Membrane Matrix Models and 3-Algebras

by

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Abstract

In this thesis we study the BPS spectrum and vacuum moduli spaces of membrane matrix models derived from dimensional reduction of the BLG and ABJM M2-brane theories. We explain how these reduced models may be mapped into each other, and describe their relationship with the IKKT matrix model. We construct BPS solutions to the reduced BLG model, and interpret them as quantized Nambu-Poisson manifolds. We study the problem of topologically twisting the reduced ABJM model, and along the way construct a new twist of the IKKT matrix model. We construct a cohomological matrix model whose partition function localizes onto the BPS moduli space of the ABJM matrix model. This partition function computes an equivariant index enumerating framed BPS states with specified R-charges.

To my mother

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Chapter 1

Introduction

In this introduction we motivate the topics to be addressed in this thesis. First, we give a brief review of M-theory, and discuss the recent M2-brane developments due to Bagger-Lambert and Gustavsson, as well as Aharony, Bergman, Jafferis, and Maldacena. We motivate the study of M2-brane BPS configurations using matrix models, and then describe the methods we will use. We conclude this chapter by summarizing the remaining chapters.

In order to describe physical phenomena, two theories are required. To explain gravitational interactions, we use the theory of general relativity, which describes the gravitational force on a large scale. Understanding the electromagnetic, weak, and strong interactions requires the use of the standard model, which is a quantum field theory with gauge group $\mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$. There is unfortunately one problem with this description of nature. While the standard model explains the three forces on a quantum level, general relativity is a classical theory which uses the differential geometry of manifolds. Attempting to quantize general relativity and thus unify the theories leads to a theory that is non-renormalizable.

String theory solves this problem mathematically by assuming that the fundamental particles are not point-like in nature, but 1-dimensional loops of energy. From this assumption, the normal laws of electrodynamics and general relativity emerge in the low energy limit. In the particular case of general relativity, it arises in a low energy limit in considering a bosonic string with periodic boundary conditions. If one considers supersymmetric strings of this form, then there are five different ways to construct a such a string: the type I theory, the type IIA theory, the type IIB theory, and the heterotic $\mathsf{E}_8 \times \mathsf{E}_8$ and $\mathsf{SO}(32)$ theories. It was later shown that these theories are related by various dualities, called s-duality, t-duality, u-duality, mirror symmetry, and the conifold transition. In 1995 at the *Strings '95* conference, Witten suggested that these theories were different perturbative limits of an underlying theory he called M-Theory.

Not much is known about M-theory. We do know that the low energy limit is described by 11-dimensional supergravity. Furthermore, it is a nonperturbative theory which contains no strings. That is, the objects of interest are 2-dimensional branes and 5-dimensional branes called *M2-branes* and *M5-branes.* Hence, it is interesting to study the properties of M2-branes and M5-branes in order to learn more about M-theory.

1.1 Recent M2-brane developments

Our understanding of the properties of M2-branes has increased significantly due to the work of Bagger, Lambert, and independently Gustavsson(BLG)[9, 49]. In these papers, they proposed a lagrangian description for a stack of two M2-branes. It was a theory with $\mathcal{N} = 8$ supersymmetry, as well as a $\mathsf{SO}(8)$ R-symmetry group. The novel feature of this theory was that for the supersymmetries to close, the matter fields need to take values in an algebraic structure called a *3-Lie algebra.* A 3-Lie algebra is a straightforward generalization of the normal Lie algebra in that it is a vector space equipped with a totally antisymmetric triple bracket obeying a generalization of the Jacobi identity called the *fundamental identity.* In order to write down a meaningful lagrangian, they required the 3-Lie algebras to be equipped with an inner product. Such 3-Lie algebras are said to be *metric 3-Lie algebras.* In order to respect unitarity, the inner product should be positive definite. It was soon found that there was only one 3-Lie algebra respecting this property, called $\mathcal{A}_4$. In order to generalize the theory, one can relax the requirement of positive-definiteness, and study 3-Lie algebras with split signature. This does not violate unitarity, as the ghost modes present in the theory decouple from the action.

The other option in seeking generalizations of this theory is to relax the amount of supersymmetry required. A theory of M2-branes with $\mathcal{N} = 6$ supersymmetry was constructed by Aharony, Bergman, Jafferis, and Maldacena(ABJM) [4]. In this theory, the matter fields take values in the bifundamental representation of a $\mathsf{U}(N) \times \mathsf{U}(N)$ gauge group. They showed that in the case of $N = 2$, and for an $\mathsf{SU}(2) \times \mathsf{SU}(2)$ gauge group, this theory was equivalent to the BLG lagrangian. Furthermore, they showed that this theory could describe an arbitrary amount of M2-branes. Following this, Bagger and Lambert showed that the ABJM theory could be reformulated in a 3-algebraic language. By dropping the requirement of total antisymmetry of the 3-bracket, they rewrote this theory using what they called *hermitian 3-algebras*.

1.2 M2-brane BPS configurations

States for which the mass is equal to one or more eigenvalues of the central charge of the supersymmetry algebra of a theory are known as *BPS states*. They are static bosonic solutions which minimize their energy, and furthermore are minima of the action. BPS states are important to study in supersymmetric theories because the dimension of the corresponding representation is an integer. This means that it cannot be changed by varying the parameters of the theory in a continuous way. In particular the BPS states receive no quantum corrections. We should therefore study M2-brane BPS configurations in order to learn more about the nonperturbative aspects of M-theory.

In this thesis we will study the BPS spectrum and vacuum moduli space of the BLG and ABJM theories by studying supersymmetric matrix models that are related to these theories. A powerful tool for the enumeration of supersymmetric vacua is provided by the Witten index since it is invariant under deformations of the continuous parameters of the field theory. However, supersymmetric gauge theories have much richer structures that are only partially captured by the Witten index; to extract more information about the field theory, we need to exploit its symmetries. In three dimensions, a generalization of the Witten index is constructed using not only the dilatation operator H, but also the $\mathsf{SO}(2)$ angular momentum J and the generators R_i of the Cartan subalgebra of the R-symmetry group; schematically this refined index is given by

$$\mathscr{I}(x,y,t) = \operatorname{Tr}_{\mathcal{H}_{\rm BPS}} (-1)^F \, x^H \, y^{2J} \, \prod_i t_i^{R_i} \tag{1.1}$$

where the fugacities x, y, t are inserted to resolve degeneracies. Like the Witten index, it can be interpreted as a Feynman path integral with euclidean action by compactifying the time direction on a circle S^1 with supersymmetric twisted boundary conditions involving the $\mathsf{SO}(2)$ rotation J and the global R-symmetry twists R_i; then H is the generator of translations along S^1 and $\mathcal{H}_{\rm BPS}$ is the Hilbert space of the theory with $\mathbb{R}^2$ regarded as the spatial slice. In the weak coupling limit $x \to 0$ where the circle decompactifies, this theory reduces to a supersymmetric quantum mechanics on the moduli space of BPS solutions; these are the models that we will study in this thesis. We also attempt to study these states directly using 3-algebraic structures. We will investigate stable bosonic solutions to the equations of motion of the related M2-brane matrix models, and attempt to make sense of their geometry.

1.3 Quantum geometry

What we know about the geometry of M2-branes has been obtained by considering appropriate lifts of D1-brane analysis to M-theory. In the string theory setting, the noncommutative geometries arising are formed from a Lie algebra structure. Lie algebras can often be regarded as the quantization of a Poisson structure which gives rise to noncommutative geometries. An important example is the Berezin quantized sphere, where the operators $\hat{x}^i$, corresponding to the euclidean coordinates x^i which satisfy $x^i x^i = 1$, form the generators of $\mathfrak{su}(2)$, $[\hat{x}^i, \hat{x}^j] = \mathrm{i}\epsilon^{ijk}\hat{x}^k$. This fuzzy sphere arises naturally in the description of D1-branes ending on D3-branes.

In Type IIB string theory, magnetic monopoles of charge N can be regarded as a stack of N D1-branes ending on a D3-brane [26]. From the perspective of the D1-brane string theory, the effective dynamics are described by the Nahm equations

$$\frac{\mathrm{d}T^i}{\mathrm{d}s} + \varepsilon^{ijk}\,[T^j, T^k] = 0\ , \tag{1.2}$$

where T^i describe fluctuations of the D1-branes parallel to the worldvolume of the D3-brane. These equations have a solution $T^i(s) = f(s)\,\tau^i$, where $f(s) = \frac{1}{s}$ and $\tau^i = \varepsilon^{ijk}\,[\tau^j, \tau^k]$, which describes the transverse scalar fields by a fuzzy 2-sphere [74, 31]. The two extra fuzzy dimensions are required to reconstruct the D3-brane from the D1-branes.

The Basu-Harvey equations [14] are conjectured to describe stacks of M2-branes ending on an M5-brane in M-theory, analogously to the Nahm equations describing stacks of D1-branes ending on a D3-brane. Reformulated, they read

$$\frac{\mathrm{d}T^i}{\mathrm{d}s} + \varepsilon^{ijkl}\,[T^j, T^k, T^l] = 0\ . \tag{1.3}$$

It should allow for a solution via factorization $T^i(s) = f(s)\,\tau^i$, where $f(s) = \frac{1}{\sqrt{2s}}$ and $\tau^i = \varepsilon^{ijkl}\,[\tau^j, \tau^k, \tau^l]$. Thus the transverse scalar fields T^i could live in the 3-Lie algebra $\mathcal{A}_4$, which describes the intersecting configuration in terms of multiple M2-branes again as a fuzzy funnel, this time with the extra three worldvolume dimensions of the M5-brane arising as a fuzzy 3-sphere.

In [28] it was demonstrated how the Nahm equations can be understood as a boundary condition for open strings. This point of view becomes insightful when examining how the worldvolume geometry of the D3-brane is deformed by a constant B-field applied in the transverse directions to the D1-branes. This induces a constant shift in the Nahm equations which can be accounted for by a noncommutative geometry on the D3-brane, described by the Heisenberg commutation relations

$$[T^i, T^j] = \mathrm{i}\,\theta^{ij}\ , \tag{1.4}$$

where θ^{ij} is a constant antisymmetric matrix whose components are related to the components of the B-field.

Analogously, the Basu-Harvey equations can be derived as a boundary condition of open membranes. By including a constant C-field on the M5-brane, the M2-

brane funnel from the M5-brane point of view can be reproduced if the Basu-Harvey equations are suitably modified [28]. This modification identifies the open membrane boundary conditions in the presence of a C-field, which describes the M5-brane worldvolume by a quantum geometry of the form

$$[T^i, T^j, T^k] = \mathrm{i}\,\Theta^{ijk} \ , \tag{1.5}$$

where Θ^{ijk} is a totally antisymmetric constant tensor whose components are related to the components of the constant C-field.

Both the Nahm equations and the Basu-Harvey equations are special cases of generalized Nahm equations built on n-Lie algebras. Just like the commutator (1.4) arises by quantizing a Poisson bracket on $\mathbb{R}^2$, it is suggested in [28] that the correct form of the 3-Lie algebra (1.5) is given by a quantization of the Nambu 3-bracket on $\mathbb{R}^3$, see e.g. [52] and references therein. One of the questions we answer in this thesis is: Does such a quantization exist? If so, are we able to make sense of the resulting geometry?

1.4 Cohomological membrane models

The final topic we consider in this thesis is formulating topologically twisted membrane matrix models that localize the dynamics onto the BPS moduli spaces. These are theories that have a fermionic scalar supersymmetry that is a twisted version of the original theory. This technique has already been applied to the IKKT matrix models [51, 71, 61] and solved using the cohomological field theory formalism. Similar twists have been constructed for the BLG theory [68]; We investigate constructing a similar twist for the ABJM matrix model.

1.5 Thesis plan

This thesis is concerned with studying M2-brane BPS states via matrix models. Specifically, we address the following questions: Do the matrix models constructed from the M2-brane theories admit BPS solutions? Can we understand their geometry in terms of quantized manifolds? Can we twist these models so that their dynamics localize on the BPS moduli spaces?

In the following chapters we address the answers to these questions. This thesis commences with a review of the algebraic structures necessary to understand the M2-brane matrix models. We review the concept of n-Lie algebras, and then discuss the particular case of $n = 3$. We present several examples of 3-Algebras relevant to the remainder of the discussion.

The next chapter contains a review of Berezin-Toeplitz quantization. We discuss the notion of a prequantization, We then introduce Berezin quantization, discuss the quantization of complex projective space, and conclude the chapter with a presentation of the basic ideas behind Toeplitz quantization.

The following chapter concerns the quantization of Nambu-Poisson structures. In it, we show that for a certain type of Nambu n-bracket, they are mapped under a modified Berezin-Toeplitz quantization to a specific n-Lie algebra. We then attempt to make sense of the resulting geometries and apply these results to the geometry of M5-branes.

The next chapter involves using these results in the study of BPS solutions of the matrix models derived from the BLG and ABJM theories. We first construct the BLG and ABJM matrix models via dimensional reduction to zero dimensions, and then show how they can be mapped into each other depending on various scaling limits, or choice of 3-algebra. We also demonstrate how these models may be mapped to the IKKT matrix model under the higgs mechanism proposed by Mukhi and Papageorgakis [73]. We then find several BPS solutions to the reduced BLG model, and interpret them as quantized geometries in the sense of the previous chapter.

We would like to use the reduced ABJM model to compute an equivariant index, so the next chapter is concerned with cohomological 3-algebra models. We review the twist of the BLG theory constructed in [68], and we investigate the effect of the Muhki-Papageorgakis map on this model. After dimensional reduction, the resulting

3-algebra model could potentially induce a cohomological deformation of the reduced ABJM model under the mappings of the previous chapter. We conclude this chapter by demonstrating how this is related to a novel twist of the IKKT matrix model.

The final chapter deals with equivariant 3-algebra models, wherein we construct by hand a cohomological matrix model with $\mathcal{N} = 2$ supersymmetry that allows us to compute the equivariant index of the reduced ABJM model index we are interested in using localization methods. We first briefly review the ideas behind localization, and then explain the construction of the cohomological model. We end the chapter with the explicit calculation of the equivariant index.

The work presented in this this appeared in the published papers [34, 35] and the preprint [36].

Chapter 2

n-algebras

In this chapter we define algebraic structures called n-algebras. The special case of n-Lie algebras were originally considered by Filippov [42] as a straightforward generalization of a Lie algebra. 3-Lie algebras have seen recent interest due to the proposal of Bagger-Lambert and Gustavsson [9, 49] for modeling two M2-branes in terms of an $\mathcal{N} = 8$ supersymmetric theory.

Hermitian 3-algebras were first studied by Bagger and Lambert [10]. They used these algebras to rewrite the ABJM theory in a language that uses ternary brackets. These algebras differ from n-Lie algebras in that one drops the requirement of total antisymmetry of the bracket.

This chapter is structured in the following way. We begin by reviewing n-Lie algebras. We discuss the particular case of the Nambu n-bracket, and a specific truncation of this bracket. We then specialize to the $n = 3$ case, where we discuss 3-Lie algebras as well as hermitian 3-algebras. We also examine several examples. Part of the review present here originally appeared in [34].

2.1 n-Lie algebras

An *n-Lie algebra* [42] is a vector space $\mathcal{A}$ equipped with a totally antisymmetric, multilinear bracket $[-, \ldots, -] : \mathcal{A}^{\wedge n} \to \mathcal{A}$, which satisfies the *fundamental identity*

$$\left[x^1, x^2, \ldots, x^{n-1}, [y^1, y^2, \ldots, y^n]\right] = \sum_{i=1}^{n} \left[y^1, \ldots, [x^1, \ldots, x^{n-1}, y^i], \ldots, y^n\right] \qquad (2.1)$$

for all x^i, $y^i \in \mathcal{A}$. The fundamental identity is a generalization of the Jacobi identity. While the adjoint action of a Lie algebra on itself generates its inner derivations, the space of inner derivations of an n-Lie algebra $\mathcal{A}$ is spanned by operators $D(x^1 \wedge \cdots \wedge x^{n-1}) \in \mathfrak{gl}(\mathcal{A})$, $x^i \in \mathcal{A}$, defined by

$$D(x^1 \wedge \cdots \wedge x^{n-1}) \cdot y := [x^1, \ldots, x^{n-1}, y] \tag{2.2}$$

for $y \in \mathcal{A}$. The inner derivations form a Lie algebra

$$\big[D(x), D(y)\big] \cdot z := D(x) \cdot \big(D(y) \cdot z\big) - D(y) \cdot \big(D(x) \cdot z\big) , \quad x, y \in \mathcal{A}^{\wedge(n-1)} , \; z \in \mathcal{A} , \tag{2.3}$$

where closure of the Lie bracket is guaranteed by the fundamental identity. We call the Lie algebra of inner derivations of an n-Lie algebra $\mathcal{A}$ its *associated Lie algebra* $\mathfrak{g}_{\mathcal{A}}$.

We can reduce an n-Lie algebra $\mathcal{A}$ to an $n-1$-Lie algebra $\mathcal{A}'$, cf. [42]. We choose an element $x_0 \in \mathcal{A}$ and identifies the vector space of $\mathcal{A}'$ with $\mathcal{A}$. The $n-1$-Lie bracket on $\mathcal{A}'$ is defined as $[x^1, \ldots, x^{n-1}]_{\mathcal{A}'} = [x^1, \ldots, x^{n-1}, x_0]$, $x^i \in \mathcal{A}$. By placing an inner product on the vector space $\mathcal{A}'$, we can moreover restrict $\mathcal{A}'$ to the orthogonal complement of x_0 in $\mathcal{A}'$. Applying this procedure $n-2$ times, we arrive at a second Lie algebra $\mathfrak{h}_{\mathcal{A}}$ starting from $\mathcal{A}$, which generally differs from $\mathfrak{g}_{\mathcal{A}}$.

2.1.1 Nambu brackets

The Nambu n-bracket is a generalization of the Poisson bracket to a bracket acting on n functions that satisfies both a generalized Leibniz rule and generalized Jacobi identity. Nambu's original goal was to define an extended hamiltonian mechanics built on these brackets. Requiring both the Leibniz rule and Jacobi identity makes the quantization of this bracket extremely difficult. These structures play important roles in recent proposals for describing M-brane configurations. Here we briefly review these brackets.

A *Nambu-Poisson structure* [75, 85] on a smooth manifold $\mathcal{M}$ is an n-ary, totally antisymmetric linear map $\{-, \ldots, -\} : \mathcal{C}^\infty(\mathcal{M})^{\wedge n} \to \mathcal{C}^\infty(\mathcal{M})$, which satisfies the *generalized Leibniz rule*

$$\{f_1 f_2, f_3, \dots, f_{n+1}\} = f_1 \{f_2, \dots, f_{n+1}\} + \{f_1, \dots, f_{n+1}\} f_2 \tag{2.4}$$

as well as the *fundamental identity*

$$\begin{aligned}\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \{\{f_1, \dots, f_{n-1}, g_1\}, \dots, g_n\} + \dots \\ + \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_n\}\}\end{aligned} \tag{2.5}$$

for $f_i, g_i \in \mathcal{C}^\infty(\mathcal{M})$. The map $\{-, \dots, -\}$ is called a *Nambu n-bracket*, the manifold $\mathcal{M}$ is called a *Nambu-Poisson manifold*, and we call the algebra of smooth functions $\mathcal{C}^\infty(\mathcal{M})$ endowed with the Nambu n-bracket a *Nambu-Poisson algebra*. The Leibniz rule and the fundamental identity imply that the manifold $\mathcal{M}$ admits an n-vector field $\varpi \in (T\mathcal{M})^{\wedge n}$ called a *Nambu-Poisson tensor*, such that

$$\{f_1, \dots, f_n\} = \varpi(\mathrm{d}f_1 \wedge \dots \wedge \mathrm{d}f_n) \tag{2.6}$$

for all $f_i \in \mathcal{C}^\infty(M)$.

In this thesis we will be predominantly interested in the case where $\mathcal{M}$ is a sphere. Recall that the canonical symplectic structure on the sphere S^2 reads as

$$\omega = \begin{pmatrix} 0 & \mathrm{vol}_\theta \\ -\mathrm{vol}_\theta & 0 \end{pmatrix} \tag{2.7}$$

in the basis given by the usual angular coordinates $\varphi = (\varphi^1, \varphi^2) := (\theta, \phi)$, where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. Here $\mathrm{vol}_\theta = \sin\theta$ is the volume element on S^2. The 2-vector field ϖ defining the Poisson or Nambu 2-bracket is obtained by inverting the matrix ω, and we have[1]

$$\{f_1, f_2\} := \varpi(\mathrm{d}f_1 \wedge \mathrm{d}f_2) = \frac{\varepsilon^{ij}}{\mathrm{vol}_\theta} \frac{\partial f_1}{\partial \varphi^i} \frac{\partial f_2}{\partial \varphi^j} \ . \tag{2.8}$$

[1]Throughout this thesis, we will always implicitly sum over repeated indices irrespective of their positions.

Analogously, we define the d-vector field ϖ yielding the Nambu d-bracket on S^d parameterized by the usual angular coordinates φ^i by

$$\{f_1, \dots, f_d\} := \varpi(\mathrm{d}f_1 \wedge \dots \wedge \mathrm{d}f_d) := \frac{\varepsilon^{i_1 \dots i_d}}{\mathrm{vol}_\varphi} \frac{\partial f_1}{\partial \varphi^{i_1}} \dots \frac{\partial f_d}{\partial \varphi^{i_d}} \ . \tag{2.9}$$

Consider now the standard embedding of the sphere S^d of radius R into $\mathbb{R}^{d+1}$, where the cartesian coordinates x^μ, $\mu = 1, \dots, d+1$ are given by

$$x^1 = R\,\cos(\varphi^1)\ ,\quad x^2 = R\,\sin(\varphi^1)\,\cos(\varphi^2)\ ,\quad x^3 = R\,\sin(\varphi^1)\,\sin(\varphi^2)\,\cos(\varphi^3)\ ,\quad \dots\ . \tag{2.10}$$

This embedding induces the volume element on S^d given in spherical coordinates by

$$\mathrm{vol}_\varphi := R^d\,\sin^{d-1}(\varphi^1)\,\sin^{d-2}(\varphi^2)\cdots\sin(\varphi^{d-1})\ . \tag{2.11}$$

We will not use vol_φ directly in the definition, but rescale it by a factor of R^{1-2d}. The Nambu d-bracket of the embedding coordinate functions $x^\mu(\varphi^i)$ can then be calculated to be

$$\left\{x^{\mu_1}(\varphi^i), \dots, x^{\mu_d}(\varphi^i)\right\} = R^{d-1}\,\varepsilon^{\mu_1 \dots \mu_d \mu_{d+1}}\,x^{\mu_{d+1}}(\varphi^i)\ . \tag{2.12}$$

We can extend this bracket to polynomials in x^μ by using the generalized Leibniz rule in the following way. Given a Nambu-Poisson bracket on a subset Υ of the algebra of smooth functions $\mathcal{C}^\infty(\mathcal{M})$ on a manifold $\mathcal{M}$, we can consistently extend this bracket to the subset $\mathbb{C}[\Upsilon] \subset \mathcal{C}^\infty(\mathcal{M})$ of polynomials in elements of Υ. We will use complete induction to verify the fundamental identity. By direct computation, we can see that the relation

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{i=1}^{n} \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_i\}, \dots, g_n\} \tag{2.13}$$

implies

$$\{f_1, \dots, f_{n-1}, \{x\,g_1, \dots, g_n\}\} = \sum_{i=1}^{n} \{x\,g_1, \dots, \{f_1, \dots, f_{n-1}, g_i\}, \dots, g_n\} \tag{2.14}$$

for an arbitrary element $x \in \Upsilon$. Furthermore, the relation (2.13) implies

$$\{x\, f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{i=1}^{n} \{g_1, \dots, \{x\, f_1, \dots, f_{n-1}, g_i\}, \dots, g_n\} \qquad (2.15)$$

as well if and only if

$$\sum_{i=1}^{n} \big(\{g_1, \dots, g_{i-1}, x, \dots, g_n\} \{f_1, \dots, f_{n-1}, g_i\} + \{g_1, \dots, g_{i-1}, f_1, \dots, g_n\} \{x, f_2, \dots, f_{n-1}, g_i\} \big) = 0 \; . \qquad (2.16)$$

The relation (2.16) is satisfied for $f_i, g_i \in \Upsilon$, as here the fundamental identity holds. Moreover, it extends trivially to $\mathbb{C}[\Upsilon]$ by complete induction. Thus the fundamental identity indeed holds on all of $\mathbb{C}[\Upsilon]$.

Let us assume that the components of the Poisson tensor ϖ on a smooth manifold $\mathcal{M}$ are given by homogeneous polynomials of degree $d(\varpi) \geq 1$ in some coordinates (x^μ). If the polynomial ring $\mathbb{C}[x^\mu]$ is furthermore a subset of $\mathcal{C}^\infty(\mathcal{M})$, then there is a truncation of the Nambu-Poisson algebra $\mathcal{C}^\infty(\mathcal{M})$ to an n-Lie algebra structure on $\mathbb{C}[x^\mu]$ [53, 27] as reviewed below.

We define for every $K \in \mathbb{N}$ a totally antisymmetric, linear n-bracket on $\mathbb{C}[x^\mu]$ according to

$$\{f_1, \dots, f_n\}_K := \begin{cases} \{f_1, \dots, f_n\} & \text{if } \; d(f_1) + \ldots + d(f_n) + d(\varpi) - n \leq K \\ 0 & \text{else} \end{cases} \; , \qquad (2.17)$$

where $f_i \in \mathbb{C}[x^\mu]$ and $d(f_i)$ denotes the degree of the polynomial f_i. It is immediately clear that the Leibniz rule cannot survive the truncation. The fundamental identity, however, does, as we show in the following, cf. [53, 27]. Let $f_i, g_i \in \mathbb{C}[x^\mu]$. We then have

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}_K\}_K = \sum_{i=1}^{n} \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_i\}_K, \dots, g_n\}_K \; . \qquad (2.18)$$

The cases $d(f_i) = 0$ or $d(g_i) = 0$ for some i are trivial, let us therefore assume that $d(f_i) > 0$ and $d(g_i) > 0$. Equation (2.18) is nontrivial if and only if the outer brackets on either side are non-vanishing, which amounts to

$$d(f_1) + \ldots + d(f_{n-1}) + d(g_1) + \ldots + d(g_n) + 2d(\varpi) - 2n \leq K \ . \tag{2.19}$$

Because of $d(\varpi) \geq 1$, it is easy to see that this condition also implies that none of the inner brackets of (2.18) vanish. Thus, whenever (2.18) is nontrivial, the brackets are given by the ordinary Nambu-Poisson brackets and thus satisfy the fundamental identity.

2.2 Ternary algebras

We now consider the case $n = 3$. We define metric 3-Lie algebras, as well as the more general hermitian 3-algebras. We explain several examples important to the remainder of this thesis.

2.2.1 Metric 3-Lie algebras

A *metric 3-Lie algebra* is a vector space $\mathcal{A}$ equipped with a positive-definite symmetric bilinear form $(\cdot,\cdot)$, along with totally antisymmetric trilinear map $[\cdot,\cdot,\cdot]$, which maps $\mathcal{A}^{\wedge 3} \to \mathcal{A}$ so that we have

$$[\tau_a, \tau_b, \tau_c] = f_{abcd}\tau_d \ , \tag{2.20}$$

for generators τ_a, and totally antisymmetric structure constants f_{abcd}. This bracket satisfies the *fundamental identity*

$$\begin{aligned} [\tau_d, \tau_e, [\tau_a, \tau_b, \tau_c]] = [[\tau_d, \tau_e, \tau_a], \tau_b, \tau_c] + [\tau_a, [\tau_d, \tau_e, \tau_b], \tau_c] + \\ + [\tau_a, \tau_b, [\tau_d, \tau_e, \tau_c]] \quad . \end{aligned} \tag{2.21}$$

We require the metric to satisfy the following *compatibility condition*

$$([\tau_a, \tau_b, \tau_c], \tau_d) = -(\tau_c, [\tau_a, \tau_b, \tau_d]) \quad , \quad \tau_a \in \mathcal{A} \ .$$

Every metric 3-Lie algebra admits an *associated Lie algebra*, $\mathfrak{g}_{\mathcal{A}}$. We define the generators of $\mathfrak{g}_{\mathcal{A}}$ to be the operators D_{ab} expressed in terms of the 3-Lie bracket as

$$D_{ab}(\tau_c) = [\tau_a, \tau_b, \tau_c] \ . \tag{2.22}$$

They form a Lie algebra with respect to the commutator given by

$$[D_{ab}, D_{cd}](\tau_e) = [\tau_a, \tau_b, [\tau_c, \tau_d, \tau_e] - [\tau_c, \tau_d, [\tau_a, \tau_b, \tau_e]] \ \ , \ \ \tau_e \in \mathcal{A} \ . \tag{2.23}$$

The closure of this bracket is guaranteed by the fundamental identity.

One can reduce a 3-Lie algebra to a Lie algebra generally different from $\mathfrak{g}_{\mathcal{A}}$ [42]. One chooses an element $\tau_0 \in \mathcal{A}$, and identifies the vector space $\mathcal{A}'$ with $\mathcal{A}$. The Lie bracket on $\mathcal{A}'$ is defined as

$$[\tau_a, \tau_b] = [\tau_a, \tau_b, \tau_0] \ \ , \ \ \tau_a \in \mathcal{A} \ . \tag{2.24}$$

Let us review some important examples of metric 3-Lie algebras. A subspace $I \subset \mathcal{A}$ is an *ideal* if $[I, \mathcal{A}, \mathcal{A}] \subset I$. A 3-Lie algebra is said to be *simple* if it has no proper ideals. There is a unique simple 3-Lie algebra over the complex numbers. With respect to a basis $\{\tau_1, \tau_2, \tau_3, \tau_4\}$, define the 3-Lie bracket as

$$[\tau_a, \tau_b, \tau_c] = \epsilon_{abcd}\tau_d \ . \tag{2.25}$$

The inner product relations read as

$$(\tau_a, \tau_b) = \delta_{ab} \ . \tag{2.26}$$

This algebra is denoted $\mathcal{A}_4$ and is the 3-Lie algebra that describes a stack of two M2-branes in the BLG theory.

What we call the *Nambu-Heisenberg 3-Lie algebra* $\mathcal{A}_{NH}$ is generated by four elements $\{\tau_1, \tau_2, \tau_3, \mathbb{1}\}$ and has defining bracket

$$[\tau_a, \tau_b, \tau_c] = \epsilon_{abc}\mathbb{1} \ . \tag{2.27}$$

The element $\mathbb{1}$ is central in the sense that

$$[\mathbb{1}, \tau_a, \tau_b] = 0 \ . \tag{2.28}$$

This 3-Lie algebra is not strictly metric. We have

$$(\mathbb{1}, \tau_a) = 0 \ , \tau_a \in \mathcal{A}_{NH} \ . \tag{2.29}$$

However, $\mathbb{1}$ is a nonzero element, so the symmetric bilinear form is degenerate.

2.2.2 Lorentzian 3-Lie algebras

It is possible to cure the degeneracy of the Nambu-Heisenberg 3-Lie algebra by considering 3-Lie algebras of split signature. A large class of 3-Lie algebras $\mathcal{A}_{\mathfrak{h}}$ with compatible metric of lorentzian signature are described as the semisimple indecomposable lorentzian 3-Lie algebras of dimension $d+2$ which are obtained by double extension from a semisimple Lie algebra $\mathfrak{h}$ of dimension d [33]. Let τ_a, $a = 1, \ldots, d$, be a set of generators for $\mathfrak{h}$ with antisymmetric structure constants f_{abc} defined by the Lie bracket $[\tau_a, \tau_b] = f_{abc}\,\tau_c$. The 3-Lie algebra $\mathcal{A}_{\mathfrak{h}}$ has generators τ_a, J and $\mathbb{1}$ with the 3-bracket relations

$$[\tau_a, \tau_b, \tau_c] = f_{abc}\,\mathbb{1} \quad , \quad [\mathsf{J}, \tau_a, \tau_b] = f_{abc}\,\tau_c \quad , \quad [\mathbb{1}, \tau_a, \tau_b] = 0 = [\mathbb{1}, \tau_a, \mathsf{J}] \tag{2.30}$$

and the inner product relations

$$\begin{aligned} &(\mathbb{1}, \mathbb{1}) = 0 \ , \quad (\mathbb{1}, \tau_a) = 0 \ , \quad (\mathbb{1}, \mathsf{J}) = -1 \ , \\ &(\mathsf{J}, \tau_a) = 0 \ , \quad (\mathsf{J}, \mathsf{J}) = \beta \ , \quad (\tau_a, \tau_b) = \delta_{ab} \ , \end{aligned} \tag{2.31}$$

where $\beta \in \mathbb{R}$ is an arbitrary constant. Note that with $Z_0 = \tau_0$, the reduced bracket (2.24) coincides with the Lie bracket of $\mathfrak{h}$ and $\mathcal{A}'_{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{R}$. On the other hand, the associated Lie algebra of $\mathcal{A}_{\mathfrak{h}}$ is the semi-direct sum

$$\mathfrak{g}_{\mathcal{A}_{\mathfrak{h}}} = \mathfrak{u}(1)^d \ltimes \mathfrak{h} \ . \tag{2.32}$$

We will often be interested in the following example of a lorentzian 3-Lie algebra. The *Nappi-Witten 3-Lie algebra* $\mathcal{A}_{NW}$, with generators $\{ \tau_1, \tau_2, \tau_3, \mathsf{J}, \mathbb{1} \}$, is defined by the relations

$$[\tau_a, \tau_b, \tau_c] = \epsilon_{abc}\mathbb{1} \quad , \quad [\mathsf{J}, \tau_a, \tau_b] = \epsilon_{abc}\tau_c \ , \quad [\mathbb{1}, \tau_a, \tau_b] = 0 \ . \tag{2.33}$$

This 3-Lie algebra is the semisimple finite dimensional indecomposable lorentzian 3-Lie algebra obtained by double extension from the Lie algebra $\mathfrak{so}(3)$. This is a metric 3-Lie algebra of lorentzian signature. The inner product relations read as

$$\begin{aligned} &(\mathbb{1}, \mathbb{1}) = 0 \ , \quad (\mathbb{1}, \tau_a) = 0 \ , \quad (\mathbb{1}, \mathsf{J}) = -1 \ , \\ &(\mathsf{J}, \tau_a) = 0 \ , \quad (\mathsf{J}, \mathsf{J}) = b \ , \qquad (\tau_a, \tau_b) = \delta^{ab} \ . \end{aligned} \tag{2.34}$$

where $b \in \mathbb{R}$. Its associated Lie algebra is

$$\mathfrak{g}_{\mathcal{A}_{NW}} \cong \mathfrak{iso}(3) \ . \tag{2.35}$$

2.2.3 Hermitian 3-algebras

We will now relax the requirement of total antisymmetry of the 3-bracket; these 3-algebras are generally called 3-Leibniz algebras. Here we are interested in the special class of 3-Leibniz algebras called hermitian 3-algebras. They comprise a complex metric 3-algebra which is a finite-dimensional complex vector space $\mathcal{A}$ equipped with a hermitian inner product $(-,-)$ and a trilinear map $[-,-;-] : \mathcal{A}^{\wedge 3} \to \mathcal{A}$. We require that the 3-bracket is antisymmetric in its first two entries only, and that it is complex linear in its first two arguments and complex antilinear in its third argument. A *complex metric hermitian 3-algebra* is a vector space $\mathcal{A}$ equipped with hermitian inner product $(\cdot,\cdot)$, along with a map $[\cdot,\cdot;\cdot]$

$$[\tau_a, \tau_b; \tau_c] = f_{abcd}\tau_d \ . \tag{2.36}$$

We require that this bracket is antisymmetric in the first two entries only. It is complex linear in the first two arguments and complex anti-linear in the third. It satisfies a version of the *fundamental identity*

$$\begin{aligned}&[[\tau_m, \tau_n; \tau_k], \tau_a; \tau_b] - [[\tau_m, \tau_a; \tau_b], \tau_n; \tau_k] - [[\tau_m, [\tau_n, \tau_a; \tau_b], \tau_k] \\ &+ [[\tau_m, \tau_n; [\tau_k, \tau_b; \tau_a]] = 0 \ .\end{aligned} \tag{2.37}$$

We require the metric to satisfy the following *compatibility conditions*

$$([\tau_a, \tau_b; \tau_c], \tau_d) = -([\tau_a, \tau_b; \tau_d], \tau_c) \ . \tag{2.38}$$

Every complex metric 3-algebra satisfying the fundamental identity admits an *associated Lie algebra* $\mathfrak{g}_{\mathcal{A}}$. The generators of $\mathfrak{g}_{\mathcal{A}}$ are defined to be operators D_{ab} expressed in terms of the 3-bracket as

$$D_{ab}(\tau_c) := [\tau_c, \tau_a; \tau_b] \ . \tag{2.39}$$

They form a Lie algebra with respect to the commutator given by

$$[D_{ab}, D_{cd}](\tau_e) = [[\tau_e, \tau_c; \tau_d], \tau_a; \tau_b] - [[\tau_e, \tau_a; \tau_b], \tau_c; \tau_d] \ . \tag{2.40}$$

The closure of this bracket is guaranteed by the fundamental identity. In this thesis we are primarily interested in the following hermitian 3-algebra. Consider the vector space $\mathcal{A} = \mathsf{Hom}\,(V_L, V_R)$ of linear maps $X : V_L \to V_R$ between two complex inner product spaces V_L and V_R. The 3-bracket defined by

$$[X, Y; Z] = \lambda\,(X\,Z^\dagger\,Y - Y\,Z^\dagger\,X) \ , \tag{2.41}$$

for an arbitrary constant $\lambda \in \mathbb{C}$, satisfies the fundamental identity (2.37). The metric on $\mathcal{A}$ given by the Schmidt inner product

$$(X, Y) = \operatorname{Tr}_{V_L}(X^\dagger Y) \tag{2.42}$$

then satisfies the compatibility conditions (2.38). This 3-algebra has associated Lie algebra $\mathfrak{g}_{\mathcal{A}} = \mathfrak{u}(V_L) \oplus \mathfrak{u}(V_R)$: An endomorphism $\phi = (\phi_L, \phi_R) \in \mathfrak{g}_{\mathcal{A}}$ acts on $X \in \mathcal{A}$ as

$$\phi X = X\, \phi_L - \phi_R\, X\ . \tag{2.43}$$

Chapter 3

Berezin-Toeplitz quantization

In this chapter we review both Berezin and Toeplitz quantization as well as geometric quantization of complex projective spaces $\mathbb{C}P^n$, as this is the approach we consider in this thesis. The original constructions are due to Kostant and Souriau [67, 66, 83]. Berezin-Toeplitz quantization is a hybrid form of geometric and deformation quantization in that it uses the Hilbert space of geometric quantization together with the relaxed correspondence principle of deformation quantization. The Hilbert space is chosen as the space of holomorphic sections of a very ample line bundle over the Kähler manifold one wishes to quantize and functions turn into endomorphisms of this Hilbert space under quantization.

