ monodromy on $\mathbb{T}^2$, where the asymmetric twisting is related to the presence of non-geometric fluxes [40, 39]. Such conformal field theories exhibit the nonassociative structure of the underlying target space as a discontinuity of the three-point functions [27]. In fact, one can read off a deformed product of three functions up to linear order in the background flux by calculating off-shell correlation functions of tachyon vertex operators [21].

However, the fact that Q-space noncommutativity corresponds to a noncommutativity relation between T-dual coordinates on the original geometric H-flux background (c.f. (1.5)), suggests that closed string non-geometry should be investigated within a framework that implements T-duality transformations at the level of target space geometry. In particular, since transition functions of a d–dimensional smooth manifold M are valued in the structure group of the tangent bundle, the existence of stringy $\mathrm{O}(d,d)$ transition functions leads to the notion of a generalized tangent bundle of rank $2d$. This doubling introduces additional geometric variables (such as winding coordinates) which provide a *bona fide* local geometric description of non-geometric spaces. The above observation has motivated two different approaches to non-geometry; namely, generalized geometry and doubled geometry, which can be used in a complimentary way to construct string actions in non-geometric frames.

In the generalized geometry approach (see e.g. [57, 56, 59]), the key observation is that the T-duality group $\mathrm{O}(d,d)$ is also the structure group of the generalized tangent bundle

$$C = TM \oplus T^*M \tag{1.8}$$

over the target space M. This bundle is equipped with a generalized metric which combines the usual Riemannian metric and the B-field in an $\mathrm{O}(d,d)$-invariant way. The non-geometric fluxes are generated by an abelian subgroup of $\mathrm{O}(d,d)$ transformations of sections of C, which are called β-transforms. Specifically, β-transforms take the standard geometric description in terms of a metric, B-field and dilaton field into a framework where the non-geometric fluxes are expressed in terms of a bivector field $\beta = \frac{1}{2}\,\beta^{ij}\,\partial_i \wedge \partial_j$. Non-geometry enters via the bivector β, which can only be defined locally using the T-duality group; however, the non-geometric fluxes are globally well defined [59]. In particular, for the case of vanishing metric flux the non-geometric fluxes are given by

$$Q^{ij}{}_k = \partial_k \beta^{ij} \qquad , \qquad R = [\beta, \beta]_S \tag{1.9}$$

where $[-,-]_S$ is the Schouten bracket which is the natural extension of the Lie bracket to multivector fields (see appendix A.3). By the second member of (1.9) the bivector β is a quasi-Poisson structure [22]; the generalized tangent bundle (1.8) together with this observation will play a prominent role in this thesis.

On the other hand, doubled geometry [73] (see also [71] for a review) takes a more direct approach to implementing $\mathrm{O}(d,d)$ invariance in string theory formalism. Here, spacetime coordinates x^i and their dual coordinates $\tilde{x}_i$, which are canonically conjugate to the winding numbers w^i are put on an equal footing. By defining $x^I := (x^i, \tilde{x}_i)$ and $\partial_I := (\partial_i, \tilde{\partial}^i := \frac{\partial}{\partial \tilde{x}_i})$, where $I = 1, \cdots, 2d$, the target space geometry is doubled. However, since the winding coordinates correspond to non-physical degrees of freedom, their dynamics should be eliminated in the low-energy effective action of any field theory on doubled space. This is accomplished by using the projection $\tilde{\partial}^i = 0$ while keeping the generalized tangent bundle (1.8). This relation between doubled space and generalized geometry and suggests that an $\mathrm{O}(d,d)$ invariant formulation of string theory should be pursued in both doubled geometry and generalized geometry.

For this, double field theory [75, 76] (see also [1, 17, 72] for reviews) uses the doubled geometry variables x^I and the generalized metric of generalized geometry to propose a manifestly $\mathrm{O}(d,d)$ invariant action on doubled space. Within this framework, a field theory on M for the non-geometric fluxes of the T-duality chain (1.4) can be obtained simply by performing a formal T-duality transformation and a field redefinition in the doubled field theory action using the β-transforms of generalized geometry [5]. In fact, it is possible to formulate a bi-invariant action, i.e. invariant under both diffeomorphisms and β-transforms, for closed strings in non-geometric flux backgrounds [23] (see also [24]). With these methods, double field theory provides a suitable context for a desired deformation quantization of non-geometric spaces. The structure of nonassociative deformations of geometry in double field theory is analysed in [25]. In this thesis we study noncommutative and nonassociative deformations of non-geometric spaces by employing and developing methods within the generalised geometry framework.

The appearance of nonassociative geometry in string theory is not new. It arises naturally in the context of open string noncommutative gauge theory when D-branes are placed in a non-constant B-filed background*. The nonassociative deformation is controlled by the three-form H-flux $H = \mathrm{d}B \neq 0$ via Kontsevich's deformation quantization of the pertinent H-twisted Poisson structure, which is reproduced by the correlation functions of open string tachyon vertex operators [41, 66, 63]. However, nonassociativity disappears in on-shell tachyon scattering amplitudes by using the Dirac–Born–Infeld field equations on the D-brane [63, 65]. This is also true for the closed string sector: Once momentum conservation in tachyon scattering amplitudes is taken into account, all traces of nonassociativity disappear and the usual crossing symmetry of correlation functions in two-dimensional conformal field theory is recovered [21].

Nonassociativity appears also in ordinary quantum mechanics when charged particles in three dimensions are placed in the field of a magnetic monopole [80]. In this case the physical momenta $\boldsymbol{\pi}$ satisfy the commutation relation $[\pi_i, \pi_j] = \mathrm{i}\,\hbar\, e\, \epsilon_{ijk}\, B^k$, where e is the particle's charge and B is the magnetic field, and together with the canonical commutation relations they define a noncommutative momentum space. In this background translations $U(\boldsymbol{a}) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\, \boldsymbol{a}\cdot\boldsymbol{\pi}}$ by a vector $\boldsymbol{a}$ do not commute. The violation of translation invariance is given by a phase equal to the magnetic flux $\Phi_{\boldsymbol{a}_1,\boldsymbol{a}_2} = \frac{1}{2}\,(\boldsymbol{a}_1 \times \boldsymbol{a}_2)\cdot \boldsymbol{B}$ through the triangle spanned by the two vectors. This phase is a *2-cocycle* of the abelian group of translations. Nonassociativity arises through a non-trivial Jacobiator for the momenta, and thus translations do not associate by a phase equal to the magnetic flux $\Phi_{\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3} = \frac{1}{6}\,\big((\boldsymbol{a}_1 \times \boldsymbol{a}_2)\cdot \boldsymbol{a}_3\big)\,\nabla\cdot \boldsymbol{B}$ through the tetrahedron spanned by the three vectors, which is a *3-cocycle* of the abelian group of translations. For $\nabla \cdot \boldsymbol{B} = 0$ the magnetic field satisfies Maxwell's equations and associativity persists. In particular, for a constant magnetic field the phase space

*A different approach is taken in [93], where gauge theory on the octonions and other nonassociative algebras is formulated in the framework of cochain twist deformations.

commutation relations in the strong field limit reproduce the non-commuting coordinates of the lowest Landau level [118]. On the other hand, if magnetic monopoles are present, then $\nabla \cdot \boldsymbol{B} \neq 0$, and nonassociativity persists unless the magnetic flux is quantized, i.e.

$$\tfrac{e}{\hbar}\,\Phi_{\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3} \ \in \ \pi\,\mathbb{Z} \ . \tag{1.10}$$

This is simply the Dirac quantization condition for magnetic charge. In this context it ensures the basic postulates of quantum mechanics, wherein associativity of operators is required. Similarly, nonassociativity in string theory should be regarded as a feature whose consistency induces constraints, such as flux quantization, and teaches us something about the nature of non-geometric string theory.

In the rest of this thesis we shall investigate the origins of noncommutative and nonassociative geometry for closed strings in R-space, and perform deformation quantization using various methods (see [101] for a review). In particular, inspired by [69, 59], we will argue in *Chapter 2* that the suitable analogue of the open string Poisson sigma-model for closed strings in flux backgrounds, is a higher version of the Poisson sigma-model called the Courant sigma-model. This is a sigma-model on an open three-dimensional *membrane* with target space M, where the boundaries of the membrane are regarded as the closed strings [99]. Its field content is valued in a Courant algebroid, which in the present case is the standard Courant algebroid $C = TM \oplus T^*M$ with the structure functions of C appropriate to the pure constant R-flux background [59, 77, 22]. We will show that the membrane sigma-model reduces to a twisted Poisson sigma-model on the boundary whose target space is the *cotangent bundle* T^*M of the original target space M. The twisting is given by a non-flat $U(1)$-gerbe on *momentum space*, and the resulting linear twisted Poisson structure coincides exactly with that proposed in [90, 91]. Our membrane sigma-model thus gives a straightforward dynamical explanation of the nonassociative phase space algebra (1.6) and also a geometric interpretation for the effective target space geometry seen by closed strings in R-flux compactifications.

With the twisted closed string boundary conditions considered in [90, 40], we will then argue that the closed string path integral is equivalent to that of an *open* string twisted Poisson sigma-model on a disk. The resulting boundary correlation functions define a quantization of the R-flux background, and in fact one can reproduce the entire setting of Kontsevich's global deformation quantization [89] for twisted Poisson structures.

In *Chapter 3* we develop Kontsevich's global deformation quantization for arbitrary R-flux, and describe the resulting nonassociative star product, the corresponding associator as well as their various derivation properties through the formality maps [99]. In the case of constant R-flux we derive explicit closed formulas which resemble the Moyal–Weyl formula. We shall also see that our formalism appears to have the right features to define a proper nonassociative quantization of Nambu–Poisson 3-brackets, at least for constant trivectors. We will further demonstrate how the three-product of fields proposed in [21] arises in special subsectors of our general formalism.

Using this approach we are also able to clarify the meaning of Seiberg–Witten maps in this setting. As discussed in [7], the deformation quantization of twisted Poisson structures leads to noncommutative gerbes. The nonassociative star product can be "untwisted" to a family of associative star products that are all related by Seiberg–Witten maps. This formulation has a large gauge symmetry given by star commutators with gauge parameters that live on phase space. In our setting we can also work directly with the nonassociative star product by restricting the class of admissible gauge fields. Seiberg–Witten maps can then be used to describe fluctuations at the boundary of the membrane (or at the endpoints of the open strings), and are closely related to quantized general coordinate transformations. We find two particularly interesting examples. Firstly, a dynamical Seiberg–Witten map from the associative canonical star product on phase space to the nonassociative R-twisted star product, which can be computed explicitly to all orders in closed form and may be the first example of its kind. Secondly, Nambu–Poisson maps that can be used to add fluctuations to the (constant) R-flux background.

In *Chapter 4* we develop an alternative approach to deformation quantization of the non-geometric R-flux background [99]. There we show that our twisted Poisson structure has the structure of a Lie 2-algebra, which is a categorified version of an ordinary Lie algebra with the Jacobi identity weakened to a natural transformation. In fact, the Lie 2-algebra that we use is related to the noncommutative Q-space background, and it can be regarded as a reduction of the structure algebra of the Courant algebroid C over a point. This Lie 2-algebra is integrated to a Lie 2-group $\mathscr{G}$ which is a categorification of the Heisenberg group that defines the double twisted torus. By using the nonassociative convolution product induced by horizontal multiplication in $\mathscr{G}$, we induce a nonassociative star product on the algebra of functions on phase space by embedding it as an algebra object in the category $\mathscr{G}$. This mapping can be regarded as a higher version of the Weyl–Wigner quantization map, which is familiar from conventional approaches to noncommutative field theory [117]. We demonstrate that this star product is identical to the nonassociative Kontsevich star product calculated in chapter 3.

In *Chapter 5* we provide a third way of quantizing non-geometric R-flux backgrounds using twist deformation techniques [100]. In particular, we construct a Hopf algebra H from the Lie algebra of symmetries acting on the phase space description of R-space (see e.g. [92]). In order to accommodate nonassociativity, we regard H as a quasi-Hopf algebra [48] with a trivial three-cocycle $\phi \in H \otimes H \otimes H$ called the *associator*, and deform it using a *cochain twist* $\mathcal{F} \in H \otimes H$. Our approach in this chapter is related to the categorical constructions of chapter 4 by the observation that every braided monoidal category whose objects are vector spaces, is equivalent to the representation category of some (quasi-)Hopf algebra H (see e.g. [92, Chap. 9] for details).

To demostrate that twisting methods naturally allow for the introduction of a 3-form R-flux in phase space, we proceed in three steps. First we construct the Hopf algebra K related to the abelian Lie algebra of translations in $2d$-dimensional phase space $\mathcal{M}$, and deform it using an abelian 2-cocycle twist $F \in K \otimes K$. The action of the twisted Hopf algebra K_F on the algebra of functions $C^\infty(\mathcal{M})$ yields the canonical Moyal–Weyl star product on phase space. We then endow $\mathcal{M}$ with a trivector R which is T-dual to the 3-form of a uniform background H-flux, and bring it into the twist quantization scheme by introducing a unique family of twist elements which are parametrised by constant momentum. The pointwise product of functions on phase space then deforms to a family of associative noncommutative Moyal–Weyl type star products on constant momentum slices. These deformed products are equal to the ones derived in chapter 3 for Q-flux backgrounds, and are related to the nonassociative star product by twists. In the context of chapter 3, these relations are described by Seiberg–Witten maps. Finally, we promote constant momenta to dynamical momenta appropriate to R-space, and twist the pertinent Hopf algebra H using a cochain twist $\mathcal{F}$, which is tantamount to an abelian cocycle twist of the canonical Moyal–Weyl product. The underlying Lie algebra of symmetries of R-space is non-abelian, nilpotent, and includes non-local Bopp shifts on phase space that mix positions with momenta. The resulting twisted Hopf algebra is a quasi-Hopf algebra whose action on $C^\infty(\mathcal{M})$ quantizes the phase space structure of constant R-flux backgrounds and yields the nonassociative star product of chapter 3.

The advantage of this method is in its algorithmic character: Once a twist is known, it is just a matter of applying the cochain twist machinery to systematically deform all geometric structures which are covariant under the symmetries of a manifold. We demonstrate this by deforming the differential calculus on phase space $\mathcal{M}$, and thus formulate nonassociative deformations of the exterior differential algebra and of the $C^\infty(\mathcal{M})$-bimodule structure on R-flux backgrounds. We then define integration on the deformed algebra of forms on phase space to be the standard integration. This is crucial for the calculation of quantum mechanical averages in chapter 6, and also for setting up a framework to study field theories on R-space in chapter 7. We find that the integral of multiple exterior star products of differential forms is not (graded) cyclic, but rather satisfies weaker notions of 2-cyclicity and 3-cyclicity that we describe. In particular, 3-cyclicity turns out to be crucial for a consistent formulation of quantum theory on R-space, and ensures that nonassociative deformations of field theory can be made consistent with the requirements of crossing symmetry of conformal field theory scattering amplitudes.

Some generalizations of our twist deformation methods to non-constant R-flux backgrounds as well as more generic R-flux string vacua are also briefly considered. In particular, we study the case of position-dependent R-fluxes in (1.6) and the conditions under which the techniques of twist deformation quantization developed in this chapter carry through at least locally. By restricting to functions of the position coordinates in M, this technique provides a framework for quantizing generic Nambu-Poisson 3-brackets determined by the trivector field R. We also consider the extension of the phase space algebra (1.6) to quasi-Poisson structures that are generic non-linear functions of the momenta. This type of structures appear in R-flux backgrounds that arise from non-parabolic monodromies of $\mathbb{T}^2$, and in particular, in the elliptic model of [90, 40]. In this case we apply Kontsevich's deformation quantization of phase space to compute the nonassociative star product and associator up to third order in a derivative expansion in the R-flux, which is used to identify the pertinent Hopf algebra of symmetries and a (non-unique) cochain twist.

In *Chapter 6* we develop a consistent formulation of nonassociative quantum mechanics on a phase space $\mathcal{M}$ endowed with a constant 3-form R-flux [100]. Since a conventional treatment in terms of linear operators on separable Hilbert spaces is not possible when associativity is lost, we achieve this by employing our phase space star product quantization. To construct states we introduce an appropriate unital composition of functions in $C^\infty(\mathcal{M})$ which is noncommutative and associative. We then investigate the role of observables without referring to their representations and show that expectation values of functions on phase space satisfy reality and positivity using the 3-cyclicity condition of the star product. In this formalism we find that a triple of operators that do not associate does not have common eigenstates, which is a clear sign of position space quantization in the presence of R-flux. This induces an uncertainty relation proportional to the transverse momentum for the measurement of a pair of position coordinates. We also find non-zero expectation values for the uncertainty of suitably defined area and volume operators in configuration space, leading to a minimal volume element. We thus provide a concrete and rigorous derivation for the uncertainty relations anticipated by [90, 27]. In this sense, our treatment is the first step towards realising more elaborate models, such as field theory or gravity, on non-geometric flux compactifications. We also study the dynamics of the theory and find that operator time evolution in the Heisenberg picture is not a derivation of the star product algebra of operators.

Finally, in *Chapter 7* we undertake the construction of nonassociative scalar field theories within the phase space description of non-geometric R-flux backgrounds [102]. Here, we propose action functionals for a single real scalar field φ and introduce interactions to study the effects of the nonassociativity, which enters via the nonassociative star product we calculated in chapters 3 and 5. We demonstrate that interactions up to φ^5 are associative at tree level using the 3-cyclicity of the star product, and find that many features which are familiar from noncommutative scalar fied theories (see e.g. [117, 47] for reviews) persist in the nonassociative case. In particular, momentum conservation along the non-commuting directions is violated due to the dynamical character of our star product, while loop corrections to the propagator exhibit the usual UV/IR mixing observed in noncommutative scalar field theories with non-constant noncommutativity parameter [108]. However, in contrast to the associative case, the phase factor induced by the deformation is not invariant under cyclic permutations of the external momenta, and thus the usual classification of the Feynman diagrams that enter the perturbation expansion into planar and non-planar diagrams is modified. We describe these modifications in detail using some considerations from graph theory, and explaining the crucial role that is assumed by 3-cyclicity in their derivation. As an application of our formalism, we study φ^4 and calculate the one-loop contributions to the propagator.

Two appendices at the end of the paper are delegated to some of the more technical aspects of our analysis. In appendix A we review in some detail all notions regarding the higher algebraic and geometric structures that are employed in the main text. In appendix B we present some technical details of the explicit computation of Kontsevich's formula.

Chapter 2

Membrane σ-models for non-geometric backgrounds

In this chapter we propose sigma-models for closed strings in R-flux backgrounds. The Poisson sigma-model with target space M describes the topological sector of string theory in two-form B-field backgrounds. To incorporate non-trivial three-form fluxes, one instead needs a coupling to *membranes*, which motivates the need for using higher mathematical structures for the twistings that arise in these instances. The effective dynamics in three-form flux backgrounds is thus provided by suitable Courant sigma-models with target space M which describe topological sectors of membrane theories. We will first review how the sigma-model appropriate to H-space can be reduced on the boundary of an open membrane to a twisted Poisson sigma-model with target space M [105, 69, 70, 30]. Then we will show that for constant R-flux the appropriate Courant sigma-model reduces to a string theory with target space the *cotangent bundle* of M with twisted Poisson structure which coincides with that found in [90, 91]. This geometric interpretation of the R-flux background is related to the doubled geometry description of non-geometric flux compactifications [73, 43].

2.1 Poisson and Courant sigma-models

AKSZ sigma-models [3] whose target spaces comprise a symplectic Lie n-algebroid E over a manifold M may be constructed using higher Chern–Simons action functionals [53] (see appendix A.6 for the relevant details concerning algebroids). A simple case is the cotangent Lie algebroid $E = T^*M$ over a Poisson manifold M with Poisson bivector $\Theta = \frac{1}{2}\,\Theta^{ij}(x)\,\partial_i \wedge \partial_j$, where $x = (x^i) \in M$ are local coordinates with $\partial_i := \frac{\partial}{\partial x^i}$. This is a symplectic Lie 1-algebroid with the canonical symplectic structure on the cotangent bundle T^*M. Let Σ_2 be a two-dimensional string worldsheet. The AKSZ construction defines a topological field theory on $C^\infty(T\Sigma_2, T^*M)$ (regarded as a space of Lie algebroid morphisms). A Poisson Lie algebroid-valued differential form on Σ_2 is given by the smooth embedding $X = (X^i) : \Sigma_2 \to M$ of the string worldsheet in target space, and an auxiliary one-form field on the worldsheet $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$. The corresponding AKSZ action is

$$S^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j\Big)\ , \tag{2.1}$$

which coincides with the action of the Poisson sigma-model [78, 114, 32, 33]. The Poisson sigma-model is the most general two-dimensional topological field theory that can be obtained from the AKSZ construction.

Note that although on-shell the bivector field Θ is required to have vanishing Schouten–Nijenhuis bracket with itself (in particular so that it defines a differential d_Θ on the algebra of multivector fields, see appendix A.3), the perturbative expansion of [32] still makes sense when Θ is a twisted Poisson bivector and reproduces the Kontsevich formality maps for nonassociative star products [34]. The topological nature of the Poisson sigma-model allows for it to be perturbatively expanded around a non-vacuum solution.

A Courant structure is the first higher analogue of a Poisson structure. The corresponding AKSZ sigma-model has target space comprising a symplectic Lie 2-algebroid with a "degree 2 symplectic form", which is the same thing as a Courant algebroid E over a manifold M [109]. In [109] it is shown that Courant algebroids

$E \to M$ are in a canonical bijective correspondence with AKSZ sigma-models on a three-dimensional membrane worldvolume Σ_3. A Courant algebroid-valued differential form on Σ_3 is given by the smooth embedding of the membrane worldvolume $X = (X^i) : \Sigma_3 \to M$ in target space, a one-form $\alpha = (\alpha^I) \in \Omega^1(\Sigma_3, X^*E)$, and an auxiliary two-form field on the worldvolume $\phi = (\phi_i) \in \Omega^2(\Sigma_3, X^*T^*M)$. The structure functions of the Lie 2-algebroid are specified by choosing a local basis of sections $\{\psi_I\}$ of $E \to M$ such that the fibre metric $h_{IJ} := \langle \psi_I, \psi_J \rangle$ is constant. We define the anchor matrix ${P_I}^i$ by $\rho(\psi_I) = {P_I}^i(x)\, \partial_i$, and the three-form $T_{IJK}(x) := [\psi_I, \psi_J, \psi_K]_E$. Then the canonical three-dimensional topological field theory associated to the Courant algebroid $E \to M$ is described by the AKSZ action

$$
\begin{aligned}
S^{(2)}_{\rm AKSZ} = \int_{\Sigma_3} \Big(\phi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, h_{IJ}\, \alpha^I \wedge \mathrm{d}\alpha^J - {P_I}^i(X)\, \phi_i \wedge \alpha^I + \\
+ \frac{1}{6}\, T_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K \Big) \ ,
\end{aligned} \tag{2.2}
$$

which is the action of the Courant sigma-model [79, 69, 111].

2.2 Sigma-models for geometric fluxes

The Courant algebroid of exclusive interest in geometric flux compactifications of string theory is the *standard Courant algebroid* $C = TM \oplus T^*M$ twisted by a closed NS–NS three-form flux $H = \frac{1}{6}\, H_{ijk}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$. The structure maps of C comprise the skew-symmetrization of the *H-twisted Courant–Dorfman bracket* given by [121]

$$
\begin{aligned}
\big[(Y_1, \alpha_1)\, ,\, (Y_2, \alpha_2)\big]_H \ := \ \big([Y_1, Y_2]_{TM} \, , \, \mathcal{L}_{Y_1}\alpha_2 - \mathcal{L}_{Y_2}\alpha_1 \\
- \tfrac{1}{2}\, \mathrm{d}\big(\alpha_2(Y_1) - \alpha_1(Y_2)\big) + H(Y_1, Y_2, -)\big)
\end{aligned} \tag{2.3}
$$

for vector fields $Y_1, Y_2 \in C^\infty(M, TM)$ and one-form fields $\alpha_1, \alpha_2 \in \Omega^1(M)$, the metric is the natural dual pairing between TM and T^*M,

$$
\big\langle (Y_1, \alpha_1)\, ,\, (Y_2, \alpha_2) \big\rangle = \alpha_2(Y_1) + \alpha_1(Y_2) \ , \tag{2.4}
$$

and the anchor map is the trivial projection $\rho : C \to TM$ onto the first factor; the map $\mathsf{d} : C^\infty(M) \to C^\infty(M, C)$ is given by $\mathsf{d}f = \frac{1}{2}\,\mathrm{d}f$. This is an *exact* Courant algebroid, i.e. it fits into the short exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\ \rho^*\ } C \xrightarrow{\ \rho\ } TM \longrightarrow 0 \ , \tag{2.5}$$

where $\rho^* : T^*M \to C^*$ is the transpose of the anchor map ρ followed by the identification $C^* \cong C$ induced by the pairing on the Courant algebroid. Every exact Courant algebroid on M is isomorphic to one of the form $C = TM \oplus T^*M$ with the structure maps given as above. The isomorphism classes are parametrized by elements $[H] \in H^3(M, \mathbb{R})$ of the degree 3 real cohomology of the target space.

To determine the structure maps of the exact Courant algebroid in a convenient basis, we suppose henceforth that the tangent bundle $TM \cong M \times \mathbb{R}^d$ is trivial, where $d = \dim(M)$. This assumption will avoid the appearance of geometric f-fluxes and other fluxes, as eventually we will want to apply triple T-duality to take us directly into the pure R-flux background. Then in local coordinates $x = (x^i)$ for M, a natural frame for $TM \oplus T^*M$ is given by

$$\varrho_i = \partial_i \qquad \text{and} \qquad \chi^i = \mathrm{d}x^i \tag{2.6}$$

for $i = 1, \dots, d$. Writing ϱ_i for $(\varrho_i, 0)$ and χ^i for $(0, \chi^i)$ for simplicity, the metric is given by

$$\langle \varrho_i, \chi^j \rangle = \delta_i{}^j \ . \tag{2.7}$$

The corresponding twisted Courant–Dorfman algebra is isomorphic to the algebra with the sole non-trivial brackets

$$[\varrho_i, \varrho_j]_H = H_{ijk}\, \chi^k \ . \tag{2.8}$$

The non-vanishing ternary brackets are given by (see appendix A.6)

$$[\varrho_i, \varrho_j, \varrho_k]_H = H_{ijk} \ . \tag{2.9}$$

As reviewed in [91], the brackets (2.8) and (2.9) for constant H-flux mimic the phase space quasi-Poisson algebra of a charged particle in the background field of a magnetic monopole [80].

We now write

$$(\alpha^I) = (\alpha^1, \dots, \alpha^{2d}) := (\alpha^1, \dots, \alpha^d, \xi_1, \dots, \xi_d) \tag{2.10}$$

where $(\alpha^i) \in \Omega^1(\Sigma_3, X^*TM)$ and $(\xi_i) \in \Omega^1(\Sigma_3, X^*T^*M)$; throughout, upper case indices $I, J, \dots \in \{1, \dots, 2d\}$ run over directions of the doubled geometry, while lower case indices $i, j, \dots \in \{1, \dots, d\}$ run over directions of the original configuration space. Then the action (2.2) becomes

$$S^{(2)}_{\rm WZ} = \int_{\Sigma_3} \Big(\phi_i \wedge \mathrm{d}X^i + \alpha^i \wedge \mathrm{d}\xi_i - \phi_i \wedge \alpha^i + \frac{1}{6}\, H_{ijk}(X)\, \alpha^i \wedge \alpha^j \wedge \alpha^k \Big) \ . \tag{2.11}$$

When $\Sigma_2 := \partial\Sigma_3 \neq \emptyset$, this is the action of the canonical open topological membrane theory [105]. In this case we can take the consistent Dirichlet boundary conditions $\alpha^i = \phi_i = 0$ on Σ_2 (we could also take $X^i = \xi_i = 0$ and hybrids thereof; see [69] for a discussion of the resulting modifications). One can also modify the action by adding a boundary term of the form

$$\begin{aligned} S^{\partial}_{\rm WZ} = \oint_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, \Theta^{ij}(X)\, \xi_i \wedge \xi_j + \Gamma^i{}_j(X)\, \xi_i \wedge \alpha^j + \\ + \frac{1}{2}\, \Xi_{ij}(X)\, \alpha^i \wedge \alpha^j \Big) \ . \end{aligned} \tag{2.12}$$

In [69, 70] only the Θ-deformation is kept, corresponding to a canonical transformation on the Courant algebroid which gives the boundary/bulk open topological membrane action

$$\begin{aligned}\widetilde{S}^{(2)}_{\mathrm{WZ}} = &\int_{\Sigma_3} \Big(\phi_i \wedge \big(\mathrm{d}X^i - \alpha^i\big) + \alpha^i \wedge \mathrm{d}\xi_i + \frac{1}{6}\, H_{ijk}(X)\, \alpha^i \wedge \alpha^j \wedge \alpha^k\Big) + \\ &+ \oint_{\Sigma_2} \frac{1}{2}\, \Theta^{ij}(X)\, \xi_i \wedge \xi_j \ .\end{aligned} \tag{2.13}$$

In this case the consistent boundary conditions require that $\Theta = \frac{1}{2}\,\Theta^{ij}(x)\,\partial_i \wedge \partial_j$ is an H-twisted Poisson bivector on M, i.e. its Schouten–Nijenhuis bracket with itself is given by

$$[\Theta, \Theta]_{\mathrm{S}} = {\textstyle\bigwedge}^3 \Theta^\sharp(H) \ , \tag{2.14}$$

and the Jacobi identity for the corresponding bracket is violated (see appendix A.3). Here ${\textstyle\bigwedge}^3\, \Theta^\sharp(H)$ denotes the natural way to turn the three-form H into a three-vector by using Θ to “raise the indices”. After integrating out the two-form fields ϕ_i we arrive at the AKSZ action

$$\begin{aligned}\widetilde{S}^{(1)}_{\mathrm{AKSZ}} = &\oint_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, \Theta^{ij}(X)\, \xi_i \wedge \xi_j\Big) + \\ &+ \int_{\Sigma_3} \frac{1}{6}\, H_{ijk}(X)\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ ,\end{aligned} \tag{2.15}$$

which is the action of the H-twisted Poisson sigma-model with target space M [105, 88]. Note that including the last term of (2.12) would result in an additional global B-field coupling $\frac{1}{2}\,\Xi_{ij}(x)\,\mathrm{d}X^i \wedge \mathrm{d}X^j$ on the string worldsheet.

2.3 Sigma-models for non-geometric fluxes

The relevance of the topological twisted Poisson sigma-model (2.15) in the effective theory of strings in R-flux backgrounds was noted in [51, 58]. Here we shall start with the general Courant sigma-model (2.2) and the argument of [59] that the appropriate theory in R-space is described by a non-topological membrane sigma-model, not a string theory. The membrane action in this case is not generally equivalent to the action of a string theory on the boundary of a membrane. This would also corroborate the observation of [51] that the R-space geometry does not seem to exist as a low-energy effective description of string theory, in the sense that open strings in R-space cannot be consistently decoupled from gravity. The absence of a topological limit and the non-decoupling of gravity for open strings in H-space was also observed in [63]. In a sense to be elucidated below, the membrane theory geometrizes the non-geometric R-flux background, in a way reminiscent of the manner in which M-theory geometrizes string dualities. Although the R-space is not even locally geometric as a Riemannian manifold [116], in this paper we work only at tree-level in the low-energy effective field theory on target space where we can treat the R-space locally as the original d-dimensional manifold M.

The Courant algebroid pertinent to the R-flux background is again the standard Courant algebroid $C = TM \oplus T^*M$, but now twisted by a trivector flux $R = \frac{1}{6} R^{ijk}(x)\, \partial_i \wedge \partial_j \wedge \partial_k$ satisfying a suitable integrability condition. The bracket on C is the skew-symmetrization of Roytenberg's R-twisting of the Courant–Dorfman bracket given by [110, 59, 22]

$$\begin{aligned} \big[(Y_1,\alpha_1)\,,\,(Y_2,\alpha_2)\big]_R \ :=\ & \big([Y_1,Y_2]_{TM} + R(\alpha_1,\alpha_2,-)\,, \\ & \mathcal{L}_{Y_1}\alpha_2 - \mathcal{L}_{Y_2}\alpha_1 - \tfrac12\,\mathrm{d}\big(\alpha_2(Y_1)-\alpha_1(Y_2)\big)\big)\ , \end{aligned} \tag{2.16}$$

while the remaining structure maps are identical to those of section 2.2.

Writing the generators of the natural frame for $TM \oplus T^*M$ as ϱ_i and χ^i as before, the corresponding Roytenberg algebra is isomorphic to the algebra with the non-trivial brackets

$$[\chi^i,\chi^j]_R = R^{ijk}\,\varrho_k \tag{2.17}$$

and the metric (2.7). When R is a constant flux this is the d-dimensional Heisenberg algebra. This mimics the commutation relations for closed string fields which are obtained by applying three T-duality transformations to the H-space $M = \mathbb{T}^3$ [90, 91], with the remaining non-trivial structure map

$$[\chi^i,\chi^j,\chi^k]_R = R^{ijk}\ . \tag{2.18}$$

Below we will recover the commutation relations of [90, 91] dynamically from an associated twisted Poisson sigma-model.

With the same splitting (2.10), the action (2.2) in the pure R-flux background becomes

$$
\begin{aligned}
S_R^{(2)} = & \int_{\Sigma_3} \Big(\phi_i \wedge \big(\mathrm{d}X^i - \alpha^i\big) + \alpha^i \wedge \mathrm{d}\xi_i + \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k\Big) + \\
& + \oint_{\Sigma_2} \frac{1}{2}\, g^{ij}(X)\, \xi_i \wedge *\xi_j \ ,
\end{aligned}
\tag{2.19}
$$

where $g^{-1} = \frac{1}{2}\, g^{ij}(x)\, \partial_i \otimes \partial_j$ is the inverse of a chosen metric tensor on target space M, and $*$ is the Hodge duality operator with respect to a chosen metric on the worldsheet $\Sigma_2 = \partial\Sigma_3$. Here, we have again chosen Dirichlet boundary conditions $\alpha^i = \phi_i = 0$ on Σ_2. As in [59], we have added a metric-dependent term on the boundary Σ_2 of the membrane, which breaks the topological symmetry of the Courant sigma-model, in order to ensure that the choice $R^{ijk} \neq 0$ is consistent with the equations of motion and also with the gauge symmetries of the field theory [69]. Note that only g^{-1} appears, not the metric g itself; it will play the role of a metric on momentum space later on. Integrating out the two-form fields ϕ_i leads to the action

$$
S_R^{(2)} = \oint_{\Sigma_2} \xi_i \wedge \mathrm{d}X^i + \int_{\Sigma_3} \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k + \oint_{\Sigma_2} \frac{1}{2}\, g^{ij}(X)\, \xi_i \wedge *\xi_j \ . \tag{2.20}
$$

We will now specialize to the case where both the R-flux and the target space metric are constant; this is the situation relevant to the considerations of [27, 90, 21, 91]. On the boundary of the membrane, the equations of motion for X^i then force $\xi_i = \mathrm{d}P_i$ to be an exact form (modulo harmonic forms on Σ_2), where $P_i \in C^\infty(\Sigma_3, X^*T^*M)$ is a section of the cotangent bundle of M restricted to Σ_3. This solution is also consistent with the equations of motion in the bulk and henceforth we restrict the configuration space for the path integral to this domain of fields. Then the action (2.20) reduces to a pure boundary action of the form

$$
S_R^{(2)} = \oint_{\Sigma_2} \Big(\mathrm{d}P_i \wedge \mathrm{d}X^i + \frac{1}{2}\, R^{ijk}\, P_i\, \mathrm{d}P_j \wedge \mathrm{d}P_k\Big) + \oint_{\Sigma_2} \frac{1}{2}\, g^{ij}\, \mathrm{d}P_i \wedge *\mathrm{d}P_j \ . \tag{2.21}
$$

This action can be recast in the form

$$S_R^{(2)} = \oint_{\Sigma_2} -\frac{1}{2}\,\Theta^{-1}_{IJ}(X)\,\mathrm{d}X^I \wedge \mathrm{d}X^J + \oint_{\Sigma_2} \frac{1}{2}\,g_{IJ}\,\mathrm{d}X^I \wedge *\mathrm{d}X^J \ , \tag{2.22}$$

where the fields

$$X = (X^I) = (X^1,\dots,X^{2d}) := (X^1,\dots,X^d,P_1,\dots,P_d) \tag{2.23}$$

embed the string worldsheet Σ_2 in the cotangent bundle of M, i.e. the effective target space is now *phase space*. Here, we have introduced the block matrix on T^*M given by

$$\Theta = \big(\Theta^{IJ}\big) = \begin{pmatrix} R^{ijk}\,p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \tag{2.24}$$

with local phase space coordinates

$$x = (x^I) = (x^1,\dots,x^{2d}) := (x^1,\dots,x^d,p_1,\dots,p_d) \ . \tag{2.25}$$

The "closed string metric"

$$\big(g_{IJ}\big) = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} \tag{2.26}$$

acts on momentum space but not on configuration space. The matrix Θ is always invertible and its inverse is given by

$$\Theta^{-1} = \big(\Theta^{-1}_{IJ}\big) = \begin{pmatrix} 0 & -\delta_i{}^j \\ \delta^i{}_j & R^{ijk}\,p_k \end{pmatrix} \ . \tag{2.27}$$

We can linearise the action (2.22) in the embedding fields $X = (X^I) : \Sigma_2 \to T^*M$ by introducing auxiliary fields $\eta_I \in \Omega^1(\Sigma_2, X^*T^*(T^*M))$ to write

$$S_R^{(2)} = \oint_{\Sigma_2} \Big(\eta_I \wedge \mathrm{d}X^I + \frac{1}{2}\,\Theta^{IJ}(X)\,\eta_I \wedge \eta_J\Big) + \oint_{\Sigma_2} \frac{1}{2}\,G^{IJ}\,\eta_I \wedge *\eta_J \ , \tag{2.28}$$

where the "open string metric"

$$\left(G^{IJ}\right) = \begin{pmatrix} g^{ij} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.29}$$

is related to (2.26) by the usual closed-open string relations [115] that involve Θ and the "B-field" Θ^{-1} (note that (g_{IJ}) is not the inverse of (G^{IJ})). This is the action of the non-topological generalized Poisson sigma-model for the embedding of the string worldsheet Σ_2 into the *cotangent bundle* T^*M of the manifold M with bivector field

$$\Theta = \tfrac{1}{2}\,\Theta^{IJ}(x)\,\partial_I \wedge \partial_J \ , \tag{2.30}$$

whose coefficient matrix Θ^{IJ} is given by (2.24), and $\partial_I := \frac{\partial}{\partial x^I}$; below we will write phase space derivatives as $\partial_i := \frac{\partial}{\partial x^i}$ and $\tilde{\partial}^i := \frac{\partial}{\partial p_i}$. For completeness, we express the action (2.28) more explicitly in phase space component form by decomposing the one-form fields

$$(\eta_I) = (\eta_1, \dots, \eta_{2d}) := (\eta_1, \dots, \eta_d, \pi^1, \dots, \pi^d) \tag{2.31}$$

and writing

$$\begin{aligned} S_R^{(2)} = &\oint_{\Sigma_2} \Big(\eta_i \wedge \mathrm{d}X^i + \pi^i \wedge \mathrm{d}P_i + \frac{1}{2}\,R^{ijk}\,P_k\,\eta_i \wedge \eta_j + \eta_i \wedge \pi^i\Big) + \\ &+ \oint_{\Sigma_2} \frac{1}{2}\,g^{ij}\,\eta_i \wedge *\eta_j \ . \end{aligned} \tag{2.32}$$

The first order action (2.32) is equivalent to the string sigma-model (2.21). Note that only the momentum space components P_i of the strings have propagating degrees of freedom in T^*M. In this sense the generalized Poisson sigma-model is still topological in the original configuration space M. Moreover, the bivector field Θ defines a twisted Poisson structure on the cotangent bundle, with twisting provided by a (trivial) non-flat $U(1)$-gerbe in *momentum space*: Computing its Schouten–

Nijenhuis bracket with itself yields

$$[\Theta, \Theta]_{\rm S} = \textstyle\bigwedge^3 \Theta^\sharp(H) \ , \tag{2.33}$$

where

$$H = \tfrac{1}{6}\, R^{ijk}\, {\rm d}p_i \wedge {\rm d}p_j \wedge {\rm d}p_k \tag{2.34}$$

is a closed three-form H-flux on the cotangent bundle T^*M. A 2-connection on this gerbe is given by the B-field

$$B = \tfrac{1}{6}\, R^{ijk}\, p_k \, {\rm d}p_i \wedge {\rm d}p_j \tag{2.35}$$

with $H = {\rm d}B$, which is gauge equivalent to the topological part of the string sigma-model (2.21). In this way our membrane sigma-model (2.19) provides a geometric interpretation of the R-flux background. This gerbe description will be exploited in chapter 3 for the derivation of Seiberg-Witten maps .

The antisymmetric brackets at linear order

$$\{x^I, x^J\}_\Theta = \Theta^{IJ}(x) \tag{2.36}$$

are given explicitly by

$$\{x^i, x^j\}_\Theta = R^{ijk}\, p_k \ , \qquad \{x^i, p_j\}_\Theta = \delta^i{}_j \qquad \text{and} \qquad \{p_i, p_j\}_\Theta = 0 \ . \tag{2.37}$$

The corresponding Jacobiator is

$$\{x^I, x^J, x^K\}_\Theta := [\Theta, \Theta]_{\rm S}(x^I, x^J, x^K) = \Pi^{IJK} \ , \tag{2.38}$$

where

$$\Pi^{IJK} = \tfrac{1}{3}\, \big(\Theta^{KL}\, \partial_L \Theta^{IJ} + \Theta^{IL}\, \partial_L \Theta^{JK} + \Theta^{JL}\, \partial_L \Theta^{KI}\big) \ . \tag{2.39}$$

The only non-vanishing components of this trivector field are

$$\{x^i, x^j, x^k\}_\Theta = R^{ijk} \ . \tag{2.40}$$

The expressions (2.37) and (2.40) are precisely the nonassociative phase space commutation relations for quantized closed string coordinates which were derived in [90, 91].

Although we are mostly interested in the case of constant R^{ijk}, we can speculate on how to extend our discussion to non-constant R-flux. By local orthogonal transformations the 3-vector R can be brought into canonical form wherein its only non-vanishing components are $R^{ijk}(x) = |R(x)|^{1/3}\,\varepsilon^{ijk}$ for $i,j,k = 1,2,3$, where ε^{ijk} is the totally antisymmetric tensor and $|R(x)|$ is the determinant of the matrix $R^{iJ}(x)$, $J=(jk)$. By a suitable coordinate transformation, R^{ijk} can thus be taken to be the constant tensor ε^{ijk}. Depending on how the remaining structure functions of the Courant algebroid $C \to M$ transform, this may then yield a reduction of the membrane sigma-model (2.19) on Σ_3 to a string sigma-model on the boundary Σ_2 as before.

In any case, the sigma-model (2.32) and its associated brackets also make sense when R^{ijk} is a general function of $x \in M$, i.e. a generic trivector field on configuration space. Quantizing these brackets thus provides a means for quantizing generic Nambu–Poisson structures on M with 3-bracket determined by the trivector R. Moreover, quantization of the membrane sigma-model provides a dynamical realization of the nonassociative geometry. This quantization is explored in the following chapters.

2.4 Boundary conditions and correlation functions

It is natural to expect that the path integral for the R-twisted Courant sigma-model provides a universal quantization formula for closed strings in R-space, regarded as the boundaries of the membranes. For the H-space open membrane sigma-model of section 2.2, it is argued in [69, 70] that the path integral defines a formal quantization for the corresponding twisted Poisson structure. There, an explicit prescription is given for quantizing appropriate current algebra and L_∞ brackets of the boundary strings from correlation functions of the open topological membrane theory. For more general deformations of the exact Courant algebroid $C = TM \oplus T^*M$, the path integral is argued to provide a universal quantization formula for generic quasi-Lie bialgebras [70]. Unfortunately, the complicated nature of the BV formalism which is necessary to quantize the open topological membrane theory obstructs a complete quantization. In particular, the Courant sigma-model with R-flux involves very complicated 2-algebroid gauge symmetries; for the general Courant sigma-model the full gauge-fixed action can be found in [111], and it involves both ghost fields and ghosts-for-ghosts.

