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Polycyclic monoids and their generalisations

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Abstract

This thesis can be split into two parts. The first was inspired by a monograph by Bratteli and Jorgensen. We study arbitrary, not necessarily transitive, strong actions of polycyclic inverse monoids P_n. We obtain some new results concerning the strong actions of P_2 on $\mathbb{Z}$ determined by the choice of one positive odd number p. We show that the structure of the representation can be explained by studying the binary representations of the numbers $\frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$. We also generalise the connection between the positively self conjugate submonoids of P_n and congruences on the free monoid A_n^* developed by Meakin and Sapir.

The second part can be seen as a generalisation of the first. Graph inverse semigroups generalise the polycyclic inverse monoids and play an important role in the theory of C^*-algebras. We provide an abstract characterisation of graph inverse semigroups and show how they may be completed to form what we call the Cuntz-Krieger semigroup of the graph — this semigroup is then the semigroup analogue of the Leavitt path algebra of the graph. We again generalise the connection of Meakin and Sapir this time to certain subsemigroups of the graph inverse semigroup and congruences on the free graph.

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Introduction

Mathematics has always been interested in symmetry. Group theory has been the branch of mathematics used for handling symmetry for the last hundred and fifty years. Take a straight-forward example, the symmetries of an equilateral triangle. The three reflections, two rotations and the identity transformation form the group of symmetries of this object that is better known as the dihedral group D_3. Group theory can handle symmetries in a wide number of different areas using similar ideas.

However there are some symmetries that groups cannot capture. These symmetries require the more general notion of an inverse semigroup. Take the equilateral triangle's fractal counterpart, the Sierpiński triangle, also know as the Sierpiński gasket [27]. The Sierpiński triangle is a two dimensional analogue of the Cantor middle third set. The distinguishing feature of it is that certain parts of the object are 'the same as' the whole object. Whatever we mean by 'the same as' we call it a *partial symmetry*. Let α be an injective map from the whole gasket to one corner. Each corner is the same as the whole gasket so an injective map can be defined. By injectivity we can define a map α^{-1} such that $\alpha^{-1}\alpha$ is the identity map on the whole gasket (here we take the convention of writing functions on the left). As the range of α is not the entire object we have that $\alpha\alpha^{-1}$ is not the identity map on the whole object. In fact, it is impossible to find a left inverse for α. This means that this symmetry can not be captured by a group. Thus the group of symmetries of the equilateral triangle and the Sierpiński triangle are the same. We don't want to discard these partial symmetries from our studies. To include these partial symmetries the definition of a group needs to be weakened. This is done by generalising what it means for an element to have an inverse

Inverse semigroups where discovered[1] in 1952. Wagner preferred the term *generalized groups* in his first paper on the topic [57]. Independently, Preston [49] wrote a paper entitled *Inverse semi-groups* in 1954. He followed this with two more papers on the topic in the same year. Possibly less wellknown is how close Gołab [15] came in 1939 while working on *pseudogroups*. Veblen and Whitehead [56] introduced transformation pseudogroups in 1932; however, they were not concerned with the abstract idea. Their definition of composition of partial maps, say ψ and φ, required the image of ψ to equal the domain of φ for the product $\varphi \circ \psi$ to exist. Gołab states that the composition of two partial maps $\varphi \circ \psi$ is defined when the image of ψ has non-empty intersection with the domain of φ. However the product if the intersection was empty was left undefined. This meant the definition of a pseudogroup was one step, the empty map, away from the definition of an inverse semigroup. Schein [54] makes the comparison to the history of the integers. First came positive numbers, then negative numbers, and finally zero, the claim being that the concept of zero and the empty map are psychologically the hardest to grasp. Whether or not things would have been significantly different had Gołab made this step we will never know.

Wagner's realisation that the empty map needed to be included came from spotting a now obvious connection. A partial map is a type of binary relation. Let ψ be a partial map on A. Then ψ can be defined a subset of $A \times A$. With this realisation Wagner could harness the power of binary relations. The multiplication of binary relations is well defined (the empty relation having the empty set as its domain and image). Now composition of partial maps was an everywhere defined associative binary operation. Thus they formed a semigroup in which every element was invertible in some sense. After the independent work of Wagner and Preston the use of algebraic concepts of

[1]Whether inverse semigroups existed before their definition was written down (or at all) is an existential question. The emphasis here is that inverse semigroups are the appropriate objects to study and not just a curious generalisation of groups.

homomorphisms, substructures, congruences, etc. could begin.

One powerful tool that inverse semigroup theorists use is another binary relation. The natural (partial) order on an inverse semigroup S. Intuitively, the natural order relates two partial maps if one is a restriction (or extension) of the other. The natural order on any group is trivial. A full discussion on why this is so highlights the extra structure of inverse semigroups. Unfortunately we don't have time for that discussion here, see [27] page 21 and [18] page 152 instead.

The representation theory of inverse semigroups is as old as inverse semigroups themselves[2]. Regular representations of groups utilise Cayley's theorem:

"Every group is isomorphic to a subgroup of a symmetric group."

Independently, Wagner and Preston generalised this theorem to inverse semigroups in their first papers on the subject. The *symmetric inverse monoid* on a set X is the set of all partial bijections on X with the multiplication as defined above [27]. It is the inverse semigroup equivalent to the symmetric group. The Wagner-Preston theorem

"Every inverse semigroup is isomorphic to an inverse subsemigroup of a symmetric inverse monoid"

is analogous to Cayley's result. Thus every inverse semigroup can be represented by partial bijections. From here it has been shown that every effective representation is the sum of effective transitive representations. The representation of an inverse semigroup by partial bijections leads naturally to the idea of partial actions. These actions are not everywhere defined and we will speak about them in more detail in chapter one.

Our attention in this thesis will focus on 1) the actions and representations of the *polycyclic inverse monoids* and 2) their generalisations.

[2]Preston's paper *Representations of inverse semigroups* appeared a mere eight pages after his initial paper on the topic of inverse semigroups [50].

1) The polycyclic inverse monoids P_n were introduced by Nivat and Perrot in 1971 [44] and are defined for all natural numbers $n \geq 2$ [3]. Cuntz rediscovered these monoids while defining the Cuntz C^*-algebras [9]; these C^*-algebras are generated by a finite set of partial isometries that satisfy the relations of the polycyclic monoids. Within C^*-algebra theory the polycyclic monoids are often called *Cuntz inverse semigroups*[4]. One way to visualise the polycyclic monoids is in terms of the Cantor set [17]. Let a_l be an injective map from the whole set to the left third and a_r from the whole set to the right third. The given domains and ranges of these maps imply the following relations;

$$a_l^{-1}a_l = 1 = a_r^{-1}a_r \text{ and } a_l^{-1}a_r = 0 = a_r^{-1}a_l$$

where 1 denotes the identity map and 0 denotes the empty map. The six maps (identity map, empty map, $a_l, a_r, a_l^{-1}, a_r^{-1}$) and these relations define P_2.

Our study of the representations of the polycyclic monoids are motivated by the monograph by Bratteli and Jorgensen [5]. Their focus is on certain classes on *branching function systems*. We will show that these systems can be interpreted as specific sorts of representations of the polycyclic monoid.

2) The presentation for the graph inverse semigroup in [46] was a starting point for the second part of this thesis. The graph inverse semigroups are the most natural generalisation of the polycyclic monoids. The polycyclic inverse monoid P_n is constructed from the free monoid on an n-letter alphabet. Such a monoid can be viewed as the free category of a directed graph consisting of one vertex and n loops. This suggests that polycyclic monoids might be generalized by replacing free monoids by free categories and this is how graph inverse semigroups arise.

This was first carried out by Ash and Hall [2] in 1975 at which point history repeated itself. In [10], Cuntz and Krieger introduced a class of C^*-algebras, constructed

[3] The bicyclic monoid can be considered as the polycyclic monoid for $n = 1$, however we will not discuss it in this work.

[4] We will not use this term to refer to the polycyclic monoids as we use it to describe another structure later.

from suitable directed graphs, now known as *Cuntz-Krieger algebras*. In fact they considered finite square matrices A with entries in $\mathbb{Z}_2$, in 1982 Watatani [58] made the connections to graphs with adjacency matrix A. Although Watatani continued with this graph approach it wasn't until 1997 when Kumjin, Pask, Raeburn and Renault published their paper [24] that the field took notice. Work on C^*-algebras associated to graphs continued and Fowler, Laca and Raeburn [13] in 2000 finally defined the C^*-algebra for arbitrary directed graphs twenty years after they were introduced. In [11] Drinen and Tomforde further develop this theory[5]. From here the inverse semigroup and groupoid approach has been developed as a means of working with the associated C^*-algebra, namely by Farthing, Muhly and Yeend [14].

However the graph inverse semigroup has received very little attention based on its own merits, more as a tool for studying other objects that an object of interest in itself. As such, much of the semigroup theoretical power is yet to be harnessed. In 2005, Abrams and Pini [1] introduced what they called *Leavitt path algebras* as the algebra analogues of the Cuntz-Krieger algebras. These are also the subject of [55]. The connection between graph inverse semigroups and the Cuntz-Krieger algebras is spelled out by Paterson [46] and, significantly for section 3.6, in the work of Lenz [39].

[5]This paper also contains a full, in-depth history to the C^*-algebra development.

Chapter 1

Background

In this chapter we discuss basic theory and state a few important definitions. The inverse semigroup results should be familiar to most readers however the representation theory may not be. Graphs and categories are barely touched upon as we will only really need the basic ideas.

1.1 Semigroups

An non-empty set S with an associative binary operation is called a *semigroup*. The product in a semigroup will usually be denoted by concatenation but sometimes we shall use $\cdot$ for emphasis; we shall also use it to denote actions. If there exists an element $1 \in S$ such that $s1 = s = 1s$ for all $s \in S$ then we call 1 an *identity* element; semigroups with identities are called monoids.

A semigroup is said to have a *zero* element 0 if $s0 = 0 = 0s$ for all $s \in S$. It is straight-forward to show that a semigroup can have at most one identity and zero. For a semigroup without an identity we define $S^1 = S \cup \{1\}$ where $s1 = s = 1s$ for all $s \in S$ and $1 \cdot 1 = 1$, for a semigroup with an identity we define $S^1 = S$. We call S^1 in both cases *S with an identity adjoined if necessary*. A non-empty subset R of S is called a *subsemigroup* of S if $ab, ba \in R$ for all $a, b \in R$, and if S is a monoid then R is a submonoid if it is a subsemigroup and $1 \in R$.

We say that a semigroup S is *regular* if for all $a \in S$ there exists an a' such that $aa'a = a$. An element e of a semigroup S is called *idempotent* if $e^2 = e$ and for any semigroup or subset R we denote the set of idempotents by $E(R)$.

We say a semigroup S acts on a set X if there exists a function $\cdot : S \times X \to X$ such that

$$(st) \cdot x = s \cdot (t \cdot x) \text{ for all } s, t \in S,\ x \in X.$$

If S is a monoid then we also require that $1 \cdot x = x$ for all $x \in X$. Page 37 of Lawson's Book [27] gives more information about monoid actions. Howie's Book [18] is a good reference for general semigroup theory.

1.2 Congruences

Congruences on semigroups are the natural equivalent to normal subgroups of groups. Let $\theta : S \to T$ be a homomorphism of semigroups. The kernel of θ is the relation $\ker\theta$ defined on S by

$$\ker\theta = \{(a, b) \in S \times S : \theta(a) = \theta(b)\}.$$

It is easy to check that $\ker\theta$ is an equivalence such that

$$(a, b), (c, d) \in \ker\theta \Rightarrow (ac, bd) \in \ker\theta.$$

Formally, a *congruence* ρ on a semigroup S is a subset of $S \times S$ with the following properties:

- $(a, a) \in \rho$ for all $a \in S$ *reflexivity*
- $(a, b) \in \rho \Leftrightarrow (b, a) \in \rho$ *symmetry*

- $(a, b), (b, c) \in \rho \Rightarrow (a, c) \in \rho$ *transitivity*

That is that ρ is an *equivalence*.

- $(a, b) \in \rho, c \in S \Rightarrow (ca, cb) \in \rho$ *left congruency*
- $(a, b) \in \rho, c \in S \Rightarrow (ac, bc) \in \rho$ *right congruency*

The two trivial examples of congruences on any semigroup S are the *equality (identity) congrence* $\iota = \{(a, b) \in S \times S : a = b\}$, and the *universal congruence* $\upsilon = S \times S$.

Let ρ be a congruence on a semigroup S. We can partition S by ρ by grouping together elements that are ρ-equivalent. We denote the set of ρ-equivalence classes (congruence classes) by S/ρ. The set S/ρ is a semigroup with respect to the binary operation of equivalence classes called the *quotient* of S by ρ.

A subset I of an semigroup S is called a right ideal if $Is \subseteq I$ for all $s \in S$. Left ideals are defined dually and two-sided ideals, or simply ideals, are ideals which are both left and right ideals. An ideal I of S is *proper* if $I \neq S$. Let I be a proper ideal of a general semigroup S. Then we can form the *Rees congruence* ρ_I as follows;

$$x \, \rho_I \, y \Leftrightarrow x = y \text{ or } x, y \in I.$$

By definition ρ_I is reflexive, symmetric, transitive and a congruence. We may take the quotient of S by the congruence ρ_I, which can be denoted as

$$S/\rho_I = (S \setminus I) \cup \{0\}.$$

For more on Rees congruences see Howie [18] page 33.

1.3 Free monoids

In this work, we shall also use the theory of strings [40]. Let A be a finite set, called in this context an *alphabet*. Then A^* is the set of all finite strings over A, including the empty string ε. This forms a monoid with respect to concatenation of strings: the *free monoid* on A. It is clear that if $|A| = n = |B|$ and $n \in \mathbb{N}$ then A^* is isomorphic to B^*. Note: We define $\mathbb{N} = \mathbb{N} \cup \{0\}$. If x is a string then $|x|$ denotes its length. By A^ω we mean the set of all right-infinite strings over A. If p is a finite string then $p^\omega = ppp\ldots$, an infinite string. A finite string is said to be *primitive* if it is not a power of another string. The strings x and y are said to be *conjugate* if we can write $x = uv$ and $y = vu$ for some strings u and v. This defines an equivalence relation on the set of all strings and so we may talk about *conjugacy classes*. A string is said to be a *Lyndon word* if it is primitive and minimal in its conjuagacy class where the order is the lexicographic order [40].

Lemma 1.3.1 (Proposition 1.3.2 [40])**.** *Two strings commute if and only if they are powers of the same string.*

Lemma 1.3.2 (Proposition 1.3.4 [40])**.** *Suppose that $xz = zy$. Then there exist strings u and v such that $x = uv$, $y = vu$ and $z \in u(vu)^*$.*

The following result is well-known [7] but we include the proof for the sake of completeness.

Lemma 1.3.3. *Let p be a primitive string. If $p^2 = upv$ then either $u = \varepsilon$ or $v = \varepsilon$.*

Proof. We may write $p = \overline{p}\check{p}$ and $p = u\overline{p}$ and $p = \check{p}v$. We have that $\overline{p}p = pv$. Thus by Lemma 1.3.2 there exist strings x and y such that $\overline{p} = xy$, $v = yx$ and $p \in x(yx)^*$. We have that $up = p\check{p}$. Thus by Lemma 1.3.2 there exist strings x' and y' such that $u = x'y'$, $\check{p} = y'x'$ and $p \in x'(y'x')^*$. Now $p = u\overline{p} = x'y'xy$, also $p = \check{p}v = y'x'yx$. Thus $x'y'xy = y'x'yx$. By length considerations $x'y' = y'x'$ and $xy = yx$. By Lemma 1.3.1, it follows that there are strings a and b such that $x = a^\alpha$ and $y = a^\beta$, and $x' = b^\gamma$ and $y' = b^\delta$. But $p \in a^\alpha(a^\beta a^\alpha)^*$. Thus p can be written as a product of a's. However p is primitive. Thus $p = a$ and so $\alpha = 1$. That is $\overline{p} = p$ and $u = \varepsilon$. $\square$

1.4 Inverse semigroups

A semigroup S is an *inverse semigroup* if it is regular and its set of idempotents commutes. It is an early result of inverse semigroup theory that for all a in an inverse semigroup S there exist an $a' \in S$ such that

$$aa'a = a \text{ and } a'aa' = a'.$$

To see that a' is unique assume $aa''a = a$ and $a''aa'' = a''$. Then

$$a'a = a'aa''a = (a'a)(a''a) = (a''a)(a'a) = a''a.$$

Similarly $aa' = aa''$. Thus

$$a'' = a''aa'' = a''aa' = a'aa' = a'.$$

We denote the *inverse* of $s \in S$ by s^{-1}.

We now introduce the *natural partial order*. Let S be an inverse semigroup. We define a relation $\leq$ on S by $a \leq b$ if there exists an idempotent e such that $a = eb$. As $a = (aa^{-1})a$ we have $a \leq a$ and the relation is reflexive. Suppose $a \leq b$ and $b \leq a$. Then there exist $e, f \in E(S)$ such that $a = eb$ and $b = fa$ and

$$a = eb = efa = fea = feeb = feb = fa = b.$$

Thus the relation is anti-symmetric. Finally, if $a \leq b, b \leq c$ then $a = eb$ and $b = fc$ thus $a = efc$. As $e, f \in E(S)$ their product is idempotent and $a \leq c$. Therefore $\leq$ is a partial order. The natural partial order is compatible with inversion and multiplication. The natural partial order will be the only partial order considered when we deal with inverse semigroups.