This chapter is organized in the following way. We first review prequantization, and then proceed to the geometric quantization of complex projective spaces. Then, we review Berezin quantization of complex projective spaces, and we conclude this chapter with a review of Toeplitz quantization. Parts of the review presented here appeared in [34].

3.1 Prequantization

Prequantization is a procedure that relates a a Kähler manifold $\mathcal{M}$, together with its algebra of smooth functions, to a hermitian line bundle equipped with a hermitian connection ∇, called a *prequantum line bundle*. The resulting line bundle is of too large a dimension, so we will see how to reduce its dimension using polarizations.

We fix a Kähler manifold $\mathcal{M}$ with complex dimension n. Let $\nabla : \Gamma(T\mathcal{M}) \times \Gamma(E) \to \Gamma(E)$ be a connection on a vector bundle $E \to \mathcal{M}$. The space of sections is denoted by $\Gamma(E)$, and the space of vector fields on the manifold is denoted by $\Gamma(T\mathcal{M})$. The curvature of this connection is a section of $\Lambda^2(T^*\mathcal{M}) \otimes \mathsf{End}\,(E)$. It is defined by

$$F(X,Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s \ , \tag{3.1}$$

for vector fields X, Y and sections s.

If ∇ is hermitian, then its curvature is a 2-form with values in $\mathsf{End}\,(E)$. If E is a complex line bundle, then we have

$$\mathsf{End}\,(E) \cong \mathcal{M} \times \mathrm{i}\mathbb{R} \ . \tag{3.2}$$

This implies that we have

$$\mathrm{i}F \in \Omega^2(\mathcal{M}, \mathbb{R}) \ , \tag{3.3}$$

so that F is a real valued 2-form.

Now consider the collection of all triples $\mathsf{M} := (L, h, \nabla)$. $L \to \mathcal{M}$ is a complex line bundle, and h is a hermitian metric. The curvature defines a map

$$\mathsf{M} \to \Omega^2(\mathcal{M}) \ , (L, h, \nabla) \to \mathrm{i}F \ . \tag{3.4}$$

In order to define a prequantization we need to consider the inverse of this map. In particular, given a Kähler manifold $(\mathcal{M}, \omega)$, one would like to find a hermitian line bundle with connection h so that we have

$$\omega = \frac{\mathrm{i}}{2\pi} F \ . \tag{3.5}$$

This is the *prequantization condition.* In geometric quantization, this condition guarantees that the correspondence principle is satisfied. For our purposes, we merely observe that (3.5) implies that L is a positive or ample line bundle and therefore that a certain power $L^{\otimes k_0}$ of this line bundle is very ample[1]. In the following, we will assume that L is already very ample, for otherwise one can make the necessary replacements $L \to L^{\otimes k_0}$, $\omega \to k_0\,\omega$, $\nabla \to \nabla^{\otimes k_0}$, and $h \to h^{\otimes k_0}$. The line bundle (L, h) is called a *quantum line bundle* for $(\mathcal{M}, \omega)$ and $(\mathcal{M}, \omega, L, h)$ is a *prequantized Hodge*[2] *manifold.*

The hermitian metric h together with the Liouville volume form $\mathrm{d}\mu = \frac{\omega^n}{n!}$ on $\mathcal{M}$ induces a metric on the space of smooth sections $\Gamma^\infty(\mathcal{M}, L)$ given by

$$(s_1|s_2) := \int_{\mathcal{M}} \mathrm{d}\mu\ h_x\big(s_1(x), s_2(x)\big)\ , \tag{3.6}$$

for $s_1, s_2 \in \Gamma^\infty(\mathcal{M}, L)$. This yields a projection from $L^2(\mathcal{M}, L)$, the L^2-completion of the space $\Gamma^\infty(\mathcal{M}, L)$, to $H^0(\mathcal{M}, L)$, the space of global holomorphic sections of L. The inner product on $L^2(\mathcal{M}, L)$ also induces an inner product on $H^0(\mathcal{M}, L)$, which we denote by the same symbol.

We are now ready to define prequantization. *Prequantization* is a linear map

$$Q : C^\infty(\mathcal{M}) \to \mathsf{Hom}\left(\Gamma(E), \Gamma(E)\right)\ , f \to Q_f\ . \tag{3.7}$$

The operator Q_f is defined by

$$Q_f(s) = \nabla_{\theta_f} s - 2\pi\mathrm{i} f s\ , \tag{3.8}$$

for all functions $f \in C^\infty(\mathcal{M})$ and sections $s \in \Gamma(E)$. θ_f denotes the hamiltonian vector field of f with respect to the Kähler form ω. This map is a map of Lie algebras in that

$$[Q_f, Q_g]s = Q_{\{f,g\}}s\ , \tag{3.9}$$

for all sections $s \in \Gamma(E)$ and functions $f, g \in C^\infty(\mathcal{M})$. This map is also skew-

[1] A line bundle is very ample if it posses enough global sections to set up an embedding of its base manifold into complex projective space

[2] We can choose an appropriate normalization such that $[\omega] \in H^2(\mathcal{M}, \mathbb{Z})$.

hermitian:

$$h(Q_f s, s') + h(s, Q_f s') = 0 \ . \tag{3.10}$$

It associates to a Kähler manifold a Hilbert space $H^0(\mathcal{M}, L)$ and to each function f on $\mathcal{M}$ a skew-hermitian operator Q_f. Unfortunately this results in a Hilbert space that is too large. For the case of a real manifold, we consider the real line $\mathbb{R}$. The corresponding phase space is the cotangent bundle $T^*\mathbb{R}$ with canonical symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$. The corresponding prequantum line bundle is $E = T^*\mathbb{R} \times \mathbb{C} \to T^*\mathbb{R}$. Therefore, the prequantum Hilbert space $H^0(T^*\mathbb{R}, L)$ is the space $L^2(T^*\mathbb{R}, \mathbb{C})$ of complex valued square integrable functions. However, quantum mechanics tells us that the correct Hilbert space is square integrable functions of one variable, not two variables. A standard solution to this problem is to use polarizations.

A *complex polarization* is a complex distribution P on a manifold $\mathcal{M}$ such that

1. For all $m \in \mathcal{M}$, $P_m \subset T_m\mathcal{M}$ is lagrangian.
2. The dimension of V_x is constant.
3. It is integrable .

A lagrangian manifold is a manifold that is maximally isotropic. A *distribution* on a complex manifold $\mathcal{M}$ is a choice of subspace V_m of each tangent space $T_m(\mathcal{M})$ which changes in a smooth way . It is *integral* if, at least locally, there is a foliation, of constant dimension, of $\mathcal{M}$ by submanifolds such that V_m is the tangent space of the submanifolds containing m. A complex manifolds has at least two polarizations, the *holomorphic polarization* spanned by vectors $\frac{\partial}{\partial z}$ and antiholomorphic polarization, spanned by vectors $\frac{\partial}{\partial \bar{z}}$. In this thesis, we identify $H^0(\mathcal{M}, L)$ with the Hilbert space $\mathscr{H} = \mathscr{H}_L$, and by doing so we choose to work with holomorphic polarization.

3.2 Berezin quantization

In this section we briefly review Berezin quantization. Previously, we considered mappings that assigned operators to functions using the geometry of the Kähler manifold. Here we consider mappings in the other direction. That is, we assign functions to operators; These functions are called the *symbol* of the operator.

3.2.1 Coherent states

Consider the total space $\mathbb{L}$ of the line bundle L, with projection $\pi : \mathbb{L} \to \mathcal{M}$, and $\mathbb{L}_o = \mathbb{L}\backslash o$, where o is the zero section. We define a function $\psi_q(s)$ which indicates how much we have to scale a section $s \in \mathscr{H}_L$ to pass through a given point $q \in \mathbb{L}_o$ via

$$s\big(\pi(q)\big) =: \psi_q(s)\, q \ . \tag{3.11}$$

By Riesz's theorem, there is a unique holomorphic section e_q such that

$$(e_q|s) = \psi_q(s) \tag{3.12}$$

for all sections $s \in \mathscr{H}_L$. The element e_q is called the *Rawnsley coherent state vector*, a generalization of the *Perelomov coherent states* appearing from a group theoretic perspective. The *Rawnsley coherent state projector* is given by

$$P_x := \frac{|e_q)(e_q|}{(e_q|e_q)} \ , \quad q \in \mathbb{L}_o \ . \tag{3.13}$$

Note that P_x only depends on $\pi(q) = x$. This is due to the scaling of ψ_q, $\psi_{c\,q} = \frac{1}{c}\,\psi_q$.

In our quantization of $\mathcal{M} = \mathbb{C}P^n$ with $L = \mathcal{O}(k)$, the Rawnsley coherent states are simply the truncated Glauber vectors $|z, k\rangle$ on $\mathbb{C}^{n+1}$ (cf. e.g. [59]) given by

$$|z\rangle = \exp\big(\bar{z}_\alpha\, \hat{a}^\dagger_\alpha\big)|0\rangle = \sum_{\vec{p}} \frac{\bar{z}^{\,\vec{p}}}{\sqrt{\vec{p}!}}\,|\vec{p}\rangle = \sum_{k=0}^{\infty} |z, k\rangle \ , \tag{3.14}$$

where

$$|z, k\rangle = \frac{1}{k!}\,\big(\bar{z}_\alpha\, \hat{a}^\dagger_\alpha\big)^k|0\rangle = \sum_{|\vec{p}|=k} \frac{\bar{z}^{\,\vec{p}}}{\sqrt{\vec{p}!}}\,|\vec{p}\rangle \ . \tag{3.15}$$

The coherent state projector takes the form

$$P_z = \frac{|z, k\rangle\langle z, k|}{\langle z, k|z, k\rangle} \ . \tag{3.16}$$

Useful relations for the computations which follow are $\hat{a}_\alpha|z,k\rangle = \bar{z}_\alpha|z,k-1\rangle$ and $\langle z,k|z,k\rangle = \frac{1}{k!}\,|z|^{2k}$.

3.2.2 Berezin quantization

The *lower* or *covariant Berezin symbol* of an operator $\hat{f} \in \mathsf{End}\,(\mathscr{H}_L)$ is defined as

$$\sigma(\hat{f}\,)(x) := \operatorname{tr}(\hat{f}\,P_x)\ . \tag{3.17}$$

The space $\sigma(\mathsf{End}\,(\mathscr{H}_L))$ is the space of quantizable functions $\Sigma \subset \mathcal{C}^\infty(\mathcal{M})$. The map σ is injective and thus we can define the Berezin quantization of a function as the inverse of σ on Σ given by

$$f \ \longmapsto\ Q(f) := \hat{f} = \sigma^{-1}(f)\ , \quad f \in \Sigma\ . \tag{3.18}$$

3.3 Toeplitz quantization

In Toeplitz quantization (see e.g. [24]), the operator $T_L(f)$ corresponding to a function f acts on an element s of the Hilbert space $\mathscr{H}_L$ by multiplying the corresponding section s and subsequent projection back to holomorphic sections via the inner product $(-|-)$. Hence

$$T_L(f)(s) := \Pi(f\,s)\ , \quad f \in \mathcal{C}^\infty(\mathcal{M})\ ,\ s \in \mathscr{H}_L\ . \tag{3.19}$$

The appropriate projector is the coherent state projector P_x and we arrive at

$$T_L(f) = \int_{\mathcal{M}} \mathrm{d}\mu\ f(x)\,P_x\ . \tag{3.20}$$

The Toeplitz quantization map is the adjoint of the Berezin quantization map with respect to the Hilbert-Schmidt norm and the L^2-measure induced by the Liouville volume form [81]. The ordering prescriptions resulting from Berezin and Toeplitz quantizations of $\mathcal{M} = \mathbb{C}P^n$ correspond to Wick and anti-Wick ordering, respectively, cf. [59].

Toeplitz quantization is of interest for various reasons. First, it converges towards geometric quantization as shown in [87]. Second, strict convergence theorems can be deduced, and in particular for $\mathcal{M} = \mathbb{C}P^n$ one has [24]

$$\lim_{k\to\infty} \Big\| \mathrm{i}\, k \left[T_{\mathcal{O}(k)}(f), T_{\mathcal{O}(k)}(g)\right] - T_{\mathcal{O}(k)}\big(\{f,g\}\big) \Big\|_{HS} = 0 \ . \tag{3.21}$$

Chapter 4

Kinematical quantization of n-Lie algebras

Quantization in physics is best understood as a recipe for passing from a classical system to some corresponding quantum system. It is expected that in the limit where Planck's constant goes to zero, the quantum system should reduce to the classical system. Over time, it has been shown that that this idea is not totally appropriate. In fact, there are several mathematical theorems that state that there is no quantization recipe that satisfies all the required quantization axioms. This is especially true in considering the quantization of n-Lie algebras. As we mentioned earlier, n-Lie algebras in physics were first considered by Nambu in attempts to generalize hamiltonian physics. Finding both a correspondence between classical and quantum observables, as well as deriving appropriate quantum dynamics, has proved to be a difficult problem.

In this chapter we consider only the problem of kinematical quantization. That is, we find an explicit map that relates truncated Nambu-Poisson brackets on a sphere to a generalization of the commutator. This chapter is structured in the following way. We first list our generalized quantization axioms and write down a deformation quantization of truncated Nambu-Poisson structures. Then, we perform Berezin-Toepiltz quantization of these brackets on a sphere, and show how they are mapped to a particular n-Lie bracket, in both the even and odd dimensional case. We also show how our quantized spheres are related to previously studied fuzzy spheres.

We then consider the quantization of Nambu-Poisson brackets on a hyperboloid. Finally, we consider the quantization of $\mathbb{R}^n$ and in particular study the case of the Nambu-Heisenberg 3-Lie algebra. We then interpret this algebra as a quantization of an M5-brane geometry in a constant C-field background. The work presented here appeared in [34].

4.1 Quantization of Nambu-Poisson structures

4.1.1 Conventional quantization

The problem of quantization splits into two parts. The first task is to establish the kinematical relationship between classical and quantum observables. The second is to deduce the dynamical laws of a quantum system from their classical counterparts.

Classically, the state space of a dynamical system is a Poisson manifold $\mathcal{M}$ and the observables are the smooth functions on $\mathcal{M}$. We will demand that the Poisson structure is non-degenerate, which requires that $\mathcal{M}$ has even dimension and turns the Poisson structure into a symplectic structure on $\mathcal{M}$. At the quantum level, the states of a physical system are given by rays in a complex Hilbert space $\mathscr{H}$ and observables are linear operators acting on $\mathscr{H}$.

The problem of finding a quantization for a given Poisson manifold is highly nontrivial and not understood in full generality. We will impose the following axioms, which yield a *full quantization* (cf. e.g. [2]):

- Q1. The map $f \mapsto \hat{f}$ is linear over $\mathbb{C}$ and maps smooth real functions on $\mathcal{M}$ to hermitian linear operators on $\mathscr{H}$.
- Q2. If f is a constant function, then $\hat{f}$ is scalar multiplication by the corresponding constant.
- Q3. The *correspondence principle*: If $\{f_1, f_2\} = g$ then $[\hat{f}_1, \hat{f}_2] = -\mathrm{i}\,\hbar\,\hat{g}$.
- Q4. The operators $\hat{x}^\mu$ and $\hat{p}_\mu$ act irreducibly on $\mathscr{H}$.

Here $f, f_i, g \in \mathcal{C}^\infty(\mathcal{M})$ and $\{-,-\}$ and $[-,-]$ denote the Poisson bracket on $\mathcal{M}$ and the commutator of elements of $\mathsf{End}\,(\mathscr{H})$, respectively. However, the Grönewoldvan Howe theorem states that there is no such quantization, see [2] or [46]. There is an analogous theorem for $\mathcal{M} = S^2$.

There are three common loopholes to this obstruction. First, we can drop irreducibility and ignore axiom Q4. Second, we could quantize a subclass of functions in $\mathcal{C}^\infty(\mathcal{M})$. Third, we could generalize the correspondence principle such that it only holds up to first order in $\hbar$. The first two approaches lead to prequantization and further to the formalism of *geometric quantization* [90], while the third approach leads to approximate operator representations and eventually to the machinery of *deformation quantization* [16, 65]. We recall that the canonical quantization prescription of Weyl, von Neumann and Dirac is not Q3, but just the corresponding condition on the coordinates of phase space, which further supports the third approach.

Our constructions are based on Berezin[1] and Toeplitz quantization, which are hybrids of geometric and deformation quantization. They both rely on the Hilbert space constructed in geometric quantization but satisfy the correspondence principle only to first order in $\hbar$. We restrict to quantizing only a subset of functions in Berezin quantization. We will therefore impose axioms Q1 and Q2, and axiom Q3 only to linear order in $\hbar$. In Berezin-Toeplitz quantization, these representations are usually irreducible. In our extension of this construction we will, however, have to allow for reducible representations as well.

We will not require that quantizing a complete set of classical observables yields a complete[2] set of quantum observables, which would establish a one-to-one correspondence between $\mathsf{End}\,(\mathscr{H})$ and $\mathcal{C}^\infty(\mathcal{M})$.

4.1.2 Generalized quantization axioms for Nambu brackets

We start by demanding that a quantization associates to a Nambu-Poisson manifold $\mathcal{M}$ a Hilbert space $\mathscr{H}$ and maps a set of *quantizable functions* $\Sigma \subset \mathcal{C}^\infty(\mathcal{M})$ on $\mathcal{M}$ to endomorphisms on $\mathscr{H}$. We impose the quantization conditions Q1, Q2, and Q4$'$,

[1] By Berezin quantization, we mean the standard constructions of fuzzy geometry. The algebra of functions is reduced to the algebra of lower Berezin symbols of $\mathsf{End}\,(\mathscr{H})$, where the product is given by the corresponding operator product.

[2] Completeness here means Schur's lemma: If an operator commutes with each element, it is proportional to the identity. Completeness in the classical case is the analogous statement involving the Poisson bracket and the constant function.

but relax Q3 in the spirit of Berezin-Toeplitz quantization. The quantization map will always be injective, and on its image $\widehat{\Sigma} \subset \mathsf{End}\,(\mathscr{H})$ we introduce its inverse σ. (In Berezin-Toeplitz quantization, σ is the lower Berezin symbol.) The axiom Q3 is then modified to

Q3′. The quantization maps a subalgebra of the Nambu-Poisson algebra on $\mathcal{M}$ to an n-Lie algebra structure on a subspace of $\mathsf{End}\,(\mathscr{H})$, which satisfies the constraint[3]

$$\lim_{\hbar \to 0} \left\| \frac{\mathrm{i}}{\hbar}\, \sigma\big([\hat{f}_1, \dots, \hat{f}_n]\big) - \{f_1, \dots, f_n\} \right\|_{L^2} = 0 \tag{4.1}$$

for all quantizable functions $f_i \in \Sigma$.

In conventional quantization, σ is bijective and therefore the correspondence principle as stated here is equivalent to the usual one formulated in terms of operators.

The canonical choice for an n-ary linear and totally antisymmetric bracket on $\mathsf{End}\,(\mathscr{H})$ in the literature (cf. e.g. [75, 85, 32]) is the totally antisymmetric operator product

$$[\hat{f}_1, \dots, \hat{f}_n] := \varepsilon^{i_1 \dots i_n}\, \hat{f}_{i_1} \dots \hat{f}_{i_n} \ . \tag{4.2}$$

This bracket neither satisfies the fundamental identity nor the Leibniz rule, in general.

A different bracket can be defined on Nambu-Poisson manifolds, on which we can truncate the Nambu-Poisson structure as discussed in §2.1.1: In the cases we are interested in, the set of quantizable functions Σ is a set of polynomials of a certain maximal degree K. On this set, an n-Lie algebra structure is given by the truncated Nambu-Poisson bracket $\{-, \dots, -\}_K$. This n-Lie algebra structure can be lifted from Σ to an n-Lie algebra structure on $\mathsf{End}\,(\mathscr{H})$: The bracket

$$[\hat{A}_1, \dots, \hat{A}_n] := \sigma^{-1}(-\mathrm{i}\hbar\{\sigma(\hat{A}_1), \dots, \sigma(\hat{A}_n)\}_K) \tag{4.3}$$

is linear, antisymmetric and satisfies the fundamental identity for arbitrary operators $\hat{A}_i \in \mathsf{End}\,(\mathscr{H})$, as $\sigma \circ \sigma^{-1} = \mathrm{id}$. We note that for $\hbar \to 0$, we have $K \to \infty$, and

[3] We assume the existence of a measure $\mathrm{d}\mu$ on $\mathcal{M}$. As we quantize Kähler manifolds exclusively, we can use the Liouville volume form $\mathrm{d}\mu = \frac{\omega^n}{n!}$, where ω is the Kähler 2-form and $\dim_{\mathbb{C}} \mathcal{M} = n$.

the truncated n-Lie algebra approaches the Nambu-Poisson algebra on $\mathcal{M}$. For this reason, the correspondence principle Q3′ is satisfied by definition. Also, in some cases this bracket will turn out to be equal to the totally antisymmetric operator product if all the arguments are linear polynomials. For $n = 2$, this bracket does not reproduce the commutator, but a deformation thereof.

4.1.3 Quantization of complex projective spaces

To quantize $\mathcal{M} = \mathbb{C}P^n$, we choose L to be the holomorphic line bundle $\mathcal{O}(k)$ of degree $k \in \mathbb{N}$ and ω the Kähler form giving rise to the Fubini-Study metric on $\mathbb{C}P^n$. For $L = \mathcal{O}(k)$, the space $\mathscr{H}_k := \mathscr{H}_L = H^0(\mathcal{M}, L)$ is finite-dimensional and spanned by homogeneous polynomials of degree k in the standard homogeneous coordinates z_α, $\alpha = 0, 1, \ldots, n$ on $\mathbb{C}P^n$. Hence

$$\begin{aligned} \mathscr{H}_k &:= \operatorname{span}_{\mathbb{C}}\big\{ z_{\alpha_1} \cdots z_{\alpha_k} \;\big|\; \alpha_i = 0, 1, \ldots, n \big\} \\ &= \operatorname{span}_{\mathbb{C}}\Big\{ z_0^{p_0}\, z_1^{p_1} \cdots z_n^{p_n} \;\Big|\; p_\alpha \in \mathbb{N}_0 \;,\; |\vec{p}\,| := \sum_{\alpha=0}^{n} p_\alpha = k \Big\} \;. \end{aligned} \tag{4.4}$$

For later convenience, we identify this space with the k-particle Hilbert space in the Fock space of $n+1$ harmonic oscillators given by

$$\mathscr{H}_k \cong \operatorname{span}_{\mathbb{C}}\Big\{ \frac{\hat{a}^\dagger_{\alpha_1} \cdots \hat{a}^\dagger_{\alpha_k}}{\mathcal{N}} |0\rangle \Big\} = \operatorname{span}_{\mathbb{C}}\Big\{ \frac{(\hat{a}^\dagger_0)^{p_0} \cdots (\hat{a}^\dagger_n)^{p_n}}{\sqrt{p_0! \cdots p_n!}} |0\rangle =: \frac{1}{\sqrt{\vec{p}\,!}}\, |\vec{p}\,\rangle \Big\} \;, \tag{4.5}$$

where $\mathcal{N} \in \mathbb{R}$ is a normalization constant. The creation and annihilation operators satisfy the usual Heisenberg-Weyl algebra $[\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta}$, and $|0\rangle$ denotes the vacuum vector with $\hat{a}_\alpha |0\rangle = 0$.

We can show that the quantization axioms are verified for $\mathcal{M} = \mathbb{C}P^n$. The map $Q : \Sigma \to \mathsf{End}\,(\mathscr{H}_k)$ is linear, and the constant function is mapped to the identity operator since from the form of the coherent state projector we find

$$Q\Big(\frac{z_{\alpha_1} \cdots z_{\alpha_k}\, \bar{z}_{\beta_1} \cdots \bar{z}_{\beta_k}}{|z|^{2k}} \Big) = \frac{1}{k!}\, \hat{a}^\dagger_{\alpha_1} \cdots \hat{a}^\dagger_{\alpha_k} |0\rangle\langle 0| \hat{a}_{\beta_1} \cdots \hat{a}_{\beta_k} \tag{4.6}$$

with $|z|^2 := \bar{z}_\alpha\, z_\alpha$, so that in particular $Q(1) = \mathbb{1}_{\mathscr{H}_k}$. To check the third quantization axiom, it is convenient to employ a "star product[4]" on $\mathbb{C}P^n$. The star product is induced by pulling back the operator product onto the set of quantizable functions to get

$$f * g := \sigma(\hat{f}\,\hat{g}) \ , \quad f, g \in \Sigma \ . \tag{4.7}$$

To obtain a particularly nice form, we need an embedding $\mathbb{C}P^n \hookrightarrow \mathbb{R}^{(n+1)^2-1}$ given by the Jordan-Schwinger transformation

$$x^M = \frac{\bar{z}_\alpha\,\lambda^M_{\alpha\beta}\,z_\beta}{|z|^2} \ , \qquad M = 1, \ldots, (n+1)^2 - 1 \ , \tag{4.8}$$

where $\lambda^M_{\alpha\beta}$ are the Gell-Mann matrices of the isometry group $\mathsf{SU}(n+1)$ of $\mathbb{C}P^n$. In terms of the coordinates x^M, we can write this star product as [12]

$$(f * g)(x) = \sum_{l=0}^{k} \frac{(k-l)!}{k!\,l!} \left(\partial_{M_1} \cdots \partial_{M_l} f(x)\right) K^{M_1 N_1} \cdots K^{M_l N_l} \left(\partial_{N_1} \cdots \partial_{N_l} g(x)\right) \ , \tag{4.9}$$

where $\partial_M := \frac{\partial}{\partial x^M}$ and

$$K^{MN} = \frac{1}{n+1}\,\delta_{MN} + \frac{1}{\sqrt{2}}\left(d^{MN}{}_K + \mathrm{i}\,f^{MN}{}_K\right) x^K - x^M\,x^N \ . \tag{4.10}$$

Here $d^{MN}{}_K$ and $f^{MN}{}_K$ are the symmetric tensor and structure constants of $\mathsf{SU}(n{+}1)$. Note that (4.9) forms an expansion in terms of $\hbar = \frac{1}{k}$ for k large. It is possible to show that the symplectic form which gives rise to the Fubini-Study metric on $\mathbb{C}P^n$ in the coordinates x^M is given by $2\,\mathrm{i}\,K^{[MN]}$ [12]. The correspondence principle therefore reads as

$$\lim_{k \to \infty} \left\| \mathrm{i}\,k\,(f * g - g * f) - 2\,\mathrm{i}\,K^{[MN]}\,(\partial_M f)\,(\partial_N g) \right\|_{L^2} = 0 \ , \tag{4.11}$$

which one verifies using (4.9).

Let us examine the case of $\mathbb{C}P^1 \cong S^2$ in some more detail. With the choice $L = \mathcal{O}(k)$, Σ corresponds to the set of spherical harmonics $Y_{\ell m}$ with angular momentum

[4]This product, sometimes called the coherent state star product, is not a formal star product.

$\ell \leq k$. The Poisson bracket is

$$\{x^\mu, x^\nu\} = R\,\varepsilon^{\mu\nu\kappa}\,x^\kappa\ , \tag{4.12}$$

where R is the radius of the sphere S^2. The quantization axiom Q3 implies that the quantizations $\hat{x}^\mu$ of the coordinates x^μ satisfy the Lie algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = -\mathrm{i}\,\hbar\,R\,\varepsilon^{\mu\nu\kappa}\,\hat{x}^\kappa\ . \tag{4.13}$$

The deformation parameter $\hbar$ here is not continuous. To compute it, we again use the Jordan-Schwinger transformation (4.8),

$$x^\mu := \frac{R}{|z|^2}\,\bar{z}_\alpha\,\sigma^\mu_{\alpha\beta}\,z_\beta\ , \tag{4.14}$$

where x^μ are coordinates on $S^2 \hookrightarrow \mathbb{R}^3$ and z_α are homogeneous coordinates on the projective line $\mathbb{C}P^1$, while σ^μ, $\mu = 1,2,3$, is the standard basis of 2×2 Pauli spin matrices for $\mathfrak{su}(2)$, see Appendix A. We work out the quantization of the coordinate functions to be

$$x^\mu \ \longmapsto\ \hat{x}^\mu = \frac{R}{k!}\,\sigma^\mu_{\alpha\beta}\,\hat{a}^\dagger_\alpha\,\hat{a}^\dagger_{\rho_1}\cdots\hat{a}^\dagger_{\rho_{k-1}}|0\rangle\langle 0|\hat{a}_\beta\,\hat{a}_{\rho_1}\cdots\hat{a}_{\rho_{k-1}} =: \frac{R}{k!}\,\sigma^\mu_{\alpha\beta}\,|\alpha,k\rangle\langle k,\beta|\ . \tag{4.15}$$

Working through the details, we find

$$\hbar = \frac{2}{k}\ . \tag{4.16}$$

The classical limit is obtained for $k\to\infty$, and (4.13) suggests that in this limit the algebra of coordinate functions (and thus the whole algebra of functions) becomes commutative.

4.2 Quantization of spheres

In this section, we will provide an extension of Berezin-Toeplitz quantization to spheres. We shall also examine in detail the n-Lie algebra structure on the arising operator algebra and compare these quantizations to previous versions of fuzzy spheres in higher dimensions.

4.2.1 Hyperspherical harmonics

Consider the space $\mathbb{R}^{d+1}$ with its usual cartesian coordinates x^μ, $\mu = 1, \ldots, d+1$. Let S^d be the sphere of radius R embedded in this space as the quadric $x^\mu\, x^\mu = R^2$. The hyperspherical harmonics $Y_{\ell \boldsymbol{m}}$ spanning the algebra of smooth functions $\mathcal{C}^\infty(S^d)$ correspond to polynomials which are of degree ℓ in the coordinates x^μ after imposing the equation $x^\mu\, x^\mu = R^2$.

There is an embedding of even-dimensional spheres S^d into $\mathbb{C}P^r$, with $r+1 := 2^{\lfloor \frac{d+1}{2} \rfloor}$ the dimension of the spinor representation of $\mathsf{SO}(d+1)$. We consider the generators γ^μ, $\mu = 1, \ldots, d+1$, of the Clifford algebra[5] $Cl(\mathbb{R}^{d+1})$ satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. If d is even, the spinor representation of $\mathsf{SO}(d+1)$ is irreducible. The relation[6]

$$[\gamma^{\mu\nu} \odot \mathbb{1}_{r+1}, \gamma^\rho \odot \gamma^\rho] = 0 \ , \tag{4.17}$$

where $\gamma^{\mu\nu} := \frac{1}{2}\,[\gamma^\mu, \gamma^\nu]$, together with Schur's lemma implies $\gamma^\rho \odot \gamma^\rho = c\, \mathbb{1}_{r+1} \odot \mathbb{1}_{r+1}$, $c \in \mathbb{C}$, for even d. Using the generators $\gamma^\mu_{\alpha\beta}$ of the Clifford algebra constructed in Appendix A yields $c = 1$, so $\gamma^\rho \odot \gamma^\rho = \mathbb{1}_{r+1} \odot \mathbb{1}_{r+1}$. Therefore, the embedding relation $x^\mu\, x^\mu = R^2$ is satisfied for

$$x^\mu := \frac{R}{|z|^2}\, \bar{z}_\alpha\, \gamma^\mu_{\alpha\beta}\, z_\beta \ , \tag{4.18}$$

which generalizes the usual Jordan-Schwinger transformation. The space of hyper-

[5] A construction of the explicit matrix representation of the Clifford algebras yielding spinor representations is given in Appendix A

[6] Here and in the following, $\odot$ denotes the normalized symmetric tensor product.

spherical harmonics $Y_{\boldsymbol{\ell m}}$ with $\ell \leq k$ is thus spanned by the functions

$$\gamma^{\mu_1}_{\alpha_1\beta_1}\cdots\gamma^{\mu_j}_{\alpha_j\beta_j}\,\delta_{\alpha_{j+1}\beta_{j+1}}\cdots\delta_{\alpha_k\beta_k}\,\bar z_{\alpha_1}\cdots\bar z_{\alpha_k}\,z_{\beta_1}\cdots z_{\beta_k}\ , \tag{4.19}$$

Our embedding $S^d \hookrightarrow \mathbb{C}P^r$ induces an injection $\rho : \mathcal{C}^\infty(S^d) \hookrightarrow \mathcal{C}^\infty(\mathbb{C}P^r)$. Polynomials in the coordinates x^μ restricted to S^d form a dense subset in $\mathcal{C}^\infty(S^d)$ and they are turned into global functions on $\mathbb{C}P^r$ via the substitution (4.18). Moreover, the Fubini-Study metric on $\mathbb{C}P^r$ induces the standard round metric on S^d, with volume form $\mathrm{d}\mu_{S^d}$, which is can be seen as the embedding is manifestly $\mathsf{SO}(d+1)$-invariant. This implies in particular that for a function $f \in \mathcal{C}^\infty(\mathcal{M})$, one has

$$\int_{\mathbb{C}P^r} \mathrm{d}\mu\ \rho(f) = \mathrm{vol}\int_{S^d} \mathrm{d}\mu_{S^d}\ f\ , \tag{4.20}$$

where vol is a constant volume factor. Therefore, the L^2-inner product on $\mathbb{C}P^r$ with respect to the Fubini-Study metric is naturally compatible with the L^2-inner product on S^d with respect to the round metric.

We will obtain odd-dimensional spheres as a reduction of even-dimensional spheres. We reduce S^{2d} to S^{2d-1} by putting $x^{2d+1} = 0$. Let us introduce $s := \frac{r+1}{2}$. Using the inductive construction of the Clifford algebra given in Appendix A, we have $\gamma^{2d+1} = \mathrm{i}^d\, \mathbb{1}_s \otimes \sigma^3$, where the gamma-matrices act on $\mathbb{C}^{r+1} = \mathbb{C}^{2s}$. In complex coordinates, the condition $x^{2d+1} = 0$ thus implies

$$\sum_{\alpha=0}^{s-1} \bar z_\alpha\, z_\alpha - \sum_{\alpha=s}^{2s} \bar z_\alpha\, z_\alpha = 0\ . \tag{4.21}$$

This condition reduces the space $\mathbb{C}P^r$, into which we embedded S^{2d}, to $\mathbb{C}P^{s-1} \times \mathbb{C}P^{s-1}$. In particular, this reduces the embedding $S^4 \hookrightarrow \mathbb{C}P^3$ to $S^3 \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$.

We can further reduce S^{2d-1} to S^{2d-2} by putting $x^{2d} = 0$. In the inductive construction, we have $\gamma^{2d} = \mathbb{1}_s \otimes \sigma^1$, which yields the condition

$$\sum_{\alpha=0}^{s-1} (\bar z_\alpha\, z_{\alpha+s} + \bar z_{\alpha+s}\, z_\alpha) = 0\ . \tag{4.22}$$

This equation is solved by putting $z_{\alpha+s} = \mathrm{i}\, z_\alpha$, which reduces $\mathbb{C}P^{s-1} \times \mathbb{C}P^{s-1}$ to the diagonal subspace $\mathbb{C}P^{s-1}$. It follows from both the reduction as well as the fact that the embedding respects the isometries that (4.20) also holds for odd-dimensional spheres.

4.2.2 Berezin quantization of even-dimensional spheres

Even-dimensional spheres S^d are straightforward to quantize, and we therefore start with them. Our goal is to construct a Hilbert space $\mathscr{H}_k$ together with a quantization map $x^\mu \mapsto \hat{x}^\mu$ taking functions on S^d to endomorphisms of $\mathscr{H}_k$ such that $\hat{x}^\mu\,\hat{x}^\mu = R_F^2\,\mathbb{1}_{\mathscr{H}_k}$, where the "fuzzy radius" R_F will be identified below. We also want to construct the bracket of a d-Lie algebra, such that ideally it satisfies the generalized quantization axiom Q3$'$. For the spheres, this implies that we are looking for a quantization map $x^\mu \mapsto \hat{x}^\mu$ together with a d-Lie bracket satisfying

$$[\hat{x}^{\mu_1}, \ldots, \hat{x}^{\mu_d}] = -\mathrm{i}\,\hbar(k)\,R^{d-1}\,\varepsilon^{\mu_1\ldots\mu_d\mu_{d+1}}\,\hat{x}^{\mu_{d+1}}\ . \tag{4.23}$$

We return to the embedding of S^d into $\mathbb{C}P^r$ and use the Hilbert space $\mathscr{H}_k$ of Berezin-quantized $\mathbb{C}P^r$ with quantum line bundle $L = \mathcal{O}(k)$. Thus $\mathscr{H}_k$ is identified as the k-particle subspace of the Fock space of $r+1$ harmonic oscillators, with

$$\hat{a}^\dagger_{\alpha_1}\cdots\hat{a}^\dagger_{\alpha_k}|0\rangle \in \mathscr{H}_k\ ,\quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta}\ ,\quad \hat{a}_\alpha|0\rangle = 0\ . \tag{4.24}$$

We define the lower Berezin symbol $\sigma_R(\hat{f}\,)$ of an operator $\hat{f} \in \mathsf{End}\,(\mathscr{H}_k)$ by the L^2-projection of the lower Berezin symbol $\sigma(\hat{f}\,) \in \Sigma \subset \mathcal{C}^\infty(\mathbb{C}P^r)$ onto $\Sigma_R \subset \mathcal{C}^\infty(S^d)$. Explicitly, this amounts to introducing the restricted coherent state projector

$$\begin{aligned} P_x^R :=& \sum_{m=0}^{k} x^{\mu_1}\cdots x^{\mu_m}\, k!\,\left(\frac{2}{R}\right)^m \gamma^{\mu_1}_{\alpha_1\beta_1}\cdots\gamma^{\mu_m}_{\alpha_m\beta_m} \\ &\times\ \hat{a}^\dagger_{\alpha_1}\cdots\hat{a}^\dagger_{\alpha_m}\,\hat{a}^\dagger_{\rho_1}\cdots\hat{a}^\dagger_{\rho_{k-m}}|0\rangle\langle 0|\hat{a}_{\beta_1}\cdots\hat{a}_{\beta_m}\,\hat{a}_{\rho_1}\cdots\hat{a}_{\rho_{k-m}} \\ =:& \sum_{m=0}^{k} x^{\mu_1}\cdots x^{\mu_m}\, k!\,\left(\frac{2}{R}\right)^m \gamma^{\mu_1}_{\alpha_1\beta_1}\cdots\gamma^{\mu_m}_{\alpha_m\beta_m}\,|\alpha_1\ldots\alpha_m, k\rangle\!\bullet\!\langle k, \beta_1\ldots\beta_m|\ , \end{aligned} \tag{4.25}$$

and with eq. (A.4) of Appendix A we conclude that $P_x^R\,P_x^R = P_x^R$. The coordinates

x^μ can be substituted again by (4.18) to obtain an expression for P_x^R in terms of homogeneous coordinates on $\mathbb{C}P^r$. The restriction of the lower Berezin symbol now reads

$$\sigma_R(\hat{f}\,)(x) := \operatorname{tr}(P_x^R\,\hat{f}\,)\ . \tag{4.26}$$

The map $\sigma_R\ :\ \mathsf{End}\,(\mathscr{H}_k)\ \to\ \Sigma_R$ is no longer injective due to the projection involved from $\Sigma \subset \mathcal{C}^\infty(\mathbb{C}P^r)$ to Σ_R. However, since $\Sigma_R \subset \Sigma$, we can use the inverse of the unrestricted Berezin symbol σ to define a quantization map

$$Q\ :\ \Sigma_R\ \longrightarrow\ \mathsf{End}\,(\mathscr{H}_k)\ ,\qquad f\ \longmapsto\ \sigma^{-1}(f)\ . \tag{4.27}$$

For the coordinate functions, this quantization yields

$$x^\mu\ \longmapsto\ \hat{x}^\mu := Q(x^\mu) = \frac{R}{k!}\,\gamma^\mu_{\alpha\beta}\,|\alpha,k\rangle\!\bullet\!\langle k,\beta|\ . \tag{4.28}$$

The operators $\hat{x}^\mu$ generate all of $\mathsf{End}\,(\mathscr{H}_k)$. This can be shown in the following way. We first note that totally antisymmetric products of $d-1$ of the operators $\hat{x}^\mu$ span the space of all operators of the form $|\alpha_1,k\rangle\!\bullet\!\langle k,\beta_1|$. A product of two such antisymmetric products decomposes into operators of the form $|\alpha_1\alpha_2,k\rangle\!\bullet\!\langle k,\beta_1\beta_2|$ and $|\alpha_1,k\rangle\!\bullet\!\langle k,\beta_1|$. In this way, we can inductively construct all of $\mathsf{End}\,(\mathscr{H}_k)$ by noncommutative polynomials in the operators $\hat{x}^\mu$ of maximal degree $k\,(d-1)$. This implies in particular that the noncommutative polynomials of degree $k\,(d-1)$ form an algebra. This agrees with the known result for the fuzzy sphere, where the algebra $\mathsf{End}\,(\mathscr{H}_k)$ consists of noncommutative polynomials of degree k.