Here we would like to develop a quantization framework that is based on the induced twisted Poisson sigma-model (2.32), which involves only Lie algebroid gauge symmetries, and whose quantization on the disk is described in [32, 33]. For this, we will interpret the membrane theory as an effective theory of *open* strings with suitable boundary conditions imposed on the string embedding fields. In [69] (see also [30]) it is proposed that the boundary $\Sigma_2 = \partial\Sigma_3$ can be taken to be an open string worldsheet in the open topological membrane theory by regarding the membrane worldvolume Σ_3 as a manifold with *corners* (see e.g. [81]), and allowing for different boundary conditions on the various components of the boundary. In the following we will take another approach that is directly related to the way in which the twisted Poisson structure originates in closed string theory on the R-flux background [90, 40] (see [91] for a review). We shall argue that the corners of the membrane worldvolume can be mimicked via branch cuts on a closed surface which give the multivalued string maps responsible for the target space noncommutativity. In this way the membrane serves to provide a sort of open/closed string duality; the analogy between closed strings in non-geometric flux backgrounds and open strings was also pointed out in [90].

The setting of [90, 40, 6] is that of closed strings on the Q-space duality frame obtained by applying two T-duality transformations to the three-torus $M = \mathbb{T}^3$

with constant NS–NS three-form flux $H = h\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. Locally, this space is a fibration of a two-torus $\mathbb{T}^2$ over a circle S^1; globally it is not well-defined as a Riemannian manifold and is the simplest example of a *T-fold* [73]. A representative class of twisted torus fibrations are provided by elliptic T-folds where the monodromies act on the fibre coordinates as rotations. The closed string worldsheet is the cylinder $\mathcal{C} = \mathbb{R} \times S^1$ with coordinates (σ^0, σ^1). The embedding field corresponding to the base direction is denoted X^3, while for the fibre directions we use complex fields denoted $Z, \overline{Z} = \frac{1}{\sqrt{2}}\,(X^1 \pm \mathrm{i}\, X^2)$. As an extended closed string wraps w^3 times around the base of the fibration, the fibre directions need only close up to a monodromy corresponding to an $SL(2,\mathbb{Z})$ automorphism of the $\mathbb{T}^2$-fibre. One thus arrives at the *twisted* boundary conditions

$$\begin{aligned} Z(\sigma^0, \sigma^1 + 2\pi) &= \mathrm{e}^{\,2\pi\,\mathrm{i}\,\theta}\, Z(\sigma^0, \sigma^1) \qquad \text{and} \\ X^3(\sigma^0, \sigma^1 + 2\pi) &= X^3(\sigma^0, \sigma^1) + 2\pi\, w^3 \ , \end{aligned} \tag{2.41}$$

where $\theta = -h\, w^3$; more precisely, one should impose asymmetric boundary conditions for the left- and right-moving fields in the fibre directions. To linear order in the flux, one can solve the equations of motion of the closed string worldsheet sigma-model in the usual way via oscillator mode expansions for the fibre coordinate fields subject to the twisted boundary conditions (2.41). As we explained in chapter 1, by standard canonical the fibre directions acquire a noncommutative deformation determined by the H-flux and the winding number (or T-dual Kaluza–Klein momentum) w^3 in the S^1-direction, in exactly the same way in which open string boundaries are deformed in the presence of a B-field. Written in terms of a real parametrization, we may express this closed string noncommutativity generally in the Q-flux background via the Poisson brackets

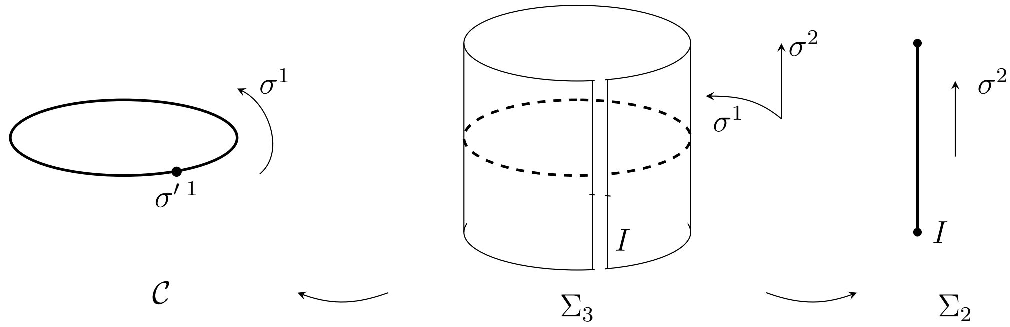
$$\{x^i, x^j\}_Q = Q^{ij}{}_k\, w^k \qquad \text{and} \qquad \{x^i, w^j\}_Q = 0 = \{w^i, w^j\}_Q \ , \tag{2.42}$$

with constant flux $Q^{ij}{}_k = -2\pi\, h\, \varepsilon^{ij}{}_k$. These brackets define a *bona fide* Poisson structure, since they are just the relations of a Heisenberg algebra, as in the defining

2-brackets of the corresponding Courant algebroid. A T-duality transformation to the R-flux background sends $Q^{ij}{}_k \mapsto R^{ijk}$ and $w^k \mapsto p_k$, and maps the Poisson brackets (2.42) to the twisted Poisson structure (2.37). This change of duality frame will be useful for some of our later considerations.

The interplay between open and closed string interpretations of the noncommutative and nonassociative flux backgrounds was noted in [28, 51]. A step towards understanding the pertinent picture was recently carried out in the context of matrix theory compactifications on twisted tori in [36], which constructs solutions with noncommutative and nonassociative cotangent bundles. To explicitly relate the closed and open string pictures, we use the fact that the boundary conditions (2.41) define a twisted sector of an orbifold conformal field theory on the quotient of $M = \mathbb{T}^3$ by the free action of a discrete abelian monodromy group; they describe closed strings on the orbifold, which can be regarded as open strings on the covering space M. When computing conformal field theory correlation functions, the monodromy can be implemented by inserting a suitable twist field at a point $\sigma'^{\,1} \in S^1$ which creates a branch cut along the temporal direction $\mathbb{R}$ for the multivalued closed string fields. We now extend the worldsheet $\mathcal{C} = \mathbb{R} \times S^1$ to the membrane worldvolume $\Sigma_3 = \mathbb{R} \times (S^1 \times \mathbb{R})$ with coordinates $(\sigma^0, \sigma^1, \sigma^2)$ such that the branch point at $\sigma'^{\,1} \in S^1$ is blown up to a branch cut $I = \{\sigma'^{\,1}\} \times \mathbb{R} \subset S^1 \times \mathbb{R}$ extended along the σ^2-direction, i.e. the branch cut on the closed string worldsheet is blown up to a “branch surface” on a closed membrane worldvolume. The membrane fields are also taken to be multivalued and non-differentiable across the branch cut I; hence Stokes’ theorem on Σ_3 receives contributions from the multivalued fields across the cut whenever integration by parts is used to reduce worldvolume integrals, as we did in section 2.3. This effectively reduces the membrane to an “open string” with worldsheet $\Sigma_2 := \partial\Sigma_3 = \mathbb{R} \times I$ and coordinates (σ^0, σ^2); classically, the mapping $\Sigma_3 \to \Sigma_2$ is a simple application of Stokes’ theorem on the equations of motion $\xi_i = \mathrm{d}P_i$. In this way the branch cut I plays the role of a “corner” separating Σ_3 into regions. The more general orbifolds of [40] can be treated in an analogous way.

We can depict these two reductions from the membrane theory to the closed and open string theories via the following schematic diagram:



We do not display the temporal direction as it plays no role. The mapping $\Sigma_3 \to \mathcal{C}$ is obtained by restriction of the domain of the membrane path integral to fields which are independent of σ^2. This gives a dimensional reduction of the membrane fields to closed string fields which is reminiscent of the Kaluza–Klein reduction of M-theory to Type IIA string theory. The mapping $\Sigma_3 \to \Sigma_2$ is a restriction of field variables in the membrane path integral to the cut I of the spatial membrane cylinder. By reparametrization of the membrane worldvolume, it also defines a map to the disk Σ_2 viewed as the complex upper half-plane with boundary the real line $\mathbb{R}$, where the endpoints of the cut at $\pm\infty$ are mapped to finite values. These two restrictions of the field domain in the membrane path integral define the open/closed string duality that we were after; in a certain sense it represents a sort of transmutation between D-branes and fluxes. It is somewhat in line with the recent analysis of [44] which demonstrates how non-geometric doubled space coordinates arise as solutions to Neumann boundary conditions in open string theory on flux backgrounds. Note that in order to ensure independence of the specific location of the branch cut I, it is important to assume that the R-flux is constant. However, this restriction is no longer needed after we take the $2+1$-dimensional Courant sigma-model as the fundamental model for closed strings in the R-flux background.

Considering that the endpoints are at $\pm\infty$, it is natural to choose the boundary conditions for the open string on the cut I to coincide with those of [32]. In this sense, the twisted boundary conditions (2.41) on Q-space can be made compatible with the Cattaneo–Felder boundary conditions for the open twisted Poisson sigma-model. In the following we will take the topological limit of (2.32) where $g \ll R$; this essentially decouples the open string modes from the closed string modes. Then the propagator of the topological sigma model is given by

$$\big\langle X^I(w)\,\eta_J(z)\big\rangle = \tfrac{\mathrm{i}\,\hbar}{2\pi}\,\delta^I{}_J\,\mathrm{d}_z\phi^h(z,w)\ , \tag{2.43}$$

where $\hbar$ is a formal expansion parameter, the harmonic angle function

$$\phi^h(z,w) := \frac{1}{2\,\mathrm{i}}\,\log\frac{(z-w)\,(z-\overline{w}\,)}{(\,\overline{z}-w)\,(\,\overline{z}-\overline{w}\,)} \tag{2.44}$$

for $z,w\in\mathbb{C}$ is the Green's function for the Laplacian on the disk with Neumann boundary conditions, and $\mathrm{d}_z := \mathrm{d}z\,\frac{\partial}{\partial z}+\mathrm{d}\overline{z}\,\frac{\partial}{\partial\overline{z}}$. In this case, the Feynman diagram expansion of suitable observables in the sigma-model reproduces Kontsevich's graphical expansion for global deformation quantization of our twisted Poisson structure [89, 32, 33, 34], which we will take as our proposal for the quantization of the R-flux background. In chapter 3 we shall compute the following schematic functional integrals, whose precise meaning will be explained later on and whose precise definitions can be found in [32, 33, 34]. For $x\in T^*M$, functions $f_i\in C^\infty(T^*M)$, and a collection of $n\geq 1$ multivector fields $\mathcal{X}_r=\frac{1}{k_r!}\,\mathcal{X}_r^{I_1\dots I_r}(x)\,\partial_{I_1}\wedge\cdots\wedge\partial_{I_{k_r}}\in C^\infty(T^*M,\bigwedge^{k_r}T(T^*M))$ of degree k_r, define

$$U_n(\mathcal{X}_1,\dots,\mathcal{X}_n)(f_1,\dots,f_m)(x) = \int\ \mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\,S_R^{(2)}}\ \frac{\mathrm{i}}{\hbar}\,S_{\mathcal{X}_1}\cdots\frac{\mathrm{i}}{\hbar}\,S_{\mathcal{X}_n}\,\mathcal{O}_x(f_1,\dots,f_m)\ , \tag{2.45}$$

where $m=2-2n+\sum_r k_r$, $S_{\mathcal{X}_r}=\oint_{\Sigma_2}\frac{1}{k_r!}\,\mathcal{X}_r^{I_1\dots I_r}(X)\,\eta_{I_1}\cdots\eta_{I_r}$, and the boundary observables $\mathcal{O}_x(f_1,\dots,f_m)$ are given by

$$\mathcal{O}_x(f_1,\dots,f_m) = \int_{X(\infty)=x}\Big[f_1\big(X(q_1)\big)\cdots f_m\big(X(q_m)\big)\Big]^{(m-2)} \tag{2.46}$$

with $1=q_1>q_2>\cdots>q_m=0$ and ∞ distinct points on the boundary of the disk $\partial\Sigma_2$. The path integrals are weighted with the full gauge-fixed action and the integrations taken over all fields including ghosts. In particular, for functions $f,g\in C^\infty(T^*M)$ one may define a star product by the functional integral

$$(f\star g)(x) = \int_{X(\infty)=x} f\big(X(1)\big)\,g\big(X(0)\big)\ \mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\,S_R^{(2)}}\ , \tag{2.47}$$

whose properties will be thoroughly investigated in what follows.

2.5 Twisted and higher Poisson structures

We close this chapter with some general remarks about the twisted Poisson structures we have derived, which will serve to help understand some of the higher structures that will arise in chapter 4. Consider the algebra $\mathcal{V}^\sharp = C^\infty(T^*M, \bigwedge^\sharp T(T^*M))$ of multivector fields on the cotangent bundle of the target space M. Let $H = \frac{1}{6} H_{IJK}(x)\, \mathrm{d}x^I \wedge \mathrm{d}x^J \wedge \mathrm{d}x^K$ be the closed three-form (2.34) on T^*M; it extends by the Leibniz rule to give a ternary bracket $[-,-,-]_H$ on $\mathcal{V}^\sharp$ of degree 1. Together with the Schouten–Nijenhuis bracket $[-,-]_{\mathrm{S}}$, it defines an L_∞-structure on $\mathcal{V}^\sharp$ with zero differential, generalizing the canonical differential graded Lie algebra structure in the case of vanishing R-flux (see appendix A for the relevant definitions and background material). On the subspace $C^\infty(T^*M)$ of smooth functions on T^*M, the H-twisted Poisson structure (2.30) naturally defines a 2-term L_∞-algebra $\big(V_1 \xrightarrow{\mathsf{d}} V_0\big)$. Here $V_1 = C^\infty(T^*M)$ and V_0 is the space of vector fields $\mathcal{X} \in C^\infty(T^*M, T(T^*M))$ which preserve Θ in the sense that $\mathcal{L}_{\mathcal{X}}\Theta = 0$, where $\mathcal{L}_{\mathcal{X}}$ is the Lie derivative along $\mathcal{X}$. The map $\mathsf{d} = \mathrm{d}_\Theta = -[-,\Theta]_{\mathrm{S}}$ is the Lichnerowicz differential which sends a function $f \in C^\infty(T^*M)$ to its Hamiltonian vector field $\mathcal{X}_f = \Theta(\mathrm{d}f, -)$ [110]. The derived bracket (A.13) on V_1 is just the quasi-Poisson bracket on $C^\infty(T^*M)$ determined by Θ as

$$\{f,g\}_\Theta := [\mathsf{d}f, g]_{\mathrm{S}} = \Theta(\mathrm{d}f, \mathrm{d}g)\ . \tag{2.48}$$

The associated Jacobiator (A.14) can be written as

$$\{f,g,h\}_\Theta = H(\mathcal{X}_f, \mathcal{X}_g, \mathcal{X}_h)\ . \tag{2.49}$$

Note that here the differential d is not nilpotent, and the right-hand side of (2.49) can be expressed in terms of $\mathsf{d}^2 \neq 0$; this is reminiscent of a covariant derivative that does not square to zero when the curvature is non-zero.

The corresponding commutation relations in the associated semistrict Lie 2-algebra $\mathscr{V}$ are (see appendix A.1)

$$\begin{aligned} [\mathcal{X},\mathcal{Y}]_{\mathscr{V}} &= [\mathcal{X},\mathcal{Y}]_{T(T^*M)} \ , \\ \big[(\mathcal{X},f)\,,\,(\mathcal{Y},g)\big]_{\mathscr{V}} &= \big([\mathcal{X},\mathcal{Y}]_{T(T^*M)}\,,\,\mathcal{X}(g)-\mathcal{Y}(f)+\{f,g\}_\Theta\big) \ , \end{aligned} \tag{2.50}$$

while the Jacobiator is

$$[\mathcal{X},\mathcal{Y},\mathcal{Z}]_{\mathscr{V}} = \big(\big[[\mathcal{X},\mathcal{Y}]_{T(T^*M)},\mathcal{Z}\big]_{T(T^*M)}\,,\,H(\mathcal{X},\mathcal{Y},\mathcal{Z})\big) \tag{2.51}$$

for $\mathcal{X},\mathcal{Y},\mathcal{Z}\in C^\infty(T^*M,T(T^*M))$ and $f,g\in C^\infty(T^*M)$. At linear order, denoting the generators $(\partial_I,0)$ and $(0,x^I)$ by $\boldsymbol{p}_I$ and $\boldsymbol{x}^I$ for simplicity, we have

$$[\boldsymbol{p}_I,\boldsymbol{p}_J]_{\mathscr{V}}=0 \ , \qquad [\boldsymbol{p}_I,\boldsymbol{x}^J]_{\mathscr{V}}=\delta_I{}^J \qquad \text{and} \qquad [\boldsymbol{x}^I,\boldsymbol{x}^J]_{\mathscr{V}}=\Theta^{IJ} \tag{2.52}$$

together with

$$[\boldsymbol{p}_I,\boldsymbol{p}_J,\boldsymbol{p}_K]_{\mathscr{V}}=H_{IJK} \ . \tag{2.53}$$

In the following we will quantize this Lie 2-algebra.

As a side observation, it is intriguing to note that the twisted Poisson brackets (2.37) on the phase space T^*M have an alternative interpretation as a higher Poisson structure on the configuration space M (see appendix A.5). In this setting we regard the momenta p_i as the degree 0 generators ∂_i of the multivector field algebra $V^\sharp=C^\infty(M,\bigwedge^\sharp TM)$. We take a degree 3 multivector field $R=R^{ijk}\,\partial_i\wedge\partial_j\wedge\partial_k$, where R^{ijk} is a constant R-flux on M. Then the non-trivial derived brackets (A.31) of R are generated by

$$\begin{aligned} \mathsf{d}x^i \ := \ \{x^i\}_R &= R^{ijk}\,\partial_j\wedge\partial_k \ , \\ \{x^i,x^j\}_R &= R^{ijk}\,\partial_k \ , \\ \{x^i,x^j,x^k\}_R &= R^{ijk} \ , \end{aligned} \tag{2.54}$$

with all other brackets vanishing at linear order in x^i and ∂_i. These higher Poisson brackets define a 2-term L_∞-algebra structure on $V^\sharp$.

Chapter 3

Global deformation quantization

As discussed in section 2.4, a suitable perturbation expansion of the membrane/string sigma-model of chapter 2 motivates an approach to the quantum geometry of the R-flux background based on deformation quantization. In [89], Kontsevich constructs a deformation quantization of an arbitrary Poisson structure, based on a graphical calculus which is reproduced by the Feynman diagram expansion of the open Poisson sigma-model on a disk [32]. In this chapter we shall follow this prescription to derive a *nonassociative* star product deformation of the usual pointwise product of functions on $\mathcal{M} = T^*M$ along the direction of a generic twisted Poisson bivector Θ, and describe its derivation properties. We then restrict to the case of constant R-flux where we derive an explicit closed formula for the star product and its associator, giving a quantization of the 2-brackets (2.37) and the 3-brackets (2.40) respectively. We apply this formalism to derive Seiberg–Witten maps relating nonassociative and associative deformations, and also add fluctuations to the R-flux background. We further explain how the 3-product proposed in [21] fits into our formalism.

3.1 Formality map and star products

Kontsevich's formalism relies on the construction of the *formality map*. The formality map is a sequence of L_∞-morphisms U_n, $n \in \mathbb{Z}_{\geq 0}$ that map tensor products of n multivector fields to m-differential operators on the manifold $\mathcal{M}$. It defines an L_∞-quasi-isomorphism between the differential graded Lie algebra of multivector fields equipped with zero differential and the Schouten–Nijenhuis bracket (see appendix A.3), and the differential graded Lie algebra of multidifferential operators equipped with the Hochschild differential and the Gerstenhaber bracket (see appendix A.2). Consider a collection of multivectors $\mathcal{X}_i$ of degree k_i for $i = 1, \dots, n$. Then $U_n(\mathcal{X}_1, \dots, \mathcal{X}_n)$ is a multidifferential operator whose degree m is determined by the relation

$$m = 2 - 2n + \sum_{i=1}^{n} k_i \ . \tag{3.1}$$

In particular, U_0 yields the usual pointwise product of functions while U_1 is the Hochschild–Kostant–Rosenberg map which takes a k-vector field to a k-differential operator defined by

$$U_1\big(\mathcal{X}^{I_1 \dots I_k}\, \partial_{I_1} \wedge \cdots \wedge \partial_{I_k}\big)(f_1, \dots, f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \mathcal{X}^{I_{\sigma(1)} \dots I_{\sigma(k)}}\, \partial_{I_{\sigma(1)}} f_1 \cdots \partial_{I_{\sigma(k)}} f_k \tag{3.2}$$

for $f_i \in C^\infty(\mathcal{M})$. When the multivector fields $\mathcal{X}_i$ are all set equal to the bivector Θ, the star product of functions $f, g \in C^\infty(\mathcal{M})$ is given by the formal power series

$$f \star g := \sum_{n=0}^{\infty} \frac{(\mathrm{i}\, \hbar)^n}{n!}\, U_n(\Theta, \dots, \Theta)(f, g) \equiv \Phi(\Theta)(f, g) \ , \tag{3.3}$$

where $\hbar$ is a formal deformation parameter and $U_n(\Theta, \dots, \Theta)$ is a bidifferential operator by (3.1).

Kontsevich introduced a convenient diagrammatic representation on the upper hyperbolic half-plane $\mathbb{H}$ that provides all possible (admissible) differential operators to each order of the expansion (3.3), and thus determines the formality map U_n. Kontsevich diagrams encode the rules for contracting indices and positioning partial derivatives. Each diagram Γ consists of:

1. Edges e that are geodesics in $\mathbb{H}$ and represent partial derivatives;
2. A set $q_1, \dots, q_m \in \mathbb{R}$ of *grounded* vertices that represent functions; and
3. A set $p_1, \dots, p_n \in \mathbb{H} \setminus \mathbb{R}$ of *aerial* vertices that represent the k_i-vector fields $\mathcal{X}_i$, and thus k_i edges may emanate from them.

Here the real line $\mathbb{R}$ is the boundary of $\mathbb{H}$. An edge emanating from a given point p_i is labelled as $e_i^{k_i}$. Edges that start from a vertex v can land on any other vertex apart from v, while the condition $2n+m-2 \geq 0$ must be satisfied. The multidifferential operator

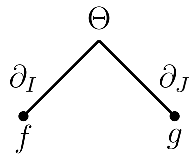
$$U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) := \sum_{\Gamma \in G_n} w_\Gamma \, D_\Gamma(\mathcal{X}_1, \ldots, \mathcal{X}_n) \tag{3.4}$$

is calculated by summing over operators $D_\Gamma(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ in the class G_n of all n-th order admissible diagrams Γ, each contributing with weight w_Γ given by the integral [87]

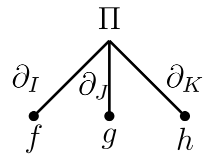
$$w_\Gamma = \frac{1}{(2\pi)^{2n+m-2}} \int_{\mathbb{H}_n} \bigwedge_{i=1}^{n} \left(\mathrm{d}\phi^h_{e_i^1} \wedge \cdots \wedge \mathrm{d}\phi^h_{e_i^{k_i}} \right) , \tag{3.5}$$

where $\mathbb{H}_n$ is the space of pairwise distinct points $p_i \in \mathbb{H}$ and the role of the harmonic angles $\phi^h_{e_i^{k_i}}$ is explained in appendix B.

In this setting, the diagrams for the bivector (quasi-Poisson bracket) $\Theta(f,g) = \frac{1}{2}\,\Theta^{IJ}\,\partial_I f\,\partial_J g$ and the trivector (Jacobiator) $\Pi(f,g,h) = \frac{1}{3}\,\Pi^{IJK}\,\partial_I f\,\partial_J g\,\partial_K h$ contributions are



and



which we will call the wedge and triple wedge respectively. The geodesics here have been drawn as straight lines for the sake of clarity. Computing (3.3) then provides the nonassociative star product deformation along the quasi-Poisson structure Θ.

Kontsevich's construction allows for one or more multivectors to be inserted in $U_n(\Theta, \ldots, \Theta)$. In our case, inserting the trivector $\Pi = [\Theta, \Theta]_{\mathrm{S}}$ that we acquired from the Schouten–Nijenhuis bracket is of particular interest since it encodes the nonassociativity of our star product. Then on functions $f, g, h \in C^\infty(\mathcal{M})$ the series (3.3) is replaced with

$$[f,g,h]_\star := \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!}\, U_{n+1}(\Pi, \Theta, \ldots, \Theta)(f,g,h) \equiv \Phi(\Pi)(f,g,h) \tag{3.6}$$

which, as we show in section 3.2, is the associator for the star product (3.3). The condition (3.1) now implies that $U_{n+1}(\Pi, \Theta, \ldots, \Theta)$ is a tridifferential operator. The map U_{n+1} is calculated as in (3.4); this time though the integrations for the diagrams that give the associated weights (3.5) are much more involved since the edges of the triple wedge can land on any other wedge. Restricting to constant R-flux R^{ijk} cures this problem, making the derivation of an explicit expression possible; this will be analysed in section 3.3.

3.2 Derivation properties and the associator

In order to define L_∞-morphisms, the maps U_n must satisfy for $n \geq 1$ the *formality conditions* [89, 94, 85, 119]

$$\begin{aligned} \mathrm{d}_{\mu_2} U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) + \frac{1}{2} \sum_{\substack{\mathcal{I} \sqcup \mathcal{J} = (1,\ldots,n) \\ \mathcal{I},\mathcal{J} \neq \emptyset}} \varepsilon_{\mathcal{X}}(\mathcal{I}, \mathcal{J}) \left[U_{|\mathcal{I}|}(\mathcal{X}_{\mathcal{I}})\,,\, U_{|\mathcal{J}|}(\mathcal{X}_{\mathcal{J}}) \right]_{\mathrm{G}} \\ = \sum_{i<j} (-1)^{\alpha_{ij}}\, U_{n-1}\big([\mathcal{X}_i, \mathcal{X}_j]_{\mathrm{S}}, \mathcal{X}_1, \ldots, \widehat{\mathcal{X}_i}, \ldots, \widehat{\mathcal{X}_j}, \ldots, \mathcal{X}_n \big) \ , \end{aligned} \tag{3.7}$$

where $\mathrm{d}_{\mu_2} U_n := -[U_n, \mu_2]_{\mathrm{G}}$ with $\mu_n : C^\infty(\mathcal{M})^{\otimes n} \to C^\infty(\mathcal{M})$ the usual commutative and associative pointwise product of n functions, $[-,-]_{\mathrm{G}}$ denotes the Gerstenhaber bracket defined in appendix A.2, and for a multi-index $\mathcal{I} = (i_1, \ldots, i_k)$ we denote $\mathcal{X}_{\mathcal{I}} := \mathcal{X}_{i_1} \wedge \cdots \wedge \mathcal{X}_{i_k}$ and $|\mathcal{I}| := k$; the sign factor $\varepsilon_{\mathcal{X}}(\mathcal{I}, \mathcal{J})$ is the "Quillen sign" associated with the partition $(\mathcal{I}, \mathcal{J})$ of the integer n, $(-1)^{\alpha_{ij}}$ is a prescribed sign rule arising from the L_∞-structure (see appendix A.1), and the hats denote omitted multivectors. Formality follows from the Ward–Takahashi identities for the Lie algebroid gauge symmetry of the Poisson sigma-model in the BV formalism. In our case of interest, the conditions (3.7) reduce to

$$\mathrm{d}_\star \Phi(\Theta) = \mathrm{i}\, \hbar\, \Phi(\mathrm{d}_\Theta \Theta) \ , \tag{3.8}$$

where the coboundary operators are $\mathrm{d}_\star = -[-,\star]_{\mathrm{G}}$ and $\mathrm{d}_\Theta = -[-,\Theta]_{\mathrm{S}}$ with $\mathrm{d}_\Theta\Theta = \Pi$. Using (3.8) and (A.15) we derive a formula for the associator (3.6) given by

$$[f,g,h]_\star = \tfrac{2\,\mathrm{i}}{\hbar}\,\big((f\star g)\star h - f\star(g\star h)\big)\ , \tag{3.9}$$

which is non-zero since the product $\star$ is not associative. This formula provides an exact formal expression which can be calculated up to any order in the deformation parameter $\hbar$ using Kontsevich diagrams.

The formality conditions give rise to derivation properties. Using (3.6) we can define a new function $\underline{f}$ for every function f by [85, 119]

$$\underline{f} = f + \frac{(\mathrm{i}\,\hbar)^2}{2}\,U_3(f,\Theta,\Theta) + \sum_{n=3}^{\infty}\frac{(\mathrm{i}\,\hbar)^n}{n!}\,U_{n+1}(f,\Theta,\dots,\Theta)\ . \tag{3.10}$$

The formality condition is then $\mathrm{d}_\star\,\underline{f} = \mathrm{i}\,\hbar\,\Phi(\mathrm{d}_\Theta f)$, which tells us that the Hamiltonian vector field $\mathrm{d}_\Theta f$ is mapped to the inner derivation

$$\mathrm{d}_\star\,\underline{f} = \tfrac{\mathrm{i}}{\hbar}\,[\,\underline{f}\,,-]_\star\ , \tag{3.11}$$

where $[f,g]_\star := f\star g - g\star f$ is the star commutator of functions $f,g\in C^\infty(\mathcal{M})$. Similarly, a new vector field $\underline{\mathcal{X}}$ for any vector field $\mathcal{X}$ is defined by

$$\underline{\mathcal{X}} = \mathcal{X} + \frac{(\mathrm{i}\,\hbar)^2}{2}\,U_3(\mathcal{X},\Theta,\Theta) + \sum_{n=3}^{\infty}\frac{(\mathrm{i}\,\hbar)^n}{n!}\,U_{n+1}(\mathcal{X},\Theta,\dots,\Theta)\ . \tag{3.12}$$

The formality condition is now $\mathrm{d}_\star\,\underline{\mathcal{X}} = \mathrm{i}\,\hbar\,\Phi\,(\mathrm{d}_\Theta\mathcal{X})$. d_Θ-closed vector fields $\mathcal{X}$ preserve the twisted Poisson structure, i.e. $\mathrm{d}_\Theta\mathcal{X} = 0$. The formality condition then implies the derivation property

$$\underline{\mathcal{X}}(f\star g) = \underline{\mathcal{X}}(f)\star g + f\star\underline{\mathcal{X}}(g) \tag{3.13}$$

for $f,g\in C^\infty(\mathcal{M})$.

Finally, we consider the formality condition

$$\mathrm{d}_\star \Phi(\Pi) = \mathrm{i}\,\hbar\, \Phi(\mathrm{d}_\Theta \Pi) \ . \tag{3.14}$$

Using (3.6), we can express the left-hand side of this expression as

$$\begin{aligned} \mathrm{d}_\star \Phi(\Pi)(f,g,h,k) =& f \star [g,h,k]_\star - [f\star g,h,k]_\star + \\ &+ [f,g\star h,k]_\star - [f,g,h\star k]_\star + [f,g,h]_\star \star k \ , \end{aligned} \tag{3.15}$$

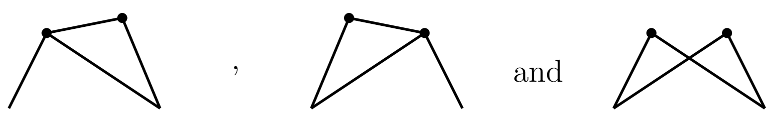
while the Schouten–Nijenhuis bracket on the right-hand side is

$$\begin{aligned} \mathrm{d}_\Theta \Pi \ := \ [\Pi,\Theta]_\mathrm{S} =& \tfrac{1}{24}\big(\Theta^{LM}\,\partial_M \Pi^{IJK} - \Theta^{IM}\,\partial_M\Pi^{JKL} + \Theta^{JM}\,\partial_M\Pi^{KLI} - \\ &- \Theta^{KM}\,\partial_M\Pi^{LIJ} + \Pi^{IJM}\,\partial_M\Theta^{KL} - \Pi^{JKM}\,\partial_M\Theta^{LI} + \\ &+ \Pi^{KLM}\,\partial_M\Theta^{IJ} - \Pi^{LIM}\,\partial_M\Theta^{JK} - \Pi^{IKM}\,\partial_M\Theta^{JL} + \\ &+ \Pi^{JLM}\,\partial_M\Theta^{KI}\big)\, \partial_I\wedge\partial_J\wedge\partial_K\wedge\partial_L \ . \end{aligned} \tag{3.16}$$

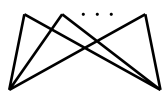
Then (3.14) relates these two expressions and gives the derivation property for $f,g,h,k \in C^\infty(\mathcal{M})$.

3.3 Nonassociative star product for constant flux

We now turn our attention to the case of constant R^{ijk} considered in chapter 2 and calculate the products we found in section 3.1 explicitly. Let us begin by computing (3.3). The zero order diagram is the usual pointwise multiplication. There is only one admissible first order diagram (the wedge) whose weight is found to be $\frac{1}{2}$ in appendix B, hence $U_1(\Theta)(f,g) = \frac{1}{2}\,\Theta^{IJ}\,\partial_I f\,\partial_J g$. The admissible second order diagrams are



The first one represents $\Theta^{KL}\,\partial_L\Theta^{IJ}\,\partial_I f\,\partial_K\partial_J g = R^{ijk}\,\partial_i f\,\partial_j\partial_k g = 0$ due to antisymmetry of R^{ijk}. The second diagram also vanishes for the same reason. Consequently, all higher order diagrams that contain these two sub-diagrams are equal to zero. The third diagram is simply the product of two wedges, therefore its weight is $\frac{1}{4}$. Hence $U_2(\Theta,\Theta)(f,g) = \frac{1}{4}\,\Theta^{IJ}\,\Theta^{KL}\,\partial_I\partial_K f\,\partial_J\partial_L g$. Since wedges that land on wedges do not contribute to $U_n(\Theta,\dots,\Theta)$, there is only one admissible diagram to each order of the form

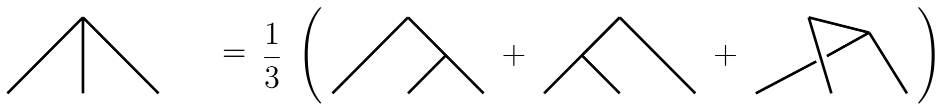


Hence the star product for the constant R-flux background is given by the Moyal type formula

$$f \star g = \mu_2\Big(\exp\big(\tfrac{\mathrm{i}\hbar}{2}\,\Theta^{IJ}\,\partial_I\otimes\partial_J\big)(f\otimes g)\Big)\ , \tag{3.17}$$

where as before μ_2 is the pointwise multiplication map of functions.

The associator (3.6) for constant R^{ijk} can be computed either by calculating Kontsevich diagrams and summing the series or by using the star product (3.17) to compute the left-hand side of (3.9). Here we will follow the second approach, but before doing so it is instructive to calculate diagrams up to third order. A method for calculating Kontsevich diagrams involving two functions for linear Poisson structures was developed in [87]; however we have found this setting unsuitable for calculations involving more than two grounded vertices and so we calculate diagrams in the usual manner. The lowest order admissible diagram is the triple wedge, whose weight is $\frac{1}{6}$ (see appendix B), thus $U_1(\Pi) = \frac{1}{6}\,\Pi^{IJK}\,\partial_I f\,\partial_J g\,\partial_K h$. In $U_2(\Pi,\Theta)$ a wedge is added, but since R^{ijk} is constant, diagrams where the wedge lands on the trivector Π are zero; thus all non-zero diagrams have weight $\frac{1}{12}$. Third order is more interesting as we now have two wedges that may land on each other. These diagrams are non-zero since the remaining edges all land on different functions. Calculating their weights (see appendix B) we find that they combine to a trivector diagram according to the formula



which when written out explicitly reproduces the formula (2.39) for the Schouten–

Nijenhuis bracket $[\Theta,\Theta]_{\rm S}$ with constant R^{ijk}.

To calculate the associator (3.9) explicitly to all orders, we first observe that due to antisymmetry of R^{ijk} the star product (3.17) factorizes as

$$f\star g=\mu_2\Big(\exp\big(\tfrac{{\rm i}\,\hbar}{2}\,R^{ijk}\,p_k\,\partial_i\otimes\partial_j\big)\,\exp\big[\tfrac{{\rm i}\,\hbar}{2}\,\big(\partial_i\otimes\tilde\partial^i-\tilde\partial^i\otimes\partial_i\big)\big](f\otimes g)\Big)=:f\star_p g\ , \tag{3.18}$$

where as before we write $\partial_i=\frac{\partial}{\partial x^i}$ and $\tilde\partial^i=\frac{\partial}{\partial p_i}$. Here we denote the nonassociative product $\star:=\star_p$ where p is the dynamical momentum variable. By replacing the dynamical variable p with a constant $\bar p$ we obtain the associative Moyal star product $\bar\star\ :=\star_{\bar p}$. Nonassociativity arises because $\star$ acts non-trivially on the p-dependence of $\star:=\star_p$ in the associator. Applying this on $(f\star g)\star h$ and $f\star(g\star h)$ using antisymmetry of R^{ijk} we find

$$\begin{aligned}(f\star g)\star h\ &:=\ (f\star_p g)\star_p h\\ &=\Big[\,\bar\star\,\Big(\exp\big(\tfrac{\hbar^2}{4}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\big)(f\otimes g\otimes h)\Big)\Big]_{\bar p\to p}\\ f\star(g\star h)\ &:=\ f\star_p(g\star_p h)\\ &=\Big[\,\bar\star\,\Big(\exp\big(-\tfrac{\hbar^2}{4}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\big)(f\otimes g\otimes h)\Big)\Big]_{\bar p\to p}\ ,\end{aligned} \tag{3.19}$$

where no ordering is required on the right-hand sides due to associativity of the Moyal product and the operation $[-]_{\bar p\to p}$ reinstates the dynamical momentum dependence. Using (3.9) we therefore find

$$[f,g,h]_\star=\frac{4\,{\rm i}}{\hbar}\,\Big[\,\bar\star\,\Big(\sinh\big(\tfrac{\hbar^2}{4}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\big)(f\otimes g\otimes h)\Big)\Big]_{\bar p\to p}\ . \tag{3.20}$$

This manner of regarding our nonassociative products is consistent with the observation of section 2.3 that only the momentum directions in the membrane sigma-model are dynamical on $\mathcal{M}$. From (3.20) it follows that our nonassociative star product is cyclic, i.e. the associator is a total derivative and for Schwartz functions $f,g,h\in C^\infty(\mathcal{M})$ we have

$$\int_{\mathcal{M}}{\rm d}^{2d}x\ [f,g,h]_\star(x)=0\ . \tag{3.21}$$

The cyclic property (3.21) also holds for the nonassociative star products derived from open string amplitudes in curved backgrounds [63, 65] (see also [51]).

We conclude by writing out the derivation property (3.15) explicitly for constant R-flux. The Schouten–Nijenhuis bracket (3.16) now vanishes, and therefore (3.15) reduces to

$$[f\star g,h,k]_\star-[f,g\star h,k]_\star+[f,g,h\star k]_\star=f\star[g,h,k]_\star+[f,g,h]_\star\star k \tag{3.22}$$

for four functions f, g, h, and k, while the remaining derivation properties we found in section 3.2 remain unaffected. We can interpret (3.22) in the following way. Just as the star commutator provides a quantization of the twisted Poisson structure defined by the antisymmetric 2-bracket $\{f,g\}_\Theta:=\Theta(\mathrm{d}f,\mathrm{d}g)$, in the sense that $[f,g]_\star=2\,\mathrm{i}\,\hbar\,\{f,g\}_\Theta+\mathcal{O}(\hbar^2)$, the associator (3.20) defines a quantization of the Nambu–Poisson structure (see appendix A.4) defined by the completely antisymmetric 3-bracket

$$\{f,g,h\}_\Theta:=\Pi(\mathrm{d}f,\mathrm{d}g,\mathrm{d}h)\ , \tag{3.23}$$

in the sense that

$$[f,g,h]_\star=6\,\mathrm{i}\,\hbar\,\{f,g,h\}_\Theta+\mathcal{O}\big(\hbar^2\big)\ . \tag{3.24}$$

To this order, the star derivation property (3.22) is just a consequence of the usual Leibniz rule

$$\{f\,g,h,k\}_\Theta=f\,\{g,h,k\}_\Theta+\{f,h,k\}_\Theta\,g \tag{3.25}$$

for the Nambu–Poisson bracket (3.23). However, it is not clear whether higher derivation properties encode the fundamental identity. We will return to this issue in chapter 4 where we will see that the equation (3.22) can also be interpreted as the star product version of the pentagon identity (A.42) for the Lie 2-group that we encounter there.

3.4 Seiberg–Witten maps

We will now apply the formalism of this chapter to analyse the effect of adding fluctuations to the membrane boundary and the open string endpoints. To start, we shall recall a few relevant facts of the open string case with an ordinary Poisson structure. We will then show how this generalizes to the case of H-twisted Poisson structures and ultimately to the membrane setting with R-flux.

By studying open strings in a closed string background, Seiberg and Witten [115] found equivalent effective descriptions in terms of ordinary as well as noncommutative gauge theories. Realizing this, they proposed the existence of maps $\hat{\mathcal{A}}(a)$ and $\hat{\Lambda}(\lambda, a)$ from an ordinary gauge potential a_μ and gauge parameter λ to their noncommutative cousins $\hat{\mathcal{A}}_\mu$ and $\hat{\Lambda}$, such that an ordinary infinitesimal gauge transformation $\delta_\lambda a_\mu = \partial_\mu \lambda$ induces its noncommutative analogue

$$\delta_{\hat{\Lambda}} \hat{\mathcal{A}}_\mu = \partial_\mu \hat{\Lambda} + \mathrm{i}\, \hat{\Lambda} \star \hat{\mathcal{A}}_\mu - \mathrm{i}\, \hat{\mathcal{A}}_\mu \star \hat{\Lambda} \ . \tag{3.26}$$

Further analysis has revealed [82, 84] that the Seiberg–Witten map can be interpreted as a special generalized change of coordinates induced by an invertible linear operator $\mathcal{D}$, which is a non-linear functional of the gauge potential a. This operator maps ordinary spacetime coordinates x^μ to covariant coordinates

$$\hat{x}^\mu = \mathcal{D}(x^\mu) = x^\mu + \Theta^{\mu\nu}\, \hat{A}_\nu(x) \ , \tag{3.27}$$

where $\Theta = \frac{1}{2}\, \Theta^{\mu\nu}\, \partial_\mu \wedge \partial_\nu$ is a Poisson bivector, and it is therefore called the *covariantizing map*. The covariantizing map transforms by a star commutator with $\hat{\Lambda}$ under gauge transformations, implying the appropriate noncommutative gauge transformation for $\hat{A}_\mu$. For simplicity, we have written here equations for abelian gauge fields and have focused on the case of constant Poisson structure Θ. We will continue to focus on the abelian case, but drop all other simplifying assumptions in the following.

The construction of the covariantizing map is based on the observation that the Seiberg–Witten equivalence between two associative star products $\star$ and $\star'$ is the quantum analogue of Moser's lemma in symplectic geometry [82]. By this lemma, two symplectic forms ω and ω' on a symplectic manifold M are related by coordinate transformations generated by the flow of a vector field. In particular, it is possible to construct a semi-classical version of this flow which is appropriate to abelian gauge theory, and can be quantized to the covariantizing map $\mathcal{D}$ via Kontsevich's formality theorem [84]. Let us briefly review these constructions, before we show how they generalize for twisted Poisson structures in our membrane model.