Let $(E, \leq)$ be a partially ordered set, or poset for short. For $x \in E$ define

$$x^{\downarrow} = \{y \in E \colon y \leq x\},$$

the *principal order ideal generated by x*, and

$$x^{\uparrow} = \{y \in E \colon y \geq x\},$$

the *principal filter generated by* x. We extend this notation to subsets $A \subseteq E$ and define

$$A^{\downarrow} = \{y \in E\colon y \leq x \text{ for some } x \in A\}$$

and $A^{\uparrow}$ dually. A subset A such that $A = A^{\downarrow}$ is called an *order ideal.* If A is a finite set then $A^{\downarrow}$ is said to be a *finitely generated* order ideal. The posets we consider will always have a smallest element 0. Such a poset X is said to be *unambiguous*[1] if for all $x, y \in X$ if there exists $0 \neq z \leq x, y$ then either $x \leq y$ or $y \leq x$. Given $e, f \in E$ we say that e *covers* f if $e > f$ and there is no $g \in E$ such that $e > g > f$. For each $e \in E$ define $\hat{e}$ to be the set of elements of E that are covered by e. A poset is said to be *pseudofinite* if whenever $e > f$ there exists $g \in \hat{e}$ such that $e > g \geq f$, and for which the sets $\hat{e}$ are always finite.

An inverse subsemigroup is a subsemigroup $R \leq S$ such that $a^{-1} \in R$ for all $a \in R$. Such a subsemigroup of S is said to be *wide* if it contains all the idempotents of S. A wide inverse subsemigroup T of a semigroup S is said to be *normal* if $sts^{-1} \in T$ for all $s \in S$ and $t \in T$.

On an inverse semigroup S we define *Green's relations* in the following way:

- $a\,\mathcal{L}b$ iff $S^1a = S^1b$,
- $a\,\mathcal{R}b$ iff $aS^1 = bS^1$,
- $a\,\mathcal{H}b$ iff $a\,\mathcal{L}b$ and $a\,\mathcal{R}b$,
- $a\,\mathcal{D}b$ iff there exist a c such that $a\,\mathcal{L}c\mathcal{R}b$,
- $a\,\mathcal{J}b$ iff $S^1aS^1 = S^1bS^1$,

There are many other formulations of these relations. One useful result it that $\mathcal{D} \subseteq \mathcal{J}$ We say a semigroup is *combinatorial* if each $\mathcal{H}$-class is a singleton. A semigroup is

[1]Strictly speaking 'unambiguous except at zero' but that is too much of a mouthful.

bisimple if it has only one $\mathcal{D}$-class and a semigroup is 0-*bisimple* if the only $\mathcal{D}$-classes are $\{0\}$ and $S \setminus \{0\}$.

An inverse semigroup S is said to have *maximal idempotents* if for each non-zero idempotent e there is an idempotent e° such that $e \leq e^\circ$ where e° is a maximal idempotent such that if $e \leq i^\circ, j^\circ$ then $i^\circ = j^\circ$. Observe that this is a special case of what might ordinarily be regarded as a semigroup having maximal idempotents. An inverse semigroup will be called a *Leech semigroup* if it has maximal idempotents and each $\mathcal{D}$-class contains a maximal idempotent. Such a semigroup is said to be a *strict* if each $\mathcal{D}$-class contains a unique maximal idempotent.

An inverse semigroup S is said to be *completely semisimple* if $s\mathcal{D}t$ and $s \leq t$ implies $s = t$ for all $s, t \in S$.

If e is an idempotent in the inverse semigroup S then eSe is called a *local submonoid*. Let S be an inverse semigroup and $e \in E(S)$. We say that S is an *enlargement of* eSe if $S = SeS$.

We say that elements s and t in an inverse semigroup are *compatible*, denoted $s \sim t$, if both $s^{-1}t$ and st^{-1} are idempotents. A subset of S is *compatible* if each pair of elements in the subset are compatible. An inverse semigroup is said to be *distributive* if the following holds. Let $\{a_1, \ldots, a_m\}$ be a finite subset of S and let $a \in S$ be any element. If $\bigvee_{i=1}^m a_i$ exists then both $\bigvee_{i=1}^m aa_i$ and $\bigvee_{i=1}^m a_i a$ exist and we have the following two equalities

$$a\left(\bigvee_{i=1}^m a_i\right) = \bigvee_{i=1}^m aa_i \text{ and } \left(\bigvee_{i=1}^m a_i\right)a = \bigvee_{i=1}^m a_i a.$$

An inverse semigroup is said to be *complete* if every finite compatible subset has a join and the semigroup is distributive. A homomorphism $\phi \colon S \to T$ is said to be *join-preserving* if for every finite subset $A \subseteq S$ the existence of $\vee A$ implies the existence of $\vee\phi(A)$ and $\vee\phi(A) = \phi(\vee A)$.

Define $\mathbf{d}(s) = s^{-1}s$ and $\mathbf{r}(s) = ss^{-1}$. A pair of elements $s, t \in S$ is said to be *orthogonal* if $s^{-1}t = 0 = st^{-1}$. Observe that s and t are orthogonal iff $\mathbf{d}(s)\mathbf{d}(t) = 0$

and $\mathbf{r}(s)\mathbf{r}(t) = 0$. A subset of S is said to be *orthogonal* iff each pair of distinct elements in it is orthogonal. If the join of a finite set of orthogonal elements exists we talk about *orthogonal joins*. An inverse semigroup with zero S will be said to be *orthogonally complete* if it has joins of all finite orthogonal subsets and multiplication distributes over finite orthogonal joins. Homomorphisms between inverse semigroups with zero map finite orthogonal subsets to finite orthogonal subsets. If orthogonal joins are preserved then we say that the homomorphism is *orthogonal join-preserving*.

The symmetric inverse monoids are (orthogonally) complete.

The proofs of the following may be found in [27].

Lemma 1.4.1. *Let S be an inverse semigroup.*

1. *For each element s in an inverse semigroup S the subset $s^{\downarrow}$ is compatible.*
2. *If s and t are compatible then $s \wedge t$ exists and $\mathbf{d}(s \wedge t) = \mathbf{d}(s)\mathbf{d}(t)$, and $\mathbf{r}(s \wedge t) = \mathbf{r}(s)\mathbf{r}(t)$*
3. *If s and t are compatible and $\mathbf{d}(s) \leq \mathbf{d}(t)$ then $s \leq t$, and if s and t are compatible and $\mathbf{r}(s) \leq \mathbf{r}(t)$ then $s \leq t$.*
4. *If $s \wedge t$ exists then $as \wedge at$ exists for any a and $a(s \wedge t) = as \wedge at$ (and $sa \wedge ta$ exists for any a and $(s \wedge t)a = sa \wedge ta$).*

The following is the finitary version of Proposition 1.4.20 [27].

Lemma 1.4.2. *Let S be a finitely complete inverse semigroup. Then the following are equivalent:*

1. *S is distributive.*
2. *$E(S)$ is a distributive lattice.*
3. *For all finite subsets $A, B \subseteq S$ if $\vee A$ and $\vee B$ both exist then $\vee AB$ exists and $(\vee A)(\vee B) = \vee AB$.*

We use the term *boolean algebra* to mean what is often referred to as a generalised boolean algebra, and a *unital boolean algebra* is what is usually termed a boolean algebra. The proof of the following can be obtained by generalising the proofs of Lemmas 2.2 and 2.3 of [34].

Lemma 1.4.3. *Let S be an orthogonally complete inverse semigroup whose semilattice of idempotents is a boolean algebra. Then S is actually complete.*

An inverse semigroup S is said to be *unambiguous* if for all non-zero idempotents e and f if $ef \neq 0$ then $e \leq f$ or vice-versa. An inverse semigroup S is said to satisfy the *Dedekind height condition* if for all non-zero idempotents e we have $\left|e^{\uparrow} \cap E(S)\right| < \infty$. We define a *Perrot semigroup* to be an inverse semigroup that is unambiguous and has the Dedekind height property.

A semilattice E is said to be 0-*disjunctive* if for each $0 \neq f \in E$ and e such that $0 \neq e < f$, there exists $0 \neq e' < f$ such that $ee' = 0$. It can be proved that an inverse semigroup S is *congruence-free* if it is 0-simple, fundamental and its semilattice of idempotents is 0-disjunctive [47]. Combinatorial inverse semigroups are always fundamental and we shall therefore not need the more general notion in this thesis.

An inverse semigroup is E^*-*unitary* if $0 \neq e \leq s$, where e is an idempotent, implies that s is an idempotent. The following is Remark 2.3 of [39] which is worth repeating since it was a surprise to many.

Lemma 1.4.4. *If S is an E^*-unitary inverse monoid then $(S, \leq)$ is a meet semilattice.*

Proof. Let $s, t \in S$. Suppose first that $a \leq s, t$ implies $a = 0$. Then in fact $s \wedge t = 0$. We shall therefore suppose that s and t have non-zero lower bounds. Let $0 \neq a \leq s, t$. Then $s^{-1}t, st^{-1}$ are both idempotents since S is E^*-unitary. It follows that s and t are compatible. By Lemma 1.4.1, this implies that $s \wedge t$ exists, as required. □

There are many naturally occurring examples of E^*-unitary inverse monoids and it is a condition that is easy to verify. In particular, the graph inverse semigroups are

E^*-unitary. More generally, an inverse semigroup is called an *inverse $\wedge$-semigroup* if each pair of elements has a meet.

Lemma 1.4.5. *Let S be an unambiguous inverse semigroup. Then the partially ordered set $(S, \leq)$ is unambiguous if and only if S is E^*-unitary.*

Proof. Let S be an E^*-unitary inverse semigroup. Let $0 \neq a \wedge b \leq a, b$. Then $0 \neq \mathbf{d}(a \wedge b) \leq \mathbf{d}(a), \mathbf{d}(b)$. By unambiguity, it follows that either $\mathbf{d}(a) \leq \mathbf{d}(b)$ or vice-versa. We assume the former without loss of generality. Thus $\mathbf{d}(a) \leq \mathbf{d}(b)$. However $a^{-1}b$ and ab^{-1} are both above non-zero idempotents. Thus from the fact that the semigroup is E^*-unitary we have that a is compatible with b. By Lemma 1.4.1, we have that $a \leq b$, as required

Let $(S, \leq)$ be an unambiguous poset. We prove that S is E^*-unitary. Let $0 \neq e \leq s$ where e is an idempotent. We prove that s is an idempotent. Clearly $e \leq s^{-1}$. Thus s and s^{-1} are comparable. If $s \leq s^{-1}$ then by taking inverses we also have that $s^{-1} \leq s$ and vice-versa. It follows that $s = s^{-1}$. Thus $s^2 = ss^{-1}$ is an idempotent. Now s and s^2 are also comparable. If $s \leq s^2$ then s is an idempotent and we are done. If $s^2 \leq s$ then $s = ss^{-1}s = s^3 \leq s^2$ and so $s \leq s^2$ and s is again an idempotent. □

For more details on inverse semigroups, the reader is directed to [27, 47].

1.5 Representations of inverse semigroups

Note that our inverse semigroups will have a zero, and we shall assume that homomorphisms preserve the zero. If $A \subseteq S$ is a subset of an inverse semigroup define

$$A^{\uparrow} = \{s \in S \colon\ a \leq s \text{ for some } a \in A\}.$$

This is referred to as the *closure* or *upper saturation* in some literature. If $A = A^{\uparrow}$ then A is said to be *closed (upwards)*. This matches up with the notation used for a filter on a poset.

We now recall the key elements of the theory of the representations of inverse semigroups by partial bijections [30, 20].

A *partial bijection* of a set X is a bijective map between two subsets of X. The *symmetric inverse monoid* $I(X)$ is the set of all partial bijections on the set X. A *representation* of an inverse semigroup by means of partial bijections is a homomorphism $\theta\colon\ S \to I(X)$. If S is a monoid we shall assume that the homomorphism is a monoid homomorphism.

A representation of an inverse semigroup in this sense leads to a corresponding notion of an *action* of the inverse semigroup S on the set X. This action is defined by $s \cdot x = \theta(s)(x)$, if this exists, where $\theta(s)$ denotes the partial bijection s is mapped to under θ and $\theta(s)(x)$ is the evaluation of this partial bijection at x. We shall use the words 'action' and 'representation' interchangeably.

If S acts on X, we shall refer to X as a *space* and its elements as *points*. A subset $Y \subseteq X$ closed under the action is called a *subspace*. Disjoint unions of actions are again actions. We shall always assume that our actions are *effective*, meaning that for each $x \in X$ there is an $s \in S$ such that $s \cdot x$ exists. Under this assumption, the action of an inverse semigroup S on the set X induces an equivalence relation $\sim$ on the set X when we define $x \sim y$ iff $s \cdot x = y$ for some $s \in S$. The action is said to be *transitive* if $\sim$ is $X \times X$. We call the induced equivalence classes *orbits*. We denote the orbit of a point $x \in X$ under the action of an inverse semigroup S by $\mathrm{Orb}_S(x)$.

Just as in the theory of permutation representations of groups, every representation of an inverse semigroup is a disjoint union of transitive representations. Let X and Y be spaces. A *morphism* from X to Y is a function $\alpha\colon\ X \to Y$ such that $s \cdot x$ exists implies that $s \cdot \alpha(x)$ exists and $\alpha(s \cdot x) = s \cdot \alpha(x)$. A morphism is said to be *strong* if it satisfies the condition that $\exists s \cdot x \Leftrightarrow \exists s \cdot \alpha(x)$. A bijective strong morphism is called an *equivalence*.

As with group actions, equivalent actions are the same except for the labelling of the points. It can be proved that the images of strong morphisms are subspaces, and strong morphisms between transitive spaces are surjective.

Fix a point $x \in X$. The *stabiliser* S_x of the point x is the set consisting of all $s \in S$ such that $s \cdot x = x$. It is an upwardly closed inverse subsemigroup of S that does not contain zero. Let $y \in X$ be any point. Then by transitivity, there is an element $s \in S$ such that $s \cdot x = y$. Then $s^{-1}s \in S_x$ and the set $(sS_x)^{\uparrow}$ is the set of all elements of S which map x to y.

Let H be a closed inverse subsemigroup of S that does not contain zero. Define a *left coset* of H to be a set of the form $(sH)^{\uparrow}$ where $s^{-1}s \in H$.

Lemma 1.5.1.

1. *Two cosets $(sH)^{\uparrow}$ and $(tH)^{\uparrow}$ are equal iff $s^{-1}t \in H$.*
2. *If $(sH)^{\uparrow} \cap (tH)^{\uparrow} \neq \emptyset$ then $(sH)^{\uparrow} = (tH)^{\uparrow}$*

We denote by S/H the set of all left cosets of H in S. The inverse semigroup S acts on the set S/H when we define

$$a \cdot (sH)^{\uparrow} = (asH)^{\uparrow} \Leftrightarrow \mathbf{d}(as) \in H.$$

This defines a transitive action.

A closed inverse subsemigroup H of an inverse semigroup S is said to be *proper* if $0 \notin H$. This is a weaker definition than that is used in [30] as we don't need the stronger idea here. The important thing for our actions is that the zero of our inverse semigroup is mapped to the empty map in the symmetric inverse monoid. This ensures that zero can not appear in any point stabiliser. This reasoning is more fully explained on page 31 of [37]

Theorem 1.5.2. *Every transitive action of the inverse semigroup with zero S is equivalent to the action of S on a space of the form S/H where H is some proper closed inverse subsemigroup of S.*

If H and K are any proper closed inverse subsemigroups of S then they determine equivalent actions if and only if there exists $s \in S$ such that

$$sHs^{-1} \subseteq K \text{ and } s^{-1}Ks \subseteq H.$$

Such a pair of closed inverse subsemigroups is said to be *conjugate*.

Proposition 1.5.3. *Let S be an inverse semigroup acting transitively on the sets X and Y, and let $x \in X$ and $y \in Y$. Let S_x and S_y be the stabilisers in S of x and y respectively. There is a morphism $\alpha\colon X \to Y$ such that $\alpha(x) = y$ iff $S_x \subseteq S_y$. If such a morphism exists then it is unique.*

1.6 Graphs, categories and groupoids

A *directed graph* $\mathcal{G}$ is a collection of *vertices* $\mathcal{G}_0$ and a collection of *edges* $\mathcal{G}_1$ together with two functions $\mathbf{d}, \mathbf{r}\colon \mathcal{G}_1 \to \mathcal{G}_0$ called the *domain* and the *range* respectively. All our graphs will be digraphs and we use the terms interchangeably. The *in-degree* of a vertex v is the number of edges x such that $\mathbf{r}(x) = v$ and the *out-degree* of a vertex v is the number of edges x such that $\mathbf{d}(x) = v$. A *sink* is a vertex whose out-degree is zero and a *source* is a vertex whose in-degree is zero. Two edges x and y *match* if $\mathbf{d}(x) = \mathbf{r}(y)$. A *path* is any sequence of edges $x_1 \dots x_n$ such that x_i and x_{i+1} match for all $i = 1, \dots, n_1$. The *length* $|x|$ of a path x is the total number of edges in it. The empty path, or path of length zero, at the vertex v is denoted by 1_v.

Throughout this paper categories will be small and objects are replaced by identities. A *category* C is a collection of *arrows* and the set of identities of C is denoted by C_0. Each arrow a has a *domain*, denoted by $\mathbf{d}(a)$, and a *codomain* denoted by $\mathbf{r}(a)$, both of these are identities and $a = a\mathbf{d}(a) = \mathbf{r}(a)a$. If $\mathbf{d}(a) = \mathbf{r}(b)$ then the arrow ba exists and the multiplication is associative. Given identities e and f the set of arrows eCf is called a *hom-set*.

An arrow a is *invertible* or an *isomorphism* if there is an arrow a^{-1} such that $a^{-1}a =$

$\mathbf{d}(a)$ and $aa^{-1} = \mathbf{r}(a)$. A category in which every arrow is invertible is called a *groupoid*. We denote the subset of invertible elements of C by $G(C)$. This forms a groupoid. If $G(C) = C_0$ then we shall say that the groupoid of invertible elements is *trivial*. We say that a category C has *trivial subgroups* if the only invertible elements in the local monoids are the identities. An identity e in a category C is said to be a *root* if for every identity f the set $eCf \neq \emptyset$. A *principal right ideal* in a category C is a subset of the form aC where $a \in C$. Principal left ideals are defined dually.