This quantization satisfies the quantization axioms Q1, Q2, and Q4′, as these properties trivially survive the projection. We will come back to the d-Lie algebra structure and the correspondence principle Q3′ shortly.

4.2.3 Toeplitz quantization of spheres

Recall that the embedding (4.18) induces an injection $\rho : \mathcal{C}^\infty(S^d) \hookrightarrow \mathcal{C}^\infty(\mathbb{C}P^r)$. We can therefore identify the Toeplitz quantization of a function $f \in \mathcal{C}^\infty(S^d)$ with the Toeplitz quantization of $\rho(f) \in \mathcal{C}^\infty(\mathbb{C}P^r)$. This means, in particular, that the convergence theorems of [24] hold on S^d as well. Recall that for $\mathcal{M} = \mathbb{C}P^r$ we have

$$\lim_{k\to\infty} \left\| T_{\mathcal{O}(k)}(f) \right\|_{HS} = \|f\|_{L^2} \tag{4.29}$$

together with (3.21). On S^d, we consider the Poisson structure which is obtained via the pull-back of the symplectic form ω along the embedding $S^d \hookrightarrow \mathbb{C}P^r$. It follows that the Poisson algebra thus obtained on S^d is embedded in the Poisson algebra on $\mathbb{C}P^r$, and the estimates (3.21) and (4.29) for S^d are just restrictions of the corresponding estimates on $\mathbb{C}P^r$.

4.2.4 d-Lie algebra structure

As discussed in §4.1, we will use the d-Lie bracket constructed out of a lift of the truncation of the Nambu-Poisson structure on Σ_R. For this, note that Σ_R consists of polynomials in the x^μ of maximal degree k, and that the components of the Nambu-Poisson tensor are homogeneous polynomials of degree 1. We can therefore endow Σ_R with the truncated Nambu-Poisson bracket $\{-,\dots,-\}_k$. Furthermore, we lift this bracket to $\mathsf{End}\,(\mathscr{H})$ as described in §4.1 The resulting d-Lie bracket satisfies the correspondence principle by definition. Note that it vanishes on operators $\hat{A} \in \mathsf{End}\,(\mathscr{H})$ with vanishing Berezin symbol $\sigma_R(\hat{A})$.

Let us now examine how this bracket is related to the totally antisymmetric operator product (4.2). First, note that

$$[\hat{x}^1, \dots, \hat{x}^d] = -\mathrm{i}\hbar \hat{x}^{d+1} \ . \tag{4.30}$$

The antisymmetric product of two operators is given by

$$\begin{aligned}\hat{x}^\mu\, \hat{x}^\nu &= \left(\frac{R}{k!}\right)^2 \left(\gamma^\mu_{\alpha\beta}\,|\alpha,k\rangle\!\langle k,\beta|\right)\left(\gamma^\nu_{\gamma\delta}\,|\gamma,k\rangle\!\langle k,\delta|\right)\\ &= \frac{R^2}{k!\,k}\,(\gamma^\mu\,\gamma^\nu)_{\alpha\beta}\,|\alpha,k\rangle\!\langle k,\beta| + \frac{R^2\,(k-1)}{k!\,k}\,\gamma^\mu_{\alpha_1\beta_1}\,\gamma^\nu_{\alpha_2\beta_2}\,|\alpha_1\alpha_2,k\rangle\!\langle k,\beta_1\beta_2| \ .\end{aligned} \tag{4.31}$$

Due to $\mathsf{SO}(d+1)$-invariance of the construction, we can focus on the expression

$[\hat{x}^1, \dots, \hat{x}^d]$. Using (4.31) we compute

$$\sum_{\mu_i=1}^{d} \varepsilon^{\mu_1 \cdots \mu_d}\, \hat{x}^{\mu_1} \cdots \hat{x}^{\mu_d} = \sum_{\mu_i=1}^{d} \varepsilon^{\mu_1 \cdots \mu_d}\, \hat{x}^{\mu_1 \mu_2} \cdots \hat{x}^{\mu_{d-1} \mu_d} \ , \tag{4.32}$$

where we introduced $\hat{x}^{\mu\nu} := \frac{1}{2}\,[\hat{x}^\mu, \hat{x}^\nu] = \frac{R^2}{k!\,k}\,(\gamma^{\mu\nu})_{\alpha\beta}\,|\alpha, k\rangle\!\bullet\!\langle k, \beta|$. From (4.31), we notice that for large k the dominant contribution to the above d-bracket is given by

$$\sum_{\mu_i=1}^{d} \varepsilon^{\mu_1 \cdots \mu_d} \left(\frac{R^2}{k!\,k}\right)^{\frac{d}{2}} \big((k-1)\,(k-1)!\big)^{\frac{d}{2}-1} \\ \times\ (\gamma^{\mu_1\mu_2})_{\alpha_1\alpha_2} \cdots (\gamma^{\mu_{d-1}\mu_d})_{\alpha_{d-1}\alpha_d}\,|\alpha_1\alpha_3 \dots \alpha_{d-1}, k\rangle\!\bullet\!\langle k, \alpha_2\alpha_4 \dots \alpha_d| \ . \tag{4.33}$$

Thus we have to study the symmetric tensor product

$$\sum_{\mu_i=1}^{d} \varepsilon^{\mu_1 \cdots \mu_d}\, \gamma^{\mu_1\mu_2} \odot \cdots \odot \gamma^{\mu_{d-1}\mu_d} \ , \tag{4.34}$$

and the desired outcome would be proportional to $\gamma^{d+1} \odot \mathbb{1}_{r+1}$ (and hence the full result to $\hat{x}^{d+1}$).

Before we can evaluate this product, we need the following result. Recall that we showed the generators γ^μ of $Cl(\mathbb{R}^{d+1})$ obey

$$\gamma^\mu \odot \gamma^\mu = \mathbb{1}_{r+1} \odot \mathbb{1}_{r+1} \tag{4.35}$$

in an irreducible representation of $\mathsf{SO}(d+1)$. Using this result, we have

$$-\sum_{\mu,\nu=1}^{d} \gamma^{\mu\nu} \odot \gamma^{\mu\nu} = (d-2)\,\mathbb{1}_{r+1} \odot \mathbb{1}_{r+1} + 2\,\gamma_{\rm ch} \odot \gamma_{\rm ch} \ . \tag{4.36}$$

We also find

$$\sum_{\mu,\nu=1}^{d} \gamma_{\rm ch}\, \gamma^{\mu\nu} \odot \gamma^{\mu\nu} = -d\, \gamma_{\rm ch} \odot \mathbb{1}_{r+1} \ ,$$

$$(\underbrace{\gamma_{\rm ch} \odot \mathbb{1}_{r+1} \odot \ldots \odot \mathbb{1}_{r+1}}_{\ell})^2 = \frac{1}{\ell}\big(\mathbb{1}_{r+1} \odot \cdots \odot \mathbb{1}_{r+1} + (\ell-1)\, \gamma_{\rm ch} \odot \gamma_{\rm ch}$$
$$\odot\, \mathbb{1}_{r+1} \odot \cdots \odot \mathbb{1}_{r+1}\big) \ . \tag{4.37}$$

We can now evaluate this product for various d. For example, for $d = 4$ we have

$$\sum_{\mu_i=1}^{4} \varepsilon^{\mu_1\mu_2\mu_3\mu_4}\, \gamma^{\mu_1\mu_2} \odot \gamma^{\mu_3\mu_4} = -\sum_{\mu_i=1}^{4} \gamma^5\, \gamma^{\mu_3\mu_4} \odot \gamma^{\mu_3\mu_4} = 4\gamma^5 \odot \mathbb{1}_4 \ . \tag{4.38}$$

Including all orders in k we find

$$[\hat{x}^{\mu_1}, \hat{x}^{\mu_2}, \hat{x}^{\mu_3}, \hat{x}^{\mu_4}] = 8R^3\, \frac{k+2}{k^3}\, \varepsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5}\, \hat{x}^{\mu_5} \ . \tag{4.39}$$

This agreement between the totally antisymmetric operator product and the d-Lie bracket on $\mathsf{End}\,(\mathscr{H})$ breaks down, however, for polynomials of higher degree: While the latter d-ary product satisfies the fundamental identity for arbitrary operators, the former does not. Also, performing the same calculation for $d = 6$, one concludes that both d-ary products do not agree here even for linear polynomials. The same feature is expected to hold for higher d. Summarizing, the d-Lie bracket agrees with the totally antisymmetric operator product for linear polynomials and $d \leq 4$.

4.2.5 Commutative limit

A nice feature of the rather explicit quantization prescription given above is that the commutative limit is intuitively very clear. Consider again the product (4.31). While the first term receives contributions from both symmetric and antisymmetric parts in μ and ν, the second term is symmetric. The first term is also relatively suppressed by a factor of $k-1$. It therefore vanishes in the limit $k \to \infty$, rendering the coordinate algebra commutative. Analogously, one can show that the nonassociativity for odd-dimensional spheres (see below) vanishes in the limit, cf. [20].

The radius of the fuzzy spheres is defined through $\hat{x}^\mu\,\hat{x}^\mu = R_F^2\,\mathbb{1}_{\mathscr{H}_k}$. By direct computation, we find

$$\hat{x}^\mu\,\hat{x}^\mu = R^2\left(1+\frac{d}{k}\right)\mathbb{1}_{\mathscr{H}_k}\ , \quad \mathbb{1}_{\mathscr{H}_k} = \frac{1}{k!}\,|k\rangle\!\!\bullet\!\!\langle k|\ . \tag{4.40}$$

In the limit $k\to\infty$, the fuzzy radius $R_F = \sqrt{1+\frac{d}{k}}\,R$ approaches the classical radius of S^d.

4.2.6 Quantized isometries

We now examine how the $\mathsf{SO}(d+1)$ isometries of the sphere translate to quantum level. Recall first that on Σ_R, the rotations act according to

$$M^{\mu\nu}\rhd f := x^\mu\partial^\nu f - x^\nu\partial^\mu f = \bar{z}^\alpha\,\gamma^{\mu\nu}_{\alpha\beta}\,\frac{\partial}{\partial\bar{z}^\beta}f - z^\alpha\,\gamma^{\mu\nu}_{\alpha\beta}\,\frac{\partial}{\partial z^\beta}f\ . \tag{4.41}$$

This action is contained in the associated Lie algebra of the d-Lie algebra Σ_R. Note that the lift of this structure to $\mathsf{End}\,(\mathscr{H})$ produces the correct action of $\mathsf{SO}(d+1)$ only for operators $\hat{A}$, for which $(\sigma^{-1}\circ\sigma)(\hat{A}) = \hat{A}$. Operators $\hat{A}$ for which $\sigma(\hat{A}) = 0$ are obviously left invariant under the action of $\mathfrak{g}_{\mathsf{End}\,(\mathscr{H})}$.

Note also that the associated Lie algebra of the d-Lie algebra Σ_R contains a subset of the diffeomorphisms, as well, which is similarly translated appropriately only to some operators in $\mathsf{End}\,(\mathscr{H})$.

4.2.7 Odd-dimensional spheres

The quantization of odd-dimensional spheres is slightly more subtle. We want to obtain the odd spheres S^{2d-1} from the even spheres S^{2d} by some kind of reduction process. A naive approach would be to translate the constraint $x^{2d+1} = 0$ to the operator equation $\hat{x}^{2d+1}|\mu\rangle = 0$ for all $|\mu\rangle \in \mathscr{H}_k$. This approach does not work,[7] as the condition is not invariant under the action of operators corresponding to other coordinates. The underlying reason is that the Hilbert space $\mathscr{H}_k$ corresponds to a subring of the homogeneous coordinate ring of $\mathbb{C}P^r$, and imposing operator

[7] Nor does the slight generalization $\hat{x}^{2d+1}\,\hat{x}^{2d+1}|\mu\rangle = 0$.

conditions on the Hilbert space corresponds to factoring by a holomorphic ideal, cf. [80]. The condition $x^{2d+1} = 0$, however, is not holomorphic.

The main problem here is that although we still have $[\gamma^{\mu\nu} \odot \mathbb{1}_{r+1}, \gamma^\rho \odot \gamma^\rho] = 0$, Schur's lemma does not apply as the representation is reducible. It is therefore necessary to restrict to a maximal set of irreducible representations on which $\hat{x}^\mu\, \hat{x}^\mu = R_F^2\, \mathbb{1}_{\mathscr{H}_k}$. The construction [48, 79] is rather technical, and so we just comment on the interpretation in terms of oscillators.

For simplicity, consider $S^3 \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \subset \mathbb{C}P^3$. We split the annihilation and creation operators of the harmonic oscillators appearing in the quantization of $\mathbb{C}P^3$, $\hat{a}_\alpha, \hat{a}_\alpha^\dagger$, $\alpha = 0,1,2,3$, into two groups of harmonic oscillators appearing in the quantization of $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\hat{b}_\beta, \hat{b}_\beta^\dagger$ and $\hat{c}_\beta, \hat{c}_\beta^\dagger$, $\beta = 0,1$. The reduced Hilbert space is spanned by the two classes of vectors

$$\hat{b}^\dagger_{\beta_1} \cdots \hat{b}^\dagger_{\beta_{s-1}}\, \hat{c}^\dagger_{\beta_s} \cdots \hat{c}^\dagger_{\beta_k} |0\rangle \in \mathcal{V}_{k,s-1}\ , \quad \hat{b}^\dagger_{\beta_1} \cdots \hat{b}^\dagger_{\beta_s}\, \hat{c}^\dagger_{\beta_{s+1}} \cdots \hat{c}^\dagger_{\beta_k} |0\rangle \in \mathcal{V}_{k,s}\ , \tag{4.42}$$

where $s = \frac{k+1}{2}$ and k is restricted to odd values. The operator product is always followed by a projection back onto this Hilbert space, which renders it nonassociative. On the irreducible representations $\mathcal{V}_{k,s}$ and $\mathcal{V}_{k,s-1}$ of $\mathsf{Spin}(4)$, the operator product $\hat{x}^\mu\, \hat{x}^\mu$ is indeed proportional to the identity operator. We need to recall that

$$\mathcal{V}_k := \bigoplus_{s=0}^{k} \mathcal{V}_{k,s} \tag{4.43}$$

is an irreducible representation of $\mathsf{Spin}(5)$. Since $(\gamma^\mu)^2 \propto \mathbb{1}_4$, it suffices to examine the eigenvalues of the operator $\mathcal{O} := \gamma^5 \odot \gamma^5 \odot \mathbb{1}_4 \odot \cdots \odot \mathbb{1}_4$, where $\gamma^5 = -\gamma^1\, \gamma^2\, \gamma^3\, \gamma^4$. We can show that the eigenvalues of $\mathcal{O}$ in the representations $\mathcal{V}_{k,s}$ and $\mathcal{V}_{k,k-s}$ are identical. Consider the quantization of $\mathbb{C}P^r$, $r = 2n-1$ with creation and annihilation operators satisfying the Heisenberg-Weyl algebra $[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}$, $\alpha, \beta = 1, \dots, 2n$. The vectors $\hat{a}_\alpha^\dagger|0\rangle$ generate the reducible spinor representation V of $\mathsf{SO}(d+1)$, for d odd. The k-fold totally symmetrized tensor product representation $V^{\odot k}$ is then generated by $\hat{a}^\dagger_{\alpha_1} \cdots \hat{a}^\dagger_{\alpha_k}|0\rangle$. The spinor representation V splits into the direct sum of two irreducible representations, $V = V_+ \oplus V_-$, where $V_\pm$ are the ± 1 eigenspaces of the chirality operator $\gamma_{\rm ch}$. The totally symmetrized tensor product representations then split according to

$$\mathcal{V}_k := V^{\odot k} = \bigoplus_{s=0}^{k} \left(V_+^{\odot s} \oplus V_-^{\odot(k-s)}\right) =: \bigoplus_{s=0}^{k} \mathcal{V}_{k,s} \ . \tag{4.44}$$

We now calculate the action of the operator $\mathcal{O} := \gamma_{\rm ch} \odot \gamma_{\rm ch} \odot \mathbb{1}_{r+1} \odot \cdots \odot \mathbb{1}_{r+1}$ on the subspace $\mathcal{V}_{k,s}$. To do this, we first split the creation and annihilation operators into two groups $(\hat{b}_i, \hat{b}_i^\dagger) = (\hat{a}_i, \hat{a}_i^\dagger)$ and $(\hat{c}_i, \hat{c}_i^\dagger) = (\hat{a}_{i+n}, \hat{a}_{i+n}^\dagger)$, where $i = 1, \ldots, n$. Vectors $|\vec{p}, s\rangle \in \mathcal{V}_{k,s}$ then take the form $\hat{b}_{i_1}^\dagger \cdots \hat{b}_{i_s}^\dagger \, \hat{c}_{i_{s+1}}^\dagger \cdots \hat{c}_{i_k}^\dagger |0\rangle$ and the operator $\mathcal{O}$ acts according to

$$\mathcal{O}|\vec{p}, s\rangle = \left(\hat{b}_{i_1}^\dagger \, \hat{b}_{i_2}^\dagger |k-2\rangle\!\bullet\!\langle k-2| \hat{b}_{i_1} \, \hat{b}_{i_2} + \hat{c}_{i_1}^\dagger \, \hat{c}_{i_2}^\dagger |k-2\rangle\!\bullet\!\langle k-2| \hat{c}_{i_1} \, \hat{c}_{i_2}\right)|\vec{p}, s\rangle \ . \tag{4.45}$$

For a vector $|\vec{p}, s\rangle \in \mathcal{V}_{k,s}$ with $k \geq 3$, we can verify that $\mathcal{O}|\vec{p}, s\rangle \propto |\vec{p}, s\rangle$, and that the eigenvalue of $\mathcal{O}$ is identical in the representations $\mathcal{V}_{k,s}$ and $\mathcal{V}_{k,k-s}$.

Moreover, on S^3, the totally antisymmetric operator product which agrees with the 3-Lie bracket at linear level should actually be modified to read as

$$[\hat{x}^\mu, \hat{x}^\nu, \hat{x}^\kappa] := -[\hat{x}^\mu, \hat{x}^\nu, \hat{x}^\kappa, \hat{x}^5] = \mathrm{i}\, \hbar(k)\, R^2\, \varepsilon^{\mu\nu\kappa\lambda}\, \hat{x}^\lambda \ , \tag{4.46}$$

which has been suggested in [14]. Because of these technicalities, we have focused our discussion on even-dimensional spheres with the extensions to odd-dimensional spheres being technical, but obvious.

4.2.8 Comparison to other fuzzy spheres

Let us now put our quantization prescription into the context of previous constructions of fuzzy spheres. First, the idea of embedding spheres into complex projective space has been used previously to construct fuzzy spheres. In particular, the fuzzy 4-sphere has been constructed from the fact that $\mathbb{C}P^3$ is a sphere bundle over S^4, $S^2 \hookrightarrow \mathbb{C}P^3 \to S^4$, cf. [70, 38, 1]. Second, a purely group theoretic approach was pursued in [48, 79].

The Hilbert space in both approaches agrees with the Hilbert space we found from a generalization of Berezin-Toeplitz quantization. The point at which the approaches differ is in the handling of *radial fuzziness*. As we showed above, the algebra of quantum operators $\hat{x}^\mu$ exhausts all of $\mathsf{End}\,(\mathscr{H}_k)$. Therefore the algebra of quantum operators is isomorphic to the algebra of lower Berezin symbols of the complex projective space $\mathbb{C}P^r$ used in the embedding $S^d \hookrightarrow \mathbb{C}P^r$, and not to the corresponding algebra for S^d. This means that at quantum level the multiplication of two quantized functions yields modes which should be interpreted as transverse or radial to the embedding $S^d \hookrightarrow \mathbb{C}P^r$.

There are two solutions to this problem in the literature. In [47, 48, 79] it was suggested to project out these modes after operator multiplication, which yields a nonassociative algebra. In [70], where fuzzy S^4 was used as a regulator for quantum field theories, it was suggested to modify the Laplace operator such that the unwanted modes are dynamically punished by a mass term, i.e. their excitation is suppressed.

As eliminating the radial modes by projecting them out after multiplication immediately yields inconsistencies in the interpretation of solutions to the Basu-Harvey equation in terms of fuzzy 3-spheres (see e.g. [77]), we insisted on keeping these modes. This allowed us to interpret fuzzy S^3 and fuzzy S^4 as quantizations of Nambu-Poisson manifolds under the assumption of a reasonable correspondence principle.

Note also that the d-Lie bracket vanishes if one of the arguments is a purely radial mode. Moreover, the d-Lie bracket always yields operators which are quantizations of a function on S^d. That is, if d-Lie brackets are exclusively used and the binary operator product is avoided, the radial modes are naturally projected out.

4.3 Quantization of hyperboloids

Our approach to quantizing spheres S^d was based on properties of the euclidean Clifford algebra $Cl(\mathbb{R}^{d+1})$. A natural question at this stage is whether it is possible to extend our quantization procedure using Clifford algebras for indefinite metrics.

The answer is affirmative if we relax our quantization axiom Q1 and allow for non-unitary representations.

4.3.1 Classical hyperboloids

Recall that a space-like direction is turned into a time-like one by multiplying the Clifford algebra generator γ^μ corresponding to this direction by i. In this way we obtain the spinor representation of the isometry group of the space $\mathbb{R}^{p,q}$ of dimension $d+1 := p+q$. Into this space we can embed the d-dimensional hyperbolic space $H^{p,q}$ as the quadric

$$x^\mu\, x^\nu\, \eta_{\mu\nu} := (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - \cdots - (x^{p+q})^2 = r\ , \tag{4.47}$$

where $\eta_{\mu\nu}$ is the metric on $\mathbb{R}^{p,q}$. We will always consider the case $r > 0$. This restriction eliminates only cones, as by multiplying the embedding equation by -1 one exchanges the roles of (p,q) and inverts the sign of the curvature. The hyperboloid $H^{p,q}$ corresponds to the coset $\mathsf{SO}(p,q)/\mathsf{SO}(p-1,q)$, and $H^{d+1,0} = S^d$. For $p = 1$, the hyperboloid splits into two sheets.

The treatment of hyperboloids proceeds analogously to the analysis of spheres. An embedding into $\mathbb{R}^{p+q}$ is obtained by substituting trigonometric functions with hyperbolic functions in (2.10), as appropriate for angles in a plane of signature $(1,1)$, and setting $R = \sqrt{r}$. The same substitution applies to the volume element (2.11). The natural Nambu brackets differ from those on the sphere only through the volume element that one divides by, and we thus define the Nambu bracket on $H^{p,q}$ by

$$\{f_1, \ldots, f_d\} := \frac{\varepsilon^{i_1\ldots i_d}}{\mathrm{vol}_\varphi}\,\frac{\partial f_1}{\partial \varphi^{i_1}} \cdots \frac{\partial f_d}{\partial \varphi^{i_d}}\ , \tag{4.48}$$

which translates into the Nambu bracket of the embedding coordinates

$$\{x^{\mu_1}, \ldots, x^{\mu_d}\} = R^{d-1}\, \varepsilon^{\mu_1\ldots\mu_d}{}_{\mu_{d+1}}\, x^{\mu_{d+1}}\ . \tag{4.49}$$

Here we have defined $\varepsilon^{\mu_1\ldots\mu_d}{}_{\mu_{d+1}} := \varepsilon^{\mu_1\ldots\mu_d\nu}\,\eta_{\nu\mu_{d+1}}$.

4.3.2 Quantization of $H^{p,q}$

As we are concerned only with the kinematical problem of quantization, which we presume to lead to an algebra of quantized functions approximating the algebra of functions on a space in a well-defined manner, we can choose to relax the quantization axiom Q1 by mapping real functions to non-hermitian operators and thus to work with non-unitary representations. This was done in [40] in order to construct a fuzzy AdS_2. This approach is a straightforward generalization of the description of quantum spheres given in §4.2, and it also fits into the deformation quantization prescription of §4.1 For a quantization of a hyperboloid using unitary representations, see e.g. [11].

To allow for an indefinite metric in the Clifford algebra, we have to allow for non-hermitian generators.[8] To quantize the hyperboloid $H^{p,q}$ embedded in $\mathbb{R}^{p,q}$, we thus multiply the generators γ^μ along the time-like directions $\mu = p+1, \ldots, p+q$ by a factor of i and follow the same steps as in the quantization of the sphere S^{p+q-1}. The factors of i guarantee that the equation $\hat{x}^\mu\, \hat{x}^\nu\, \eta_{\mu\nu} = R_F^2\, \mathbb{1}_{\mathscr{H}_k}$ is satisfied for the indefinite metric $\eta_{\mu\nu}$. We introduce again the d-Lie algebra bracket by the lift of the truncated Nambu-Poisson structure on the set of lower Berezin symbols to the operator algebra. It is only for $d \leq 4$ that this bracket agrees with the totally antisymmetric operator product

$$[\hat{x}^{\mu_1}, \ldots, \hat{x}^{\mu_d}] = -\mathrm{i}\, \hbar\, R^{d-1}\, \varepsilon^{\mu_1 \ldots \mu_d}{}_{\mu_{d+1}}\, \hat{x}^{\mu_{d+1}} \tag{4.50}$$

at linear level. This bracket on its own forms the d-Lie algebra $A_{p,q}$. Recall that every simple d-Lie algebra over $\mathbb{R}$ is isomorphic to a $d+1$-dimensional d-Lie algebra $A_{p,q}$, for some (p, q) with $d = p + q - 1$, cf. e.g. [41].

As the technical details of the construction (e.g. the restriction to certain irreducible representations for odd-dimensional hyperboloids) work exactly as for spheres, we refrain from going into further details. We stress, however, that while the quantization of spheres is intimately related to harmonic analysis in the sense that $\mathsf{End}\,(\mathscr{H}_k)$ was related to certain hyperspherical harmonics, this is not the case for

[8] Recall that the square of a hermitian matrix always has positive eigenvalues.

the quantum hyperboloids. Thus their quantization is somewhat different in spirit from the standard examples of noncommutative spaces, such as the noncommutative torus.

Strictly speaking, we actually quantize the one-point compactifications of the hyperboloids, as there is still an embedding of this compactified hyperboloid into the complex projective space appearing in the construction. Here a point $\varphi = (\varphi^1, \ldots, \varphi^d)$ on $H^{p,q}$ is mapped to a point φ' on the sphere S^d with the same angular coordinates and subsequently embedded into $\mathbb{C}P^r$ via the Jordan-Schwinger transform (4.8). In this embedding, the point corresponding to infinity on the hyperboloid is also mapped to a point of S^d. It is in this sense that we quantize the compactifications of the hyperboloids.

4.4 Quantization of $\mathbb{R}^n$ by foliations

As a final set of examples, we will now look at the implications of the quantization axioms for the quantization of $\mathbb{R}^n$. The relevant n-Lie algebras at linear level correspond to Nambu-Heisenberg n-Lie algebras, which in turn suggest a quantization of $\mathbb{R}^n$ in terms of foliations by fuzzy spheres S_F^{n-1} or noncommutative hyperplanes $\mathbb{R}_\theta^{n-1}$. We also briefly study an extension of this quantization by adding an extra outer automorphism to the Nambu-Heisenberg n-Lie algebra, which describes a twisting of the n-Lie algebra and a dimensional oxidation of the quantization of $\mathbb{R}^n$.

4.4.1 Nambu-Poisson structures on $\mathbb{R}^n$ and Nambu-Heisenberg n-Lie algebras

The natural Nambu n-bracket on $\mathbb{R}^n$ is defined by the linear extension (via the generalized Leibniz rule) and completion (with respect to the canonical L^2-norm) of the bracket

$$\{x^{\mu_1}, \ldots, x^{\mu_n}\} = \varepsilon^{\mu_1 \cdots \mu_n} \ . \tag{4.51}$$

This Nambu-Poisson structure is naturally $\mathsf{SO}(n)$-invariant. Additionally, we can impose further Nambu-Poisson structures on $\mathbb{R}^n$ with Nambu $n-1$-brackets. The

$\mathsf{SO}(n)$ symmetry suggests to add the Nambu-Poisson structure of a foliation of $\mathbb{R}^n$ by spheres[9] S^{n-1}, with bracket

$$\{x^{\mu_1}, \dots, x^{\mu_{n-1}}\} = R^{n-2}\, \varepsilon^{\mu_1 \dots \mu_{n-1} \mu_n}\, x^{\mu_n} \ . \tag{4.52}$$

Alternatively, one could break the $\mathsf{SO}(n)$ invariance to $\mathsf{SO}(n-1)$ and introduce the Nambu-Poisson structure of a foliation of $\mathbb{R}^n$ by hyperplanes $\mathbb{R}^{n-1}$, with bracket

$$\left\{x^{\check{\mu}_1}, \dots, x^{\check{\mu}_{n-1}}\right\} = \varepsilon^{\check{\mu}_1 \dots \check{\mu}_{n-1}} \ , \quad \check{\mu}_i = 1, \dots, n-1 \ . \tag{4.53}$$

In the latter case, we can continue and introduce additionally a Nambu-Poisson structure with a Nambu $n-2$-bracket, and so on. We denote the space $\mathbb{R}^n$ endowed with $k \leq n-2$ successive hyperplane foliations and one spherical foliation by $\mathbb{R}^n_k$. In the case $k = n-2$ there is no spherical foliation, while for $k = 0$ there is only the spherical foliation.

The components of the Nambu-Poisson tensor are constants, so that the truncation of the Nambu-Poisson structure as presented in §2.1.1 unfortunately does not work here. We will therefore restrict to an n-Lie algebra structure which is non-trivial only at linear level and there agrees with the totally antisymmetric operator product. Correspondingly, the quantization axiom Q3′ can only be satisfied at linear level. Thus, the Nambu-Poisson structure (4.51) has to turn under quantization into the n-Lie algebra $\mathcal{A}_{\mathrm{NH}}$ with bracket

$$[\hat{x}^{\mu_1}, \dots, \hat{x}^{\mu_n}] = -\mathrm{i}\,\hbar\, \varepsilon^{\mu_1 \dots \mu_n}\, \mathbb{1} \ , \tag{4.54}$$

where the vector space $\mathcal{A}_{\mathrm{NH}}$ is spanned by the operators $\hat{x}^\mu$, $\mu = 1, \dots, n$, and $\mathbb{1}$. This algebra is called the *Nambu-Heisenberg n-Lie algebra*. The nested foliations yield additional $n-1$-Lie algebra structures on $\mathcal{A}_{\mathrm{NH}}$. We will study these quantizations in the following, starting from the quantizations of $\mathbb{R}^3_0$ and $\mathbb{R}^3_1$.

[9]In the case of $\mathbb{R}^{p,q}$, one would instead use the hyperboloids $H^{p,q}$.

4.4.2 Quantization of $\mathbb{R}^3_0$ and $\mathbb{R}^3_1$

The 3-Lie algebra $\mathcal{A}_{\rm NH}$ was examined in the original paper [75], as well as in [85]. Recall that it has the defining relation (2.27). This relation is a consistency constraint for a quantization of both $\mathbb{R}^3_0$ and $\mathbb{R}^3_1$ according to our generalized quantization axioms.

To realize the quantization map on the endomorphism algebra of some Hilbert space $\mathscr{H}$, we assume that the generator $\mathbb{1}$ appearing on the right-hand side of (2.27) is central in this algebra and acts on vectors of the Hilbert space $\mathscr{H}$ as multiplication by a complex number. This implies that its commutator with any other endomorphism vanishes. From the definition of the 3-bracket as a totally antisymmetrized operator product,

$$[\hat{A},\hat{B},\hat{C}] := \begin{cases} \hat{A}\,[\hat{B},\hat{C}] + \hat{B}\,[\hat{C},\hat{A}] + \hat{C}\,[\hat{A},\hat{B}] & \text{for} \quad \hat{A},\hat{B},\hat{C} \in \operatorname{span}(\hat{x},\hat{y},\hat{z},\mathbb{1}) \\ 0 & \text{else} \end{cases} , \tag{4.55}$$

it is clear that a central element of the 2-Lie bracket will not, in general, be a central element in the 3-Lie algebra. Thus we will have the relations

$$[\mathbb{1},\hat{A},\hat{B}] = \alpha\,[\hat{A},\hat{B}] \ , \quad \alpha \in \mathbb{C}^\times \tag{4.56}$$

for all $\hat{A},\hat{B}$, rather than $[\mathbb{1},\hat{A},\hat{B}] = 0$.

The possibilities of realizing the relation (2.27) as a totally antisymmetric operator product have been listed in [75]. Nambu employs the Lie algebras of $\mathsf{SU}(2)$, $\mathsf{SO}(1,2) \cong \mathsf{SL}(2,\mathbb{R})$, the euclidean group in two dimensions, and the galilean group in one dimension. Here we restrict ourselves to the three-dimensional cases. We will show below that the first three cases correspond to quantizations of $\mathbb{R}^3_0$, $\mathbb{R}^{1,2}_0$, and $\mathbb{R}^3_1$, respectively.

$\mathbb{R}^3_0$

In the first case of $\mathsf{SU}(2)$, the Lie algebra yielding (2.27) corresponds to the coordinate algebra of the fuzzy sphere S^2_F. The radial restriction $\hat{x}^\mu\,\hat{x}^\mu = \rho\,\mathbb{1}_{\mathscr{H}}$ for a constant $\rho \in \mathbb{C}^\times$, however, is missing. We thus obtain a foliation of $\mathbb{R}^3$ by fuzzy spheres. This space is usually denoted $\mathbb{R}^3_\lambda$ in the literature [50, 15]. Recall that on a fuzzy sphere built on the Hilbert space $\mathscr{H}_k = H^0(\mathbb{C}P^1, \mathcal{O}(k))$, the 3-bracket is given by

$$[\hat{x}^1, \hat{x}^2, \hat{x}^3] = \left(\frac{R}{k!}\right)^3 \big((k-1)!\big)^2\, k\, (\varepsilon^{\mu\nu\kappa}\, \sigma^\mu\, \sigma^\nu\, \sigma^\kappa)_{\alpha\beta}\, |\alpha, k\rangle\langle k, \beta| = -\mathrm{i}\, \frac{6R^3}{k}\, \mathbb{1}_{\mathscr{H}_k}\ , \quad (4.57)$$

and the fuzzy radius is $R_F = R_{F,k} := R\sqrt{1+\frac{2}{k}}$. As $R^2_{F,k}\,\mathbb{1}_{\mathscr{H}_k} = \hat{x}^\mu\,\hat{x}^\mu$ is not fixed, the relation (2.27) admits fuzzy spheres of various radii. For given deformation parameter $\hbar$, we have $\hbar = \frac{6R^3}{k}$ from (4.57) and consequently a quantization of the radius of the fuzzy sphere

$$R_{F,k} = \sqrt{1+\frac{2}{k}}\,\sqrt[3]{\frac{\hbar\, k}{6}} \quad (4.58)$$

built on the Hilbert space $\mathscr{H}_k$.

We now introduce the Hilbert space $\mathscr{H} := \bigoplus_{k\in\mathbb{N}} \mathscr{H}_k$ together with the algebra of "quantum functions" $\mathcal{A} := \bigoplus_{k\in\mathbb{N}} \mathsf{End}\,(\mathscr{H}_k)$. This corresponds to a "discrete foliation" of $\mathbb{R}^3$ by fuzzy spheres with radii $R_{F,k}$. The quantization of a polynomial in the coordinates x^μ corresponding to a function on $\mathbb{R}^3$ is given by a quantization of this coordinate function on each fuzzy sphere. The 3-bracket is non-vanishing only on those elements of $\mathcal{A}$ which are all at most linear elements of the same subalgebra $\mathsf{End}\,(\mathscr{H}_k)$. The geometry corresponding to the noncommutative algebra of functions $\mathcal{A}$ is the space $\mathbb{R}^3_\lambda$, with $\lambda = \sqrt{2\hbar/3R}$.

Let us now examine how the associated Lie algebra $\mathfrak{g}_\mathcal{A}$ is related to the isometries of $\mathbb{R}^3_\lambda$. A priori, there is no reason to expect a direct connection, as the "fundamental" object in this quantization is the Lie bracket of the quantized coordinate functions $\hat{x}^\mu$. The associated Lie algebra of this 2-Lie algebra is the 2-Lie algebra itself, i.e. $\mathfrak{su}(2)$, which indeed corresponds to the (continuous) isometries of $\mathbb{R}^3_\lambda$.