Moser's lemma states that for a family of non-degenerate closed symplectic forms ω_t, where $t \in [0,1]$, there exists a family of diffeomorphisms ρ_t such that the pullback by ρ_0 is the identity map and $(\rho_t)^*\omega_t = \omega_0$. The diffeomorphisms ρ_t are generated by the flow of a vector field X, i.e. X satisfies the equation

$$\partial_t \rho_t = X(\rho_t) \ , \tag{3.28}$$

where $\partial_t := \frac{\partial}{\partial t}$. In fact, equation (3.28) can be integrated to the flow

$$(\rho_t)^* = \mathrm{e}^{\,\partial_t + X} \ \mathrm{e}^{\,-\partial_t}|_{t=0} \ . \tag{3.29}$$

By differentiating $(\rho_t)^*\omega_t = \omega_0$ with respect to t, and using the property $\big(\partial_t(\rho_t)^*\big)\omega_t = (\rho_t)^*\mathcal{L}_X\omega_t$, where $\mathcal{L}_X$ denotes the Lie derivative along X, we arrive at the condition

$$(\rho_t)^* \left(\partial_t\omega_t + \mathrm{d}\,\iota_X\omega_t\right) = 0 \ , \tag{3.30}$$

where Cartan's formula, and the fact that ω_t is closed have been used. It follows from (3.30) that $\partial_t\omega_t$ is closed, and thus by Poincaré's lemma it is also exact, i.e.

$$\partial_t\omega_t = \mathrm{d}a \tag{3.31}$$

for some 1-form a. The vector field is then completely determined by the equation

$$\iota_X\omega_t + a = 0 \ . \tag{3.32}$$

Consider now a gauge transformation $a \mapsto a + \mathrm{d}\lambda$; under this transformation $X \mapsto X + X_\lambda$, where X_λ is a Hamiltonian vector field satisfying the condition

$$\iota_{X_\lambda}\omega_t + \mathrm{d}\lambda = 0 \ , \tag{3.33}$$

as it can be seen from equation (3.32). Since X_λ is a Hamiltonian vector field, it induces a Poisson bracket on M corresponding to the Poisson structure ω_t by the relation

$$\{\lambda, g\}_t = \omega_t\big(X_g, X_\lambda\big) = \mathrm{d}g(X_\lambda) = \mathcal{L}_{X_\lambda} g \ , \tag{3.34}$$

where $\lambda, g \in C^\infty(M)$. Then the gauge transformation and the diffeomorphisms give the total transformation [82]

$$g \xmapsto{\mathrm{d}\lambda} g + \{\lambda, g\}_t \xmapsto{a} \rho^* g + \rho^*\{\lambda, g\}_t = \rho^* g + \{\rho^*\lambda, \rho^* g\} \ , \tag{3.35}$$

where $\{-,-\}$ is the Poisson structure given by $\omega_0 = \omega$.

This formalism is used in [85] (see also [84]) to construct a semi-classical version of the Seiberg–Witten map. The construction employs the Poisson bivector $\Theta = \frac{1}{2}\,\Theta^{ij}\,\partial_i \wedge \partial_j$ dual to the Poisson structure ω, i.e. $\Theta^{ij}\omega_{jk} = \delta^i{}_k$. Motivated by the role of the gauge potential 1-form a in Moser's lemma, the authors of [85] introduce the vector field

$$a_\Theta = \Theta(a, -) = \Theta^{ij}\, a_j\, \partial_i \ , \tag{3.36}$$

and the bivector field corresponding to the 2-form field strength $f = \mathrm{d}a$

$$f_\Theta = \mathrm{d}_\Theta a_\Theta = -\frac{1}{2}\,\Theta^{ij}\, f_{jk}\, \Theta^{kl}\, \partial_i \wedge \partial_l \ , \tag{3.37}$$

where $f_{ij} = \partial_i\, a_j - \partial_j\, a_i$ and $\mathrm{d}_\Theta = -[-,\Theta]_S$, with $[-,]_S$ the Schouten–Nijenhuis bracket (see appendix A.3). The Poisson bivector is perturbed by introducing by a one-parameter family of bivectors Θ_t, where $t \in [0,1]$, which correspond to ω_t. Equation (3.31) then becomes

$$\partial_t\Theta_t = \mathrm{d}_{\Theta_t} a_{\Theta_t} = f_{\Theta_t} \ , \tag{3.38}$$

with $\Theta_{t=0} = \Theta$, and its solution is given by the geometric series

$$\Theta_t = \Theta - t\,\Theta\, f\Theta + t^2\Theta\, f\Theta\, f\Theta + \mathcal{O}(t^3) = \frac{\Theta}{1 + t\, f\, \Theta} \ , \tag{3.39}$$

where the bivector Θ corresponding to the Poisson structure $\omega_0 = \omega$ is "twisted" by the two-form field strength to $\Theta_{t=1} = \Theta'$, which corresponds to the Poisson structure $\omega_1 = \omega'$. The flow ρ_a^* generated by a_{Θ_t} is obtained by integrating (3.38), which of course yields (3.29) with X substituted by a_{Θ_t}.

The Seiberg–Witten map in this setting is given by the semi-classical generalized gauge potential

$$A_a = \rho_a^* - \mathrm{id} \ . \tag{3.40}$$

As before, under the gauge transformation $a \mapsto a + \mathrm{d}\lambda$ the vector field a_Θ transforms as

$$a_\Theta \longmapsto a_\Theta + \mathrm{d}_\Theta \lambda \ , \tag{3.41}$$

where $\mathrm{d}_\Theta\lambda$ is a Hamiltonian vector field. Then for a function g and $t = 1$, the total transformation (3.35) becomes

$$g \xmapsto{\mathrm{d}\lambda} g + \{\lambda, g\}_{\Theta'} \xmapsto{a} \rho_a^* g + \rho_a^*\{\lambda, g\} = \rho_a^* g + \{\hat{\lambda}, \rho_a^* g\} \ , \tag{3.42}$$

with

$$\hat{\lambda}(\lambda, a) = \sum_{n=0}^{\infty} \frac{(a_{\Theta_t} + \partial_t)^n(\lambda)}{(n+1)!}\Big|_{t=0} \ , \tag{3.43}$$

and the gauge potential transforms as

$$A_{a+\mathrm{d}\lambda} = A_a + \mathrm{d}_\Theta \hat{\lambda} + \{\hat{\lambda}, A_a\} \ , \tag{3.44}$$

which is the semi-classical analog of (3.26). Equations (3.40) and (3.43) are the semi-classical versions of the Seiberg–Witten map, while the flow ρ_a^* is the semi-classical covariantizing map.

In the quantum case, the covariantizing map $\mathcal{D}_a$ is similarly obtained as the "flow" of the differential operator $\underline{a_\Theta}$, which is given by the deformation quantization of a_Θ using equation (3.12) [84, 85]. The pertinent bidifferential operator is given by

$$\underline{f_\Theta} = \mathrm{d}_\star \underline{a_\Theta} \ , \tag{3.45}$$

where $\mathrm{d}_\star = -[-,\star]_G$ is the coboundary operator and $[-,-]_G$ is the Gerstenhaber brackets (see appendix A.2); this equation is the image of (3.37) under the formality map. The differential operators $\underline{a_\Theta}$ and $\underline{f_\Theta}$ satisfy quantum equations that mimic their semi-classical counterparts, and thus substituting vectors, bivectors and functions with their deformed versions yields the covariantizing map

$$\mathcal{D}_a = \exp\big(\partial_t + \underline{a_\Theta}\big)\ \exp\big(-\partial_t\big)|_{t=0} \ , \tag{3.46}$$

as the "flow" of the differential operator $\underline{a_\Theta}$.

It is now straightforward to obtain the quantum Seiberg–Witten map. For this, consider the generalized noncommutative gauge potential

$$\mathcal{A}_a = \mathcal{D}_a - \mathrm{id} \ , \tag{3.47}$$

and an infinitesimal gauge transformation $a \mapsto a + \mathrm{d}\lambda$, under which the differential operator $\underline{a_\Theta}$ transforms as

$$\underline{a_\Theta} \longmapsto \underline{a_\Theta} + \frac{1}{\mathrm{i}\,\hbar}\,\mathrm{d}_\star \underline{\lambda} \ , \tag{3.48}$$

where $\mathrm{d}_\star\underline{\lambda}$ is the deformation of the Hamiltonian vector field $\mathrm{d}_\Theta\lambda$, and $\underline{\lambda}$ is given by (3.10). In (3.41), which is the classical analogue of this transformation, $\delta a_\Theta = \mathrm{d}_\Theta\lambda$ is a Hamiltonian vector field, and thus the gauge transformation induces a canonical transformation. In the quantum case, the formality condition following (3.12) maps $\mathrm{d}_\Theta\lambda$ to an inner derivation of the star product, and thus it induces a noncommutative gauge transformation $\delta_{\hat\Lambda}$, for which

$$\mathcal{A}_{a+\mathrm{d}\lambda} = \mathcal{A}_a + \frac{1}{\mathrm{i}\,\hbar}\Big(\mathrm{d}_\star\hat\Lambda + [\hat\Lambda, \mathcal{A}_a]_\star\Big) \ , \tag{3.49}$$

where the Seiberg–Witten map $\hat{\Lambda}(\lambda, a)$ is obtained from (3.43) by substituting a_{Θ_t} with $\underline{a_{\Theta_t}}$. The transformation chain (3.42) now becomes

$$g \overset{\mathrm{d}\lambda}{\longmapsto} g + [\underline{\lambda}, g]_{\star'} \overset{a}{\longmapsto} \mathcal{D}_a g + \mathcal{D}_a[\underline{\lambda}, g]_{\star'} = \mathcal{D}_a g + [\hat{\Lambda}, \mathcal{D}^*_a g]_\star \ , \tag{3.50}$$

from which it can be seen that the covariantizing map $\mathcal{D}_a$ relates the star products $\star$ and $\star'$ via $\mathcal{D}_a(f \star' g) = \mathcal{D}_a f \star \mathcal{D}_a g$, i.e. it is an isomorphism of associative algebras where noncommutative gauge transformations are realised as inner automorphisms. For further technical details of the construction, we refer to [82, 84, 85, 86]. In the following, we will describe several covariantizing maps and the corresponding gauge symmetries for twisted Poisson structures, which relevant for strings in the R-flux background.

The presence of a non-trivial closed three-form background H leads to a twisted Poisson structure: The bivector Θ fails to fulfill the Jacobi identity and its Schouten–Nijenhuis bracket is consequently non-zero: $[\Theta, \Theta]_{\mathrm{S}} = \hbar \bigwedge^3 \Theta^\sharp(H)$, where we have introduced a factor $\hbar$ to ensure formal convergence of all expressions in the ensuing contruction. (Such a factor is understood to be implicitly included in Θ in the rest of this section.) From the point of view of background fields and fluctuations, the structure we are dealing with is a *gerbe*: Given a suitable covering of the target space manifold by contractible open patches (labelled by Greek indices $\alpha, \beta, \ldots$), we can write H in terms of local two-form fields B_α as $H = \mathrm{d}B_\alpha$ on each patch. On the overlap of two patches, the difference $B_\beta - B_\alpha =: F_{\alpha\beta}$ is closed, hence exact and can be expressed in terms of one-form fields $a_{\alpha\beta}$ as $F_{\alpha\beta} = \mathrm{d}a_{\alpha\beta}$. On triple overlaps we then encounter local gauge parameters $\lambda_{\alpha\beta\gamma}$ that satisfy a suitable integrability condition. This hierarchical description in terms of forms has a dual description in terms of multivector fields that is suitable for deformation quantization and leads to noncommutative gerbes in the sense described in [7]: The twisted Poisson bivector Θ can be locally untwisted by the two-form fields B_α, leading to bona fide Poisson bivectors $\Theta_\alpha = \Theta\,(1 - \hbar\, B_\alpha\, \Theta)^{-1}$. These local Poisson tensors Θ_α and the corresponding associative star products $\star_\alpha$ are related by covariantizing maps computed from $a_{\alpha\beta}$.

As mentioned in section 2.3, the relevant geometric structure in R-space is a gerbe in momentum space, with curvature (2.34) and 2-connection (2.35). Here we are dealing with a topologically trivial setting, so the forms and multivector fields are all globally defined. Nevertheless, the constructions of twisted noncommutative gauge theory and Seiberg–Witten maps are non-trivial and interesting. On $\mathcal{M} = T^*M$ the patch index α is replaced by a constant momentum vector $\bar{p}$ that parametrizes a degree of freedom in the choice of Poisson structure $\Theta_{\bar{p}}$ and two-form background field $B_{\bar{p}}$. In matrix form, the pertinent bivector and two-form fields are

$$\Theta = \begin{pmatrix} \hbar\, R^{ijk}\, p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \ , \qquad \Theta_{\bar{p}} = \begin{pmatrix} \hbar\, R^{ijk}\, \bar{p}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \tag{3.51}$$

and

$$B_{\bar{p}} = \begin{pmatrix} 0 & 0 \\ 0 & R^{ijk}\, (p_k - \bar{p}_k) \end{pmatrix} \ . \tag{3.52}$$

They satisfy

$$H = \mathrm{d}B_{\bar{p}} \ , \qquad [\Theta,\Theta]_{\mathrm{S}} = \hbar \textstyle\bigwedge^3 \Theta^\sharp(H) \qquad \text{and} \qquad [\Theta_{\bar{p}},\Theta_{\bar{p}}]_{\mathrm{S}} = 0 \ , \tag{3.53}$$

together with

$$\Theta = \Theta_{\bar{p}}\, (1 + \hbar\, B_{\bar{p}}\, \Theta_{\bar{p}})^{-1} \qquad \text{and} \qquad \Theta_{\bar{p}} = \Theta\, (1 - \hbar\, B_{\bar{p}}\, \Theta)^{-1} \ . \tag{3.54}$$

The corresponding 1-connection is given by $a_{\bar{p},\bar{p}'} = R^{ijk}\, p_i\, (\bar{p}_k - \bar{p}'_k)\, \mathrm{d}p_j$. Note that we cannot choose $\hbar\, B$ to be equal to Θ^{-1} as that would add terms of order $\hbar^{-1}$ to $H = \mathrm{d}B$, which is incompatible with (3.53), and it would lead to convergence problems for the geometric series in (3.54). The deformation quantizations along Θ and $\Theta_{\bar{p}}$ yield the nonassociative star product $\star$ and the associative star product $\star_{\bar{p}}$ respectively. For the special case $\bar{p} = 0$, Θ_0 and $\star_0$ are respectively the canonical Poisson structure and associative star product on phase space. For fixed three-form H, choices for B can differ by any closed (and hence exact) two-form $F = \mathrm{d}A$. The corresponding choices of Poisson structures and star products are related by covariantizing maps constructed from gauge potentials

$$A = A_I(x)\,\mathrm{d}x^I = a_i(x,p)\,\mathrm{d}x^i + \tilde{a}^i(x,p)\,\mathrm{d}p_i\ . \tag{3.55}$$

Associated to these covariantizing maps are Seiberg–Witten maps as explained before. Gauge transformations $\delta_\lambda A = \mathrm{d}\lambda$, where $\lambda = \lambda(x,p)$, induce a change of the covariantizing maps by a star commutator with $\hat{\Lambda}(\lambda, a)$.

So far we have discussed ordinary Seiberg–Witten maps for bona fide Poisson bivectors. It would appear to be more natural to find also a construction based directly on the twisted Poisson bivector Θ. Terms involving the non-zero Jacobiator (Schouten–Nijenhuis bracket) usually spoil such a construction. In the present case it turns out, however, that for the class of gauge potentials of the form $A = (A_I) = (0, \tilde{a}^i(x,p))$ (i.e. with $a_i(x,p) = 0$) the unwanted terms drop out, because $([\Theta,\Theta]_S)^{IJK}\,A_K$ is proportional to $R^{ijk}\,a_k = 0$. The restriction thus imposed on the class of admissible gauge potentials leads to a corresponding restriction on the class of covariantizing maps. The admissible class of maps is, however, still very large and actually quite interesting: Evaluating $\Theta(A,-) = \Theta^{IJ}\,A_J\,\partial_I = \delta^i{}_j\,\tilde{a}^j(x,p)\,\partial_i$ shows that any map generated by a vector field of the form $\tilde{a}^i(x,p)\,\partial_i$, which acts on configuration space and may even depend on the momentum variables, is admissible. The associated class of ordinary and noncommutative gauge transformations is more restricted: The gauge parameters λ and $\hat{\Lambda}$ may only depend on the momenta p.

An interesting subclass of the covariantizing maps just described are generated by gauge potentials of the form $A = R(a_2, -)$, where a_2 is a two-form on configuration space and we have used the natural map $\bigwedge^2 T^*M \to TM$ induced by the three-vector R; in components $\tilde{a}^i(x) = R^{ijk}\,(a_2)_{jk}(x)$. The semi-classical version of the resulting maps has been discussed in the context of Nambu–Poisson structures on p-branes in [83], where a_2 plays the role of a two-form gauge potential, and (noncommutative) gauge transformations are computed using Nambu–Poisson brackets and x-dependent one-form gauge parameters. (The restriction to gauge parameters that depend only on momenta is not needed here.) Section 3.3 provides a quantization of these Nambu–Poisson maps for membranes ($p = 2$) with a constant

Nambu–Poisson trivector R.

Another interesting special example concerns the relationship between $\star_0$ and $\star_{\bar{p}}$: The corresponding covariantizing map $\mathcal{D}_{\bar{p}}$ is constructed from $F_{\bar{p}} = \mathrm{d}A_{\bar{p}} = B_0 - B_{\bar{p}}$, with a gauge potential defined by

$$A_{\bar{p}} = \tilde{a}^j(p)\,\mathrm{d}p_j = \tfrac{1}{2}\,R^{ijk}\,p_i\,\bar{p}_k\,\mathrm{d}p_j \tag{3.56}$$

up to gauge transformations $\delta_{\tilde{\lambda}}\tilde{a}^j(p) = \tilde{\partial}^j\tilde{\lambda}(p)$. It satisfies $f \star_{\bar{p}} g = \mathcal{D}_{\bar{p}}^{-1}\,(\mathcal{D}_{\bar{p}}f \star_0 \mathcal{D}_{\bar{p}}g)$. Formally replacing the constant $\bar{p}$ by the dynamical momentum variable p in this equation gives a Seiberg–Witten map from the associative canonical star product $\star_0$ to the nonassociative star product $\star := \star_p$ as

$$f \star g = \big[\mathcal{D}_{\bar{p}}^{-1}\,(\mathcal{D}_{\bar{p}}f \star_0 \mathcal{D}_{\bar{p}}g)\,\big]_{\bar{p}\to p}\ . \tag{3.57}$$

In view of the foregoing discussion, this can also be written as

$$f \star g = \big[\mathcal{D}_{\bar{p}}(f \star g)\big]_{\bar{p}\to p} = \big[\mathcal{D}_{\bar{p}}f \star_0 \mathcal{D}_{\bar{p}}g\big]_{\bar{p}\to p}\ , \tag{3.58}$$

since the underlying vector field $\Theta(A_{\bar{p}}, -)$ from (3.56) vanishes when $\bar{p} \to p$ and the covariantizing map becomes trivial. In the given gauge, the vector field $\Theta(A_{\bar{p}}, -)$ receives no quantum corrections from deformation quantization and the Seiberg–Witten map can thus be computed explicitly in closed form. This is one of the very rare cases where this is possible.

From the point of view of noncommutative gauge theory as well as noncommutative string geometry, gauge transformations preserve star products. Expressed in terms of gauge fields, gauge transformations correspond to different choices of one-form potentials A that preserve the curvature two-form $F = \mathrm{d}A$. From the membrane point of view, however, the three-form H is the fundamental global quantity and gauge transformations correspond to different choices of two-form potentials B that preserve the gerbe curvature $H = \mathrm{d}B$. The role of the gauge parameter is taken by a one-form gauge potential A. As we have discussed, such one-forms generate covariantizing maps $\mathcal{D}$. These maps preserve associativity (as well as nonassociativity). The collection of these maps describes the gauge degrees of freedom of our system. Concretely, our construction for the twisted Poisson structure Θ yielded covariantizing maps $\mathcal{D}_\xi$ for all vector fields ξ on configuration space. This evidently generates a huge gauge degree of freedom generated by quantized general coordinate transformations. This point adds some credibility to the terminology "nonassociative gravity" that was coined in [27] to describe the quantum geometry of closed strings in non-geometric flux backgrounds.

3.5 Closed string vertex operators and 3-product

We close this chapter by comparing our associator with the ternary product for closed strings propagating in a background with constant R-flux which was proposed in [21]. Here the authors perform a linearized conformal field theory analysis of the three-point function of tachyon vertex operators in a flat background with constant H-flux. After applying three T-dualities they arrive at a nonassociative algebra of closed string vertex operators in the R-flux background, from which they propose a deformation of the pointwise product of functions via a 3-product of the form

$$f \bullet g \bullet h = f\, g\, h + R^{ijk}\, \partial_i f\, \partial_j g\, \partial_k h + \mathcal{O}\big(R^2\big) \ . \tag{3.59}$$

In the light of the above analysis, we are able to explain this result analytically and relate it to our expressions. Expanding either of the two bracketings (3.19) to linear order we have

$$f \star g \star h = f\, g\, h + R'^{\,ijk} \big[\partial_i f\, \bar\star\, \partial_j g\, \bar\star\, \partial_k h\big]_{\bar p \to p} + \mathcal{O}\big(R'^{\,2}\big) \ , \tag{3.60}$$

from which we conclude that (3.59) agrees with (3.60) to first order in $R' := \pm\, \frac{\hbar^2}{4}\, R$, but without the Moyal star product between the derivatives of f, g and h, and without the dependence on the dynamical variable p. For functions that are independent of p, the two formulas agree to linear order. The main difference between the two formulas stems from our consideration of the cotangent bundle of M as the effective target space geometry of closed strings in the R-flux compactification.

This also explains the proposal of [21] that the binary product of functions is the usual pointwise multiplication, as for closed strings only three and higher point correlation functions experience the effect of the flux background. Setting $p = \bar{p} = 0$ corresponds to the sector of zero winding number in the T-dual Q-space frame. It truncates phase space to the original configuration space M and recovers the usual commutative pointwise product $f \,\bar{\star}\, g = f\, g = f \star g$, consistent with the fact that only extended closed strings with non-trivial winding number (dual momentum) are sensitive to the noncommutative deformation in the Q-flux background (see section 2.4); nevertheless, this sector still retains a non-trivial associator (3.20) of fields in the nonassociative R-flux background as in [27, 21]. Moreover, as in [21], higher order associators are not simply related to successive applications of (3.20). Together with the cyclic property (3.21), we see therefore that in this sector our deformation quantization approach agrees with the 3-product of [21]. The authors of [21] also conjecture an all orders ternary product obtained by exponentiation of the trivector R as a straightforward generalization of the Moyal–Weyl formula. Our results confirm this conjecture insofar that the exponential of R is indeed part of the correct all order expressions (3.19).

As we discussed in section 2.5, the twisted Poisson structures we have found naturally give rise to an L_∞-structure on the algebra $C^\infty(\mathcal{M})$. In [41] it is shown that correlators of *open* string vertex operators in a non-constant H-flux background endow the Kontsevich deformation of the algebra of functions on M with the structure of an A_∞-algebra (see appendix A.1), or more precisely an A_∞-space, which are the natural algebras that appear in generic open-closed string field theories; in particular, the corresponding star commutator algebra is an L_∞-algebra. The correlators of closed string vertex operators computed in [21] also exhibit dilogarithmic singularities analogous to those found in [41] (see also [63]), and it would be interesting to see if they lead to an analogous A_∞-structure; indeed in [2] it is shown that the reflection identity for the Rogers dilogarithm relates four-point correlation functions to two-point correlators in a manner reminiscent of an associator, while the pentagonal identity is related to a factorization property of five-point functions which is reminiscent of the higher coherence relation for the associator. The similarity between open and closed string correlators is also noted in [19]. In chapter 4 we will see such structures emerging rather directly in the full quantized algebra of functions.

Chapter 4

Strict deformation quantization

In [87], Kathotia compares the two canonical deformation quantizations of the linear Kirillov–Poisson structure on the vector space $W = \mathfrak{g}^*$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra. These quantizations are provided by the Kontsevich formalism and the associated Lie group convolution algebra. Let us briefly recall how this latter quantization scheme proceeds [107]. One should first Fourier transform functions on W to obtain elements in $C^\infty(\mathfrak{g})$. The Lie algebra $\mathfrak{g}$ is then identified with its integrating Lie group G in a neighbourhood of the identity element via the exponential mapping. On G, the convolution product between functions induced by the group multiplication and the Baker–Campbell–Hausdorff formula can be used. By performing the inverse operations to pullback the result a star product on $C^\infty(W)$ is obtained. For nilpotent Lie algebras, the exponential map between $\mathfrak{g}$ and G is a global diffeomorphism. In this case, the above construction is equivalent to both Kontsevich's deformation quantization and quantization via the universal enveloping algebra of $\mathfrak{g}$ [87].

Since our twisted Poisson structure (2.37) is linear for constant R-flux, it is natural to ask if there is an analogous approach which would provide an alternative quantization framework to the combinatorial approach we took in chapter 3. In this chapter we shall develop such an approach based on integrating a suitable Lie 2-algebra to a Lie 2-group which will define a convolution algebra object in a braided monoidal category (see Appendices A.1 and A.7 for the precise definitions), and demonstrate that it is equivalent to the quantization of chapter 3 which was based on our proposed membrane sigma model. Here we focus for definiteness on the case of configuration space $M = \mathbb{T}^d$ which is a d-dimensional torus with constant R-flux. This approach will then further clarify how the R-space nonassociativity is realized by a 3-cocycle associated to a nonassociative representation of the translation group, as arises in the presence of a magnetic monopole [80], and its relation to the topological nonassociative tori studied in [28].

4.1 Lie 2-algebras for non-geometric backgrounds

Let $V \cong \mathbb{R}^{2d}$ be a vector space of dimension $2d$ with a fixed choice of basis elements which we denote by

$$(\hat{x}^I) = (\hat{x}^1, \dots, \hat{x}^{2d}) = (\hat{x}^1, \dots, \hat{x}^d, \hat{p}_1, \dots, \hat{p}_d) \ , \tag{4.1}$$

where throughout this chapter we use hats to distinguish abstract vector space and categorical elements from the concrete coordinate functions we used in previous sections. We define a bracket $[-,-]_R : V \wedge V \to V$ by the relations

$$[\hat{x}^i, \hat{x}^j]_R = \mathrm{i}\, R^{ijk}\, \hat{p}_k \ , \quad [\hat{p}_i, \hat{p}_j]_R = 0 \quad \text{and} \quad [\hat{x}^i, \hat{p}_j]_R = \mathrm{i}\, \hbar\, \delta^i_j = -[\hat{p}_j, \hat{x}^i]_R \ , \tag{4.2}$$

which is just an abstract presentation of the twisted Poisson brackets (2.37). This bracket defines a *pre-Lie algebra* structure on V, i.e. it is antisymmetric but does not satisfy the Jacobi identity; it leads to the non-vanishing Jacobiator

$$[\hat{x}^i, \hat{x}^j, \hat{x}^k]_R := \tfrac{1}{3} \left(\big[[\hat{x}^i, \hat{x}^j]_R, \hat{x}^k\big]_R + \big[[\hat{x}^k, \hat{x}^i]_R, \hat{x}^j\big]_R + \big[[\hat{x}^j, \hat{x}^k]_R, \hat{x}^i\big]_R \right) = \hbar\, R^{ijk} \ , \tag{4.3}$$

and all other Jacobiators vanish. Hence the bracket naturally defines a Lie 2-algebra $\mathscr{V}$ (see appendix A.1). For this, we set $V_0 = V$, $V_1 = V$, and let $\mathsf{d} : V_1 \to V_0$ be the identity map id_V. Let $[-,-] : V_0 \wedge V_0 \to V_0$ and $[-,-] : V_0 \otimes V_1 \to V_1$ be the bracket (4.2) of V, and let $[-,-,-] : V_0 \wedge V_0 \wedge V_0 \to V_1$ be the Jacobiator (4.3) of V. Then $\big(V_1 \xrightarrow{\mathsf{d}} V_0\big)$ is a 2-term L_∞-algebra canonically associated to the pre-Lie algebra V.

We can identify the twisted Poisson structure (2.37) on the algebra of functions $C^\infty(\mathcal{M})$ with the natural twisted Poisson structure on the dual V^* of the pre-Lie algebra V as follows. We first identify linear functions on V^* with elements of V, and define $\{\hat{v}_1, \hat{v}_2\}_R(x) := \langle x, [\hat{v}_1, \hat{v}_2]_R \rangle$, where $\hat{v}_1, \hat{v}_2 \in V$, $x \in V^*$ and $\langle -, - \rangle : V^* \otimes V \to \mathbb{R}$ denotes the dual pairing. By imposing the Leibniz identity, this defines a quasi-Poisson bracket that extends to polynomial functions on V^*, which in turn are dense in $C^\infty(V^*)$.

As we discuss in appendix A.7, there is no general construction of Lie 2-groups from Lie 2-algebras, but we can build a suitable integration map with some intuition provided from our considerations in chapter 2. For this, we will write down an equivalent Lie 2-algebra for which a corresponding Lie 2-group can be "guessed". We start by replacing the pre-Lie algebra V with a quadratic Lie algebra $\mathfrak{g}$ whose generators $\hat{x}^i, \hat{\bar{p}}_j$, $i, j = 1, \dots, d$ have the Lie brackets

$$[\hat{x}^i, \hat{x}^j]_Q = \mathrm{i}\, R^{ijk}\, \hat{\bar{p}}_k \qquad \text{and} \qquad [\hat{x}^i, \hat{\bar{p}}_j]_Q = 0 = [\hat{\bar{p}}_i, \hat{\bar{p}}_j]_Q \ , \tag{4.4}$$

together with the non-degenerate inner product defined by

$$\langle \hat{x}^i, \hat{\bar{p}}_j \rangle = \delta^i{}_j \qquad \text{and} \qquad \langle \hat{x}^i, \hat{x}^j \rangle = 0 = \langle \hat{\bar{p}}_i, \hat{\bar{p}}_j \rangle \tag{4.5}$$

which is invariant under the adjoint action and is of split signature. There are two ways to think about this Lie algebra. Firstly, it is the reduction of the Courant algebroid of section 2.3 over a point; we may regard $\mathfrak{g} \cong \mathbb{R}^d \oplus (\mathbb{R}^d)^*$ as the cotangent bundle $T^*\mathbb{R}^d$ with its canonical symplectic structure. Secondly, it is an abstract version of the Q-space Poisson brackets (2.42), and in particular it coincides with the $\hbar = 0$ limit of the brackets given by (4.2) and (4.3); in this way we will mimic the dynamical quantization of section 3.3 by first integrating the d-dimensional Heisenberg algebra (4.4) involving the "non-dynamical momenta" $\hat{\bar{p}}_i$, and then making the momenta "dynamical" $\hat{\bar{p}}_i \to \hat{p}_i$ to recover the T-dual pre-Lie algebra (4.2) appropriate to the R-space frame with the non-trivial Jacobiator (4.3).

Associated to the quadratic Lie algebra $\mathfrak{g}$ is a Lie 2-algebra $\tilde{\mathscr{V}}$ corresponding to the 2-term L_∞-algebra

$$\tilde{V} \ = \ \big(\tilde{V}_1 = \mathbb{R} \ \xrightarrow{\tilde{\mathsf{d}}} \ \tilde{V}_0 = \mathfrak{g}\big) \tag{4.6}$$

which is *skeletal*, i.e. $\tilde{\mathsf{d}} = 0$, with brackets $[-,-] : \tilde{V}_0 \wedge \tilde{V}_0 \to \tilde{V}_0$ given by the Lie bracket (4.4) of $\mathfrak{g}$ and $[-,-] : \tilde{V}_0 \otimes \tilde{V}_1 \to \tilde{V}_1$ given by $[\hat{v}, c] = 0$ for $\hat{v} \in \mathfrak{g}$, $c \in \mathbb{R}$, and Jacobiator $[-,-,-] : \tilde{V}_0 \wedge \tilde{V}_0 \wedge \tilde{V}_0 \to \tilde{V}_1$ given by $[\hat{v}_1, \hat{v}_2, \hat{v}_3] = \langle [\hat{v}_1, \hat{v}_2]_Q, \hat{v}_3\rangle$ for $\hat{v}_i \in \mathfrak{g}$; this is just the reduction over a point of the Lie 2-algebra structure (A.40) canonically associated to the exact Courant algebroid $C \to M$ of section 2.3. The corresponding classifying triple is $(\mathfrak{g}, \mathbb{R}, j)$ where $\mathbb{R}$ is the trivial representation of $\mathfrak{g}$ and the 3-cocycle $j : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R}$ is given by

$$j(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \big\langle [\hat{v}_1, \hat{v}_2]_Q \, , \, \hat{v}_3 \big\rangle \ . \tag{4.7}$$

The cocycle condition (or equivalently the pentagonal coherence relation (A.7)) follows from adjoint-invariance of the inner product and since $\mathfrak{g}$ acts trivially on $\mathbb{R}$; note that its only non-trivial values on generators are given by

$$j(\hat{x}^i, \hat{x}^j, \hat{x}^k) = R^{ijk} \tag{4.8}$$

as in (4.3). The cohomology of the Heisenberg Lie algebra (4.4) is described in [113]; in particular for degree 3 one has

$$\dim H^3(\mathfrak{g}, \mathbb{R}) = D := \tfrac{1}{6}\, d\, (d-1)\, (d-2) - d \tag{4.9}$$

and the space of 3-cocycles

$$Z^3(\mathfrak{g}, \mathbb{R}) = \textstyle\bigwedge^3 \big(\hat{x}^*_1, \dots, \hat{x}^*_d\big) \tag{4.10}$$

is the vector space of homogeneous elements of degree 3 of the Grassmann algebra over the dual basis to $\hat{x}^1, \dots, \hat{x}^d$. It follows that the Jacobiator (4.7) gives rise to a generator $[j]$ of $H^3(\mathfrak{g}, \mathbb{Z}) = \mathbb{Z}^D$, and all generators are obtained via a choice of basis for the space of totally antisymmetric 3-vectors as in (4.8) (modulo linear redefinitions of the central elements $\hat{\tilde{p}}_1, \dots, \hat{\tilde{p}}_d$).

4.2 The integrating Lie 2-groups

The classifying data $(\mathfrak{g}, \mathbb{R}, j)$ of the Lie 2-algebra $\tilde{\mathscr{V}}$, with $\mathbb{R} = \mathfrak{u}(1)$ regarded as the one-dimensional abelian Lie algebra, can be straightforwardly exponentiated to a triple $(G, U(1), \varphi)$ corresponding to a special Lie 2-group $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1)$ (see appendix A.7), modulo one subtlety. The universal 2-step nilpotent Lie algebra $\mathfrak{g}$ of rank d integrates to the non-compact simply connected d-dimensional Heisenberg group G, the associated free 2-step nilpotent Lie group. In order to exponentiate the generator $[j] \in H^3(\mathfrak{g}, \mathbb{R})$ induced by the Jacobiator (4.7) of $\tilde{\mathscr{V}}$ to a *compact* element $[\varphi] \in H^3(G, U(1))$, it is necessary to restrict the space of 3-cocycles (4.10) to a lattice $\Lambda \cong \mathbb{Z}^d$ of maximal rank in the linear span of the generators $\hat{x}^1, \dots, \hat{x}^d$. This lattice injects into a cocompact lattice Γ in G; the resulting quotient G/Γ is a Heisenberg nilmanifold or "double twisted torus", familiar in $d = 3$ dimensions as the doubled space of the geometric T-dual to the three-torus with H-flux [75]. We assume that the lattice is equipped with a non-degenerate inner product which is given in a suitable basis by $\eta = (\eta_{ab}) : \Lambda \otimes_{\mathbb{Z}} \Lambda \to \mathbb{R}$, $a, b = 1, \dots, d$, with inverse $\eta^{-1} = (\eta^{ab}) : \Lambda^* \otimes_{\mathbb{Z}} \Lambda^* \to \mathbb{R}$, and a non-degenerate dual pairing $\Sigma = ({\Sigma_a}^i) : \Lambda \otimes_{\mathbb{R}} (\mathbb{R}^d)^* \to \mathbb{R}$ which is a vielbein for the inner product, i.e. ${\Sigma_a}^i \, \delta_{ij} \, {\Sigma_b}^j = \eta_{ab}$.

With these restrictions understood, the Lie 2-algebra $\tilde{\mathscr{V}}$ given by (4.6) integrates to the Lie 2-group

$$\mathscr{G}_1 = G \times U(1) \overset{\mathsf{s}}{\underset{\mathsf{t}}{\rightrightarrows}} \mathscr{G}_0 = G \tag{4.11}$$

having $U(1)$ as the group of automorphisms of its unit object 1 in G, in which the source and target maps s, t are both projections onto the first factor, vertical multiplication is given by $(g, \zeta) \circ (g, \zeta') = (g, \zeta \, \zeta')$ for $g \in G$ and $\zeta, \zeta' \in U(1)$, and horizontal multiplication $\otimes$ given by group multiplication. The associator

$$\mathscr{P}_{g,h,k} \, : \, (g \otimes h) \otimes k \ \longrightarrow \ g \otimes (h \otimes k) \tag{4.12}$$

is the automorphism given by

$$\mathscr{P}_{g,h,k} = \big(g\, h\, k \, , \, \varphi(g,h,k) \big) \ , \tag{4.13}$$

where we have integrated the Lie algebra 3-cocycle (4.7) to the smooth normalised Lie group 3-cocycle $\varphi : G \times G \times G \to U(1)$ with

$$j(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \frac{\partial^3}{\partial t_1\, \partial t_2\, \partial t_3}\bigg|_{t_i=0} \varphi\big(\exp t_1\, \hat{v}_1 \, , \, \exp t_2\, \hat{v}_2 \, , \, \exp t_3\, \hat{v}_3 \big) \tag{4.14}$$

for all $\hat{v}_i \in \Lambda$. All other structure maps of the Lie 2-group $\mathscr{G}$ are identity isomorphisms. Finally, to make the transformation to "dynamical" momentum variables $\hat{\tilde{p}}_i \to \hat{p}_i$, and hence integrate our original Lie 2-algebra $\mathscr{V}$ with brackets (4.2) and (4.3), we endow $\mathscr{G}$ with a braiding

$$\mathscr{B}_{g,h} \, : \, g \otimes h \ \longrightarrow \ h \otimes g \tag{4.15}$$

which is the automorphism given by

$$\mathscr{B}_{g,h} = \big(g\, h \, , \, \beta(g,h) \big) \ , \tag{4.16}$$

where we have integrated the inner product (4.5) to the smooth normalised map $\beta : G \times G \to U(1)$ with

$$\langle \hat{v}_1, \hat{v}_2 \rangle = \frac{\partial^2}{\partial t_1\, \partial t_2}\bigg|_{t_i=0} \beta\big(\exp t_1\, \hat{v}_1, \exp t_2\, \hat{v}_2 \big) \tag{4.17}$$

for all $\hat{v}_i \in \Lambda$. The braided monoidal category $\mathscr{G}$ is then the Lie 2-group that integrates the Lie 2-algebra $\mathscr{V}$.

We can make this construction somewhat more concrete and explicit in a way that will be suitable to our ensuing constructions. For this, we formally exponentiate the Lie 2-algebra generators to define

$$\hat{Z}^a = \exp\big(2\pi\,\mathrm{i}\,(\Sigma^{-1})_i{}^a\,\hat{x}^i\big) \qquad \text{and} \qquad \hat{P}_\xi = \exp\big(\mathrm{i}\,\xi^i\,\hat{p}_i\big) \tag{4.18}$$

for $a = 1,\dots,d$ and $\xi = (\xi^i) \in \mathbb{R}^d$. We may compute exterior products $\otimes : \mathscr{G}\times\mathscr{G}\to\mathscr{G}$ of the elements (4.18) in the Lie 2-group $\mathscr{G}$ by formally applying the Baker–Campbell–Hausdorff formula using the brackets (4.2) and (4.3); since the bracket functor in this case is nilpotent, the Hausdorff series is still applicable to the sole finite non-vanishing order that we require it without any need of the Jacobi identity. The commutation relations are then given by

$$\hat{Z}^a\otimes\hat{Z}^b = \hat{P}_{\xi_R^{ab}}\otimes\hat{Z}^b\otimes\hat{Z}^a\ , \tag{4.19}$$

$$\hat{Z}^a\otimes\hat{P}_\xi = \mathrm{e}^{\,2\pi\,\mathrm{i}\,\hbar\,(\Sigma^{-1})_i{}^a\,\xi^i}\ \hat{P}_\xi\otimes\hat{Z}^a\ , \tag{4.20}$$

$$\hat{P}_\xi\otimes\hat{P}_{\xi'} = \hat{P}_{\xi'}\otimes\hat{P}_\xi\ , \tag{4.21}$$

where $\xi_R^{ab}\in\mathbb{R}^d$ is given by

$$\big(\xi_R^{ab}\big)^i = -4\pi^2\,(\Sigma^{-1})_j{}^a\,R^{ijk}\,(\Sigma^{-1})_k{}^b\ . \tag{4.22}$$

In (4.20) we recognize the non-trivial braiding isomorphism $\mathscr{B}_{\hat{Z}^a,\hat{P}_\xi}$ on 2-group objects given by the map $\beta:\mathbb{R}^d\times\Lambda^*\to U(1)$ whose only non-trivial values are

$$\beta(\xi,m) = \mathrm{e}^{\,2\pi\,\mathrm{i}\,\hbar\,\xi^i\,(\Sigma^{-1})_i{}^a\,m_a} \tag{4.23}$$

for $\xi = (\xi^i)\in\mathbb{R}^d$ and $m = (m_a)\in\Lambda^*\cong\mathbb{Z}^d$, while the remaining commutation relations in (4.19)–(4.21) are those of the rank d Heisenberg group G. The non-trivial associators follow by applying the Baker–Campbell–Hausdorff formula once more to find

$$\big(\hat{Z}^a\otimes\hat{Z}^b\big)\otimes\hat{Z}^c = \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\hbar\,R^{abc}}\ \hat{Z}^a\otimes\big(\hat{Z}^b\otimes\hat{Z}^c\big)\ , \tag{4.24}$$

where

$$R^{abc} = 2\pi^2\,R^{ijk}\,(\Sigma^{-1})_i{}^a\,(\Sigma^{-1})_j{}^b\,(\Sigma^{-1})_k{}^c \tag{4.25}$$

are the dimensionless nonassociativity R-flux parameters. This expression is the Lie 2-group version of the "cyclic double commutator" that was calculated in [27], which we recognise as the action of the non-trivial associator isomorphism $\mathscr{P}_{\hat{Z}^a,\hat{Z}^b,\hat{Z}^c}$ on 2-group objects. The corresponding 3-cocycle can be regarded as a group homomorphism or tricharacter $\varphi : \Lambda^* \times \Lambda^* \times \Lambda^* \to U(1)$ defined by

$$\varphi(m,n,q) = \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\hbar\,R^{abc}\,m_a\,n_b\,q_c} \ . \tag{4.26}$$

This map is normalised, i.e. $\varphi(m,n,q)=1$ if either of m, n, or q is 0; this implies that the two obvious maps from $\hat{Z}^a \otimes (1 \otimes \hat{Z}^b) = \mathscr{P}\big((\hat{Z}^a \otimes 1) \otimes \hat{Z}^b\big)$ to $\hat{Z}^a \otimes \hat{Z}^b$ are consistent. It is also skew-symmetric, i.e. $\varphi(m,n,q) = \varphi(n,m,q)^{-1} = \varphi(m,q,n)^{-1} = \varphi(q,n,m)^{-1}$, and it obeys the required pentagonal cocycle identity

$$\varphi(m,n,q)\,\varphi(m,n+q,r)\,\varphi(n,q,r) = \varphi(m+n,q,r)\,\varphi(m,n,q+r) \tag{4.27}$$

for $m,n,q,r \in \Lambda^*$, which is equivalent to the pentagon identity (A.42) of the category $\mathscr{G}$. The pentagon identity can also be derived explicitly by iterating the above calculations to find the non-trivial higher nonassociativity relations

$$\begin{aligned}
\hat{Z}^a \otimes \big(\hat{Z}^b \otimes (\hat{Z}^c \otimes \hat{Z}^d)\big) &= \mathrm{e}^{\,2\pi\,\mathrm{i}\,\hbar\,R^{bcd}}\ \hat{Z}^a \otimes \big((\hat{Z}^b \otimes \hat{Z}^c) \otimes \hat{Z}^d\big) \\
&= \mathrm{e}^{\,2\pi\,\mathrm{i}\,\hbar\,(R^{abc}+R^{abd})}\ \big(\hat{Z}^a \otimes \hat{Z}^b\big) \otimes \big(\hat{Z}^c \otimes \hat{Z}^d\big) \\
&= \mathrm{e}^{\,2\pi\,\mathrm{i}\,\hbar\,(R^{acd}+R^{abd}+R^{bcd})}\ \big(\hat{Z}^a \otimes (\hat{Z}^b \otimes \hat{Z}^c)\big) \otimes \hat{Z}^d \\
&= \mathrm{e}^{\,2\pi\,\mathrm{i}\,\hbar\,(R^{abc}+R^{acd}+R^{abd}+R^{bcd})}\ \big((\hat{Z}^a \otimes \hat{Z}^b) \otimes \hat{Z}^c\big) \otimes \hat{Z}^d \ .
\end{aligned} \tag{4.28}$$

As discussed in appendix A.7, MacLane's coherence theorem implies that these relations automatically imply all higher associativity relations in the category $\mathscr{G}$. This is particularly interesting from the perspective of the quantization of Nambu–Poisson structures that we discussed in section 3.3: As the fundamental identity (A.26) should be encoded in the coherence relations involving five objects, our categorical approach automatically encodes its quantization. This should therefore help to alleviate at least some of the difficulties that arise in implementing the fundamental identity for Nambu–Poisson brackets at the quantum level (see e.g. [45] for a discussion).