Categories can be seen as a generalisation of monoids. Every monoid is merely a category with one object. The *free category* $\mathcal{G}^*$ generated by the directed graph $\mathcal{G}$ is the set of all paths equipped with concatenation as the partial multiplication. If $x, y \in \mathcal{G}^*$ are such that either $x = yz$ or $y = xz$ for some path z then we say that x and y are *prefix-comparable*. The *n-rose* is a graph with one vertex and n edges, also know as the *bouquet of n-circles*. The free category of the n-rose is the free monoid on n generators. This is the key idea that connects chapter two to chapter three; free categories are generalisations of free monoids.

A *functor* between two categories is a map that preserves identities and respects the multiplication. We say a functor is *full* if it is surjective, and *faithful* if it is injective when restricted to the hom-sets. A functor $F : C \to D$ is *essentially surjective* if every identity in D is isomorphic to the image under F of an identity in C. Two categories C, D and *equivalent* if there is a full, faithful and essentially surjective functor between them.

Two arrows e, f with a common range have a *pullback* if there exists $P, p_1, p_2 \in (C_0 \times C_1 \times C_1)$ such that $ep_1 = fp_2$. Moreover, for any other such triple (Q, q_1, q_2) for which $eq_1 = fq_2$, there must exist a unique $u : Q \to P$ (called mediating morphism) such that $q_1 = p_1 u$ and $q_2 = p_2 u$.

Given a directed graph $\mathcal{G}$, we define $\mathcal{G}^\omega$ to be the set of all right-infinite paths in the graph $\mathcal{G}$. Such paths have the form $w = w_1 w_2 w_3 \ldots$ where the w_i are edges in the graph and $\mathbf{d}(w_i) = \mathbf{r}(w_{i+1})$. If $x \in \mathcal{G}^*$, that is a finite path in $\mathcal{G}$, we write $x\mathcal{G}^\omega$ to mean the set of all right-infinite paths in $\mathcal{G}^\omega$ that begin with x as a finite prefix.

Lemma 1.6.1. *If $x\mathcal{G}^\omega \cap y\mathcal{G}^\omega \neq \emptyset$ then x and y are prefix comparable and so either $x\mathcal{G}^\omega \subseteq y\mathcal{G}^\omega$ or $y\mathcal{G}^\omega \subseteq x\mathcal{G}^\omega$.*

If $\mathcal{G}$ has any vertices of in-degree 0, that is, *sources*, then a finite path may get stuck and we may not be able to continue it to an infinite path. For this reason, we shall require that our directed graphs have the property that the in-degree of each vertex is at least 1. There is a map $\mathcal{G}^* \to \mathcal{G}^\omega$ given by $x \mapsto x\mathcal{G}^\omega$. It need not be injective but it will be useful to us to have a sufficient condition when it is.

Lemma 1.6.2. *Let $\mathcal{G}$ be a directed graph in which the in-degree of each vertex is at least 2. Then if x and y are finite paths in the free category on $\mathcal{G}$ such that $x\mathcal{G}^\omega = y\mathcal{G}^\omega$ then $x = y$.*

Proof. The finite paths x and y must be prefix comparable and have the same target vertex v. Therefore to show that they are equal, it is enough to prove that they have the same length. Without loss of generality assume that $|x| < |y|$. Then $y = xz$ for some finite path z. Denote the source vertex of x by u. Suppose that $z = a\bar{z}$ where a is one edge with target u. By assumption, there is at least one other edge with target u; call this edge b. We may extend b by means of an infinite path ω. Thus by assumption $xb\omega = yb\omega'$ for some infinite string ω'. But this implies that $b = a$ which is a contradiction. It follows that x and y have the same length and so must be equal. $\square$

For any digraph $\mathcal{G}$ we define $\mathcal{G}^{-1}$ to be the *opposite* graph. The vertices in $\mathcal{G}^{-1}$ are the vertices of $\mathcal{G}$ and for every edge f in $\mathcal{G}$ there exist an edge f^{-1} in $\mathcal{G}_1^{-1}$ such that $\mathbf{d}(f) = \mathbf{r}(f^{-1})$ and $\mathbf{r}(f) = \mathbf{d}(f^{-1})$. To put simply, we obtain the opposite graph by reversing the direction of all the arrows. We define a *cycle* to be a non-trivial path p such that $\mathbf{d}(p) = \mathbf{r}(p)$ and a *tree* is a connected graph without cycles. A graph $\mathcal{G}$ is said to be *strongly connected* if for each vertex there is a path to every other vertex.

Chapter 2

Polycyclic monoids and their representation

The main motivation for this chapter is the monograph by Bratteli and Jorgensen [5]. The results in the chapter are based on the author's paper with Lawson [20]. The monograph deals with certain special representations of the Cuntz C^*-algebras, however on the strength of the connection between polycyclic monoids and Cuntz C^*-algebras, it can also be regarded as a contribution to the representation theory of the polycyclic monoids by means of partial permutations.

This was made explicit in Lawson's paper [30] which showed that Kawamura's classification of the branching function systems, introduced in [5], inducing irreducible representations of the Cuntz C^*-algebra, could be interpreted as a classification of the so-called *primitive* representations of the corresponding polycyclic monoid. The term 'primitive' generalises the use of the word in the theory of permutation representations of groups. This suggested to us that we try to reinterpret as much as possible of [5] in terms of the representation theory of the polycyclic monoids.

Specifically, we define an inverse submonoid of P_n, denoted by G_n, which corresponds to the 'gauge invariant subalgebra' defined in [5]. Much of Bratelli and Jorgensen's monograph can then be interpreted as studying the relationship between the representations of P_n and G_n defined in a precise way on certain abelian groups. The chapter also expands upon the connection between the wide inverse submonoids of P_n and relations on the free monoid. This work is motivated by [42, 31].

2.1 Basic properties

For each $n \geq 2$, the *polycyclic monoid* P_n is defined as a monoid with zero by the presentation

$$P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} : a_i^{-1}a_i = 1 \text{ and } a_i^{-1}a_j = 0, i \neq j \rangle.$$

It can be shown that the non-zero elements of P_n are of the form xy^{-1} where x, y are elements of A_n^*, the free monoid on the set of generators $A_n = \{a_1, \ldots, a_n\}$. Multiplication then takes the following form.

$$xy^{-1} \cdot uv^{-1} = \begin{cases} xzv^{-1} & \text{if } u = yz \text{ for some string } z \\ x\,(vz)^{-1} & \text{if } y = uz \text{ for some string } z \\ 0 & \text{otherwise.} \end{cases}$$

If $u = yz$ or $y = uz$ then y and u will be said to be *prefix comparable*. We defined this term earlier for categories, this is simply the special case of the definition on the free category of an n-rose. It is clear from the multiplication that the idempotents of P_n are all the elements of the form xx^{-1}.

Lemma 2.1.1. *$xy^{-1} \leq uv^{-1} \Leftrightarrow \exists p \in A_n^*$ such that $x = up$ and $y = vp$.*

Proof. Let $xy^{-1} \leq uv^{-1}$ be non-zero elements of P_n. Then $xy^{-1} = zz^{-1}uv^{-1}$ for some $zz^{-1} \in E(P_n)$. It is enough to show that there exists some $p \in A_n^*$ such that $xy^{-1} = (up)(vp)^{-1}$. As xy^{-1} is non-zero z and u must be prefix comparable. If $u = zr$ then

$$xy^{-1} = zz^{-1}uv^{-1} = zz^{-1}(zr)v^{-1} = (zr)v^{-1} = uv^{-1}.$$

Setting $p = \epsilon$ we have found an appropriate p as $\epsilon \in A_n^*$. If $z = up$ then

$$xy^{-1} = zz^{-1}uv^{-1} = up(up)^{-1}uv^{-1} = up(vp)^{-1}$$

as required.

Let xy^{-1}, uv^{-1} be non-zero elements of P_n such that $\exists p \in A_n^*$ with $x = up, y = vp$. Then

$$xx^{-1}uv^{-1} = x(up)^{-1}uv^{-1} = x(vp)^{-1} = xy^{-1}.$$

Thus $xy^{-1} \leq uv^{-1}$. □

An important property of the idempotents in P_n is the following: if $xx^{-1}yy^{-1} \neq 0$ then xx^{-1} and yy^{-1} are comparable. The polycyclic monoid P_n is finitely generated, combinatorial, E^*-unitary, 0-bisimple and congruence-free.

Representations of the P_n by partial bijections are in principle easy to construct. Choose an infinite set X. Let $X_1, \ldots X_n, Y$ be pairwise disjoint subsets of X whose union is X. The subsets X_i have the same cardinality as X. For each i choose a bijection $\alpha_i \colon X \to X_i$. With this data, we can define a representation of P_n in $I(X)$ by mapping a_i to α_i and a_i^{-1} to α_i^{-1} and then extending to the whole of P_n. This is a well-defined representation of P_n and every representation can be obtained in this way. To see that all representations are obtained in this way we take any representation $P_n \times X \to X$. This gives us a map θ from P_n to $I(X)$. But then the partial bijection corresponding to any idempotent under θ has to be a the identity map on some subset of X. As $a_1a_1^{-1}a_ia_i^{-1} = 0$ for all $1 < i \leq n$ we have that the subset $\theta(a_1a_1^{-1})$ is an identity on must be disjoint from the subset $\theta(a_ia_i^{-1})$ is an identity on. When we focus on the idempotent $a_1a_1^{-1}$ we notice that the subset $\theta(a_1a_1^{-1})$ is an identity on has to contain the subsets that $\theta(a_1a_1(a_1a_1)^{-1}), \theta(a_na_1(a_na_1)^{-1}), \ldots, \theta(a_na_1(a_na_1)^{-1})$ are identities on. We may extended this to any idemponent of the form $a_1x(a_1x)^{-1}$. Thus each subset is infinite and has the same cardinality as X.

Those representations in which $Y = \emptyset$ are particularly interesting and are said to be *strong*. It should be noted that this 'strong' is a completely different idea to that of a 'strong morphism' between representations.

Strong representations are identical to what Bratteli and Jorgensen [5] call *branching function systems*. Such a system consists of a non-empty set X and n injective functions, $f_i : X \to X$, $1 \leq i \leq n$, whose images are disjoint and partition X. Given such a system $(X, f_1, \ldots, f_n)$ we can construct a strong representation $P_n \to I(X)$ by mapping $a_i \mapsto f_i$ and then extending to arbitary elements of P_n in the obvious way. Conversely, given a strong representation $P_n \to I(X)$, we can construct a branching function system $(X, f_1, \ldots, f_n)$ where f_i is the map induced on X by the element a_i of P_n: namely, $x \mapsto a_i x$.

In his two papers, Kawamura [22, 23] classified the *cyclic* branching function systems in terms of finite and infinite strings. In [30], Lawson showed that Kawamura's work was a consequence of the theory of transitive representations of the polycyclic monoids. Specifically, he classified all the closed inverse submonoids of P_n up to conjugacy. This result suggested that inverse semigroup theory might be profitably employed in studying the applications of the polycyclic monoids in mathematics. This brings us to [5]. This long paper is almost entirely devoted to the theory of strong representations of the polycyclic monoids, although it contains no explicit reference to inverse semigroup theory.

2.2 Wide inverse submonoids of P_n

In this section we generalise the connection between positively self conjugate submonoids of P_n and congruences on A_n^* introduced by Meakin and Sapir [42]. These generalisations are motivated firstly by Lawson in [30] and secondly by an interesting submonoid that arises from this result. We expand upon the connection both in terms of congruence properties and submonoid properties.

The following result, a generalisation of a result by Meakin and Sapir, was proved by Lawson

Theorem 2.2.1. *There is a bijection between right congruences on A_n^* and the wide inverse submonoids of P_n.*

A very simple right congruence, also a left congruence, defined on A_n^* is the following. For $x, y \in A_n^*$ define

$$x \cong y \Leftrightarrow |x| = |y| \, .$$

By the above theorem, this gives rise to a wide inverse submonoid of P_n, which from now on will be denoted by G_n, and called the *gauge inverse monoid of rank n*. Explicitly,

$$G_n = \{xy^{-1} \in P_n \colon \ |x| = |y|\} \cup \{0\}.$$

The goal of this section is to relate the properties of the wide inverse submonoids of P_n to their associated right congruences. A wide inverse subsemigroup T of an inverse semigroup S is said to be *normal* if $sts^{-1} \in T$ for all $s \in S$ and $t \in T$.

Lemma 2.2.2. *G_n is a normal inverse submonoid of P_n.*

Proof. By construction G_n is a wide inverse submonoid of P_n. Let $s = xy^{-1} \in P_n$ and $t = uv^{-1} \in G_n$. If sts^{-1} is non-zero then the pairs y and u, and y and v are prefix comparable. Because $|u| = |v|$, the four possible equalities that result reduce to just two: either $u = v$, which implies that t is an idempotent, and so its conjugate is an idempotent; or $u = yw$ and $v = yz$ for some strings w and z. Since u and v have the same length then so too do w and z, and

$$xy^{-1}(uv^{-1})yx^{-1} = xw(xz)^{-1} \in G_n.$$

□

Instead of looking at further examples of right congruences or wide inverse submonoids we shall continue the discussion in the general setting. The following is a slicker proof of the original theorem by Meakin and Sapir. We define a wide inverse submonoid R of P_n to be *positively self conjugate* if

$$pRp^{-1} \subseteq R \text{ for all } p \in A_n^*.$$

Proposition 2.2.3. *There is a bijection between congruences on A_n^* and the positively self conjugate inverse submonoids of P_n.*

Proof. Let ρ be a congruence on A_n^*. Let

$$P_\rho = \{xy^{-1} \in P_n : x \,\rho\, y\} \cup \{0\}.$$

We shall call a set constructed in this way the *set associated to ρ*. As ρ is a right congruence we know that P_ρ is a wide inverse submonoid of P_n by Theorem 3.3 of [30]. We shall now show P_ρ is positively self conjugate. Let $xy^{-1} \in P_\rho$. Then $px \,\rho\, py$ for all $p \in A_n^*$ as ρ is a left congruence and $x \,\rho\, y$. Thus

$$(px)(py)^{-1} = p(xy^{-1})p^{-1} \in P_\rho$$

and so $pP_\rho p^{-1} \subseteq P_\rho$.

Let S be a positively self conjugate inverse submonoid of P_n. Define a relation ρ_S on A_n^* by

$$x \,\rho_S\, y \Leftrightarrow xy^{-1} \in S.$$

We shall call a relation constructed in this way the *relation associated to S*. As S is a wide inverse monoid we know that ρ_S is a right congruence on A_n^*, again by Theorem 3.3 of [30]. We shall show that ρ_S is also a left congruence. As S is positively self conjugate we know $pSp^{-1} \subseteq S$. Suppose $x \,\rho_S\, y$ and $p \in A_n^*$ arbitrary. By assumption $xy^{-1} \in S$. Because S is positively self conjugate

$$p(xy^{-1})p^{-1} = (px)(py)^{-1} \in S.$$

Thus $px \,\rho_S\, py$ and ρ is a left congruence. □

We will now extend the theorem to normal inverse submonoids of P_n. This is motivated by the result that G_n is normal in P_n. We define a wide inverse submonoid R of P_n to be *negatively self conjugate* if

$$p^{-1}Rp \subseteq R \text{ for all } p \in A_n^*.$$

Lemma 2.2.4. *A submonoid S of P_n is positively and negatively self conjugate if and only if it is normal.*

Proof. Let S be a positively and negatively self conjugate submonoid of P_n. Thus $pSp^{-1} \subseteq S$ and $q^{-1}Sq \subseteq S$ for all strings $p, q \in A_n^*$. Thus

$$xy^{-1}Syx^{-1} = x(y^{-1}Sy)x^{-1} \subseteq xSx^{-1} \subseteq S$$

for all $xy^{-1} \in P_n$.

Now suppose S is normal in P_n. Then $xy^{-1}Syx^{-1} \subseteq S$ for all $xy^{-1} \in P_n$. If we let y be the empty string then we see that $xSx^{-1} \subseteq S$ for all strings x and S is positively self conjugate. Similarly letting x be the empty string gives us S is negatively self conjugate. $\square$

Theorem 2.2.5. *There is a bijection between congruences ρ of A_n^* such that A_n^*/ρ is left cancellative and normal inverse submonoids of P_n.*

Proof. Let ρ on A_n^* be a congruence such that A_n^*/ρ is left cancellative. Let P_ρ be the set associated to ρ which is a positively self conjugate submonoid of P_n by Proposition 2.2.3. We shall prove P_ρ is a negatively self conjugate inverse submonoid of P_n by showing $p^{-1}P_\rho p \subseteq P_\rho$ for an arbitary $p \in A_n^*$. Let $p \in A_n^*$ and let $xy^{-1} \in p^{-1}P_\rho p$ be non-zero. Then there exists a $uv^{-1} \in P_\rho$ such that $xy^{-1} = p^{-1}uv^{-1}p$. Thus $u = px$ and $v = py$. As $u \,\rho\, v$ and A_n^*/ρ is left cancellative we have $x \,\rho\, y$. Thus $xy^{-1} \in P_\rho$ and $p^{-1}P_\rho p \subseteq P_\rho$. Therefore P_ρ is positively and negatively self conjugate and so by Lemma 2.2.4 it is normal.

Let S be a normal submonoid of P_n. By Lemma 2.2.4 our submonoid S is positively self conjugate and by Proposition 2.2.3 the relation associated to S, ρ_S, is a congruence. We also have that S is negatively self conjugate. We shall show that A_n^*/ρ is left cancellative. Suppose $x\,\rho\,y$ with $x = ps$ and $y = pt$. By assumption $xy^{-1} \in S$. Because S is negatively self conjugate $p^{-1}(xy^{-1})p \in S$ for all $p \in A_n^*$. Then we have

$$p^{-1}(xy^{-1})p = (p^{-1}x)(p^{-1}y)^{-1} = st^{-1} \in S$$

thus $s\,\rho\,t$ and A_n^*/ρ is left cancellative. $\square$

Our aim was to translate the normality property of inverse submonoids of P_n into a property of a corresponding congruence on A_n^*. We can now try to see which other properties translate in this way.