The associated Lie algebra $\mathfrak{g}_\mathcal{A}$ is of dimension six with generators $D_{\mu\nu} := D(\hat{x}^\mu \wedge$

$\hat{x}^\nu$), $\mu,\nu = 0,1,2,3$, where $\hat{x}^0 := -\mathrm{i}\,\hbar\,\mathbb{1}$. In the basis

$$\begin{aligned} X^1 &= D_{12} - D_{30} \ , \quad X^2 = D_{23} - D_{10} \ , \quad X^3 = D_{13} + D_{20} \ , \\ Y^1 &= D_{12} + D_{30} \ , \quad Y^2 = D_{23} + D_{10} \ , \quad Y^3 = D_{13} - D_{20} \ , \end{aligned} \tag{4.59}$$

the non-vanishing commutation relations read

$$[X^i, X^j] = 2\varepsilon^{ijk}\, X^k \ , \quad [Y^i, Y^j] = 2\varepsilon^{ijk}\, Y^k \ , \quad i,j,k = 1,2,3 \ . \tag{4.60}$$

Thus the associated Lie algebra is $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$, as expected since $\mathcal{A} \cong A_4$ in this case. The generators $X^i - Y^i$ generate the $\mathfrak{su}(2)$ isometries on $\mathbb{R}^3_\lambda$. The remaining generators transform the operator $\rho\,\mathbb{1}$, which corresponds to a (scalar) radius function in the geometric picture

$\mathbb{R}^{1,2}_0$

An analogous construction holds for the 3-bracket built on the Lie algebra $\mathsf{SO}(1,2) \cong \mathsf{SL}(2,\mathbb{R})$. Here the fuzzy spheres are replaced by the fuzzy hyperboloids $H^{1,2}_F$ (or $H^{2,1}_F$) constructed in §4.3. This defines the noncommutative space $\mathbb{R}^{1,2}_\lambda$. We thus obtain a foliation of $\mathbb{R}^3$ by fuzzy hyperboloids in this case.

$\mathbb{R}^3_1$

In the third case, the euclidean group in two dimensions, we start from the Lie algebra

$$[\hat{x}^1, \hat{x}^2] = -\mathrm{i}\,\xi\,\hat{x}^3 \ , \quad [\hat{x}^3, \hat{x}^1] = [\hat{x}^3, \hat{x}^2] = 0 \tag{4.61}$$

with a constant $\xi \in \mathbb{C}$. This algebra breaks the explicit $\mathsf{SO}(3)$ invariance down to $\mathsf{SO}(2)$. Since $\hat{x}^3$ is a central element of this algebra we can assume it acts as $\alpha\,\mathbb{1}$, $\alpha \in \mathbb{C}$ on any irreducible representation, and thus we can put $\xi = \frac{\hbar}{\alpha^2}$. The 3-bracket defined from the antisymmetric operator product is then given by

$$[\hat{x}^1, \hat{x}^2, \hat{x}^3] = \hat{x}^3\,[\hat{x}^1, \hat{x}^2] = -\mathrm{i}\,\hbar\,\mathbb{1} \ . \tag{4.62}$$

The quantum geometry behind this algebra $\mathcal{A}$ is thus a foliation of $\mathbb{R}^3$ in terms of standard noncommutative planes $\mathbb{R}^2_\theta$ extending in the directions parameterized by x^1 and x^2. The eigenvalues of $\hat{x}^3$ corresponding to the x^3 position of the noncommutative plane determine the noncommutativity parameter $\theta = \frac{\hbar}{x^3}$. This implies that the plane through $x^3 = 0$ is somewhat ill-defined. As $\mathsf{SO}(3)$-invariance is broken by the Nambu-Poisson structure here, one can equally well interpret the eigenvalues of $(\hat{x}^3)^{-1}$ as the position of the noncommutative plane. In this case, we obtain a commutative plane $\mathbb{R}^2$ through the origin. The noncommutative space with coordinate algebra $\mathcal{A}$ in this case is denoted $\mathbb{R}^3_{1,\theta}$.

The associated Lie algebra $\mathfrak{g}_\mathcal{A}$ is again spanned by the six generators $D_{\mu\nu} := D(\hat{x}^\mu \wedge \hat{x}^\nu)$, $\mu, \nu = 0, 1, 2, 3$ satisfying the non-vanishing commutation relations

$$\begin{aligned} [D_{12}, D_{13}] &= -D_{10} \ , \quad [D_{10}, D_{20}] = -D_{30} \ , \\ [D_{12}, D_{23}] &= -D_{20} \ , \quad [D_{10}, D_{12}] = -D_{13} \ , \\ [D_{23}, D_{13}] &= +D_{30} \ , \quad [D_{20}, D_{13}] = -D_{23} \ . \end{aligned} \tag{4.63}$$

This is an indecomposable simple Lie algebra. The isometries of $\mathbb{R}^3_{1,\theta}$, however, span the Lie algebra $\mathbb{R} \oplus \mathfrak{so}(2)$, corresponding to translations along the x^3 direction and rotations in the foliating planes. As the $\mathfrak{so}(2)$ rotations act as outer derivations of the Heisenberg algebra $[\hat{x}^1, \hat{x}^2] = -\mathrm{i}\,\theta\,\mathbb{1}$, there is no relation between the isometries and the associated Lie algebra. Worthy of note is the maximal subalgebra of the associated Lie algebra given by

$$[D_{12}, D_{23}] = -D_{20} \ , \quad [D_{12}, D_{20}] = D_{23} \ , \quad [D_{30}, -] = 0 \ , \tag{4.64}$$

which is isomorphic to $\mathfrak{iso}(2) \ltimes \mathbb{R}$. We conclude that the associated Lie algebra only describes non-geometric symmetries, and hence purely quantum isometries of the space $\mathbb{R}^3_{1,\theta}$ in the sense explained above. Note that as the operators appearing in the construction of $\mathbb{R}^2_\theta$ are not trace-class, we cannot use the trick (**??**) to render $\mathbb{1}$ a central element of the 3-Lie algebra of coordinate functions in this case.

4.4.3 Quantum geometry of M5-branes

We have thus found a geometric interpretation of the equations

$$[\hat{X}^\mu, \hat{X}^\nu, \hat{X}^\kappa] = -\mathrm{i}\,\hbar\,\Theta^{\mu\nu\kappa}\,\mathbb{1} \quad \text{and} \quad [\mathbb{1}, -, -] = 0 \tag{4.65}$$

found by Chu and Smith in [28] describing the quantum geometry of an M5-brane in a constant C-field background, where

$$\Theta^{\mu\nu\kappa} = \begin{cases} \varepsilon^{\mu\nu\kappa}\, C_1 \ , & \mu, \nu, \kappa = 0, 1, 2 \\ \varepsilon^{\mu\nu\kappa}\, C_2 \ , & \mu, \nu, \kappa = 3, 4, 5 \\ 0 & \text{otherwise} \end{cases} \tag{4.66}$$

and C_1, C_2 are constants related to the components of the C-field. They correspond to the quantizations of $\mathbb{R}^{1,2} \times \mathbb{R}^3$ with foliations by either fuzzy hyperboloids and spheres or noncommutative planes. We may heuristically regard the foliating noncommutative geometries as the dimensional reductions of the M5-brane configuration in the presence of a C-field to a configuration of D-branes in the appropriate B-field background.

Chapter 5

Membrane matrix models

Dimensional reductions of ten-dimensional maximally supersymmetric Yang-Mills theory lead to interesting zero-dimensional and one-dimensional matrix models, called respectively the IKKT [57] and BFSS [13] models. The IKKT matrix model is conjecturally a non-perturbative completion of type IIB string theory, while the BFSS matrix quantum mechanics is dual to M-theory in discrete light-cone quantization on flat space. Their classical solutions describe brane configurations which have also found interpretations in terms of noncommutative geometry.

In string theory, fuzzy spheres appear as classical solutions to D0-brane equations of motion in the presence of an external Ramond-Ramond flux [74]. In the IKKT matrix model description they arise as solutions to the classical equations of motion if one adds a Chern-Simons term representing the coupling to the external field [58]. The corresponding modification of the BFSS model is a massive matrix model with Chern-Simons term, called the BMN matrix model [19], which conjecturally describes the discrete light-cone quantization of M-theory on a supersymmetric pp-wave background and lifts the flat directions of the BFSS model. In this case both fuzzy spheres and fuzzy hyperboloids appear as half-BPS solutions [11, 78], and describe static large M2-branes or static large longitudinal M5-branes.

In this chapter we describe an analogous treatment of the BLG and ABJM membrane theories. We consider a dimensional reduction of these theories to a zero-dimensional 3-Lie algebra model and matrix model, respectively. We first introduce the reduced BLG and ABJM models by describing how to construct them from dimensional reduction. We show how both can be mapped to the IKKT matrix model using the Muhki-Papageorgakis map. We then show how to map between this models by taking various scaling limits, or by making a choice of 3-algebra. We end this chapter by finding several BPS solutions to the 3-Lie algebra model, and interpreting them as Nambu-Poisson manifolds in the sense of the previous chapter.

5.1 Reduced 3-Lie algebra model

5.1.1 BLG theory

The BLG theory [9, 49] is an $\mathcal{N}=8$ supersymmetric Chern-Simons-matter theory in three dimensions with matter fields taking values in a metric 3-Lie algebra $\mathcal{A}$ and a connection one-form taking values in the associated Lie algebra $\mathfrak{g}_{\mathcal{A}}$. The matter fields consist of eight scalar fields X^I, $I=1,\ldots,8$ and their superpartners, which can be combined into a Majorana spinor Ψ of $\mathsf{SO}(1,10)$ satisfying $\Gamma_{012}\Psi=-\Psi$; throughout we denote $\Gamma_{M_1\cdots M_k}:=\frac{1}{k!}\,\Gamma_{[M_1}\cdots\Gamma_{M_k]}$. The Chern-Simons term is constructed using the alternative cyclic invariant form $(\!(-,-)\!)$ available on $\mathfrak{g}_{\mathcal{A}}$ which is induced by the inner product $(-,-)$ on the 3-Lie algebra $\mathcal{A}$. Altogether the action reads

$$\begin{aligned} S_{\rm BLG}=\int \mathrm{d}^3x\,\Big(&-\tfrac12\left(\nabla_\mu X^I,\nabla^\mu X^I\right)+\tfrac{\mathrm{i}}{2}\left(\bar\Psi,\Gamma^\mu\,\nabla_\mu\Psi\right)+\tfrac{\mathrm{i}}{4}\left(\bar\Psi,\Gamma_{IJ}[X^I,X^J,\Psi]\right)\\ &-\tfrac{1}{12}\left([X^I,X^J,X^K],[X^I,X^J,X^K]\right)+\tfrac12\,\epsilon^{\mu\nu\lambda}\,(\!(A_\mu,\partial_\nu A_\lambda+\tfrac13\,[A_\nu,A_\lambda])\!)\Big)\ , \end{aligned} \tag{5.1}$$

where $\mu,\nu,\lambda=0,1,2$ are indices for euclidean coordinates on $\mathbb{R}^{1,2}$. The matrices Γ^μ, together with Γ^I, form the generators of the Clifford algebra $C\ell(\mathbb{R}^{1,10})$. The covariant derivatives act according to

$$\nabla_\mu X^I=\partial_\mu X^I+A_\mu\,X^I:=\partial_\mu X^I+A_\mu^{ab}\,D(\tau_a,\tau_b)X^I:=\partial_\mu X^I+A_\mu^{ab}\,[\tau_a,\tau_b,X^I]\ , \tag{5.2}$$

where τ_a are generators of the 3-Lie algebra $\mathcal{A}$.

5.1.2 Dimensional reduction

The reduced 3-Lie algebra model presented in [35] is the zero dimensional reduction of the BLG lagrangian. We reduce the covariant derivatives ∇_μ to an action of the gauge potential A_μ, which yields

$$\begin{aligned} \mathcal{S}_{\rm BLG} = \tfrac{1}{6}\,\epsilon^{\mu\nu\lambda}\ {\rm Tr}_{\,\mathfrak{g}_\mathcal{A}}\big(A_\mu\,[A_\nu,A_\lambda]\big) - \tfrac{1}{2}\,(A_\mu X^I, A^\mu X^I) + \tfrac{\rm i}{2}\,(\bar\Psi,\Gamma^\mu A_\mu\Psi) \\ + \tfrac{\rm i}{4}\,(\bar\Psi,\Gamma_{IJ}[X^I,X^J,\Psi]) - \tfrac{1}{12}\,([X^I,X^J,X^K],[X^I,X^J,X^K])\ . \end{aligned} \tag{5.3}$$

This action is invariant under the $\mathcal{N}=8$ supersymmetry transformations

$$\begin{aligned} \delta X^I &= {\rm i}\,\bar\varepsilon\,\Gamma^I\,\Psi\ , \\ \delta\Psi &= A_\mu X^I\,\Gamma^\mu\,\Gamma_I\,\varepsilon - \tfrac{1}{6}\,[X^I,X^J,X^K]\,\Gamma_{IJK}\,\varepsilon\ , \\ \delta A_\mu &= {\rm i}\,\bar\varepsilon\,\Gamma_\mu\,\Gamma_I\,[X^I,\Psi,-]\ . \end{aligned} \tag{5.4}$$

It is also invariant under the gauge transformations generated by $\Lambda\in\mathfrak{g}_\mathcal{A}$ given as

$$A_\mu\ \longmapsto\ -[A_\mu,\Lambda]\ , \qquad X^I\ \longmapsto\ [\Lambda,X^I]\ , \qquad \Psi\ \longmapsto\ \Lambda\Psi\ . \tag{5.5}$$

The vacuum moduli space $\mathfrak{M}^{\rm BLG}_\mathcal{A}$ of the 3-Lie algebra model is defined by setting $A_\mu = 0 = \Psi$ and

$$\big[X^I,X^J,X^K\big] = 0 \tag{5.6}$$

in order to satisfy the BPS equations implied by (5.4). For the 3-Lie algebra $\mathcal{A}=A_4$, the moduli space is given by [37]

$$\mathfrak{M}^{\rm BLG}_{A_4} = \big(\mathbb{R}^8/\mathbb{Z}_2\big)\times\big(\mathbb{R}^8/\mathbb{Z}_2\big)\ . \tag{5.7}$$

5.1.3 Reduction to the IKKT matrix model

If we assume that the BLG theory describes M2-branes, then we ought to be able to reduce the BLG theory to the effective description of D2-branes which is given by maximally supersymmetric Yang-Mills theory in three dimensions. In the paper [73], Mukhi and Papageorgakis proposed such a reduction procedure for the BLG theory with 3-Lie algebra $\mathcal{A}_4$, which reduces to $\mathcal{N}=8$ supersymmetric Yang-Mills theory with gauge group $\mathsf{SU}(2)$. Below we briefly review this reduction by going through the corresponding procedure for the dimensionally reduced model.

We start from our model (5.3) with 3-Lie algebra $\mathcal{A}_4$, whose generators are denoted e_i, $i=1,2,3,4$, and assume that one of the scalar fields, corresponding to the M-theory direction, develops a vacuum expectation value (vev) which is proportional to the radius R of the M-theory circle. Using the $\mathsf{SO}(4)$-invariance of $\mathcal{A}_4$, we can align this vev in the e_4 direction so that

$$\langle X^8 \rangle = -\tfrac{R}{\ell_p^{3/2}}\, e_4 = -g_{\rm YM}\, e_4 \ , \tag{5.8}$$

where ℓ_p and $g_{\rm YM}$ are the 11-dimensional Planck length and the Yang-Mills coupling constant, respectively. We now expand the action (5.3) around this vev by rewriting

$$X^8 = -g_{\rm YM}\, e_4 + Y^8 \ , \tag{5.9}$$

where $Y^8 \in \mathcal{A}$ still has components along the e_4 direction. The 3-brackets containing X^8 reduce according to

$$[A,B,X^8] = g_{\rm YM}\,[A,B,e_4] + [A,B,Y^8] \ , \quad A,B\in\mathcal{A} \ , \tag{5.10}$$

and in the strong coupling limit, i.e. for large values of $g_{\rm YM}$, 3-brackets containing X^8 reduce to the Lie bracket of $\mathfrak{so}(3)$ due to $[e_i,e_j,-e_4]=\varepsilon_{ijk4}\,e_k$. It is easy to see that the potential terms in (5.3) containing matter fields reduce to the corresponding terms of the IKKT matrix model for $\gamma\to\infty$ and $\mu=0$.

The reduction of the terms involving the gauge potential is slightly more involved. We consider the splitting $\mathfrak{g}_{\mathcal{A}_4}=\mathfrak{so}(4)\cong\mathfrak{so}(3)\oplus\mathfrak{so}(3)$. Specifically, we decompose the gauge field into terms involving the e_4 direction and those independent of that direction.

$$A_\mu = A_\mu^{ij}\, D(e_i, e_j) = A_\mu^i\, D(e_i, e_4) + B_\mu^i\, \tfrac{1}{2}\, \varepsilon_{ijk}\, D(e_j, e_k)\ , \tag{5.11}$$

where we use the notation

$$A_\mu^{i4} D(e_j, e_4) = A_\mu^i D(e_i, e_4)\ . \tag{5.12}$$

In the action (5.3), the field B_μ^i appears in the strong coupling limit only algebraically, and its equation of motion reads

$$B_\mu^i = \tfrac{1}{2g_{\rm YM}}\, \eta_{\mu\nu}\, \varepsilon^{\nu\rho\lambda}\, \varepsilon^{ijk}\, A_\rho^j\, A_\lambda^k - \tfrac{1}{2g_{\rm YM}}\, \varepsilon^{ijk}\, A_\mu^j\, X^{8\,k}\ , \tag{5.13}$$

where $\eta_{\mu\nu}$ denotes the Minkowski metric on $\mathbb{R}^{1,2}$. Altogether, the reduction (5.10) together with the splitting (5.11) and the equation of motion (5.13) reduce the action (5.3) with $\gamma \to \infty$ and $\mu = 0$ to the action of the IKKT matrix model with gauge group $\mathfrak{su}(2) \cong \mathfrak{so}(3)$,

$$S_{\rm IKKT} = -\tfrac{1}{4} \left([\mathscr{X}_M, \mathscr{X}_N], [\mathscr{X}^M, \mathscr{X}^N] \right) + \tfrac{\mathrm{i}}{2} \left(\bar{\Psi}, \Gamma^M [\mathscr{X}_M, \Psi] \right)\ , \tag{5.14}$$

where we combined the fields (A_μ, X^I) with $\mu = 0, 1, 2$ and $I = 1, \ldots, 7$ into $\mathscr{X}^M$ with $M = 0, 1, \ldots, 9$, and absorbed the coupling $g_{\rm YM}$ into a rescaling of fields. Here the invariant bilinear inner product coincides with the Cartan-Killing form on the Lie algebra $\mathfrak{su}(2)$, $(\mathscr{X}, \mathscr{Y}) = \mathrm{Tr}(\mathscr{X}\, \mathscr{Y})$. This matrix model possesses $\mathcal{N} = 2$ supersymmetry.

5.2 Reduced ABJM model

5.2.1 ABJM theory

The Van Raamsdonk formulation of the ABJM model [88] is a matrix valued action describing stacks of M2-branes with gauge group $\mathsf{U}(N_L) \times \mathsf{U}(N_R)$[1] With a gauge group isomorphic to $\mathsf{SU}(2) \times \mathsf{SU}(2)$, it is equivalent to the BLG lagrangian. The Z_A are bifundamental scalar fields, and ψ_A are bifundamental spinor fields with $A = 1, .., 4$. The covariant derivative is defined to be

$$D_\mu Z := \partial_\mu Z + A^R_\mu Z - Z A^L_\mu \quad . \tag{5.15}$$

The gauge transformations of the model are

$$\begin{aligned} &Z^A \to g_R Z^A g_L^{-1} \ , \quad A_\mu^{(L,R)} \to g_{(L,R)} A_\mu g^{-1}_{(L,R)} + g_{(L,R)} \partial_\mu g^{-1}_{(L,R)} \ , \\ &\psi^A \to g_R \psi^A g_L^{-1} \ . \end{aligned} \tag{5.16}$$

The lagrangian is

$$\begin{aligned} \mathcal{L} = &\tfrac{k}{4\pi} \epsilon^{\mu\nu\lambda} (A^L_\mu \partial_\nu A^L_\lambda + \tfrac{2\mathrm{i}}{3} A^L_\mu A^L_\nu A^L_\lambda - A^R_\mu \partial_\mu A^R_\lambda - \tfrac{2\mathrm{i}}{3} A^R_\mu A^R_\nu A^R_\lambda) \\ &- D_\mu Z^\dagger_A D^\mu Z^A - i \bar{\psi}^A \gamma^\mu D_\mu \psi_A + \tfrac{2\pi i}{k} (\bar{Z}_A Z^A \bar{\psi}^B \psi_B - \bar{\psi}^B Z^A Z^\dagger_A \psi_B \\ &- 2 Z^\dagger_A Z^B \bar{\psi}^A \psi_B + 2 \bar{\psi}^B Z^A \bar{Z}_B \psi_A - \epsilon^{ABCD} Z^\dagger_A \psi_B Z^\dagger_C \psi_D \\ &+ \epsilon_{ABCD} Z^A \bar{\psi}^B Z^C \bar{\psi}^C) + \tfrac{4\pi^2}{3k^2} (Z^A Z^\dagger_A Z^B Z^\dagger_B Z^C Z^\dagger_C \\ &+ Z^\dagger_A Z^A Z^\dagger_B Z^B Z^\dagger_C Z^C + 4 Z^A Z^\dagger_B Z^C Z^\dagger_A Z^B Z^\dagger_C - 6 Z^A Z^\dagger_B Z^B Z^\dagger_A Z^C Z^\dagger_C) \ . \end{aligned} \tag{5.17}$$

[1]Strictly speaking, this is the ABJ model [3]. We will still refer to this model as as the ABJM model for simplicity, although we refer to the specialization $N_L = N_R$ as the "ABJM limit ".

The supersymmetry transformations read as

$$\begin{aligned}
\delta Z^A &= \mathrm{i}\omega^{AB}\psi_B\ ,\\
\delta Z^\dagger_A &= \mathrm{i}\bar\psi^B\omega_{AB}\ ,\\
\delta\psi_A &= -\gamma^\mu\omega_{AB}\nabla_\mu Z^B + \tfrac{2\pi}{k}\Big(-\omega_{AB}(Z^C Z^\dagger_C Z^B - Z^B Z^\dagger_C Z^C) + 2\omega_{CD}Z^C Z^\dagger_A Z^D\Big)\ ,\\
\delta\bar\psi^A &= \nabla_\mu Z^\dagger_B\omega^{AB}\gamma^\mu + \tfrac{2\pi}{k}\Big(-(Z^\dagger_B Z^C Z^\dagger_C - Z^\dagger_C Z^C Z^\dagger_B)\omega^{AB} + 2Z^\dagger_D Z^A Z^\dagger_C\omega^{CD}\Big)\ ,\\
\delta A^L_\mu &= \tfrac{\pi\mathrm{i}}{k}(-Z^A\bar\psi^B\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu\psi_A Z^\dagger_B)\ ,\\
\delta A^R_\mu &= \tfrac{\pi\mathrm{i}}{k}(-\bar\psi^A Z^B\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu Z^\dagger_A\psi_B)\ .
\end{aligned} \tag{5.18}$$

5.2.2 Dimensional reduction

We perform a dimensional reduction which modifies the gauge fields and covariant derivatives change in the following way: Gauge fields $A^{L,R}_\mu$ become scalar fields $A^{(L,R)}_i$, $i = 1, ..., 3$. Covariant derivatives reduce to the following

$$\begin{aligned}
D_\mu Z^A &\to A^R_i Z^A - Z^A A^L_i\ \ ,\quad D_\mu Z^\dagger_A \to A^L_i Z^\dagger_A - Z^\dagger_A A^R_i\ \ ,\\
D_\mu\psi^A &\to A^R_i\psi^A - \psi^A A^L_i\ \ ,\quad D_\mu\psi^\dagger_A \to A^L_i\psi^\dagger_A - \psi^\dagger_A A^R_i\ \ .
\end{aligned} \tag{5.19}$$

The gauge transformations are changed so that we have

$$\begin{aligned}
&Z^A \to g_R Z^A g^\dagger_L\ ,\quad A^{(L,R)}_i \to g_{(L,R)}A_i g^\dagger_{(L,R)}\ .\\
&\psi^A \to g_R\psi^A g^\dagger_L\ .
\end{aligned} \tag{5.20}$$

The reduced lagrangian reads as

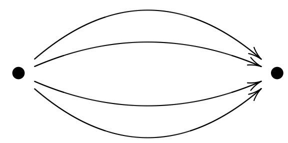
$$\begin{aligned}
\mathcal{S}_{\mathrm{ABJM}} = \ &\mathrm{Tr}_V\Big(\tfrac{2\mathrm{i}}{3}\,\kappa\,\epsilon^{\mu\nu\lambda}\left(A^L_\mu A^L_\nu A^L_\lambda - A^R_\mu A^R_\nu A^R_\lambda\right)\\
&-2A^L_\mu Z^\dagger_i A^R_\mu Z^i + A^L_\mu Z^\dagger_i Z^i A^L_\mu + A^R_\mu Z^i Z^\dagger_i A^R_\mu - \mathrm{i}\,\bar\psi_i\gamma^\mu A^R_\mu\psi^i\\
&+\mathrm{i}\,\bar\psi_i\gamma^\mu\psi^i A^L_\mu + \tfrac{\mathrm{i}}{2\kappa}\left(Z^\dagger_i Z^i\bar\psi_j\psi^j - \bar\psi_j Z^i Z^\dagger_i\psi^j - 2Z^\dagger_i Z^j\bar\psi^i\psi_j\right. \qquad (5.21)\\
&\left.+2\bar\psi^j Z^i Z^\dagger_j\psi_i - \epsilon^{ijkl}Z^\dagger_i\psi_j Z^\dagger_k\psi_l + \epsilon_{ijkl}Z^i\bar\psi^j Z^k\bar\psi^l\right)\\
&+\tfrac{1}{12\kappa^2}\left(Z^i Z^\dagger_i Z^j Z^\dagger_j Z^k Z^\dagger_k + Z^\dagger_i Z^i Z^\dagger_j Z^j Z^\dagger_k Z^k\right.\\
&\left.+4Z^i Z^\dagger_j Z^k Z^\dagger_i Z^j Z^\dagger_k - 6Z^i Z^\dagger_j Z^j Z^\dagger_i Z^k Z^\dagger_k\right)\Big)\ .
\end{aligned} \tag{5.22}$$

The corresponding supersymmetry transformations are

$$
\begin{aligned}
\delta Z^i &= \mathrm{i}\,\omega^{ij}\,\psi_j\ , \\
\delta Z^\dagger_i &= \mathrm{i}\,\psi^\dagger_j\,\omega_{ij}\ , \\
\delta\psi_i &= -\gamma^\mu\,\omega_{ij}\left(Z^j\,A^L_\mu - A^R_\mu\,Z^j\right) - \tfrac{1}{2\kappa}\Big(\omega_{ij}\left(Z^k\,Z^\dagger_k\,Z^j - Z^j\,Z^\dagger_k\,Z^k\right) - 2\omega_{kl}\,Z^k\,Z^\dagger_i\,Z^l\Big)\ , \\
\delta\bar\psi^i &= \left(Z^\dagger_j\,A^R_\mu - A^L_\mu\,Z^\dagger_j\right)\omega^{ij}\,\gamma^\mu - \tfrac{1}{2\kappa}\Big(\left(Z^\dagger_j\,Z^k\,Z^\dagger_k - Z^\dagger_k\,Z^k\,Z^\dagger_j\right)\omega^{ij} - 2Z^\dagger_l\,Z^i\,Z^\dagger_k\,\omega^{kl}\Big)\ , \\
\delta A^L_\mu &= -\tfrac{\mathrm{i}}{4\kappa}\left(Z^i\,\psi^\dagger_j\,\gamma_\mu\,\omega_{ij} - \omega^{ij}\,\gamma_\mu\,\psi_i\,Z^\dagger_j\right)\ , \\
\delta A^R_\mu &= -\tfrac{\mathrm{i}}{4\kappa}\left(\psi^\dagger_i\,Z^j\,\gamma_\mu\,\omega_{ij} - \omega^{ij}\,\gamma_\mu\,Z^\dagger_i\,\psi_j\right)\ ,
\end{aligned}
\tag{5.23}
$$

where ω^{ij} are $\mathcal{N}=6$ supersymmetry transformation parameters obeying $\omega^{ij} = (\omega_{ij})^* = -\frac{1}{2}\,\epsilon^{ijkl}\,\omega_{kl}$.

We can also regard the ABJM matrix model in terms of a quiver. The ABJM quiver is constructed by adding arrows to the A_1 quiver[2] to give the ABJM quiver



(5.24)

We deform the generic $\mathcal{N} = 2$ supersymmetric Chern-Simons quiver matrix model ((C.10)) by adding a suitable quartic superpotential of the chiral superfields $\boldsymbol{\Phi}^i$ [44, 17] which reads as

$$\mathcal{W}(\boldsymbol{\Phi}) = \tfrac{\kappa}{4!}\,\epsilon^{ijkl}\ \mathrm{Tr}_V\big(\boldsymbol{\Phi}_i\,\boldsymbol{\Phi}_j^\dagger\,\boldsymbol{\Phi}_k\,\boldsymbol{\Phi}_l^\dagger\big)\ . \tag{5.25}$$

The extrema of the superpotential define the relations of the double quiver associated to the ABJM quiver (5.24). The BPS equations of the ABJM theory were derived in [62]; here we present them for the dimensionally reduced model. They are determined by the quantities

$$\mathcal{Z}_i^{jk} := Z^j\,Z_i^\dagger\,Z^k - Z^k\,Z_i^\dagger\,Z^j \tag{5.26}$$

for $j < k$. We set the fermions equal to zero. The BPS equations for the supersymmetric solutions of the matrix model then follow from the fermionic supersymmetry variations in (5.23) using the independence of the gamma-matrices as a basis of the

[2] See appendix (C.14) .

Clifford algebra, and are given by

$$\begin{aligned}
\left[A^L_\mu, A^L_\nu\right] = 0 = \left[A^R_\mu, A^R_\nu\right] \ , \\
A^R_1\, Z^1 - Z^1\, A^L_1 - \mathrm{i}\left(A^R_2\, Z^1 - Z^1\, A^L_2\right) = 0 \ , \\
A^R_\mu\, Z^i - Z^i\, A^L_\mu = 0 \quad (i \neq 1 \ , \ \mu = 1,2) \ , \\
A^R_0\, Z^2 - Z^2\, A^L_0 - \mathrm{i}\, \mathcal{Z}^{21}_1 = 0 \ , \\
A^R_0\, Z^3 - Z^3\, A^L_0 - \mathrm{i}\, \mathcal{Z}^{13}_1 = 0 \ , \\
A^R_0\, Z^4 - Z^4\, A^L_0 - \mathrm{i}\, \mathcal{Z}^{14}_1 = 0 \ , \\
\mathcal{Z}^{31}_3 = \mathcal{Z}^{41}_4 = \mathcal{Z}^{21}_3 \ , \\
\mathcal{Z}^{43}_4 = \mathcal{Z}^{34}_3 = \mathcal{Z}^{32}_3 = 0 = \mathcal{Z}^{42}_4 = \mathcal{Z}^{23}_2 = \mathcal{Z}^{24}_4 \ , \\
\mathcal{Z}^{jk}_i = 0 \quad (i \neq j \neq k) \ .
\end{aligned} \tag{5.27}$$

5.2.3 Reduction to the IKKT matrix model

Let us now extend the Mukhi-Papageorgakis map to reduce the ABJM model (5.22). Here we work within the ABJM limit. We break the product gauge group $G = \mathsf{U}(N) \times \mathsf{U}(N)$ to a diagonal $\mathsf{U}(N)$ subgroup by taking an axial combination of the gauge fields

$$A^L_\mu = A_\mu + \mathrm{i}\, B_\mu \ , \qquad A^R_\mu = A_\mu - \mathrm{i}\, B_\mu \ . \tag{5.28}$$

We write the real and imaginary parts of the scalars and the spinors as

$$Z^i = X^i + \mathrm{i}\, X^{i+4} \ , \qquad \psi^i = \chi^i + \mathrm{i}\, \chi^{i+4} \tag{5.29}$$

for $i = 1, 2, 3, 4$. We further decompose the fields into the generators of the $\mathsf{U}(N)$ gauge group as

$$Z^i = X^i_0\,\tau_0 + \mathrm{i}\,X^{i+4}_0\,\tau_0 + X^i_a\,\tau_a + \mathrm{i}\,X^{i+4}_a\,\tau_a \ ,$$
$$\psi^i = \psi^i_0\,\tau_0 + \mathrm{i}\,\psi^{i+4}_0\,\tau_0 + \psi^i_a\,\tau_a + \mathrm{i}\,\psi^{i+4}_a\,\tau_a \ . \tag{5.30}$$

We expand the scalar fields around a fixed vacuum configuration proportional to a coupling constant g. Using the $\mathsf{SU}(4)$ invariance of the matrix model, we can select the scalar field Z^4 to expand around so that

$$Z^i = \mathrm{i}\,g\,\delta^{i,4}\,\mathbb{1} + X^i_0\,\tau_0 + X^i_a\,\tau_a + \mathrm{i}\,X^{i+4}_0\,\tau_0 + \mathrm{i}\,X^{i+4}_a\,\tau_a \ . \tag{5.31}$$

We first investigate the effect of the scaling limit on the Chern-Simons matrix action from the first line of (5.22). The various terms of the action separate into $\mathsf{U}(1)$ and $\mathsf{SU}(N)$ components, and in the strong coupling limit $g \to \infty$ the $\mathsf{U}(1)$ terms decouple so we will ignore them from now on. When we make the gauge field replacement (5.28), the Chern-Simons term reads as in (5.56), which in the scaling limit will reduce to (5.57).

The contributing terms to the reduction of the second line of (5.22) give

$$\begin{aligned}\mathcal{S}_{\rm k} = \ & \mathrm{Tr}_V\Big(-\sum_{i=1}^4 \big([A_\mu, X^i]^2 + [A_\mu, X^{i+4}]^2\big) - 4g\,[A_\mu, X^8]\,B^\mu - 4g^2\,B_\mu\,B^\mu \\ & - \mathrm{i}\sum_{i=1}^4 \big(\bar{\chi}_i\,\gamma^\mu\,[A_\mu, \chi^i + \mathrm{i}\,\chi^{i+4}] + \mathrm{i}\,\bar{\chi}_{i+4}\,\gamma^\mu\,[A_\mu, \chi^i + \mathrm{i}\,\chi^{i+4}]\big)\Big) \ .\end{aligned} \tag{5.32}$$

Combining (5.57) and (5.32), we can integrate out B_μ using its equation of motion

$$B_\mu = -\tfrac{1}{g}\,[A_\mu, X^8] + \tfrac{\kappa}{g^2}\,\epsilon_{\mu\nu\lambda}\,[A^\nu, A^\lambda] \ . \tag{5.33}$$

This causes the scalar field X^8 to decouple from the action. We write the minimal spinor of $\mathsf{SO}(1,2) \times \mathsf{SO}(7)$ for the reduced theory as in (5.59), where each χ^I is also a two-component Majorana spinor. Then the action (5.32) reduces to

$$\begin{aligned} \mathcal{S}_{\rm k} = \ & {\rm Tr}_V\Big(-\sum_{a=1}^3 \big([A_\mu, X^a]^2 + [A_\mu, X^{a+4}]^2\big) - [A_\mu, X^4]^2 \\ & - 8\kappa^2\,[A_\mu, A_\nu]^2 - {\rm i}\,\bar\psi\,\gamma^\mu\,[A_\mu, \psi]\Big) \ . \end{aligned} \tag{5.34}$$

We now investigate the potential terms from the last four lines of (5.22). The surviving terms from the bosonic potential are of the form

$$\begin{aligned} \mathcal{V}_{\rm b}(X) = \ & -\tfrac{1}{8\kappa^2}\ {\rm Tr}_V\Big(\sum_{a,b=1}^3 \big([X^a, X^b]^2 + [X^{a+4}, X^{b+4}]^2\big) \\ & + 2\sum_{a=1}^3 \big([X^a, X^4]^2 + 2[X^{a+4}, X^4]^2\big)\Big) \ . \end{aligned} \tag{5.35}$$

The fermions produce a potential that reads as

$$\mathcal{V}_{\rm f}(X,\psi) = -\tfrac{1}{\kappa}\ {\rm Tr}_V\Big(\sum_{a=1}^3 \big(\bar\psi\,\gamma_a\,[X^a,\psi] + \bar\psi\,\gamma_{a+4}\,[X^{a+4},\psi]\big) + \bar\psi\,\gamma_4\,[X^4,\psi]\Big) \tag{5.36}$$

for a suitable basis of $\mathsf{SO}(7)$ gamma-matrices $\gamma^a, \gamma^4, \gamma^{a+4}$, $a = 1,2,3$.

Finally, we rescale the fields as

$$B_\mu \longrightarrow g\,B_\mu \ , \qquad X_0^i \longrightarrow \tfrac{1}{g}\,X_0^i \ , \qquad \psi_0^i \longrightarrow \tfrac{1}{g}\,\psi_0^i \tag{5.37}$$

and then the full reduced action takes the form

$$\begin{aligned} \mathcal{S}_{\rm red} = \ & \tfrac{1}{g^2}\ {\rm Tr}_V\Big(-\sum_{a=1}^3 \big([A_\mu, X^a]^2 + [A_\mu, X^{a+4}]^2\big) - [A_\mu, X^4]^2 - 8\kappa^2\,[A_\mu, A_\nu]^2 \\ & - \tfrac{1}{8\kappa^2}\sum_{a,b=1}^3 \big([X^a, X^b]^2 + [X^{a+4}, X^{b+4}]^2\big) - \tfrac{1}{4\kappa^2}\sum_{a=1}^3 \big([X^a, X^4]^2 + [X^{a+4}, X^4]^2\big) \\ & - \tfrac{1}{\kappa}\sum_{a=1}^3 \big(\bar\psi\,\gamma_a\,[X^a,\psi] + \bar\psi\,\gamma_{a+4}\,[X^{a+4},\psi]\big) - \tfrac{1}{\kappa}\,\bar\psi\,\gamma_4\,[X^4,\psi] - {\rm i}\,\bar\psi\,\gamma^\mu\,[A_\mu,\psi]\Big) \ . \end{aligned} \tag{5.38}$$

We can combine the bosonic fields into a single field $X^M = (2A_\mu, X^a, X^4, X^{a+4})$ with $M = 1,\dots,10$. Then this action, along with the choice of Chern-Simons coupling constant $\kappa = \frac{1}{2}$, produces the action of the ten-dimensional IKKT matrix model.