4.3 Categorified Weyl quantization

We will now apply this categorical formalism to the deformation quantization of the algebra of functions $C^\infty(\mathcal{M})$ on $\mathcal{M} = T^*M = \mathbb{T}^d \times (\mathbb{R}^d)^*$, regarded as the algebra $C^\infty(V^*)$ as explained before. Here $\mathbb{T}^d = \mathbb{R}^d/\Lambda$, and the $d \times d$ invertible matrix $\Sigma = ({\Sigma_a}^i)$ defines the periods of the directions of the d-torus $M = \mathbb{T}^d$, i.e. $x^i \sim x^i + {\Sigma_a}^i$, $a = 1, \dots, d$ for each $i = 1, \dots, d$; in particular, the (inverse) metric of $\mathbb{T}^d$ is given by ${\Sigma_a}^i\, \delta^{ab}\, {\Sigma_b}^j = g^{ij}$. We embed $C^\infty(\mathcal{M})$ as an algebra object $\mathcal{A}$ of the Lie 2-group $\mathscr{G}$ via a categorification of the Weyl quantization map, see e.g. [117]; it is defined as the linear isomorphism on $C^\infty(\mathcal{M})$ given on the dense set of plane waves by

$$\mathscr{W}\big(\,\mathrm{e}^{\,\mathrm{i}\,k_I\,x^I}\big) = \hat{W}(m,\xi) := \exp\big(\,\mathrm{i}\,k_I\,\hat{x}^I\big)\ , \tag{4.29}$$

and extended by linearity; here

$$(k_I) = (k_1, \dots, k_{2d}) = (k_1, \dots, k_d, \xi^1, \dots, \xi^d) \tag{4.30}$$

with

$$k_i = 2\pi\, (\Sigma^{-1})_i{}^a\, m_a\ , \qquad m = (m_a) \in \Lambda^* \tag{4.31}$$

the quantized Fourier momenta appropriate to smooth single-valued functions on $\mathbb{T}^d$. We regard (4.29) as an object in a suitable enrichment of the Lie 2-group $\mathscr{G}$ to a linear category over $\mathbb{C}$, which we think of as an analogue of a convolution group algebra generated by the operators (4.18). This map can be applied to an arbitrary Schwartz function f on $\mathbb{T}^d \times (\mathbb{R}^d)^*$ by expanding f in its Fourier transformation

$$f(x,p) = \sum_{m\in\Lambda^*}\ \mathrm{e}^{\,2\pi\,\mathrm{i}\,(\Sigma^{-1})_i{}^a\, m_a\, x^i}\ \int_{\mathbb{R}^d} \frac{\mathrm{d}^d\xi}{(2\pi)^d}\ f_m(\xi)\ \mathrm{e}^{\,\mathrm{i}\,\xi^i\, p_i}\ , \tag{4.32}$$

where the inverse Fourier transform is given by

$$f_m(\xi) = \frac{1}{|\det \Sigma|} \int_{\mathbb{T}^d} \mathrm{d}^d x \ \mathrm{e}^{-2\pi\,\mathrm{i}\,(\Sigma^{-1})_i{}^a\, m_a\, x^i} \int_{(\mathbb{R}^d)^*} \mathrm{d}^d p \ \mathrm{e}^{-\,\mathrm{i}\,\xi^i\, p_i}\, f(x,p) \ . \tag{4.33}$$

We then set

$$\mathscr{W}(f) := \sum_{m\in\Lambda^*} \int_{\mathbb{R}^d} \frac{\mathrm{d}^d\xi}{(2\pi)^d}\ f_m(\xi)\ \hat{W}(m,\xi) \ . \tag{4.34}$$

The convolution product $\circledast$ of two functions $f,g\in C^\infty(\mathcal{M})$ is defined via the horizontal product of two quantized functions as

$$\mathscr{W}(f\circledast g) := \mathscr{W}(f)\otimes\mathscr{W}(g) \tag{4.35}$$

in the 2-group $\mathscr{G}$ and the inverse map $\mathscr{W}^{-1}$ from (4.29). Another straightforward application of the Baker–Campbell–Hausdorff formula as in (4.19)–(4.21) yields the 2-group multiplication law

$$\begin{aligned} \hat{W}(m,\xi)\otimes\hat{W}(n,\lambda) =\,& \mathrm{e}^{\,\pi\,\mathrm{i}\,\hbar\,(\Sigma^{-1})_i{}^a\,(m_a\,\lambda^i-n_a\,\xi^i)} \\ & \times \hat{W}\big(m+n\,,\,\xi+\lambda-R^{abc}\,m_a\,n_b\,\Sigma_c\big) \ , \end{aligned} \tag{4.36}$$

and we obtain

$$\begin{aligned} (f\circledast g)(x,p) = \sum_{m,n\in\Lambda^*} & \int_{\mathbb{R}^d} \frac{\mathrm{d}^d\xi}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\mathrm{d}^d\lambda}{(2\pi)^d}\ f_n(\lambda)\, g_{m-n}(\xi-\lambda) \\ & \times\ \mathrm{e}^{-\pi\,\mathrm{i}\ (\Sigma^{-1})_i{}^a\,(\hbar\,(m_a\,\lambda^i-n_a\,\xi^i)-2\pi\,(\Sigma^{-1})_j{}^b\,m_a\,n_b\,R^{ijk}\,p_k)} \\ & \times\ \mathrm{e}^{\,2\pi\,\mathrm{i}\,m_a\,(\Sigma^{-1})_i{}^a\,x^i+\mathrm{i}\,\xi^i\,p_i} \ . \end{aligned} \tag{4.37}$$

After introducing a factor of $\hbar$ as in (3.53), this formula is identical to the star product (3.18) that we found by formal deformation quantization along the twisted Poisson structure Θ, and hence the two quantizations are equivalent in this particular case. This result is a Lie 2-algebra version of Kathotia's theorem [87, section 5] which asserts the equivalence between Kontsevich's deformation quantization and the group convolution algebra quantization of the dual of a nilpotent Lie algebra.

The crux of this theorem does not rely on the Jacobi identity, and is easily applied to our pre-Lie algebra: By a trivial relabelling of the generators, the commutation relations (4.2) satisfy the conclusions of [87, Theorem 5.2.1]. It is tempting to conjecture that the Lie 2-group convolution algebra quantization that we have developed in this section is equivalent to Kontsevich's deformation quantization along the linear twisted Poisson bivector field on the dual of any nilpotent pre-Lie algebra. It would be interesting to similarly characterise the nonassociative quantizations of generic semistrict nilpotent Lie 2-algebras, but these questions lie beyond the scope of this thesis.

We conclude by establishing that the algebra of functions $\mathcal{A} = C^\infty(\mathbb{T}^d \times (\mathbb{R}^d)^*)$ endowed with the nonassociative product $\circledast$ is really an algebra object of the Lie 2-group $\mathscr{G}$, i.e. it satisfies the associativity relation (A.44) of the category. Using the multiplication law (4.36) of the 2-group $\mathscr{G}$ we compute triple products of the operators (4.29) to get

$$
\begin{aligned}
&\big(\hat{W}(m,\xi) \otimes \hat{W}(n,\lambda)\big) \otimes \hat{W}(q,\eta) \\
&\qquad = \ \mathrm{e}^{\pi\,\mathrm{i}\,\hbar\,R^{abc}\,m_a\,n_a\,q_c}\ \mathrm{e}^{\pi\,\mathrm{i}\,\hbar\,(\Sigma^{-1})_i{}^a\,(m_a\,\lambda^i - n_a\,\xi^i + (m+n)_a\,\eta^i - q_a\,(\xi+\lambda)^i)} \\
&\qquad\qquad \times \hat{W}\big(m+n+q\,,\,\xi+\lambda+\eta - R^{abc}\,m_a\,n_b\,\Sigma_c - R^{abc}\,(m+n)_a\,q_b\,\Sigma_c\big)\ .
\end{aligned}
\tag{4.38}
$$

A completely analogous calculation for the other ordering shows that

$$
\begin{aligned}
\hat{W}(m,\xi) \otimes \big(\hat{W}(n,\lambda) \otimes \hat{W}(q,\eta)\big) &= \varphi(m,n,q)\ \big(\hat{W}(m,\xi) \otimes \hat{W}(n,\lambda)\big) \otimes \hat{W}(q,\eta) \\
&= \mathscr{P}\big[\big(\hat{W}(m,\xi) \otimes \hat{W}(n,\lambda)\big) \otimes \hat{W}(q,\eta)\big]\ ,
\end{aligned}
\tag{4.39}
$$

where φ is the 3-cocycle (4.26) and we have used (4.24) to identify the application of the associator isomorphism $\mathscr{P}$ to 2-group objects (4.29); this formula is extended to operators (4.34) in the usual way using linearity. Using (4.38) and the quantization map (4.29), (4.35) we now compute the triple convolution product of functions

$f,g,h\in C^\infty(\mathcal{M})$ to get

$$\begin{aligned}\big(f\circledast(g\circledast h)\big)(x,p) \;=\; & \sum_{m,n,q\in\Lambda^*} \mathrm{e}^{-\pi\,\mathrm{i}\,\hbar\,R^{abc}\,m_a\,n_b\,q_c}\ \mathrm{e}^{\,2\pi\,\mathrm{i}\,(\Sigma^{-1})_i{}^a\,m_a\,x^i} \int_{\mathbb{R}^d}\frac{\mathrm{d}^d\xi}{(2\pi)^d}\ \mathrm{e}^{\,\mathrm{i}\,\xi^i\,p_i} \\ & \times \int_{\mathbb{R}^d}\frac{\mathrm{d}^d\lambda}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\mathrm{d}^d\eta}{(2\pi)^d}\ f_{m-n-q}(\xi-\lambda-\eta)\,g_n(\lambda)\,h_q(\eta) \\ & \times\ \mathrm{e}^{\,\pi\,\mathrm{i}\,\hbar\,(\Sigma^{-1})_i{}^a\,((m-q)_a\,\lambda^i-n_a\,(\xi-\eta)^i+m_a\,\eta^i-q_a\,\xi^i)} \\ & \times\ \mathrm{e}^{\,\mathrm{i}\,R^{abc}\,((m-q)_a\,n_b+m_a\,q_b)\,\Sigma_c{}^i\,p_i}\ , \end{aligned} \tag{4.40}$$

which agrees with the corresponding formula of (3.19). From (4.39) it follows that

$$\big(f\circledast(g\circledast h)\big)(x,p)=\mathscr{P}\big((f\circledast g)\circledast h\big)(x,p) \tag{4.41}$$

as required, where here $\mathscr{P}((f\circledast g)\circledast h)$ is short-hand notation for the composition of morphisms on the right-hand side of (A.44) applied to $(f\otimes g)\otimes h$.

4.4 Monopole backgrounds and topological nonassociative tori

We conclude this chapter by comparing our noncommutative and nonassociative deformation of the cotangent bundle $\mathcal{M}=\mathbb{T}^d\times(\mathbb{R}^d)^*$ with some other appearances of nonassociativity in the literature. The relations (4.36) (or (4.19)) are reminiscent of those obeyed by the gauge invariant operators which generate a projective representation of the translation group in the background field of a Dirac monopole [80] (see also [104]), where the projective phase is a 2-cochain determined by the magnetic flux through a 2-simplex; in our case this flux is proportional to $\xi_R(m,n)\in\mathbb{R}^d$ where

$$\xi_R(m,n)^i=-R^{abc}\,m_a\,n_b\,\Sigma_c{}^i\ , \tag{4.42}$$

and it arises as the gerbe 2-holonomy of the B-field (2.35) through the triangle at p formed by the lattice vectors $m,n\in\Lambda^*\subset(\mathbb{R}^d)^*$ in momentum space. The triple product relation (4.39) (or (4.24)) is reminiscent of the nonassociativity relation which arises from the 3-cocycle proportional to the flux through the 3-simplex enclosing the monopole; in our case the 3-cocycle (4.26) is determined by the gerbe H-flux (2.34) through the tetrahedron at p formed by the lattice vectors m, n and q in momentum space. See [51] for an open string realization of the monopole background in terms of D0-branes in H-space, or equivalently D3-branes in R-space.

Let us now compare our construction with the nonassociative tori discussed in [28, 29, 60]. For $n, q \in \Lambda^*$, we use the 3-cocycle (4.26) to define unitary operators $\hat{U}_{n,q}$ on the Hilbert space $\ell^2(\Lambda^*)$ of square-summable sequences f_m on the momentum lattice Λ^* of $M = \mathbb{T}^d$ by

$$\big(\hat{U}_{n,q} f\big)_m = \varphi(m,n,q)\, f_m \ . \tag{4.43}$$

These operators obey the composition law

$$\varphi(m,n,q)\, \hat{U}_{m,n}\, \hat{U}_{m+n,q} = \alpha_m\big(\hat{U}_{n,q}\big)\, \hat{U}_{m,n+q} \ , \tag{4.44}$$

where α_m is the adjoint action by the regular representation $f_n \mapsto f_{n+m}$ of lattice translations by $m \in \Lambda^*$. One then defines the twisted convolution product

$$(f \circledast_\varphi g)_m = \sum_{n\in\Lambda^*} f_n\, \alpha_n(g_{m-n})\, \hat{U}_{n,m-n} \tag{4.45}$$

on the algebra $C^\infty(\Lambda^*, \mathcal{K})$, where $\mathcal{K} = \mathcal{K}\big(\ell^2(\Lambda^*)\big)$ is the algebra of compact operators on $\ell^2(\Lambda^*)$. This defines a nonassociative twisted crossed-product algebra $\mathcal{K}\big(\ell^2(\Lambda^*)\big) \rtimes_\varphi \Lambda^*$ which is identified with the algebra of functions on the nonassociative torus. When $\varphi = 1$ ($R = 0$), the operators $\hat{U}_{n,q}$ all act as the identity operator on $\ell^2(\Lambda^*)$ and α_m can be taken to be the identity; then $\circledast_{\varphi=1}$ is just the usual convolution product on the algebra $C^\infty(\mathbb{T}^d) \otimes \mathcal{K}$ of stabilized functions on the torus $\mathbb{T}^d$, which is Morita equivalent to the usual commutative algebra $C^\infty(\mathbb{T}^d)$. In the general case, by [28, Proposition 3.1] the twisted convolution product $\circledast_\varphi$ satisfies (A.44) and hence makes $\mathcal{K}\big(\ell^2(\Lambda^*)\big) \rtimes_\varphi \Lambda^*$ an algebra object of the tensor category $\mathscr{G}$; in [60] it is shown that this defines a *strict* (i.e. non-formal) nonassociative deformation quantization.

We can identify a covariant representation of $\big(\Lambda^*\,,\,\mathcal{K}(\ell^2(\Lambda^*))\big)$ by using the commutation relations (4.20) to identify the generators of translations in the lattice Λ^* as the operators $\hat W(m,0)$ for $m\in\Lambda^*$. From (4.36) we may then identify operators through the 2-group multiplication law

$$\hat W(m,0)\otimes\hat W(n,0):=\hat U_{m,n}\otimes\hat W(m+n,0)\ , \tag{4.46}$$

where

$$\hat U_{m,n}=\hat P_{\xi_R(m,n)} \tag{4.47}$$

and we have used the Baker–Campbell–Hausdorff formula together with antisymmetry of R^{abc}. By (4.39), it follows from [28, section 3] that these operators coincide with the ones introduced in (4.43). This correspondence is completely analogous to that found in [51, section 5.2] via an open string analysis of D3-branes in R-space; in particular, our representation of the operators $\hat U_{m,n}$ is determined by the surface holonomy (4.42) of the pertinent B-field as in [51]. However, in our picture, the meaning of the stabilization by the algebra of compact operators $\mathcal{K}$ is clear: It represents precisely the additional cotangent degrees of freedom through the unitary momentum operators $\hat P_\xi\in\mathcal{K}$ from (4.18).

We close by commenting on how our nonassociative algebras may be related to associative ones which can be represented as operator algebras on separable Hilbert spaces, hence justifying some of the constructions above. In the context of open strings in non-trivial H-flux backgrounds, it was shown in [68, 66] how to map the nonassociative algebra of functions equipped with the Kontsevich star product to an associative algebra by enlarging the deformed configuration space to a deformed phase space; the resulting algebra is interpreted as an algebra of pseudo-differential operators as now both coordinates x^I and derivatives ∂_I appear. This mapping is the analogue of the Bopp shift which maps the Heisenberg commutation relations onto trivial commuting variables when viewed as a subalgebra of extended canonical phase space commutation relations. In [36] such Bopp shifts are used to map noncommutative twisted tori onto commutative tori with the same phase space nonassociativity. In our case, the resulting algebra should be compared with the Lie 2-algebra constructed in section 2.5 which has underlying associative coordinate algebra. The construction of [68, 66] is simply a physical implementation of MacLane's coherence theorem, which states that any monoidal category is equivalent to a strict monoidal category in which the associativity isomorphism (4.12) is simply the obvious identification by rebracketing $(g \otimes h) \otimes k \mapsto g \otimes (h \otimes k)$. In the present case, it is shown in [29] that the equivalence functor is obtained by applying $\mathscr{P}^{-1}$ to (4.45) and it takes an algebra object $\mathcal{A}$ to the associative crossed product algebra $\mathcal{A} \rtimes \Lambda^*$; this augmented algebra is in a sense the "exponentiation" of the extended algebras of [68, 66].

Chapter 5

Twist deformation quantization

In this chapter we describe a third way of quantizing non-geometric R-flux backgrounds using twist deformation techniques. The terminology "twist" refers to a deformation of a Hopf algebra H which is constructed from the universal enveloping algebra of a Lie algebra of symmetries acting on the phase space description of R-space. Such deformations are typically provided by a 2-cocycle $F \in H \otimes H$ called a *twist* (see e.g. [92]); gauge and gravity theories on a noncommutative space as deformations of their classical counterparts using twisting techniques can be found e.g. in [8, 10] (see also [11]). The advantage of twist deformation quantization is that it accommodates nonassociativity in a natural and concrete way which overcomes the difficulties encountered in quantizing nonassociative algebras using (higher) Lie algebraic methods, such as Baker-Campbell-Hausdorff quantization: One simply requires that the usual coassociativity of the Hopf algebra H holds only up to a 3-cocycle $\phi \in H \otimes H \otimes H$ called the *associator*; this yields a quasi-Hopf algebra [48]. If ϕ is trivial, i.e. it is the coboundary of a 2-cochain $F \in H \otimes H$, then the twisting is provided by F. Once a twist is known, it is just a matter of applying the cochain twist machinery to deform all geometric structures which are covariant under the symmetries of a manifold; such cochain twists were employed in [16] to describe nonassociative differential calculus and in [93] to formulate gauge theory on nonassociative algebras (see also [15]).

The use of trivial 3-cocycles as sources of nonassociativity first appeared in the physics literature in the description of magnetic translations of charged particles in the background of a magnetic monopole, where it was shown that demanding associativity yields the Dirac quantization condition [80]. In this case one finds an associative representation of the global translations, even though the Jacobi identity for the infinitesimal generators continues to fail. This point of view is taken in the description of non-geometric toroidal flux backgrounds within the framework of Matrix theory compactifications in [37]. Thus although one finds a non-trivial Jacobiator (1.7) on R-space, demanding associativity of global quantities may teach us something about the structure of non-geometric fluxes, such as flux quantization; indeed, the on-shell worldsheet tachyon scattering amplitudes computed in [21] exhibit no violations of associativity once momentum conservation is taken into account, in accord with the standard crossing symmetry of correlation functions in two-dimensional conformal field theory. This point of view of nonassociative R-space is addressed in the context of double field theory in [25], while the parallels between nonassociative parabolic R-flux string models and the dynamics of charged particles in uniform magnetic charge distributions is elucidated in [14].

5.1 Hopf algebras and deformation quantization

Twist deformation techniques provide a very precise and systematic way of quantizing any algebraic structure acted upon by a (quasi-)Hopf algebra. Such is the case for the algebra of functions on a space acted upon by a Lie group of symmetries which will be our main application in this paper. In this section we briefly review standard deformation quantization by cocycle twists as well as the more general case of cochain twists which is our main case of interest.

5.1.1 Cocycle twist quantization

We begin by defining some of the basic algebraic structures that we will encounter in the following. We then describe deformation quantization via a Drinfel'd cocycle twist.

A *bialgebra* H over $\mathbb{C}$ is an associative unital algebra with a counital coalgebra structure that satisfies the properties

$$(\mathrm{id}_H \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_H) \circ \Delta \ , \tag{5.1}$$

$$(\mathrm{id}_H \otimes \varepsilon) \circ \Delta = \mathrm{id}_H = (\varepsilon \otimes \mathrm{id}_H) \circ \Delta \ , \tag{5.2}$$

where $\varepsilon : H \to \mathbb{C}$ is the counit and $\Delta : H \to H \otimes H$ is the coproduct. The relation (5.1) means that the coalgebra is *coassociative*. Throughout we use the usual Sweedler notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ with $h, h_{(1)}, h_{(2)} \in H$ and suppress the summation.

A *quasi-triangular bialgebra* is a pair $(H, \mathcal{R})$ where H is a bialgebra, and $\mathcal{R} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)} \in H \otimes H$ is an invertible element which obeys

$$(\Delta \otimes \mathrm{id}_H)(\mathcal{R}) = \mathcal{R}_{13}\, \mathcal{R}_{23} \ , \quad (\mathrm{id}_H \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\, \mathcal{R}_{12} \ , \quad (\tau \circ \Delta)(h) = \mathcal{R}\, \Delta(h)\, \mathcal{R}^{-1} \tag{5.3}$$

for all $h \in H$, where $\mathcal{R}_{12} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)} \otimes 1_H$, $\mathcal{R}_{13} = \mathcal{R}_{(1)} \otimes 1_H \otimes \mathcal{R}_{(2)}$, $\mathcal{R}_{23} = 1_H \otimes \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)}$ with 1_H the unit of H, and we abbreviate the product map $\mu : H \otimes H \to H$ by $\mu(h \otimes h') = h\, h'$ for all $h, h' \in H$. Here we have defined the *transposition map* $\tau : H \otimes H \to H \otimes H$ as

$$\tau(h \otimes h') := h' \otimes h \tag{5.4}$$

for all $h, h' \in H$.

Through the transposition map the *co-opposite coproduct* $\Delta^{\mathrm{op}} : H \to H \otimes H$ is defined by

$$\Delta^{\mathrm{op}}(h) := (\tau \circ \Delta)(h) = h_{(2)} \otimes h_{(1)} \ . \tag{5.5}$$

Then H is a cocommutative coalgebra if $\Delta^{\mathrm{op}}(h) = \Delta(h)$ for all $h \in H$. If $(H, \mathcal{R})$ is a quasi-triangular bialgebra then cocommutativity simply means that $\Delta(h)\, \mathcal{R} = \mathcal{R}\, \Delta(h)$ for all $h \in H$ as can be easily seen from (5.3); in general the element $\mathcal{R}$ intertwines the action of the coproduct Δ with the co-opposite coproduct Δ^{op}.

A *Hopf algebra* over $\mathbb{C}$ is a bialgebra H equipped with an algebra anti-automorphism

$S : H \to H$ called the antipode satisfying

$$\mu \circ (\mathrm{id}_H \otimes S) \circ \Delta = \eta_{1_H} \circ \varepsilon = \mu \circ (S \otimes \mathrm{id}_H) \circ \Delta \ , \tag{5.6}$$

where $\eta_h : \mathbb{C} \to H$ is the unit homomorphism with $\eta_h(1) = h$ for $h \in H$. A quasi-triangular Hopf algebra $(H, \mathcal{R})$ consists of a quasi-triangular structure $\mathcal{R}$ on the underlying bialgebra of H.

In this thesis we will be primarily interested in the large class of Hopf algebras H which arise as universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$. The algebra $U(\mathfrak{g})$ is constructed by taking the quotient of the tensor algebra $T(\mathfrak{g}) = \bigoplus_{k\geq 0} \mathfrak{g}^{\otimes k} = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$ by the two-sided ideal $\mathcal{I}$ generated by elements of the form $x \otimes y - y \otimes x - [x, y]$, where $x, y \in \mathfrak{g}$. Next we equip $U(\mathfrak{g})$ with the symmetric coalgebra structure

$$\begin{aligned} \Delta \,:\, U(\mathfrak{g}) \;&\longrightarrow\; U(\mathfrak{g}) \otimes U(\mathfrak{g}) \ , \qquad && \Delta(x) = x \otimes 1 + 1 \otimes x \ , \\ \varepsilon \,:\, U(\mathfrak{g}) \;&\longrightarrow\; \mathbb{C} \ , && \varepsilon(x) = 0 \ , \\ S \,:\, U(\mathfrak{g}) \;&\longrightarrow\; U(\mathfrak{g}) \ , && S(x) = -x \end{aligned} \tag{5.7}$$

defined on primitive elements $x \in \mathfrak{g}$ and extended to all of $U(\mathfrak{g})$ as algebra (anti-)homomorphisms. The desired Hopf algebra H is then $U(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}$ with these structure maps. Finally, we further equip H with the trivial quasi-triangular structure

$$\mathcal{R}_0 = 1 \otimes 1 \ , \qquad 1_H := 1 \tag{5.8}$$

which turns it into a cocommutative quasi-triangular Hopf algebra.

A Hopf algebra H can act on a complex vector space V to give a representation of H on V. In particular, a *left action* of H on V is a pair (λ, V), where $\lambda : H \otimes V \to V$ is a linear map, $\lambda(h \otimes v) =: \lambda_h(v)$, such that $\lambda_{h\,g}(v) = \lambda_h(\lambda_g(v))$ and $\lambda(1_H \otimes v) = v$, where $g, h \in H$ and $v \in V$. It is customary to denote the action of H on V by $\triangleright$ and write the above relations as

$$h \triangleright v \ \in \ V \ , \qquad (h\,g) \triangleright v = h \triangleright (g \triangleright v) \ , \qquad 1_H \triangleright v = v \ . \tag{5.9}$$

Such a vector space is called a *left H-module*. If V carries additional structure, for example if it is an algebra (A, μ_A) where $\mu_A : A \otimes A \to A$ is the product on A, or a coalgebra (C, Δ_C) where $\Delta_C : C \to C \otimes C$ is the coproduct on C, then we demand that the action of H is covariant in the sense that it preserves the additional structure of V. Thus we say that a unital algebra (A, μ_A) over $\mathbb{C}$ is a *left H-module algebra* if A is a left H-module and

$$h \triangleright (a\, b) = h \triangleright \mu_A(a \otimes b) = \mu_A\big(\Delta(h) \triangleright (a \otimes b)\big) = \big(h_{(1)} \triangleright a\big)\, \big(h_{(2)} \triangleright b\big)\ , \tag{5.10}$$

where $h \in H$ and $a, b \in A$. Here we abbreviate $\mu_A(a \otimes b) = a\, b$; we use this notation throughout when no confusion arises. Likewise, a counital coalgebra (C, Δ_C) is a *left H-module coalgebra* if C is a left H-module and

$$(h \triangleright c)_{(1)} \otimes (h \triangleright c)_{(2)} = \Delta_C(h \triangleright c) = \Delta(h) \triangleright \Delta_C(c) = \big(h_{(1)} \triangleright c_{(1)}\big) \otimes \big(h_{(2)} \triangleright c_{(2)}\big)\ , \tag{5.11}$$

where $h \in H$ and $c \in C$. Similar definitions hold for right H-modules, and right H-module algebras and coalgebras.

From the above definitions it is understood that a left or right H-module is a representation by an algebra A or a coalgebra C of the Hopf algebra H; if H is modified then so is the representation. Such modifications were introduced by Drinfel'd. Let $H[[\hbar]]$ denote the $\hbar$-adic completion of H consisting of all formal H-valued power series in a deformation parameter $\hbar$. A *Drinfel'd twist* is an invertible element $F = F_{(1)} \otimes F_{(2)} \in H[[\hbar]] \otimes H[[\hbar]]$ that satisfies the two conditions

$$(F \otimes 1_H)\, (\Delta \otimes \mathrm{id}_H)(F) = (1_H \otimes F)\, (\mathrm{id}_H \otimes \Delta)(F)\ , \tag{5.12}$$

$$(\varepsilon \otimes \mathrm{id}_H)(F) = 1_H = (\mathrm{id}_H \otimes \varepsilon)(F)\ . \tag{5.13}$$

We will further demand that $F = 1_H \otimes 1_H + \mathcal{O}(\hbar)$ is a deformation of the trivial twist, which is always formally invertible for sufficiently small $\hbar$. By these two conditions F is a counital 2-cocycle which can be used to define a new Hopf algebra H_F with the same underlying algebra as $H[[\hbar]]$ but with a twisted coalgebraic structure given by the twisted coproduct

$$\Delta_F(h) = F\,\Delta(h)\,F^{-1} \tag{5.14}$$

and the twisted antipode

$$S_F(h) = U_F\,S(h)\,U_F^{-1} \qquad \text{where} \quad U_F = \mu\circ(\mathrm{id}_H\otimes S)(F) \tag{5.15}$$

for $h\in H$. This new bialgebra H_F is called a *twisted Hopf algebra*; coassociativity and counitality of (5.14) follow respectively from the 2-cocycle condition (5.12) and the counital condition (5.13). If $(H,\mathcal{R})$ is a quasi-triangular Hopf algebra then the quasi-triangular structure is also twisted by the formula

$$\mathcal{R}_F := \tau(F)\,\mathcal{R}\,F^{-1} = F_{21}\,\mathcal{R}\,F^{-1} \ , \tag{5.16}$$

where $F_{21} = F_{(2)}\otimes F_{(1)}$ in Sweedler notation. The Hopf algebra H_F need not be cocommutative even if H is cocommutative; this may be checked by calculating the twisted co-opposite coproduct

$$\Delta_F^{\mathrm{op}}(h) = \mathcal{R}_F\,\Delta_F(h)\,\mathcal{R}_F^{-1} \ , \tag{5.17}$$

where $h\in H$.

For the twisted Hopf algebra H_F to act covariantly on a left H-module algebra (A,μ_A) we need to twist (deform) the binary product $\mu_A : A\otimes A\to A$ to a new product defined by

$$a\star b = \mu_A\big(F^{-1}\triangleright(a\otimes b)\big) = \big(F^{-1}_{(1)}\triangleright a\big)\,\big(F^{-1}_{(2)}\triangleright b\big) \ , \tag{5.18}$$

for $a,b\in A$. The deformed product is called a *star product* and $(A[[\hbar]],\star)$ is a deformation quantization of (A,μ_A); indeed one has $a\star b = a\,b+\mathcal{O}(\hbar)$. The twist cocycle condition (5.12) ensures associativity of the star product (5.18), while the counital condition (5.13) implies that if (A,μ_A) is unital with unit 1_A then $(A[[\hbar]],\star)$

is also unital with the same unit.

5.1.2 Quasi-Hopf algebras and cochain twist quantization

In this thesis we are concerned with nonassociative twist deformations, therefore we will be using an appropriate generalisation of a Hopf algebra, called a quasi-Hopf algebra [48]. To explain what a quasi-Hopf algebra is let us begin by defining the notion of a *quasi-bialgebra*. This is simply a bialgebra H where coassociativity is required to hold only up to a 3-cocycle ϕ, i.e. the condition (5.1) is substituted by

$$(\mathrm{id}_H \otimes \Delta) \circ \Delta(h) = \phi \left[(\Delta \otimes \mathrm{id}_H) \circ \Delta(h) \right] \phi^{-1} \ , \tag{5.19}$$

where $h \in H$ and $\phi = \phi_{(1)} \otimes \phi_{(2)} \otimes \phi_{(3)} \in H \otimes H \otimes H$ is an invertible 3-cocycle (see e.g. [92]) in the sense that

$$(1_H \otimes \phi) \left[(\mathrm{id}_H \otimes \Delta \otimes \mathrm{id}_H)(\phi) \right] (\phi \otimes 1_H) = \left[(\mathrm{id}_H \otimes \mathrm{id}_H \otimes \Delta)(\phi) \right] \left[(\Delta \otimes \mathrm{id}_H \otimes \mathrm{id}_H)(\phi) \right] . \tag{5.20}$$

We say that ϕ is counital if it additionally satisfies the condition

$$(\varepsilon \otimes \mathrm{id}_H \otimes \mathrm{id}_H)(\phi) = (\mathrm{id}_H \otimes \varepsilon \otimes \mathrm{id}_H)(\phi) = (\mathrm{id}_H \otimes \mathrm{id}_H \otimes \varepsilon)(\phi) = 1_H \otimes 1_H \ . \tag{5.21}$$

These two conditions on ϕ ensure that all distinct orderings of higher coproducts by insertions of ϕ yield the same result and are consistent with the counital condition (5.2).

The definition of a quasi-triangular quasi-bialgebra is that of a quasi-triangular bialgebra with the two first axioms of (5.3) modified by ϕ to

$$(\Delta \otimes \mathrm{id}_H)(\mathcal{R}) = \phi_{321}\, \mathcal{R}_{13}\, \phi_{132}^{-1}\, \mathcal{R}_{23}\, \phi \ , \qquad (\mathrm{id}_H \otimes \Delta)(\mathcal{R}) = \phi_{231}^{-1}\, \mathcal{R}_{13}\, \phi_{213}\, \mathcal{R}_{12}\, \phi^{-1} \tag{5.22}$$

in the notation of section 5.1.1 with $\phi_{abc} := \phi_{(a)} \otimes \phi_{(b)} \otimes \phi_{(c)}$, while the third axiom of (5.3) remains unchanged.

A *quasi-Hopf algebra* $\mathcal{H} = (H, \phi)$ is a quasi-bialgebra H equipped with an antipode that consists of two elements $\alpha, \beta \in H$ and an algebra anti-automorphism $S : H \to H$ obeying

$$S(h_{(1)})\, \alpha\, h_{(2)} = \varepsilon(h)\, \alpha \ , \qquad h_{(1)}\, \beta\, S(h_{(2)}) = \varepsilon(h)\, \beta \ , \tag{5.23}$$

$$\phi_{(1)}\, \beta\, S(\phi_{(2)})\, \alpha\, \phi_{(3)} = 1_H \ , \qquad S\big(\phi^{-1}_{(1)}\big)\, \alpha\, \phi^{-1}_{(2)}\, \beta\, S\big(\phi^{-1}_{(3)}\big) = 1_H \ , \tag{5.24}$$

for all $h \in H$. The antipode is determined uniquely only up to the transformations

$$S'(h) = u\, S(h)\, u^{-1} \ , \qquad \alpha' = u\, \alpha \ , \qquad \beta' = \beta\, u^{-1} \ , \tag{5.25}$$

for any invertible element $u \in H$ and any $h \in H$. When $\phi = 1_H \otimes 1_H \otimes 1_H$ is the trivial 3-cocycle, the conditions (5.24) imply that $\alpha\, \beta = \beta\, \alpha = 1_H$, and the symmetry (5.25) allows us to suppose without loss of generality that $\alpha = \beta = 1_H$. Then (5.19) reduces to the coassociativity condition (5.1) and (5.23) to the usual definition of an antipode given by (5.6), thus the quasi-Hopf algebra $\mathcal{H}$ becomes a coassociative Hopf algebra.

A useful way to construct a quasi-Hopf algebra is to start with a Hopf algebra H and an invertible twist element $F \in H[[\hbar]] \otimes H[[\hbar]]$ that does not satisfy the cocycle condition (5.12). In particular, if $(H, \phi, \mathcal{R})$ is a quasi-triangular quasi-Hopf algebra and F is an arbitrary invertible element in $H[[\hbar]] \otimes H[[\hbar]]$ obeying (5.13), then $(H_F, \phi_F, \mathcal{R}_F)$ defined as follows is also a quasi-triangular quasi-Hopf algebra. It has the same algebra and counit as H, with twisted coproduct and quasi-triangular structure defined by the same formulas (5.14) and (5.16), with twisted antipode

$$S_F = S \ , \qquad \alpha_F = S\big(F^{-1}_{(1)}\big)\, \alpha\, F^{-1}_{(2)} \ , \qquad \beta_F = F_{(1)}\, \beta\, S\big(F_{(2)}\big) \ , \tag{5.26}$$

and with twisted 3-cocycle given by the coboundary

$$\phi_F = \partial^* F := F_{23}\, \big[(\mathrm{id}_H \otimes \Delta)(F)\big]\, \phi\, \big[(\Delta \otimes \mathrm{id}_H)\big(F^{-1}\big)\big]\, F^{-1}_{12} \ , \tag{5.27}$$

where $F_{23} = 1_H \otimes F$, $F^{-1}_{12} = F^{-1} \otimes 1_H$ and $\phi_F \in H[[\hbar]] \otimes H[[\hbar]] \otimes H[[\hbar]]$ is called the *associator* (see e.g. [93]). A Hopf algebra H viewed as a trivial quasi-Hopf algebra has $\alpha = \beta = 1_H$ and the symmetry (5.25). Twisting H with the counital 2-cochain twist F then provides a quasi-Hopf algebra $\mathcal{H}_F = (H_F, \phi_F)$ with $S_F = S$, $\alpha_F = \mu \circ (S \otimes \mathrm{id}_H)(F^{-1})$ and $\beta_F = \mu \circ (\mathrm{id}_H \otimes S)(F) = \alpha_F^{-1}$ which by (5.25) is equivalent to (5.15). The twisted coproduct Δ_F fails to satisfy (5.1), and in particular (5.19) is a consequence of this definition.

A left H-module algebra (A, μ_A) is then twisted to a *nonassociative* algebra $(A[[\hbar]], \star)$ by the same formula (5.18) with the associator appearing when we re-bracket products of three elements as

$$(a \star b) \star c = \big(\phi_{F\,(1)} \triangleright a\big) \star \big[\big(\phi_{F\,(2)} \triangleright b\big) \star \big(\phi_{F\,(3)} \triangleright c\big)\big] \ , \tag{5.28}$$

for $a, b, c \in A$. The cocycle condition (5.20) on ϕ_F ensures that the distinct ways of re-bracketing higher order products by inserting ϕ_F all yield the same result.

5.1.3 Twist quantization functor

A natural way to deal with both noncommutative and nonassociative structures arising as above is through the formalism of braided monoidal categories. The algebras encountered above are "braided-commutative" and "quasi-associative", in the sense that they are noncommutative and nonassociative but in a controlled way by means of a braiding and a multiplicative associator, respectively. This means that the algebras *are* commutative and associative when regarded as objects of a suitable braided monoidal category which is different from the usual category of complex vector spaces. The twist deformation quantization described above can then be regarded as a functor that yields algebras in such a braided monoidal category, and at the same time quantises all other covariant structures with respect to a symmetry. We briefly review this framework here as we will make reference to it later on, and because it connects with some of the constructions of chapter 4.

Recall that a monoidal category $\mathscr{C}$ consists of a collection of objects $X, Y, Z, \dots$ with a tensor product between any two objects and a natural associativity isomorphism $\mathscr{P}_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ for any three objects obeying the pentagon identity, which states that the two ways of re-bracketing morphisms

$\big((X\otimes Y)\otimes Z\big)\otimes W \to X\otimes\big(Y\otimes(Z\otimes W)\big)$ are the same. Then MacLane's coherence theorem states that all different ways of inserting associators $\mathscr{P}$ as needed to make sense of higher order re-bracketed expressions yield the same result. A braiding on $\mathscr{C}$ is a natural commutativity isomorphism $\mathscr{B}_{X,Y} : X\otimes Y \to Y\otimes X$ for any pair of objects which is compatible with the associativity structure in a natural way. If $\mathscr{C}$ is the category of complex vector spaces, then the associator $\mathscr{P}$ is the identity morphism and the braiding $\mathscr{B}$ is the transposition morphism.

If $\mathcal{H} = (H,\phi)$ is a quasi-Hopf algebra, we take $\mathscr{C}$ to be the category ${}_H\mathscr{M}$ of left H-modules. This is a monoidal category with tensor product defined via the coproduct Δ and with associator given by

$$\mathscr{P}_{X,Y,Z}\big((x\otimes y)\otimes z\big) = \big(\phi_{(1)}\triangleright x\big)\otimes\big[\big(\phi_{(2)}\triangleright y\big)\otimes\big(\phi_{(3)}\triangleright z\big)\big] \tag{5.29}$$

for all $x\in X$, $y\in Y$ and $z\in Z$. If in addition H is quasi-triangular then there is a braiding defined by

$$\mathscr{B}_{X,Y}(x\otimes y) = \big(\mathcal{R}_{(1)}\triangleright y\big)\otimes\big(\mathcal{R}_{(2)}\triangleright x\big) \tag{5.30}$$

for all $x\in X$ and $y\in Y$.

Given a cochain twist $F\in H[[\hbar]]\otimes H[[\hbar]]$, the constructions of this chapter determine a functorial isomorphism of braided monoidal categories

$$\mathscr{F}_F \,:\, {}_H\mathscr{M} \longrightarrow {}_{H_F}\mathscr{M} \tag{5.31}$$

which acts as the identity on objects and morphisms, but intertwines the tensor, braiding and associativity structures. In particular, the covariance condition (5.10) means that the product map μ_A is a morphism in the category ${}_H\mathscr{M}$; hence $\mathscr{F}_F$ functorially deforms H-module algebras into H_F-module algebras, and in this sense it may be regarded as a "twist quantization functor".

In our main case of interest in this chapter, we will take $H = U(\mathfrak{g})$ to be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ of symmetries acting on a manifold $\mathcal{M}$; then the Hopf algebra H acts on the algebra of smooth functions $A = C^\infty(\mathcal{M})$, and by functoriality of the twist deformation it is also quantized to a generically noncommutative and nonassociative algebra A_F, which is in fact commutative and associative in the category ${}_{H_F}\mathscr{M}$. Similarly, the exterior algebra of differential forms $\Omega^\bullet(\mathcal{M})$ is quantized to $\Omega^\bullet_F(\mathcal{M})$ as a differential calculus on A_F; in this way any geometry can be systematically quantized with respect to a symmetry and a choice of 2-cochain F.

5.2 Cochain twist quantization of non-geometric flux backgrounds

In this section we employ the formalism of section 5.1 to study nonassociative deformations of certain non-geometric closed string backgrounds. In order to make contact with the key ideas of chapter 2 we will initially study standard deformation quantization on the cotangent bundle of a closed string vacuum and subsequently add a constant background R-flux. This approach has the advantage of illuminating pertinent non-local and non-geometric symmetry transformations analogous to the ones which arise on T-folds induced by parabolic monodromies.

5.2.1 Quantum phase space

Let us begin by considering a manifold M of dimension d with trivial cotangent bundle $\mathcal{M} := T^*M \cong M \times (\mathbb{R}^d)^*$ and coordinates $x^I = (x^i, p_i)$, where $I = 1, \ldots, 2d$, $(x^i) \in M$, $(p_i) \in (\mathbb{R}^d)^*$ and $i = 1, \ldots, d$. Throughout we use upper case indices for the full phase space while lower case indices will be reserved for position or momentum space individually. Consider the abelian Lie algebra $\mathfrak{h} = \mathbb{R}^d \oplus (\mathbb{R}^d)^*$ of dimension $2d$ generated by P_i and $\tilde{P}^i$. It is realised on $\mathcal{M}$ by its action on the algebra of smooth complex functions $C^\infty(\mathcal{M})$ which we take to be given by the vector fields

$$P_i \triangleright f := \partial_i f \qquad \text{and} \qquad \tilde{P}^i \triangleright f := \tilde{\partial}^i f \ , \tag{5.32}$$

where $f \in C^\infty(\mathcal{M})$, $\partial_i = \frac{\partial}{\partial x^i}$ and $\tilde{\partial}^i = \frac{\partial}{\partial p_i}$. For constant vectors $a = (a^i) \in \mathbb{R}^d$ and $a = (\tilde{a}_i) \in (\mathbb{R}^d)^*$ we can define $P_a = a^i\, P_i$ and $\tilde{P}_{\tilde{a}} = \tilde{a}_i\, \tilde{P}^i$ which as vector fields on

$\mathcal{M}$ translate x^i and p_i by a^i and $\tilde{a}_i$ respectively, hence $\mathfrak{h}$ is the classical phase space translation algebra on M.

We can now construct the related Hopf algebra K that acts on an algebra (A, μ_A) in the usual way, i.e. we consider the universal enveloping algebra $K = U(\mathfrak{h})$ and equip it with the coalgebra structure from section 5.1.1. In particular, the action of K on $A = C^\infty(\mathcal{M})$ is given by (5.32) extended covariantly to all elements in K using linearity and the Leibniz rule for the vector fields ∂_i and $\tilde{\partial}^i$.