Lemma 2.2.6. *There is a bijection between congruences ρ such that A_n^*/ρ is right cancellative and positively self conjugate inverse submonoids S of P_n such that $S \setminus \{0\}$ is upwardly closed in P_n.*

Proof. Let ρ be a congruence such that A_n^*/ρ is right cancellative and let P_ρ be the set associated to ρ. As ρ is an congruence we know that P_ρ is a positively self conjugate inverse submonoid of P_n by Proposition 2.2.3. We shall now show $P_\rho \setminus \{0\}$ is closed. Let $xy^{-1} \in P_\rho \setminus \{0\}$ and let $st^{-1} \geq xy^{-1}$. By Lemma 2.1.1 we have that $x = sr$ and $y = tr$. Then $s\,\rho\,t$ as A_n^*/ρ is a right cancellative and so $st^{-1} \in P_\rho$. Therefore $st^{-1} \in P_\rho$ if $xy^{-1} \in P_\rho$ and $xy^{-1} \leq st^{-1}$. Thus $P_\rho \setminus \{0\}$ is upwardly closed.

Let S be a positively self conjugate inverse submonoid of P_n such that $S \setminus \{0\}$ is upwardly closed. Let ρ_S be the relation associated to S. As S is a positively self conjugate inverse monoid we know that ρ_S is an congruence on A_n^*. We shall show that A_n^*/ρ is also right cancellative. Suppose $x\,\rho_S\,y$ with $x = sr$ and $y = tr$. By Lemma 2.1.1 we have $xy^{-1} \leq rs^{-1}$. As S is upwardly closed we have that $rs^{-1} \in S$ and as such $r\,\rho_S\,s$. Thus A_n^*/ρ_S is right cancellative. $\square$

Lemma 2.2.7. *Let ρ be a right congruence on A_n^* and P_ρ the associated wide inverse submonoid. There is a bijection between A_n^*/ρ and $P_\rho/\mathcal{D}$.*

Proof. First we shall establish how Green's $\mathcal{D}$ relation is defined on P_ρ. Let $xy^{-1}\mathcal{D}uv^{-1}$ in P_ρ. Then there exists a $ab^{-1} \in P_\rho$ such that $P_\rho uv^{-1} = P_\rho ab^{-1}$ and $xy^{-1}P_\rho = ab^{-1}P_\rho$. Thus $v = b$ and $a = x$ and $x\,\rho\,v$ as $a\,\rho\,b$.

Now let $x\,\rho\,v$ and xy^{-1}, uv^{-1} be elements of P_ρ. Then $xy^{-1}P_\rho = xv^{-1}P_\rho$ and $P_\rho xv^{-1} = P_\rho uv^{-1}$. Thus $xy^{-1}\mathcal{R}xv^{-1}\mathcal{L}uv^{-1}$ so $xy^{-1}\mathcal{D}uv^{-1}$. Therefore $xy^{-1}\mathcal{D}uv^{-1}$ in P_ρ iff $x\,\rho\,v$.

Let $[xy^{-1}]$ denote the equivalence class containing xy^{-1} in $P_\rho/\mathcal{D}$ and let $[x]$ denote the equivalence class containing x in A_n^*/ρ. Define $\theta : P_\rho/\mathcal{D} \to A_n^*/\rho$ by $\theta([xy^{-1}]) = [x]$. Clearly this map is surjective as for all $[x] \in A_n^*/\rho$ we can find $[xy^{-1}] \in P_\rho/\mathcal{D}$. Let $[xy^{-1}] = [uv^{-1}]$ in $P_\rho/\mathcal{D}$. Then $x\,\rho\,v\,\rho\,u$ so

$$\theta([xy^{-1}]) = [x] = [u] = \theta([uv^{-1}]).$$

Finally we prove injectivity. Let $\theta([xy^{-1}]) = \theta([uv^{-1}])$. Then $[x] = [y]$ and $x\,\rho\,u$. As $u\,\rho\,v$ and as ρ is transitive we have $x\,\rho\,v$. Thus $xy^{-1}\mathcal{D}uv^{-1}$ and $[xy^{-1}] = [uv^{-1}]$.

Therefore there is a bijection between A_n^*/ρ and $P_\rho/\mathcal{D}$. □

2.3 The gauge inverse monoids G_n

In this section, we analyse the wide inverse submonoid G_n of P_n which [5] used to determine the orbits of P_n in certain cases. Our analysis of this new inverse monoid goes beyond what we need for our immediate purposes since it seems to be an interesting object in its own right.

Define a function μ from $G_n^* = G_n \setminus \{0\}$ to $\mathbb{N}$ by

$$\mu(xy^{-1}) = |x|\,.$$

We shall refer to μ as the *weight function* on G_n.

Lemma 2.3.1. *For all xy^{-1}, uv^{-1} in G_n, the function μ has the following properties.*

1. $\mu(xy^{-1}) = \mu((xy^{-1})^{-1})$,

2. *If $xy^{-1}uv^{-1} \neq 0$ then $\mu(xy^{-1}uv^{-1}) = \max(\mu(xy^{-1})), (\mu(uv^{-1}))$.*

Proof. The proof of (1) is clear. We prove (2). Suppose that $|y| \geq |u|$. Then $y = uz$, for some string z. We have that $xy^{-1}uv^{-1} = x(vz)^{-1}$ and so $\mu(xy^{-1}uv^{-1}) = \mu(xy^{-1})$. A similar argument assuming that $|y| < |u|$ then proves the claim. □

Recall that a function $\theta\colon S \to T$ between two inverse semigroups with zero is a *prehomomorphism* if $ss' \neq 0$ implies that $\theta(ss') = \theta(s)\theta(s')$. The set $(\mathbb{N}, \wedge)$ is a semilattice when we define $m \wedge n = \max(m, n)$. It follows from Lemma 2.3.1(2), that μ is a prehomomorphism from the inverse semigroup G_n to the semilattice $(\mathbb{N}, \wedge)$.

Lemma 2.3.2. *G_n is E^*-unitary.*

Proof. Let $xx^{-1} \in E(G_n)$, $uv^{-1} \in G_n$ such that the product $xx^{-1}uv^{-1}$ is a non-zero idempotent. We will show that uv^{-1} is also idempotent. As the product is non-zero we have x and u are prefix comparable. If $x = ur$ then $xx^{-1}uv^{-1} = x(vr)^{-1}$. By assumption $x(vr)^{-1}$ is idempotent and so $x = vr$. Thus $ur = x = vr$, so $u = v$ and uv^{-1} is idempotent.

If $u = xr$ then $xx^{-1}uv^{-1} = (xr)v^{-1} = uv^{-1}$. By assumption the product of xx^{-1} and uv^{-1} is idempotent. Thus uv^{-1} is idempotent.

□

Green's relations $\mathcal{L}$ and $\mathcal{R}$ in G_n have the same form as in P_n.

Lemma 2.3.3. *In the inverse monoid G_n, we have the following:*

1. $xy^{-1} \,\mathcal{L}\, uv^{-1}$ *iff* $y = v$.

2. $xy^{-1} \mathcal{R} uv^{-1}$ *iff* $x = u$.

3. $xy^{-1} \mathcal{H} uv^{-1}$ *iff* $xy^{-1} = uv^{-1}$.

4. $xy^{-1} \mathcal{D} uv^{-1}$ *iff* $\mu(xy^{-1}) = \mu(uv^{-1})$.

5. $\mathcal{D} = \mathcal{J}$.

Proof. (1) Let $xy^{-1} \mathcal{L} uv^{-1}$. Then there exists an $rs^{-1} \in G_n$ such that $rs^{-1}xy^{-1} = uv^{-1}$. Thus y is a prefix of u. By symmetry u is a prefix of y and as such they are equal.

A dual argument proves (2) and then (1) and (2) together prove (3).

(4) Let $xy^{-1} \mathcal{D} uv^{-1}$. Then there exists $wz^{-1} \in G_n$ such that

$$xy^{-1} \mathcal{L} wz^{-1} \mathcal{R} uv^{-1}.$$

Thus $y = z$ and $w = u$. It follows that

$$\mu(xy^{-1}) = |x| = |y| = |z| = |w| = |u| = \mu(uv^{-1}).$$

Conversely, let $\mu(xy^{-1}) = \mu(uv^{-1})$. Then $uy^{-1} \in G_n$ and

$$xy^{-1} \mathcal{L} uy^{-1} \mathcal{R} uv^{-1}.$$

Thus $xy^{-1} \mathcal{D} uv^{-1}$, in G_n.

(5) The inclusion $\mathcal{D} \subseteq \mathcal{J}$ always holds. We prove that $\mathcal{J} \subseteq \mathcal{D}$. Let $xy^{-1} \mathcal{J} uv^{-1}$. Then we can write

$$xy^{-1} = st^{-1}uv^{-1}wz^{-1}$$

for some elements $st^{-1}, wz^{-1} \in G_n$. By Lemma 2.3.1, we have that $\mu(xy^{-1}) \geq \mu(uv^{-1})$. By symmetry, we deduce that $\mu(xy^{-1}) = \mu(uv^{-1})$. $\square$

It follows that the number of $\mathcal{D}$-classes in G_n is countably infinite which contrasts strikingly with P_n which has exactly two.

For each $r \geq 0$, define

$$G_n^{\geq r} = \{xy^{-1} \in G_n \colon \ \mu(xy^{-1}) \geq r\} \cup \{0\}$$

all the elements of G_n whose weights are at least r together with zero.

Proposition 2.3.4. *The subset $G_n^{\geq r}$ is an ideal of G_n and every non-zero ideal is of this form.*

Proof. Let $xy^{-1} \in G_n^{\geq r}$ and $uv^{-1} \in G_n$. If the product $xy^{-1} \cdot uv^{-1}$ is zero then it belongs to $G_n^{\geq r}$. If the product $xy^{-1} \cdot uv^{-1}$ is non-zero then its weight is at least r, by Lemma 2.3.1, and so also belongs to $G_n^{\geq r}$. The same is true for the product $uv^{-1} \cdot xy^{-1}$. Thus $G_n^{\geq r}$ is an ideal.

Now let I be an ideal of G_n. If $I = G_n$ then $I = G_n^{\geq 0}$. We may therefore assume in what follows that the identity is not contained in I. Let xy^{-1} be in I such that $\mu(xy^{-1}) = r$ is minimal. By Lemma 2.3.3, all elements of G_n with weight r belong to I. Let $uv^{-1} \in G_n$ such that $\mu(uv^{-1}) = p > r$. Let z be any string of length $p - r$. Then $(xz)(yz)^{-1} \in G_n$ has weight p. By Lemma 2.1.1 we have $(xz)(yz)^{-1} \leq xy^{-1}$. Every ideal of an inverse semigroup is automatically an order ideal with respect to the natural partial order and so $(xz)(yz)^{-1} \in I$. Thus I contains elements of weight p. But, again by Lemma 2.3.3, it must contain all elements of weight p and so contains uv^{-1}. $\square$

It follows that the number of ideals in G_n is countably infinite and so G_n is far from being congruence-free.

We have now classified all ideals of G_n. We can take this one step further by classifying all congruences of G_n. To do this we will use the concept of Rees congruences.

Lemma 2.3.5. *Let ρ be a congruence on G_n such that $xy^{-1} \,\rho\, 0$ for some element xy^{-1} of weight t. Then for each string ab^{-1} whose weight is at least t we have that $ab^{-1} \,\rho\, 0$.*

Proof. Let $ab^{-1} \in G_n$ such that $\mu(ab^{-1}) = (t + i)$ for some $i \in \mathbb{N}$. Let $z \in A_n^*$ such that $|z| = i$. Then $a(xz)^{-1}, (yz)b^{-1} \in G_n$ and

$$ab^{-1} = a(xz)^{-1}(xy^{-1})(yz)b^{-1} \,\rho\, a(xz)^{-1}0(yz)b^{-1} = 0.$$

Therefore $ab^{-1} \,\rho\, 0$. □

The following lemma is especially useful as the property holds in all wide inverse submonoids of P_n.

Lemma 2.3.6. *Let $xx^{-1} < yy^{-1}$. Then there is an idempotent zz^{-1} such that $zz^{-1}yy^{-1} \neq 0$ but $zz^{-1}xx^{-1} = 0$. That is $E(G_n)$ is 0-disjunctive.*

Proof. By Lemma 2.1.1 $x = yu$ for some non-empty string u. Let the first letter of u be a and let $b \neq a$ be any other letter. Put $z = yb$. Then $zz^{-1} < yy^{-1}$ and so their product is non-zero. On the other hand, $z(z^{-1}x)x^{-1} = zb^{-1}y^{-1}yux^{-1} = z(b^{-1}u)x^{-1} = 0$. □

Lemma 2.3.7. *Let ρ be a congruence on G_n which is not the equality congruence. Then there is a non-zero idempotent $xx^{-1} \in G_n$ such that $xx^{-1} \,\rho\, 0$.*

Proof. Because the congruence is not equality there are pairs of elements such that $xy^{-1} \,\rho\, uv^{-1}$ and $xy^{-1} \neq uv^{-1}$. Let xy^{-1}, uv^{-1} be such a pair. As ρ is a congruence we have $yx^{-1} \,\rho\, vu^{-1}$ and so

$$xx^{-1} = xy^{-1}yx^{-1} \,\rho\, uv^{-1}vu^{-1} = uu^{-1}.$$

Similarly, $yy^{-1} \,\rho\, vv^{-1}$. If $x = u$ and $y = v$ then $xy^{-1} = uv^{-1}$. As $xy^{-1} \neq uv^{-1}$ we have either $xx^{-1} \neq uu^{-1}$ or $yy^{-1} \neq vv^{-1}$ so we can assume, without loss of generality, that we have $xx^{-1} \,\rho\, uu^{-1}$ and $xx^{-1} \neq uu^{-1}$. Left multiplying each side of the relation by xx^{-1} gives

$$xx^{-1} = xx^{-1}xx^{-1} \,\rho\, xx^{-1}uu^{-1}.$$

If $xx^{-1}uu^{-1} = 0$ then we are done. Otherwise $xx^{-1}uu^{-1} \neq 0$. If $xx^{-1}uu^{-1} \neq 0$ then x and u are prefix comparable and so either $xx^{-1} < uu^{-1}$ or $xx^{-1} > uu^{-1}$. Without loss of generality we assume that $xx^{-1} < uu^{-1}$. However, by Lemma 2.3.6 there then exists an idempotent zz^{-1} such that $zz^{-1}xx^{-1} = 0$ but $zz^{-1}uu^{-1} \neq 0$. It follows that $zz^{-1}uu^{-1} \,\rho\, 0$ and we are done. $\square$

Let ρ be a congruence on G_n that is not the equality congruence. Then in the light of Lemma 2.3.7 the following definition makes sense. The *weight* of ρ is the smallest $\mu(xx^{-1})$ such that $xx^{-1} \,\rho\, 0$.

Lemma 2.3.8. *Let ρ be a non-equality congruence on G_n of weight t. Let xy^{-1} and uv^{-1} be elements of G_n such that $\mu(xy^{-1}), \mu(uv^{-1}) < t$ and $xy^{-1} \,\rho\, uv^{-1}$. Then $xy^{-1} = uv^{-1}$.*

Proof. Suppose first that $\mu(xy^{-1}) = \mu(uv^{-1})$. Observe that $ux^{-1} \cdot xy^{-1} \cdot yv^{-1} = uv^{-1}$ holds in G_n. Thus $uv^{-1} = ux^{-1} \cdot xy^{-1} \cdot yv^{-1} \,\rho\, ux^{-1} \cdot uv^{-1} \cdot yv^{-1}$. As $\mu(uv^{-1}) < t$, the weight of ρ, we have that uv^{-1} cannot be congruent to zero. It follows that $ux^{-1} \cdot uv^{-1} \cdot yv^{-1} \neq 0$. Hence $x = u$ and $v = y$, and so $xy^{-1} = uv^{-1}$.

Now suppose that $\mu(xy^{-1}) \neq \mu(uv^{-1})$. Without loss of generality, we may assume that $\mu(xy^{-1}) < \mu(uv^{-1}) < t$. That is $\mu(xy^{-1}) \leq (t-2)$. We saw in the proof of Lemma 2.3.7 that $xx^{-1} \,\rho\, uu^{-1}$ if $xy^{-1} \,\rho\, uv^{-1}$. If we left multiply both sides of $xx^{-1} \,\rho\, uu^{-1}$ by xx^{-1} we have

$$xx^{-1} = xx^{-1}xx^{-1} \,\rho\, xx^{-1}uu^{-1}.$$

Observe that $xx^{-1}uu^{-1} \neq 0$ as this would contradict the weight of t. Thus xx^{-1} and uu^{-1} are comparable and as $\mu(xy^{-1}) < \mu(uv^{-1})$ we have $uu^{-1} < xx^{-1}$. By Lemma 2.3.6 we have $zz^{-1} \in G_n$ such that $zz^{-1}uu^{-1} = 0$ and $zz^{-1}xx^{-1} \neq 0$. In the proof of Lemma 2.3.6 we construct such a z. Following that construction the length of z is $(|x| + 1)$. This gives

$$\mu(zz^{-1}) = |z| = |x| + 1 = \mu(xx^{-1}) + 1 \leq (t-2) + 1 < t.$$

Therefore $\mu(zz^{-1}xx^{-1}) = \mu(zz^{-1}) < t$ and

$$zz^{-1}xx^{-1} \,\rho\, zz^{-1}uu^{-1} = 0.$$

A contradiction on the weight of ρ. Thus $\mu(xy^{-1}) = \mu(uv^{-1})$ and so $xy^{-1} = uv^{-1}$. $\square$

Theorem 2.3.9. *The only congruences on G_n are Rees congruences.*

Proof. The equality congruence is the Rees congruence associated with the ideal $\{0\}$. Let ρ be a congruence which is not the equality congruence having weight t. Let ρ_t be the Rees congruence associated with the ideal $G_n^{\geq t}$.

We first show that $\rho \subseteq \rho_t$. Let $xy^{-1}\,\rho\,uv^{-1}$. If $\mu(xy^{-1}), \mu(uv^{-1}) < t$, then by Lemma 2.3.8 we have that $uv^{-1} = xy^{-1}$ and so $xy^{-1}\,\rho_t\,uv^{-1}$. Note that if $\mu(uv^{-1}) < t$, $\mu(xy^{-1}) \geq t$, or visa versa, then

$$uv^{-1}\,\rho\,xy^{-1}\,\rho\,0$$

which contradicts the assumption that ρ has weight t.