For later use, we note the similarity between the BPS equations of the ABJM and IKKT matrix models. In the case of the ABJM model the BPS equations are given by (5.27), while in the case of the IKKT model the BPS equations are determined by commuting matrices

$$\big[X^M, X^N\big] = 0 \; . \tag{5.39}$$

However, the 3-algebra form (5.46) of the ABJM equations does not map to the IKKT equations (5.39) under the scaling limit described here. This is due to the removal of the gauge fields from (5.27): In the axial limit (5.28) of the gauge fields, the field B_μ causes a bosonic degree of freedom to decouple from the action in the scaling limit in order that one may combine the gauge fields with the scalars in the appropriate way.

5.3 Hermitian 3-algebra Model

Alternative 3-algebra models can be written down if one relaxes the requirements of maximal supersymmetry and of total antisymmetry of the 3-bracket. We first break the $\mathsf{SO}(8)$ R-symmetry group of the maximally supersymmetric theory to $\mathsf{SU}(4) \times \mathsf{U}(1)$. The supercharges transform under $\mathsf{SU}(4) \cong \mathsf{SO}(6)$, whilst the $\mathsf{U}(1)$ factor provides an additional global symmetry. Introduce four complex 3-algebra valued scalar fields Z^i, $i = 1, 2, 3, 4$. Denote the corresponding four fermions by ψ^i; they are two-component Dirac spinors of $\mathsf{SO}(1,2)$. We select a real set of gamma-matrices γ_μ, with $\gamma_{012} = \mathbb{1}$. The Majorana condition is $\bar\varepsilon = \varepsilon^\top \gamma_0$. For a generic hermitian 3-algebra $\mathcal{A}$, the analog of our 3-Lie algebra model (5.3) is given by

$$\begin{aligned} \mathcal{S}^{\mathbb{C}}_{\rm BLG} = {} & \tfrac{1}{6}\, \epsilon^{\mu\nu\lambda}\ \mathrm{Tr}_{\,\mathfrak{g}_{\mathcal{A}}} \big(A_\mu\, [A_\nu, A_\lambda]\big) - (A_\mu Z_i^\dagger, A^\mu Z^i) + \mathrm{i}\, (\bar\psi_i, \gamma^\mu\, A_\mu \psi^i) - \mathcal{V}(Z) \\ & - (\bar\psi^i, [\psi_i, Z^j; Z_j]) + 2\, \mathrm{i}\, (\bar\psi^i, [\psi_j, Z^j; Z_i]) + \tfrac{\mathrm{i}}{2}\, \epsilon_{ijkl}\, (\bar\psi^i, [Z^k, Z^l; \psi^j]) \\ & - \tfrac{\mathrm{i}}{2}\, \epsilon_{ijkl}\, (Z^l, [\bar\psi^i, \psi^j; Z^k]) \ , \end{aligned} \tag{5.40}$$

where the sextic potential is given by

$$\mathcal{V}(Z) = \tfrac{2}{3}\left(\Upsilon_i^{jk}(Z)\,,\;\Upsilon_i^{jk}(Z)^\dagger\right) \tag{5.41}$$

with

$$\Upsilon_i^{jk}(Z) = [Z^j, Z^k; Z_i] - \tfrac{1}{2}\,\delta_i^j\,[Z^l, Z^k; Z_l] + \tfrac{1}{2}\,\delta_i^k\,[Z^l, Z^j; Z_l]\ . \tag{5.42}$$

The supersymmetry transformations of this model read

$$\begin{aligned}
\delta Z^i &= \mathrm{i}\,\bar{\varepsilon}^{ij}\,\psi_j\ ,\\
\delta\psi^i &= -\gamma^\mu\,A_\mu Z_j\,\varepsilon^{ij} + [Z_j, Z^k; Z_k]\,\varepsilon^{ij} + [Z^k, Z^l; Z^i]\,\varepsilon_{kl}\ ,\\
\delta A_\mu &= -\mathrm{i}\,[-, Z_i; \psi_j]\,\gamma_\mu\,\varepsilon^{ij} + \mathrm{i}\,\bar{\varepsilon}^{ij}\,\gamma_\mu\,[-, \psi_j; Z_i]\ .
\end{aligned} \tag{5.43}$$

5.4 Matrix model mappings

In this section we describe relationships between our membrane matrix models. The various formulations of these models are related to each other, and it is possible to pass between them when certain constraints are placed on the relevant 3-algebras. For a particular 3-algebra, we show that it is possible to pass from a certain reduced 3-Lie algebra model to our ABJM matrix model. Furthermore, in a certain scaling limit, one can reach the 3-algebra model from the ABJM matrix model from, again for a particular 3-Lie algebra.

5.4.1 Mapping to the ABJM matrix model

We will now parallel the construction of [10] to demonstrate that our reduced model, for a particular choice of hermitian 3-algebra $\mathcal{A}$ and gauge group, yields the $\mathcal{N} = 6$ ABJM matrix model. Let $\mathcal{A} = \mathsf{Hom}\,(V_L, V_R)$ with 3-bracket (2.41) and inner product (2.42). The gauge group is the product $\mathsf{U}(N_L) \times \mathsf{U}(N_R)$, corresponding to the associated Lie algebra $\mathfrak{g}_{\mathcal{A}} = \mathfrak{u}(N_L) \oplus \mathfrak{u}(N_R)$. With these choices, the action

(5.40) becomes

$$\begin{aligned} \mathcal{S}^{\mathbb{C}}_{\rm BLG} = \ & {\rm Tr}_V\Big(A^L_\mu\, Z_i^\dagger\, Z^i\, A^L_\mu + A^R_\mu\, Z^i\, Z_i^\dagger\, A^R_\mu - 2A^L_\mu\, Z_i^\dagger\, A^R_\mu\, Z^i - {\rm i}\,\bar\psi^i\,\gamma^\mu\, A^R_\mu\,\psi_i \\ & + {\rm i}\,\bar\psi^i\,\gamma^\mu\,\psi_i\, A^L_\mu + \tfrac16\,\epsilon^{\mu\nu\lambda}\,\big(A^L_\mu\,\big[A^L_\nu, A^L_\lambda\big] - A^R_\mu\,\big[A^R_\nu, A^R_\lambda\big]\big) - \mathcal{V}(Z) \\ & - {\rm i}\,\lambda\,\big(\bar\psi^i\,\psi_i\, Z_j^\dagger\, Z^j + \bar\psi^i\, Z^j\, Z_j^\dagger\,\psi_i - 2\bar\psi^i\,\psi_j\, Z_i^\dagger\, Z^j + \bar\psi^i\, Z^j\, Z_i^\dagger\,\psi_j\big) \\ & + {\rm i}\,\lambda\,\big(\epsilon_{ijkl}\,\bar\psi^i\, Z^k\,\bar\psi^j\, Z^l - \epsilon^{ijkl}\, Z_l^\dagger\,\psi_i\, Z_k^\dagger\,\psi_j\big)\Big)\ . \end{aligned} \tag{5.44}$$

Our choice of 3-bracket is antisymmetric in the first two entries. This lets us rewrite the potential (5.41) as

$$\mathcal{V}(Z) = \ {\rm Tr}_V\big(- \tfrac23\,\big[Z^i, Z^j; Z^k\big]\,\big[Z_i^\dagger, Z_j^\dagger; Z_k^\dagger\big] + \tfrac12\,\big[Z^k, Z^i; Z^i\big]\,\big[Z_k^\dagger, Z_j^\dagger; Z_j^\dagger\big]\big)\ . \tag{5.45}$$

The global minima of $\mathcal{V}(Z)$ are described by the equations

$$\mathcal{Z}_i^{jk} := \big[Z^j, Z^k; Z_i\big] = 0\ , \tag{5.46}$$

which are just the BPS equations (5.27) with $A_\mu = 0$. These equations coincide with the extrema of the superpotential (5.25), and hence define the relations of the double of the ABJM quiver (5.24). We can evaluate the 3-brackets explicitly, and then the potential assumes the manifestly $\mathsf{SU}(4)$-invariant form

$$\begin{aligned} \mathcal{V}(Z) = \ & -\tfrac{2\lambda^2}{3}\ {\rm Tr}_V\big(2Z^k\, Z_j^\dagger\, Z^i\, Z_k^\dagger\, Z^j\, Z_i^\dagger + 2Z^k\, Z_j^\dagger\, Z^i\, Z_i^\dagger\, Z^j\, Z_k^\dagger \\ & + \tfrac12\, Z^i\, Z_i^\dagger\, Z^k\, Z_k^\dagger\, Z^j\, Z_j^\dagger + \tfrac12\, Z_i^\dagger\, Z^i\, Z_j^\dagger\, Z^j\, Z_k^\dagger\, Z^k\big)\ . \end{aligned} \tag{5.47}$$

For the choice of constant $\lambda = \frac{1}{2\kappa}$, we recover the $\mathcal{N} = 6$ ABJM matrix model (5.22).

Note that the BPS equations (5.46) and their conjugates imply that the collection $Z_i^\dagger\, Z^j$ of endomorphisms of V_L for $i, j = 1, 2, 3, 4$ form a mutually commuting set of $N_L \times N_L$ matrices; similarly $Z^j\, Z_i^\dagger$ are a mutually commuting set of $N_R \times N_R$ matrices. In the ABJM limit $N_L = N_R = N$, the operators $Z_i^\dagger\, Z^j$ and $Z^j\, Z_i^\dagger$ moreover have the same spectra, and the vacuum moduli space $\mathfrak{M}_N^{\rm ABJM}$ is therefore given by the

N-th symmetric product orbifold

$$\mathfrak{M}_N^{\rm ABJM} = \big(\mathbb{C}^4\big)^N \,/\, \mathfrak{S}_N \tag{5.48}$$

where $\mathfrak{S}_N$ is the Weyl group of $\mathsf{U}(N)$ acting by permuting the components of N-vectors. As we will make use of this result later, let us derive it explicitly. For this, we note that the BPS equations in this case are solved by commuting matrices $[Z^i, Z^j] = 0$, $i, j = 1, 2, 3, 4$. Then Z^i can be put simultaneously into their Jordan normal forms, with k eigenvalues $\zeta_1^i, \dots, \zeta_k^i$ of each endomorphism Z^i, i.e. for each fixed $i \in \{1, 2, 3, 4\}$, each ζ_l^i, $l = 1, \dots, k$, corresponds to a Jordan block; doing so breaks the $\mathsf{U}(N) \times \mathsf{U}(N)$ gauge symmetry to a diagonal $\mathsf{U}(N)$ subgroup. To every Jordan block one associates its dimension λ_l, independently of $i \in \{1, 2, 3, 4\}$ because Z^i mutually commute. The collection $\lambda = (\lambda_1, \dots, \lambda_k)$ of dimensions satisfies

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \ , \qquad \sum_{l=1}^k \lambda_l = N \ , \tag{5.49}$$

and thus defines a linear partition of the rank N of length k. Then the isomorphism (5.48) is generated by the map

$$\big(Z^1, Z^2, Z^3, Z^4\big) \ \longmapsto \ \sum_{l=1}^k \lambda_l \, \vec{z}_l \ \in \ \big(\mathbb{C}^4\big)^N \,/\, \mathfrak{S}_N \ , \tag{5.50}$$

where $\big\{\vec{z}_l = (\zeta_l^1, \zeta_l^2, \zeta_l^3, \zeta_l^4)\big\}_{l=1,\dots,k}$ is a set of k points in $\mathbb{C}^4$ with multiplicities given by the linear partition λ.

5.4.2 Mapping to the lorentzian Lie algebra model

Following [55], we shall now demonstrate how a particular contraction relates the lorentzian version of the 3-Lie algebra model (5.3) with the ABJM matrix model (5.22). The first step is to construct the *lorentzian Lie algebra model.* We fix a semisimple Lie algebra $\mathfrak{h}$ and expand the fields of the reduced 3-Lie algebra model in terms of the generators of $\mathcal{A}_{\mathfrak{h}}$ satisfying the 3-bracket relations (2.33) as

$$
\begin{aligned}
X^I &= X^I_{\rm c}\,\mathbb{1} + X^I_0\,\tau_0 + X^I_a\,\tau_a \ , \\
\Psi &= \Psi_{\rm c}\,\mathbb{1} + \Psi_0\,\tau_0 + \Psi_a\,\tau_a \ , \\
A^\mu &= A^\mu_{0a}\,D_{0a} + A^\mu_{ab}\,D_{ab} \ .
\end{aligned} \tag{5.51}
$$

It is convenient to make the field definitions

$$
\hat X^I = X^I_a\,\tau_a \ , \qquad \hat\Psi = \Psi_a\,\tau_a \ , \qquad \hat A^\mu = A^\mu_{0a}\,\tau_a \ , \qquad B^\mu = f_{abc}\,A^\mu_{ab}\,\tau_c \ . \tag{5.52}
$$

We insert these expansions into (5.3), and denote the inner product (2.34) by $\mathrm{Tr}_{\,\mathfrak{h}}$ here. Using (2.24), 3-brackets involving the generator τ_0 induce the Lie bracket of $\mathcal{A}'_{\mathfrak{h}}$ through

$$
\big[X^I, X^J, \tau_0\big] = \big[\hat X^I, \hat X^J\big] \ . \tag{5.53}
$$

A similar reduction occurs for the brackets involving fermions. For the terms involving the gauge fields, we use (2.34) to infer that terms proportional to the central element $\mathbb{1}$ decouple from the gauge interactions, and in fact completely from the action. In this way we find the lorentzian Lie algebra model

$$
\begin{aligned}
\mathcal{S}_{\mathfrak{h}} = \ & \mathrm{Tr}_{\,\mathfrak{h}}\Big(\big(\tfrac12\,[\hat A^\mu, \hat X^I] + B^\mu\,X^I_0\big)^2 + \tfrac14\,(X^K_0)^2\,[\hat X^I, \hat X^J]^2 - \tfrac12\,\big(X^I_0\,[\hat X^I, \hat X^J]\big)^2 \\
& + \tfrac{\mathrm{i}}{2}\,\bar{\hat\Psi}\,\Gamma^\mu\,[\hat A_\mu, \hat\Psi] - \mathrm{i}\,\bar\Psi_0\,\Gamma^\mu\,B_\mu\,\hat\Psi - \tfrac12\,\bar\Psi_0\,\hat X^I\,[\hat X^J, \Gamma_{IJ}\hat\Psi] + \tfrac12\,\bar{\hat\Psi}\,X^I_0\,[\hat X^J, \Gamma_{IJ}\hat\Psi] \\
& + \tfrac12\,\epsilon^{\mu\nu\lambda}\,[\hat A_\mu, \hat A_\nu]\,B_\lambda\Big) \ .
\end{aligned} \tag{5.54}
$$

It is invariant under the supersymmetry transformations

$$
\begin{aligned}
& \delta\hat\Psi = \big([\hat A^\mu, \hat X^I] + B^\mu\,X^I_0\big)\,\Gamma_\mu\,\Gamma_I\,\varepsilon - \tfrac12\,X^K_0\,[\hat X^I, \hat X^J]\,\Gamma_{IJK}\,\varepsilon \ , \qquad \delta\Psi_0 = 0 \ , \\
& \delta\hat X^I = \mathrm{i}\,\Gamma^I\,\bar\varepsilon\,\hat\Psi \ , \qquad \delta X^I_0 = \mathrm{i}\,\Gamma^I\,\bar\varepsilon\,\Psi_0 \ , \\
& \delta B^\mu = \mathrm{i}\,\bar\varepsilon\,\Gamma^\mu\,\Gamma_I\,[\hat X^I, \hat\Psi] \ , \qquad \delta\hat A^\mu = \mathrm{i}\,\bar\varepsilon\,\Gamma^\mu\,\Gamma_I\,\hat X^I\,\Psi_0 + \mathrm{i}\,\bar\varepsilon\,\Gamma^\mu\,\Gamma_I\,X^I_0\,\hat\Psi \ .
\end{aligned} \tag{5.55}
$$

In the following we show how this model is related to the ABJM matrix model (5.22): For a particular choice of gauge symmetry breaking and scaling limit, we show that one can recover the lorentzian Lie algebra model (5.54) from (5.22). As we will make use of similar reductions throughout this paper, we describe it here in detail.

For this, we consider the ABJM limit $N_L = N_R = N$. To take the scaling limit, we first make the gauge field redefinitions (5.28) which breaks the gauge symmetry to a diagonal $\mathsf{U}(N)$ subgroup of $G = \mathsf{U}(N) \times \mathsf{U}(N)$. With this replacement, the Chern-Simons term from the first line of (5.22) reads

$$\mathcal{S}_{\rm g} = \kappa\, \epsilon^{\mu\nu\lambda}\ {\rm Tr}_V \big(B_\mu\, [A_\nu, A_\lambda] - \tfrac{1}{3}\, B_\mu\, B_\nu\, B_\lambda\big)\ . \tag{5.56}$$

We write the real and imaginary parts of the scalars and fermions as (5.29).We decompose the scalars and fermions further into trace and traceless components as (5.30)

In this decomposition we have identified τ_0 with the generator of $\mathfrak{u}(1)$, and τ_a, $a = 1, \dots, d = N^2 - 1$, are the generators of $\mathfrak{su}(N)$. We scale the fields as (5.37) with all other fields unchanged, and the coupling constant as $\kappa \to \frac{1}{g}\,\kappa$. Taking the limit $g \to 0$ we find that the Chern-Simons term (5.56) reduces to

$$\mathcal{S}_{\rm g} = \kappa\, \epsilon^{\mu\nu\lambda}\ {\rm Tr}_V \big(B_\mu\, [A_\nu, A_\lambda]\big)\ , \tag{5.57}$$

while the second line of (5.22) becomes

$$\mathcal{S}_{\rm k} = -\ {\rm Tr}_V \Big(\big([A_\mu, X^I] + 2B_\mu\, X^I_0\big)^2 + {\rm i}\, \bar\psi\, \gamma^\mu\, [A_\mu, \psi] - 2\bar\psi\, \gamma^\mu\, B_\mu\, \psi_0 - 2\bar\psi_0\, \gamma^\mu\, B_\mu\, \psi \Big)\ . \tag{5.58}$$

In this reduction we have combined the indices i and $i + 4$ for $i = 1, 2, 3, 4$ into an index $I = 1, \dots, 8$, and the components of the spinors into a single Majorana fermion

$$\psi = \big(\chi^1, \dots, \chi^8\big)^\top\ . \tag{5.59}$$

Now we consider the bosonic sextic potential. In the scaling limit, the surviving terms from the potential contain four trace components and eight real traceless components. Using $\mathsf{SU}(4)$ R-symmetry we arrange them as

$$Z^i = \delta^{i,1}\left(X^i_0 + \mathrm{i}\,X^{i+4}_0\right)\tau_0 + \left(X^i_a + \mathrm{i}\,X^{i+4}_a\right)\tau_a \ . \tag{5.60}$$

If we combine the trace components as

$$X^I_0 = \left(X^1_0, 0, 0, 0, X^5_0, 0, 0, 0\right) \ , \tag{5.61}$$

then the reduced bosonic potential reads

$$\mathcal{V}_{\rm b}(X) = -\tfrac{1}{2\kappa^2}\ \mathrm{Tr}_{\,V}\Big(\tfrac14\left(X^K_0\right)^2\left[X^I, X^J\right]^2 - \tfrac12\left(X^I_0\left[X^I, X^J\right]\right)^2\Big) \ . \tag{5.62}$$

We finally consider the quartic Yukawa potential. In this scaling limit, the surviving term of this potential has contributions from two bosonic trace components and two traceless bosonic components. We arrange them as in (5.60) and the spinor components into a Majorana fermion as in (5.59). The resulting potential reads

$$\mathcal{V}_{\rm f}(X,\psi) = -\tfrac{1}{\kappa}\ \mathrm{Tr}_{\,V}\big(\bar\psi\, X^I_0\left[X^J, \gamma_{IJ}\,\psi\right]\big) \tag{5.63}$$

for suitable antisymmetrized products of 8×8 gamma-matrices γ_{IJ} (see e.g. [55, App. A]).

The fully contracted theory thus reads

$$\begin{aligned}\mathcal{S}_{\rm red} = \ & -\mathrm{Tr}_{\,V}\Big(\big([A_\mu, X^I] + 2B_\mu\, X^I_0\big)^2 + \mathrm{i}\,\bar\psi\,\gamma^\mu\,[A_\mu,\psi] - 2\bar\psi\,\gamma^\mu\,B_\mu\,\psi_0 - 2\bar\psi_0\,\gamma^\mu\,B_\mu\,\psi \\ & -\tfrac{1}{2\kappa^2}\left(\tfrac14\,(X^K_0)^2\,[X^I,X^J]^2 - \tfrac12\,X^I_0\,[X^I,X^J]^2\right) - \tfrac1\kappa\,\bar\psi\,X^I_0\,[X^J,\gamma_{IJ}\,\psi] \\ & + \kappa\,\epsilon^{\mu\nu\lambda}\,[A_\mu,A_\nu]\,B_\lambda\Big) \ .\end{aligned} \tag{5.64}$$

This is the original lorentzian Lie algebra model (5.54) with $\mathfrak{h} = \mathfrak{su}(N)$ and inner product (2.42).

5.5 Solutions to 3-Lie algebra model

In this section we study the equations of motion of the reduced 3-Lie algebra model. We introduce deformation terms, construct various BPS solutions, and interpret them as quantized Nambu-Poisson manifolds in the sense of the previous chapter.

5.5.1 Deformations

We introduce deformations consisting of mass and Myers-like flux terms given respectively by

$$\begin{aligned} S_{\rm mass} &= \int \mathrm{d}^3x \left(-\tfrac{1}{2} \sum_{I=1}^{8} \mu_{1,I}^2 \left(X^I, X^I\right) + \tfrac{\mathrm{i}}{2}\, \mu_2 \left(\bar{\Psi}, \Gamma_{3456}\Psi\right)\right) , \\ S_{\rm flux} &= \int \mathrm{d}^3x\ H^{IJKL} \left([X^I, X^J, X^K], X^L\right) , \end{aligned} \tag{5.65}$$

where H^{IJKL} is totally antisymmetric and can be thought of as originating from a four-form flux. A particularly interesting deformation is given by

$$\mu_{1,I} = \mu_2 = \mu \quad \text{and} \quad H^{IJKL} = -\tfrac{\mu}{6} \begin{cases} \varepsilon^{IJKL} & I,J,K,L \leq 4 \\ \varepsilon^{(I-4)(J-4)(K-4)(L-4)} & I,J,K,L \geq 5 \\ 0 & \text{otherwise} \end{cases} . \tag{5.66}$$

This deformation was studied first in [45], see also [82, 56]. It is closely related to the deformation giving rise to the BMN matrix model [19] and homogeneous gravitational wave backgrounds, as we will discuss later on. It explicitly breaks the R-symmetry group $\mathsf{SO}(8)$ down to $\mathsf{SO}(4) \times \mathsf{SO}(4)$, but preserves all 16 supersymmetries if the matter fields live in a 3-Lie algebra. The complete deformed 3-Lie algebra model reads as

$$
\begin{aligned}
S = &-\tfrac{1}{2}\left(A_\mu X^I, A^\mu X^I\right) + \tfrac{\mathrm{i}}{2}\left(\bar{\Psi}, \Gamma^\mu\, A_\mu \Psi\right) \\
&- \tfrac{1}{2}\sum_{I=1}^{8} \mu_{1,I}^2\left(X^I, X^I\right) + \tfrac{\mathrm{i}}{2}\,\mu_2\left(\bar{\Psi}, \Gamma_{3456}\Psi\right) + H^{IJKL}\left([X^I, X^J, X^K], X^L\right) \\
&+ \tfrac{\mathrm{i}}{4}\left(\bar{\Psi}, \Gamma_{IJ}[X^I, X^J, \Psi]\right) - \tfrac{1}{12}\left([X^I, X^J, X^K], [X^I, X^J, X^K]\right) \\
&+ \tfrac{1}{6}\,\epsilon^{\mu\nu\lambda}\ \mathrm{Tr}_{\,\mathfrak{g}_{\mathcal{A}}}\left(A_\mu\left[A_\nu, A_\lambda\right]\right)\ .
\end{aligned}
\tag{5.67}
$$

This deformed model has the same amount of supersymmetry as the original unreduced field theory. However, it is only invariant under the group $\mathsf{SO}(1,2)\times\mathsf{SO}(8)$ instead of the desired 11-dimensional Lorentz group $\mathsf{SO}(1,10)$, which is due to the dichotomy of gauge and matter fields in the original BLG theory. This is in marked contrast to the IKKT matrix model which arises from dimensional reduction of maximally supersymmetric Yang-Mills theory to zero dimensions, and therefore exhibits manifest $\mathsf{SO}(1,9)$ invariance.

We also consider similar deformations of the IKKT matrix model.In the strong coupling limit, the Myers-like term in (5.67) is reduced according to

$$
H^{IJKL}\left([X^I, X^J, X^K], X^L\right) \quad\longrightarrow\quad 4g_{\mathrm{YM}}\, H^{IJK8}\left([X^I, X^J], X^K\right)\ , \tag{5.68}
$$

and this is the Myers term appearing in the deformation of the BFSS model to the BMN matrix model [19]. Including the mass terms, the deformation terms reduce to

$$
\begin{aligned}
S_{\text{mass+flux}} = &-\tfrac{1}{2}\sum_{I=1}^{7}\mu_{1,I}^2\left(\mathscr{X}^{I+2}, \mathscr{X}^{I+2}\right) + \tfrac{\mathrm{i}}{2}\,\mu_2\left(\bar{\Psi}, \Gamma_{3456}\Psi\right) \\
&+ 4g_{\mathrm{YM}}\sum_{I,J,K=1}^{7} H^{IJK8}\left([\mathscr{X}^{I+2}, \mathscr{X}^{J+2}], \mathscr{X}^{K+2}\right)\ .
\end{aligned}
\tag{5.69}
$$

5.5.2 Equations of motion

The classical equations of motion of this model with a metric 3-Lie algebra $\mathcal{A}$ read

$$
\begin{aligned}
A_\mu\, A^\mu\, X^I - \mu_{1,I}^2\, X^I - \mathrm{i}\,[\bar\Psi, X^J, \Gamma_{IJ}\Psi] \\
+\tfrac{1}{2}\,\big[X^J, X^K, [X^J, X^K, X^I]\big] + 4H^{IJKL}\,[X^J, X^K, X^L] &= 0\ , \\
\Gamma^\mu\, A_\mu\Psi + \mu_2\,\Gamma_{3456}\Psi + \tfrac{1}{2}\,\Gamma_{IJ}[X^I, X^J, \Psi] &= 0\ , \\
\tfrac{1}{2}\,\epsilon^{\mu\nu\lambda}\,[A_\nu, A_\lambda] - \tfrac{1}{\gamma^2}\,\big[A_\nu, [A^\nu, A^\mu]\big] - D(X^I, A^\mu X^I) + \tfrac{\mathrm{i}}{2}\,D(\bar\Psi, \Gamma^\mu\Psi) &= 0\ .
\end{aligned}
\tag{5.70}
$$

The classical equations of motion of the IKKT matrix model (5.14), i.e. the strong coupling limit of the deformed 3-Lie algebra model (5.67), read

$$
\begin{aligned}
\big[\mathscr{X}_N, [\mathscr{X}^N, \mathscr{X}^M]\big] - \tfrac{\mathrm{i}}{2}\,\Gamma^\mu_{\alpha\beta}\{\Psi^\beta, \bar\Psi^\alpha\} + \Delta^M &= 0\ , \\
[\mathscr{X}_M, \Psi] + \mu_2\,\Gamma_{3456}\Psi &= 0\ ,
\end{aligned}
\tag{5.71}
$$

where α, β are spinor indices of a Majorana-Weyl spinor of $\mathsf{SO}(1,9)$ and the deformation contribution is

$$
\Delta^M = \begin{cases} -\mu_{1,M-2}^2\,\mathscr{X}^M + 12 g_{\mathrm{YM}} \displaystyle\sum_{I,J=1}^{7} H^{IJ(M-2)8}\,[\mathscr{X}^{I+2}, \mathscr{X}^{J+2}] & \text{for} \quad 3 \leq M \leq 9 \\ 0 & \text{for} \quad M = 0,1,2 \end{cases}\ .
\tag{5.72}
$$

In the following we will study solutions to these equations and examine their classical stability.

5.5.3 Fuzzy spheres

As it is the most prominent 3-Lie algebra, let us start with a solution involving $\mathcal{A}_4$. For this, we choose the supersymmetric deformation (5.66) to obtain a natural $\mathsf{SO}(4)$ symmetry group, which matches the associated Lie group of $\mathcal{A}_4$. We put $A_\mu = \Psi = 0$. As our scalar fields, we choose

$$
X^i = \alpha\, e_i\ , \quad X^{i+4} = 0\ , \quad \text{with} \quad \alpha^4 + \tfrac{4}{3}\,\mu\,\alpha^2 + \tfrac{1}{3}\,\mu^2 = 0\ ,
\tag{5.73}
$$

where e_i, $i = 1,2,3,4$ are generators of $\mathcal{A}_4$. This solution corresponds to fuzzy three-sphere. We can compute the hessian of the action $\frac{\delta^2 S}{\delta X^{i\,a}\,\delta X^{j\,b}}$, where $\delta X^{i\,a}$ describes the variation of X^i in the 3-Lie algebra direction e_a. We find a 16×16 matrix with eigenvalues $(0,2,6)$ occurring in multiplicities $(6,9,1)$. The six flat directions correspond to variations rotating the fuzzy sphere. (The other eigenvalues correspond to "squashing" the fuzzy sphere in various ways.) We conclude that the solution (5.73) is indeed a stable stationary point of the action (5.67). Moreover, like the ground states used in [45], our solutions are invariant under the full set of 16 supersymmetries of the deformed action. This can be checked explicitly by noting that the supersymmetry transformation for $A_\mu = 0$ reads [45]

$$\delta_\varepsilon X^I = \mathrm{i}\,\bar{\varepsilon}\,\Gamma^I \Psi \ , \qquad \delta_\varepsilon \Psi = -\tfrac{1}{6}\,[X^I, X^J, X^K]\,\Gamma^{IJK}\varepsilon - \mu\,\Gamma_{3456}\,\Gamma^I\,X^I \varepsilon \ , \tag{5.74}$$

where ε is a constant Majorana spinor of $\mathsf{SO}(1,10)$ satisfying $\Gamma_{012}\varepsilon = \varepsilon$, and hence our fuzzy three-sphere background satisfies the supersymmetry condition $\delta_\varepsilon X^I = 0 = \delta_\varepsilon \Psi$.

We can now apply the Higgs mechanism. We assume that one of the scalar fields acquires a vev and perform a strong coupling expansion. Let us choose $X^4 = g_{\rm YM}\, e_4 + Y^4$ and take a double scaling limit $g_{\rm YM}, \mu \to \infty$ with $\hat{\mu} = \frac{\mu}{g_{\rm YM}}$ fixed. The equations of motion reduce to

$$\begin{aligned} \big[X^j, [X^i, X^j]\big] - 2\hat{\mu}\,\varepsilon^{ijk}\,[X^j, X^k] = 0 \ , \\ \big[X^j, X^k, [X^j, X^k, X^4]\big] + 2\hat{\mu}\,\varepsilon^{4jkl}\,[X^j, X^k, X^l] = 0 \ , \end{aligned} \tag{5.75}$$

for $i = 1,2,3$. The first equation is the equation of motion of the IKKT model with a Myers term and its solution is a fuzzy two-sphere, i.e. the matrices X^i take values in $\mathfrak{su}(2)$. The second equation requires the Lie algebra $\mathfrak{su}(2)$ to be consistently embedded in $\mathcal{A}_4$. Altogether, we see that the fuzzy two-sphere originates as the strong coupling limit of the fuzzy three-sphere. Geometrically, we reduced the fuzzy three-sphere to its equator with radius $g_{\rm YM}$, which corresponds to the fuzzy two-sphere solution. This is *not* the projection of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$.

Note that our deformation is very similar to that of the BMN model, which can be considered as the BFSS model on a non-trivial pp-wave background. The fuzzy two-sphere solution is in that case interpreted as giant gravitons, i.e. M2-branes wrapping the fuzzy S^2 with certain kinematical properties. The supersymmetric deformation (5.66) has been holographically linked in [45] to the matrix model description of the maximally supersymmetric type IIB plane wave in discrete light-cone quantization; this Hpp-wave background is a ten-dimensional Cahen-Wallach symmetric space with metric

$$\mathrm{d}s^2 = 2\,\mathrm{d}x^+\,\mathrm{d}x^- + \sum_I \left(\mathrm{d}x_I^2 - \tfrac{1}{4}\,\mu^2\,x_I^2\,(\mathrm{d}x^+)^2\right)\,, \tag{5.76}$$

and constant null self-dual Ramond-Ramond five-form flux $H_{\mathrm{RR}} = \mu\,\mathrm{d}x^+ \wedge (\mathrm{d}x^{1234} + \mathrm{d}x^{5678})$, where the sum runs over $I = 1, \dots, 8$ and $\mathrm{d}x^{IJKL} := \mathrm{d}x^I \wedge \mathrm{d}x^J \wedge \mathrm{d}x^K \wedge \mathrm{d}x^L$, which arises as a Penrose limit of the near horizon black hole geometry $\mathrm{AdS}_5 \times S^5$ in type IIB supergravity [22]. The fuzzy three-sphere solution obtained here was identified in [45] with longitudinal D3-brane giant gravitons in this background.

5.5.4 $\mathbb{R}^3_\lambda$ and the noncommutative plane

In the (undeformed) IKKT matrix model, the simplest classical solution is given by operators $\mathscr{X}^1 = \lambda_1$ and $\mathscr{X}^2 = \lambda_2$, where λ_1 and λ_2 are the generators of the Heisenberg algebra $[\lambda_1, \lambda_2] = \theta\,\mathbb{1}$, $\theta \in \mathbb{R}$. The D-brane interpretation of this solution involves D(-1)-branes described by the scalar fields in a background B-field proportional to θ^{-1} which are smeared out into a D1-brane, whose worldvolume is the noncommutative space $\mathbb{R}^2_\theta$. This solution can be evidently extended to direct sums of $\mathbb{R}^2_\theta$, by demanding that further pairs of scalar fields satisfy the Heisenberg algebra. Note, however, that there is an issue with the normalizability of the central element $\mathbb{1}$, as the Heisenberg algebra only has infinite-dimensional unitary representations.

The classical vacuum state of the reduced model with action (5.14) is given by commuting matrices $\mathscr{X}^M$. Noncommutative spacetime arises instead as a vacuum configuration of the *twisted* reduced model with action

$$\widetilde{S}_{\rm IKKT} = {\rm Tr}\Big(-\tfrac{1}{4}\big([\mathscr{X}_M,\mathscr{X}_N] - \theta_{MN}\,\mathbb{1}\big)\big([\mathscr{X}^M,\mathscr{X}^N] - \theta^{MN}\,\mathbb{1}\big) + \tfrac{\rm i}{2}\,\bar\Psi\,\Gamma^M[\mathscr{X}_M,\Psi]\Big)\ , \tag{5.77}$$

where the "twist" θ_{MN} is generically a 10×10 constant antisymmetric real matrix; in the special case considered above only $\theta_{12}=\theta$ is nonzero. The solutions with $\mathscr{X}^M = \lambda_M$, $[\lambda_M,\lambda_N] = \theta_{MN}\,\mathbb{1}$ correspond to BPS-saturated backgrounds which preserve half the $\mathcal{N}=2$ supersymmetry. Upon introducing the covariant coordinates

$$\mathscr{X}_M = \lambda_M + \theta_{MN}\,\mathscr{A}^N\ , \tag{5.78}$$

corresponding to expansion around the infinitely-extended D-branes in the original IKKT model, we obtain the action for $\mathsf{U}(1)$ noncommutative supersymmetric Yang-Mills theory with 16 supercharges [6] and trivial vacuum state $\mathscr{A}^M=0$; the gauge fields $\mathscr{A}^M$ are interpreted as dynamical fluctuations about the noncommutative spacetime. To obtain the action for noncommutative Yang-Mills theory with $\mathsf{U}(m)$ gauge group, corresponding to the background of m coincident D-branes, we expand around the vacuum $\mathscr{X}^M = \lambda_M\otimes\mathbb{1}_m$. Exactly the same sort of configurations arise in our model. The configuration $X^i=\tau_i$ for $i=1,2,3$ and $X^j=0$ for $j=4,5,6,7,8$, where $\tau_1,\tau_2,\tau_3,\mathbb{1}$ are generators of the Nambu-Heisenberg 3-Lie algebra $\mathcal{A}_{\rm NH}$,

$$[\tau_1,\tau_2,\tau_3] = \theta\,\mathbb{1}\ , \qquad [\mathbb{1},\tau_i,\tau_j] = 0\ , \tag{5.79}$$

forms a solution to our equations of motion (5.70) in the absence of fluxes and for $A_\mu=\Psi=0$. Recently it was derived as a boundary condition on the geometry of an M5-brane in the M2–M5 brane system in a constant background C-field [28]. It has associated Lie algebra $\mathfrak{g}_{\mathcal{A}_{\rm NH}}\cong\mathbb{R}^6$.

The solution $X^I=\tau_I$, $[\tau_I,\tau_J,\tau_K] = \Theta_{IJK}\,\mathbb{1}$, with Θ^{IJK} a constant real three-form flux, describes the vacuum state of the "twisted" version of the scalar potential of the action (5.67) based on the 3-Lie algebra $\mathcal{A}=\mathcal{A}_{\rm NH}$ in the absence of masses and fluxes, which generically reads

$$\widetilde{V}(X) = -\tfrac{1}{12}\left([X^I, X^J, X^K] - \Theta^{IJK}\, \mathbb{1}, [X^I, X^J, X^K] - \Theta^{IJK}\, \mathbb{1}\right) . \tag{5.80}$$

In fact, this solution preserves 16 supersymmetries. This follows from the general fact that the model (5.67) based on a 3-Lie algebra $\mathcal{A}$ with central element $\mathbb{1}$ for the configuration (5.66) possesses an additional 16 kinematical supersymmetries [45]

$$\tilde{\delta}_\xi X^I = 0 , \qquad \tilde{\delta}_\xi \Psi = \xi\, \mathbb{1} , \tag{5.81}$$

where ξ is a constant spinor of $\mathsf{SO}(1,10)$ satisfying $\Gamma_{012}\xi = -\xi$. Setting $X^I = \tau_I$, $\mu = 0$ and $\xi = \frac{1}{6}\,\Theta_{IJK}\,\Gamma^{IJK}\varepsilon$ in the supersymmetry transformations (5.74) and (5.81), we find the relations

$$(\delta_\varepsilon + \tilde{\delta}_\xi)X^I = 0 , \qquad (\delta_\varepsilon + \tilde{\delta}_\xi)\Psi = 0 , \tag{5.82}$$

and hence half of the 32 supersymmetries are preserved in these backgrounds. This is consistent with the calculation of [86] which shows that the one-loop vacuum energy of these backgrounds vanishes.