Phase space quantization is carried out simply by twisting K in the manner described in section 5.1.1. A suitable abelian twist $F \in K[[\hbar]] \otimes K[[\hbar]]$ is given by

$$F = \exp\left[-\tfrac{\mathrm{i}\,\hbar}{2}\left(P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_i\right)\right] , \tag{5.33}$$

where $\hbar$ is the deformation parameter. In this simple case $\Delta_F = \Delta$, where the coproduct Δ is defined in (5.7), and thus the twisted Hopf algebra K_F is cocommutative. The twisted quasi-triangular structure is easily calculated from (5.16) and is given by

$$\mathcal{Q} = F^{-2} = \exp\left[\,\mathrm{i}\,\hbar\left(P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_i\right)\right] . \tag{5.34}$$

We may now deform any left (or right) K-module algebra (A, μ_A) using (5.18) and the relevant action. Let us do this for the algebra of functions on $\mathcal{M}$, i.e. we set $A = C^\infty(\mathcal{M})$ and $\mu_A(f \otimes g) = f\,g$ the pointwise multiplication of functions, and derive its deformation quantization $(C^\infty(\mathcal{M})[[\hbar]], \star)$; the star product given by (5.18) is

$$f \star g = \mu_A\Big(\exp\left[\tfrac{\mathrm{i}\,\hbar}{2}\left(\partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\right)\right](f \otimes g)\Big) , \tag{5.35}$$

where the action (5.32) has been used. This *noncommutative* star product is the canonical associative Moyal-Weyl star product familiar from quantum mechanics. With its use, the usual quantum phase space commutation relations are calculated as

$$[x^i, p_j]_\star = \mathrm{i}\,\hbar\,\delta^i{}_j \ , \qquad [x^i, x^j]_\star = 0 = [p_i, p_j]_\star \ , \tag{5.36}$$

where $[f, g]_\star := f \star g - g \star f$ for all $f, g \in C^\infty(\mathcal{M})$.

5.2.2 Noncommutative quantum phase space

Let us endow M with a constant trivector $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$ which is T-dual to the background H-flux of a non-trivial closed string B-field. To bring R into the twist quantization scheme we introduce a family of antisymmetric linear combinations of the generators of $\mathfrak{h}$ as

$$\bar{M}_{ij} := M^{\bar{p}}_{ij} = \bar{p}_i \, P_j - \bar{p}_j \, P_i \ , \tag{5.37}$$

which we will regard as parametrized by constant momentum surfaces $\bar{p} = (\bar{p}_i) \in (\mathbb{R}^d)^*$. The generators $\bar{M}_{ij}$ are unique in the sense that they are the only rank two tensors constructed by primitive elements in $\mathfrak{h}$ to lowest order that can be non-trivially contracted with the constant antisymmetric trivector components R^{ijk}. The restriction to constant momentum surfaces here will ensure (co)associativity, but will be relaxed in the next subsection.

The unique twist element that can be constructed in this way from generators of $\mathfrak{h}$ is the abelian twist $\bar{F}_R \in K[[\hbar]] \otimes K[[\hbar]]$ given by

$$\bar{F}_R = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{8} \, R^{ijk} \, (\bar{M}_{ij} \otimes P_k - P_i \otimes \bar{M}_{jk})\big) \ , \tag{5.38}$$

where $\hbar$ is the deformation parameter. The Hopf algebra K is twisted to a new Hopf algebra $K_{\bar{F}_R}$ which is cocommutative with quasi-triangular structure given by

$$\bar{\mathcal{R}} = \bar{F}_R^{\,-2} \ , \tag{5.39}$$

and the algebra of functions $(C^\infty(\mathcal{M}), \mu_A)$ is quantized to $(C^\infty(\mathcal{M})[[\hbar]], \bar{\star} := \star_{\bar{p}})$. The star product $\bar{\star}$ is calculated by (5.18) as

$$f \, \bar{\star} \, g = \mu_A\Big(\exp\big[\tfrac{\mathrm{i}\,\hbar}{2} \, R^{ijk} \, \bar{p}_k \, \partial_i \otimes \partial_j\big](f \otimes g)\Big) \tag{5.40}$$

for all $f, g \in C^\infty(\mathcal{M})$, and it is a *family of Moyal-Weyl products*. This can be seen by calculating the $\bar{\star}$-commutators $[f,g]_{\bar{\star}} := f \, \bar{\star} \, g - g \, \bar{\star} \, f$ on phase space coordinate functions where we find

$$[x^i, x^j]_{\bar{\star}} = \mathrm{i}\,\hbar\,\theta^{ij}(\bar{p}) \ , \qquad [x^i, p_j]_{\bar{\star}} = 0 = [p_i, p_j]_{\bar{\star}} \ , \tag{5.41}$$

where

$$\theta^{ij}(\bar{p}) = R^{ijk}\,\bar{p}_k \ . \tag{5.42}$$

This reveals that upon twisting, the cotangent bundle $\mathcal{M} = T^*M$ is deformed to $M_{\theta(\bar{p})} \times (\mathbb{R}^d)^*$; hence configuration space is quantized to a family of noncommutative spaces $M_{\theta(\bar{p})}$ with constant noncommutativity parameters proportional to the constant background R-flux and parametrized by the surfaces of constant momentum $\bar{p} = (\bar{p}_i) \in (\mathbb{R}^d)^*$. This is similar to what happens in the associative Q-flux T-duality frame with parabolic monodromy, where the configuration space noncommutativity is proportional to the winding numbers $w \in \mathbb{Z}^d$ of closed strings, i.e. $[x^i, x^j] = \mathrm{i}\,\hbar\,Q^{ij}{}_k\,w^k$.

The above construction is further extended to give a noncommutative quantum phase space if we use the abelian twist

$$\begin{aligned}\bar{\mathcal{F}} &:= \bar{F}_R\,F = F\,\bar{F}_R \\ &= \exp\big[-\tfrac{\mathrm{i}\hbar}{2}\big(\tfrac{1}{4}\,R^{ijk}\,(\bar{M}_{ij}\otimes P_k - P_i\otimes\bar{M}_{jk}) + P_i\otimes\tilde{P}^i - \tilde{P}^i\otimes P_i\big)\big] \ .\end{aligned} \tag{5.43}$$

The quasi-triangular Hopf algebra $(K, \mathcal{R}_0)$ is twisted to the cocommutative quasi-triangular Hopf algebra $(K_{\bar{\mathcal{F}}}, \bar{\mathcal{Q}})$ with quasi-triangular structure

$$\bar{\mathcal{Q}} = \bar{\mathcal{F}}^{-2} = \mathcal{Q}\,\bar{\mathcal{R}} = \bar{\mathcal{R}}\,\mathcal{Q} \ , \tag{5.44}$$

and quantization of $(C^\infty(\mathcal{M}), \mu_A)$ is given by $(C^\infty(\mathcal{M})[[\hbar]], \bar{\star} := \star_{\bar{p}})$, where

$$f\,\bar{\star}\,g = \mu_A\Big(\exp\big[\tfrac{\mathrm{i}\hbar}{2}\big(R^{ijk}\,\bar{p}_k\,\partial_i\otimes\partial_j + \partial_i\otimes\tilde{\partial}^i - \tilde{\partial}^i\otimes\partial_i\big)\big](f\otimes g)\Big) \ , \tag{5.45}$$

for all $f, g \in C^\infty(\mathcal{M})$. In this case the full cotangent bundle $\mathcal{M}$ becomes a noncommutative quantum phase space with commutation relations

$$[x^i, x^j]_{\bar\star} = \mathrm{i}\,\hbar\,\theta^{ij}(\bar p) \qquad , \qquad [x^i, p_j]_{\bar\star} = \mathrm{i}\,\hbar\,\delta^i{}_j \qquad , \qquad [p_i, p_j]_{\bar\star} = 0 \tag{5.46}$$

where $[f,g]_{\bar\star} := f\,\bar\star\,g - g\,\bar\star\,f$ for all $f, g \in C^\infty(\mathcal{M})$. In particular, the zero momentum surface $\bar p = 0$ recovers the canonical Moyal-Weyl product $\star = \star_0$ on phase space from subsection 5.2.1.

The star product (5.45) is exactly the one that appeared in section 3.3 as the restriction of the nonassociative star product (3.18) to slices of constant momentum in phase space. However, in the context of cocycle twist quantization its origin can be traced back to the unique choice of contraction of the R-flux R^{ijk} which is compatible with the "minimal" (translation) symmetries of $\mathcal{M}$ and associativity. We will see below how this choice naturally extends to dynamical momentum and leads to the nonassociative star product (3.17), thus further clarifying the operations of restricting to constant momentum and of reinstating dynamical momentum dependence that were used in section 3.3 for the derivation of the quantized associator.

5.2.3 Nonassociative quantum phase space

As discussed in chapter 2, the trivector R has no natural geometric interpretation on configuration space except via T-duality. On the other hand, it is a 3-form on phase space $\mathcal{M}$ which is in fact the curvature of a non-flat $U(1)$ gerbe in momentum space. The presence of this 3-form enhances the symmetries of $\mathcal{M} = T^*M$ and thus the abelian Lie algebra $\mathfrak{h}$ should be enlarged in order to accommodate the new symmetries. For this, we extend $\mathfrak{h}$ to the non-abelian nilpotent Lie algebra $\mathfrak{g}$ of dimension $\frac{1}{2}\,d\,(d+3)$ generated by P_i, $\tilde P^i$ and $M_{ij} = -M_{ji}$ with commutation relations given by

$$[\tilde P^i, M_{jk}] = \delta^i{}_j\,P_k - \delta^i{}_k\,P_j \ , \tag{5.47}$$

while all other commutators are equal to zero. The actions of P_i and $\tilde{P}^i$ on $C^\infty(\mathcal{M})$ are still given by (5.32) from which we find the action of M_{ij} on $C^\infty(\mathcal{M})$ to be

$$M_{ij} \triangleright f := p_i\,\partial_j f - p_j\,\partial_i f \tag{5.48}$$

for all $f \in C^\infty(\mathcal{M})$. Introducing elements $M_\sigma = \frac{1}{2}\,\sigma^{ij}\,M_{ij} \in \mathfrak{g}$, where $\sigma^{ij} = -\sigma^{ji} \in \mathbb{R}$, we see that M_σ generates the non-local coordinate transformations

$$x^i \longmapsto x^i + \sigma^{ij}\,p_j\ , \qquad p_i \longmapsto p_i \tag{5.49}$$

that mix positions and momenta, which in quantum mechanics are called *Bopp shifts*. This symmetry is reminiscent of those encountered in T-folds (Q-space), where diffeomorphism symmetries include T-duality transformations that mix positions with winding numbers which are T-dual to the conjugate momenta. This points to the use of doubled geometry,* while here we are on a phase space of $2d$ coordinates. In this sense the symmetries (5.49) can be regarded as the analogue of T-duality transformations in our algebraic framework.

One may now proceed as we did previously to find the related quasi-triangular Hopf algebra $(H, \mathcal{R}_0) = (U(\mathfrak{g}), \Delta, \varepsilon, S, \mathcal{R}_0)$, i.e. we equip the universal enveloping algebra $U(\mathfrak{g})$ with the structure maps (5.7) and the quasi-triangular structure (5.8). Then H can be twisted by the abelian Drinfel'd twist

$$F_R = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{8}\,R^{ijk}\,(M_{ij}\otimes P_k - P_i \otimes M_{jk})\big)\ . \tag{5.50}$$

The result is a cocommutative twisted Hopf algebra H_{F_R} with quasi-triangular structure $\mathcal{R} = {F_R}^{-2}$. We can use H_{F_R} to twist the algebra of functions $(C^\infty(\mathcal{M}), \mu_A)$, and the resulting star product has the form

$$f \star_p g := f \star g = \mu_A\Big(\exp\big[\tfrac{\mathrm{i}\,\hbar}{2}\,R^{ijk}\,p_k\,\partial_i \otimes \partial_j\big](f\otimes g)\Big) \tag{5.51}$$

*It is possible to extend our star products below to T-duality covariant star products defined on double phase space, as in [14]; a field theory written in this formalism is manifestly $O(d,d)$-invariant. However, in order to avoid overly cumbersome equations with essentially the same generic features, for simplicity we write all formulas below only in the R-flux duality frame.

for all $f,g\in C^\infty(\mathcal{M})$, and it is a noncommutative, associative Moyal-Weyl type star product similar to the one found in subsection 5.2.2. The algebra $(C^\infty(\mathcal{M}),\mu_A)$ is hence quantized to $(C^\infty(\mathcal{M})[[\hbar]],\star_p:=\star)$ and $\mathcal{M}$ acquires a spatial noncommutativity since

$$[x^i,x^j]_\star = \mathrm{i}\,\hbar\,\theta^{ij}(p)\ , \qquad [x^i,p_j]_\star = 0 = [p_i,p_j]_\star\ , \tag{5.52}$$

where $[f,g]_\star := f\star g - g\star f$ for all $f,g\in C^\infty(\mathcal{M})$ and $\theta^{ij}(p)$ is defined by (5.42).

We can incorporate quantum phase space in this description using the method of subsection 5.2.2. The pertinent non-abelian twist $\mathcal{F}\in H[[\hbar]]\otimes H[[\hbar]]$ is

$$\mathcal{F} := F_R\,F = F\,F_R = \exp\big[-\tfrac{\mathrm{i}\,\hbar}{2}\,\big(\tfrac14\,R^{ijk}\,(M_{ij}\otimes P_k - P_i\otimes M_{jk}) + P_i\otimes\tilde{P}^i - \tilde{P}^i\otimes P_i\big)\big]\ , \tag{5.53}$$

where we have used the antisymmetry of R^{ijk}. Equivalently, instead of (5.53) we can use the action of (5.43) on $C^\infty(\mathcal{M})$ to write

$$\mathcal{F} = [\bar{\mathcal{F}}]_{\bar{p}\to p} \tag{5.54}$$

where the operation $[-]_{\bar{p}\to p}$ denotes the change to dynamical momentum. †

The twist $\mathcal{F}$ is an invertible counital 2-cochain, hence $H_{\mathcal{F}}$ defines a quasi-Hopf algebra $\mathcal{H}=(H_{\mathcal{F}},\phi)$ where the associator $\phi=\phi_{\mathcal{F}}$ calculated from (5.27) is

$$\phi = \exp\big(\tfrac{\hbar^2}{2}\,R^{ijk}\,P_i\otimes P_j\otimes P_k\big)\ . \tag{5.55}$$

Its coproduct $\Delta_{\mathcal{F}}: H_{\mathcal{F}}\to H_{\mathcal{F}}\otimes H_{\mathcal{F}}$ is given by (5.14); calculating this on the generating primitive elements we get

$$\begin{aligned}\Delta_{\mathcal{F}}(P_i) &= \Delta(P_i)\ ,\\ \Delta_{\mathcal{F}}(\tilde{P}^i) &= \Delta(\tilde{P}^i) + \tfrac{\mathrm{i}\,\hbar}{2}\,R^{ijk}\,P_j\otimes P_k\ ,\\ \Delta_{\mathcal{F}}(M_{ij}) &= \Delta(M_{ij}) - \mathrm{i}\,\hbar\,(P_i\otimes P_j - P_j\otimes P_i)\ .\end{aligned} \tag{5.56}$$

† We can also restrict to constant momentum by taking a double scaling limit $\hbar\to 0$, $R^{ijk}\to\infty$ with $\bar{R}^{ijk}:=\hbar\,R^{ijk}$ held constant. This limit is equivalent to the restriction $\bar{F}_R=[F_R]_{p\to\bar{p}}$ when acting on $C^\infty(\mathcal{M})$.

In particular, $\mathcal{H}$ is a non-cocommutative quasi-Hopf algebra with quasi-triangular structure $\underline{\mathcal{R}} = \mathcal{F}^{-2}$, as a straightforward calculation of the co-opposite coproduct $\Delta_{\mathcal{F}}^{\rm op}$ on primitive elements using (5.17) reveals.

A left (or right) H-module algebra (A, μ_A) can now be deformed using (5.18) and the relevant action. Let us do this for the algebra of functions on $\mathcal{M}$, i.e. we set $A = C^\infty(\mathcal{M})$ and $\mu_A(f \otimes g) = f\, g$, and derive its deformation quantization $(C^\infty(\mathcal{M})[[\hbar]], \star := \star_p)$ with the star product given by (5.18). We find

$$f \star g = \mu_A\Big(\exp\big[\tfrac{{\rm i}\hbar}{2}\,\big(R^{ijk}\,p_k\,\partial_i \otimes \partial_j + \partial_i \otimes \tilde\partial^i - \tilde\partial^i \otimes \partial_i\big)\big](f \otimes g)\Big)\ , \tag{5.57}$$

where the actions (5.32) and (5.48) have been used. This is a *nonassociative star product* and hence $(C^\infty(\mathcal{M})[[\hbar]], \star)$ is a nonassociative algebra, i.e. the product of three functions is associative only up to the associator (5.55). This is expressed by (5.28) which in this case can be written in the more explicit form

$$\begin{aligned}(f \star g) \star h = f \star (g \star h) + \sum_{n=1}^\infty \frac{1}{n!}\,\Big(\frac{\hbar^2}{2}\Big)^n\, R^{i_1 j_1 k_1} \cdots R^{i_n j_n k_n}\,(\partial_{i_1} \cdots \partial_{i_n} f) \star \\ \star\, \big((\partial_{j_1} \cdots \partial_{j_n} g) \star (\partial_{k_1} \cdots \partial_{k_n} h)\big)\ .\end{aligned} \tag{5.58}$$

From (5.56) we find the (modified) Leibniz rules

$$\begin{aligned}\partial_i(f \star g) &= (\partial_i f) \star g + f \star (\partial_i g)\ , \\ \tilde\partial^i(f \star g) &= (\tilde\partial^i f) \star g + f \star (\tilde\partial^i g) + \tfrac{{\rm i}\hbar}{2}\,R^{ijk}\,(\partial_j f) \star (\partial_k g)\ ,\end{aligned} \tag{5.59}$$

and in particular ∂_i is a derivation of the star product. These relations simply reflect the fact that the nonassociative R-flux background exhibits space translation symmetry, but not momentum translation symmetry, due to the dynamical nonassociativity. The loss of translation invariance in momentum space is related to the violation of Bopp shift symmetry through

$$\Delta_{\mathcal{F}}(\tilde P^i) - \Delta(\tilde P^i) = -\tfrac{1}{4}\,R^{ijk}\,\big(\Delta_{\mathcal{F}}(M_{jk}) - \Delta(M_{jk})\big)\ , \tag{5.60}$$

indicating that the Bopp shift generators M_{ij} are observables which detect effects of nonassociativity.

The nonassociative star product (5.57) coincides with the star products (3.17) and (4.37) which we calculated in sections 3.3 and 4.3 respectively for functions on $\mathcal{M}$. To see that (5.58) coincides with the associator (4.26), it is enough to compute it on the position space plane waves $U_{\tilde a} = \mathrm{e}^{\,\mathrm{i}\,\tilde a_i\, x^i}$ where $\tilde a = (\tilde a_i) \in (\mathbb{R}^d)^*$ are constant vectors. One easily finds

$$\big(U_{\tilde a} \star U_{\tilde b}\big) \star U_{\tilde c} = \varphi_R\big(\tilde a\,,\,\tilde b\,,\,\tilde c\big)\ U_{\tilde a} \star \big(U_{\tilde b} \star U_{\tilde c}\big)\ , \tag{5.61}$$

where

$$\varphi_R\big(\tilde a\,,\,\tilde b\,,\,\tilde c\big) := \exp\big(-\tfrac{\mathrm{i}\,\hbar^2}{2}\, R^{ijk}\,\tilde a_i\,\tilde b_j\,\tilde c_k\,\big) \tag{5.62}$$

is the 3-cocycle of the group of translations that was obtained in section 4.3 (see also [14]).

Using (5.57) we can calculate the $\star$-commutation relations on the coordinate functions of $\mathcal{M}$. We find

$$[x^i, x^j]_\star = \mathrm{i}\,\hbar\,\theta^{ij}(p)\ , \qquad [x^i, p_j]_\star = \mathrm{i}\,\hbar\,\delta^i{}_j\ , \qquad [p_i, p_j]_\star = 0 \tag{5.63}$$

where $[f,g]_\star := f\star g - g\star f$ for all $f,g\in C^\infty(\mathcal{M})$ and $i,j,k = 1,\dots,d$. Nonassociativity of (5.57) can also be seen through the failure of the Jacobi identity for this $\star$-commutator, analogously to (1.7).

5.2.4 Integral formulas

The star product formula (5.57) that we found in the previous subsection is written in the language of differential operators, in practise though for explicit calculations it is useful to employ an integral representation of the star product. A Fourier integral formula for this product was derived in section 4.3 (see also [14, section 4.2]), however a related but more useful formula can be easily derived by expressing $g(x)$ in terms of its Fourier transform $\hat g(k)$, where $k\in\mathcal{M}^*$; here $\mathcal{M} := T^*M \cong M\times(\mathbb{R}^d)^* \cong \mathbb{R}^d\times(\mathbb{R}^d)^*$ and $\mathcal{M}^*\cong(\mathbb{R}^d)^*\times\mathbb{R}^d$. Let us recall that the star product (5.57) can be written in upper case index notation as in equation (3.17). The derivative ∂_J turns into multiplication by $\mathrm{i}\, k_J$ and we can interpret the exponential $\exp\big(-\frac{\hbar}{2}\,\Theta^{IJ}\,k_J\,\partial_I\big)$ as a shift operator to rewrite (3.17) as

$$(f\star g)(x) = \int_{\mathcal{M}^*} \mathrm{d}^{2d}k\ f\big(x-\tfrac{\hbar}{2}\,\Theta k\big)\ \hat{g}(k)\ \mathrm{e}^{\,\mathrm{i}\,k_I\,x^I}\ , \tag{5.64}$$

where $(\Theta k)^I := \Theta^{IJ}\,k_J$. Using the inverse Fourier transformation, this becomes

$$(f\star g)(x) = \frac{1}{(2\pi)^{2d}} \int_{\mathcal{M}^*} \mathrm{d}^{2d}k \int_{\mathcal{M}} \mathrm{d}^{2d}x'\ f\big(x-\tfrac{\hbar}{2}\,\Theta k\big)\ g(x'\,)\ \mathrm{e}^{\,\mathrm{i}\,k_I\,(x-x'\,)^I}\ . \tag{5.65}$$

Since the matrix Θ is unimodular, we can use the inverse matrix (B-field)

$$\Theta^{-1} = \big(\Theta^{-1}_{IJ}\big) = \begin{pmatrix} 0 & -\delta^i{}_j \\ \delta_i{}^j & R^{ijk}\,p_k \end{pmatrix} \tag{5.66}$$

to change variables to $z=-\frac{\hbar}{2}\,\Theta k$ and $w=x'-x$, and in this way we finally obtain a nice expression in terms of a twisted convolution product

$$(f\star g)(x) = \Big(\frac{1}{\pi\,\hbar}\Big)^{2d} \int_{\mathcal{M}} \mathrm{d}^{2d}z \int_{\mathcal{M}} \mathrm{d}^{2d}w\ f(x+z)\,g(x+w)\ \mathrm{e}^{-\frac{2\,\mathrm{i}}{\hbar}\,z^I\,\Theta^{-1}_{IJ}\,w^J} \tag{5.67}$$

that is often more convenient for computations than (3.17); it is also well-defined as a non-perturbative formula on the larger class of Schwartz functions on phase space $\mathcal{M}$. One should not forget that the matrix Θ^{-1} in (5.67) depends on p_i and hence on x^I, but otherwise the expression is again formally identical to the standard twisted convolution product formula for the Moyal-Weyl product (see e.g. [117] for a review).

There are also integral formulas available for our twists. Changing the sign of Θ and dropping the multiplication operator μ_2 in (3.17), we can similarly derive an integral formula for the action of the twist on a pair of functions f and g (evaluated at x and y respectively) given by

$$\begin{aligned}\big(\mathcal{F}\triangleright(f\otimes g)\big)(x,y)=&\Big(\frac{1}{\pi\,\hbar}\Big)^{2d}\int_{\mathcal{M}}\mathrm{d}^{2d}z\ \times\\ &\times\int_{\mathcal{M}}\mathrm{d}^{2d}w\,\big(\,\mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\,z^I\,\Theta^{-1}_{IJ}\,w^J}\,f\big)\otimes\big(\,\mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\,z^I\,\Theta^{-1}_{IJ}\,w^J}\,g\big)(x+z,y+w)\ .\end{aligned} \tag{5.68}$$

Introducing the shift operator $(\mathcal{T}_a f)(x) := f(x+a)$ for any $2d$-vector a, the twist element acting on $C^\infty(\mathcal{M})$ thus becomes

$$\mathcal{F}=\Big(\frac{1}{\pi\,\hbar}\Big)^{2d}\int_{\mathcal{M}}\mathrm{d}^{2d}z\ \int_{\mathcal{M}}\mathrm{d}^{2d}w\ \big(\,\mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\,z^I\,\Theta^{-1}_{IJ}\,w^J}\,\mathcal{T}_z\big)\otimes\big(\,\mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\,z^I\,\Theta^{-1}_{IJ}\,w^J}\,\mathcal{T}_w\big)\ . \tag{5.69}$$

5.3 Nonassociative differential calculus on non-geometric flux backgrounds

The approach of section 5.2 has the great virtue of enabling a systematic development of exterior differential calculus on these deformations. In this section we develop the basic ingredients necessary for both an investigation of nonassociative quantum mechanics within a phase space quantization formalism, which we undertake in chapter 6, as well as a putative formulation of field theories on the nonassociative R-flux backgrounds which we discuss in chapter 7.

5.3.1 Covariant differential calculus

In this subsection we will use the cochain twist (5.53) to deform the exterior algebra of differential forms $(\Omega^\bullet,\mu_\wedge,\mathrm{d})$, where $\Omega^\bullet := \bigoplus_{n\geq 0}\Omega^n$ with $\Omega^n=\Omega^n(\mathcal{M})$ the vector space of complex smooth n-forms on $\mathcal{M}$, $\mu_\wedge:\Omega^n\otimes\Omega^m\to\Omega^{n+m}$ the usual exterior product $\mu_\wedge(\omega\otimes\omega'):=\omega\wedge\omega'$ and $\mathrm{d}:\Omega^n\to\Omega^{n+1}$ the exterior derivative with $\mathrm{d}^2=0$.

We demand that the action of the Hopf algebra $H=U(\mathfrak{g})$ from subsection 5.2.3 on $(\Omega^\bullet,\mu_\wedge,\mathrm{d})$ is covariant in the sense that (c.f. (5.10))

$$h\triangleright(\omega\wedge\omega')=\big(h_{(1)}\triangleright\omega\big)\wedge\big(h_{(2)}\triangleright\omega'\big) \tag{5.70}$$

and that d is equivariant under the action of H in the sense that

$$h \triangleright (\mathrm{d}\omega) = \mathrm{d}(h \triangleright \omega) \ , \tag{5.71}$$

for all $\omega, \omega' \in \Omega^\bullet$ and $h \in H$; in the framework of subsection 5.1.3, these two conditions respectively mean that the structure maps $\mu_\wedge$ and d are both morphisms in the category ${}_H\mathscr{M}$ of left H-modules. The action of H on $\Omega^\bullet$ can be determined by finding the action on Ω^1 and extending it to $\Omega^\bullet$ as an algebra homomorphism using the Leibniz rule

$$\mathrm{d}(\omega \wedge \omega'\,) = \mathrm{d}\omega \wedge \omega' + (-1)^{\deg \omega}\, \omega \wedge \mathrm{d}\omega' \tag{5.72}$$

for all $\omega, \omega' \in \Omega^\bullet$. Employing (5.71), the action of H on Ω^0 given by (5.32) and (5.48), and the fact that d commutes with the Lie derivative along any vector field, we conclude that the action of H on $\Omega^\bullet$ is given by the Lie derivative $\mathcal{L}_h$ along elements $h \in H$. As previously the action is defined on primitive elements of H as an algebra homomorphism, i.e. $\mathcal{L}_{\xi\,\xi'} := \mathcal{L}_\xi \circ \mathcal{L}_{\xi'}$ for $\xi, \xi' \in \mathfrak{g} \subset U(\mathfrak{g})$, and it extends to a left action via linearity of the Lie derivative and the Leibniz rule to get $\mathcal{L}_{h\,h'} = \mathcal{L}_h \circ \mathcal{L}_{h'}$ for all $h, h' \in H$. Calculating this action on the generating 1-forms gives

$$M_{ij} \triangleright \mathrm{d}x^k := \mathcal{L}_{M_{ij}}\big(\mathrm{d}x^k\big) = \delta_j{}^k\, \mathrm{d}p_i - \delta_i{}^k\, \mathrm{d}p_j \ , \tag{5.73}$$

with all other generators $\mathrm{d}x^I$ invariant under the action of H.

Following the methods of section 5.1 (with $A = \Omega^\bullet$ and $\mu_A = \mu_\wedge$), we ensure that $\Omega^\bullet$ is covariant under the action of $\mathcal{H} = (H_{\mathcal{F}}, \phi)$ by introducing a deformed exterior product $\wedge_\star$ on $\Omega^n[[\hbar]] \otimes \Omega^m[[\hbar]] \to \Omega^{n+m}[[\hbar]]$ given by the formula

$$\omega \wedge_\star \omega' = \mu_\wedge\big(\mathcal{F}^{-1} \triangleright (\omega \otimes \omega'\,)\big) = \big(\mathcal{F}^{-1}_{(1)} \triangleright \omega\big) \wedge \big(\mathcal{F}^{-1}_{(2)} \triangleright \omega'\,\big) \ , \tag{5.74}$$

for all $\omega, \omega' \in \Omega^\bullet$. The exterior derivative is still a derivation for the deformed exterior product and thus we call the twisted exterior algebra $(\Omega^\bullet[[\hbar]], \wedge_\star, \mathrm{d})$ the *nonassociative exterior differential calculus*. Using (5.74) on the generating 1-forms we find the relations

$$\mathrm{d}x^I \wedge_\star \mathrm{d}x^J = -\mathrm{d}x^J \wedge_\star \mathrm{d}x^I = \mathrm{d}x^I \wedge \mathrm{d}x^J \ , \tag{5.75}$$

where $I, J \in \{1, \dots, 2d\}$. We can again write (5.28) in a more enlightening form for the case at hand as

$$\begin{aligned}(\omega \wedge_\star \omega'\,) \wedge_\star \omega'' =& \omega \wedge_\star (\omega' \wedge_\star \omega''\,) + \sum_{n=1}^\infty \frac{1}{n!} \Big(\frac{\hbar^2}{2}\Big)^n R^{i_1 j_1 k_1} \cdots R^{i_n j_n k_n} \times \\ & \times \mathcal{L}_{i_1} \cdots \mathcal{L}_{i_n}(\omega) \ \wedge_\star \ \big(\mathcal{L}_{j_1} \cdots \mathcal{L}_{j_n}(\omega'\,) \wedge_\star \mathcal{L}_{k_1} \cdots \mathcal{L}_{k_n}(\omega''\,)\big)\end{aligned} \tag{5.76}$$

for all $\omega, \omega', \omega'' \in \Omega^\bullet$, where we have abbreviated $\mathcal{L}_i := \mathcal{L}_{\partial_i}$. When $\omega, \omega', \omega'' \in \Omega^1$, this formula takes an even simpler form given by

$$\begin{aligned}(\omega \wedge_\star \omega'\,) \wedge_\star \omega'' =& \ \omega \wedge_\star (\omega' \wedge_\star \omega''\,) + \sum_{n=1}^\infty \frac{1}{n!} \Big(\frac{\hbar^2}{2}\Big)^n R^{i_1 j_1 k_1} \cdots R^{i_n j_n k_n} \times \\ & \times (\partial_{i_1} \cdots \partial_{i_n} \omega_L)\, \mathrm{d}x^L \ \wedge_\star \big((\partial_{j_1} \cdots \partial_{j_n} \omega'_M)\, \mathrm{d}x^M \wedge_\star (\partial_{k_1} \cdots \partial_{k_n} \omega''_N)\, \mathrm{d}x^N\big) \ ,\end{aligned} \tag{5.77}$$

where $i_l, j_l, k_l \in \{1, \dots, d\}$ and $L, M, N \in \{1, \dots, 2d\}$. It follows that

$$\big(\mathrm{d}x^I \wedge_\star \mathrm{d}x^J\big) \wedge_\star \mathrm{d}x^K = \mathrm{d}x^I \wedge_\star \big(\mathrm{d}x^J \wedge_\star \mathrm{d}x^K\big) =: \mathrm{d}x^I \wedge_\star \mathrm{d}x^J \wedge_\star \mathrm{d}x^K \ , \tag{5.78}$$

where $I, J, K \in \{1, \dots, 2d\}$.

The exterior product provides an A-bimodule structure on $\Omega^\bullet$, where $A = (C^\infty(\mathcal{M}), \mu_A)$, with right and left action given by the pointwise multiplication of an n-form by a function. Let us denote this action by $\blacktriangleright$ and $\blacktriangleleft$ respectively; then covariance of the bimodule under the action of the Hopf algebra H means

$$h \triangleright (f \blacktriangleright \omega) = \big(h_{(1)} \triangleright f\big) \blacktriangleright \big(h_{(2)} \triangleright \omega\big) \ , \qquad h \triangleright (\omega \blacktriangleleft f) = \big(h_{(1)} \triangleright \omega\big) \blacktriangleleft \big(h_{(2)} \triangleright f\big) \ , \tag{5.79}$$

for all $h \in H$, $f \in C^\infty(\mathcal{M})$ and $\omega \in \Omega^\bullet$. To ensure that the bimodule is covariant under the action of the quasi-Hopf algebra $\mathcal{H}$ we must replace its action by the deformed right and left actions given respectively by the formulas

$$f \blacktriangleright_\star \omega = \lambda_\blacktriangleright\big(\mathcal{F}^{-1} \triangleright (f \otimes \omega)\big) \ , \qquad \omega \,{}_\star\!\blacktriangleleft f = \lambda_\blacktriangleleft\big(\mathcal{F}^{-1} \triangleright (\omega \otimes f)\big) \ , \tag{5.80}$$

where $\lambda_\blacktriangleright(f \otimes \omega) = f \blacktriangleright \omega = f\,\omega$ for all $\omega \in \Omega^\bullet$ and $f \in C^\infty(\mathcal{M})$, and similarly for $\lambda_\blacktriangleleft : \Omega^\bullet \otimes A \to \Omega^\bullet$, which yields the deformed $A_\star$-bimodule, where $A_\star = (C^\infty(\mathcal{M})[[\hbar]], \star)$. Since $f \blacktriangleright \omega = \omega \blacktriangleleft f$, here and throughout we will abuse notation for the sake of simplicity by denoting $\blacktriangleright_\star$ and ${}_\star\!\blacktriangleleft$ by $\star$ where no confusion arises. A short calculation then reveals the non-trivial bimodule relations between coordinates and 1-forms given by

$$x^i \star \mathrm{d}x^j = \mathrm{d}x^j \star x^i + \tfrac{\mathrm{i}\,\hbar}{2}\, R^{ijk}\, \mathrm{d}p_k \ , \tag{5.81}$$

while all other left and right $A_\star$-actions coincide.

5.3.2 Integration

To compute quantum mechanical averages, and also to set up a Lagrangian formalism for field theory, we need a suitable definition of integration $\int$ on $(\mathcal{S}(\mathcal{M})[[\hbar]], \star)$, where $\mathcal{S}(\mathcal{M}) \subset C^\infty(\mathcal{M})$ is the subalgebra of Schwartz functions on $\mathcal{M} = T^*M$.

Let us first notice that the star product (5.57) satisfies

$$f \star g = f\,g + \text{total derivative} \ . \tag{5.82}$$

This can be easily verified if we write the star product in the form (3.17), and keep in mind that a total derivative in phase space includes both position and momentum derivatives. The order $\hbar^n$ term can then be written as

$$\Theta^{I_1 J_1} \cdots \Theta^{I_n J_n}\, \big(\partial_{I_1} \cdots \partial_{I_n} f\big)\, \big(\partial_{J_1} \cdots \partial_{J_n} g\big) = \partial_{I_1} \cdots \partial_{I_n} \big(\Theta^{I_1 J_1} \cdots \Theta^{I_n J_n}\, f\, \partial_{J_1} \cdots \partial_{J_n} g\big) \tag{5.83}$$

since no momentum derivatives act on the upper left block of Θ, which means that (5.82) is satisfied to all orders in $\hbar$. Then the usual integration on $\mathcal{M}$ satisfies the

2-cyclicity condition

$$\int_{\mathcal{M}} \mathrm{d}^{2d}x\ f \star g = \int_{\mathcal{M}} \mathrm{d}^{2d}x\ g \star f = \int_{\mathcal{M}} \mathrm{d}^{2d}x\ f\, g \tag{5.84}$$

for all $f, g \in \mathcal{S}(\mathcal{M})$, i.e. the standard integral on $(\mathcal{S}(\mathcal{M})[[\hbar]], \star)$ is 2-cyclic.

In addition to the 2-cyclicity condition, the standard integral on $(\mathcal{S}(\mathcal{M})[[\hbar]], \star)$ satisfies a cyclicity condition on the associator derived from the property

$$f \star (g \star h) = (f \star g) \star h + \text{total derivative}\ , \tag{5.85}$$

which easily follows from (5.58) and (5.59). Hence the standard integral also satisfies the property

$$\int_{\mathcal{M}} \mathrm{d}^{2d}x\ f \star (g \star h) = \int_{\mathcal{M}} \mathrm{d}^{2d}x\ (f \star g) \star h =: \int_{\mathcal{M}} \mathrm{d}^{2d}x\ f \star g \star h \tag{5.86}$$

for all $f, g, h \in \mathcal{S}(\mathcal{M})$, which we call the *3-cyclicity condition*; this property is identical to equation(3.21) which was derived using Kontsevich's global deformation quantization.

The 2-cyclicity condition does *not* generally guarantee the usual cyclicity property involving integration of n-fold star products of functions. This is because the star product is nonassociative and thus a bracketing for the star product of n functions has to be specified. Once this is done one cannot freely move functions cyclically under integration using (5.84) as one would normally do in the associative case; instead the 3-cyclicity condition (5.86) can be used to re-bracket the integrated expression and to investigate its equivalence with expressions involving different bracketings. In general, the total number of ways to bracket a star product of n functions is given by the Catalan number C_{n-1}, where

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!} \tag{5.87}$$

for $n \geq 0$. Starting from the integral

$$\int_{\mathcal{M}} \mathrm{d}^{2d}x \; f_1 \star \big(f_2 \star (f_3 \star (\cdots \star f_n)\cdots)\big) \;, \tag{5.88}$$

one can prove that it is equal to a number of different bracketings but always inequivalent to any other bracketing of the form $f_1 \star$(bracketed expression), i.e. under 3-cyclicity the distinct ways of bracketing an integrated n-fold star product of functions are organised into C_{n-2} classes, one for each different bracketing where f_1 is free at the front. For example, for $n = 4$ there are five different bracketings out of which two are of the form (5.88) and thus we have two different classes of equivalent bracketings, namely

$$\begin{aligned} \int_{\mathcal{M}} \mathrm{d}^{2d}x \; f_1 \star \big(f_2 \star (f_3 \star f_4)\big) &= \int_{\mathcal{M}} \mathrm{d}^{2d}x \; (f_1 \star f_2) \star (f_3 \star f_4) \\ &= \int_{\mathcal{M}} \mathrm{d}^{2d}x \; \big((f_1 \star f_2) \star f_3\big) \star f_4 \end{aligned} \tag{5.89}$$

and

$$\int_{\mathcal{M}} \mathrm{d}^{2d}x \; f_1 \star \big((f_2 \star f_3) \star f_4\big) = \int_{\mathcal{M}} \mathrm{d}^{2d}x \; \big(f_1 \star (f_2 \star f_3)\big) \star f_4 \;. \tag{5.90}$$

A graded 2-cyclicity condition could also be derived for the deformed exterior product, provided that we can write an equation similar to (5.82) for it. This seems complicated since generally (5.74) cannot be written explicitly in closed form, but fortunately there is a way around this problem: We can use the result of [9] where it was shown that if the identity

$$U_{\mathcal{F}} = \mathcal{F}_{(1)}\, S\big(\mathcal{F}_{(2)}\big) = 1_H \tag{5.91}$$

holds, where $U_{\mathcal{F}} = \mu \circ (\mathrm{id}_H \otimes S)(\mathcal{F})$ and S is the antipode, then standard integration on $(\Omega^\bullet[[\hbar]], \wedge_\star, \mathrm{d})$ is graded 2-cyclic. This is always true for abelian twists but does not hold in general; however, in our case the twisted antipode $S_{\mathcal{F}}$ coincides with S. It is then straightforward to demonstrate (5.91) on $\mathcal{S}(\mathcal{M})$ by using the representation of primitive elements of H on functions in (5.53) and antisymmetry of R^{ijk}. Hence we conclude that for the nonassociative exterior differential calculus $(\Omega^\bullet[[\hbar]], \wedge_\star, \mathrm{d})$ the graded 2-cyclicity condition

$$\int_{\mathcal{M}} \omega \wedge_\star \omega' = (-1)^{\deg(\omega)\deg(\omega')} \int_{\mathcal{M}} \omega' \wedge_\star \omega = \int_{\mathcal{M}} \omega \wedge \omega' \tag{5.92}$$

is indeed satisfied.

The 3-cyclicity condition (5.86) can also be generalized by noticing that similarly to ∂_i being a derivation for the nonassociative star product by (5.59), the Lie derivative $\mathcal{L}_i$ is a derivation of the deformed exterior product $\wedge_\star$ by (5.56) and the discussion that followed (5.72), and since $[\mathcal{L}_i, \mathcal{L}_j] = \mathcal{L}_{[\partial_i,\partial_j]} = 0$ by (5.76) one has

$$\omega \wedge_\star (\omega' \wedge_\star \omega'') = (\omega \wedge_\star \omega') \wedge_\star \omega'' + \text{total Lie derivative} . \tag{5.93}$$

Since $\int_{\mathcal{M}} \mathcal{L}_i(\omega) = 0$ for all $\omega \in \Omega^\bullet$ we thus get the generic 3-cyclicity condition

$$\int_{\mathcal{M}} \omega \wedge_\star (\omega' \wedge_\star \omega'') = \int_{\mathcal{M}} (\omega \wedge_\star \omega') \wedge_\star \omega'' =: \int_{\mathcal{M}} \omega \wedge_\star \omega' \wedge_\star \omega'' \tag{5.94}$$

for all $\omega, \omega', \omega'' \in \Omega^\bullet$, generalizing (5.86) which is the Ω^0 case.

5.4 Generic non-geometric fluxes

We conclude by briefly discussing some preliminary steps towards extending the analysis of this chapter to more complicated non-geometric R-flux compactifications. We consider, in particular, the two separate cases in turn where the constant 3-tensor R^{ijk} is replaced with a general function of the position coordinates $x \in M$ and where the 2-tensor $\theta^{ij}(p) = R^{ijk}\, p_k$ is replaced by a general function of the conjugate momenta $p \in (\mathbb{R}^d)^*$. The former type of generalisation has been discussed recently in the context of double field theory in [25], while the latter type of generalisations arise in [90, 40].

5.4.1 Nambu–Poisson three-brackets

The extension of our results to non-constant R-fluxes is closely related to the problem of quantization of generic Nambu–Poisson structures (see appendix A.4). A Nambu–Poisson 3-bracket is a skew-symmetric ternary bracket defined on the space of smooth functions $C^\infty(M)$ on a manifold M, which generalizes the Poisson 2-bracket and can be expressed in terms of a trivector field $\Pi \in C^\infty\big(M, \bigwedge^3 TM\big)$ as $\{f,g,h\} = \Pi(\mathrm{d}f, \mathrm{d}g, \mathrm{d}h)$. The bracket is used to define a Nambu multi-Hamiltonian flow [103]

$$\frac{\mathrm{d}f}{\mathrm{d}t} = X_{H_1,H_2} f := \{f, H_1, H_2\} \tag{5.95}$$

with Nambu Hamiltonian vector field X_{H_1,H_2} for any two smooth functions H_1 and H_2. For a Nambu–Poisson structure, one requires that the vector fields X_{H_1,H_2} act as a derivation on the bracket, so that

$$X_{H_1,H_2}\{f,g,h\} = \{X_{H_1,H_2}f, g, h\} + \{f, X_{H_1,H_2}g, h\} + \{f, g, X_{H_1,H_2}h\} \ . \tag{5.96}$$

This implies that the linear span of Nambu Hamiltonian vector fields defines a Lie algebra with Lie bracket

$$[X_{f,g}, X_{f',g'}] = X_{X_{f,g}f',\,g'} + X_{f',\,X_{f,g}g'} \ . \tag{5.97}$$

The condition (5.96), when expressed solely in terms of brackets, is known as the fundamental identity [120]. It is the generalization of the Jacobi identity for Poisson brackets, which is a differential condition on a Poisson bivector. For 3-brackets, the fundamental identity is a differential as well as an algebraic condition on the 3-vector field Π. The algebraic condition implies that Π is a decomposable trivector

$$\Pi = X_1 \wedge X_2 \wedge X_3 \ . \tag{5.98}$$

The vector fields X_1, X_2 and X_3 are linearly independent (unless $\Pi = 0$) and in view of (5.97) they define an involutive distribution. This implies that the local as well as the global Frobenius theorem applies and in particular that around each point of the manifold M there exists a coordinate chart $(U; x^1, x^2, x^3)$ such that

$$\Pi = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \ . \tag{5.99}$$

This expression can be multiplied with a scalar pre-factor (e.g. a constant) without spoiling the properties of a Nambu–Poisson structure.