Now we only need to consider the case where $\mu(xy^{-1}) \geq t$ and $\mu(uv^{-1}) \geq t$. Thus $xy^{-1}\,\rho\,0$ and $uv^{-1}\,\rho\,0$ and so $xy^{-1}\,\rho_t\,0$ and $uv^{-1}\,\rho_t\,0$ by the definition of ρ_t giving $xy^{-1}\,\rho_t\,uv^{-1}$, by transitivity.

Now we show that $\rho_t \subseteq \rho$. Let $xy^{-1}\rho_t uv^{-1}$. If $\mu(xy^{-1}) < t$ then $uv^{-1} = xy^{-1}$ by the definition of ρ_t and so $xy^{-1}\,\rho\,uv^{-1}$. If $\mu(xy^{-1}) \geq t$ then $\mu(uv^{-1}) \geq t$ by the definition of ρ_t. Thus $xy^{-1}\,\rho\,0$ and $uv^{-1}\,\rho\,0$ by Lemma 2.3.5. By transitivity $xy^{-1}\,\rho\,uv^{-1}$, as required. $\square$

For each $r \geq 0$ define G_n^r to consist of all elements of weight r together with the zero element. By Lemma 2.3.1, the product of two elements of weight r is either zero or again of weight r and the inverses of elements of weight r are again of weight r. Thus G_n^r is an inverse subsemigroup of G_n. For each r, p with $r < p$, define

$$G_n^{r,p} = \bigcup_{i=r}^{p} G_n^i.$$

This too is an inverse subsemigroup of G_n and consists of zero and all elements whose weights lie between r and p inclusive.

Let the inverse subsemigroup $G_n^{0,p}$ be denoted by $G_n^{\leq p}$. Then $G_n^{\leq 0} \subseteq G_n^{\leq 1} \subseteq G_n^{\leq 2} \subseteq \ldots$ and $G_n = \bigcup_{r \in \mathbb{N}} G_n^{\leq r}$.

Lemma 2.3.10. $|G_n^r| = n^{2r} + 1$ *and* $|G_n^{r,p}| = 1 + \sum_{i=r}^{p} n^{2i}$

Proof. To prove the first equality, observe that each string xy^{-1} of weight r consists of a pair of arbitrary strings both of length r. Each string is a product of the n generators, thus we have n possibilities for each of the $2r$ positions. We add one for the zero element. The second equality is now immediate. □

An inverse monoid is *locally finite* if all of its finitely generated inverse submonoids are finite.

Proposition 2.3.11. *The inverse monoid G_n is locally finite. In particular, G_n is not finitely generated as an inverse monoid.*

Proof. Let $X = \{x_1 y_1^{-1}, \ldots, x_q y_q^{-1}\}$ be any finite subset of G_n. Let p be the maximum of all the weights of elements in X. The inverse submonoid of G_n generated by the set X must be contained in the finite inverse subsemigroup $G_n^{\leq p}$. Thus the inverse submonoid of G_n generated by any finite subsemigroup is always finite. □

It is worth summarising what we have found out about G_n as a counterpoint to P_n.

Proposition 2.3.12. *The inverse monoid G_n is combinatorial (Lemma 2.3.3 (3)) and E^*-unitary (Lemma 2.3.2) but it is not finitely generated. The number of $\mathcal{D}$-classes and the number of ideals is countably infinite and every congruence is a Rees congruence.*

2.4 Actions of polycyclic monoids

We begin by describing how the general theory outlined earlier works out in the case of polycyclic inverse monoids. The detailed proofs can be found in [30].

The *natural action of P_n on A_n^** is defined as follows. If $xy^{-1} \in P_n$ and $u \in A_n^*$ then

$$xy^{-1} \cdot u = \begin{cases} xp & \text{if } u = yp \text{ for some string } p \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This is a transitive action and the stabiliser of any point is finite.

Explicitly, we call an action $P_n \times X \to X$ *strong* if

$$X = \bigcup_{i=1}^{n} a_i \cdot X.$$

This should not be confused with the idea of a strong morphism between actions.

The following is Proposition 5.1 of [30].

Proposition 2.4.1. *A transitive action of P_n which is not strong is equivalent to the natural action of P_n on the set of finite strings.*

We can therefore concentrate on the transitive strong actions. By Theorem 1.5.2 these are classified by the corresponding proper closed inverse submonoids of P_n. Let $xx^{-1} \in E(G_n)$. Then

$$(xx^{-1})^{\uparrow} = \{yy^{-1} \in G_n \colon \ y \text{ is a prefix of } x\}.$$

Proposition 2.4.2. *The set $(xx^{-1})^{\uparrow}$ is a finite closed inverse submonoid of G_n that does not contain zero. Every finite closed inverse monoid of G_n that does not contain zero is of this form.*

Proof. First we show for any $x \in \mathcal{G}^*$ that $(xx^{-1})^{\uparrow}$ is a finite closed inverse submonoid of G_n that does not contain zero. Being upwardly closed and finiteness are clear and as all the elements of $(xx^{-1})^{\uparrow}$ are idempotents we don't need to worry about inverses. Therefore we need only show that it is closed under multiplication. Let $zz^{-1}, yy^{-1} \in (xx^{-1})^{\uparrow}$. Then z and y are both prefixes of x. Therefore z and y are prefix comparable. Without loss of generality we assume $z = yr$. Then

$$zz^{-1}yy^{-1} = (yr)(yr)^{-1}yy^{-1} = (yr)(yr)^{-1} = zz^{-1}.$$

Similarly, $yy^{-1}zz^{-1} = zz^{-1}$ and as such $(xx^{-1})^{\uparrow}$ is closed under multiplication.

Let H be a finite closed inverse submonoid of G_n that does not contain zero. As H is finite we can find $xx^{-1} \in E(H)$ such that $\mu(xx^{-1})$ is maximal. Then $xx^{-1} \cdot yy^{-1} \in H$ is a non-zero idempotent for any $yy^{-1} \in E(H)$. It follows that x and y are prefix comparable. But the length of x must be at least that of y and so $xx^{-1} \leq yy^{-1}$. Thus $(xx^{-1})^{\uparrow} \subseteq H$ is the set of idempotents of H. Let $uv^{-1} \in H$ be an arbitrary element. Then $uv^{-1} \cdot uv^{-1} \in H$ is non-zero. Thus v and u are prefix comparable and therefore equal. It follows that every element of H is an idempotent. Hence $H = (xx^{-1})^{\uparrow}$. $\square$

The next two results connect Section 2.2 to the rest of the chapter. The proposition is motivation for the theorem that follows it. The relation *induced* by an action of an inverse semigroup on the set X is the set

$$\{(x, y) \in X \times X : \exists s \in S \text{ with } x = s \cdot y\}.$$

Proposition 2.4.3. *The restriction to G_n of the natural action of P_n on the free monoid induces the right congruence $\cong$ on the free monoid A_n^* (where $\cong$ is defined by $x \cong y$ iff $|x| = |y|$). The finite closed inverse submonoids of G_n are the point stabilisers for this action.*

Proof. Let $x, y \in A_n^*$ such that $|x| = |y|$. Then $yx^{-1} \cdot x = y$. Conversely, suppose that $yx^{-1} \cdot u = v$. Then $u = xz$ and $v = yz$ for some string z. Thus

$$|v| = |yz| = |y| + |z| = |x| + |z| = |xz| = |u|$$

as required.

Fix $x \in A_n^*$. Then by Theorem 1.5.2 and Proposition 2.4.2 the stabiliser of this point is $(xx^{-1})^{\uparrow}$. Thus all the finite closed inverse submonoids occur as stabilisers for the above action. $\square$

Theorem 2.4.4. *Let P_ρ be the wide inverse monoid associated to the right congruence ρ on A_n^* as defined in Proposition 2.2.3. Then the restriction to P_ρ of the natural action of P_n on the free monoid induces the right congruence ρ.*

Proof. Let $C_\rho(x) = \{y \in A_n^* : (x, y) \in \rho\}$ be the equivalence class containing x and let $\mathrm{Orb}_{\mathrm{P}_\rho}(x)$ be the orbit of x under the action of P_ρ on A_n^*. We first show that $C_\rho(x) \subseteq \mathrm{Orb}_{\mathrm{P}_\rho}(x)$. Let $y \in C_\rho(x)$. Then $xy^{-1}, yx^{-1} \in P_\rho$. If we look at their respective actions on the strings y and x we see that $y \in \mathrm{Orb}_{\mathrm{P}_\rho}(x)$.

Let $y \in \mathrm{Orb}_{\mathrm{P}_\rho}(x)$. Then there exists $uv^{-1} \in P_\rho$ such that $uv^{-1} \cdot x = y$ and $vu^{-1} \cdot y = x$. Thus v is a prefix of x, u is a prefix of y and there exists a $z \in A_n^*$ such that $x = vz$ and $y = uz$. As $uv^{-1} \in P_\rho$ we have $(u, v) \in \rho$, which is a right congruence, so $(y, x) = (uz, vz) \in \rho$. Thus $y \in C_\rho(x)$. $\square$

We now return to the proper closed inverse submonoids of P_n. The next few results are taken from [30]. As they are not the author's work they appear without proof. The results provide a logical structure which is followed later in the chapter and are vital to set up Theorem 2.4.15, this result plays a vital part in connecting our work to Bratteli and Jorgensen's work.

The following is Lemma 4.1 of [30].

Proposition 2.4.5. *Let x and p be strings such that p is non-empty and where x and p have no non-trivial suffix in common. The smallest closed inverse submonoid of P_n containing the element $x(xp)^{-1}$ is*

$$P_n^{x,p} = \{xp^r\bar{p}(xp^s\bar{p})^{-1} : \ r, s \geq 0, \bar{p} \text{ is a prefix of } p\} \cup \{\bar{x}\bar{x}^{-1} : \ \bar{x} \text{ is a prefix of } x\}.$$

The idempotents of this semigroup are the elements of the form yy^{-1} where y is a prefix of the string xp^ω.

We have that $P_n^{x,p} \subseteq P_n^{y,q}$ if and only if $x = y$ and $p = q^s$ for some $s \geq 0$ with equality iff $x = y$ and $p = q$.

Notation We write P_n^p instead of $P_n^{\varepsilon,p}$.

If we look at the idempotents of any $P_n^{x,p}$ we get another proper closed inverse submonoid. Explicitly,

$$E(P_n^{x,p}) = \{yy^{-1} \in P_n : y \text{ is a prefix of } xp^\omega\}.$$

It is clear this is an inverse submonoid. This set doesn't contain zero and it is straightforward to see that it is upwardly closed. We note that if we take all the prefixes of a non-ultimately periodic right infinite string then this also forms a proper closed inverse submonoid. These infinite chains of idempotents can also be seen as generalisation of the finite closed inverse submonoids.

The following is Theorem 4.3 of [30].

Theorem 2.4.6. *Each proper closed inverse submonoid of P_n belongs to exactly one of the following classes:*

1. Finite chain type*: it consists of a finite chain of idempotents.*
2. Infinite chain type*: it consists of an infinite chain of idempotents.*
3. Cycle type*: it is of the form $P_n^{x,p}$ where $p \neq \varepsilon$ and where x and p have no non-trivial suffix in common. If $x = \varepsilon$ we say that P_n^p is of* pure cycle type.

Let H be a proper closed inverse submonoid. If $E(H)$ is finite then $E(H) = H$ by Proposition 2.4.2 and the *type* of H is the finite string w with the property that the idempotents of H are precisely those elements of the form uu^{-1} where u is a prefix of w. If $E(H)$ is infinite then the *type* of H is the infinite string w with the property that the idempotents of H are precisely those elements of the form uu^{-1} where u is a prefix of w. We say that H is *ultimately periodic* if its type is an ultimately periodic infinite string of the form xp^w, where x, p are finite strings. We say that it is *aperiodic* if its type is an infinite string which is not ultimately periodic.

We now set about classifying the proper closed inverse submonoids of P_n up to conjugacy.

The following is Theorem 4.4 of [30].

Theorem 2.4.7.

1. *Let H be a proper closed inverse submonoid of S of finite chain type. Then all closed inverse submonoids conjugate to it are of finite chain type, and all submonoids of finite chain type are conjugate.*

2. *Let H be a proper closed inverse submonoid of infinite chain type. The only closed inverse submonoids conjugate to H are also of infinite chain type. Two closed inverse submonoids of infinite chain type are conjugate if and only if there are idempotents $vv^{-1} \in H$ and $uu^{-1} \in K$ such that for all strings p we have that $vp(vp)^{-1} \in H$ iff $up(up)^{-1} \in K$. It follows that they are conjugate iff their types differ in only a finite number of places.*

3. *Let H be a proper closed inverse submonoid of cycle type. The only closed inverse submonoids conjugate to H are also of cycle type. Furthermore $P_n^{x,p}$ is conjugate to $P_n^{y,q}$ if and only if p and q are conjugate strings.*

For strong representations the orbits are of two types: if the type of the stabiliser of a point is *ultimately periodic* we shall say that the orbit containing that point is *rational*, whereas if it is *aperiodic* we shall say that the orbit is *irrational*.

We shall say that a transitive action of a polycyclic monoid is *primitive* if the stabiliser of any point is a maximal proper closed inverse submonoid of P_n.

The following is essentially Theorem 4.5 of [30].

Theorem 2.4.8. *A proper closed inverse submonoid H of P_n is maximal if either:*

1. *H is of infinite chain type and the type of H is aperiodic;*

2. *$H = P_n^p$ with the additional condition that p is a primitive string.*

Every proper closed inverse submonoid of the polycyclic monoid P_n which corresponds to a primitive action is conjugate to a closed inverse submonoid of one of these two types.

We conclude with some results that lead to a definition important for actions which are non-transitive. Define an action of P_n on A_n^ω as follows:

$$xy^{-1} \cdot u = \begin{cases} xp & \text{if } u = yp \text{ for some infinite string } p \\ \text{undefined} & \text{otherwise} \end{cases}$$

We call this the *natural action of P_n on A_n^ω*. This action is no longer transitive but we shall study the orbits of the action, each of which gives rise to a transitive action of P_n.

Observe that if xp^ω is an ultimately periodic string we can assume that x and p have no suffix in common, because if they did we could write $x = \bar{x}y$ and $p = \bar{p}y$, where y is as long as possible, and then $xp^\omega = \bar{x}(y\bar{p})^\omega$ with $\bar{x}$ and $y\bar{p}$ having no non-trivial suffix in common. Further we can assume that p is primitive because if $p = q^s$ then $xp^\omega = xq^\omega$.

The following is Proposition 4.7 of [30].

Proposition 2.4.9. *With respect to the natural action of P_n on A_n^ω we have the following.*

1. *The ultimately periodic string xp^ω, where p is primitive and x and p have no non-trivial suffix in common, has the stabiliser $P_n^{x,p}$.*
2. *The infinite aperiodic string w has the stabiliser $(ww^{-1})^\uparrow$.*

We have that the cyclic and aperiodic submonoids are the only maximal proper closed inverse submonoids and as such determine the transitive action of the polycyclic monoids. It follows that the natural action of the polycyclic monoid on the set of infinite strings is equivalent to the disjoint union of each of the primitive transitive representations of the polycyclic monoid with each such representation occurring exactly once.

The following are Lemmas 5.3 and 5.4 of [30].

Lemma 2.4.10. *Let P_n act strongly on the set X. Let $x \in X$. Then for each natural number m there is a unique string u of length m and a point y such that $x = u \cdot y$ (which is equivalent to saying that $\exists u^{-1} \cdot x$). Suppose that $u^{-1} \cdot x$ and $v^{-1} \cdot x$ are defined and $|u| \geq |v|$ then v is a prefix of u.*

Morphisms between strong representations behave well.

Lemma 2.4.11. *Let P_n act strongly on both X and Y. Then every morphism $\alpha \colon X \to Y$ is strong.*

The following is just our version of the coding map of [5] and is Proposition 5.5 of [30].

Proposition 2.4.12. *Let P_n act strongly on X. Then there is a strong morphism*

$$\sigma \colon\ X \to A_n^{\omega}$$

such that for any $m \in \mathbb{N}$ the finite prefix of $\sigma(x)$ of length m is the unique string u of length m such that $u^{-1} \cdot x$ is defined.

We call the map σ the *coding morphism.* If the coding morphism is injective then Bratteli and Jorgensen [5] define the action to be *multiplicity-free.*

It follows that a multiplicity-free strong action is equivalent to a subspace of the natural action of P_n on infinite strings. Thus a strong action is multiplicity-free if and only if it is a disjoint union of primitive strong actions each of which occurs at most once, thus providing a completely algebraic characterisation of this notion. If, in addition, the orbits are all rational then the action is classified by listing the Lyndon words that represent each of the primitive strings that occur.

We now introduce some terminology which will enable us to connect with the work in [5]. An orbit of a strong action of a polycyclic monoid is called a *cycle.* A rational cycle of a primitive action is said to be *atomic*. By Theorem 2.4.8, an atomic cycle is determined by a Lyndon word. The length of that Lyndon word will be called the *atomic weight* of the cycle. If all cycles are atomic we shall also say that the action is *atomic*. We therefore have the following.

Theorem 2.4.13. *A multiplicity-free, atomic action of a polycyclic monoid P_n is determined up to equivalence by a set of pairwise inequivalent Lyndon words.*

The case of the above theorem we are interested in is when there are only a finite set of cycles.

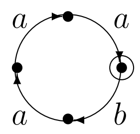
We shall now show how to construct a strong action directly from a Lyndon word. It is enough to construct a branching function system.