We can interpret this solution as a quantized Nambu-Poisson manifold. If we assume that X^3 acquires a vev proportional to a coupling constant, then in the strong coupling limit the Nambu-Heisenberg algebra reduces to the ordinary Heisenberg algebra. In this sense, the noncommutative plane $\mathbb{R}^2_\theta$ can be regarded as the strong coupling limit of $\mathbb{R}^3_\lambda$. Again, we can extend our solution to the direct sum $\mathbb{R}^3_\lambda \oplus \mathbb{R}^3_\lambda$ by demanding that three more of the scalar fields form another copy of the Nambu-Heisenberg 3-Lie algebra; This is the quantized geometry relevant to an M5-brane in a constant C-field background [28, 34]. As in the case of the IKKT matrix model, there is a problem with the normalizability of the 3-central element $\mathbb{1}$; The compatibility condition (2.22) forbids us to assign finite norm to $\mathbb{1}$.

5.5.5 Homogeneous plane wave backgrounds

The homogeneous plane wave with metric (5.76), and supported by a Neveu-Schwarz flux, can be constructed as the group manifold of the twisted Heisenberg group whose Lie algebra is an extension of the Heisenberg algebra by one additional generator J defined by

$$[\lambda_M, \lambda_N] = \theta_{MN}\, \mathbb{1} \ , \qquad [J, \lambda_M] = \theta_{MN}\, \lambda_N \ , \qquad [\mathbb{1}, \lambda_M] = [\mathbb{1}, J] = 0 \ . \tag{5.83}$$

The simplest case is $\theta_{MN} = \varepsilon_{MN}$, $M, N = 1, 2$ corresponding to the Nappi-Witten algebra [76], which is a non-semisimple lorentzian Lie algebra of dimension four. The Lie brackets (5.83) are then those of the universal central extension of $\mathfrak{iso}(2)$.

Let us now consider the mass and flux deformations of the IKKT model (5.69) given by

$$\mu_{1,6} = \mu_{1,7} = \mu \ , \qquad H^{5678} = h \ , \tag{5.84}$$

where all other mass terms and components of H vanish. We choose the ansatz

$$\mathscr{X}^6 = \alpha\, \mathbb{1} \ , \qquad \mathscr{X}^7 = \beta\, J \ , \qquad \mathscr{X}^8 = \gamma\, \lambda_1 \ , \qquad \mathscr{X}^9 = \gamma\, \lambda_2 \ , \tag{5.85}$$

with $\mathscr{X}^M = 0 = \Psi$ for $M = 0, 1, 2, 3, 4, 5$, for our solution. Then the equations of motion (5.71) are satisfied if

$$\mu^2 = 576 g_{\rm YM}^2\, h^2 \quad \text{and} \quad \beta = -24 g_{\rm YM}\, h \ , \tag{5.86}$$

while the parameters α and γ are arbitrary. These solutions are not supersymmetric.

This noncommutative background can be regarded as a linear Poisson structure on a four-dimensional Hpp-wave. The invariant, non-degenerate symmetric bilinear forms on the Nappi-Witten Lie algebra are parametrized by a real number b and are defined by

$$(\lambda_i, \lambda_j) = \delta_{ij} \ , \qquad (\mathbb{1}, J) = 1 \ , \qquad (J, J) = b \tag{5.87}$$

for $i, j = 1, 2$, with all other pairings vanishing. Then the group manifold possesses a homogeneous bi-invariant lorentzian metric defined by the pairing of the left-invariant Cartan-Maurer one-forms as

$$\mathrm{d}s_4^2 = (g^{-1}\,\mathrm{d}g, g^{-1}\,\mathrm{d}g) \ . \tag{5.88}$$

We can parametrize group elements g as

$$g = \exp\left(\mathrm{e}^{\mathrm{i}\,\beta\,x^+/2}\,z\,\mathscr{Z}_+ + \mathrm{e}^{-\mathrm{i}\,\beta\,x^+/2}\,\overline{z}\,\mathscr{Z}_-\right)\,\exp\left(x^-\,\mathscr{X}^6 + x^+\,\mathscr{X}^7\right)\ , \tag{5.89}$$

where $\mathscr{Z}_\pm = \mathscr{X}^8 \pm \mathrm{i}\,\mathscr{X}^9$, $x^\pm \in \mathbb{R}$ and $z \in \mathbb{C}$. Then the metric in these global coordinates reads

$$\mathrm{d}s_4^2 = 2\alpha\,\beta\,\mathrm{d}x^+\,\mathrm{d}x^- + \gamma^2\,|\mathrm{d}z|^2 - \tfrac{1}{4}\,\beta^2\left(\gamma^2\,|z|^2 - b\right)(\mathrm{d}x^+)^2\ , \tag{5.90}$$

which is the standard form of the plane wave metric of a four-dimensional Cahen-Wallach symmetric spacetime in Brinkman coordinates. This spacetime is further supported by a constant null Neveu-Schwarz three-form flux

$$H_{\mathrm{NS}} = -\tfrac{1}{3}\left(g^{-1}\,\mathrm{d}g, \mathrm{d}(g^{-1}\,\mathrm{d}g)\right) = 2\,\mathrm{i}\,\beta\,\gamma^2\,\mathrm{d}x^+ \wedge \mathrm{d}z \wedge \mathrm{d}\overline{z}\ , \tag{5.91}$$

which is proportional to the flux deformation h of the matrix model.

The hessian for this solution is a 16×16 matrix with eigenvalues $(0, 1728, 576, 1152)\,h$ of multiplicities $(6, 1, 1, 8)$. The six flat directions corresponds to the following symmetries of the matrix model defined by (5.14) and (5.69) with the appropriate inner product (5.87). One direction corresponds to the $U(1)$ subgroup of the plane wave isometry group rotating the transverse space $z \in \mathbb{C}$. Three directions correspond to translations of the Nappi-Witten generators by multiples of the central element $\mathbb{1}$. Of these, only shifts of the generator J are inner automorphisms of the Lie algebra (5.83); In particular, the automorphism $J \mapsto J - b\,\mathbb{1}$ can be used to set the parameter b to 0 in (5.87), which is equivalent to the redefinition $x^- \to x^- - \frac{1}{8}\,\frac{\gamma^2\,\beta}{\alpha}\,b\,x^+$ in the plane wave metric (5.90). The shifts in λ_i are isometries which translate the transverse space along the null direction x^+. Another direction corresponds to scale transformations $\mathbb{1} \to \mathrm{e}^{\zeta}\,\mathbb{1}$, which becomes a Lie algebra automorphism after redefining $\lambda_i \to \mathrm{e}^{\zeta/2}\,\lambda_i$. The final symmetry of the action corresponds to the simultaneous scale transformations $J \to \mathrm{e}^{-\zeta}\,J$, $\lambda_i \to \mathrm{e}^{\zeta}\,\lambda_i$.

This Hpp-wave background is thus a stable solution of the deformed IKKT matrix model. It arises as a Penrose limit of the maximally supersymmetric black hole solution with near horizon geometry $\mathrm{AdS}_2 \times S^2$ in four-dimensional toroidal compactification of string theory and M-theory, or alternatively of the near horizon region of NS5-branes [22]. Extending this solution by an additional noncommutative plane gives a Cahen-Wallach space which is a Penrose limit of the near horizon geometry $\mathrm{AdS}_3 \times S^3$ of the self-dual string in six dimensions [22].

We can find a similar solution to our 3-Lie algebra model if we consider the Nappi-Witten 3-Lie algebra (2.33), if we choose the background (5.65) with mass and flux terms

$$\mu_{1,6} = \mu_{1,7} = \mu_{1,8} = \mu \ , \qquad H^{5678} = h \ , \tag{5.92}$$

and all other mass terms and components of H are zero. The obvious generalization of the ansatz (5.85) to the 3-Lie algebra model reads

$$X^4 = \alpha\,\mathbb{1} \ , \qquad X^5 = \beta\,J \ , \qquad X^6 = \gamma\,\tau_1 \ , \qquad X^7 = \gamma\,\tau_2 \ , \qquad X^8 = \gamma\,\tau_3 \ , \tag{5.93}$$

with $A_\mu = 0 = \Psi$ and $X^I = 0$ for $I = 1,2,3$, and from the equations of motion we obtain conditions on the parameters

$$\mu^2 = 64h^2 \ , \qquad \beta = -\tfrac{8h}{\gamma} \ , \tag{5.94}$$

while the parameters α and γ are again arbitrary. It is natural to associate this solution with the extension of the pp-wave geometry (5.90) by an additional transverse direction $y \in \mathbb{R}$,

$$\mathrm{d}s_5^2 = 2\alpha\,\beta\,\mathrm{d}x^+\,\mathrm{d}x^- + \gamma^2\,\big(|\mathrm{d}z|^2 + \mathrm{d}y^2\big) - \tfrac14\,\beta^2\,\big(\gamma^2\,(|z|^2 + y^2) - b\big)\,(\mathrm{d}x^+)^2 \ . \tag{5.95}$$

This five-dimensional Cahen-Wallach space arises as a Penrose limit of an $\mathrm{AdS}_2 \times S^3$

background, which corresponds to the near horizon geometry of black hole solutions for $\mathcal{N} = 2$ supergravity in five dimensions [22].

The hessian of this solution is a 25×25 matrix with eigenvalues $(0, 192, 64, 128, 256, 320)\, h$ of multiplicity $(8, 5, 3, 3, 3, 3)$. Again the eight flat directions correspond to the $\mathsf{SO}(3)$ subgroup of the plane wave isometry group generating rotations of the transverse space $(z, y) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$, to null translations of the transverse space, to automorphisms $J \mapsto J - b\,\mathbb{1}$ of the Nappi-Witten 3-Lie algebra, and to conformal rescalings of the 3-central element $\mathbb{1}$. This background is thus a stable solution of the 3-Lie algebra reduced model (5.3).

Chapter 6

Cohomological 3-algebra models

In what follows we shall be interested in the exact computation of the partition function of the ABJM matrix model using localization techniques. For this, we shall need to deform the model in suitable ways in order to obtain a theory with equivariant cohomological symmetries that will enable the localization procedure to be applied exactly. In this chapter we shall study cohomological versions of our membrane matrix models that are obtained by a topological twisting procedure, and point out various ensuing difficulties. The possible inequivalent twists of Chern-Simons-matter theories in three dimensions with $\mathcal{N} \geqslant 4$ supersymmetry were classified in [64]. In the case of an $\mathcal{N} = 8$ theory with R-symmetry group $\mathsf{SO}(8)$, restricting the supercharges to the vector representation does not generate any additional twists. However, letting the supercharges transform in the spinor representation via the triality of the R-symmetry group does allow for two additional twists. One of these new twists was constructed in [68]; in this section we investigate the effect of applying the Mukhi-Papageorgakis map to this topologically twisted theory. After dimensional reduction, the ensuing 3-algebra model can potentially induce a cohomological deformation of the ABJM matrix model under the mappings of §5.4 which is dual to a novel topological twisting of the ten-dimensional IKKT model.

We begin this chapter by reviewing a topological twist of the BLG theory. We then show how this theory is mapped to the $\mathcal{N} = 4$ equivariant extension of the Blau-Thompson model under the Mukhi-Papageorgakis map. Applying this mechanism to the dimensionally reduced twisted BLG theory, we find a novel topological twist of the ten-dimensional IKK matrix model. We then discuss lifting this twist to the ABJM matrix model using the mappings of §5.4.

6.1 Topologically twisted BLG theory

We begin by briefly reviewing the topologically twisted theory constructed in [68]. In the conventions of §5.1.2, the BLG action without deformations in euclidean space reads

$$\begin{aligned} S_{\rm BLG} = \int {\rm d}^3x\ \Big(&\tfrac{\rm i}{2}\,\epsilon^{\mu\nu\lambda}\,{\rm Tr}_{\,\mathfrak{g}_{\mathcal{A}}}\big(A_\mu\,\partial_\nu A_\lambda - \tfrac13\,A_\mu\,[A_\nu,A_\lambda]\big) + \tfrac12\,(\nabla_\mu X^I,\nabla^\mu X^I) \\ &- \tfrac{\rm i}{2}\,(\bar\Psi,\Gamma^\mu\,\nabla_\mu\Psi) + \tfrac{\rm i}{4}\,(\bar\Psi,\Gamma_{IJ}\,[X^I,X^J,\Psi]) + \tfrac{1}{12}\,([X^I,X^J,X^K],[X^I,X^J,X^K])\Big)\ . \end{aligned} \tag{6.1}$$

This action is invariant under the 16 supersymmetries generated by

$$\begin{aligned} \delta X^I &= {\rm i}\,\bar\varepsilon\,\Gamma^I\,\Psi\ , \\ \delta\Psi &= \nabla_\mu X^I\,\Gamma^\mu\,\Gamma_I\,\varepsilon - \tfrac16\,[X^I,X^J,X^K]\,\Gamma_{IJK}\,\varepsilon\ , \\ \delta A_\mu &= {\rm i}\,\bar\varepsilon\,\Gamma_\mu\,\Gamma_I\,[X^I,\Psi,-]\ . \end{aligned} \tag{6.2}$$

The main difference from the split signature case is that the euclidean action involves only the holomorphic part of the spinor, so that we must make the definition

$$\bar\Psi := \Psi^\top\,\mathcal{C}\ , \tag{6.3}$$

where $\mathcal{C}$ is the charge conjugation matrix satisfying

$$\mathcal{C}\,\Gamma^M\,\mathcal{C}^{-1} = \big(-\Gamma^M\big)^\top\ , \qquad \mathcal{C}^\top = -\mathcal{C}\ , \tag{6.4}$$

and M is the 11-dimensional vector index which decomposes into $\mu = 1,2,3$ and $I = 4,\dots,11$.

Consider now the rotational symmetry breaking $\mathsf{Spin}(11) \to \mathsf{Spin}(3)\times\mathsf{Spin}(3)\times$

$\mathsf{Spin}(5)$, under which the corresponding gamma-matrices can be decomposed as

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma^3 \ , \Gamma^{\mu+3} = \mathbb{1} \otimes \gamma^\mu \otimes \mathbb{1} \otimes \gamma^1 \ , \qquad \Gamma^{i+6} = \mathbb{1} \otimes \mathbb{1} \otimes \gamma^i \otimes \gamma^2 \tag{6.5}$$

where γ^μ, $\mu = 1,2,3$, are Pauli spin matrices and γ^i, $i = 1,\dots,5$, are 4×4 gamma-matrices in five euclidean dimensions. The charge conjugation matrix decomposes as

$$\mathcal{C} = \mathrm{i}\,\gamma^2 \otimes \mathrm{i}\,\gamma^2 \otimes C \otimes \mathbb{1} \ , \tag{6.6}$$

where C is the five-dimensional charge conjugation matrix. The $\mathsf{SO}(8)$ chirality matrix is

$$\Gamma^{123} = -\mathrm{i}\,\Gamma^{4\cdots 11} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathrm{i}\,\gamma^3 \ . \tag{6.7}$$

This means that the spinors have four indices: two for the $\mathsf{SO}(3)$ factors, one for the $\mathsf{SO}(5)$ factor, and one for $\mathsf{SO}(8)$ chirality. The twist is constructed by replacing an $\mathsf{SO}(3)$ factor with the diagonal subgroup of $\mathsf{Spin}(3)\times\mathsf{Spin}(3)$. Then we can expand the twisted spinors

$$\Psi = (\psi, \chi^\mu) \tag{6.8}$$

into an $\mathsf{SO}(3)$ scalar and vector. We also decompose the bosons

$$X^I = (X^\mu, Y^i) \tag{6.9}$$

into an $\mathsf{SO}(3)$ vector and five scalars.

The resulting twisted BLG action is the sum of a topological action

$$\begin{aligned} S_{\rm top} = \int \mathrm{d}^3x\ \Big(&\tfrac{\mathrm{i}}{2}\,\epsilon^{\mu\nu\lambda}\ \mathrm{Tr}_{\,\mathfrak{g}_{\mathcal{A}}}\big(A^+_\mu\,\partial_\nu A^+_\lambda + \tfrac{1}{3}\,A^+_\mu\,[A^+_\nu, A^+_\lambda]\big) \\ &- \tfrac{1}{2}\,\epsilon^{\mu\nu\lambda}\,\big(\bar\chi_\mu, \nabla^+_\nu\chi_\lambda - \mathrm{i}\,\gamma_i\,[\chi_\nu, X_\lambda, Y^i]\big)\Big) \end{aligned} \tag{6.10}$$

plus a metric-dependent cohomological action

$$S_{\rm m} = \int \mathrm{d}^3x \, \Big(\tfrac14 \, (\nabla^\mu X^\nu - \nabla^\nu X^\mu , \nabla^\mu X^\nu - \nabla^\nu X^\mu) + \tfrac12 \, (\nabla^+_\mu Y^i , \nabla^-_\mu Y^i) \tag{6.11}$$

$$+ \tfrac12 \, (\nabla_\mu X^\mu + \tfrac{\mathrm{i}}{6} \, \epsilon_{\mu\nu\lambda} \, [X^\mu, X^\nu, X^\lambda], \nabla_\mu X^\mu + \tfrac{\mathrm{i}}{6} \, \epsilon_{\mu\nu\lambda} \, [X^\mu, X^\nu, X^\lambda])$$

$$+ \tfrac12 \, ([Y^i, Y^j, Y^k], [Y^i, Y^j, Y^k]) + \tfrac12 \, ([X^\mu, Y^j, Y^k], [X^\mu, Y^j, Y^k])$$

$$+ (\bar\psi, \nabla^-_\mu \chi^\mu + \mathrm{i} \, \gamma_i \, [Y^i, X^\mu, \chi_\mu] + \tfrac{\mathrm{i}}{4} \, \gamma_{ij} \, [Y^i, Y^j, \psi])$$

$$+ \tfrac{\mathrm{i}}{4} \, (\bar\chi^\mu, \gamma_{ij} \, [Y^i, Y^j, \chi_\mu]) \Big) \ ,$$

where the gauge fields and covariant derivatives have been complexified so that

$$A^\pm_\mu := A_\mu \mp \tfrac{\mathrm{i}}{2} \, \epsilon_{\mu\nu\lambda} \, [X^\nu, X^\lambda, -] \ , \qquad \nabla^\pm_\mu := \nabla_\mu \pm \tfrac{\mathrm{i}}{2} \, \epsilon_{\mu\nu\lambda} \, [X^\nu, X^\lambda, -] \ . \tag{6.12}$$

The total action $S_{\rm top} + S_{\rm m}$ is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta X^\mu &= \bar\varepsilon \, \chi^\mu \ , \\
\delta Y^i &= \bar\varepsilon \, \gamma^i \, \psi \ , \\
\delta\psi &= -\big(\nabla_\mu X^\mu + \tfrac{\mathrm{i}}{6} \, \epsilon_{\mu\nu\lambda} \, [X^\mu, X^\nu, X^\lambda]\big) \, \varepsilon \ , \\
\delta\chi_\mu &= \epsilon_{\mu\nu\lambda} \, \nabla^\nu X^\lambda \, \varepsilon + \nabla^+_\mu Y^i \, \gamma_i \, \varepsilon + \tfrac{\mathrm{i}}{2} \, [Y^i, Y^j, X_\mu] \, \gamma_{ij} \, \varepsilon \ , \\
\delta A_\mu &= \mathrm{i} \, \bar\varepsilon \, \big(- [X_\mu, \psi, -] + \epsilon_{\mu\nu\lambda} \, [X^\nu, \chi^\lambda, -] + \gamma_i \, [Y^i, \chi_\mu, -] \big) \ .
\end{aligned} \tag{6.13}$$

Setting the fermions equal to zero in (6.13), one finds that the corresponding BPS equations for the supersymmetric solutions of the field theory are

$$\begin{aligned}
\nabla^+_\mu X^\mu - \tfrac{\mathrm{i}}{3} \, \epsilon_{\mu\nu\lambda} \, [X^\mu, X^\nu, X^\lambda] = 0 &= \nabla^+_\mu X_\nu - \nabla^+_\nu X_\mu \ , \\
\nabla^+_\mu Y^i = 0 &= F^+_{\mu\nu} \ , \\
[Y^i, Y^j, Y^k] = 0 &= [Y^i, Y^j, X^\mu]
\end{aligned} \tag{6.14}$$

where the twisted field strength is defined by

$$F^{+}_{\mu\nu} := F_{\mu\nu} - \mathrm{i}\,\epsilon_{\nu\lambda\rho}\,[\nabla_\mu X^\lambda, X^\rho, -] + \mathrm{i}\,\epsilon_{\mu\lambda\rho}\,[\nabla^{+}_\nu X^\lambda, X^\rho, -] \tag{6.15}$$

with $F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$.

6.1.1 Mapping to the Blau-Thompson model

Let us consider the metric 3-Lie algebra $\mathcal{A} = A_4$ and apply the higgsing procedure to the twisted BLG theory. We proceed by letting the scalar fields Y^i have classical values proportional to fixed 3-Lie algebra elements. Using $\mathsf{SO}(5)$ symmetry we can assume that only Y^1 acquires a vacuum expectation value, and by $\mathsf{SO}(4)$ invariance we can align this value in the 3-Lie algebra direction τ_4. Hence we make the replacement

$$Y^1 \ \longrightarrow \ -g\,\tau_4 + Y^1\ , \tag{6.16}$$

where g is a gauge coupling constant. The reduction of the gauge fields works in the usual way: With respect to the splitting $\mathfrak{g}_{\mathcal{A}} = \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, we make the replacements

$$A^{\pm}_\mu \ \longrightarrow \ A^{\pm}_\mu \pm \tfrac{1}{2}\,B_\mu\ , \tag{6.17}$$

where now we regard $A^{\pm}_\mu, B_\mu \in \mathfrak{so}(3)$. In the strong coupling limit $g \to \infty$, 3-brackets containing Y^1 reduce to the brackets $[X^\mu, X^\nu] := [X^\mu, X^\nu, -\tau_4]$ of the Lie algebra $\mathcal{A}' = \mathfrak{so}(3)$; We denote the invariant form on either factors of $\mathfrak{so}(3) = \mathfrak{su}(2)$ by $\mathrm{Tr}_{\mathcal{A}'}$, which coincides with the Cartan-Killing form. We also define a modified field strength

$$\widetilde{F}_{\mu\nu} := F_{\mu\nu} - \mathrm{i}\,\epsilon_{\nu\lambda\rho}\,[\nabla_\mu X^\lambda, X^\rho] + \mathrm{i}\,\epsilon_{\mu\lambda\rho}\,[\nabla_\nu X^\lambda, X^\rho]\ . \tag{6.18}$$

By inserting this combination of gauge fields into the total action $S_{\rm top} + S_{\rm m}$, we find that in the strong coupling limit the field B_μ only interacts with the Chern-

Simons and scalar kinetic terms algebraically. Its equation of motion reads as

$$B_\mu = \tfrac{1}{2g}\,\epsilon_{\mu\nu\lambda}\,\widetilde{F}^{\nu\lambda} - \tfrac{1}{2}\,\nabla_\mu Y^1 \ , \tag{6.19}$$

where we keep only those terms that will remain in the strong coupling limit. Integrating out the field B_μ, we find that the Chern-Simons term from the first line of (6.10) reduces to the modified Yang-Mills term $\int \mathrm{d}^3x\ \mathrm{Tr}_{\mathcal{A}'}\big((\widetilde{F}_{\mu\nu})^2\big)$. We further find that the field Y^1 decouples from the remaining terms of the total action, so we introduce a new index $a = 1,2,3,4$. After suitable rescaling of the fields we find that the reduced action is given by

$$\begin{aligned} S_{\rm red} = \int \mathrm{d}^3x\ \ \mathrm{Tr}_{\mathcal{A}'}\big(&\tfrac{1}{2}\,(\widetilde{F}_{\mu\nu})^2 - \tfrac{\mathrm{i}}{2}\,\epsilon^{\mu\nu\lambda}\,\bar{\chi}_\mu\,(\nabla_\nu\chi_\lambda + [\chi_\nu, X_\lambda]) \\ &+ \tfrac{1}{2}\,[X^\mu, X^\nu]^2 + [X^\mu, Y^a]^2 + \tfrac{1}{2}\,[Y^a, Y^b]^2 \\ &+ \tfrac{1}{2}\,(\nabla_\mu X^\nu)^2 - \nabla_\mu X^\nu\,\nabla_\nu X^\mu + \tfrac{1}{2}\,(\nabla_\mu X^\mu)^2 + \tfrac{1}{2}\,\nabla^\mu Y^a\,\nabla_\mu Y^a \\ &+ \bar{\psi}\,\nabla_\mu\chi^\mu - \mathrm{i}\,\bar{\psi}\,[X^\mu, \chi_\mu] - \tfrac{\mathrm{i}}{2}\,\bar{\psi}\,\gamma_a\,[Y^a, \psi] - \tfrac{\mathrm{i}}{2}\,\bar{\chi}^\mu\,\gamma_a\,[Y^a, \chi_\mu]\big) \ . \end{aligned} \tag{6.20}$$

As with the original twisted BLG action, the reduced action is the sum of a topological term and a metric dependent cohomological action.

In [64] it was shown that the Mukhi-Papageorgakis map is compatible with the topological twisting procedure. We thus expect that the reduced model is some topological twist of $\mathcal{N} = 8$ supersymmetric Yang-Mills theory in three dimensions. The possible twists for this gauge theory were classified in [23]: We can either arrive at a twisted $\mathcal{N} = 2$ supersymmetric BF-theory, or a twisted $\mathcal{N} = 4$ equivariant extension of the Blau-Thompson model. Comparing our lagrangian (6.20) with those listed in [23], we find that we have obtained the on-shell formulation of the $\mathcal{N} = 4$ equivariant extension of the Blau-Thompson model; it can be realized as the worldvolume gauge theory of D2-branes wrapping supersymmetric three-cycles in Type IIA string theory. Maximally supersymmetric Yang-Mills gauge theories on S^3 are also considered in [43].

6.1.2 Cohomological IKKT matrix model

As the equivariantly extended Blau-Thompson model is a twist of $\mathcal{N} = 8$ supersymmetric Yang-Mills theory, its dimensional reduction should yield some topological twist of the IKKT matrix model. The zero-dimensional reduction of the action (6.20) becomes

$$\begin{aligned} \mathcal{S}_{\rm BT} = \ & {\rm Tr}_{\mathcal{A}'}\big(\tfrac{1}{2}\,([A_\mu, A_\nu] - {\rm i}\,\epsilon_{\nu\lambda\rho}\,[[A_\mu, X^\lambda], X^\rho] + {\rm i}\,\epsilon_{\mu\lambda\rho}\,[[A_\nu, X^\lambda], X^\rho])^2 \\ & + \tfrac{1}{2}[A_\mu, X^\nu]^2 - [A_\mu, X^\nu]\,[A_\nu, X^\mu] + \tfrac{1}{2}\,[A_\mu, X^\mu]^2 \\ & + \tfrac{1}{2}\,[X^\mu, X^\nu]^2 + [X^\mu, Y^a]^2 + \tfrac{1}{2}\,[Y^a, Y^b]^2 + \tfrac{1}{2}\,[A_\mu, Y^a]^2 \\ & - \tfrac{\rm i}{2}\,\epsilon^{\mu\nu\lambda}\,\bar\chi_\mu\,([A_\nu, \chi_\lambda] + [\chi_\nu, \chi_\lambda]) - \tfrac{\rm i}{2}\,\bar\chi^\mu\,\gamma_a\,[Y^a, \chi_\mu] \\ & + \bar\psi\,([A_\mu, \chi^\mu] - {\rm i}\,\bar\psi\,[X^\mu, \chi_\mu] - \tfrac{\rm i}{2}\,\bar\psi\,\gamma_a\,[Y^a, \psi])\big) \ . \end{aligned} \tag{6.21}$$

This matrix model defines an $\mathcal{N} = 4$ equivariant extension of the usual IKKT matrix model in ten dimensions, which can be solved exactly by using localization techniques. It possesses a nilpotent $\mathcal{N} = 2$ topological symmetry which acts on the fields as

$$\begin{aligned} \delta A_\mu &= \bar\varepsilon\,\chi_\mu \ , \\ \delta X_\mu &= {\rm i}\,\bar\varepsilon\,\chi_\mu \ , \\ \delta\chi^\mu &= {\rm i}\,\epsilon^{\mu\nu\lambda}\,[A_\mu, A_\nu]\,\varepsilon \ , \\ \delta\bar\chi_\mu &= -\,\gamma_a\,[A_\mu, Y^a]\,\varepsilon \ , \\ \delta\psi &= 0 \ , \\ \delta\bar\psi &= -\,[A_\mu, X^\mu]\,\varepsilon - {\rm i}\,\gamma_{ab}\,[Y^a, Y^b]\,\varepsilon \ , \\ \delta Y^a &= -\,2\,{\rm i}\,\bar\varepsilon\,\gamma^a\,\psi \ . \end{aligned} \tag{6.22}$$

In §5.4 we showed how the reduced ABJM and BLG models are related. We could thereby hope to lift the cohomological deformation (6.21) of the IKKT matrix model to obtain an analogous twist of the ABJM matrix model which would enable the exact computation of the deformed partition function using localization techniques. However, it was shown in [64] that for three-dimensional $\mathcal{N} = 6$ Chern-Simons-matter theories the only possible twists involve vector supercharges, and hence it is not possible to directly obtain such a cohomological deformation of the ABJM theory. In the next chapter we shall alleviate this problem by constructing a cohomological matrix model by hand which explicitly localizes onto the BPS equations of the ABJM matrix model; hence it computes an equivariant index for the model explicitly, and moreover possesses the same qualitative features as the matrix model (6.21) under the Mukhi-Papageorgakis map.

Chapter 7

Equivariant 3-Algebra Models

In this final chapter we shall relate the computation of partition function of the ABJM matrix model to those of a certain cohomological matrix model. Cohomological matrix models comprise a certain type of topological field theory which are constructed by specifying a set of fields, a set of equations, and a set of symmetries; the correlation functions constructed from this data compute intersection numbers on the moduli space of solutions to the equations modulo the symmetries [21]. They have actions of the form

$$\mathcal{S}(\mathbf{\Phi}) = \mathcal{Q}\,\mathcal{V}(\mathbf{\Phi}) \ , \tag{7.1}$$

where $\mathcal{Q}$ is the nilpotent BRST charge of the model acting on a gauge-invariant functional $\mathcal{V}(\mathbf{\Phi})$ of the field content $\mathbf{\Phi}$. Matrix models of this type have appealing properties. For example, they are often exactly solvable by using localization methods. A prominent example of this type of theory is due to Moore, Nekrasov and Shatashvili [71]: They computed the path integral for the Yang-Mills matrix model by constructing a related cohomological field theory, and then solving the cohomological deformation using localization techniques. This formalism was generalized to a large class of quiver matrix models in [30]. Since our dimensionally reduced membrane models and the IKKT matrix model are related via the Mukhi-Papageorgakis map, we could expect that the deformation approach of [71] can be lifted to our model; in this section we will apply this approach to the ABJM matrix model by constructing a related cohomological matrix model and then computing the path integral using localization methods. The deformation of the matrix integral is accomplished using the global $\mathsf{SU}(4) = \mathsf{Spin}(6)$ R-symmetry of the model, and it preserves $\mathcal{N} = 2$ supersymmetry. It involves a choice of a generic element in the Cartan subalgebra of the R-symmetry group, which enables one to construct well-defined matrix integrals.

The structure of this chapter is as follows. We begin with a review of equivariant localization. Then, we discuss the localization of $\mathcal{N} = 2$ Chern-Simons quiver matrix models, and the related difficulties. We then proceed to construct a cohomological matrix model that will localizes onto the BPS equations of the ABJM matrix model. We conclude with an analysis of the vacuum moduli space, and an explicit computation of the partition function.

7.1 Equivariant localization

We begin by summarizing the main features involved in equivariant localization, in a form that we shall employ it. Localization is a technique used in supersymmetric quantum field theory by which a path integral over an infinite-dimensional field domain is reduced to a finite-dimensional integral; here we apply it to reduce the partition functions for our quiver matrix models to integrals over the critical point locus of some matrix functional. For this, we perturb the action $\mathcal{S}(\mathbf{\Phi})$ of our model and consider the deformed partition function

$$\mathscr{Z}_t = \int \mathrm{d}\mathbf{\Phi}\ \mathrm{e}^{-\mathcal{S}(\mathbf{\Phi})-t\,\mathcal{Q}\,\mathcal{V}(\mathbf{\Phi})}\ , \tag{7.2}$$

where $\mathrm{d}\mathbf{\Phi}$ is a suitably normalized, supersymmetry-invariant measure on field space and $t \in \mathbb{R}$ parameterizes a continuous family of partition functions such that $\mathscr{Z} := \mathscr{Z}_0$ is the partition function of the original matrix model. Since the action $\mathcal{S}(\mathbf{\Phi})$ is supersymmetric, $\mathcal{Q}\,\mathcal{S}(\mathbf{\Phi}) = 0$, and the scalar supercharge $\mathcal{Q}$ is nilpotent on gauge-invariant operators, we have

$$\frac{\partial \mathscr{Z}_t}{\partial t} = -\int \mathrm{d}\mathbf{\Phi}\ \mathcal{Q}\,\mathcal{V}(\mathbf{\Phi})\,\mathrm{e}^{-\mathcal{S}(\mathbf{\Phi})-t\,\mathcal{Q}\,\mathcal{V}(\mathbf{\Phi})} = -\int \mathrm{d}\mathbf{\Phi}\ \mathcal{Q}\big(\mathcal{V}(\mathbf{\Phi})\,\mathrm{e}^{-\mathcal{S}(\mathbf{\Phi})-t\,\mathcal{Q}\,\mathcal{V}(\mathbf{\Phi})}\big) = 0\ , \tag{7.3}$$

where in the last step we have integrated by parts using the derivation property of the BRST operator $\mathcal{Q}$ with $\mathcal{Q}\mathcal{S}(\mathbf{\Phi}) = 0$, and used invariance of the measure $\mathrm{d}\mathbf{\Phi}$ on field space under the BRST symmetry. This means that the original partition function $\mathscr{Z} = \mathscr{Z}_0$ is computed by (7.2) at any value of t. In the limit $t \to \infty$, the partition function often simplifies; in particular, if $\mathcal{Q}\,\mathcal{V}(\mathbf{\Phi})$ is positive definite then the contributions to the integral in this limit come from the minima $\mathbf{\Phi}_0$ in field space where $\mathcal{Q}\,\mathcal{V}(\mathbf{\Phi}_0) = 0$. The partition function (7.2) can then be evaluated by applying the method of steepest descent. The differences between contributions from $\mathbf{\Phi}_0$ and a generic point $\mathbf{\Phi}$ in field space are exponentially suppressed as $t \to \infty$; the dominant contributions to this integral therefore come from points in a neighbourhood $\mathscr{N}(\mathbf{\Phi}_0)$ of $\mathbf{\Phi}_0$. Assuming that $\mathrm{e}^{-\mathcal{S}(\mathbf{\Phi})}$ varies slowly with respect to $\mathrm{e}^{-t\,\mathcal{Q}\,\mathcal{V}(\mathbf{\Phi})}$, the partition function reduces to

$$\mathscr{Z} = \int_{\mathcal{Q}\,\mathcal{V}(\mathbf{\Phi})=0} \mathrm{d}\mathbf{\Phi}_0\ \mathrm{e}^{-\mathcal{S}(\mathbf{\Phi}_0)} \int_{\mathscr{N}(\mathbf{\Phi}_0)} \mathrm{d}\mathbf{\Phi}'\ \mathrm{e}^{-t\,\mathcal{Q}\,\mathcal{V}(\mathbf{\Phi}')} \tag{7.4}$$

with $\mathbf{\Phi}' \in \mathscr{N}(\mathbf{\Phi}_0)$ denoting fluctuations around the minima $\mathbf{\Phi}_0$; here we have dropped higher order terms using nilpotency of the supersymmetry variations. The t-dependence of the fluctuation integral in (7.4) cancels by supersymmetry of the measure $\mathrm{d}\mathbf{\Phi}'$ when one performs the bosonic and fermionic integrations. Note that for cohomological matrix models with actions of the form (7.1), we can apply this argument directly to the integral $\int \mathrm{d}\mathbf{\Phi}\ \mathrm{e}^{-t\,\mathcal{S}(\mathbf{\Phi})}$ itself, so that (7.4) is given by an integral over minima of the original action $\mathcal{S}(\mathbf{\Phi})$ with $\mathcal{S}(\mathbf{\Phi}_0) = 0$.

7.1.1 Localization of $\mathcal{N} = 2$ Chern-Simons quiver matrix models

Let us apply this formalism to the matrix models having actions (C.10) with a positive definite quadratic form $\mathrm{Tr}_{\mathfrak{g}}$; to ensure convergence of the matrix integral, here we set $A_0 = \mathrm{i}\, A_3$ with A_3 hermitian and $\gamma^0 = \mathrm{i}\,\gamma^3$. For the cohomological deformation of this action we take

$$\mathcal{Q}\mathcal{V} = \mathcal{Q}\,\bar{\mathcal{Q}}\,\mathrm{Tr}_{\mathfrak{g}}\big(\tfrac{1}{2}\,\bar{\lambda}\,\lambda - 2D\,\sigma\big) \ , \tag{7.5}$$

where the supercharge $\mathcal{Q}$ generates the nilpotent supersymmetry transformations (C.6) with $\eta = \varepsilon$ and the spinor normalization $\bar{\varepsilon}\,\varepsilon = 1$. The deformation term then reads explicitly as

$$\mathcal{Q}\mathcal{V} = \mathrm{Tr}_{\mathfrak{g}}\big(- \tfrac{1}{2}\,[A_\mu, A_\nu]^2 - [A_\mu, \sigma]^2 + D^2 + \tfrac{\mathrm{i}}{2}\,\bar{\lambda}\,\gamma^\mu\,[A_\mu, \lambda] + \mathrm{i}\,[\bar{\lambda}, \sigma]\,\lambda\big) \ . \tag{7.6}$$

Writing $X^I = (A^\mu, \sigma)$, $\Psi = (\lambda^1, \lambda^2)$, $\Gamma^I = (\gamma^\mu, \mathrm{i}\,\mathbb{1})$, and $F = D$, this is just the action of the four-dimensional Yang-Mills matrix model (B.5) (with $g = 1$). The localization locus $\mathcal{Q}\mathcal{V} = 0$ is given by

$$[A_\mu, A_\nu] = 0 = [A_\mu, \sigma] \ , \qquad D = 0 = \lambda = \bar{\lambda} \tag{7.7}$$

for the gauge sector, which coincides with the BPS equations (C.11). By noting that the matter part of the action (C.8) is itself a BRST-exact term

$$\mathcal{S}_{\mathrm{m}} = \mathcal{Q}\,\bar{\mathcal{Q}}\,\mathrm{Tr}_{\mathfrak{g}}\big(\bar{\psi}\,\psi - 2Z^\dagger\,\sigma\,Z\big) \ , \tag{7.8}$$

we may choose the localization locus

$$Z = F = 0 = \psi \tag{7.9}$$

for the matter interactions. Then the action (C.10) vanishes at the critical points. For gauge group $G = \mathsf{U}(N)$, the fixed point locus thus coincides with the moduli variety of quadruples (A_μ, σ) of commuting matrices; for $G = \mathsf{U}(N_L) \times \mathsf{U}(N_R)$, it is a subvariety of the vacuum moduli space of the ABJM matrix model defined by the BPS equations (5.27). While the analogous localization procedure works nicely in the field theory setting to provide exact results for supersymmetric Chern-Simons-matter theories on S^3 [60, 69] and their dimensional reductions to a point [7, 54], in our dimensionally reduced model the result of the localization integral (7.4) comes out to involve terribly divergent integrals over the Cartan subalgebra of $\mathfrak{g}$ which are beyond regularization; the partition function in our case is not well-defined because the action lacks supersymmetric mass terms for the scalars. Below we shall cure this problem by constructing a cohomological matrix model whose fixed point locus provides a rigorous definition of the same moduli variety via a further equivariant deformation parametrized by the R-symmetry group of the matrix model. We follow the method of [30] to compute a supersymmetric equivariant index using localization techniques. Although the localization integral still formally diverges, the presence of twisted masses enables one to define it via a suitable prescription that we explain in detail.