The central object of interest of this thesis is a constant trivector R-flux $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$. For appropriately chosen coordinates, the decomposition (5.99) implies that a Nambu–Poisson tensor Π is in fact such a constant trivector, at least locally, and most of the results of this chapter thus apply, including the formalism of twist deformation quantization. Conversely, if R extends locally in a three-dimensional submanifold of M, then R is a Nambu–Poisson tensor. The parts of section 5.3 dealing with integration apply to the particular class of Nambu–Poisson structures for which (5.99) holds globally.

5.4.2 Non-parabolic monodromies

When a three-torus $\mathbb{T}^3$ in the Q-flux duality frame is viewed as a $\mathbb{T}^2$-fibration over S^1, a periodic translation along the base must act on the local complex structure modulus τ of a fibre $\mathbb{T}^2$ as an $SL(2,\mathbb{Z})$ Möbius transformation, in order to end up with an automorphic fibre. These transformations define the monodromy properties of the fibration and fall into conjugacy classes of $SL(2,\mathbb{Z})$ [74]; the case of trivial monodromies corresponds to geometric spaces (manifolds), while non-trivial classes correspond to non-geometric spaces (T-folds). Parabolic monodromies are of infinite order and act as discrete shifts $\tau \mapsto \tau + n$, where $n \in \mathbb{Z}$. As discussed in chapter 1, under T-duality the T-fold is mapped to the parabolic R-flux backgrounds characterized by the phase space relations (1.6); this algebra provides one of the simplest examples of nonassociativity and may be regarded as the analogue of the Moyal-Weyl background that arises in open string theory with a constant B-field (see e.g. [117]). The case of elliptic monodromies, which are of finite order and act as $\mathbb{Z}_N$-transformations on the $\mathbb{T}^2$ coordinates, were also considered in [90, 40] where it was shown that the position coordinate commutator in (1.6) is generalised to a particular non-linear function $\vartheta^{ij}(p)$ of momentum and R-flux. We briefly describe here how to extend the setting of section 5.2 to allow for twist deformations that correspond to a class of quasi-Poisson structures Θ_e which are generic functions of momentum.

The class of generalizations of (2.24) that we are interested in are obtained by substituting (5.42) with a generic function of momentum $\vartheta^{ij}(\bar{p})$ to get a bivector $\Theta_e = \frac{1}{2}\,\Theta_e^{IJ}\,\partial_I \wedge \partial_J$ on phase space $\mathcal{M}$ given by

$$\Theta_e = \big(\Theta_e^{IJ}\big) = \begin{pmatrix} \vartheta^{ij}(p) & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} . \tag{5.100}$$

As in the case of chapter 2, a computation of the Schouten-Nijenhuis bracket of this bivector with itself reveals that it defines an H-twisted Poisson structure on $\mathcal{M}$, where H is the 3-form

$$H = \tfrac{1}{6}\,\tilde{\partial}^i \vartheta^{jk}(p)\,\mathrm{d}p_i \wedge \mathrm{d}p_j \wedge \mathrm{d}p_k \tag{5.101}$$

which is the curvature of a twisting $U(1)$ gerbe on momentum space. The corresponding Jacobiator $J = \bigwedge^3 \Theta_e^\sharp(H)$ is the 3-vector whose only non-vanishing components are given by

$$J^{ijk}(p) = \tfrac{1}{3}\,\big(\tilde{\partial}^i\vartheta^{jk}(p) + \tilde{\partial}^j\vartheta^{ki}(p) + \tilde{\partial}^k\vartheta^{ij}(p)\big) . \tag{5.102}$$

Kontsevich's deformation quantization of a generic (quasi-)Poisson structure is *a priori* quite involved as the number of weights that have to be calculated at each order of the diagrammatic expansion of the star product increases geometrically. A nonassociative star product up to third order in a derivative expansion of a generic B-field was calculated in [65] by using a twisted Poisson sigma-model to determine the weights of Kontsevich graphs; from the topological sigma-model formalism, the Kontsevich formula inherits an invariance under the involution which interchanges functions and maps $\Theta_e \mapsto -\Theta_e$. By applying the open/closed string duality argument of section 2.4, we can transport their results to our closed string case and quantize the R-flux background via the star product

$$
\begin{aligned}
f \star g = [f \mathbin{\bar\star} g]_{\bar p \to p} &- \tfrac{\hbar^2}{12}\, \tilde\partial^k \vartheta^{ij} \big([\partial_k\partial_i f \mathbin{\bar\star} \partial_j g]_{\bar p\to p} + [\partial_j f \mathbin{\bar\star} \partial_k \partial_i g]_{\bar p \to p} \big) \\
&- \tfrac{\mathrm{i}\,\hbar^3}{48}\, \tilde\partial^l \tilde\partial^k \vartheta^{ij} \big([\partial_l\partial_k\partial_i f \mathbin{\bar\star} \partial_j g]_{\bar p\to p} - [\partial_j f \mathbin{\bar\star} \partial_l\partial_k\partial_i g]_{\bar p\to p} \big) \\
&+ \tfrac{\hbar^4}{288} \big(\tilde\partial^l \vartheta^{mn}\big) \big(\tilde\partial^k \vartheta^{ij}\big) \big([\partial_l\partial_m\partial_k\partial_i f \mathbin{\bar\star} \partial_n\partial_j g]_{\bar p\to p} \\
&\qquad + 2\, [\partial_l\partial_m\partial_j f \mathbin{\bar\star} \partial_n\partial_k\partial_i g]_{\bar p\to p} + [\partial_n\partial_j f \mathbin{\bar\star} \partial_l\partial_m\partial_k\partial_i g]_{\bar p\to p} \big) \\
&+ \mathcal{O}\big(\tilde\partial^3\vartheta, (\tilde\partial\vartheta)^3\big)
\end{aligned}
\tag{5.103}
$$

for $f,g \in C^\infty(\mathcal{M})$, where as before the operation $[-]_{\bar p\to p}$ denotes the change from constant to dynamical momentum and

$$
f \mathbin{\bar\star} g = \mu_A\Big(\exp\big[\tfrac{\mathrm{i}\,\hbar}{2} \big(\vartheta^{ij}(\bar p)\, \partial_i \otimes \partial_j + \partial_i \otimes \tilde\partial^i - \tilde\partial^i \otimes \partial_i \big) \big] (f\otimes g) \Big) \tag{5.104}
$$

is an associative Moyal-Weyl type product on $C^\infty(\mathcal{M})$. One can check that this star product is nonassociative and that it reduces to the star product (5.57) in the parabolic case $\vartheta^{ij}(p) = \theta^{ij}(p) = R^{ijk}\, p_k$, by antisymmetry of the R-flux components. In particular, by substituting f and g with phase space coordinates we find the quantum phase space relations

$$
[x^i, x^j]_\star = \mathrm{i}\,\hbar\, \vartheta^{ij}(p) \ , \qquad [x^i, p_j]_\star = \mathrm{i}\,\hbar\, \delta^i{}_j \ , \qquad [p_i, p_j]_\star = 0 \tag{5.105}
$$

while the quantized Jacobiator is

$$
[[x^i, x^j, x^k]]_\star = 3\, \hbar^2\, J^{ijk}(p) \ . \tag{5.106}
$$

We may now construct the pertinent Hopf algebra of symmetries of the closed string background. Consider the non-abelian Lie algebra $\mathfrak{g}_e$ generated by $\tilde P^i$ and $P_i^{(f)} := f(p)\, P_i$, where $f(p) \in C^\infty((\mathbb{R}^d)^*)$, with the only non-trivial commutation relations given by

$$
\big[\tilde P^i \,,\, P_j^{(f)} \big] = P_j^{(\tilde\partial^i f)} \ . \tag{5.107}
$$

The generators of $\mathfrak{g}_e$ are realised on phase space $\mathcal{M}$ by the action (5.32) on $C^\infty(\mathcal{M})$; they respectively generate momentum translations and position translations together with a momentum-dependent scaling by $f(p)$. In particular, this infinite-dimensional Lie algebra contains the Lie subalgebra of translations and Bopp shifts in phase space $\mathfrak{g}$ that we used in section 5.2. The pertinent Hopf algebra H_e is the universal enveloping algebra $U(\mathfrak{g}_e)$ equipped with the coalgebra structure (5.7).

A suitable (but not unique) twist $\mathcal{F}_e \in H_e[[\hbar]] \otimes H_e[[\hbar]]$ that reproduces the star product (5.103) is given by

$$\begin{aligned}
\mathcal{F}_e = [\bar{\mathcal{F}}_e]_{\bar{p}\to p} &- \tfrac{\hbar^2}{12}\,\tilde{\partial}^k\vartheta^{ij}(p)\,[\bar{\mathcal{F}}_e]_{\bar{p}\to p}\big(P_k\,P_i \otimes P_j + P_j \otimes P_k\,P_i\big) \\
&- \tfrac{\mathrm{i}\,\hbar^3}{48}\,\tilde{\partial}^l\tilde{\partial}^k\vartheta^{ij}(p)\,[\bar{\mathcal{F}}_e]_{\bar{p}\to p}\big(P_l\,P_k\,P_i \otimes P_j - P_j \otimes P_l\,P_k\,P_i\big) \\
&+ \tfrac{\hbar^4}{288}\,\big(\tilde{\partial}^l\vartheta^{mn}(p)\big)\,\big(\tilde{\partial}^k\vartheta^{ij}(p)\big)\,[\bar{\mathcal{F}}_e]_{\bar{p}\to p}\,\times \\
&\qquad \times\big(P_l\,P_m\,P_k\,P_i \otimes P_n\,P_j + 2\,P_l\,P_m\,P_j \otimes P_n\,P_k\,P_i + \\
&\qquad\quad + P_n\,P_j \otimes P_l\,P_m\,P_k\,P_i\big) \\
&+ \mathcal{O}\big(\tilde{\partial}^3\vartheta, (\tilde{\partial}\vartheta)^3\big)\ ,
\end{aligned} \tag{5.108}$$

where

$$\bar{\mathcal{F}}_e = \exp\big[-\tfrac{\mathrm{i}\,\hbar}{2}\,\big(\vartheta^{ij}(\bar{p})\,P_i \otimes P_j + P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_i\big)\big]\ . \tag{5.109}$$

The expression (5.108) defines a 2-cochain on $H_e[[\hbar]]$ with coboundary $\phi_e = \partial^*\mathcal{F}_e$ given by

$$\begin{aligned}
\phi_e = 1 \otimes 1 \otimes 1 &+ \tfrac{\hbar^2}{2}\,J^{ijk}(p)\,[\bar{\mathcal{F}}_{12}\,\bar{\mathcal{F}}_{23}]_{\bar{p}\to p}\,(P_i \otimes P_j \otimes P_k) \\
&+ \tfrac{\mathrm{i}\,\hbar^3}{8}\,\tilde{\partial}^l J^{ijk}(p)\,[\bar{\mathcal{F}}_{12}\,\bar{\mathcal{F}}_{23}]_{\bar{p}\to p}\big(P_l\,P_i \otimes P_j \otimes P_k - P_j \otimes P_k \otimes P_l\,P_i\big) \\
&+ \tfrac{\hbar^4}{8}\,\big(\tilde{\partial}^l\vartheta^{mn}(p)\big)\,J^{ijk}(p)\,[\bar{\mathcal{F}}_{12}\,\bar{\mathcal{F}}_{23}]_{\bar{p}\to p}\,\times \\
&\quad \times\big(P_l\,P_m\,P_i \otimes P_k \otimes P_n\,P_j + P_l\,P_m\,P_i \otimes P_n\,P_k \otimes P_j + \\
&\qquad + P_i\,P_n \otimes P_l\,P_m\,P_k \otimes P_j + P_i \otimes P_l\,P_m\,P_k \otimes P_n\,P_j + \\
&\qquad + 2\,P_l\,P_i \otimes P_m\,P_k \otimes P_n\,P_j + 2\,P_n\,P_i \otimes P_m\,P_k \otimes P_l\,P_j + \\
&\qquad + P_i \otimes P_n\,P_k \otimes P_l\,P_m\,P_j + P_i\,P_n \otimes P_k \otimes P_l\,P_m\,P_j\big) \\
&+ \mathcal{O}\big(\tilde{\partial}^3\vartheta, (\tilde{\partial}\vartheta)^3\big)\ .
\end{aligned} \tag{5.110}$$

It is straightforward to check that the expression (5.110) satisfies the cocycle condition (5.20) order by order in $\hbar$, and hence yields the counital associator 3-cocycle for the quasi-Hopf algebra obtained from twisting H_e by $\mathcal{F}_e$. Note that each term in (5.110) involves the classical Jacobiator (5.102).

The generality of this setting now allows for deformation quantization of the geometry of elliptic R-flux backgrounds up to third order in the R-flux. However, in this case cyclicity of the nonassociative star product (5.103) is a more delicate issue and requires a more sophisticated definition of integration on non-parabolic R-spaces; see [65] for a detailed analysis of this problem in the context of open string theory, and [25] for an investigation in the context of double field theory.

Chapter 6

Nonassociative quantum mechanics

The standard formulation of quantum mechanics is based on linear operators acting on a separable Hilbert space and the corresponding operator algebras are by construction associative. Nevertheless, it turns out that the mathematical tools and structures that we have developed in chapter 5 do in fact allow for a direct quantitative discussion of nonassociativity in quantum mechanics, adding to the more qualitative arguments that can already be found in the literature. The lack of associativity alters the theory of quantum mechanics drastically, but against all odds a consistent formulation is apparently indeed possible.

For this, we employ our phase space star product quantization, and constuct states by introducing an appropriate unital composition of functions in $C^\infty(\mathcal{M})$ which is noncommutative and associative. We then use the 3-cyclicity condition to show that the expectation values of functions (observables) on $\mathcal{M}$ satisfy the desired properties of reality and positivity. Within this framework, we find clear sign of position space quantization in the presence of R-flux, and compute the non-trivial uncertainty relation between position coordinates. We also define suitable area and volume operators in configuration space, which possess non-vanishing uncertainties and lead to a minimal volume element; therefore our formalism provides a rigorous derivation for the uncertainty relations anticipated by [90, 27]. Finally, we study the dynamics of the theory and find that operator time evolution in the Heisenberg picture is not a derivation of the star product algebra of operators.

6.1 Nambu-Heisenberg brackets, star products and compositions

The task of formulating a nonassociative version of quantum mechanics is closely related to the quest of quantizing Nambu-Poisson brackets. A natural choice would be the Jacobiator of operators, but it obviously vanishes for associative operator algebras. As a work-around, a Nambu-Heisenberg bracket was introduced by Nambu as half of a Jacobiator [103]

$$[A,B,C]_{\mathrm{NH}} = A\,B\,C + C\,A\,B + B\,C\,A - B\,A\,C - A\,C\,B - C\,B\,A \ . \tag{6.1}$$

It is straightforward to evaluate the Nambu-Heisenberg bracket on coordinate functions for any of our star products.* For instance, for the associative constant $\bar p$ star product (5.45) we find

$$[x^i,x^j,x^k]_{\,\bar\star\text{-NH}} = \mathrm{i}\,\hbar\,\big(R^{ijl}\,\bar p_l\,x^k + R^{jkl}\,\bar p_l\,x^i + R^{kil}\,\bar p_l\,x^j\big) \ . \tag{6.2}$$

Nambu suggested to consider nonassociative algebras for the quantization of his bracket. We do have the tools now to study this proposal. In the nonassociative case, we need to specify which operators are multiplied first. We choose by default the first pair and write

$$[A,B,C]_{\mathrm{NH}} = [A,B]\,C + [C,A]\,B + [B,C]\,A \ , \tag{6.3}$$

where $[A,B]\,C := (A\,B)\,C - (B\,A)\,C$. For the nonassociative star product (5.57), evaluated on a triple of coordinate functions, this gives

$$[x^i,x^j,x^k]_{\star\text{-NH}} = \mathrm{i}\,\hbar\,\big(R^{ijl}\,p_l\star x^k + R^{jkl}\,p_l\star x^i + R^{kil}\,p_l\star x^j\big) \ . \tag{6.4}$$

*Nambu-Heisenberg brackets have been previously investigated in the context of phase space quantum mechanics based on *associative* star products in [125].

The opposite Nambu-Heisenberg bracket

$$[A,B,C]'_{\mathrm{NH}} = C\,[B,A] + B\,[A,C] + A\,[C,B] \tag{6.5}$$

is in general no longer equal to minus the original Nambu-Heisenberg bracket. Their sum gives the Jacobiator

$$[A,B,C] := [[A,B],C] + [[C,A],B] + [[B,C],A] = [A,B,C]_{\mathrm{NH}} + [A,B,C]'_{\mathrm{NH}} \ . \tag{6.6}$$

For the nonassociative star product (5.57), evaluated on a triple of coordinate functions, we obtain the non-zero Jacobiator (c.f. (1.7))

$$[x^i,x^j,x^k]_\star = \mathrm{i}\,\hbar\,\big(R^{ijl}\,[p_l,x^k]_\star + R^{jkl}\,[p_l,x^i]_\star + R^{kil}\,[p_l,x^j]_\star\big) = 3\,\hbar^2\,R^{ijk} \tag{6.7}$$

as a more convincing candidate than (6.2) or (6.4) for a quantized Nambu-Poisson bracket.

An indirect approach to nonassociative quantum mechanics can be based on the family of associative star products $\bar\star$ for constant $\bar p$-slices and the mappings that link them, and to $\star$ by twists, very much in the spirit of describing a general manifold in terms of Euclidean spaces by local coordinate charts and transition functions. A regular operator/Hilbert space approach to nonassociative quantum mechanics can in fact be based on standard canonical quantization and the twist (5.50) from the Moyal-Weyl product (5.35) to the nonassociative product (5.57); after quantization, the twist is expressed in terms of operators acting on a suitable Hilbert space.

Instead of these solid but indirect approaches to nonassociative quantum mechanics, we shall pursue a more direct approach: The phase space formulation of quantum mechanics [98] is powerful enough to study nonassociative quantum mechanics *in situ* (see e.g. [124] and references therein). Observables are implemented as real functions on phase space, states are represented by pseudo-probability Wigner-type density functions, and noncommutativity of operators enters via a star product of functions, which is the deformation quantization of a classical Poisson structure.

Let us start by introducing some convenient notation and conventions. We introduce the *compositions* $\circ$ and $\bar{\circ}$ by

$$(A \circ B) \star C := A \star (B \star C) \ , \qquad C \star (A \,\bar{\circ}\, B) := (C \star A) \star B \tag{6.8}$$

for all $A, B, C \in C^\infty(\mathcal{M})[[\hbar]]$. The compositions are related by complex conjugation: $(A \circ B)^* = B^* \,\bar{\circ}\, A^*$ and *vice versa*. We choose the convention that $\bar{\circ}$ is evaluated before $\circ$ in all expressions that involve both compositions, so that†

$$(A \circ B) \star (C \,\bar{\circ}\, D) := \big((A \circ B) \star C\big) \star D = \big(A \star (B \star C)\big) \star D \ . \tag{6.9}$$

The compositions can be extended to an arbitrary number of functions and are by construction associative. Like the star products that we are considering in this thesis, the compositions are noncommutative and unital: $1 \circ A = A = A \circ 1$. For an associative algebra, $\circ$ would just be the product in that algebra. However, in the nonassociative case $A \circ B$ cannot even generally be replaced by some suitable element of the algebra $(C^\infty(\mathcal{M})[[\hbar]], \star)$, because if this were possible then $(A \circ B) \star 1 = A \star (B \star 1) = A \star B$ would imply $A \circ B = A \star B$ and thus $(A \star B) \star C = (A \circ B) \star C = A \star (B \star C)$ for all $C \in C^\infty(\mathcal{M})[[\hbar]]$. In a nonassociative algebra this is obviously not true for all $A, B \in C^\infty(\mathcal{M})[[\hbar]]$. There are, however, some notable exceptions, e.g. $x^i \circ x^i = x^i \star x^i = (x^i)^2$ and $p_i \circ p_i = p_i \star p_i = (p_i)^2$.

6.2 States, operators and eigenvalues

States map observables to numbers, which are interpreted as expectation values and link theory to experiment. For this purpose one requires convexity, reality, unit trace, and positivity properties. The latter property is particularly difficult to implement in a nonassociative setting. A definition that ultimately fulfills all these

†This convention looks asymmetric, but as long as we are just computing expectation values, it gives the same results as the alternative convention as a consequence of 3-cyclicity. Physically this is a remnant of operator-state duality. In the context of time-evolution and similar transformations this duality is, however, no longer a symmetry in the nonassociative setting.

requirements is as follows. A *state* ρ is an expression of the form

$$\rho = \sum_{\alpha=1}^{n} \lambda_\alpha \, \psi_\alpha \,\bar{\circ}\, \psi_\alpha^* \ , \tag{6.10}$$

where $n \geq 1$, $\lambda_\alpha > 0$, $\sum_{\alpha=1}^{n} \lambda_\alpha = 1$, and ψ_α are complex-valued *phase space wave functions*, which are normalized as

$$\int_{\mathcal{M}} \mathrm{d}^{2d}x \ |\psi_\alpha|^2 = 1 \ , \tag{6.11}$$

but are not necessarily orthogonal.[‡] For any two states ρ_1 and ρ_2, the convex linear combination $\rho_3 = \lambda\,\rho_1 + (1-\lambda)\,\rho_2$ with $\lambda \in [0,1]$ is again a state. The space of states is thus a convex set, whose extrema we call pure states. A necessary (but not sufficient) condition for a state to be pure is that it is of the form $\rho = \psi \,\bar{\circ}\, \psi^*$. Given a state ρ, the expectation value of a function on phase space ("operator") A is obtained by the phase space integral

$$\begin{aligned} \langle A \rangle &:= \int_{\mathcal{M}} \mathrm{d}^{2d}x \ A \star \rho \\ &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ A \star (\psi_\alpha \,\bar{\circ}\, \psi_\alpha^*) \\ &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (A \star \psi_\alpha) \star \psi_\alpha^* \\ &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ \psi_\alpha^* \star (A \star \psi_\alpha) = \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ \psi_\alpha^* \, (A \star \psi_\alpha) \ , \end{aligned} \tag{6.12}$$

where we have used 2-cyclicity. Using 3-cyclicity and the fact that complex conjugation acts anti-involutively on the star product, $(A \star \psi)^* = \psi^* \star A^*$, we find

[‡]Using the familiar language of quantum mechanics, we refer to complex-valued functions on phase space that are multiplied by star products as "operators" and to real-valued functions on phase space that are associated to something that can in principle be measured as "observables". The phase space wave functions ψ_α and their complex conjugates ψ_α^* should not be confused with state vector kets or bras. The corresponding objects in ordinary quantum mechanics are normalized but otherwise arbitrary operators that are not necessarily related to rank one projectors.

$$\langle A\rangle^* = \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (A\star\psi_\alpha)^*\star\psi_\alpha = \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ \psi_\alpha^*\star(A^*\star\psi_\alpha) = \langle A^*\rangle \ . \tag{6.13}$$

Observables (i.e. real-valued functions on phase space $A^* = A$) therefore have real expectation values as desired. We will later show that 3-cyclicity also ensures reality of eigenvalues. Thanks to 3-cyclicity, our approach to nonassociative quantum mechanics is thus not affected by a previously proposed no-go theorem [49].

Expectation values can also be computed for star products of functions (because star products of functions are again functions). The definition of expectation value can be further extended to compositions of operators as

$$\begin{aligned}\langle A_1\circ A_2\circ\cdots\circ A_k\rangle &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (A_1\circ A_2\circ\cdots\circ A_k)\star\psi_\alpha \,\bar{\circ}\, \psi_\alpha^* \\ &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ \big[A_1\star\big(A_2\star\big(\cdots\star(A_k\star\psi_\alpha)\cdots\big)\big)\big]\star\psi_\alpha^* \ .\end{aligned} \tag{6.14}$$

Positivity is a tricky concept in the nonassociative setting. In terms of our definition of a state, it is realized for any state ρ and any function on phase space A as

$$\begin{aligned}\langle A^*\circ A\rangle &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ \psi_\alpha^*\star\big(A^*\star(A\star\psi_\alpha)\big) \\ &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (\psi_\alpha^*\star A^*)\star(A\star\psi_\alpha) \\ &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (A\star\psi_\alpha)^*\,(A\star\psi_\alpha) \\ &= \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ |A\star\psi_\alpha|^2 \geq 0 \ ,\end{aligned} \tag{6.15}$$

where we have used 2-cyclicity, 3-cyclicity and anti-involutivity with respect to complex conjugation. With a similar computation we see that

$$(A,B) := \langle A^*\circ B\rangle = \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (A\star\psi_\alpha)^*\,(B\star\psi_\alpha) \tag{6.16}$$

defines a semi-definite sesquilinear form for any given state ρ. This will be the basis of the derivation of uncertainty relations below, because it implies the Cauchy-Schwarz inequality

$$\big|(A,B)\big|^2 \leq (A,A)\,(B,B)\ . \tag{6.17}$$

Using 3-cyclicity, the expectation value (6.12) of a single operator (no compositions) can be rewritten in terms of a *state function* $S_\rho = \sum_{\alpha=1}^n \lambda_\alpha\, \psi_\alpha \star \psi_\alpha^*$ as

$$\langle A\rangle = \sum_{\alpha=1}^n \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x\ (A\star\psi_\alpha)\star\psi_\alpha^* = \sum_{\alpha=1}^n \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x\ A\star(\psi_\alpha\star\psi_\alpha^*) = \int_{\mathcal{M}} \mathrm{d}^{2d}x\ A\,S_\rho\ . \tag{6.18}$$

The state function S_ρ is a real function on phase space that is normalized as

$$\langle 1\rangle = \int_{\mathcal{M}} \mathrm{d}^{2d}x\ S_\rho = \sum_{\alpha=1}^n \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x\ |\psi_\alpha|^2 = 1\ , \tag{6.19}$$

but is not necessarily non-negative everywhere. It plays the role of a quasi-probability distribution function, like the Wigner function in the associative case. However, unlike the associative case, we cannot formulate the theory entirely in terms of the state function S_ρ, but rather we also need to frequently refer to the phase space wave functions ψ_α.

A function ("operator") A can have eigenfunctions f (with respect to the star product multiplication) with eigenvalues $\lambda \in \mathbb{C}$: $A\star f = \lambda\, f$. Complex conjugation implies $f^*\star A^* = \lambda^*\, f^*$. We can show that real functions $A = A^*$ have real eigenvalues, but this fact is not quite as straightforward as in the associative case. We have

$$f^*\star(A\star f) - (f^*\star A)\star f = (\lambda-\lambda^*)\,(f^*\star f)\ . \tag{6.20}$$

The left-hand side of this equation is non-zero in general, but it vanishes after integrating over phase space and using 3-cyclicity. We obtain

$$(\lambda-\lambda^*)\int_{\mathcal{M}} \mathrm{d}^{2d}x\ f^*\star f = (\lambda-\lambda^*)\int_{\mathcal{M}} \mathrm{d}^{2d}x\ |f|^2 = 0\ . \tag{6.21}$$

The integral is non-zero unless f is identically equal to zero and therefore $\lambda =$

λ^* as desired. Using similar manipulations, we can show that eigenfunctions are orthogonal if they correspond to distinct eigenvalues. In the nonassociative case we need to distinguish eigen-state functions and eigen-wave functions (unless we integrate and use 3-cyclicity): $A \star \psi = \lambda\, \psi$ does not necessarily imply $A \star S_\rho = \lambda\, S_\rho$, where $\rho = \psi \,\bar{\circ}\, \psi^*$ and $S_\rho = \psi \star \psi^*$, because $(A \star \psi) \star \psi^* \neq A \star (\psi \star \psi^*)$ in general.

All definitions here and in the following are consistent with the associative limit of phase space quantum mechanics. The nonassociative case is more restrictive and in a way it teaches us also something about ordinary phase space quantum mechanics. We have attempted to keep all definitions as general as possible. Depending on the intended application, further restrictions may be necessary; for example, it is natural to require states to be symmetric: $\rho = \rho'$, where ρ is as given in (6.10) and $\rho' = \sum_{\alpha=1}^n \lambda_\alpha\, \psi_\alpha^* \,\bar{\circ}\, \psi_\alpha$.

6.3 Uncertainty relations, area and volume operators

A pair of operators that do not commute cannot have a complete set of common eigenstates; a pair of operators with a central non-zero commutator do not have any simultaneous eigenstates. These well-known facts of quantum mechanics are important for measurements and can also be verified for nonassociative phase space quantum mechanics. A new feature is that analogous statements hold for any triple of operators that do not associate. Let us illustrate this for phase space coordinate functions $x^I \in \{x^1, \dots, x^d, p_1, \dots, p_d\}$ with commutator and associator

$$x^I \star x^J - x^J \star x^I = \mathrm{i}\, \hbar\, \Theta^{IJ} \ , \qquad (x^I \star x^J) \star x^K - x^I \star (x^J \star x^K) = \tfrac{\hbar^2}{2}\, R^{IJK} \ , \tag{6.22}$$

respectively, where $R^{IJK} := \partial_K \Theta^{IJ}$ is constant and non-zero (and then equal to R^{ijk}) only for (selected) configuration space coordinates, c.f. (2.24) and (5.58). Let us assume that a pair of phase space coordinates x^I and x^J with $I \neq J$ have a common (normalized) eigen-state function S: $x^I \star S = \lambda^I\, S$ and $x^J \star S = \lambda^J\, S$.

Using 3-cyclicity, this implies

$$\big\langle [x^I, x^J]_\star \big\rangle = \int_{\mathcal{M}} \mathrm{d}^{2d}x \ \big(x^I \star (x^J \star S) - x^J \star (x^I \star S)\big) = \lambda^I\, \lambda^J - \lambda^J\, \lambda^I = 0 \ , \quad (6.23)$$

and hence x^I and x^J with $\big\langle \Theta^{IJ} \big\rangle \neq 0$ cannot have a common eigen-state function S. Let us now assume that a triple of phase space coordinates x^I, x^J, and x^K have a common eigen-state function S with eigenvalues λ^I, λ^J, and λ^K. Using 3-cyclicity repeatedly we find

$$\begin{aligned} \int_{\mathcal{M}} \mathrm{d}^{2d}x \ \big((x^I \star x^J) \star x^K\big) \star S &= \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (x^I \star x^J) \star (x^K \star S) \\ &= \lambda^K \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (x^I \star x^J) \star S \\ &= \lambda^K \int_{\mathcal{M}} \mathrm{d}^{2d}x \ x^I \star (x^J \star S) = \lambda^K\, \lambda^J\, \lambda^I \ , \end{aligned} \quad (6.24)$$

while using 2-cyclicity, 3-cyclicity and the fact that λ^I must be real we find similarly

$$\int_{\mathcal{M}} \mathrm{d}^{2d}x \ \big(x^I \star (x^J \star x^K)\big) \star S = \int_{\mathcal{M}} \mathrm{d}^{2d}x \ (x^J \star x^K) \star (S \star x^I) = \lambda^I\, \lambda^K\, \lambda^J \ . \quad (6.25)$$

Taking the difference of the two expressions implies

$$\tfrac{\hbar^2}{2}\, R^{IJK} = \lambda^K\, \lambda^J\, \lambda^I - \lambda^I\, \lambda^K\, \lambda^J = 0 \quad (6.26)$$

and we arrive at the striking result that coordinates x^i, x^j and x^k which do not associate, i.e. for which $R^{ijk} \neq 0$, cannot have a common eigen-state function S; whence they cannot be measured simultaneously with arbitrary precision. This is a clear sign of a coarse-graining (quantization) of space in the presence of R-flux.

Let us now turn to the study of uncertainties. In the definition of the uncertainties, we *a priori* face the problem of having to make a choice between using expectation values based on phase space wave functions (with the advantage of the availability of inequalities) and state functions (with computational advantages). In the computation of uncertainties for phase space coordinates, this luckily does not play a role because $x^I \circ x^I = x^I \star x^I$ and thus

$$0 \leq \sum_{\alpha=1}^{n} \lambda_\alpha \int_{\mathcal{M}} \mathrm{d}^{2d}x \; \psi_\alpha^* \star \big(x^I \star (x^I \star \psi_\alpha)\big) = \int_{\mathcal{M}} \mathrm{d}^{2d}x \; (x^I \star x^I) \star S_\rho = \big\langle (x^I)^{\star 2} \big\rangle \; . \tag{6.27}$$

Without ambiguity, we can therefore define the uncertainty as usual in terms of the expectation value of the square of the shifted coordinate $\widetilde{x}^I := x^I - \langle x^I \rangle$ as

$$\Delta x^I := \sqrt{\big\langle (\widetilde{x}^I)^{\star 2} \big\rangle} = \sqrt{\big\langle (x^I)^{\star 2} \big\rangle - \big\langle x^I \big\rangle^2} \; . \tag{6.28}$$

This uncertainty is zero for eigen-state functions ($x^I \star S = \lambda \, S$) as well as for eigen-wave functions ($x^I \star \psi = \lambda \, \psi$). The function x^I is real and the uncertainty can thus be rewritten as

$$(\Delta x^I)^2 = \big\langle \widetilde{x}^I \star \widetilde{x}^I \big\rangle = \big\langle \widetilde{x}^I \circ \widetilde{x}^I \big\rangle = \big(\widetilde{x}^I \, , \widetilde{x}^I \big) \; . \tag{6.29}$$

Using the Cauchy-Schwarz inequality (6.17), and decomposing into imaginary and real parts we get

$$(\Delta x^I)^2 \, (\Delta x^J)^2 \geq \big| (\widetilde{x}^I, \widetilde{x}^J) \big|^2 = \tfrac{1}{4} \, \big| \big\langle [x^I, x^J]_\circ \big\rangle \big|^2 + \tfrac{1}{4} \, \big| \big\langle \{\widetilde{x}^I, \widetilde{x}^J\}_\circ \big\rangle \big|^2 \; , \tag{6.30}$$

where $[A, B]_\circ := A \circ B - B \circ A$ and $\{A, B\}_\circ := A \circ B + B \circ A$. Ignoring the last term yields a Born-Jordan-Heisenberg-type uncertainty relation

$$\Delta x^I \, \Delta x^J \geq \tfrac{1}{2} \, \big| \big\langle [x^I, x^J]_\circ \big\rangle \big| \; . \tag{6.31}$$

To proceed from here, we need to distinguish several cases: Whenever one of the phase space coordinates is a momentum p_i, nonassociativity does not play a role in the sense that then "$\circ = \star$", i.e. $[p_i, p_j]_\circ = [p_i, p_j]_\star = 0$ and $[p_i, x^j]_\circ = [p_i, x^j]_\star = -\, \mathrm{i} \, \hbar \, \delta_i{}^j$, and therefore

$$\Delta p_i \, \Delta p_j \geq 0 \; , \qquad \Delta x^i \, \Delta p_j \geq \tfrac{\hbar}{2} \, \delta^i{}_j \tag{6.32}$$

as in ordinary quantum mechanics. The non-trivial uncertainty relation for a pair of coordinates x^i and x^j is new and requires a more complicated computation. With the help of the associator (5.58) we can express $[x^i, x^j]_\circ$ in terms of a star commutator and obtain the surprising result

$$\begin{aligned}
[x^i, x^j]_\circ \star \psi :=& \, x^i \star (x^j \star \psi) - x^j \star (x^i \star \psi) \\
=& \, [x^i, x^j]_\star \star \psi - \hbar^2 \, R^{ijk} \, \partial_k \psi \\
=& \, \mathrm{i} \, \hbar \, R^{ijk} \, \big(p_k \star \psi + \mathrm{i} \, \hbar \, \partial_k \psi \big) \\
=& \, \mathrm{i} \, \hbar \, R^{ijk} \, \big(p_k \, \psi - \tfrac{\mathrm{i} \hbar}{2} \, \partial_k \psi + \mathrm{i} \, \hbar \, \partial_k \psi \big) \, = \, \mathrm{i} \, \hbar \, R^{ijk} \, \psi \star p_k \ ,
\end{aligned} \tag{6.33}$$

that is, the momentum operator ends up on the "wrong" side of ψ. Using this result, we obtain an uncertainty relation for position measurements

$$\Delta x^i \, \Delta x^j \geq \tfrac{\hbar}{2} \, \big| R^{ijk} \, \langle p_k \rangle' \big| \ , \tag{6.34}$$

where $\langle p_k \rangle'$ is the expectation value of p_k computed with respect to the *opposite* state ρ'. Only for symmetric or antisymmetric states, i.e. $\rho' = \pm \, \rho$, will this be equal to the standard expectation value $\langle p_k \rangle$, and one should consider adding this requirement to the definition of a state. The new uncertainty relation (6.34) features uncertainties for position measurements in directions transverse to momentum, while the usual Heisenberg uncertainty relation relates uncertainties of position and momentum in the same direction.

It is tempting to interpret the left-hand side of (6.34) as an area uncertainty that grows linearly with transverse momentum, but this is misleading: The position uncertainty relation makes a prediction for the average outcome of many identically prepared experiments in which *either* x^i *or* x^j is measured. In none of these (Gedanken-)experiments are positions in two different directions measured simultaneously (or one shortly after the other), but this would be required for a genuine area uncertainty. An analogous criticism applies to the superficial interpretation of Heisenberg uncertainty as an uncertainty of phase space areas in ordinary quantum mechanics. To remedy the situation, we shall define an area operator whose expectation value can be computed and interpreted as fundamental area measurement uncertainty (or minimal area). The approach generalizes to higher dimensional objects and we will also derive a fundamental volume measurement uncertainty, which results from the nonassociativity of coordinate functions.

The oriented area spanned by two segments $\delta\vec{r}_1$ and $\delta\vec{r}_2$ in three-dimensional Euclidean space is given by the vector product $\delta\vec{r}_1 \times \delta\vec{r}_2$, while the volume spanned by three segments $\delta\vec{r}_1$, $\delta\vec{r}_2$ and $\delta\vec{r}_3$ is the triple scalar product $\delta\vec{r}_1 \cdot (\delta\vec{r}_2 \times \delta\vec{r}_3)$. This can easily be generalized to higher dimensional parallelepipeds and to embedding spaces of arbitrary dimension. The most convenient description of these areas and (higher dimensional) volumes for our purposes is in terms of antisymmetrized sums of products of components of displacement vectors $\delta\vec{r}$. For the sake of generality we might as well consider displacements in phase space. For the description of quantum uncertainties we replace all of the displacement vectors by the single displacement vector (of operators) $\vec{\widetilde{x}} = \vec{x} - \langle \vec{x}\, \rangle$, and promote commutative pointwise multiplication to the noncommutative and nonassociative star product $\star$. Furthermore, we would like to construct observables, i.e. real functions on phase space. Taking all this into consideration, the appropriate area (uncertainty) operator in directions x^I, x^J is

$$A^{IJ} = \mathfrak{Im}\big([\widetilde{x}^I, \widetilde{x}^J]_\star\big) = -\,\mathrm{i}\,\big(\widetilde{x}^I \star \widetilde{x}^J - \widetilde{x}^J \star \widetilde{x}^I\big) \tag{6.35}$$

and the volume (uncertainty) operator in directions x^I, x^J, x^K is (c.f. (6.4) and (6.7))

$$V^{IJK} = \mathfrak{Re}\big([\widetilde{x}^I, \widetilde{x}^J, \widetilde{x}^K]_{\star\text{-NH}}\big) = \tfrac{1}{2}\,\big[[\widetilde{x}^I, \widetilde{x}^J, \widetilde{x}^K]\big]_\star\ . \tag{6.36}$$

The expectation values of these (oriented) area and volume operators are easily computed to be

$$\big\langle A^{IJ}\big\rangle = \hbar\,\big\langle \Theta^{IJ}\big\rangle\ , \qquad \big\langle V^{IJK}\big\rangle = \tfrac{3}{2}\,\hbar^2\,R^{IJK}\ , \tag{6.37}$$

with three interesting special cases

$$\big\langle A^{x^i,p_j}\big\rangle = \hbar\,\delta^i{}_j\ , \qquad \big\langle A^{ij}\big\rangle = \hbar\,R^{ijk}\,\langle p_k\rangle\ , \qquad \big\langle V^{ijk}\big\rangle = \tfrac{3}{2}\,\hbar^2\,R^{ijk}\ . \tag{6.38}$$

The first expression describes phase space cells with area $\hbar$. The second expression illustrates an area uncertainty proportional to the magnitude of the transverse momentum. The third expression indicates a minimal resolvable volume of order $\frac{3}{2}\,\hbar^2\,|R|$ due to nonassociativity-induced position measurement uncertainties (here $|R|$ is a generalized determinant of the antisymmetric 3-tensor R^{ijk}). Uncertainties similar to (6.38) have appeared previously in [90, 27], and here we have provided a concrete and rigorous derivation of them as expectation values of area and volume operators.

6.4 Dynamics and transformations

Let us close this chapter with some remarks on dynamics, and similar state and operator transformations in nonassociative quantum mechanics. Time evolution and other transformations should leave the structure of the theory intact. In particular notions of positivity, normalization of probabilities and reality should be preserved. Observables (i.e. real functions on phase space) should be mapped to observables and (pure) states to (pure) states. As in ordinary quantum mechanics, there are two approaches that fulfill all these requirements. In the nonassociative case the two approaches are, however, no longer equivalent.

A Schrödinger-type approach focuses on evolution equations for the phase space wave functions. The starting point is the phase space Schrödinger equation

$$\mathrm{i}\,\hbar\,\frac{\partial\psi}{\partial t}=\mathcal{H}\star\psi\ , \tag{6.39}$$

which applies to all ψ_α and ψ_α^* in the state ρ (c.f. (6.10)), and where the Hamiltonian $\mathcal{H}$ is a real function on phase space. Observationally, only the time evolution $\frac{\partial}{\partial t}\langle A\rangle$ of expectation values is relevant. It can be computed either from the Schrödinger equation (6.39) or equivalently from the time evolution equation for operators and compositions of operators given by

$$\frac{\partial\alpha}{\partial t}=\frac{\mathrm{i}}{\hbar}\,[\mathcal{H},\alpha]_\circ\ , \tag{6.40}$$

where $\alpha = A$, or $A \circ B$, or $A \circ B \circ C$, etc. The $\circ$-commutator in (6.40) is a $\circ$-derivation,

$$[\mathcal{H}, A \circ B]_\circ = [\mathcal{H}, A]_\circ \circ B + A \circ [\mathcal{H}, B]_\circ \ , \tag{6.41}$$

and thus $\frac{\partial}{\partial t}(A \circ B) = \frac{\partial A}{\partial t} \circ B + A \circ \frac{\partial B}{\partial t}$. For stationary states, wave functions simply change by a time-dependent phase and we can study energy eigenvalues E via the time-independent Schrödinger equation

$$\mathcal{H} \star \psi = E\, \psi \ . \tag{6.42}$$

A Heisenberg-type approach focuses on $\star$-commutator based evolution equations for operators given by

$$\frac{\partial A}{\partial t} = \frac{\mathrm{i}}{\hbar}\, [H, A]_\star \ . \tag{6.43}$$

This time evolution equation again fulfills all our requirements. It can be applied to single functions ("operators") as well as to star products of functions, *but it is not a derivation of* $\star$ since

$$[H, A]_\star \star B + A \star [H, B]_\star \neq [H, (A \star B)]_\star \tag{6.44}$$

This surprising fact should be seen as an interesting feature of the theory, not as a mistake. We can still compute the time-dependence of any operator that we are interested in, but we cannot determine it from the time-dependence of its constituent parts. Similarly to the Schrödinger-type approach, there is an alternative equivalent way to compute the time-dependence of expectation values, in this case by the evolution equation for phase space state functions

$$\frac{\partial S}{\partial t} = \frac{1}{\mathrm{i}\,\hbar}\, [H, S]_\star \ . \tag{6.45}$$

Stationary state functions $\star$-commute with the Hamiltonian function H and we can study energy eigenvalues E via

$$H \star S = E\, S \ . \tag{6.46}$$

The only major difference between this equation and (6.42) is that S should be a real function while there is *a priori* no such requirement for ψ.