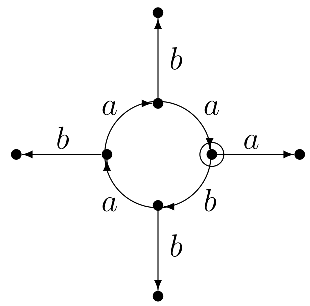
Let p be a primitive string over the alphabet $A_n = \{a_1, \ldots, a_n\}$. Put $X = A_n^* p^\omega$ a subset of A_n^ω. Each element of X can be written uniquely in the form $x\hat{p}p^\omega$ where x is a finite string of smallest possible length and $\hat{p}$ is a suffix of p. For each $a_i \in A_n$, we define a function from X to itself by $w \mapsto a_i w$. This is a branching function system. The associated action of P_n is transitive and the stabiliser of the point p^ω is P_n^p.

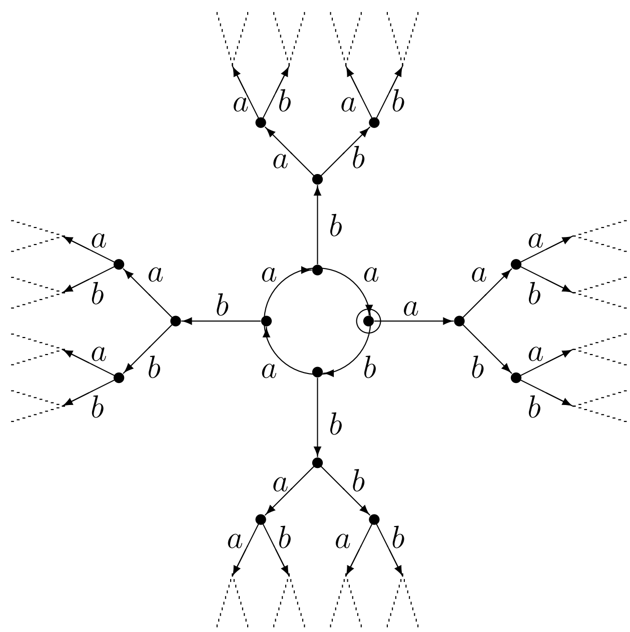
Example 2.4.14. We construct an example of a branching function system from a primitive string. We use the alphabet $A_2 = \{a, b\}$. Let $p = aaab$ be the primitive string. The set of points X of our branching function system is the set of vertices of the final one of the three directed graphs below; the first two show the steps in constructing the third. This branching function system corresponds to a transitive strong representation of P_2 on the set X. The arrows on the graphs represent the injective functions of the branching function system and are labeled by the generator of P_2 that they correspond to. The atomic weight of this cycle is 4 since $4 = |aaab|$. We have marked one of the points as an arbitrary base point which is not part of the branching function system. The element $aaab$ of P_2 fixes this point and there is no string of smaller length that fixes it. For any point in the directed graph there is a unique path from the base point to that point of smallest length. These are of the form

$$\varepsilon, b, ab, aab, xa \cdot \varepsilon, xb \cdot b, xb \cdot ab, xb \cdot aab$$

where x is any finite string.







The following theorem is the key to equating our notion of atomic weight with the notion of the 'number of atoms' used in [5].

Theorem 2.4.15. *Let p be a primitive string in A_n^*. Then the restriction to G_n of the action of P_n on $A_n^* p^\omega$ leads to an action with $|p|$ orbits.*

Proof. Let $up^\omega, vp^\omega \in A_n^* p^\omega$ where u and v are finite strings in A_n^*. We prove that up^ω and vp^ω are in the same orbit under the action of G_n if and only if $|u| \equiv |v| \pmod{|p|}$.

Let $|u| \equiv |v| \pmod{|p|}$. If $|u| = |v|$ then the result is immediate. Assume $|u| > |v|$. Then $|u| = r\,|p| + |v|$ for some $r \geq 1$. The element $u(vp^r)^{-1} \in G_n$, and $u(vp^r)^{-1} \cdot vp^\omega = up^\omega$. A similar argument holds for $|u| < |v|$.

We now prove the converse. Suppose that $xy^{-1} \cdot up^\omega = vp^\omega$ where $xy^{-1} \in G_n$. If u and v have the same length then there is nothing to prove. Assume that $|u| > |v|$. Choose α sufficiently large so that y is a prefix of up^α. Thus $yz = up^\alpha$. Then

$$xy^{-1} \cdot up^\omega = xy^{-1} \cdot up^\alpha p^\omega = xy^{-1} \cdot yzp^\omega = xzp^\omega = vp^\omega.$$

There are now three possibilities.

First, suppose that xz is a proper prefix of v. Then $v = xzw$ for some string $w \neq \varepsilon$. Then

$$|v| = |xzw| > |xz| = |yz| \geq |u|$$

which contradicts our assumption that $|u| > |v|$.

Second, suppose that v is a proper prefix of xz. Then $xz = vw$ where $w \neq \varepsilon$ and $w = p^\beta \bar{p}$ where $\bar{p}$ is a prefix of p. If $\bar{p} = \varepsilon$ then the desired conclusion follows. So we suppose that $\bar{p} \neq \varepsilon$ and of course $\bar{p} \neq p$. Then

$$xzp^\omega = vp^\beta \bar{p} p^\omega = vp^\omega$$

and so

$$p^\beta \bar{p} p^\omega = p^\omega$$

which gives

$$\bar{p} p^\omega = p^\omega.$$

Now p^2 is a proper prefix of $\bar{p}p^2$. Thus $\bar{p}pp = ppt$ where t is not empty. Let $p = \bar{p}s$ where s is not empty. Then $p^2 = spt$ where neither s nor t is empty. But this contradicts Lemma 1.3.3.

Third and finally, it follows that we must have $xz = v$. But then

$$|v| = |xz| = |yz| = |up^\alpha|$$

from which the desired conclusion follows. $\square$

In [5], each orbit of G_n is called an *atom*. The atomic weight of a cycle is just the number of atoms of the restriction of the action to G_n that that cycle yields.

2.5 Proper closed inverse subsemigroups of G_n

By Theorem 1.5.2 we know that the transitive actions of G_n are equivalent to actions on its proper closed inverse subsemigroups. We aim to classify these subsemigroups up to conjugacy so that we may compare the actions of G_n to those of P_n.

First we will look at those subsemigroups that do not contain zero. It is clear from Proposition 2.4.2 that the finite proper closed inverse submonoids are of the form

$$(xx^{-1})^\uparrow = \{yy^{-1} \in G_n \colon\ y \text{ is a prefix of } x\},$$

for all $x \in \mathcal{G}^*$.

To classify the primitive actions of G_n we also need to know which subsemigroups are proper and maximal. Clearly no finite submonoid can be maximal. To find a maximal closed inverse submonoid we need to construct an infinite equivalent.

Let $w \in A_n^\omega$ be any string. Define

$$(ww^{-1})^\uparrow = \{yy^{-1} \in G_n \colon\ y \text{ is a prefix of } w\}.$$

Proposition 2.5.1. *The set* $(ww^{-1})^{\uparrow}$ *is an infinite closed inverse submonoid of* G_n *that does not contain zero. Every infinite closed inverse monoid of* G_n *that does not contain zero is of this form.*

Proof. First we show that $(ww^{-1})^{\uparrow}$ is an infinite closed submonoid that does not contain zero. The finite version of this claim is shown in the proof of Proposition 2.4.2 and can easily be generalised to the infinite case.

Let H be an infinite proper closed inverse submonoid of G_n and let $xx^{-1}, yy^{-1} \in H$. Then since their product is non-zero they must be prefix comparable. Because H is infinite we can find idempotents in H of arbitrary weight. It follows that there is an infinite string w such that if $xx^{-1} \in H$ then x is a prefix of w. To see this simply assume that no such w exists. In addition, if y is a prefix of w then $yy^{-1} \in H$. Because there must be an idempotent zz^{-1} in H of greater weight than yy^{-1} we have y is a prefix of z and so by closure $yy^{-1} \in H$. It follows that the idempotents of H are precisely $(ww^{-1})^{\uparrow}$. By the same argument as for the finite case we can prove that there are no non-idempotent elements of H and so $H = (ww^{-1})^{\uparrow}$. $\square$

If H is a closed inverse submonoid containing zero then things are easy. As zero is the least element with respect to the partial order $H = G_n$.

We now classify our closed inverse submonoids without zero up to conjugation.

Lemma 2.5.2.

1. *Let* $(xx^{-1})^{\uparrow}$ *and* $(yy^{-1})^{\uparrow}$ *be two finite closed inverse submonoids (without zero) of* G_n*. Then they are conjugate if and only if* $\mu(xx^{-1}) = \mu(yy^{-1})$*.*
2. *Let* $(w_1w_1^{-1})^{\uparrow}$ *and* $(w_2w_2^{-1})^{\uparrow}$ *be two infinite closed inverse submonoids of* G_n *without zero. Then they are conjugate if and only if* w_1 *and* w_2 *differ in only a finite number of places. That is there exist* $p_1, p_2 \in A_n^*$ *with* $|p_1| = |p_2|$ *and* $w' \in A_n^{\omega}$ *such that* $w_1 = p_1w'$ *and* $w_2 = p_2w'$*.*

Proof. (1) Suppose that $(xx^{-1})^{\uparrow}$ and $(yy^{-1})^{\uparrow}$ are conjugate. Then there is $uv^{-1} \in G_n$ such that $uv^{-1}(xx^{-1})^{\uparrow}vu^{-1} \subseteq (yy^{-1})^{\uparrow}$ and $vu^{-1}(yy^{-1})^{\uparrow}uv^{-1} \subseteq (xx^{-1})^{\uparrow}$. We deduce that

$$yy^{-1} \leq uv^{-1} \cdot xx^{-1} \cdot vu^{-1}$$

and

$$xx^{-1} \leq vu^{-1} \cdot yy^{-1} \cdot uv^{-1}.$$

Thus xx^{-1} and yy^{-1} have the same weight and so x and y have the same length.

Conversely, suppose that x and y have the same length. Then $yx^{-1} \cdot xx^{-1} \cdot xy^{-1} = yy^{-1}$. Let $uu^{-1} \geq xx^{-1}$. Then $x = uz$ for some string z. Thus $yx^{-1} \cdot uu^{-1} \cdot xy^{-1} = yy^{-1}$. It follows that $yx^{-1}(xx^{-1})^{\uparrow}xy^{-1} \subseteq (yy^{-1})^{\uparrow}$, the reverse inclusion follows similarly. Hence $(xx^{-1})^{\uparrow}$ and $(yy^{-1})^{\uparrow}$ are conjugate.

(2) Suppose that $(w_1w_1^{-1})^{\uparrow}$ and $(w_2w_2^{-1})^{\uparrow}$ are conjugate. Then there exists $uv^{-1} \in G_n$ such that $uv^{-1}(w_1w_1^{-1})^{\uparrow}vu^{-1} \subseteq (w_2w_2^{-1})^{\uparrow}$ and $vu^{-1}(w_2w_2^{-1})^{\uparrow}uv^{-1} \subseteq (w_1w_1^{-1})^{\uparrow}$. As $|u| = |v|$ we need to show that there exist $w' \in A_n^{\omega}$ such that $w_1 = vw'$ and $w_2 = uw'$. We can find $r_1r_1^{-1} \in (w_1w_1^{-1})^{\uparrow}$ and $r_2r_2^{-1} \in (w_2w_2^{-1})^{\uparrow}$ such that $\mu(r_1r_1^{-1}) = \mu(r_2r_2^{-1}) = t > \mu(uv^{-1})$. Thus $uv^{-1} \cdot r_1r_1^{-1} \cdot vu^{-1}$ and $vu^{-1} \cdot r_2r_2^{-1} \cdot uv^{-1}$ also have weight t and

$$uv^{-1} \cdot r_1r_1^{-1} \cdot vu^{-1} \in (w_2w_2^{-1})^{\uparrow} \text{ and } vu^{-1} \cdot r_2r_2^{-1} \cdot uv^{-1} \in (w_1w_1^{-1})^{\uparrow}.$$

As both $(w_1w_1^{-1})^{\uparrow}$ and $(w_2w_2^{-1})^{\uparrow}$ have only one element of each weight we deduce that

$$uv^{-1} \cdot r_1r_1^{-1} \cdot vu^{-1} = r_2r_2^{-1} \text{ and } vu^{-1} \cdot r_2r_2^{-1} \cdot uv^{-1} = r_1r_1^{-1}.$$

So $r_1 = vr$ and $r_2 = ur$ for some $r \in A_n^*$. As this holds for all $t > \mu(uv^{-1})$ there exist $w' \in A_n^{\omega}$ such that $w_1 = vw'$ and $w_2 = uw'$.

Now suppose there exists $p_1, p_2 \in A_n^*$ with $|p_1| = |p_2|$ and $w' \in A_n^{\omega}$ such that $w_1 = p_1w'$ and $w_2 = p_2w'$. Then $p_1p_2^{-1}, p_2p_1^{-1} \in G_n$. We will show $p_2p_1^{-1}(w_1w_1^{-1})^{\uparrow}p_1p_2^{-1} \subseteq (w_2w_2^{-1})^{\uparrow}$. Let $r_1r_1^{-1} \in (w_1w_1^{-1})^{\uparrow}$. If $\mu(r_1r_1^{-1}) \leq \mu(p_1p_2^{-1})$ then r_1 is a prefix of p_1 and $p_2p_1^{-1}(r_1r_1^{-1})p_1p_2^{-1} = p_2p_2^{-1}$. If $\mu(r_1r_1^{-1}) > \mu(p_1p_2^{-1})$ then there exist a prefix s of w' such that $r_1 = p_1s$ and

$$p_2p_1^{-1}(r_1r_1^{-1})p_1p_2^{-1} = (p_2s)(p_2s)^{-1}.$$

As both p_2 and $p_2 s$ are prefixes of w_2 we have $p_2 p_1^{-1}(w_1 w_1^{-1})^{\uparrow} p_1 p_2^{-1} \subseteq (w_2 w_2^{-1})^{\uparrow}$. By symmetry we have $p_1 p_2^{-1}(w_2 w_2^{-1})^{\uparrow} p_2 p_1^{-1} \subseteq (w_1 w_1^{-1})^{\uparrow}$ and thus $(w_1 w_1^{-1})^{\uparrow}$ and $(w_2 w_2^{-1})^{\uparrow}$ are conjugate. $\square$

We shall now obtain concrete models of actions of G_n that have finite stabilisers. The following Lemma will become useful later in this section.

Lemma 2.5.3. *With respect to the restriction to G_n of the natural action of P_n on A_n^{ω} the stabiliser of an infinite string w is the infinite closed subsemigroup $(ww^{-1})^{\uparrow}$. The orbits of this action are the conjugacy classes of the infinite closed inverse submonoids.*

Proof. Let $xy^{-1} \cdot w = w$. Then x and y both have to be prefixes of w. As $|x| = |y|$ we have $x = y$ and $xy^{-1} \in (ww^{-1})^{\uparrow}$.

Now let $xx^{-1} \in (ww^{-1})^{\uparrow}$. Then x is a prefix of w and thus $xx^{-1} \cdot w = w$. Therefore $(ww^{-1})^{\uparrow}$ is the stabiliser of w with respect to the restriction to G_n of the natural action of P_n on A_n^{ω}.

Let $(w_1 w_1^{-1})^{\uparrow}$ and $(w_2 w_2^{-1})^{\uparrow}$ be conjugate. Then by Lemma 2.5.2 there exist $p_1, p_2 \in A_n^*$ with $|p_1| = |p_2|$ and $w' \in A_n^{\omega}$ such that $w_1 = p_1 w'$ and $w_2 = p_2 w'$. Thus $p_1 p_2^{-1}, p_2 p_1^{-1} \in G_n$ and we have $p_2 p_1^{-1} \cdot w_1 = w_2$ and $p_1 p_2^{-1} \cdot w_2 = w_1$. Therefore w_1 and w_2 are in the same orbit under the restriction to G_n of the natural action of P_n on A_n^{ω}.

Now let w_1 and w_2 be in the same orbit. Then there exists $uv^{-1} \in G_n$ such that $uv^{-1} \cdot w_1 = w_2$ and $vu^{-1} \cdot w_2 = w_1$. Therefore there exists $w' \in A_n^{\omega}$ such that $w_1 = vw'$ and $w_2 = uw'$. By Lemma 2.5.2 $(w_1 w_1^{-1})^{\uparrow}$ and $(w_2 w_2^{-1})^{\uparrow}$ are conjugate. $\square$

We are now in a position to construct primitive representations of G_n. First we will look at the non-maximal transitive representations. These representations come about as actions on $G_n/(xx^{-1})^{\uparrow}$, where $(xx^{-1})^{\uparrow}$ is finite. If $(xx^{-1})^{\uparrow}$ is finite then it cannot be maximal and therefore the action on the associated quotient cannot be primitive.

Let $(rs^{-1}(xx^{-1})^{\uparrow})^{\uparrow} \in G_n/(xx^{-1})^{\uparrow}$ and $uv^{-1} \in G_n$. The action is defined by

$$uv^{-1} \cdot (rs^{-1}(xx^{-1})^{\uparrow})^{\uparrow} = (uv^{-1}rs^{-1}(xx^{-1})^{\uparrow})^{\uparrow} \Leftrightarrow \mathbf{d}(uv^{-1}rs^{-1}) \in (xx^{-1})^{\uparrow}.$$

Lemma 2.5.4. *All cosets of $G_n/(xx^{-1})^{\uparrow}$ have a representative of the form ux^{-1}.*

Proof. Let $(rs^{-1}(xx^{-1})^{\uparrow})^{\uparrow}$ be a coset in $G_n/(xx^{-1})^{\uparrow}$. Thus s is a prefix of x as $sr^{-1}rs^{-1} \in (xx^{-1})^{\uparrow}$. Let t be the suffix of x such that $st = x$. Then

$$sr^{-1}(rt)(st)^{-1} = sr^{-1}rtt^{-1}s^{-1} = (st)(st)^{-1} = xx^{-1} \in (xx^{-1})^{\uparrow}.$$

As $(rs^{-1})^{-1}(rt)(st)^{-1} \in (xx^{-1})^{\uparrow}$ the cosets $(rs^{-1}(xx^{-1})^{\uparrow})^{\uparrow}$ and $((rt)(st)^{-1}(xx^{1})^{\uparrow})^{\uparrow}$ are equal. □

We can now describe the primitive actions of G_n. Recall that a proper transitive action is primitive if its stabilisers are maximal proper closed inverse subsemigroups.