7.1.2 Cohomological matrix model formalism

As only theories with $\mathcal{N} \geqslant 4$ supersymmetry can be twisted to produce deformed scalar supercharges, we focus our attention henceforth on the $\mathcal{N} = 6$ ABJM matrix model from §5.2.2 for definiteness; we construct a cohomological matrix model which localizes onto the BPS equations. In view of our discussion from §7.1.1, here we consider instead the localization locus with $A_\mu^{L,R} = 0$ as the gauge fields do not themselves transform under the R-symmetry; the BPS equations (5.27) then reduce to the relations (5.46) of the double of the ABJM quiver (5.24). Put differently, we localize the partition function of the matrix model onto the F-term constraints rather than the D-term constraints. We localize the matrix integral with respect to the equivariant BRST operator in the gauge group $G = \mathsf{U}(N_L) \times \mathsf{U}(N_R)$, twisted by the toric action of the maximal torus $\mathbb{T}^4$ of the R-symmetry group $\mathsf{SU}(4)$ of the matrix model; this deforms the nilpotent BRST charge to a differential of $\mathsf{SU}(4)$-equivariant cohomology. We denote the generators of this torus by $\epsilon_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, and set $t_i = \mathrm{e}^{\mathrm{i}\,\epsilon_i}$ with the $\mathsf{SU}(4)$-constraint

$$\sum_{i=1}^{4} \epsilon_i = 0 \tag{7.10}$$

on the toric parameters. The full symmetry group of the equivariant model is thus $\mathsf{U}(N_L) \times \mathsf{U}(N_R) \times \mathbb{T}^4$. The transformation properties of the fields and equations of motion under the toric action of $\mathbb{T}^4$ are given by

$$Z^i \longmapsto \mathrm{e}^{-\mathrm{i}\,\epsilon_i}\, Z^i \ , \qquad \mathcal{Z}_i^{jk} \longmapsto \mathrm{e}^{-\mathrm{i}\,(\epsilon_j+\epsilon_k-\epsilon_i)}\, \mathcal{Z}_i^{jk} \ . \tag{7.11}$$

In order to construct a supersymmetric matrix model we assign superpartners to these fields to give multiplets (Z^i, ψ^i) with BRST transformations

$$\mathcal{Q}Z^i = \psi^i \ , \qquad \mathcal{Q}\psi^i = \phi_R\, Z^i - Z^i\, \phi_L - \epsilon_i\, Z^i \ , \tag{7.12}$$

where the hermitian gauge parameters $\phi_{L,R} \in \mathsf{End}\,(V_{L,R})$ transform in the adjoint representation of the factors $\mathsf{U}(N_{L,R})$ of the gauge group. (There is no sum over i in the second equation.) We now add the Fermi multiplet of auxiliary fields (χ_i^{jk}, H_i^{jk}) related to the BPS equations, where the antighosts are defined as maps $\chi_i^{jk} \in \mathsf{Hom}\,(V_L, V_R)$ with transformations that read as

$$\mathcal{Q}H_i^{jk} = \phi_R\, \chi_i^{jk} - \chi_i^{jk}\, \phi_L - (\epsilon_j + \epsilon_k - \epsilon_i)\, \chi_i^{jk} \ , \qquad \mathcal{Q}\chi_i^{jk} = H_i^{jk} \ . \tag{7.13}$$

To these fields we include the gauge multiplet $(\phi_{L,R}, \bar\phi_{L,R}, \eta_{L,R})$ which is necessary to close the BRST algebra off-shell; these fields have transformations

$$\mathcal{Q}\phi_{L,R} = 0 \ , \qquad \mathcal{Q}\bar\phi_{L,R} = \eta_{L,R} \ , \qquad \mathcal{Q}\eta_{L,R} = \big[\phi_{L,R}, \bar\phi_{L,R}\big] \ . \tag{7.14}$$

In order to obtain a localization onto a well-defined moduli space of matrices that can be described as a non-singular quotient of a critical locus by the gauge group G, we incorporate additional fields $\varphi, I_{L,R}$ into the collection of bosonic fields, together with their superparters $\zeta, \rho_{L,R}$ into the collection of fermions. The new field $\varphi \in \mathsf{Hom}\,(V_L, V_R)$ transforms in the bifundamental representation of the $\mathsf{U}(N_L) \times \mathsf{U}(N_R)$ gauge group and in the determinant representation of the R-symmetry, and hence is invariant under the toric action of $\mathbb{T}^4$ by (7.10). The fields $I_{L,R} \in V_{L,R} = \mathbb{C}^{N_{L,R}}$ are also taken to be invariant under the action of the torus $\mathbb{T}^4$ for simplicity, and they transform as vectors under the actions of the left and right gauge groups $\mathsf{U}(N_{L,R})$; in what follows we shall refer to the fundamental matter fields $I_{L,R}$ as "framing vectors". The equations of motion for these additional fields are given by

$$\varphi\, I_L = 0 = \varphi^\dagger\, I_R \tag{7.15}$$

and they ensure stability of the vacua of our quiver matrix model, as we discuss in detail later on. Their BRST transformations are

$$\mathcal{Q}\varphi = \zeta \ , \qquad \mathcal{Q}I_{L,R} = \rho_{L,R} \ , \qquad \mathcal{Q}\zeta = \phi_R\, \varphi - \varphi\, \phi_L \ , \qquad \mathcal{Q}\rho_{L,R} = \phi_{L,R}\, I_{L,R} \ . \tag{7.16}$$

We now add the corresponding antighost and auxiliary fields $\xi_{L,R} \in V_{L,R}^*$ and $h_{L,R}$ with the BRST transformations

$$\mathcal{Q}\xi_{L,R} = h_{L,R} \ , \qquad \mathcal{Q}h_{L,R} = -\xi_{L,R}\, \phi_{L,R} \ . \tag{7.17}$$

The BRST symmetry $\mathcal{Q}$ squares to a gauge transformation twisted by a $\mathbb{T}^4$ rotation of the fields.

Following the treatment of §7.1.1, we will now write down a cohomological Yang-Mills type matrix model that has this field content, equations of motion, and BRST transformations. It is given by the $\mathcal{N}=2$ action

$$\begin{aligned}\mathcal{S}_{\rm coh} = \ & \mathcal{Q}\,{\rm Tr}_{\,V}\big((\chi_i^{jk\,\dagger}\,(\tfrac{g}{2}\,H_i^{jk} - {\rm i}\,[Z^j,Z^k;Z_i]) + \psi^i\,(\phi_L\,Z_i^\dagger - Z_i^\dagger\,\bar\phi_R) + \eta_L\,[\phi_L,\bar\phi_L] \\ & - \eta_R\,[\phi_R,\bar\phi_R] + \xi_L^\dagger\otimes(g'\,h_L - I_R^\dagger\,\varphi) - \xi_R^\dagger\otimes(g'\,h_R - I_L^\dagger\,\varphi^\dagger) + \bar\phi_L\,\rho_L\otimes I_L^\dagger \\ & - \bar\phi_R\,\rho_R\otimes I_R^\dagger + (\bar\phi_L\,\varphi^\dagger - \varphi^\dagger\,\bar\phi_R)\,\zeta + \tfrac{g_1}{2}\,(Z^i\,\psi_i^\dagger + Z_i^\dagger\,\psi^i) + \tfrac{g_2}{2}\,(I_L\otimes\rho_L^\dagger - I_R\otimes\rho_R^\dagger) \\ & + \tfrac{g_3}{2}\,(\varphi\,\zeta^\dagger + \varphi^\dagger\,\zeta) + \text{hermitian conjugates}\big)\ , \end{aligned} \tag{7.18}$$

where we used the canonical identifications $\mathsf{End}\,(V_{L,R}) = V_{L,R}\otimes V_{L,R}^*$. The deformation by the last three BRST-exact terms in (7.18) removes flat directions from the matrix integral for the partition function (see [30] for details); the equivariant deformation further has the effect of generating mass terms for all bosonic fields, which as we will see yields a well-defined matrix integral. Note that the relevant bosonic part of the action from the first line of (7.18) is ${\rm Tr}_{\,V}\big(\frac{g}{2}\,H_i^{jk\,\dagger}\,H_i^{jk} - {\rm i}\,H_i^{jk\,\dagger}\,\mathcal{Z}_i^{jk}\big)$; integrating out H_i^{jk} gives the bosonic potential energy $\frac{1}{2g}\,{\rm Tr}_{\,V}\big(\mathcal{Z}_i^{jk\,\dagger}\,\mathcal{Z}_i^{jk}\big)$ and supersymmetry, and thus the path integral of the matrix model localizes onto the configurations where $\mathcal{Z}_i^{jk}=0$, as desired.

Since this matrix model is cohomological, it is independent of the couplings g,g',g_1,g_2,g_3 in the action (7.18). We can compute the partition function by taking various limits of these couplings. The first step is to use the $\mathsf{U}(N_L)\times\mathsf{U}(N_R)$ gauge symmetry to diagonalize the gauge generators $\phi_{L,R}$; we denote their eigenvalues by ϕ_L^a, $a=1,\dots,N_L$, and ϕ_R^b, $b=1,\dots,N_R$. This change of variables produces Vandermonde determinants $\prod_{a<b}\,\big(\phi_L^b-\phi_L^a\big)^2$ and $\prod_{a<b}\,\big(\phi_R^b-\phi_R^a\big)^2$ in the path integral measure. Let us now take the limit $g\to\infty$. The dominant part of the action is

$$\tfrac{g}{2}\,{\rm Tr}_{\,V}\Big(H_i^{jk\,\dagger}\,H_i^{jk} + \chi_i^{jk\,\dagger}\,\big(\phi_R\,\chi_i^{jk} - \chi_i^{jk}\,\phi_L - (\epsilon_j+\epsilon_k-\epsilon_i)\,\chi_i^{jk}\big)\Big)\ . \tag{7.19}$$

The auxiliary BRST field integrals should not affect the partition function, so we fix their integration measures such that

$$\int {\rm d}H_i^{jk}\ {\rm d}H_i^{jk\,\dagger}\ \exp\Big({\rm Tr}_{\,V}\big(H_i^{jk\,\dagger}\,H_i^{jk}\big)\Big) = 1\ . \tag{7.20}$$

Integrating over the fermions gives a factor of the form $\prod_{a,b}\ \prod_i\ \prod_{j<k}\ \big(\phi_L^b-\phi_R^a+\epsilon_j+\epsilon_k-\epsilon_i\big)$. Now we take the limit $g_1\to\infty$. The relevant part of the action reads as

$$g_1\sum_{i=1}^4\ \mathrm{Tr}_V\big(\psi^i\,\psi_i^\dagger+Z^i\,(Z_i^\dagger\,\phi_R-\phi_L\,Z_i^\dagger-\epsilon_i\,Z_i^\dagger)\big)\ . \tag{7.21}$$

Performing the matter integrations puts a term in the localized matrix integral of the form $\prod_{a,b}\ \prod_i\ \big(\phi_L^b-\phi_R^a-\epsilon_i-\mathrm{i}\,0\big)^{-1}$, where we have added a small imaginary part to the generic real parameters ϵ_i to ensure convergence of the gaussian integrations. Next we treat the stabilizing fields $I_{L,R},\varphi$ and their superpartners. We first take the limit $g'\to\infty$. The dominant part of the action is

$$g'\ \mathrm{Tr}_V\big(h_L^\dagger\otimes h_L-h_R^\dagger\otimes h_R-\xi_L^\dagger\otimes\xi_L\,\phi_L+\xi_R^\dagger\otimes\xi_R\,\phi_R\big)\ . \tag{7.22}$$

The fields $h^{L,R}$ can be trivially integrated out, while performing the left and right fermionic integrations puts terms in the path integral of the form $\prod_a\ \phi_L^a\ \prod_b\ \phi_R^b$. Finally, performing the path integral in the large g_2 limit gives terms of the form $\big(\prod_a\ \phi_L^a\big)^{-1}\big(\prod_b\ \phi_R^b\big)^{-1}$, while performing the integrations in the limit $g_3\to\infty$ gives terms of the form $\big(\prod_{a,b}(\phi_L^b-\phi_R^a)\big)^{-1}$.

Combining all of the above evaluations, the final result for the localization of the cohomological matrix integral can be written in terms of integrations over the left and right gauge generators in the Cartan torus of the gauge group as

$$\begin{aligned}\mathscr{Z}^{\mathrm{ABJM}}_{N_L,N_R}(\epsilon)=&\oint\ \prod_{a=1}^{N_R}\frac{\mathrm{d}\phi_R^a}{2\pi\,\mathrm{i}}\ \prod_{b=1}^{N_L}\frac{\mathrm{d}\phi_R^b}{2\pi\,\mathrm{i}}\ \frac{\prod\limits_{a<b}\ \big(\phi_L^b-\phi_L^a\big)^2\ \prod\limits_{a<b}\ \big(\phi_R^b-\phi_R^a\big)^2}{\prod\limits_{a=1}^{N_R}\ \prod\limits_{b=1}^{N_L}\ \big(\phi_L^b-\phi_R^a-\mathrm{i}\,0\big)}\\ &\times\prod_{a=1}^{N_R}\ \prod_{b=1}^{N_L}\ \prod_{i=1}^{4}\ \frac{\prod\limits_{j<k}\ \big(\phi_L^b-\phi_R^a+\epsilon_i-\epsilon_j-\epsilon_k\big)}{\phi_L^b-\phi_R^a-\epsilon_i-\mathrm{i}\,0}\ .\end{aligned} \tag{7.23}$$

As a Lebesgue integral, this expression formally diverges. Hence we define it via an analytic continuation to a suitable contour integral prescription in the complex plane which picks up the poles of the integrand; the precise choice of contour keeps track of the auxiliary multiplet of fields that have been eliminated by taking the large coupling limits above. It is straightforward to see that the poles occur precisely on the supersymmetric solutions of the cohomological matrix model. For this, we consider the critical points of the action (7.18) where the fermions are set equal to zero. They are determined by the zeroes of the BRST charge. By (7.12) and (7.16) the fixed point equations are then

$$Z^i_{ab}\left(\phi^b_L-\phi^a_R-\epsilon_i\right)=0=\varphi_{ab}\left(\phi^b_L-\phi^a_R\right)=0\ , \qquad I^a_R\,\phi^a_R=0=I^b_L\,\phi^b_L \tag{7.24}$$

for each $i=1,2,3,4$, $a=1,\dots,N_R$, and $b=1,\dots,N_L$.

We can evaluate the integral (7.23) explicitly in dimensions $N_L=N_R=1$. As its integrand depends only on the combination $\phi:=\phi_L-\phi_R$ in this case, it can be evaluated from the residue theorem by picking up the contributions from the simple poles at $\phi=0$ and $\phi=\epsilon_i$, $i=1,2,3,4$, to get

$$\begin{aligned}\mathscr{Z}^{\rm ABJM}_{1,1}(\epsilon)=&\prod_{i=1}^4\frac{1}{\epsilon_i}\prod_{j<k}(\epsilon_i-\epsilon_j-\epsilon_k)\\&+\sum_{i=1}^4\frac{1}{\epsilon_i}\prod_{l\neq i}\frac{1}{\epsilon_i-\epsilon_l}\prod_{i'=1}^4\prod_{j<k}(\epsilon_{i'}+\epsilon_i-\epsilon_j-\epsilon_k)\ .\end{aligned}\tag{7.25}$$

On the other hand, for $N_R=0$ one finds that the contour integral vanishes for $N_L\geqslant 5$; more generally, the integral vanishes for $|N_L-N_R|$ sufficiently large, in agreement with recent analysis of the ABJM theory through the partition function of the $\mathsf{U}(N_L)\times\mathsf{U}(N_R)$ lens space matrix model [8]. However, for higher dimensions an explicit evaluation of (7.23) becomes increasingly intractable.

In the remainder of this section we shall develop an alternative local model for the fluctuation integrals in (7.4) through a geometric analysis of the neighbourhoods $\mathscr{N}(\mathbf{\Phi}_0)$ around the fixed point subset of the critical point locus with respect to the action of the R-symmetry torus $\mathbb{T}^4$. In particular, we compute an equivariant index

$$\mathscr{I}_{N_L,N_R}(t)=\ {\rm Tr}_{\,\mathcal{H}_{\rm BPS}}\,(-1)^F\prod_{i=1}^4 t_i^{R_i} \tag{7.26}$$

whose infinitesimal limit $\epsilon_i\to 0$ explicitly evaluates the contour integrals (7.23); here $\mathcal{H}_{\rm BPS}$ is the Hilbert space of framed BPS states of the cohomological field theory and R_i are the generators of the Cartan subalgebra of the global symmetry group $\mathsf{SU}(4) = \mathsf{SO}(6)$. In writing (7.26) we have used the fact that the hamiltonian H vanishes in any cohomological field theory, and set the fugacity $y = 1$ for $\mathsf{SO}(2)$ rotations as we take $A_\mu = 0$.

7.1.3 Vacuum moduli space and fixed point analysis

The partition function (7.23) can be regarded as computing a regularised volume of the non-compact vacuum moduli space $\mathfrak{M}_{N_L,N_R}$ [72], which we now define explicitly. For this, we recall the equations of motion (7.15) which imply that the vector I_L sits in the kernel and the vector I_R in the cokernel of φ. The presence of the bifundmental field φ also implies that the quotient of the fixed point locus $\mathcal{Z}_i^{jk} = 0$ by the gauge group G is equivalent to a quotient by the action of the complexified gauge group $G_{\mathbb{C}}$. Then the moduli space can be represented as a quasi-projective variety

$$\mathfrak{M}_{N_L,N_R} = \left\{ (\mathcal{Z}_i^{jk})^{-1}(0) \right\} /\!\!/ \, \mathsf{GL}(N_L, \mathbb{C}) \times \mathsf{GL}(N_R, \mathbb{C}) \ , \tag{7.27}$$

where the GIT quotient on the right is taken by removing the points at which the action of $G_{\mathbb{C}}$ is not free. Such a quotient can be defined by imposing an additional stability condition on the data $(Z^i, I_{L,R}, \varphi)$; a suitable notion of stability for our purposes can be given as follows: We say that a datum $(Z^i, I_{L,R}, \varphi)$ is stable if there are no non-trivial proper subspaces $W_{L,R} \subsetneq V_{L,R}$ which contain the vectors $I_{L,R}$ and which are invariant under the bilinear commuting operators $Z_i^\dagger \, Z^j$, $Z^j \, Z_i^\dagger$ for all $i, j = 1, 2, 3, 4$, respectively. Let us demonstrate that the gauge group $G_{\mathbb{C}}$ acts freely on stable data. Suppose that $(Z^i, I_{L,R}, \varphi)$ is fixed by $(g_L, g_R) \in G_{\mathbb{C}}$. Then $g_R \, Z^i = Z^i \, g_L$, $g_L \, Z_j^\dagger = Z_j^\dagger \, g_R$, and $g_{L,R} \, I_{L,R} = I_{L,R}$, which respectively imply that the subspaces $W_{L,R} = \ker(\mathbb{1} - g_{L,R})$ have $Z_i^\dagger \, Z^j(W_L) \subset W_L$, $Z^j \, Z_i^\dagger(W_R) \subset W_R$ and $I_{L,R} \in W_{L,R}$. It follows by stability that $g_{L,R} = \mathbb{1}$, and hence the $G_{\mathbb{C}}$-action is free. The corresponding quotient (7.27) defines a suitable moduli space of solutions to the BPS equations (5.46) modulo gauge equivalence.

Let us now characterize the fixed points of this moduli space. A fixed point $\Pi = (Z^i, I_{L,R}, \varphi) \in \mathfrak{M}_{N_L,N_R}^{\mathbb{T}^4}$ with respect to the action of $\mathbb{T}^4 \subset \mathsf{SU}(4)$ is characterized by the condition that an equivariant rotation is equivalent to a gauge transformation of the fields, so that

$$g_R\, Z^i\, g_L^{-1} = t_i^{-1}\, Z^i \ , \qquad g_{L,R}\, I_{L,R} = I_{L,R} \ , \qquad g_R\, \varphi = \varphi\, g_L \ . \tag{7.28}$$

Under the $\mathbb{T}^4$-action the vector spaces $V_{L,R}$ admit the weight space decompositions

$$V_{L,R} = \bigoplus_{\alpha\in\mathbb{Z}^4} V_{L,R}(\alpha) \tag{7.29}$$

with

$$V_{L,R}(\alpha) = \big\{ v \in V_{L,R} \ \big| \ g_{L,R}^{-1}\, v = t_1^{\alpha_1}\, t_2^{\alpha_2}\, t_3^{\alpha_3}\, t_4^{\alpha_4}\, v \big\} \tag{7.30}$$

for $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^4$. It is a straightforward consequence of (7.28) that the nonvanishing components of the maps $(Z^i, I_{L,R}, \varphi)$ are given by

$$Z^i \ : \ V_L(\alpha) \ \longrightarrow \ V_R(\alpha - e_i) \ , I_{L,R} \in V_{L,R}(0) \ , \varphi \ : \ V_L(\alpha) \ \longrightarrow \ V_R(\alpha) \ , \tag{7.31}$$

where $e_i \in \mathbb{Z}^4$, $i = 1, 2, 3, 4$, is the vector with 1 in its i-th component and 0 elsewhere. With the weight space decompositions (7.30) and (7.31), it is also easy to show that the solution of the fixed point equations (7.24) is given by setting the eigenvalues of the gauge parameter matrices $\phi_{L,R}$ in this basis equal to

$$\phi_{L,R}^{\alpha^{L,R}} = \sum_{i=1}^{4} \epsilon_i\, \alpha_i^{L,R} \ , \tag{7.32}$$

and $Z^i = 0 = I_{L,R}$ except for the components $Z^i_{\alpha-e_i,\alpha}$ and $I^0_{L,R}$. Moreover, the only non-trivial components of the BPS equations (5.27) are given by

$$\begin{aligned} & Z^j_{\alpha+e_i-e_j-e_k,\alpha+e_i-e_k}\, \big(Z_i^\dagger\big)^{\alpha+e_i-e_k,\alpha-e_k}\, Z^k_{\alpha-e_k,\alpha} \\ &= \ Z^k_{\alpha+e_i-e_j-e_k,\alpha+e_i-e_j}\, \big(Z_i^\dagger\big)^{\alpha+e_i-e_j,\alpha-e_j}\, Z^j_{\alpha-e_j,\alpha} \ , \end{aligned} \tag{7.33}$$

and for the conjugates of these equations one has

$$\begin{aligned} & \big(Z_j^\dagger\big)^{\alpha+e_j+e_k-e_i,\alpha+e_k-e_i}\, Z^i_{\alpha+e_k-e_i,\alpha+e_k}\, \big(Z_k^\dagger\big)^{\alpha+e_k,\alpha} \\ = \ & \big(Z_k^\dagger\big)^{\alpha+e_j+e_k-e_i,\alpha+e_j-e_i}\, Z^i_{\alpha+e_j-e_i,\alpha+e_j}\, \big(Z_j^\dagger\big)^{\alpha+e_j,\alpha} \ . \end{aligned} \tag{7.34}$$

We can describe the graded components of the $\mathbb{T}^4$-module decomposition (7.30) explicitly in terms of the fixed point maps as follows. Recalling the discussion at the end of §5.3, we unambiguously define subspaces of $V_{L,R}$ by

$$W_L = \bigoplus_{n_{ij}\geq 0}\ \prod_{i,j=1}^4 \big(Z_i^\dagger\, Z^j\big)^{n_{ij}}\, I_L \ , \qquad W_R = \bigoplus_{n_{ij}\geq 0}\ \prod_{i,j=1}^4 \big(Z^j\, Z_i^\dagger\big)^{n_{ij}}\, I_R \ . \tag{7.35}$$

Clearly $I_{L,R}\in W_{L,R}$, the subspace W_L is $Z_i^\dagger\, Z^j$-invariant, and W_R is $Z^j\, Z_i^\dagger$-invariant for all i,j. Whence $W_{L,R}=V_{L,R}$ by stability, and hence

$$V_L(\alpha) = \bigoplus_{\sum_j\,(n_{ij}-n_{ji})=\alpha_i}\ \prod_{i,j=1}^4 \big(Z_i^\dagger\, Z^j\big)^{n_{ij}}\, I_L \, , V_R(\alpha) = \bigoplus_{\sum_j\,(n_{ij}-n_{ji})=\alpha_i}\ \prod_{i,j=1}^4 \big(Z^j\, Z_i^\dagger\big)^{n_{ij}}\, I_R \ . \tag{7.36}$$

Note that the constraints on the sums in (7.36) imply that the weights must satisfy

$$\sum_{i=1}^4 \alpha_i = 0 \ . \tag{7.37}$$

We define finite sets of lattice points $\Pi_{L,R}\subset\mathbb{Z}^4$ by

$$\Pi_{L,R} = \big\{\alpha\in\mathbb{Z}^4 \ \big| \ V_{L,R}(\alpha)\neq 0\big\} \ , \tag{7.38}$$

with $\big|\Pi_{L,R}\big| = N_{L,R}$ nodes; the meaning of the restrictions $\Pi_{L,R}\subset\mathbb{Z}^3$ implied by (7.37) will be elucidated below. The vertices of these lattices are related by the actions of commuting matrices through the commutative diagrams

$$\begin{array}{ccc} V_L(\alpha) & \xrightarrow{\ Z_i^\dagger\, Z^j\ } & V_L(\alpha+e_i-e_j) \\ {\scriptstyle Z_k^\dagger\, Z^l}\big\downarrow & & \big\downarrow{\scriptstyle Z_k^\dagger\, Z^l} \\ V_L(\alpha+e_k-e_l) & \xrightarrow[\ Z_i^\dagger\, Z^j\ ]{} & V_L(\alpha+e_i+e_k-e_j-e_l) \end{array} \tag{7.39}$$

and

$$\begin{array}{ccc} V_R(\alpha) & \xrightarrow{\ Z^j\, Z_i^\dagger\ } & V_R(\alpha + e_i - e_j) \\ {\scriptstyle Z^l\, Z_k^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle Z^l\, Z_k^\dagger} \\ V_R(\alpha + e_k - e_l) & \xrightarrow[\ Z^j\, Z_i^\dagger\ ]{} & V_R(\alpha + e_i + e_k - e_j - e_l) \end{array} \tag{7.40}$$

We can gain a better combinatorial understanding of the sets (7.38) by employing some machinery from the theory of quiver representations (see e.g. [18]); in this setting we identify torus-invariant framed BPS states Π in the cotangent bundle of the moduli space of framed representations of the ABJM quiver (5.24) with fixed dimension vector (N_L, N_R). In fact, many interesting features of BPS states in three-dimensional supersymmetric gauge theories find natural realizations within the quiver framework. For example, there is a conjectural Seiberg duality for Chern-Simons gauge theories with $\mathcal{N} \geqslant 2$ supersymmetry (see e.g. [3]); in the present context this duality is realized as a mutation of quivers, which is a tilting procedure that therefore yields an equivalence of the corresponding derived categories of quiver representations [89].

A quiver representation is the same thing as a module for the path algebra $\mathscr{A}$ of the ABJM quiver (5.24) with relations (5.46). The path algebra $\mathscr{A}$ is generated by acting with arrows Z^i, $Z_i^\dagger$, $i = 1, 2, 3, 4$, on the framing vectors $I_{L,R}$, as in (7.35); we refer to such quiver representations as cyclic modules. In this setting we replace our definition of stable points Π above with the more natural notion of θ-stability appropriate to moduli spaces of quiver representations [63]. By regarding the conjugate fields $Z_i^\dagger$ as independent arrows, our quiver moduli problem is then formally equivalent to that of the conifold quiver whose path algebra is a noncommutative crepant resolution of the conifold singularity in six dimensions [84], except that we use multiple framings as in [29] in order to preserve the left/symmetry inherent in the original ABJM matrix model. This provides us with a concrete geometrical description of the vacuum moduli space. The R-symmetry torus $\mathbb{T}^4$ acts on the arrows Z^i, $i = 1, 2, 3, 4$; hence it acts on the whole path algebra $\mathscr{A}$ and leaves the relations (5.46) invariant. The diagonal torus $\mathbb{T}^2$ of the gauge group G induces an action of

$\mathbb{T} = \mathsf{U}(1)$ on the arrows via overall rescaling; this can be used to set e.g. $\epsilon_4 = 0$. Modding out by this gauge group action, the overall torus action is $\mathbb{T}_Q \cong \mathbb{T}^3$. We shall now argue that the $\mathbb{T}_Q$-fixed points are isolated and are parametrized by certain filtrations of the finite pyramid partitions of the conifold quiver. For this, we note that the $\mathbb{T}_Q$-fixed points in the moduli space of framed cyclic modules correspond bijectively to $\mathbb{T}_Q$-fixed ideals in the path algebra $\mathscr{A}$. There is a one-to-one correspondence between $\mathbb{T}_Q$-fixed modules of the path algebra $\mathscr{A}$ with relations and the $\mathbb{T}_Q$-fixed annihilator $\mathtt{A}$ of the framing vectors $I_{L,R} \in V_{L,R}$ consisting of stabilizing bifundamental fields φ which satisfy (7.15); the finite-dimensional annihilator $\mathtt{A}$ is a left ideal of the path algebra and it is generated by linear combinations of elements of the same weight. We claim that $\mathtt{A}$ is generated by monomials of the path algebra, such that its class $[\mathtt{A}]$ is an isolated $\mathbb{T}_Q$-fixed point in the moduli space of cyclic representations with dimension vector (N_L, N_R). For this, note that $\mathtt{A}$ is generated by linear combinations of path monomials of the same weights. Given a torus weight $t_1^{\alpha_1}\, t_2^{\alpha_2}\, t_3^{\alpha_3}$, we can find finitely many monomial paths p_l emanating from the nodes $V_{L,R}$. Elements of $\mathtt{A}$ with weight $t_1^{\alpha_1}\, t_2^{\alpha_2}\, t_3^{\alpha_3}$ are most generally written as finite sums of paths $\sum_l \xi_l\, p_l$ for some $\xi_l \in \mathbb{C}$; if $\xi_{l'} \neq 0$, then $p_{l'}$ should be included as one of the monomial generators of the $\mathbb{T}_Q$-fixed annihilator $\mathtt{A}$, since each $p_{l'}$ is a linear map from the framing vectors $I_{L,R}$ to different vector spaces. By exhausting all monomial generators in this way, we conclude that the torus fixed point $\mathtt{A}$ is generated by monomials and hence corresponds to an isolated point in the moduli space of quiver representations.

The problem of parametrizing finite-dimensional cyclic $\mathscr{A}$-modules (up to isomorphism) is now equivalent to the problem of parametrizing finite-codimensional ideals of $\mathscr{A}$ (up to $\mathscr{A}$-module isomorphism). Following [29], they are classified in terms of filtered pyramid partitions of length two empty room configurations. Recall [84] that a pyramid partition consists of two types of layers of stones, labelled L (coloured white) and R (coloured black), which denote one-dimensional subspaces $V_{L,R}(\alpha)$ of given toric weights α from (7.36). For $i \geq 0$, there are $(i+1)^2$ L-type stones on layer $2i$, and $(i+1)\,(i+2)$ R-type stones on layer $2i+1$. A finite subset

Π of this combinatorial arrangement is a pyramid partition if, for every stone of Π, the two stones immediately above it (of different colour) are also in Π.

In the ABJM limit $N_L = N_R = N$, we can make this description of the vacuum moduli space somewhat more explicit. Then the stability condition implies that the moduli space is a resolution of the N-th symmetric product orbifold (5.48) provided by the Hilbert scheme $(\mathbb{C}^4)^{[N]}$ of N points in $\mathbb{C}^4$, which parametrizes zero-dimensional subschemes of $\mathbb{C}^4$ of length N. The map $(Z^i, I) \mapsto \sum_l \lambda_l \, \vec{z}_l$ from (5.50) gives the Hilbert-Chow map

$$\left(\mathbb{C}^4\right)^{[N]} \longrightarrow \left(\mathbb{C}^4\right)^N / \mathfrak{S}_N \tag{7.41}$$

which is constructed in detail in [39]. Following the derivation in [30], the $\mathbb{T}^4$-fixed points in this case are parametrized by three-dimensional solid partitions [5] of the positive integer N; they are specified by height functions $\Pi(\iota) \in \mathbb{Z}$ on a cubic lattice with sites $\iota \in \mathbb{Z}^3$, such that $\Pi(\iota) \geq 0$ are decreasing functions in each of the three lattice directions satisfying

$$\sum_{\iota \in \mathbb{Z}^3} \Pi(\iota) = N \ . \tag{7.42}$$

7.1.4 Equivariant index for the ABJM quiver

The localization formula allows one to calculate the contribution to the partition function from each fixed point; as we have discussed, the sum over fixed points is captured by applying the residue theorem to write the contour integral (7.23) as a sum over simple poles at the critical points (7.32). As the explicit form of the residue formula is difficult to handle, we generalize the technique of [25] to extract the eigenvalues of the superdeterminants of the BRST operator $\mathcal{Q}$, arising in the fluctuation integrals (7.4), from the character of the tangent space to the moduli space at each critical point. Let Q be the fundamental representation of $\mathbb{T}^4$ with weight $(1, 1, 1, 1)$; the dual module Q^* has weight $(-1, -1, -1, -1)$. The local geometry of the moduli space of BPS solutions $\mathfrak{M}_{N_L, N_R}$ near a particular fixed point

$\Pi = (Z^i, I_{L,R}, \varphi)$ can be described by the complex of vector spaces

$$\begin{matrix} & & \mathsf{Hom}\,(V_L,V_R)\ \otimes\ Q & & \\ \mathsf{End}\,(V_L) & & \oplus & & \mathsf{Hom}\,(V_L,V_R)\ \otimes\ \big(Q^*\otimes{\textstyle\bigwedge}^2 Q\big) \\ \oplus & \xrightarrow{\ \mathrm{d}_1^\Pi\ } & V_L\ \oplus\ V_R & \xrightarrow{\ \mathrm{d}_2^\Pi\ } & \oplus \\ \mathsf{End}\,(V_R) & & \oplus & & V_L\ \oplus\ V_R \\ & & \mathsf{Hom}\,(V_L,V_R) & & \end{matrix} \tag{7.43}$$

where the map d_1^Π is an infinitesimal gauge transformation

$$\mathrm{d}_1^\Pi \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} = \begin{pmatrix} \phi_R\,Z^i - Z^i\,\phi_L \\ \phi_L\,I_L \\ \phi_R\,I_R \\ \phi_R\,\varphi - \varphi\,\phi_L \end{pmatrix} , \tag{7.44}$$

while the map d_2^Π is the differential of the equations (5.46) and (7.15) that define the vacuum moduli space so that

$$\mathrm{d}_2^\Pi \begin{pmatrix} Y^i \\ v_L \\ v_R \\ Y \end{pmatrix} = \begin{pmatrix} \big[Y^j,Z^k;Z_i\big] + \big[Z^j,Y^k;Z_i\big] + \big[Z^j,Z^k;Y_i\big] \\ \varphi^\dagger\,v_R + Y^\dagger\,I_R \\ \varphi\,v_L + Y\,I_L \end{pmatrix} . \tag{7.45}$$

The first cohomology $\ker(\mathrm{d}_2^\Pi)/\mathrm{im}(\mathrm{d}_1^\Pi)$ parametrizes deformations and provides a local model for the tangent space $T_\Pi\mathfrak{M}_{N_L,N_R}$ at the fixed point Π. As supersymmetric ground states are in one-to-one correspondence with cohomology classes of $\mathfrak{M}_{N_L,N_R}$, the total cohomology of this complex is identified with the Hilbert space $\mathcal{H}_{\mathrm{BPS}}$ of framed BPS states of the cohomological field theory.

The complex (7.43) has a natural meaning in the local geometry of the moduli space of representations of the framed ABJM quiver. Write V for a given representation of the ABJM quiver (5.24) with fixed dimension vector (N_L, N_R), and $\mathsf{Ext}^p(-,-)$ for the extension groups in the abelian category of modules for the path algebra $\mathscr{A}$. Then the first term of (7.43) is the space $\mathsf{Ext}^0(V,V) = \mathsf{Hom}\,(V,V)$ of nodes of the quiver (5.24), the second term is the space $\mathsf{Ext}^1(V,V)$ of arrows including the framing, and the third term is the vector space $\mathsf{Ext}^2(V,V)$ of all relations; as there are no relations among the F-term relations (5.46), in our case $\mathsf{Ext}^p(V,V)=0$ for all $p\geq 3$ and the deformation complex contains only three terms. Note that since here the $\mathbb{T}^4$ action leaves invariant the F-term relations (5.46) but not the superpotential (5.25) itself, the deformation complex (7.43) is neither symmetric nor self-dual; as a consequence, the local weight of a fixed point Π is not simply a sign $(-1)^{\dim T_\Pi \mathfrak{M}_{N_L,N_R}}$ but is rather a rational function of the equivariant deformation parameters ϵ_i, $i=1,2,3,4$. In the following we compute the equivariant Euler character of the deformation complex (7.43) for the ABJM quiver. Via our deformation of the nilpotent BRST operator, the equivariant Euler character can still be interpreted as a Witten index in the topologically twisted supersymmetric quantum mechanics on the moduli space $\mathfrak{M}_{N_L,N_R}$ of supersymmetric vacua.