We have found two sets of inequivalent but equally consistent transformation equations. Which approach should be used for what is ultimately a question of the physics that we would like to describe. The evolution equations of the Heisenberg-type approach close in the algebra of operators and appear therefore predestined to define active transformations like time evolution (i.e. dynamics), while the Schrödinger-type expressions could then still be useful to describe certain symmetries of the theory.

Chapter 7

Nonassociative scalar field theory

Noncommutative deformations of spacetime have inspired the formulation of noncommutative field theory (see e.g. [47, 117] for a review) and noncommutative gravity (see e.g. [8]). In this chapter we employ the results obtained in chapter 5 to develop Euclidean scalar field theories on nonassociative constant R-flux backgrounds.

Nonassociative deformations are by construction noncommutative, and thus nonassociative field theories retain features that are familiar from ordinary noncommutative scalar field theories. For example, momentum conservation is violated due to the non-constant deformation parameter (2.24), while one-loop corrections to the propagator suffer from UV/IR mixing [96]. Still nonassociative field theories exhibit novelties that are not encountered in their associative counterparts. In particular, nonassociativity of the star product suggests the introduction of an interaction term for each different way of bracketing an n-fold product of fields. The various terms are classified under 3-cyclicity of the star product, and induce a modification to the usual classification of Feynman diagrams into planar and non-planar graphs. However, it turns out that, for $n = 3, 4, 5$ the 2-cyclic and 3-cyclic properties of the star product eliminate this effect at tree level and one needs to perform perturbative calculations–at least up to one-loop–to detect it.

Another feature particular to field theories on R-space is that they are defined on phase space $\mathcal{M} := T^*M$ rather than the manifold M, however the complete physical interpretation of quantum field theory on phase space is at the moment unclear. A statistical approach is taken in [4] where it was proposed that fields on phase space acquire a physical meaning via an association to Wigner functions, but an interpretation related to deformation theory is still lacking. As the phase space formalism naturally arises in the geometrization of R-space, it is tempting to think of this model along the lines of double field theory. It would then be interesting to understand if there is some analogue of the section condition reducing the phase space field theory to field theory on the original configuration manifold M.

7.1 Action functionals

The action functional of a free Euclidean scalar field theory on R-space is given by

$$S_0[\varphi] = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^{2d}x\, \left(\partial_I\varphi \star \partial^I\varphi + m^2\, \varphi\star\varphi\right)\ , \tag{7.1}$$

where $\varphi \in \mathcal{S}(\mathcal{M}) \subset C^\infty(\mathcal{M})$ is a real scalar field of mass m, $I \in \{1,\dots,2d\}$ and the nonassociative star product is given by (3.17) or equivalently (5.57). By the 2-cyclicity condition (5.84) the star products in (7.1) reduce to the usual pointwise multiplication of functions, and S_0 becomes equal to the standard free scalar field theory action on $\mathcal{M} := T^*M$. This implies that the bare propagator is not affected by the nonassociative deformation, and thus in a similar way to the usual noncommutative field theories, we should consider interactions in order to probe nonassociative effects (see e.g. [117]).

When introducing interactions the nonassociativity of the star product suggests to include all possible bracketings of the product of n fields in the action. However, as we discussed in subsection 5.3.2, all possible integrated n-fold products of functions are classified under 3-cyclicity into C_{n-1} equivalence classes, where C_n are the Catalan numbers, and thus we keep only one representative term for each class. For a single scalar field it follows from 2-cyclicity (5.84) and 3-cyclicity (5.86) that some of the classes are equivalent, while for $n = 3,4,5$ it can be shown that all possible bracketings of the interaction term are equal. For example

$$S^{(3)}_{\rm int}[\varphi] \propto \int_{\mathcal{M}} \mathrm{d}^{2d}x\, \varphi\star(\varphi\star\varphi) = \int_{\mathcal{M}} \mathrm{d}^{2d}x\, (\varphi\star\varphi)\star\varphi \equiv \int_{\mathcal{M}} \mathrm{d}^{2d}x\, \varphi\star\varphi\star\varphi \tag{7.2}$$

and

$$\begin{aligned} S_{\text{int}}^{(4)}[\varphi] \propto \int_{\mathcal{M}} \mathrm{d}^{2d}x\, \varphi \star \big(\varphi \star (\varphi \star \varphi)\big) &= \int_{\mathcal{M}} \mathrm{d}^{2d}x\, \varphi \star \big((\varphi \star \varphi) \star \varphi\big) \\ = \int_{\mathcal{M}} \mathrm{d}^{2d}x\, (\varphi \star \varphi) \star (\varphi \star \varphi) &= \int_{\mathcal{M}} \mathrm{d}^{2d}x\, \big((\varphi \star \varphi) \star \varphi\big) \star \varphi \\ = \int_{\mathcal{M}} \mathrm{d}^{2d}x\, \big(\varphi \star (\varphi \star \varphi)\big) \star \varphi &\equiv \int_{\mathcal{M}} \mathrm{d}^{2d}x\, \varphi \star \varphi \star \varphi \star \varphi \ , \end{aligned} \tag{7.3}$$

where the final integrals in the above expressions are symbolically written without brackets to indicate the equality of all possible bracketings. This means that the φ^3, φ^4 and φ^5 theories are associative at tree level and thus one should study loop corrections in order to detect nonassociativity. The first encounter of nonassociativity at tree level is for the φ^6 theory, where four inequivalent interactions appear [64].

Nonassociative interactions turn out to be rather tricky to deal with as they feature novelties that are not present in their associative counterparts. Since the geometrization of R-space is a phase space, Fourier modes for both configuration and momentum spaces have to be considered in the field expansions. We denote the Fourier momenta corresponding to the phase space coordinates $x^I = (x^i, p_i)$ by the $2d$-vector $k_I = (k_i, \xi^i)$ and we take $\mathcal{M} := T^*M \cong M \times (\mathbb{R}^d)^* \cong \mathbb{R}^d \times (\mathbb{R}^d)^*$ and $\mathcal{M}^* = (\mathbb{R}^d)^* \times \mathbb{R}^d$ for simplicity. In this notation the standard Fourier transform is then given by

$$f(x) = \int_{\mathcal{M}^*} \frac{\mathrm{d}^{2d}k}{(2\pi)^{2d}}\, \tilde{f}(k)\ \mathrm{e}^{\,\mathrm{i}\, k_I x^I} = \int_{\mathcal{M}^*} \frac{\mathrm{d}^d k\, \mathrm{d}^d \xi}{(2\pi)^{2d}}\, \tilde{f}(k_i, \xi^i)\ \mathrm{e}^{\,\mathrm{i}\,(k_i x^i + \xi^i p_i)} \ , \tag{7.4}$$

where

$$\tilde{f}(k) = \int_{\mathcal{M}} \mathrm{d}^{2d}x\, f(x)\ \mathrm{e}^{-\,\mathrm{i}\, k_I x^I} = \int_{\mathcal{M}} \mathrm{d}^d x\, \mathrm{d}^d p\, f(x^i, p_i)\ \mathrm{e}^{-\,\mathrm{i}\,(k_i x^i + \xi^i p_i)} \ , \tag{7.5}$$

and the variation of the free field action (7.1) yields a Klein-Gordon equation with its solution being the free field propagator

$$\Delta(k) = \frac{1}{k^2 + m^2} \ , \tag{7.6}$$

where $k^2 = (k_i)^2 + (\xi^i)^2$.

It is now straightforward to use the Fourier transform (7.4) on each of the C_{n-2} interaction terms and calculate the vertex phase factor. For example, in the case of the equivalence class given by the interaction term

$$\begin{aligned} S^{(n)}_{\rm int} &= \frac{g}{n!}\int_{\mathcal{M}} \mathrm{d}^{2d}x\ \big[\cdots\big[\big((\varphi\star\varphi)\star\varphi\big)\star\varphi\big]\star\cdots\star\varphi\big] \\ &= \frac{g}{n!}\int_{\mathcal{M}} \mathrm{d}^{2d}x\int_{\mathcal{M}^*}\prod_{a=1}^{n}\frac{\mathrm{d}^{2d}k^{(a)}}{(2\pi)^{2d}}\ \varphi\big(k^{(a)}\big)\ V(x,k^{(a)})\ , \end{aligned} \tag{7.7}$$

where

$$V(x,k^{(a)}) = \Big(\cdots\Big[\Big(\big(\mathrm{e}^{\,\mathrm{i}\,k^{(1)}_I x^I}\star \mathrm{e}^{\,\mathrm{i}\,k^{(2)}_I x^I}\big)\star \mathrm{e}^{\,\mathrm{i}\,k^{(3)}_I x^I}\Big)\star \mathrm{e}^{\,\mathrm{i}\,k^{(4)}_I x^I}\Big]\star\cdots\star \mathrm{e}^{\,\mathrm{i}\,k^{(n)}_I x^I}\Big)\ . \tag{7.8}$$

Using 2-cyclicity (5.84), the star product (3.17) and the Baker–Campbell–Hausdorff formula we arrive at the phase factor

$$\begin{aligned} V(x,k^{(a)}) = \exp\Big(\,\mathrm{i}\sum_{a=1}^{n} k^{(a)}_I x^I &- \frac{\mathrm{i}\,\hbar}{2}\sum_{1\le a<b\le n)} k^{(a)}_I k^{(b)}_J\ \Theta^{IJ}(x) \\ &- \frac{\mathrm{i}\,\hbar^2}{4}\sum_{1\le a<b<c\le n} k^{(a)}_I k^{(b)}_J k^{(c)}_L\ R^{IJL}\Big)\ , \end{aligned} \tag{7.9}$$

where $\Theta^{IJ}(x)$ is the phase space deformation parameter matrix given in (2.24) and

$$\big(R^{IJK}\big) = \begin{pmatrix} R^{ijk} & 0 \\ 0 & 0 \end{pmatrix} \tag{7.10}$$

is the Jacobiator matrix. The first two terms of (7.9) are the familiar phase factor modification to the Feynman rules that appear in associative noncommutative φ^n theories, while the last term is due to the nonassociative deformation.

An interesting feature of (7.9) is that it induces violation of momentum conservation at the vertex. This is due to the x^I dependence of the deformation parameter and it is completely analogous to what occurs in the usual noncommutative field theories with spacetime varying noncommutativity parameter [108]. For this, we substitute (7.9) in (7.7) and perform the integral over $\mathcal{M}$ to obtain the momentum relations

$$\sum_{a=1}^{n} k_i^{(a)} = 0 \qquad \text{and} \qquad \sum_{a=1}^{n} \xi^{(a)\,k} = \frac{\hbar}{2} \sum_{1\leq a<b\leq n} k_i^{(a)}\, k_j^{(b)}\, R^{ijk} \ , \tag{7.11}$$

where the first equation is the usual momentum conservation on configuration space while the second equation exhibits violation of momentum conservation along the noncommutative momentum space directions. As we will see in section 7.3 this violation introduces significant complications to the calculation of loop corrections even for the relatively simple case of φ^4 theory.

7.2 Classification of Feynman diagrams

Another intriguing feature of nonassociative scalar field theories is the fact that the phase factor (7.9) is not invariant under cyclic permutations of the Fourier momenta. This is not obvious at first sight, but taking a cyclic permutation of the indices and using (7.11) reveals that the R-dependent block of the deformation parameter combines with ξ-momentum to violate cyclicity of the interaction. This novel feature is particular to nonassociative interactions and has drastic effects on the classification of Feynman diagrams.

For this, regard a connected Feynman diagram as an abstract connected graph G realised by its embedding in an orientable surface of genus γ. The vertices of the graph represent spacetime points and the edges represent propagators. For commutative scalar field theories all vertices are indistinguishable and so are all edges, therefore edge crossings can always be avoided and all possible Feynman diagrams are given by planar graph embeddings, i.e. they can be drawn on a surface of genus $\gamma = 0$. Counting all different ways that vertices and edges can be combined on a plane, provides the number of topologically equivalent diagrams, which enters as the symmetry factor of the graph in the perturbation series.

For the usual noncommutative deformations the interchange of two edges is no longer permitted and thus planarity cannot always be accomplished. However, some symmetry remains as the phase factor of the interaction is invariant under cyclic permutations of the edges. The Feynman diagrams are then classified by the minimal genus of the surfaces in which they are embedded. In fact, it is the cyclicity of the phase factor which guarantees that every way of embedding a graph G into an orientable surface Σ_γ of genus γ is equivalent. This is beautifully captured in matrix models ribbon graphs (see e.g. [18]) and was employed by Filk in [52] to classify the Feynman diagrams in noncommutative field theories into planar and non-planar diagrams.

For nonassociative scalar field theories the vertex interaction is no longer cyclic, which suggests that the different ways of embedding a graph into Σ_γ are no longer equivalent. This is supported by the *rotational embedding scheme* proposed by Edmonds [50] and discussed in detail by Youngs [123].

Theorem 1. (The rotational embedding scheme.)
Let G be a non-trivial connected graph whose set of vertices is $V(G) = \{v_1, v_2, \dots, v_n\}$. For each 2-cell embedding* of G on a surface Σ_γ of genus γ, there exists a unique m-tuple $(\sigma_1, \sigma_2, \dots, \sigma_m)$, where $\sigma_i : V(i) \to V(i)$ is a counter-clockwise cyclic permutation of the edges connected to the i-th vertex of G and $i = 1, 2, \dots, m$. Conversely, for each such m-tuple $(\sigma_1, \sigma_2, \dots, \sigma_m)$, there exists a 2-cell embedding of G on some surface Σ_γ such that for $i = 1, 2, \dots, m$, the subscripts of the vertices adjacent to v_i and in counter-clockwise order about v_i are given by σ_i.

The proof can be found in [35] (see also section 2 of [97]). For this, the graph G is promoted to a directed graph D (digraph) by taking both orientations of its edges, i.e. for each edge $(v_i, v_j) \in E(G)$ we add an edge $(v_j, v_i) \in E(D)$, where $E(D)$ is the set of edges of D. The boundaries of the 2-cell regions are given by the orbits of the permutation map $\pi : E(D) \to E(D)$, where

$$\pi((v_i, v_j)) = (v_j, v_{\sigma_j(i)}) \ , \tag{7.12}$$

while their orientation is determined by the orientation of π (clockwise or counter-clockwise). The embedding surface Σ_γ is acquired by gluing together the 2-cells on

*A 2-cell is a surface for which any closed smooth curve can be continuously contracted to a point.

their boundaries and thus its orientation is determined by the orientation of the 2-cell regions. The genus γ of the surface Σ_γ is given by

$$\gamma = 1 - \frac{1}{2}\big(||V(D)|| - ||E(D)|| + ||O(\sigma_i)||\big) \ , \tag{7.13}$$

where $||\cdot||$ denotes the cardinality and $O(\sigma_i)$ is the set of orbits for the particular m-tuple of cyclic permutations under consideration. Equation (7.13) is of course the familiar Euler formula

$$2 - 2\gamma = m - n + f \ , \tag{7.14}$$

for a Feynman diagram with m vertices, n edges and f faces. The minimal embedding of the graph is given by the rotation systems which provide the maximum number of orbits.

It is now clear that if the interaction vertex is not cyclic, then all different rotational schemes that embed a graph into Σ_γ are potentially inequivalent. However, cyclic permutations are also maps between classes of integrated n-fold products of fields, i.e. between the edges connected to a vertex. Recall from section 5.3.2 that these classes are given by the Catalan number C_{n-2}. We can use this residual symmetry to give a classification for the Feynman diagrams of nonassociative field theories: It is the standard noncommutative classification into planar and non-planar graphs where every Feynman diagram of genus γ with m vertices and n legs is subdivided into $m\,C_{n-2}$ inequivalent diagrams.

We close this section by summarizing the Feynman rules for nonassociative scalar field theories in phase space.

1. For each internal propagator we have $\int \frac{\mathrm{d}^{2d}\lambda}{(2\pi)^{2d}}\,\frac{1}{\lambda^2+m^2}$, where λ is the internal momentum.

2. For each vertex we have $\frac{g}{n!\,(2\pi)^{2d}}\int \mathrm{d}^{2d}x\,V(x,k^{(a)})$, where the phase factor is given by (7.9).

3. Each diagram carries a symmetry factor given by its noncommutative counterpart multiplied by $\frac{1}{C_{n-2}}$.

In the following we will use these rules to calculate the one-loop corrections to the two-point function of φ^4 nonassociative scalar field theory.

7.3 Nonassociative φ^4 scalar field theory

As an application of the formalism developed above, we set $n = 4$ in (7.7) and study φ^4 scalar field theory on flat R-space. The momentum relations (7.11) take the form

$$\sum_{a=1}^{4} k_i^{(a)} = 0 \tag{7.15}$$

and

$$\sum_{a=1}^{4} \xi^{k,\,(a)} = \frac{\hbar}{2} \sum_{1\le a<b\le 3} k_i^{(a)} k_j^{(b)}\, R^{ijk} = \frac{\hbar}{2} \big(k_i^{(1)}k_j^{(2)} + k_i^{(3)}k_j^{(4)}\big)\, R^{ijk} \ . \tag{7.16}$$

where (7.15) has been used, while the phase factor (7.9) becomes

$$\begin{aligned} V(x,k^{(a)}) = \exp\bigg(\,\mathrm{i}\sum_{a=1}^{4} k_I^{(a)} x^I - \frac{\mathrm{i}\,\hbar}{2} \sum_{1\le a<b\le 3} k_I^{(a)} k_J^{(b)}\, \Theta^{IJ}(x) - \\ - \frac{\mathrm{i}\,\hbar^2}{4}\, k_I^{(1)} k_J^{(2)} k_K^{(3)}\, R^{IJK}\bigg) \ , \end{aligned} \tag{7.17}$$

where both (7.15) and (7.16) have been used.

As we discussed in section 7.1, there are $C_2 = 2$ classes of interaction terms which in this case are equal. By taking cyclic permutations of the indices in (7.17) it can be easily verified that their phase factors are related by

$$V_\sigma(x,k^{\sigma(a)}) = \exp\big[\tfrac{\mathrm{i}\,\hbar}{2}\, k_I^{(1)} k_J^{(2)} k_K^{(3)}\, R^{IJK}\big]\ V(x,k^{(a)}) \ , \tag{7.18}$$

where $\sigma \in S_4$ is a cyclic permutation of the four indices. It follows that each phase factor is invariant under the composition of two cyclic permutations, so that each diagram is subdivided into $2v$ classes. This suggests that there are two different 2-cell embeddings for each vertex, i.e. the off-shell field theory has two types of vertices, and thus planar and non-planar diagrams will accordingly split into subclasses determined by which permutation is used to embed each vertex.

7.3.1 One-loop corrections

Let us now calculate the one-loop corrections to the two-point function in this model. In this case the contributions from the two subclasses are equal as the R-dependent term vanishes due to antisymmetry of R^{IJK}.

The planar part is given by (see figure 7.1)

$$\Gamma^{(2)}_{pl} = \frac{g}{3(2\pi)^{4d}} \int_{\mathcal{M}} \mathrm{d}^{2d}x \int_{\mathcal{M}^*} \frac{\mathrm{d}^{2d}\lambda}{\lambda^2+m^2}\, V(x,k^{(2)},k^{(3)}) \ , \tag{7.19}$$

where $k^{(2)}$, $k^{(3)}$ are the external momenta, λ is the loop momentum and the phase factor is calculated from equation (7.17), and is given by

$$\begin{aligned} V(x,k^{(2)},k^{(3)}) = \exp\Big(\, & \mathrm{i}\,(k^{(2)}-k^{(3)})_I\, x^I - \\ & - \frac{\mathrm{i}\,\hbar}{2}\,\big(\lambda_I(k^{(2)}-k^{(3)})_J - k^{(2)}_I k^{(3)}_J\big)\,\Theta^{IJ}(x) + \\ & + \frac{\mathrm{i}\,\hbar^2}{4}\,\lambda_I k^{(2)}_J k^{(3)}_K\, R^{IJK}\Big) \ . \end{aligned} \tag{7.20}$$

Integrating over x and using the momentum relations (7.15) and (7.16) we find

$$\Gamma^{(2)}_{pl} = \frac{g}{3(2\pi)^{2d}}\, \delta^{2d}\big(k^{(2)}-k^{(3)}\big) \int_{\mathcal{M}^*} \frac{\mathrm{d}^{2d}\lambda}{\lambda^2+m^2} \ . \tag{7.21}$$

This amplitude can be turned into a Gaussian integral by using the Schwinger parametrization

$$\frac{1}{k^2+m^2} = \int_0^\infty \mathrm{d}s\ \mathrm{e}^{\,-s\,(k^2+m^2)} \ . \tag{7.22}$$

After integrating over the loop momentum we find

$$\Gamma^{(2)}_{pl} = \frac{g\,\pi^d}{3(2\pi)^{2d}}\, \delta^{2d}\big(k^{(2)}-k^{(3)}\big) \int_0^\infty \mathrm{d}s\ \frac{\exp(-s\,m^2)}{s^d} \ , \tag{7.23}$$

which is divergent for $s \to 0$ (large k) and thus an ultraviolet cutoff Λ has to be introduced to regularise the integral. By multiplying the integrand with $\exp[-1/(s\,\Lambda^2)]$ and using the formula

$$\int_0^\infty \mathrm{d}s\ \frac{\exp(-sm^2 - \frac{1}{\Lambda^2 s})}{s^d} = 2(m\Lambda)^{d-1}\, K_{d-1}\big(\frac{2m}{\Lambda}\big) \ , \tag{7.24}$$

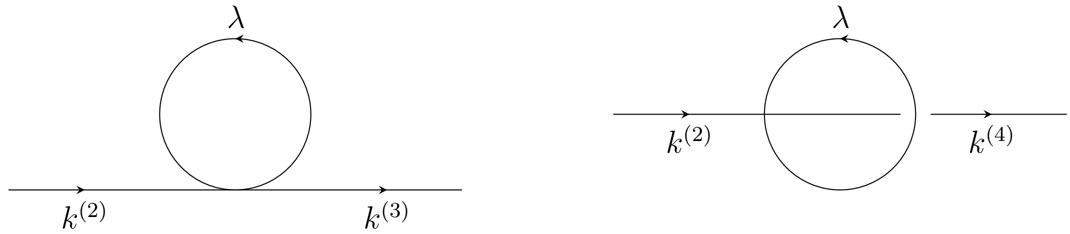


Figure 7.1: 2-point function to 1-loop: the planar and non-planar Feynman diagrams.

where K_n is the modified Bessel function of the second kind of order n, we arrive at the result

$$\Gamma^{(2)}_{pl} = \frac{g}{3(2\pi)^d}\,\delta^{2d}\big(k^{(2)} - k^{(3)}\big)\Big(\frac{m\Lambda}{2}\Big)^{d-1} K_{d-1}\Big(\frac{2m}{\Lambda}\Big)\ , \tag{7.25}$$

which is the familiar result of commutative φ^4.

Let us now turn our attention to the non-planar diagram in figure 7.1 for which the phase factor (7.17) becomes

$$V(x, k^{(2)}, k^{(3)}) = \exp\Big[\,\mathrm{i}\,(k^{(2)} - k^{(4)})_I x^I + \mathrm{i}\,\hbar\, k^{(2)}_I \lambda_J\, \Theta^{IJ}(x)\Big]\ . \tag{7.26}$$

In this case it is convenient to postpone the integral over $\mathcal{M}$ as it gives a delta-function constraint which is singular and cannot be used to evaluate the remaining integrals. Instead we introduce a Schwinger parameter and integrate over the loop momentum which yields

$$\begin{aligned}\Gamma^{(2)}_{\rm np} \ = \ & \frac{g\,\pi^d}{6(2\pi)^{4d}} \int_0^\infty \frac{\mathrm{d}^d s}{s^d}\ \exp\Big(-s\,m^2 - \frac{\hbar^2\,\big(k^{(2)}\big)^2}{4s}\Big) \times \\ & \times \int_M \mathrm{d}^d x\ \exp\big(\,\mathrm{i}\,(k^{(2)} - k^{(4)})_i\, x^i\ \big) \times \\ & \times \int_{(\mathbb{R}^d)^*} \mathrm{d}^d p\ \exp\Big(-\frac{\hbar^2}{4s}\, A^{mn}\, p_m\, p_n + \mathrm{i}\,\big(\xi^{(2)} - \xi^{(4)}\big)^i\, p_i\Big)\ ,\end{aligned} \tag{7.27}$$

where

$$A^{mn} := \sum_{k=1}^d k_i\, k_j\, R^{ikm}\, R^{jkn} \tag{7.28}$$

is a singular symmetric $d \times d$ matrix.

In this expression, the integral over M yields a delta function while the momentum space integral is Gaussian. Since A is singular, it has $p \geq 1$ zero eigenvalues and

$d-p$ non-zero real eigenvalues $\rho_1,\dots,\rho_{d-p}$. Thus only $d-p$ independent momentum space integrals are Gaussian while the rest yield delta-functions. With these manipulations we can integrate over x and p in (7.27) to obtain

$$\Gamma^{(2)}_{np} = \frac{g\,2^{-\frac{3d-p}{2}}}{6(2\pi)^{\frac{3d-p}{2}}}\Big(\frac{2}{\hbar}\Big)^{d-p}\,\frac{\delta^{(d)}\big(k_i^{(2)}-k_i^{(4)}\big)}{\sqrt{\rho_1\cdots\rho_{d-p}}}\,\prod_{j=1}^{p}\delta\Big(\big(\xi^{(2)}-\xi^{(4)}\big)\cdot e_j\Big)\times$$
$$\times\int_0^\infty\frac{\mathrm{d}^d s}{s^{\frac{d-p}{2}}}\;\exp\left(-\,s\left[m^2+\frac{4}{\hbar^2}\sum_{j=n+1}^{d}\frac{\big(\xi^{(2)}-\xi^{(4)}\big)\cdot e_j}{\rho_j}\right]-\frac{\hbar^2(k^{(2)})^2}{4\,s}\right)\,, \tag{7.29}$$

where $e_1,\cdots,e_d$ are the eigenvectors of A. We now introduce an ultraviolet cutoff and integrate over the Schwinger parameter to obtain

$$\Gamma^{(2)}_{\rm np} = \frac{g\;\delta^{(d)}\big(k^{(2)}-k^{(4)}\big)}{6\hbar^{d-p}\,2^p\,(2\pi)^{\frac{3d-p}{2}}\sqrt{\rho_1\cdots\rho_{d-p}}}\,\prod_{j=1}^{p}\delta^{(d)}\big(\big(\xi^{(2)}-\xi^{(4)}\big)\cdot e_j\big)\times$$
$$\times\,\Big(\frac{m_{\rm eff}\,\Lambda_{\rm eff}}{2}\Big)^{\frac{d-p}{2}-1}\;K_{\frac{d-p}{2}-1}\big(\tfrac{2m_{\rm eff}}{\Lambda_{\rm eff}}\big)\,, \tag{7.30}$$

where the effective cutoff is given by

$$\frac{1}{\Lambda^2_{\rm eff}}=\frac{1}{\Lambda^2}+\Big(\frac{\hbar\,k^{(2)}}{2}\Big)^2\,, \tag{7.31}$$

and the effective mass is given by

$$m^2_{\rm eff}=m^2+\frac{4}{\hbar^2}\sum_{j=p+1}^{d}\frac{\big(\xi^{(2)}-\xi^{(4)}\big)\cdot e_j}{\rho_j}\,. \tag{7.32}$$

It is not surprising that the one-loop contribution (7.30) is qualitatively similar to the one calculated for ordinary noncommutative scalar field theory with non-constant deformation parameter; it even exhibits the usual UV/IR mixing pathologies [108]. As we discussed in section 7.1, for $n=3,4,5$ the nonassociative φ^n theories are on-shell associative, and therefore, the only nonassociative effect that they exhibit is in the subdivision of their vertices into C_{n-2} classes. Although this was not observed in our example, it can be seen in higher loops and even in the one-loop correction to the four-point function of the φ^4 theory.

Chapter 8

Summary and outlook

In this thesis we have studied non-geometric backgrounds emerging from closed string flux compactifications, and their non-commutative and nonassociative structure. Our main results comprise the geometrization of these backgrounds, their deformation quantization using three distinct approaches (which we show to be complementary), the consistent formulation of nonassociative quantum mechanics, and the construction of nonassociative scalar field theory.

In particular, we proposed a Courant sigma-model on an open three-dimensional membrane as the topological sector of closed string dynamics in the non-geometric R-flux background, and reduced it to a twisted Poisson sigma-model on the boundary of the membrane. The target space of the boundary theory is a phase space equipped with a twisted Poisson structure, which is provided by a $U(1)$ gerbe in momentum space and reproduces the nonassociative phase space algebra conjectured in [90]. Therefore, a geometric interpretation for the effective target space geometry seen by closed strings in R-space as well as a straightforward dynamical explanation of (1.6) emerged naturally from our membrane description.

We also explained how our membrane sigma-model facilitates the open/closed string duality observed in [90], by which we asserted that the closed string path integral is equivalent to that of an open string twisted Poisson sigma-model on a disk. This suggests that Kontsevich's global deformation quantization is an appropriate quantization scheme, wherein the nonassociative star product and the corresponding associator were calculated. For the case of constant R-flux, we derived explicit closed formulas which feature the three-product of fields proposed in [21] as a restriction of the deformed associator on configuration space. We then developed Seiberg–Witten maps which relate our nonassociative star product to a family of associative star products parametrized by constant momentum.

Our second approach to deformation quantization of R-space was based on the observation that our twisted Poisson structure has the structure of a Lie 2-algebra, which we integrated to a Lie 2-group $\mathscr{G}$. Within this framework, we constructed a categorification of the Weyl–Wigner quantization map by embedding the algebra of functions on phase space as an object in the category $\mathscr{G}$. Using this map, we demonstrated that the nonassociative convolution product induced by horizontal multiplication in $\mathscr{G}$, is identical to the nonassociative star product we obtained via Kontsevich's global deformation quantization.

For our third approach, we developed twist deformation quantization techniques appropriate to nonassociative deformations of R-space geometry, which we explicitly related to our categorical framework. We found that the symmetries of R-space form a non-abelian algebra of translations and Bopp shifts, and give rise to a Hopf algebra, which we deformed to a quasi-Hopf algebra using a cochain twist. With this quasi-Hopf algebra, we deformed the algebra of functions and the exterior differential calculus on R-space, and thus provided the first step towards nonassociative deformations of gravity. We defined integration on the deformed algebra of forms on R-space to be the standard integration, which introduced the notions of 2-cyclicity and 3-cyclicity. We also considered extensions of this formalism to non-constant R-flux backgrounds as well as non-parabolic R-flux string vacua. In the first case we found that twisting techniques provide a framework for quantizing generic Nambu-Poisson 3-brackets determined by the trivector field R, while in the second case we were able to identify the pertinent Hopf algebra and a cochain twist up to third order in the R-flux.

Using our phase space star product quantization, we proposed a consistent formulation of nonassociative quantum mechanics on R-space. Within its framework, we demonstrated that a triple of operators that do not associate does not have common eigenstates, which suggests position space quantization in the presence of R-flux, and calculated the pertinent uncertainty relation. We then defined area and volume operators in configuration space, and provided a concrete and rigorous derivation for the uncertainty relations anticipated by [90, 27].

Finally, we constructed nonassociative scalar field theories on R-space and demonstrated that interactions up to φ^5 are on-shell associative. We observed familiar noncommutative effects, such as UV/IR mixing and violation of momentum conservation along the non-commuting directions, which is induced by the dynamical character of our star product. However, we also discovered that nonassociative interactions are not invariant under cyclic permutations of the external momenta, which is a purely nonassociative effect, and proposed a new classification of the Feynman diagrams that enter the perturbation expansion.

Let us conclude this thesis with a brief discussion on possible directions that would extend our present work, and the challenges that they present.

Membrane sigma-models and T-duality.

As a natural extension to our membrane sigma-model, we would consider the simultaneous presence of both geometric and non-geometric fluxes, i.e. all the T-dual fluxes that appear in the T-duality chain (1.4). At the membrane level, these fluxes are related by gauge transformations [69, 70] (see also [30] and references therein), which should be realised as T-duality transformations on the boundary of the membrane. However, it is not clear how the boundary string theory would be accessed since the presence of all fluxes in (2.2) hinders the application of Stokes' theorem. A possible solution would be to consider a doubled generalized bundle and impose a suitable constraint to eliminate the extra degrees of freedom. Such approaches could provide a rigorous derivation of the star products proposed in [14], and relate our membrane model with the methods of [23, 24].

Twist deformations of double field theory.

The simultaneous presence of all T-dual fluxes could also be investigated by transferring the twist deformation techniques we have developed here to the setting of double field theory. It is not difficult to imagine how this could be accomplished;

for example, one could employ the doubled phase space of [14] and examine its symmetries, searching for a suitable cochain twist. However, our formalism does not offer an analogue of the strong constraint of double field theory, therefore it is not obvious how one would define truncations from doubled space to the string theory target space.

Nonassociative field theories.

Our twist deformation framework can *a priori* be extended to deformations of connections on $\mathcal{M} = \mathcal{T}^*\mathcal{M}$, and thus more elaborate models on non-geometric spaces could be constructed, such as gauge theory or gravity. However, one should realise such theories in configuration space M in order to acquire a meaningful field theory. We recognise here the same problem that we encountered above: a suitable truncation scheme from T^*M to M is at the moment elusive.

Spherical backgrounds.

Non-geometric flux compactifications of spherical backgrounds were recently considered in [106], where T-duality was expressed in terms of the field strength $H = \mathrm{d}B$ rather that the B-field, and a Q-fluxed T-fold was constructed. An analysis from a membrane perspective could be performed to explore possible geometric interpretations of such backgrounds.

Appendix A

Higher Lie algebra structures

In this appendix we collect the pertinent mathematical material on higher structures which are used extensively in the main text.

A.1 Lie 2-algebras

Homotopy Lie algebras. An L_∞-*algebra* or *strong homotopy Lie algebra* is a graded vector space V together with a collection of totally (graded) antisymmetric n-brackets $[-,\dots,-] : \bigwedge^n V \to V$, $n \geq 1$ of degree $n-2$ satisfying the *higher* or *homotopy Jacobi identities*

$$\sum_{i=1}^{n} \sum_{\sigma\in\mathrm{Sh}(i,n-i)} (-1)^{\alpha(\sigma)} \big[[v_{\sigma(1)},\dots,v_{\sigma(i)}], v_{\sigma(i+1)},\dots,v_{\sigma(n)}\big] = 0 \tag{A.1}$$

for each $n \geq 1$. Here $(-1)^{\alpha(\sigma)}$ is a prescribed sign rule for permuting homogeneous elements $v_1,\dots,v_n \in V$, while $\mathrm{Sh}(i,n-i)$ is the set of permutations $\sigma \in S_n$ which preserve the orders of the first i elements and of the last $n-i$ elements, i.e. $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$ for $i = 1,\dots,n$.

Denote the 1-bracket by $\mathsf{d} := [-]$. It has degree -1 and the generalized Jacobi identity (A.1) for $n = 1$ reads

$$\mathsf{d}^2 = 0 \ , \tag{A.2}$$

which implies that $\mathsf{d} : V \to V$ is a differential making V into a chain complex. For

$n = 2$ one has

$$\mathsf{d}[v,w] = [\mathsf{d}v,w] + (-1)^{|v|}\,[v,\mathsf{d}w]\ , \tag{A.3}$$

which implies that d is a derivation with respect to the antisymmetric 2-bracket $[-,-] : V \wedge V \to V$. The bracket $[-,-]$ satisfies the usual Jacobi identity only up to a homotopy correction; from (A.1) with $n = 3$ we obtain

$$\begin{aligned}(-1)^{|v|\,|u|}\,\big[[v,w],u\big] + (-1)^{|w|\,|u|}\,\big[[u,v],w\big] + (-1)^{|v|\,|w|}\,\big[[w,u],v\big] = \\ = (-1)^{|v|\,|u|+1}\,\big(\mathsf{d}[v,w,u] + [\mathsf{d}v,w,u] + (-1)^{|v|}\,[v,\mathsf{d}w,u] + (-1)^{|v|+|w|}\,[v,w,\mathsf{d}u]\big)\ ,\end{aligned} \tag{A.4}$$

which implies that the *Jacobiator* $[-,-,-] : V \wedge V \wedge V \to V$ is a chain homotopy map. For $n > 3$, the identities (A.1) impose extra coherence relations on this homotopy and all higher homotopies.

If V has trivial grading, then an L_∞-algebra is simply an ordinary Lie algebra. More generally, an L_∞-algebra with vanishing n-brackets for all $n \geq 3$ is a differential graded Lie algebra.

A 2*-term* L_∞*-algebra* is a strong homotopy Lie algebra with underlying graded vector space $V = V_0 \oplus V_1$ concentrated in degrees 0 and 1; it has vanishing n-brackets for $n > 3$ and the only non-trivial identities in (A.1) occur for $n = 1, 2, 3, 4$. It may be regarded as a 2-term chain complex $V = \big(V_1 \xrightarrow{\ \mathsf{d}\ } V_0\big)$ whose bracket $[-,-] : V_i \otimes V_j \to V_{i+j}$, $i+j = 0,1$, is a chain map and whose Jacobiator $[-,-,-] : V_0 \wedge V_0 \wedge V_0 \to V_1$ is a chain homotopy from the chain map

$$V_0 \wedge V_0 \wedge V_0 \longrightarrow V_1\ , \qquad v \wedge w \wedge u \longmapsto \big[v,[w,u]\big] \tag{A.5}$$

to the chain map

$$V_0 \wedge V_0 \wedge V_0 \longrightarrow V_1\ , \qquad v \wedge w \wedge u \longmapsto \big[[v,w],u\big] + \big[w,[v,u]\big] \tag{A.6}$$

satisfying the coherence condition

$$\begin{aligned}&\big[v,[w,u,s]\big]+\big[v,[w,u],s\big]+\big[v,u,[w,s]\big]+\big[[v,w,u],s\big]+\big[u,[v,w,s]\big] \\ &\quad= \big[v,w,[u,s]\big]+\big[[v,w],u,s\big]+\big[w,[v,u,s]\big]+\big[w,[v,u],s\big]+\big[w,u,[v,s]\big]\ .\end{aligned} \tag{A.7}$$

This higher Jacobi identity relates the two ways of using the Jacobiator to re-bracket the expression $[[[s,v],w],u]$.

A related notion is that of an A_∞*-algebra*, or *homotopy associative algebra*, which is a graded vector space A endowed with a family of n-multiplication operations $\mu_n : A^{\otimes n} \to A$ of degree $n-2$, $n \geq 1$ obeying the higher or homotopy associativity relations

$$\sum_{j+k+l=n} (-1)^\sigma\, \mu_n \circ \big(\mathrm{id}_{A^{\otimes j}} \otimes \mu_k \otimes \mathrm{id}_{A^{\otimes l}}\big) = 0\ . \tag{A.8}$$

The first two relations

$$\mathsf{d}^2 = 0 \qquad \text{and} \qquad \mathsf{d}\mu_2(a,b) = \mu_2(\mathsf{d}a,b) + (-1)^{|a|}\,\mu_2(a,\mathsf{d}b) \tag{A.9}$$

for $a,b \in A$ make A into a chain complex with differential $\mathsf{d} := \mu_1$ which is a graded derivation of the binary product μ_2. The third relation states that the product μ_2 is associative up to the homotopy μ_3, and so on. From an A_∞-algebra structure on A one constructs an L_∞-algebra structure through the antisymmetric n-brackets

$$[a_1,\dots,a_n] := \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\,\mu_n(a_{\sigma(1)},\dots,a_{\sigma(n)}) \tag{A.10}$$

for $a_1,\dots,a_n \in A$. However, in general there is no converse enveloping algebra type procedure to construct an A_∞-structure from an L_∞-structure.

Lie 2-algebras. 2-term L_∞-algebras are the same things as Lie 2-algebras [12, Theorem 36], which are categorified versions of Lie algebras in which the Jacobi identity is replaced by a Jacobiator isomorphism. For this, recall that a 2-*vector space* is a linear category $\mathscr{V} = (\mathscr{V}_0, \mathscr{V}_1)$ consisting of a vector space of objects $\mathscr{V}_0$ and a vector space of morphisms $\mathscr{V}_1$, together with source and target maps $\mathsf{s},\mathsf{t} : \mathscr{V}_1 \rightrightarrows \mathscr{V}_0$ sending a morphism to its domain and range, and an inclusion map $\mathbb{1} : \mathscr{V}_0 \to \mathscr{V}_1$, $v \mapsto \mathbb{1}_v$, sending an object to its identity morphism; the set of composable morphisms is $\mathscr{V}_1 \times_{\mathscr{V}_0} \mathscr{V}_1 = \{(v_1, w_1) \in \mathscr{V}_1 \times \mathscr{V}_1 \mid \mathsf{s}(w_1) = \mathsf{t}(v_1)\}$. These maps are all linear and compatible in the usual sense with the composition $\circ : \mathscr{V}_1 \times_{\mathscr{V}_0} \mathscr{V}_1 \to \mathscr{V}_1$ in the category.

A *Lie* 2*-algebra* is a 2-vector space $\mathscr{V}$ together with an antisymmetric bilinear bracket functor $[-,-]_{\mathscr{V}} : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ and a natural antisymmetric trilinear Jacobiator isomorphism on objects satisfying a higher Jacobi identity. A Lie 2-algebra $\mathscr{V}$ is *strict* if its Jacobiator is the identity isomorphism; in that case both $\mathscr{V}_0$ and $\mathscr{V}_1$ are Lie algebras, and each operation on the category is a homomorphism of Lie algebras. Otherwise $\mathscr{V}$ is *semistrict*; this is the case of relevance to this paper.

Given a 2-term L_∞-algebra $V = \big(V_1 \xrightarrow{\mathsf{d}} V_0\big)$, we construct a 2-vector space $\mathscr{V}$ with vector spaces of objects and morphisms given by $\mathscr{V}_0 = V_0$ and $\mathscr{V}_1 = V_0 \oplus V_1$. A morphism $f = (v_0, v_1)$ in $\mathscr{V}_1$ with $v_0 \in V_0$ and $v_1 \in V_1$ has source and target given by $\mathsf{s}(v_0, v_1) = v_0$ and $\mathsf{t}(v_0, v_1) = v_0 + \mathsf{d}v_1$, while the object inclusion is $\mathbb{1}_v = (v, 0)$. The composition of two morphisms $f = (v_0, v_1)$ and $f' = (v_0 + \mathsf{d}v_1, v_1')$ in $\mathscr{V}_1$ is $f \circ f' := (v_0, v_1 + v_1')$. The bracket functor $[-,-]_{\mathscr{V}} : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ is defined on objects $v, v' \in \mathscr{V}_0$ by $[v, v']_{\mathscr{V}} = [v, v']$, where $[-,-]$ denotes the bracket in the L_∞-algebra V. The bracket of morphisms $f = (v_0, v_1)$ and $f' = (v_0', v_1')$ in $\mathscr{V}_1$ is given by

$$
\begin{aligned}
[f, f']_{\mathscr{V}} &= \big([v_0, v_0'] \,,\, [v_1, v_0'] + [v_0 + \mathsf{d}v_1, v_1']\big) \\
&= \big([v_0, v_0'] \,,\, [v_0, v_1'] + [v_1, v_0' + \mathsf{d}v_1']\big) \ .
\end{aligned}
\tag{A.11}
$$

Finally, the Jacobiator for $\mathscr{V}$ is defined on $v, w, u \in \mathscr{V}_0$ by

$$
[v, w, u]_{\mathscr{V}} := \big(\big[[v, w], u\big] \,,\, [v, w, u]\big) \ , \tag{A.12}
$$

with source $\mathsf{s}([v, w, u]_{\mathscr{V}}) = [[v, w], u]$ and target $\mathsf{t}([v, w, u]_{\mathscr{V}}) = [v, [w, u]] + [[v, u], w]$ by (A.4).