Lemma 2.5.5. *All infinite closed inverse subsemigroups are proper and maximal.*

Proof. By definition the $(ww^{-1})^{\uparrow}$ do not contain zero and therefore are all proper. We now show that they are maximal. Let $(w_1w_1^{-1})^{\uparrow} = H_1$ and $(w_2w_2^{-1})^{\uparrow} = H_2$ be infinite closed inverse submonoids. Suppose $H_1 \subseteq H_2$. Let $xx^{-1} \in H_2 \setminus H_1$ with $\mu(xx^{-1}) = k$ and $w_1(k)$ the prefix of w_1 of length k. Then $xx^{-1} \neq w_1(k)w_1(k)^{-1}$ both have the same weight and are both contained in H_2, a contradiction. Thus no such xx^{-1} exists so $H_1 = H_2$ and all infinite closed inverse subsemigroups are maximal. □

2.6 Computing orbits for strong actions of P_n

A *branching function system* consists of a non-empty set X and n injective functions, $f_i : X \to X$, $1 \leq i \leq n$, where images are disjoint and partition X. We saw earlier that these determine and are determined by the strong representations of P_n.

Lemma 2.6.1. *Let $(X, f_1, \ldots, f_n)$ be a branching function system. Then $\delta = \bigcup_{i=1}^{n} f_i^{-1}$ is a well-defined function from X to itself and $\delta f_i = 1_X$ for $1 \leq i \leq n$.*

Proof. By the definition of a branching function system we have that $1_X = \bigcup_{i=1}^{n} f_i f_i^{-1}$ is a disjoint union. As a consequence, if $x \in X$ there is a unique $1 \leq j \leq n$ such that $\exists f_j f_j^{-1}$. Thus there is a unique $1 \leq j \leq n$ such that $\exists f_j^{-1}$. We may therefore define $\delta : X \to X$ by $\delta(x) = y$ if $x \in \mathrm{im}(f_j)$ and $f_j(y) = x$. It follows that δ is a well-defined function. Finally

$$\begin{aligned}
\delta f_i &= \left(\bigcup_j f_j^{-1} \right) f_i \\
&= \bigcup_j f_j^{-1} f_i \\
&= f_i^{-1} f_i \quad \text{since } f_j^{-1} f_i = 0 \text{ if } i \neq j \\
&= 1_X
\end{aligned}$$

□

We call (X, δ) the *dynamical system associated with the branching function system.* It follows that for every strong representation $P_n \to I(X)$ there is an associated dynamical system (X, δ). In [5], the authors show that the dynamical system (X, δ) can be used to analyse the strong action $P_n \to I(X)$. To show how, we shall construct a new monoid from a given strong representation $P_n \to I(X)$.

First, we need some notation. Let $p, q \in \mathbb{N}$. Define

$$p \dot{-} q = \begin{cases} p - q & \text{if} \quad p > q \\ 0 & \text{if} \quad p \leq q. \end{cases}$$

Let $x \in A_n^*$ and $j \in \mathbb{N}$. Define

$$\mathrm{pref}_j(x) = \begin{cases} y & \text{if} \quad 0 \leq j \leq |x|, \; x = yz \text{ and } |y| = j \\ x & \text{if} \quad j > |x|. \end{cases}$$

If $x = yz$, define $y^{-1}x = z$. We call this operation *string cancellation.* On the set $A_n^* \times \mathbb{N}$ define the following binary operation

$$(x,p)(y,q) = (x \ \mathrm{pref}_p(y)^{-1}y, (p \dot{-} |y|) + q).$$

This definition is equivalent to the following

$$(x,p)(y,q) = \begin{cases} (x, p+q-|y|) & \text{if} \quad |y| \leq p \\ (x \ \mathrm{pref}_p(y)^{-1}y, q) & \text{if} \quad |y| > p. \end{cases}$$

To ease notation, we make the following definitions. Define $\mathbb{N} \times A_n^* \to A_n^*$ by $p \cdot y = \mathrm{pref}_p(y)^{-1}y$. Define $\mathbb{N} \times A_n^* \to \mathbb{N}$ by $p|_y = p \dot{-} |y|$. The binary operation we have defined assumes the following form using this notation

$$(x,p)(y,q) = (x(p \cdot y), p|_y + q).$$

The set $A_n^* \times \mathbb{N}$ equipped with this product is denoted by $A_n^* \bowtie \mathbb{N}$.

Proposition 2.6.2. *$A_n^* \bowtie \mathbb{N}$ is a monoid.*

Proof. First we will show that the product is associative. Let $x, y, z \in A_n^*$ and $p, q, r \in \mathbb{N}$. Assume $|y| \leq p$. Then

$$\begin{aligned} (x,p)(y,q)(z,r) &= (x, p+q-|y|)(z,r) \\ &= (x((p+q-|y|) \cdot z), (p+q-|y|)|_z + r) \\ &= (x(p \cdot (y(q \cdot z))), (p-|y|)|_{q \cdot z} + q|_z + r) \\ &= (x(p \cdot (y(q \cdot z))), p|_{y(q \cdot z)} + q|_z + r) \\ &= (x,p)(y(q \cdot z), q|_z + r) \\ &= (x,p)(y,q)(z,r). \end{aligned}$$

Now let $|y| > p$. Then

$$\begin{aligned}(x,p)(y,q)(z,r) &= (x\ \mathrm{pref}_p(y)^{-1}y, q)(z,r)\\ &= (x\ \mathrm{pref}_p(y)^{-1}y(q\cdot z), q|_z + r)\\ &= (x(p\cdot y)(q\cdot z), p|_{(y(q\cdot z))} + q|_z + r)\\ &= (x(p\cdot(y(q\cdot z))), p|_{y(q\cdot z)} + q|_z + r)\\ &= (x,p)(y(q\cdot z), q|_z + r)\\ &= (x,p)(y,q)(z,r).\end{aligned}$$

We conclude by observing that $(\epsilon, 0)$ is the identity for this product. □

To understand where $A_n^* \bowtie \mathbb{N}$ comes from, we prove the following.

Proposition 2.6.3. *The monoid given by the monoid presentation*

$$\langle a_1, \ldots, a_n, \delta : \delta a_i = 1 \textit{ for } 1 \leq i \leq n\rangle$$

is isomorphic to $A_n^* \bowtie \mathbb{N}$

Proof. We will denote the monoid with the above presentation by S. Firstly we note that the normal form of elements in S is $x\delta^p$, where $x \in A_n^*$ and $p \in \mathbb{N}$ and multiplication is defined by

$$x\delta^p y\delta^q = x(p\cdot y)\delta^{p|_y + q}.$$

We will now construct an isomorphism from S to $A_n^* \bowtie \mathbb{N}$. Let $\theta : S \to A_n^* \bowtie \mathbb{N}$ be a homomorphism define by $\theta(x\delta^p) = (x,p)$. For $x\delta^p$ and $y\delta^q$ we have

$$\theta(x\delta^p y\delta^q) = \theta(x(p\cdot y)\delta^{p|_y + q}) = (x(p\cdot y), p|_y + q) = (x,p)(y,q) = \theta(x\delta^p)\theta(y\delta^q).$$

Now we will prove θ is surjective and injective. Let $(x,p) \in A_n^* \bowtie \mathbb{N}$. Then we can find $x\delta^p$ in S such that $\theta(x\delta^p) = (x,p)$. Let $x\delta^p, y\delta^q \in S$ with $\theta(x\delta^p) = \theta(y\delta^q)$. Then $(x,p) = (y,q)$ and thus $x\delta^p = y\delta^q$. □

We shall write $BJ_n = A_n^* \bowtie \mathbb{N}$ and call it the *Bratteli-Jorgensen monoid on $n+1$ generators*. It is, in fact, an example of a Zappa-Szép product of two free monoids. It was defined in Scholium 2.5 of [5]. The authors assert that it is a semigroup with left inverses, which is, in fact, incorrect. The actions $BJ_n \times X \to X$ are just ordinary monoid actions, as introduced in Section 1.1, and not by partial bijections because BJ_n is not inverse. See [18] page 252 for more information on monoid actions. We shall say that such an action is *strong* if it satisfies the following two conditions:

1. $X = \bigcup_{i=1}^{n} a_i X$
2. $a_i \cdot x = a_j \cdot x \Rightarrow i = j$.

Proposition 2.6.4. *There is a bijective correspondence between strong representations of $P_n \to I(X)$ and the strong actions $BJ_n \times X \to X$.*

Proof. It is enough to show that there is a bijective correspondence between branching function systems $(X, f_1, \ldots, f_n)$ and the strong actions of BJ_n on X. Let $(X, f_1, \ldots, f_n)$ be a branching function system. Define $\delta = \bigcup f_i^{-1}$. Then by Lemma 2.6.1, the submonoid $\langle f_1, \ldots, f_n, \delta \rangle \subseteq T(X)$ is a homomorphic image of BJ_n. Thus we have a monoid homomorphism $BJ_n \to T(X)$ and so a monoid action $BJ_n \times X \to X$. To show that this is a strong action, we have to show that conditions (1) and (2) hold. Condition (1) clearly holds. Condition (2) holds because $\mathrm{im}(f_i) \cap \mathrm{im}(f_j) = \emptyset$ if $i \neq j$. We have therefore defined a strong action of BJ_n on X.

To prove the converse, let $BJ_n \times X \to X$ be a strong action. Define $f_i : X \to X$ by $x \mapsto a_i \cdot x$. These functions are injective because if $f_i(x) = f_j(y)$ then $a_i \cdot x = a_i \cdot y$. But then $\delta a_i \cdot x = \delta a_i \cdot y$ and so $x = y$, since $1 \cdot x = x$ for all $x \in X$. Suppose now that $\mathrm{im}(f_i) \cap \mathrm{im}(f_j) \neq \emptyset$. Then $f_i(x) = f_j(y)$. Thus $a_i \cdot x = a_j \cdot y$. Hence $\delta a_i \cdot x = \delta a_j \cdot y$ and so $x = y$. Thus $a_i \cdot x = a_j \cdot x$. By condition (2), $i = j$. From this, and condition (1), it follows that $(X, f_1, \ldots, f_n)$ forms a branching function system. ☐

Proposition 2.6.5. *Let $P_n \to I(X)$ be a strong representation and let (X, δ) be the associated dynamical system. Let $\sim$ be the equivalence relation on X induced by the* *strong action of P_n. Then*

$$u \sim v \Leftrightarrow \delta^p(u) = \delta^q(v) \textit{ for some } p, q \in \mathbb{N}.$$

Proof. We know that $u \sim v$ iff $\exists x\delta^p \in BJ_n$ such that $(x\delta^p) \cdot u = v$. Let $|x| = q$. Then $\delta^q(x\delta^p u) = \delta^q v$. But $\delta^q x = 1$. Thus $\delta^p(u) = \delta^q(v)$ as required.

Conversely, suppose that $\delta^p(u) = \delta^q(v)$ for some $p, q \in \mathbb{N}$. By assumption, $v \in \bigcup_{i=1}^n a_i X$. This implies that we can write $v = x \cdot v'$ for some $x \in A_n^*$ such that $|x| = q$. But then $\delta^q(v) = \delta^q(x \cdot v') = v'$. Thus $\delta^p(u) = v'$. Now apply x to both sides to get $x\delta^p u = v$, as required. □

It follows from the proposition that the orbits of a strong action of P_n on X are defined by the behaviour of the dynamical system (X, δ) constructed for the action.

Let $P_n \times X \to X$ be a strong action. We call the orbits of the action *cycles* and denote the cycle containing x by $[x]^{\sim}$. There is an induced action of G_n on X the orbits of which are called *atoms* and the atom containing x is defined by $[x]^{\approx}$.

Proposition 2.6.6. *Let $P_n \times X \to X$ be a strong action, (X, δ) the associated dynamical system and $A_n^* \bowtie \mathbb{N}$ the BJ_n-monoid associated with this action. Put*

$$BJ_n^{=} = \{(x, p) \in A_n^* \times \mathbb{N} : |x| = p\}.$$

1. *$BJ_n^{=}$ is a submonoid of BJ_n.*

2. *For the restriction to $BJ_n^{=}$ of the action of BJ_n on X the orbits are precisely the atoms.*

3. *$u \approx v \Leftrightarrow \delta^p(u) = \delta^p(v)$ for some $p \geq 0$, where $\approx$ in the equivalence associated to the restriction to $BJ_n^{=}$ of the action of BJ_n on X.*

Proof.

1. Let $(x,p),(y,q)$ be elements of $BJ_n^=$. Then

$$(x,p)(y,q) = \begin{cases} (x, p+q-|y|) & \text{if} \quad |y| \le p \\ (x \ \mathrm{pref}_p(y)^{-1}y, q) & \text{if} \quad |y| > p. \end{cases}$$

   Firstly we note that $p+q-|y| = p$ if $|y| = q$. Thus $(x,p)(y,q) = (x,p)$ if $|y| \le p$. We also have that $|(x \ \mathrm{pref}_p(y)^{-1}y)| = |x| + |y| - |\mathrm{pref}_p(y)| = |y|$ if $|x| = p$ and $|y| > p$. Thus $BJ_n^=$ is closed under products. Finally we see that $|\epsilon| = 0$ thus $(\epsilon, 0) \in BJ_n^=$.

2. Let $x \in X$ and suppose $y \in [x]^{\approx}$. Then there exists $uv^{-1} \in G_n$ such that $uv^{-1} \cdot x = y$. This implies there exists $p \in \mathbb{N}$ with $\delta^p(x) = \delta^p(y)$. Finally we see that $u\delta^p(x) = y$ and as such x and y share an orbit under the action of $BJ_n^=$.

   Conversely, let x and y be in the same orbit under the action of $BJ_n^=$. Then there exists $u\delta^p \in BJ_n^=$ such that $u\delta^p \cdot x = y$. Multiplying on the left by δ^p gives $\delta^p(x) = (\delta^p u)\delta^p(x) = \delta^p(y)$. There exist a $v \in A_n^*$ such that $v\delta^p(x) = x$, that is $v^{-1} \cdot x = \delta^p(x)$. Finally we see that $uv^{-1} \cdot x = u\delta^p(x) = y$ and as such $y \in [x]^{\approx}$.

3. We know that $u \approx v$ iff $\exists x\delta^p \in BJ_n^=$ such that $(x\delta^p) \cdot u = v$. Multiplying on the left by δ^p gives $(\delta^p x)\delta^p \cdot u = \delta^p v$. But $\delta^p x = 1$ as $x\delta^p \in BJ_n^=$. Thus $\delta^p(u) = \delta^p(v)$ as required.

   Conversely, suppose that $\delta^p(u) = \delta^p(v)$ for some $p \ge 0$. By assumption, $v \in \bigcup_{i=1}^n a_i X$. This implies that we can write $v = x \cdot v'$ for some $x \in A_n^*$ such that $|x| = p$. But then $\delta^p(v) = \delta^p(x \cdot v') = v'$. Thus $\delta^p(u) = v'$. Now apply x to both sides to get $x\delta^p u = v$, as required.

$\square$

Up to this point, we have described most of the ideas we need from pages 1-11 of [5].

Let $P_n \times X \to X$ be a strong action. We say that a point $u \in X$ is *rational* if the sequence $(u, \delta(u), \delta^2(u), \ldots)$ is *ultimately periodic* meaning $\exists p \ge 0$, $r \ge 1$ such that $\delta^p(u) = \delta^{p+r}(u)$. Otherwise, we say that $u \in X$ is *irrational*.

Lemma 2.6.7. *Let $P_n \times X \to X$ be a strong action. The point $u \in X$ is rational iff the stabiliser of u in P_n contains a non-empty element xy^{-1} with $|x| \neq |y|$.*

Proof. Suppose $u \in X$ is rational. We will construct an element xy^{-1} with $|x| \neq |y|$ that stabilises u. Then $\delta^p(u) = \delta^{p+r}(u)$ for some $p \geq 0$ and $r \geq 1$. Let $u = s \cdot u'$ where $|s| = p$. Then $\delta^p(u) = \delta^p(s \cdot u') = u'$. Thus $u' = \delta^{p+r}(s \cdot u') = \delta^r(u')$. Let $u' = t \cdot u''$ where $|t| = r$. Then $u' = \delta^r(u') = u''$. Now $u = s \cdot u'$ and $u = st \cdot u''$. Thus $s^{-1} \cdot u = (st)^{-1} \cdot u$. Letting $x = st$, $y = s$ we see $xy^{-1} \cdot u = (st)s^{-1} \cdot u = u$ and $|x| \neq |y|$ as $|t| \geq 1$.

Conversely, suppose $xy^{-1} \cdot u = u$ where $|x| = p$, $|y| = p + r$ for some $r \geq 1$. Then $y^{-1} \cdot u = x^{-1} \cdot u$. That is $\delta^{p+r}(u) = \delta^p(u)$. Thus u is rational, as required. □

Lemma 2.6.8. *If u is rational and $u \sim v$ then v is rational.*

Proof. As u is rational $\exists p, r \in \mathbb{N}$ such that $\delta^p(u) = \delta^{p+r}(u)$. Thus there exist a $x \in A_n^*$ with $|x| = p$ such that $u = x\delta^p(u) = x\delta^{p+r}(u)$. Therefore $u = xu'$ where $u' = \delta^p(u) = \delta^{p+r}(u)$, that is $u' = \delta^r(u')$. Let $u \sim v$. Then there exist $s, t \in \mathbb{N}$ such that $\delta^s(u) = \delta^t(v)$. Suppose $s \leq |x|$. Then $\delta^s(u) = yu'$, where y is the appropriate suffix of x. Let $z \in A_n^*$ such that $|z| = t$ and $v = z\delta^t(v)$. Thus

$$v = z\delta^t(v) = z\delta^s(u) = zyu' = (zy)u'.$$

Therefore $\delta^m(v) = \delta^{m+r}(v)$ where $m = |zy|$.