The equivariant character of the complex (7.43) can be calculated from its cohomology which is given by an alternating sum of the weights of the various $\mathbb{T}^4$ representations. In the representation ring of the torus group $\mathbb{T}^4$, one has $Q=\sum_i t_i^{-1}$ and $\bigwedge^2 Q=\sum_{i<j} t_i\, t_j$, and we obtain the virtual sum

$$\begin{aligned}\mathsf{ch}_\Pi^{\mathbb{T}^4}(t) = {}& V_L^*\otimes V_L + V_R^*\otimes V_R - \Big(\big(V_L^*\otimes V_R\big)\sum_{i=1}^4 t_i^{-1} + V_L + V_R + V_L^*\otimes V_R\Big) \\ & + \big(V_L^*\otimes V_R\big)\sum_{i=1}^4 t_i \sum_{j<k} t_j^{-1}\, t_k^{-1} + V_L + V_R\ , \end{aligned} \tag{7.46}$$

where we use the weight decompositions of the vector spaces

$$V_{L,R} = \sum_{\alpha^{L,R}\in\Pi_{L,R}} \prod_{i=1}^4 t_i^{\alpha_i^{L,R}} = \sum_{\alpha^{L,R}\in\Pi_{L,R}} \prod_{i=1}^3 t_i^{\alpha_i^{L,R}+\alpha_1^{L,R}+\alpha_2^{L,R}+\alpha_3^{L,R}} \tag{7.47}$$

as $\mathbb{T}^4$ representations, and the second equality here follows from the constraints (7.10) and (7.37); the dual involution acts on the weights as inversion $(t_i)^*=t_i^{-1}$. Inserting this decomposition into the character formula (7.46) and using the $\mathsf{SU}(4)$-

constraint $t_1\,t_2\,t_3\,t_4=1$ we find

$$
\begin{aligned}
\mathsf{ch}_\Pi^{\mathbb{T}^4}(t) = \Big(\sum_{j\neq k} t_j\,t_k^2 + 2\,\sum_{j=1}^4 t_j^{-1} - 1\Big) \sum_{\alpha^{L,R}\in\Pi_{L,R}}\ \prod_{i=1}^4 t_i^{\alpha_i^R-\alpha_i^L} \\
+ \sum_{\alpha^L,\beta^L\in\Pi_L}\ \prod_{i=1}^4 t_i^{\alpha_i^L-\beta_i^L} + \sum_{\alpha^R,\beta^R\in\Pi_R}\ \prod_{i=1}^4 t_i^{\alpha_i^R-\beta_i^R}\ . \qquad (7.48)
\end{aligned}
$$

The corresponding top form then gives the equivariant version of the fluctuation integral over the normal bundle $\mathscr{N}(\Pi)$ in (7.4) at each fixed point Π of the vacuum moduli space $\mathfrak{M}_{N_L,N_R}$. As the second cohomology of the complex (7.43) is non-vanishing, there is a non-trivial obstruction theory for the moduli space and the localization formula computes the equivariant Euler character of the virtual tangent bundle over $\mathfrak{M}_{N_L,N_R}$, i.e. the difference in K-theory between the tangent and normal bundles at each fixed point of the moduli space. By summing over all fixed points Π we arrive at an explicit combinatorial expression for the contour integral (7.23) given by the finite sum

$$
\begin{aligned}
\mathscr{Z}^{\rm ABJM}_{N_L,N_R}(\epsilon) = \sum_{\Pi\in\mathfrak{M}^{\mathbb{T}^4}_{N_L,N_R}} \frac{\prod_{\alpha^L,\beta^L\in\Pi_L}\Big(\sum_{i=1}^4\big(\alpha_i^L-\beta_i^L\big)\,\epsilon_i\Big)\ \prod_{\alpha^R,\beta^R\in\Pi_R}\Big(\sum_{i=1}^4\big(\alpha_i^R-\beta_i^R\big)\,\epsilon_i\Big)}{\prod_{\alpha^{L,R}\in\Pi_{L,R}}\Big(\sum_{i=1}^4\big(\alpha_i^R-\alpha_i^L\big)\,\epsilon_i\Big)} \\
\times \prod_{\alpha^{L,R}\in\Pi_{L,R}}\ \prod_{j=1}^4\Big(\big(\alpha_j^R-\alpha_j^L-1\big)\,\epsilon_j + \sum_{i\neq j}\big(\alpha_i^R-\alpha_i^L\big)\,\epsilon_i\Big)^2 \qquad (7.49) \\
\times \prod_{j\neq k}\Big(\big(\alpha_j^R-\alpha_j^L+1\big)\,\epsilon_j + \big(\alpha_k^R-\alpha_k^L+2\big)\,\epsilon_k + \sum_{i\neq j,k}\big(\alpha_i^R-\alpha_i^L\big)\,\epsilon_i\Big)\ .
\end{aligned}
$$

Consistently with the fact that it computes an equivariant index, the partition function $\mathscr{Z}^{\rm ABJM}_{N_L,N_R}(\epsilon)$ is a Laurent series in the deformation parameters $(\epsilon_1,\epsilon_2,\epsilon_3)$ with rational coefficients. The partition weights $\alpha^{L,R}\in\Pi_{L,R}$ in this formula are naturally interpreted as R-charges of framed BPS particles of the three-dimensional supersymmetric gauge theory.

Chapter 8

Summary of results

In chapter 4 we considered the quantization of Nambu-Poisson structures. We described an extension of the usual quantization axioms in which Nambu-Poisson structures are translated to n-Lie algebras for both spheres and hyperboloids. We interpreted the Nambu-Heisenberg n-Lie algebra in terms of foliations of $\mathbb{R}^n$ by fuzzy spheres and fuzzy hyperboloids. We then applied this result to the quantum geometry of M5-branes in M-theory.

In chapter 5 we constructed our membrane matrix models from dimensional reduction of the BLG and ABJM theories. We showed how these models map to the IKKT matrix model under the Mukhi-Papageorgakis map. We demonstrated how these models are related to each other by using specific scaling limits, or through particular choice of 3-algebra. We then found several stable BPS solutions to the 3-Lie algebra model, and interpreted them as quantized Nambu-Poisson manifolds.

In chapter 6 we studied cohomological 3-algebra models in order to derive a twist for the reduced ABJM model. We studied a particular twist of the BLG theory, and showed that under the Mukhi-Papageorgakis map it reduces to the on-shell $\mathcal{N} = 4$ equivariant extension of the Blau-Thompson model. For the dimensionally reduced case, we derived a novel twist of the IKKT matrix model, with the hope that this twist could be lifted to the ABJM matrix model via the mappings of the previous chapter. We explain why this is not possible.

In chapter 7 we avoided the the problem of twisting the ABJM matrix model by constructing a cohomological matrix model by hand which localizes onto the BPS equations of the ABJM matrix model. We presented the construction of the cohomological matrix model. We then analyzed its vacuum moduli space by characterizing its fixed points. We concluded by explicitly calculating its partition function, which computes an equivariant index which enumerates framed BPS states.

Appendix A

Generators of Clifford algebras

If γ^i, $i = 1, \ldots, 2d-1$ generate the Clifford algebra $Cl(\mathbb{R}^{2d-1})$, then the $2d$-tuple

$$(\gamma^\mu) = (\gamma^i \otimes \sigma^2, \mathbb{1}_s \otimes \sigma^1)\ , \quad s = 2^{d-1}\ , \quad \mu = 1, \ldots, 2d \tag{A.1}$$

generates $Cl(\mathbb{R}^{2d})$. On the other hand, we just add $\gamma_{\mathrm{ch}} := \mathrm{i}^d\, \gamma^1 \cdots \gamma^{2d}$ to the generators of $Cl(\mathbb{R}^{2d})$ to obtain a set of generators of $Cl(\mathbb{R}^{2d+1})$. We can start the induction from the usual Pauli matrices σ^i, which generate $Cl(\mathbb{R}^3)$ and satisfy $[\sigma^i, \sigma^j] = -2\,\mathrm{i}\,\varepsilon^{ijk}\,\sigma^k$. In this case, all the generators are hermitian and we have $\gamma_{\mathrm{ch}} = \mathrm{diag}(\mathbb{1}_s, -\mathbb{1}_s)$. In our chapter on quantization, we use the basis of Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \sigma^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\ , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{A.2}$$

Recall that for even $d+1$, there is a set of generators λ^a, $a = 1, \ldots, r^2$ of $\mathfrak{u}(r)$, $r = 2^{\frac{d-1}{2}}$ given by

$$\frac{1}{\sqrt{r}}\,\mathbb{1}_r\ , \quad \frac{2}{r}\,\gamma^\mu\ , \quad \frac{2\,\mathrm{i}}{r}\,\gamma^{\mu\nu}\ , \quad \frac{2\,\mathrm{i}}{r}\,\gamma^{\mu\nu\rho}\ , \quad \frac{2}{r}\,\gamma^{\mu\nu\rho\sigma}\ , \quad \ldots\ , \tag{A.3}$$

where $\gamma^{\mu_1 \cdots \mu_k}$ is the normalized antisymmetric product of gamma-matrices $\gamma^{\mu_1}, \ldots, \gamma^{\mu_k}$.

With this normalization, they satisfy the Fierz identity

$$\lambda^a_{\alpha\beta}\,\lambda^a_{\gamma\delta} = \delta_{\alpha\delta}\,\delta_{\beta\gamma}\ . \tag{A.4}$$

As these generators of $\mathfrak{u}(r)$ form an orthogonal set with respect to the Hilbert-Schmidt norm, we conclude that all of them are traceless except for the identity matrix.

Appendix B

IKKT matrix model

In this appendix we briefly review the construction of the four dimensional IKKT matrix model as a dimensional reduction of supersymmetric Yang-Mills theory [57].

In 4 dimensions, the $\mathcal{N} = 1$ Yang-Mills lagrangian reads as

$$\mathcal{L} = \tfrac{1}{4} F_{ij} F_{ij} - \tfrac{\mathrm{i}}{2} \bar{\psi} \gamma^i \nabla_i \psi \ . \tag{B.1}$$

The fields of this model are the gauge fields A_μ and the fermions ψ_α. The index i runs from 1 to 4. The fields are taken to be in the adjoint representation of a gauge group G. The field strength is defined as

$$F_{ij} = \partial_i A_j - \partial_j A_i + \mathrm{i}[A_i, A_j] \ . \tag{B.2}$$

The covariant derivative reads as

$$\nabla_i \psi = \partial_i \psi + \mathrm{i}[A_i, \psi] \ . \tag{B.3}$$

The relevant gauge transformations are

$$\psi \to g \psi g^{-1} \ , A_i \to g A_i g^{-1} - \mathrm{i}(\partial_i g) g^{-1} \ . \tag{B.4}$$

We obtain a matrix model from this action by assuming that the fields are independent of space and time. This effectively means we drop terms involving partial derivatives. The resulting Yang-Mills matrix model reads as

$$\mathcal{L} = -\mathrm{Tr}\big(\tfrac{1}{4}[A_i, A_j][A_i, A_j] + \tfrac{1}{2}\bar{\psi}\gamma^i[A_i, \psi]\big) \ . \tag{B.5}$$

Appendix C

Dimensional Reduction of supersymmetric Chern-Simons Matter Theories

This appendix concerns reduced supersymmetric Chern-Simons matter models. The resulting model is a quiver matrix model, which we use in the main body of this thesis provide an alternative construction of the ABJM matrix model. First, we review the general $\mathcal{N} = 2$ Chern-Simons theory coupled to matter, and then consider the dimensional reduction of this theory to zero dimensions. Then, we show how in the special case of a product gauge group, this model and its supersymmetries may be mapped to the $\mathcal{N} = 1$ 4-dimensional IKKT matrix model under the Mukhi-Papageorgakis map.

C.1 $\mathcal{N} = 2$ Chern-Simons quiver matrix models

The field content for the $\mathcal{N} = 2$ supersymmetric Chern-Simons gauge multiplet $\boldsymbol{V}$ in three-dimensional flat space $\mathbb{R}^{1,2}$ consists of a gauge field A_μ, $\mu = 0, 1, 2$, two auxiliary scalar fields D and σ, and a two-component complex auxiliary fermion field λ. The fields are valued in the Lie algebra $\mathfrak{g}$ of a matrix gauge group G. The action is given by

$$S_{\rm g} = \int {\rm d}^3x \ \kappa \ {\rm Tr}_{\,\mathfrak{g}}\Big(\epsilon^{\mu\nu\lambda}\, \big(A_\mu\, \partial_\nu A_\lambda + \tfrac{2\,{\rm i}}{3}\, A_\mu\, A_\nu\, A_\lambda\big) - \bar\lambda\,\lambda + 2D\,\sigma\Big) \ , \tag{C.1}$$

where $\kappa \in \mathbb{R}$ is a coupling constant and ${\rm Tr}_{\,\mathfrak{g}}$ is an invariant quadratic form on the Lie algebra $\mathfrak{g}$. The generators of the Clifford algebra $C\ell(\mathbb{R}^{1,2})$ are the gamma-matrices γ^μ which satisfy the anticommutation relations

$$\big\{\gamma^\mu, \gamma^\nu\big\} = 2\eta^{\mu\nu} \tag{C.2}$$

and are taken to be Pauli spin matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \gamma^2 = \begin{pmatrix} 0 & -{\rm i} \\ {\rm i} & 0 \end{pmatrix} \ , \tag{C.3}$$

while the spinor adjoint is

$$\bar\lambda = \lambda^\dagger\,\gamma^0 \ . \tag{C.4}$$

We perform a dimensional reduction to zero dimensions in which the gauge fields A_μ become a collection of $\mathfrak{g}$-valued scalar fields, and similarly for the other fields of $\boldsymbol{V}$. The reduced action is

$$\mathcal{S}_{\rm g} = \kappa \ {\rm Tr}_{\,\mathfrak{g}}\big(\tfrac{2\,{\rm i}}{3}\, \epsilon^{\mu\nu\lambda}\, A_\mu\, A_\nu\, A_\lambda - \bar\lambda\,\lambda + 2D\,\sigma\big) \ . \tag{C.5}$$

This action is invariant under the $\mathcal{N}=2$ supersymmetry transformations

$$\begin{aligned}
\delta A_\mu &= \tfrac{\mathrm{i}}{2}\,(\bar\eta\,\gamma_\mu\,\lambda - \bar\lambda\,\gamma_\mu\,\varepsilon)\ ,\\
\delta\sigma &= \tfrac{\mathrm{i}}{2}\,(\bar\eta\,\lambda - \bar\lambda\,\varepsilon)\ ,\\
\delta D &= \tfrac{\mathrm{i}}{2}\,(\bar\eta\,\gamma^\mu\,[A_\mu,\lambda] + [A_\mu,\bar\lambda]\,\gamma^\mu\,\varepsilon) + \tfrac{\mathrm{i}}{2}\,(\bar\eta\,[\lambda,\sigma] + [\bar\lambda,\sigma]\,\varepsilon)\ ,\\
\delta\lambda &= -\,\mathrm{i}\,\big(\tfrac12\,\gamma^{\mu\nu}\,[A_\mu,A_\nu] + D + \gamma^\mu\,[A_\mu,\sigma]\big)\,\varepsilon\ ,\\
\delta\bar\lambda &= \mathrm{i}\,\bar\eta\,\big(-\tfrac12\,\gamma^{\mu\nu}\,[A_\mu,A_\nu] + D + \gamma^\mu\,[A_\mu,\sigma]\big)\ ,
\end{aligned} \tag{C.6}$$

where η and ε are two independent Dirac spinors of $\mathsf{SO}(1,2)$ and $\gamma^{\mu\nu} := \frac12\,[\gamma^\mu,\gamma^\nu]$. The two supersymmetry transformations generated by η or ε alone commute. The commutator of an η-supersymmetry with an ε-supersymmetry generates a sum of a gauge transformation, a Lorentz rotation, a dilatation, and an R-rotation.

This action can be extended to include supersymmetric matter fields. The matter content is a chiral multiplet $\mathbf{\Phi}$ with component fields $\mathbf{\Phi} = (Z, Z^\dagger, \psi, \bar\psi, F, F^\dagger)$, which are also valued in the Lie algebra $\mathfrak{g}$. The field Z is a complex matter field, F is an auxiliary complex scalar field, and ψ is a two-component Dirac spinor field. The action reads

$$\begin{aligned}
S_{\mathrm{m}} = \int \mathrm{d}^3x\ \ \mathrm{Tr}_{\,\mathfrak{g}}\big(&\nabla_\mu Z^\dagger\,\nabla^\mu Z - Z^\dagger\,\sigma^2\,Z + Z^\dagger\,D\,Z + F^\dagger\,F\\
&+\mathrm{i}\,\bar\psi\,\gamma^\mu\,\nabla_\mu\psi - \bar\psi\,\sigma\,\psi - \mathrm{i}\,\bar\psi\,\lambda\,Z + \mathrm{i}\,Z^\dagger\,\bar\lambda\,\psi\big)\ ,
\end{aligned} \tag{C.7}$$

where the gauge covariant derivatives act as $\nabla_\mu Z := \partial_\mu Z + \mathrm{i}\,[A_\mu, Z]$. We perform a dimensional reduction as above, so that the reduced matter action reads as

$$\begin{aligned}
\mathcal{S}_{\mathrm{m}} = \ \mathrm{Tr}_{\,\mathfrak{g}}\big(&-[A_\mu,Z^\dagger]\,[A^\mu,Z] - Z^\dagger\,\sigma^2\,Z + Z^\dagger\,D\,Z + F^\dagger\,F\\
&-\bar\psi\,\gamma^\mu\,[A_\mu,\psi] - \bar\psi\,\sigma\,\psi - \mathrm{i}\,\bar\psi\,\lambda\,Z + \mathrm{i}\,Z^\dagger\,\bar\lambda\,\psi\big)\ .
\end{aligned} \tag{C.8}$$

The supersymmetry transformations are given by

$$\begin{aligned}
\delta Z &= \bar{\eta}\,\psi \ , \\
\delta Z^\dagger &= \bar{\psi}\,\varepsilon \ , \\
\delta\psi &= \mathrm{i}\,\big(\gamma^\mu\,[A_\mu, Z] - \sigma\,Z\big)\,\varepsilon + F\,\varepsilon^* \ , \\
\delta\bar{\psi} &= \mathrm{i}\,\bar{\eta}\,\big(\gamma^\mu\,[A_\mu, Z^\dagger] + Z^\dagger\,\sigma\big) \ , \\
\delta F &= \bar{\eta}^*\,\big(\gamma^\mu\,[A_\mu, \psi] + \mathrm{i}\,\lambda\,Z + \sigma\,\psi\big) \ .
\end{aligned} \tag{C.9}$$

The complete action of the reduced $\mathcal{N} = 2$ Chern-Simons-matter theory thus reads as

$$\begin{aligned}
\mathcal{S} = \ &\mathrm{Tr}_{\,\mathfrak{g}}\Big(\kappa\,\big(\tfrac{2\,\mathrm{i}}{3}\,\epsilon^{\mu\nu\lambda}\,A_\mu\,A_\nu\,A_\lambda - \bar{\lambda}\,\lambda + 2D\,\sigma\big) - [A_\mu, Z^\dagger]\,[A^\mu, Z] - Z^\dagger\,\sigma^2\,Z + Z^\dagger\,D\,Z \\
&+ F^\dagger\,F - \bar{\psi}\,\gamma^\mu\,[A_\mu, \psi] - \bar{\psi}\,\sigma\,\psi - \mathrm{i}\,\bar{\psi}\,\lambda\,Z + \mathrm{i}\,Z^\dagger\,\bar{\lambda}\,\psi\Big) \ .
\end{aligned} \tag{C.10}$$

The BRST transformations imply that the supersymmetric configurations satisfy

$$[A_\mu, A_\nu] = 0 = [A_\mu, \sigma] \ , \qquad [A_\mu, Z] = 0 = [A_\mu, Z^\dagger] \ , \qquad D = 0 = F \ . \tag{C.11}$$

When the gauge group is a product of unitary groups

$$G = \prod_{a=1}^{r} \mathsf{U}(N_a) \ , \tag{C.12}$$

we decompose the reduced vector multiplet as $\boldsymbol{V} = \bigoplus_a \boldsymbol{V}^a$ where $\boldsymbol{V}^a \in \mathsf{End}\,(V_a)$ are regarded as linear transformations of complex inner product spaces $V_a = \mathbb{C}^{N_a}$ for $a = 1, \dots, r$, while the reduced matter multiplet is decomposed as $\boldsymbol{\Phi} = \bigoplus_{a,b} \boldsymbol{\Phi}^{ab}$ with $\boldsymbol{\Phi}^{ab} \in \mathsf{Hom}\,(V_a, V_b)$ and $\boldsymbol{\Phi}^\dagger_{ab} \in \mathsf{Hom}\,(V_b, V_a)$ for $a, b = 1, \dots, r$; then $\mathrm{Tr}_{\,\mathfrak{g}}$ refers to the trace in the fundamental representation of G which is possibly graded over the factors of G. In this case the supersymmetric Chern-Simons-matter theory reduces to a quiver matrix model, which defines a finite-dimensional representation of the double of the quiver with r nodes that carry the gauge degrees of freedom A^a_μ (plus their superpartners and auxiliary fields) transforming in the adjoint representation of $\mathsf{U}(N_a)$, and with an arrow from node a to node b for every non-zero matter field Z^{ab} (plus their superpartners and auxiliary fields) transforming in the bifundamental representation of $\mathsf{U}(N_a) \times \mathsf{U}(N_b)$, along with an arrow in the opposite direction for the adjoint $Z^\dagger_{ab}$. The double quiver is further equiped with a set of relations among the arrows that follow from the BPS equations of the supersymmetric gauge theory, which define a system of static quiver vortices; geometrically, representations of the double quiver are cotangent to representations of the original quiver.

C.1.1 A_1 quiver matrix model

The simplest example of the above construction is with a product gauge group

$$G = \mathsf{U}(N_L) \times \mathsf{U}(N_R) \ . \tag{C.13}$$

The matter content $\mathbf{\Phi}$ of the theory provides a representation of the double of the A_1 quiver

$$\bullet \longrightarrow \bullet \tag{C.14}$$

We place complex inner product spaces $V_L = \mathbb{C}^{N_L}$ and $V_R = \mathbb{C}^{N_R}$ at the left and right nodes of the quiver (C.14), respectively. The matter field is regarded as a linear map $Z : V_L \to V_R$ representing the arrow of the quiver (C.14), with hermitian conjugate $Z^\dagger : V_R \to V_L$. The matrices Z, F and ψ are bifundamental fields, i.e. they transform in the fundamental representation of $\mathsf{U}(N_R)$ and in the anti-fundamental representation of $\mathsf{U}(N_L)$. The vector multiplet has field content $\boldsymbol{V} = \big(A_\mu^{L,R}, \sigma^{L,R}, \lambda^{L,R}, \bar\lambda^{L,R}, D^{L,R}\big)$. The matrices $A_\mu^{L,R} \in \mathsf{End}\,(V_{L,R})$ for $\mu = 0,1,2$ transform in the adjoint representation of $\mathsf{U}(N_{L,R})$, $\lambda^{L,R}$ are two-component complex fermionic matrices, while $\sigma^{L,R}$ and $D^{L,R}$ are auxiliary matrix fields. The invariant quadratic form is given by $\mathrm{Tr}_{\,\mathfrak{g}} = \mathrm{Tr}_{\,V_L} \oplus (-\mathrm{Tr}_{\,V_R})$, and the action of the quiver matrix model takes the form

$$
\begin{aligned}
\mathcal{S}_{A_1} = \ & \mathrm{Tr}_V \bigg(\kappa \left(\tfrac{2\,\mathrm{i}}{3} \, \epsilon^{\mu\nu\lambda} \big(A^L_\mu \, A^L_\nu \, A^L_\lambda - A^R_\mu \, A^R_\nu \, A^R_\lambda \big) - \bar\lambda^L \, \lambda^L + \bar\lambda^R \, \lambda^R + 2D^L \, \sigma^L \right. \\
& \left. - \, 2D^R \, \sigma^R \right) - \big(A^L_\mu \, Z^\dagger - Z^\dagger \, A^R_\mu \big) \big(A^R_\mu \, Z - Z \, A^L_\mu \big) - \bar\psi \, \gamma^\mu \, \big(A^R_\mu \, \psi - \psi \, A^L_\mu \big) \\
& + F^\dagger \, F + Z \, D^L \, Z^\dagger - Z^\dagger \, D^R \, Z - \mathrm{i} \, \bar\psi \, Z \, \lambda^L + \mathrm{i} \, \bar\psi \, \lambda^R \, Z + \mathrm{i} \, \bar\lambda^L \, Z^\dagger \, \psi - \mathrm{i} \, Z^\dagger \, \bar\lambda^R \, \psi \\
& + Z^\dagger \, Z \, \sigma_L^2 - Z^\dagger \, \sigma_R^2 \, Z + 2Z^\dagger \, \sigma^R \, Z \, \sigma^L - \bar\psi \, \psi \, \sigma^L + \bar\psi \, \sigma^R \, \psi \bigg) \ , \qquad \text{(C.15)}
\end{aligned}
$$

where the trace is taken over $V = V_L$ or $V = V_R$ where appropriate. The supersymmetry transformations of this matrix model are given by

$$
\begin{aligned}
\delta A^{L,R}_\mu &= \tfrac{\mathrm{i}}{2} \big(\bar\eta \, \gamma_\mu \, \lambda^{L,R} - \bar\lambda^{L,R} \, \gamma_\mu \, \varepsilon \big) \ , \\
\delta \sigma^{L,R} &= \tfrac{\mathrm{i}}{2} \big(\bar\eta \, \lambda^{L,R} - \bar\lambda^{L,R} \, \varepsilon \big) \ , \\
\delta D^{L,R} &= \tfrac{\mathrm{i}}{2} \, \bar\eta \, \gamma^\mu \, \big[A^{L,R}_\mu , \lambda^{L,R} \big] + \tfrac{\mathrm{i}}{2} \, \big[A^{L,R}_\mu , \bar\lambda^{L,R} \big] \, \gamma^\mu \, \varepsilon + \tfrac{\mathrm{i}}{2} \, \bar\eta \, \big[\lambda^{L,R} , \sigma^{L,R} \big] + \tfrac{\mathrm{i}}{2} \, \big[\bar\lambda^{L,R} , \sigma^{L,R} \big] \, \varepsilon \ , \\
\delta \lambda^{L,R} &= \mathrm{i} \, \big(\tfrac{1}{2} \, \gamma^{\mu\nu} \, \big[A^{L,R}_\mu , A^{L,R}_\nu \big] - D^{L,R} - \gamma^\mu \, \big[A^{L,R}_\mu , \sigma^{L,R} \big] \big) \, \varepsilon \ , \\
\delta Z &= \bar\eta \, \psi \ , \\
\delta Z^\dagger &= \bar\psi \, \varepsilon \ , \\
\delta \psi &= \mathrm{i} \, \gamma^\mu \, \big(Z \, A^L_\mu - A^R_\mu \, Z \big) \, \varepsilon - \mathrm{i} \, \varepsilon \, \big(Z \, \sigma^L - \sigma^R \, Z \big) + F \, \varepsilon^* \ , \\
\delta \bar\psi &= \mathrm{i} \, \bar\eta \, \gamma^\mu \, \big(Z^\dagger \, A^R_\mu - A^L_\mu \, Z^\dagger \big) + \mathrm{i} \, \bar\eta \, \big(\sigma^L \, Z^\dagger - Z^\dagger \, \sigma^R \big) \ , \\
\delta F &= \bar\eta^{\,*} \, \Big(\gamma^\mu \, \big(\psi \, A^L_\mu - A^R_\mu \, \psi \big) + \mathrm{i} \, \big(Z \, \lambda^L - \lambda^R \, Z \big) + \big(\psi \, \sigma^L - \sigma^R \, \psi \big) \Big) \ . \qquad \text{(C.16)}
\end{aligned}
$$

We begin with the simplest example for which the reduction is relatively straightforward to construct. We consider the dimensionally reduced $\mathcal{N} = 2$ Chern-Simons-matter theory (C.15), and show that under the Mukhi-Papageorgakis map it reduces to the four-dimensional IKKT matrix model with $\mathcal{N} = 1$ supersymmetry and gauge group $\mathsf{SU}(N)$. We work with the Clifford algebra $C\ell(\mathbb{R}^{1,2})$, and use Dirac spinors. Our gamma-matrices are the Pauli spin matrices, and the Majorana conditions read

$$
\bar\varepsilon \, \lambda = \bar\lambda \, \varepsilon \ , \qquad \bar\varepsilon \, \gamma^\mu \, \lambda = - \bar\lambda \, \gamma^\mu \, \varepsilon \ . \qquad \text{(C.17)}
$$

As previously, we break the gauge symmetry to a $\mathsf{U}(N)$ subgroup by making the field replacements (5.28). We also restrict the matter field Z to be hermitian. We decompose Z into components $Z' \in \mathfrak{su}(N)$ and $Z_0 \in \mathfrak{u}(1)$, and expand it around a classical value proportional to the identity with a coupling constant g as

$$Z = g\,\mathbb{1} + Z_0 + Z'\ . \tag{C.18}$$

Using global $\mathsf{U}(1)$ symmetry, we may take $g \in \mathbb{R}$. For the gaugino and auxiliary fields, we take a diagonal limit in which

$$\lambda^L = -\lambda^R =: \lambda\ , \qquad D^L = D^R =: D\ , \qquad \sigma^L = \sigma^R =: \sigma\ , \tag{C.19}$$

and further couple the gauge and matter sectors of the model together by the requirements

$$\lambda = -g\,\psi\ , \qquad \sigma = g\,Z\ , \qquad D = -g\,F\ . \tag{C.20}$$

With these gauge field replacements, and diagonal limits of the gauginos and auxiliary fields, we find that the pure Chern-Simons component from the first line of the action (C.15) reduces to (5.56). For the remaining matter contributions in (C.15), by inserting the field identifications above and expanding around the vacuum value we obtain

$$\begin{aligned}\mathcal{S}_{\rm m} = \ & \mathrm{Tr}_{\,V}\big(-[A_\mu, Z'\,]^2 - 4g^2\, B_\mu\, B^\mu \\ & + \mathrm{i}\,\bar\psi\,\gamma^\mu\,[A_\mu,\psi] - \bar\psi\,\gamma^\mu\,\{B_\mu,\psi\} + \mathrm{i}\,g\,\bar\psi\,[Z',\psi] + F^2\big)\ .\end{aligned} \tag{C.21}$$

We now scale the fields appropriately and take the strong coupling limit $g \to \infty$. We can integrate out the auxiliary field B_μ using its equation of motion

$$B_\mu = \tfrac{\kappa}{g^2}\,\epsilon_{\mu\nu\lambda}\,[A^\nu, A^\lambda]\ . \tag{C.22}$$

In deriving this equation we have ignored cubic and higher order interactions involving B_μ that become suppressed in the strong coupling limit. Inserting (C.22) into the pure Chern-Simons action (5.56), we find

$$\mathcal{S}_{\rm g} = -\tfrac{4\kappa^2}{g^2}\ {\rm Tr}_V\big([A_\mu, A_\nu]^2\big)\ . \tag{C.23}$$

We scale the matter field Z by the factor $\frac{1}{g}$, and similarly for the matter fermion (and its adjoint) and the auxilliary field F. Replacing B_μ by its equation of motion (C.22), we find that the matter action (C.21) reduces in the strong coupling limit to

$$\mathcal{S}_{\rm m} = {\rm Tr}_V\big(-\tfrac{1}{g^2}\,[A_\mu, Z'\,]^2 - \tfrac{4\kappa^2}{g^2}\,[A_\mu, A_\nu]^2 + \tfrac{\rm i}{g^2}\,\bar\psi\,\gamma^\mu\,[A_\mu,\psi] + \tfrac{\rm i}{g^2}\,\bar\psi\,[Z',\psi] + \tfrac{1}{g^2}\,F^2\big)\ . \tag{C.24}$$

We combine the scalar and gauge fields into a single field

$$X^I = (X^\mu, X^3) = (A^\mu, Z'\,) \tag{C.25}$$

where $I = 0,1,2,3$. Then with $\kappa = \frac{1}{4}$, the sum of (C.23) with the first two terms of (C.24) can be written as $-\frac{1}{2g^2}\ {\rm Tr}_V\big([X^I, X^J]^2\big)$, which is the bosonic potential of the IKKT model. For the last three terms of (C.24), we define a four-dimensional Majorana spinor of the Clifford algebra $C\ell(\mathbb{R}^{1,3})$ by

$$\Psi = \big(\psi^1, \psi^2\big)^\top\ , \tag{C.26}$$

where each real component ψ^1, ψ^2 of the Dirac spinor ψ is a two-component Majorana spinor. We then construct a set of four-dimensional gamma-matrices from our three-dimensional Pauli spin matrices as

$$\Gamma^\mu = {\rm i}\begin{pmatrix} 0 & \gamma^\mu \\ -\gamma^\mu & 0 \end{pmatrix}\ , \qquad \Gamma^3 = -{\rm i}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\ . \tag{C.27}$$

For the chirality and charge conjugation matrices, we take

$$\Gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \ , \qquad C = \begin{pmatrix} -\mathrm{i}\,\gamma^2 & 0 \\ 0 & \mathrm{i}\,\gamma^2 \end{pmatrix} \ . \tag{C.28}$$

We can then combine the last three terms of (C.24) as $\frac{1}{g^2}\,\mathrm{Tr}_{\,V}\big(-\bar\Psi\,\Gamma^I\,[X_I,\Psi]+F^2\big)$, which is the fermionic term of the IKKT model together with an auxiliary field.

Altogether the A_1 quiver matrix model action is reduced under the Mukhi-Papageorgakis map to the action of the four-dimensional IKKT model

$$\mathcal{S}_{\rm IKKT} = \tfrac{1}{g^2}\,\mathrm{Tr}_{\,V}\big(-\tfrac{1}{2}\,[X^I,X^J]^2 - \bar\Psi\,\Gamma^I\,[X_I,\Psi] + F^2\big) \ . \tag{C.29}$$

C.1.2 Supersymmetry reduction

We will now show explicitly how the supersymmetry transformations of the A_1 quiver matrix model map to those of the IKKT model under the Mukhi-Papageorgakis map. The original matrix model has $\mathcal{N}=2$ supersymmetry, while the IKKT model in four dimensions has $\mathcal{N}=1$ supersymmetry. Hence the scaling limit must reduce the supersymmetry; we do this by identifying the infinitesimal supersymmetry generators in (C.16) so that $\varepsilon=\eta$ are no longer independent. We demonstrate the reduction on each field transformation of (C.16) individually.

For the transformations of the gauge fields A_μ in (C.16), we make the gauge field identifications, identify the supersymmetry generators with each other, and scale the spinor. In four dimensions we write the fermions as four-component Majorana spinors obeying (C.17), and along with the four-dimensional gamma-matrices (C.27) we can write

$$\delta A_\mu = \bar\varepsilon\,\Gamma_\mu\,\lambda \ . \tag{C.30}$$

Following a similar process for the supersymmetry transformations of the matter field Z, the requirement (C.19) reveals the Majorana spinor condition (C.17). After expanding Z around its classical value and scaling, we can combine its supersymmetry transformation with (C.30) to get

$$\delta X^I = \bar{\varepsilon}\,\Gamma^I\,\Psi\ . \tag{C.31}$$

For the supersymmetry variation of the auxiliary field D in (C.16), we identify the supersymmetry generators with each other, scale the fields and expand around the classical value, so that the resulting supersymmetry transformation reads as

$$\delta D = \tfrac{\mathrm{i}}{2}\left(\bar{\varepsilon}\,\gamma^\mu\,[A_\mu,\lambda] + [A_\mu,\bar{\lambda}]\,\gamma^\mu\,\varepsilon\right) + \tfrac{\mathrm{i}}{2}\left(\bar{\varepsilon}\,[\lambda,Z'\,] + [\bar{\lambda},Z'\,]\,\varepsilon\right)\ . \tag{C.32}$$

Identifying the four-dimensional gamma-matrices (C.27) and (C.28), applying the Majorana spinor identity (C.17), and combining Z' with A_μ as in (C.25) results in the transformation

$$\delta D = \mathrm{i}\,\bar{\varepsilon}\,\Gamma^5\,\Gamma^I\,[X_I,\Psi]\ . \tag{C.33}$$

A similar modification occurs for the supersymmetry transformation of the auxiliary field F. We take an axial combination (5.28) of the gauge fields in (C.16) which reduces the interaction of the fermion and the gauge field to a commutator, at the cost of introducing the field B_μ, so that after making the field replacements (C.19) we arrive at

$$\delta F = \bar{\varepsilon}^*\big(\gamma^\mu\,[A_\mu,\psi] + \mathrm{i}\,\gamma^\mu\,\{B_\mu,\psi\} + [\sigma,\psi] + \mathrm{i}\,\{\lambda,Z\}\big)\ . \tag{C.34}$$

Expanding around the vacuum, and taking the appropriate scaling limit, the B_μ contribution decouples. After combining the gauge and matter fields, and rewriting the spinor and gamma-matrices, the reduction (C.34) coincides with (C.33).

Finally, we consider the spinor supersymmetry transformations. For the gaugino variation $\delta\lambda$ in (C.16), we make the usual field identifications and scalings, and combine the terms involving A_μ and Z to get

$$\delta\Psi = \ -\mathrm{i}\,\Gamma_{IJ}\,[X^I,X^J]\,\varepsilon - F\,\varepsilon\ . \tag{C.35}$$

For the matter fermions in (C.16), we take the axial limit of the gauge fields and make the field replacements to get

$$\delta\psi = \mathrm{i}\,\gamma^{\mu}\,\varepsilon\,\big([A_\mu, Z] + \mathrm{i}\,\{B_\mu, Z\}\big) + F\,\varepsilon^* \ . \tag{C.36}$$

Inserting the equation of motion (C.22) for B_μ and taking the scaling limit we find

$$\delta\psi = \mathrm{i}\,\gamma^{\mu}\,\varepsilon\,\big([A_\mu, Z] + 2\,\mathrm{i}\,\kappa\,\epsilon_{\mu\nu\lambda}\,[A^\nu, A^\lambda]\big) + F\,\varepsilon^* \ . \tag{C.37}$$

By setting $\kappa = \frac{1}{4}$ and using the Pauli spin matrix identity

$$\tfrac{\mathrm{i}}{2}\,\epsilon^{\mu\nu\lambda}\,\gamma_\lambda = \gamma^{\mu\nu} \ , \tag{C.38}$$

we find that (C.37) coincides with (C.35).

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