The skew-symmetric bracket

$$[v_1, v_1'\,] = [\mathsf{d}v_1, v_1'\,] = [v_1, \mathsf{d}v_1'\,]\ , \tag{A.13}$$

defined on elements $v_1, v_1' \in V_1$ which figures in the formula (A.11), is called the *derived bracket*. It satisfies the Jacobiator identity

$$\big[v_1, [v_1', v_1''\,]\big] - \big[[v_1, v_1'\,], v_1''\,\big] - \big[v_1', [v_1, v_1''\,]\big] = [\mathsf{d}v_1, \mathsf{d}v_1', \mathsf{d}v_1''\,]\ . \tag{A.14}$$

Classification of Lie 2-algebras. There is a bijective correspondence between semistrict Lie 2-algebras and certain classifying "Postnikov data" [12], analogous to the Faulkner construction of 3-Lie algebras. The data in question are triples $(\mathfrak{g}, W, j)$ consisting of a Lie algebra $\mathfrak{g}$, a representation of $\mathfrak{g}$ on a vector space W, and a 3-cocycle j on $\mathfrak{g}$ with values in W; the isomorphism classes are parametrized by elements $[j] \in H^3(\mathfrak{g}, W)$ of the degree 3 Lie algebra cohomology .

For a Lie 2-algebra $\mathscr{V}$ obtained from a 2-term L_∞-algebra $V = \big(V_1 \xrightarrow{\mathsf{d}} V_0\big)$, the corresponding triple $(\mathfrak{g}, W, j)$ is constructed by firstly setting $\mathfrak{g} = \ker(\mathsf{d}) \subseteq V_1$; since $\mathsf{d} = 0$ on $\mathfrak{g}$ the 2-bracket of the L_∞-structure satisfies the Jacobi identity exactly and makes $\mathfrak{g}$ into a Lie algebra. Now let $W = \mathrm{coker}(\mathsf{d}) \subseteq V_0$, and use the 2-bracket to define an action $\mathfrak{g} \otimes W \to W$ by $g \triangleright w = [g, w]$ for $g \in \mathfrak{g}$, $w \in W$; in this correspondence W is regarded as the abelian Lie algebra of endomorphisms of the zero object of $\mathscr{V}$. Finally, the Jacobiator of the L_∞-structure gives a map $[-,-,-] : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to W$ which is a Chevalley–Eilenberg 3-cocycle j whose cohomology class $[j] \in H^3(\mathfrak{g}, W)$ is the obstruction to $\mathscr{V}$ being functorially equivalent to a strict Lie 2-algebra, or equivalently to a differential $\mathbb{Z}_2$-graded Lie algebra.

A.2 Gerstenhaber brackets

Consider the Hochschild complex $H^n(\mathcal{A}, \mathcal{A}) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}^{\otimes n}, \mathcal{A})$ of an algebra $\mathcal{A}$ with product $\star \in H^2(\mathcal{A}, \mathcal{A})$. The space of n-cochains $C^n(\mathcal{A}, \mathcal{A}) = \mathrm{Hom}_{\mathbb{C}}(\bigwedge^n \mathcal{A}, \mathcal{A})$ is constructed by antisymmetrization, and the Hochschild coboundary operator $\mathrm{d}_\star$:

$C^n(\mathcal{A},\mathcal{A}) \to C^{n+1}(\mathcal{A},\mathcal{A})$ is defined by

$$\begin{aligned} \mathrm{d}_\star \mathcal{C}(f_1,\dots,f_{n+1}) &= f_1 \star \mathcal{C}(f_2,\dots,f_{n+1}) + \sum_{i=1}^{n} (-1)^i\, \mathcal{C}(f_1,\dots,f_i \star f_{i+1},\dots,f_{n+1}) \\ &\quad + (-1)^{n+1}\, \mathcal{C}(f_1,\dots,f_n) \star f_{n+1} \end{aligned} \tag{A.15}$$

for $\mathcal{C} \in C^n(\mathcal{A},\mathcal{A})$ and $f_i \in \mathcal{A}$. From the product on $\mathcal{A}$ we construct a cup product $\star : H^{n_1}(\mathcal{A},\mathcal{A}) \otimes H^{n_2}(\mathcal{A},\mathcal{A}) \to H^{n_1+n_2}(\mathcal{A},\mathcal{A})$ by

$$(\mathcal{C}_1 \star \mathcal{C}_2)(f_1,\dots,f_{n_1+n_2}) := \mathcal{C}_1(f_1,\dots,f_{n_1}) \star \mathcal{C}_2(f_{n_1+1},\dots,f_{n_1+n_2}) \tag{A.16}$$

for $\mathcal{C}_1 \in C^{n_1}(\mathcal{A},\mathcal{A})$ and $\mathcal{C}_2 \in C^{n_2}(\mathcal{A},\mathcal{A})$.

The *Gerstenhaber bracket* of $\mathcal{C}_1 \in C^{n_1}(\mathcal{A},\mathcal{A})$ and $\mathcal{C}_2 \in C^{n_2}(\mathcal{A},\mathcal{A})$ is defined by

$$[\mathcal{C}_1,\mathcal{C}_2]_{\mathrm{G}} = \mathcal{C}_1 \circ \mathcal{C}_2 - (-1)^{(n_1+1)\,(n_2+1)}\, \mathcal{C}_2 \circ \mathcal{C}_1 \tag{A.17}$$

in $C^{n_1+n_2-1}(\mathcal{A},\mathcal{A})$, where the composition product is defined as

$$\begin{aligned} &(\mathcal{C}_1 \circ \mathcal{C}_2)(f_1,\dots,f_{n_1+n_2-1}) \\ &\quad = \mathcal{C}_1\big(\mathcal{C}_2(f_1,\dots,f_{n_2}), f_{n_2+1},\dots,f_{n_1+n_2-1}\big) \\ &\qquad + \sum_{i=1}^{n_1-2} (-1)^{i\,n_2}\, \mathcal{C}_1\big(f_1,\dots,f_i,\mathcal{C}_2(f_{i+1},\dots,f_{i+n_2}), f_{i+n_2+1},\dots,f_{n_1+n_2-1}\big) \\ &\qquad\quad + (-1)^{(n_1+1)\,(n_2+1)}\, \mathcal{C}_1\big(f_1,\dots,f_{n_1-1},\mathcal{C}_2(f_{n_1},\dots,f_{n_1+n_2-1})\big) \end{aligned} \tag{A.18}$$

for $f_i \in \mathcal{A}$. The coboundary operator is then given by

$$\mathrm{d}_\star \mathcal{C} = -[\mathcal{C},\star]_{\mathrm{G}}\ . \tag{A.19}$$

The associativity of the product $\star \in C^2(\mathcal{A},\mathcal{A})$ may be expressed by using

$$[\star,\star]_{\mathrm{G}}(f,g,h) = 2\,\big((f \star g) \star h - f \star (g \star h)\big)\ . \tag{A.20}$$

Associativity is thus equivalent to $\mathrm{d}_\star \star = [\star,\star]_{\mathrm{G}} = 0$ or $\mathrm{d}_\star^2 = 0$; in that case, the *Gerstenhaber algebra* $\big(C^\sharp(\mathcal{A},\mathcal{A}), \mathrm{d}_\star, [-,-]_{\mathrm{G}}\big)$ is a differential graded Lie algebra.

A.3 Schouten–Nijenhuis brackets

Let $\mathcal{V}^\sharp = C^\infty(\mathcal{M}, \bigwedge^\sharp T\mathcal{M})$ be the graded-commutative algebra of multivector fields on a smooth manifold $\mathcal{M}$; notice that $\mathcal{V}^\sharp$ contains the associative algebra $\mathcal{V}^0 = C^\infty(\mathcal{M})$ of smooth complex functions on $\mathcal{M}$. The usual Lie bracket of vector fields $[-,-]_{T\mathcal{M}}$ extends to the canonical *Schouten–Nijenhuis bracket* $[-,-]_{\rm S}$ on $\mathcal{V}^\sharp$. It gives $\mathcal{V}^\sharp$ the structure of a differential graded Gerstenhaber algebra with vanishing differential, i.e. $[-,-]_{\rm S}$ is a graded Lie bracket of degree -1 satisfying the graded Leibniz rule with respect to the associative (graded-commutative) exterior product. Given homogeneous multivector fields $\mathcal{X} = \mathcal{X}^{I_1\dots I_{|\mathcal{X}|}}\,\partial_{I_1}\wedge\cdots\wedge\partial_{I_{|\mathcal{X}|}}$ and $\mathcal{Y} = \mathcal{Y}^{I_1\dots I_{|\mathcal{Y}|}}\,\partial_{I_1}\wedge\cdots\wedge\partial_{I_{|\mathcal{Y}|}}$, it is defined by

$$[\mathcal{X},\mathcal{Y}]_{\rm S} = (-1)^{|\mathcal{X}|-1}\,\mathcal{X}\diamond\mathcal{Y} - (-1)^{|\mathcal{X}|\,(|\mathcal{Y}|-1)}\,\mathcal{Y}\diamond\mathcal{X} \tag{A.21}$$

in $\mathcal{V}^{|\mathcal{X}|+|\mathcal{Y}|-1}$, where

$$\begin{aligned}\mathcal{X}\diamond\mathcal{Y} := \sum_{l=1}^{|\mathcal{X}|}(-1)^{l-1}\,\mathcal{X}^{I_1\dots I_{|\mathcal{X}|}}\,\partial_l\mathcal{Y}^{J_1\dots J_{|\mathcal{Y}|}}\,\partial_{I_1}\wedge\cdots\wedge\widehat{\partial_{I_l}}\wedge\cdots\wedge\partial_{I_{|\mathcal{X}|}}\wedge \\ \wedge\,\partial_{J_1}\wedge\cdots\wedge\partial_{J_{|\mathcal{Y}|}}\end{aligned} \tag{A.22}$$

and the hat indicates an omitted derivative.

The condition for a bivector $\Theta = \frac12\,\Theta^{IJ}\,\partial_I\wedge\partial_J$ to define a Poisson structure on $C^\infty(\mathcal{M})$ can be expressed through

$$[\Theta,\Theta]_{\rm S} = \tfrac{1}{3!}\,\big(\Theta^{IL}\,\partial_L\Theta^{JK} + \Theta^{JL}\,\partial_L\Theta^{KI} + \Theta^{KL}\,\partial_L\Theta^{IJ}\big)\,\partial_I\wedge\partial_J\wedge\partial_K\ . \tag{A.23}$$

The corresponding antisymmetric bracket $\{f,g\}_\Theta := \Theta({\rm d}f,{\rm d}g)$ for $f,g\in C^\infty(\mathcal{M})$ satisfies the Jacobi identity on $C^\infty(\mathcal{M})$ if and only if $[\Theta,\Theta]_{\rm S}=0$, and thus defines a Poisson bracket. In terms of the Lichnerowicz coboundary operator ${\rm d}_\Theta:\mathcal{V}^n\to\mathcal{V}^{n+1}$ defined by

$${\rm d}_\Theta = -[-,\Theta]_{\rm S}\ , \tag{A.24}$$

the Poisson condition can be expressed as $\mathrm{d}_\Theta\Theta = 0$ or $\mathrm{d}_\Theta^2 = 0$. The Poisson bracket extends to the cotangent bundle $T^*\mathcal{M}$ where it encodes the Schouten–Nijenhuis bracket of multivector fields.

A.4 Nambu–Poisson structures

Let $\mathcal{M}$ be a a smooth d–dimensional manifold. A Nambu–Poisson structure of order n, $3 \leq n \leq d$ on $\mathcal{M}$ is a totally antisymmetric map $\{-,\dots,-\} : C^\infty(M)^{\wedge n} \to C^\infty(M)$, which satisfies the generalized Leibniz rule

$$\{f\, g, h_1, \dots, h_{n-1}\} \;= f\,\{g, h_1, \dots, h_{n-1}\} + \{f, h_1, \dots, h_{n-1}\}\, g \ , \tag{A.25}$$

and a generalized Jacobi identity called the *fundamental identity* [120]

$$\begin{aligned}\big\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\big\} = \big\{\{f_1, \dots, f_{n-1}, g_1\}, \dots, g_n\big\} + \cdots +&\\ + \big\{g_1, \dots, \{f_1, \dots, f_{n-1}, g_n\}\big\} \ ,&\end{aligned} \tag{A.26}$$

where $f, g, h \in C^\infty(\mathcal{M})$.

By the generalized Leibniz rule, the Nambu–Poisson n-bracket acts as a vector field, which implies that it is determined by a Nambu–Poisson n-vector $\Pi = \frac{1}{n!}\,\Pi^{i_1\cdots i_n}(x)\,\partial_{i_1}\wedge\cdots\wedge\partial_{i_n}$ as

$$\{f_1, \dots, f_n\} = \Pi(\mathrm{d}f_1, \dots, \mathrm{d}f_n) = \Pi^{i_1\cdots i_n}(x)\,\partial_{i_1}f_1\cdots\partial_{i_n}f_n \ . \tag{A.27}$$

A Hamiltonian vector field of $(n-1)$ functions is given by

$$X_{f_{n-1}} = \textstyle\bigwedge^n\Pi^\sharp(\mathrm{d}f_{n-1}) \ , \tag{A.28}$$

where $\bigwedge^n\Pi^\sharp$ denotes the natural way of using Π to "raise indices". Then the fundamental identity means that Hamiltonian vector fields are derivations of the Nambu–

Poisson bracket, which is preserved by the flow

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \{g, f_1, \dots, f_{n-1}\} \ . \tag{A.29}$$

The Nambu 3-bracket was introduced in [103] as a generalization of the Poisson bracket, and was later extended to the Nambu–Poisson n-bracket [120]. In recent years, it has attracted a lot of attention due to its intriguing 3-Lie (n-Lie) algebra structure, which appears in the Bagger–Lambert–Gustavsson description of multiple M2-branes in M-theory (see e.g. [67] for a review). In this thesis, our prominent examples are the flat backgrounds $M = \mathbb{R}^3$ or $M = T^3$ equipped with the Nambu–Poisson 3-bracket, which is defined on coordinate functions in terms of a constant trivector $R = \frac{1}{6}\, R^{ijk}\, \partial_i \wedge \partial_j \wedge \partial_k$ by

$$\{x^i, x^j, x^k\} = R^{ijk} \ , \tag{A.30}$$

and extended by linearity and the generalized Leibniz rule. Its quantization gives the *Nambu–Heisenberg algebra*. For further details about the quantization of generic Nambu–Poisson structures, see e.g. [45] and references therein.

A.5 Higher derived brackets

Let $\Pi \in \mathcal{V}^\sharp = C^\infty(\mathcal{M}, \bigwedge^\sharp T\mathcal{M})$ be a multivector field satisfying $[\Pi, \Pi]_{\mathrm{S}} = 0$. Following [122], we define the *n-th derived bracket* of Π as

$$\{\mathcal{X}_1, \dots, \mathcal{X}_n\}_\Pi := [\cdots [\, [\Pi, \mathcal{X}_1]_{\mathrm{S}}, \mathcal{X}_2]_{\mathrm{S}}, \dots, \mathcal{X}_n]_{\mathrm{S}} \tag{A.31}$$

for $\mathcal{X}_i \in \mathcal{V}^\sharp$ and $n \geq 1$. Then the sequence of brackets $\{-, \dots, -\}_\Pi$ defines a *higher Poisson structure* on $\mathcal{V}^\sharp$. Each derived bracket strictly obeys a generalized Leibniz rule with respect to the exterior product on $\mathcal{V}^\sharp$, i.e. $\{-, \dots, -\}_\Pi$ is a derivation in each argument. By [122, Corollary 1], this sequence of higher Poisson brackets gives $\mathcal{V}^\sharp$ the structure of an L_∞-algebra; the full countable tower of homotopy Jacobi identities is equivalent to the requirement $[\Pi, \Pi]_{\mathrm{S}} = 0$.

In this correspondence we use a parity $\mathbb{Z}_2$-grading defined as the multivector degree modulo 2, and then apply the parity reversion functor. Hence we introduce the total $\mathbb{Z}_2$-grading $\mathcal{V}^\sharp = \mathcal{V}_0 \oplus \mathcal{V}_1$ where $\mathcal{V}_0 = C^\infty(\mathcal{M}, \bigwedge^{\rm odd} T\mathcal{M})$ and $\mathcal{V}_1 = C^\infty(\mathcal{M}, \bigwedge^{\rm even} T\mathcal{M})$. Owing to the generalized Leibniz rule, in examples it suffices to display the bracket at linear order in the generators of $\mathcal{V}^\sharp$, with $|1| = 1 = |x^I|$ and $|\partial_I| = 0$.

A.6 Courant algebroids

Lie algebroids. A *Lie algebroid* over a smooth manifold $\mathcal{M}$ is a vector bundle $E \to \mathcal{M}$ endowed with a Lie bracket $[-,-]_E$ on smooth sections of E and a bundle morphism $\rho : E \to T\mathcal{M}$, called the *anchor map*, which is compatible with the Lie bracket on sections, i.e. the tangent map to ρ is a Lie algebra homomorphism,

$$\mathrm{d}\rho_{[\psi_1,\psi_2]_E} = [\mathrm{d}\rho_{\psi_1}, \mathrm{d}\rho_{\psi_2}]_{T\mathcal{M}} \ , \qquad \psi_1, \psi_2 \in C^\infty(\mathcal{M}, E) \ , \tag{A.32}$$

and a Leibniz rule is satisfied when multiplying sections of E by smooth functions on $\mathcal{M}$,

$$[\psi_1, f\,\psi_2]_E = f\,[\psi_1,\psi_2]_E + \rho_{\psi_1}(f)\,\psi_2 \ , \qquad \psi_1,\psi_2 \in C^\infty(\mathcal{M}, E) \ , \qquad f \in C^\infty(\mathcal{M}) \ . \tag{A.33}$$

Equivalently, a Lie algebroid is a vector bundle $E \to \mathcal{M}$ endowed with a differential d_E of degree $+1$ on the free graded-commutative algebra $\bigwedge^\sharp_{C^\infty(\mathcal{M})} C^\infty(\mathcal{M}, E)^*$ over $C^\infty(\mathcal{M})$. For $\omega \in \bigwedge^{n-1}_{C^\infty(\mathcal{M})} C^\infty(\mathcal{M}, E)^*$ and $\psi_i \in C^\infty(\mathcal{M}, E)$, the differential d_E is given here by

$$\begin{aligned} \mathrm{d}_E\,\omega(\psi_1,\dots,\psi_n) \ &= \ \sum_{\sigma\in S_n} \Big(\rho_{\psi_{\sigma(1)}}\big(\omega(\psi_{\sigma(2)},\dots,\psi_{\sigma(n)})\big) \\ &\qquad\qquad + \omega\big([\psi_{\sigma(1)},\psi_{\sigma(2)}]_E, \psi_{\sigma(3)},\dots,\psi_{\sigma(n)}\big)\Big) \ . \end{aligned} \tag{A.34}$$

This defines a differential graded algebra

$$\mathtt{CE}(E) = \big(\textstyle\bigwedge^\sharp_{C^\infty(\mathcal{M})} C^\infty(\mathcal{M}, E)^* \,,\, \mathrm{d}_E \big) \tag{A.35}$$

which dually has the structure of a Gerstenhaber algebra with the Lie bracket on $C^\infty(\mathcal{M}, E)$ extended as a biderivation with $[\psi, f]_E = \psi(\mathrm{d}_E f)$ for $\psi \in C^\infty(\mathcal{M}, E)$ and $f \in C^\infty(\mathcal{M})$; this bracket generalizes the Schouten–Nijenhuis bracket of multivector fields. The pair (A.35) is called the *Chevalley–Eilenberg algebra* of the Lie algebroid. It is the complex which computes Lie algebroid cohomology.

A Lie algebroid over a point is just a Lie algebra (with trivial anchor map), and (A.35) is the usual Chevalley–Eilenberg algebra which computes Lie algebra cohomology. More generally, Lie algebra bundles provide natural examples of Lie algebroids.

The tangent Lie algebroid over a manifold $\mathcal{M}$ is $E = T\mathcal{M}$ with the identity anchor map $\rho = \mathrm{id}_{T\mathcal{M}}$ and the usual Lie bracket on vector fields. In this case $\mathtt{CE}(T\mathcal{M}) = \big(\Omega^\sharp(\mathcal{M}) \,,\, \mathrm{d}\big)$ is the usual de Rham complex.

Any bivector field Θ on $\mathcal{M}$ induces a map $\Theta^\sharp : T^*\mathcal{M} \to T\mathcal{M}$ via contraction together with a bracket on $C^\infty(\mathcal{M}, T^*\mathcal{M}) = \Omega^1(\mathcal{M})$ called the Koszul bracket

$$[\alpha, \beta]_\Theta := \mathcal{L}_{\Theta^\sharp(\alpha)}\beta - \mathcal{L}_{\Theta^\sharp(\beta)}\alpha - \mathrm{d}\Theta(\alpha, \beta) \tag{A.36}$$

for $\alpha, \beta \in \Omega^1(\mathcal{M})$, where $\mathcal{L}$ denotes the Lie derivative. Then $E = T^*\mathcal{M}$, $\rho = \Theta^\sharp$, and $[-,-]_E = [-,-]_\Theta$ defines a Lie algebroid on $\mathcal{M}$ if and only if the Schouten–Nijenhuis bracket of Θ vanishes, i.e. Θ defines a Poisson structure on $\mathcal{M}$. In this case $\mathrm{d}_{T^*\mathcal{M}} = \mathrm{d}_\Theta = [\Theta, -]_{\mathrm{S}}$ is the Lichnerowicz differential and the Chevalley–Eilenberg algebra (A.35) computes the Poisson cohomology of $\mathcal{M}$.

Courant algebroids. The higher structures which arise in this paper, such as twisted Poisson structures, require a higher extension of the notion of Lie algebroid. For this, consider a vector bundle $E \to \mathcal{M}$ over a smooth manifold $\mathcal{M}$ equipped with a metric $\langle -, - \rangle$ and an antisymmetric bracket $[-,-]_E : C^\infty(\mathcal{M}, E) \wedge C^\infty(\mathcal{M}, E) \to C^\infty(\mathcal{M}, E)$, together with an anchor map $\rho : E \to T\mathcal{M}$. We define the *Jacobiator*

$J : C^\infty(\mathcal{M}, E) \wedge C^\infty(\mathcal{M}, E) \wedge C^\infty(\mathcal{M}, E) \to C^\infty(\mathcal{M}, E)$ by

$$J(\psi_1, \psi_2, \psi_3) = \big[[\psi_1, \psi_2]_E \, , \, \psi_3\big]_E + \big[[\psi_2, \psi_3]_E \, , \, \psi_1\big]_E + \big[[\psi_3, \psi_1]_E \, , \, \psi_2\big]_E \ , \tag{A.37}$$

a ternary map $[-,-,-]_E : C^\infty(\mathcal{M}, E) \wedge C^\infty(\mathcal{M}, E) \wedge C^\infty(\mathcal{M}, E) \to C^\infty(\mathcal{M})$ by

$$[\psi_1, \psi_2, \psi_3]_E = \tfrac{1}{3!} \left(\big\langle [\psi_1, \psi_2]_E \, , \, \psi_3 \big\rangle + \big\langle [\psi_2, \psi_3]_E \, , \, \psi_1 \big\rangle + \big\langle [\psi_3, \psi_1]_E \, , \, \psi_2 \big\rangle \right) \ , \tag{A.38}$$

and the pullback $\mathsf{d} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}, E)$ of the exterior derivative d via the adjoint map ρ^* by

$$\langle \mathsf{d}f, \psi \rangle = \rho_\psi(f) \ , \tag{A.39}$$

where $f \in C^\infty(\mathcal{M})$ and $\psi, \psi_i \in C^\infty(\mathcal{M}, E)$; this map defines a flat connection, $\mathsf{d}^2 = 0$.

Such a vector bundle is called a *Courant algebroid* if the following conditions are satisfied:

(i) The Jacobi identity holds up to an exact expression:
$J(\psi_1, \psi_2, \psi_3) = \mathsf{d}[\psi_1, \psi_2, \psi_3]_E$;

(ii) The anchor map ρ is compatible with the bracket: $\rho_{[\psi_1,\psi_2]_E} = [\rho_{\psi_1}, \rho_{\psi_2}]_{T\mathcal{M}}$;

(iii) There is a Leibniz rule: $[\psi_1, f\,\psi_2]_E = f\,[\psi_1, \psi_2]_E + \rho_{\psi_1}(f)\,\psi_2 - \frac{1}{2}\,\langle \psi_1, \psi_2 \rangle\,\mathsf{d}f$;

(iv) $\langle \mathsf{d}f, \mathsf{d}g \rangle = 0$;

(v) $\rho_\psi\big(\langle \psi_1, \psi_2 \rangle\big) = \big\langle [\psi, \psi_1]_E + \frac{1}{2}\,\mathsf{d}\langle \psi, \psi_1 \rangle \, , \, \psi_2 \big\rangle + \big\langle \psi_1 \, , \, [\psi, \psi_2]_E + \frac{1}{2}\,\mathsf{d}\langle \psi, \psi_2 \rangle \big\rangle$;

where $\psi, \psi_i \in C^\infty(\mathcal{M}, E)$ and $f, g \in C^\infty(\mathcal{M})$.

The graded differential Lie algebra (A.35) is now generalized to a Lie 2-algebra: The structure maps $\mathsf{d}, [-,-]_E, [-,-,-]_E$ of the Courant algebroid $E \to \mathcal{M}$ on the complex

$$C^\infty(\mathcal{M}) \ \xrightarrow{\ \mathsf{d}\ } \ C^\infty(\mathcal{M}, E) \ , \tag{A.40}$$

extended as $[\psi, f]_E := \frac{1}{2}\langle \mathsf{d}f, \psi\rangle$ for $\psi \in C^\infty(\mathcal{M}, E)$ and $f \in C^\infty(\mathcal{M})$, define a 2-term L_∞-algebra [112].

A.7 Lie 2-groups

A group is a monoid in which every element has an inverse; 2-groups are categorifications of groups. For this, recall that a *tensor* or *monoidal category* is a category $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1)$ equipped with an exterior product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ together with an identity object $\mathbb{1} \in \mathscr{C}_0$ and three natural functorial isomorphisms: The unity isomorphisms $\mathbb{1}_X := \mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$ in $\mathscr{C}_1$ for all objects $X \in \mathscr{C}_0$, and the associator isomorphisms

$$\mathscr{P} = \mathscr{P}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\approx} X \otimes (Y \otimes Z) \tag{A.41}$$

for all objects $X, Y, Z \in \mathscr{C}_0$. They satisfy the pentagon identities

$$(\mathbb{1}_X \otimes \mathscr{P}_{Y,Z,W}) \circ \mathscr{P}_{X,Y\otimes Z,W} \circ \mathscr{P}_{X,Y,Z\otimes \mathbb{1}_W} = \mathscr{P}_{X,Y,Z\otimes W} \circ \mathscr{P}_{X\otimes Y,Z,W} \tag{A.42}$$

which state that the five ways of bracketing four objects commutes, and also the triangle identities which state that the associator isomorphism with $Y = \mathbb{1}$ is compatible with the unity isomorphims. For morphisms $\mathscr{F} : X \to Y$ and $\mathscr{F}' : X' \to Y'$, their exterior product is the morphism $\mathscr{F} \otimes \mathscr{F}' : X \otimes X' \to Y \otimes Y'$ in $\mathscr{C}_1$. By MacLane's coherence theorem, these identities ensure that all higher associators are consistent.

We call $\mathscr{C}$ *braided* when there are natural functorial isomorphisms

$$\mathscr{B} = \mathscr{B}_{X,Y} : X \otimes Y \xrightarrow{\approx} Y \otimes X \tag{A.43}$$

for any pair of objects $X, Y \in \mathscr{C}_0$, called commutativity relations. The braiding $\mathscr{B}_{X,Y}$ satisfies two conditions, one expressing $\mathscr{B}_{X\otimes Y,Z}$ in terms of associativity relations $\mathrm{id}_X \otimes \mathscr{B}_{Y,Z}$ and $\mathscr{B}_{Z,X} \otimes \mathrm{id}_Y$, and a similar one for $\mathscr{B}_{X,Y\otimes Z}$.

An object $\mathcal{A} \in \mathscr{C}_0$ in a tensor category $\mathscr{C}$ is an *algebra* or *monoid object* if there is a "multiplication" morphism $\circledast : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, written $a \otimes b \mapsto a \circledast b$, which is associative in the category, i.e. it satisfies the associativity condition

$$\circledast \circ (\circledast \otimes \mathrm{id}_{\mathcal{A}}) = \circledast \circ (\mathrm{id}_{\mathcal{A}} \otimes \circledast) \circ \mathscr{P}_{\mathcal{A},\mathcal{A},\mathcal{A}} \tag{A.44}$$

as maps $(\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \to \mathcal{A}$. By MacLane's coherence theorem, we can deal with nonassociative algebras in this way by expressing usual algebraic operations as compositions of maps and doing the same in the monoidal category with the relevant associator $\mathscr{P}$ inserted between any three objects as needed in order to make sense of expressions. If in addition $\mathscr{C}$ is braided, then $\mathcal{A}$ is *commutative* if its product morphism obeys

$$\circledast \circ \mathscr{B}_{\mathcal{A},\mathcal{A}} = \circledast \tag{A.45}$$

as maps $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. A *group object* is a monoid object $\mathcal{A}$ together with a "unit" morphism $1_{\mathcal{A}} : \mathbb{1} \to \mathcal{A}$ satisfying the unit condition

$$\circledast \circ (1_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}}) = \mathrm{id}_{\mathcal{A}} = \circledast \circ (\mathrm{id}_{\mathcal{A}} \otimes 1_{\mathcal{A}}) \ , \tag{A.46}$$

such that every element of $\mathcal{A}$ has an inverse with respect to the product morphism $\circledast$ and the identity $1_{\mathcal{A}}$.

A 2-*group* is a monoidal category in which every object and morphism has an inverse. A *Lie* 2-*group* is a pair $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1)$ of objects in the category of smooth manifolds and smooth maps, with source and target maps $\mathsf{s}, \mathsf{t} : \mathscr{G}_1 \rightrightarrows \mathscr{G}_0$, and a vertical multiplication $\circ : \mathscr{G}_1 \times \mathscr{G}_1 \to \mathscr{G}_1$ of morphisms. In addition there is a horizontal multiplication functor $\otimes : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ on objects and morphisms, an identity object 1, and a contravariant inversion functor $(-)^{-1} : \mathscr{G} \to \mathscr{G}$ together with natural isomorphisms provided by the associator $\mathscr{P}_{g,h,k} : (g \otimes h) \otimes k \to g \otimes (h \otimes k)$, the left and right units $1 \otimes g \cong g \cong g \otimes 1$, and the units and counits $g \otimes g^{-1} \cong 1 \cong g^{-1} \otimes g$ obeying pentagon, triangle and zig-zag identities; see [13, section 7] for details. If the structure morphisms are all identity isomorphisms, the Lie 2-group $\mathscr{G}$ is called *strict*; otherwise $\mathscr{G}$ is *semistrict*.

A Lie 2-group $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1)$ is *special* if its source and target morphisms $\mathsf{s}, \mathsf{t} : \mathscr{G}_1 \rightrightarrows \mathscr{G}_0$ are equal, and the units and counits are all identity isomorphisms. There is a bijective correspondence between special Lie 2-groups and triples (G, H, φ) consisting of a Lie group G, an action of G as automorphisms of an abelian group H, and a normalized smooth 3-cocycle $\varphi : G \times G \times G \to H$; the isomorphism classes are parametrized by elements $[\varphi] \in H^3(G, H)$ in the degree 3 group cohomology with smooth cocycles. Given a triple (G, H, φ), the corresponding semistrict Lie 2-group $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1)$ has the Lie group $\mathscr{G}_0 = G$ as the space of objects, the semi-direct product Lie group $\mathscr{G}_1 = G \ltimes H$ as the space of morphisms, and the associator $\mathscr{P}$ is given by the action of φ; the source and target maps $\mathsf{s}, \mathsf{t} : \mathscr{G}_1 \rightrightarrows \mathscr{G}_0$ are both projection onto the first factor of $G \times H$, while the cocycle condition on φ is equivalent to the pentagon identities (A.42). In this correspondence the abelian group H is the group of automorphisms of the identity object 1 in the monoidal category $\mathscr{G}$.

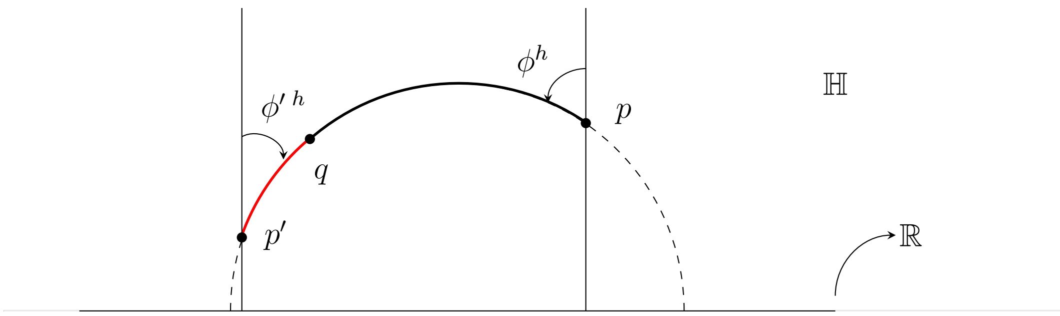
The exponential map takes an ordinary Lie algebra to its integrating simply connected Lie group, while the tangent space at the identity of an ordinary Lie group is the corresponding infinitesimal Lie algebra. In marked contrast, there are no general constructions relating Lie 2-algebras and Lie 2-groups. Integration/differentiation between *strict* Lie 2-algebras and *strict* Lie 2-groups is described in [13, 12]; a general procedure for integrating L_∞-algebras is described in [54, 62]. In the semistrict cases of interest to us in this paper, given a triple (G, H, φ) representing a special Lie 2-group $\mathscr{G}$ (with H an abelian Lie group), by differentiation we obtain a triple $(\mathfrak{g}, W, j)$ representing a 2-term L_∞-algebra V (with W regarded as an abelian Lie algebra); in this case we call the Lie 2-group $\mathscr{G}$ an *integration* of the Lie 2-algebra $\mathscr{V}$ corresponding to V.

Appendix B

Weights of Kontsevich diagrams

In this appendix we explain in some detail how to calculate the weights (3.5) of the diagrams that enter into Kontsevich's formula (3.3) and present some representative examples of the computations.

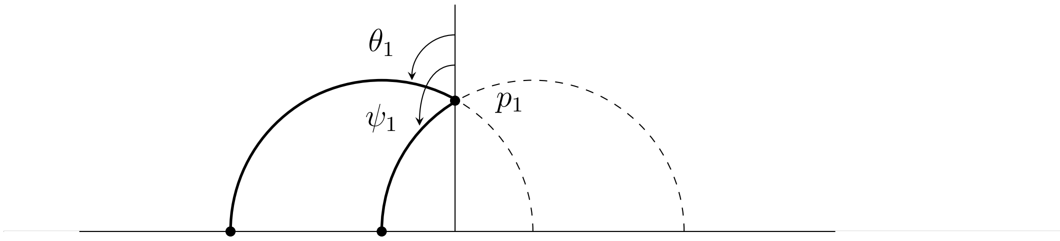
The edges of a generic diagram Γ between two vertices $p, q \in \mathbb{H}$ lie on semicircular geodesics $\ell(p,q)$ in the hyperbolic upper half-plane $\mathbb{H}$. The *harmonic angle* $\phi^h = \phi^h(p,q)$ is defined to be the angle between an edge $\ell(p,q)$ and the directed geodesic $\ell(p,\infty)$ at p; it may be integrated to provide the weight w_Γ with which each multidifferential operator contibutes to the star product (3.3). This is depicted in the following diagram:



Angles in $\mathbb{H}$ are defined in the usual manner; thus $\phi^h, \phi'^{\,h} \in [0,\pi]$ as points p and p' run along the semicircle from the real axis $\mathbb{R}$ (the boundary of $\mathbb{H}$) to q in $\mathbb{H}$. It is important to note that the harmonic angle is measured *counterclockwise*. This means that $\phi^h \in [0, 2\pi]$ as we cross q to integrate over $\mathbb{H}$ along the semicircle.

Bivector diagrams. As an example, let us calculate the weight of the wedge which corresponds to the twisted Poisson bracket $\Theta^{IJ}\,\partial_I f\,\partial_J g$; here we denote $\phi^h_{e^1_1}$

by θ_1 and $\phi^h_{e_1^2}$ by ψ_1:

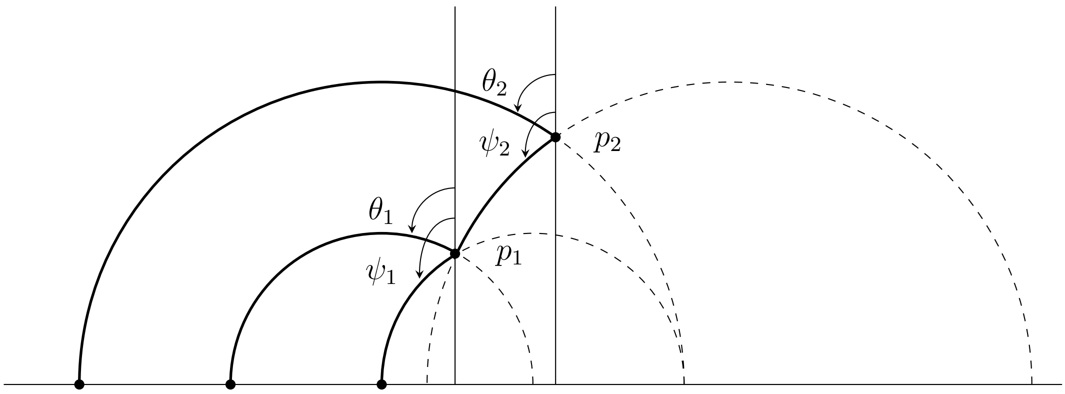


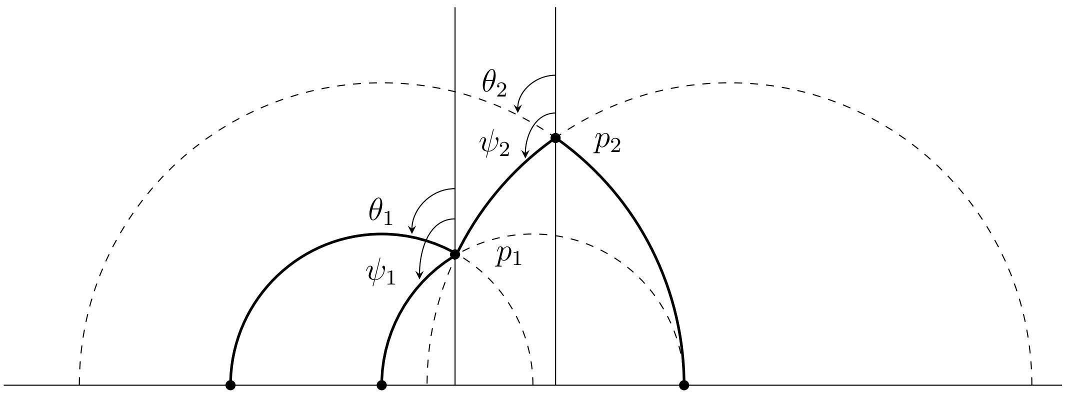
Integrating the two-form $\mathrm{d}\theta_1 \wedge \mathrm{d}\psi_1$ over $\mathbb{H}$, keeping in mind that $\psi_1 > \theta_1$, is straightforward and gives the weight

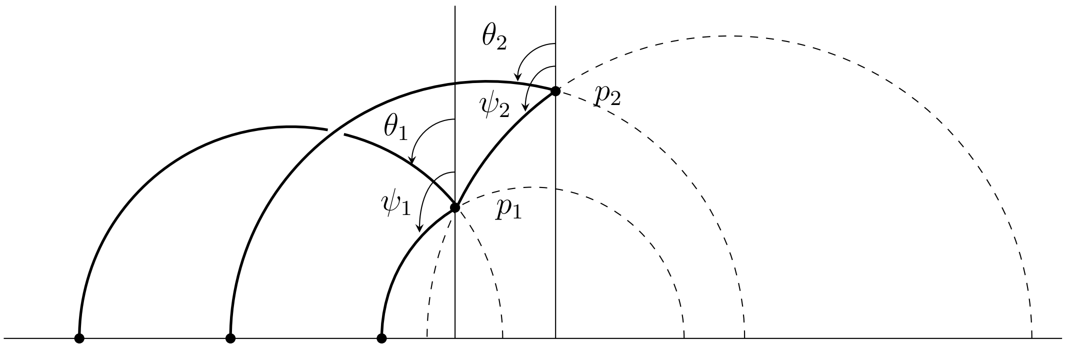
$$\frac{1}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\psi_1 \int_0^{\psi_1} \mathrm{d}\theta_1 = \frac{1}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\psi_1 \; \psi_1 = \frac{1}{2} \; . \tag{B.1}$$

It is important to note here that changing the order of integration produces a minus sign since $\mathrm{d}\psi_1 \wedge \mathrm{d}\theta_1 = -\mathrm{d}\theta_1 \wedge \mathrm{d}\psi_1$. This means that the topologically equivalent *tractable* wedge has weight equal to $-\frac{1}{2}$: A tractable diagram is one that has the derivatives assigned to its edges reversed, i.e. the tractable wedge corresponds to $\Theta^{IJ}\,\partial_J f\,\partial_I g$.

Triple product diagrams. Let us now calculate the weights of the following three diagrams which appear at order $\hbar^2$ when we star multiply three functions:







For the first two diagrams we have $\psi_2 > \theta_2$ and $\psi_2 > \psi_1 > \theta_1$, and thus we get the weights

$$w_1 = \frac{1}{(2\pi)^4} \int_0^{2\pi} \mathrm{d}\psi_1 \int_0^{\psi_1} \mathrm{d}\theta_1 \int_{\psi_1}^{2\pi} \mathrm{d}\psi_2 \int_0^{\psi_2} \mathrm{d}\theta_2 = \frac{1}{8} \tag{B.2}$$

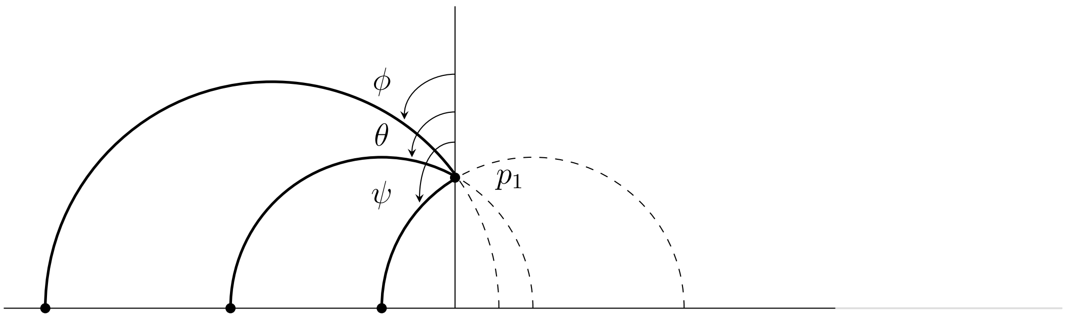
for the first diagram and

$$w_2 = \frac{1}{(2\pi)^4} \int_0^{2\pi} \mathrm{d}\psi_1 \int_0^{\psi_1} \mathrm{d}\theta_1 \int_{\psi_1}^{2\pi} \mathrm{d}\psi_2 \int_0^{\psi_2} -\mathrm{d}\theta_2 = -\frac{1}{8} \tag{B.3}$$

for the second diagram (which is tractable). The third diagram has $\psi_2 > \theta_2 > \theta_1$ and $\psi_2 > \psi_1 > \theta_1$ which gives the weight

$$w_3 = \frac{1}{(2\pi)^4} \int_0^{2\pi} \mathrm{d}\psi_1 \int_0^{\psi_1} \mathrm{d}\theta_1 \int_0^{2\pi} \mathrm{d}\psi_2 \int_{\theta_1}^{\psi_2} \mathrm{d}\theta_2 = \frac{1}{12} \,. \tag{B.4}$$

Trivector diagrams. Finally, we calculate the weight of the following trivector diagram that enters the associator $\Phi(\Pi)$:



Here $\psi > \theta > \phi$ and the formula (3.5) for the diagram weight gives [89]

$$\begin{aligned} w &= \frac{1}{(2\pi)^3} \int_{\mathbb{H}_3} \mathrm{d}\phi \wedge \mathrm{d}\theta \wedge \mathrm{d}\psi \; H(\psi - \theta)\, H(\theta - \phi)\, H(\psi - \phi) \\ &= \frac{1}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}\psi \int_0^{\psi} \mathrm{d}\theta \int_0^{\theta} \mathrm{d}\phi \;=\; \frac{1}{6} \;, \end{aligned} \tag{B.5}$$

where H denotes the Heaviside step function.

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