Now suppose $s > |x|$. Then $\delta^s(u) = y'u'$, where y' is the appropriate suffix of $u'|_r$. We can assume $|y'| < r$ as $\delta^r(u') = u'$. By this we know that $\delta^s(u)$ is also a ultimately periodic string. Let $z \in A_n^*$ such that $|z| = q$ and $v = z\delta^t(v)$. Therefore

$$v = z\delta^t(v) = z\delta^s(u) = z(y'u') = (zy')u'.$$

Therefore $\delta^m(v) = \delta^{m+r}(v)$ where $m = |zy'|$. □

It follows that we can refer to a cycle as being rational or irrational. Given a strong action $P_n \times X \to X$, we may ask the following questions

1. Is the action multiplicity-free?

2. Are there a finite number of cycles?

3. Are all the cycles rational?

In [5] they are primarily concerned with multiplicity-free actions with a finite number of cycles all of which are rational.

The following is the key notion used in [5]. Let $P_n \times X \to X$ be a strong action. We say that this action is *contracting* if there is a finite set $B \subseteq X$ such that for all $x \in X$, $\exists n_x \geq 1$ such that for all $n \geq n_x$ we have that $\delta^n(x) \in B$. Put

$$B_\infty = \{x \in X : x = \delta^p(x) \text{ for some } p \geq 1\}$$

i.e. the rational points which are *periodic.*

Proposition 2.6.9. *Let $P_n \times X \to X$ be a strong action. Then it is a contracting action iff there are only a finite number of cycles and they are all rational. In a contracting action, $B_\infty \subseteq B$ and the number of atoms is equal to $|B_\infty|$.*

Proof. Firstly we will show that any contracting strong action has only a finite number of cycles and they are all rational. Let $P_n \times X \to X$ be a contracting strong action and $x \in X$. Then there exists $n_x \geq 1$ such that $\delta^n(x) \in B$ for all $n \geq n_x$ for a designated finite set $B \subseteq X$. As $x \sim \delta^n(x)$ for all $n \geq n_x \geq 1$ every cycle contains an element from B and therefore there are only finitely many cycles. As $\delta^n(x) \in B$ for all $n \geq n_x$ and B finite we have that $\exists r \geq 1$, $n \geq n_x$ such that $\delta^{n+r}(x) = \delta^n(x)$. Thus x is rational and as such all cycles are rational.

We shall now prove the converse. Let $P_n \times X \to X$ be a strong action for which there are only finitely many cycles, all of which are rational. As the action is strong and all cycles are rational we have $\forall x \in X$ there exists $n \geq 0$, $r \geq 1$ such that $\delta^{n+r}(x) = \delta^n(x)$. For each x we can define

$$B_x = \{\delta^m(x) : n \leq m \text{ where } \delta^{n+r}(x) = \delta^n(x)\}.$$

Clearly $|B_x|$ divides r for each x and we will prove $B_x = B_y$ if $x \sim y$. To see this let $x \sim y$. Then $\delta^{p_x+r_x}(x) = \delta^{p_x}(x)$, $\delta^{p_y+r_y}(y) = \delta^{p_y}(y)$ and $\delta^s(x) = \delta^t(y)$ for appropriate p_x, r_x, p_y, r_y, s, t. Assume $s - t = z \geq 0$ and let $q_z = \max\{p_x, p_y, s, t\}$. Then $\delta^{q+z}(x) = \delta^q(y)$ for all $q \geq q_z$. Now let $\delta^m(y) \in B_y$ such that $m \geq q_z$. Then $q_z + b = m$ and

$$\delta^m(y) = \delta^{q_z+b}(y) = \delta^{q_z+z+b}(x) \in B_x.$$

Thus $B_y \subseteq B_x$ and by a dual argument $B_y = B_x$.

As we have only finitely many cycles the union of all these B_x is finite. Setting B to be a finite subset of X containing all B_x gives us that the action is contracting.

Now we will prove that in a contracting action $B_\infty \subseteq B$. Let $x \in B_\infty$. As x is periodic there exists $r \geq 1$ such that $\delta^r(x) = x$. But the action is contracting so there also exists $n_x \geq 1$ such that $\delta^n(x) \in B$ for all $n \geq n_x$. We can write $n_x = mr + s$, where $s < r$. Then $(m+1)r > n_x$, thus $\delta^{(m+1)r}(x) \in B$. But $\delta^{(m+1)r}(x) = x$ and so $x \in B$.

Finally we will prove that there are $|B_\infty|$ atoms. We will do this by proving that every element of B_∞ is associated to an atom and then showing this element is unique. As $P_n \times X \to X$ is a strong action we see that every element of B_∞ belongs to an atom. Let $[x]^\approx$ be an atom. We will see that every atom contains a periodic element. As all cycles, and therefore atoms, are rational we have that $\exists p \geq 0$, $r \geq 1$ such that $\delta^{p+r}(x) = \delta^p(x)$. We can find an $a \in \mathbb{N}$ such that $ar > p$. Then

$$\delta^{ar}(x) = \delta^{ar+ar}(x) = \delta^{ar}(\delta^{ar}(x)).$$

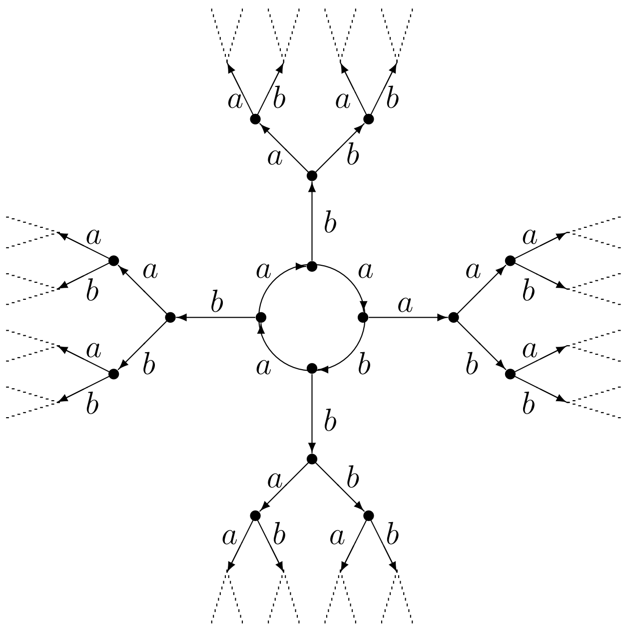
Thus $x \approx \delta^{ar}(x)$ and $\delta^{ar} \in B_\infty$.

Now let $y \in [x]^\approx$ also be periodic, that is $y = \delta^s(y)$ for some $s \geq 1$. As y and $\delta^{ar}(x)$ are in the same atom there exists $n_y \geq 1$ such that $\delta^n(y) = \delta^n(\delta^{ar}(x))$ for all $n \geq n_y$. Let $b \geq 1$ such that $bsr \geq n_y$. Then

$$y = \delta^{bsr}(y) = \delta^{bsr}(\delta^{ar}(x)) = \delta^{ar}(x).$$

Therefore each atom contains exactly one periodic element. $\square$

Example 2.6.10. *We construct an example of a branching function system. We use the alphabet $A_2 = \{a, b\}$. Let $p = abaa$ be a primitive string. The set of points of X is the set of vertices* (infinite) *of the directed graph below. This branching function system corresponds to the transitive strong representations of P_2 on the set X. It is thus a single cycle. There are four atoms* $(= |p|)$.



This is how we visualise the action of P_2 and G_2 on $p^\omega A_2^*$. For each four letter primitive string in A_2^* we obtain the same diagram and action. Therefore we can use Lyndon words to help classify transitive actions of P_n.

2.7 Cycles and atoms

In this section, we shall restrict our attention to strong representations on $\mathbb{Z}$, the free abelian group of rank 1. The subgroups of finite index are of the form $n\mathbb{Z}$ where $n \geq 2$. For fixed n, each choice of n integers $s_1, \ldots, s_n$ which are pairwise non-congruent modulo n gives rise to a strong representation of P_n on $\mathbb{Z}$. Bratteli and Jorgensen [5] prove two important results right away: first, these representations are always multiplicity-free (Proposition 3.1 [5]) and second, the number of orbits is always finite (Lemma 3.4 [5]).

It turns out that each orbit (that is, cycle) is atomic. It follows by Proposition 2.6.9 that the actions are contracting. We may therefore classify them by means of a finite set of distinct Lyndon words. The detailed analysis of this case is carried out in Sections 8.1, 8.2 and 8.3 of [5].

There are two functions that help simplify this analysis. First, define the function $\alpha_m\colon \mathbb{Z} \to \mathbb{Z}$ by $\alpha_m(x) = x + m$. Second, define the map $\iota\colon \mathbb{Z} \to \mathbb{Z}$ by $x \mapsto -x$. In the special case $n = 2$, we can use the above two maps to show every such representation is equivalent to one where we choose $s_1 = 0$ and $s_2 > 0$ and odd. Thus such actions are parametrised by an positive odd integer p. The case $p = 1$ is handled separately in Proposition 8.1 of [5] and we will not discuss it here. Thus in what follows we assume that $p \geq 3$.

There are two fundamental questions we would like to answer about such actions:

1. How many cycles are there?

2. What is the nature of each cycle in terms of Lyndon words?

Proposition 8.2 of [5] states that for each choice of $p \in \{3, 5, 7, \ldots\}$ there is always an equivalence class containing all the positive multiples of p and another class containing all the negative multiples of p. As such we shall not consider the action of P_2 on the whole of $\mathbb{Z}$ but instead on the set $X_p = \mathbb{Z} \setminus p\mathbb{Z}$. We now summarise the class of actions of P_2 we shall be studying: it is the branching function system $(X_p, \sigma_0, \sigma_1)$ where $\sigma_0(m) = 2m$ and $\sigma_1(m) = 2m + p$. It can of course be shown directly that this is a branching function system and so induces a strong representation of P_2 on X_p. If $x = x_1 \ldots x_r$ is a binary string we write σ_x for $\sigma_{x_1} \ldots \sigma_{x_r}$.

To analyze the actions (P_2, X_p) we shall use the following. Given p we are interested in the binary representations of the $p-1$ fractions:

$$\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, \frac{p-1}{p}.$$

These fractions will have infinite purely periodic binary representations because p is odd. We shall be interested in the non-conjugate periodic blocks that occur. By definition these are primitive binary strings. We shall show that there is a connection between such binary fractions and the orbits of P_2. Here are some motivating examples.

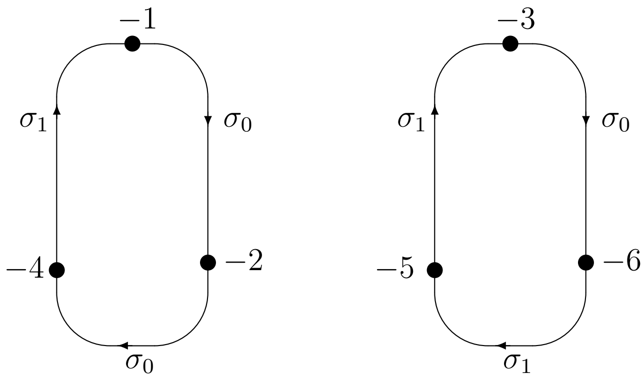
Example 2.7.1. *The case $p = 7$. In binary $\frac{1}{7} = 0.\overline{001}$. The other sevenths are as follows:*

Fraction	*Binary*	*Cycle pattern*
1/7	$0.\overline{001}$	*001*
2/7	$0.\overline{010}$	*010*
3/7	$0.\overline{011}$	*011*
4/7	$0.\overline{100}$	*100*
5/7	$0.\overline{101}$	*101*
6/7	$0.\overline{110}$	*110*

We say that $\frac{i}{7}$ and $\frac{j}{7}$ are equivalent *if their cycle patterns are conjugate. We see that there are two equivalence classes $\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$ and $\{\frac{3}{7}, \frac{5}{7}, \frac{6}{7}\}$ that correspond to the strings* 001 *and* 011*. But there is more information to be gleaned from this. Reverse the cycle pattern corresponding to the fraction $\frac{a}{7}$ to obtain a binary string x and then calculate $\sigma_x(-a)$. The results are tabulated below.*

$1/7 = 0.\overline{001}$	$\sigma_1\sigma_0\sigma_0(-1) = -1$
$2/7 = 0.\overline{010}$	$\sigma_0\sigma_1\sigma_0(-2) = -2$
$3/7 = 0.\overline{011}$	$\sigma_1\sigma_1\sigma_0(-3) = -3$
$4/7 = 0.\overline{100}$	$\sigma_0\sigma_0\sigma_1(-4) = -4$
$5/7 = 0.\overline{101}$	$\sigma_1\sigma_0\sigma_1(-5) = -5$
$6/7 = 0.\overline{110}$	$\sigma_0\sigma_1\sigma_1(-6) = -6$

The diagrams below show what is going on.



Comparing these results with the table on page 36 of [5], we see that our first table enables us to completely characterize the action of P_2 on X_7. We have two cycles that correspond to the conjugacy classes of the cycle patterns and the strings 100 *and* 110 *describe the nature of the two cycles.*

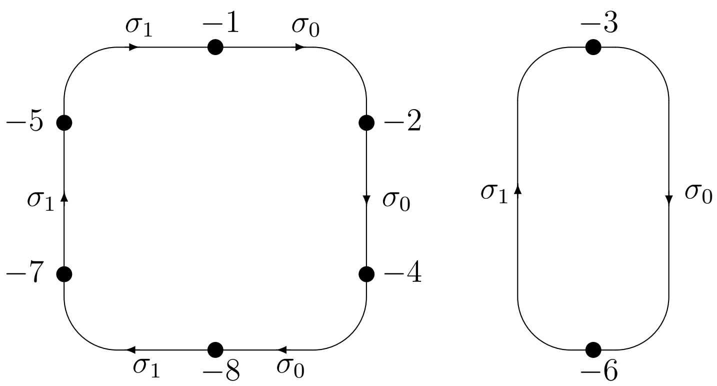
Example 2.7.2. The case $p = 9$.

Fraction	Binary
1/9	$0.\overline{000111}$
2/9	$0.\overline{001110}$
3/9	$0.\overline{01}$
4/9	$0.\overline{011100}$
5/9	$0.\overline{100011}$
6/9	$0.\overline{10}$
7/9	$0.\overline{011100}$
8/9	$0.\overline{111000}$

We see that there are two equivalence classes $\{\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}\}$ and $\{\frac{3}{9}, \frac{6}{9}\}$. The table of fixed points is as follows.

$1/9 = 0.\overline{000111}$	$\sigma_1\sigma_1\sigma_1\sigma_0\sigma_0\sigma_0(-1) = -1$
$2/9 = 0.\overline{001110}$	$\sigma_0\sigma_1\sigma_1\sigma_1\sigma_0\sigma_0(-2) = -2$
$3/9 = 0.\overline{01}$	$\sigma_1\sigma_0(-3) = -3$
$4/9 = 0.\overline{011100}$	$\sigma_0\sigma_0\sigma_1\sigma_1\sigma_1\sigma_0(-4) = -4$
$5/9 = 0.\overline{100011}$	$\sigma_1\sigma_0\sigma_0\sigma_0\sigma_1\sigma_1(-5) = -5$
$6/9 = 0.\overline{10}$	$\sigma_0\sigma_1(-6) = -6$
$7/9 = 0.\overline{110001}$	$\sigma_1\sigma_0\sigma_0\sigma_0\sigma_1\sigma_1(-7) = -7$
$8/9 = 0.\overline{111000}$	$\sigma_0\sigma_0\sigma_0\sigma_1\sigma_1\sigma_1(-8) = -8$

The same information can be presented diagrammatically.



Thus the action of classified by the strings 111000 and 10.

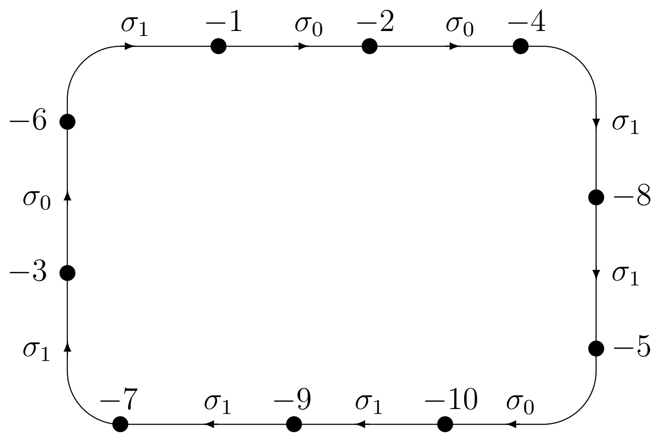
Example 2.7.3. *The case $p = 11$.*

Fraction	*Binary*
1/11	$0.\overline{0001011101}$
2/11	$0.\overline{0010111010}$
3/11	$0.\overline{0100010111}$
4/11	$0.\overline{0101110100}$
5/11	$0.\overline{0111010001}$
6/11	$0.\overline{1000101110}$
7/11	$0.\overline{1010001011}$
8/11	$0.\overline{1011101000}$
9/11	$0.\overline{1101000101}$
10/11	$0.\overline{1110100010}$

We see that in this case the fractions form a single equivalence class. The table of fixed points is given by the following.

$1/11 = 0.\overline{0001011101}$	$\sigma_1\sigma_0\sigma_1\sigma_1\sigma_1\sigma_0\sigma_1\sigma_0\sigma_0\sigma_0(-1) = -1$
$2/11 = 0.\overline{0010111010}$	$\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1\sigma_1\sigma_0\sigma_1\sigma_0\sigma_0(-2) = -2$
$3/11 = 0.\overline{0100010111}$	$\sigma_1\sigma_1\sigma_1\sigma_0\sigma_1\sigma_0\sigma_0\sigma_0\sigma_1\sigma_0(-3) = -3$
$4/11 = 0.\overline{0101110100}$	$\sigma_0\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1\sigma_1\sigma_0\sigma_1\sigma_0(-4) = -4$
$5/11 = 0.\overline{0111010001}$	$\sigma_1\sigma_0\sigma_0\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1\sigma_1\sigma_0(-5) = -5$
$6/11 = 0.\overline{1000101110}$	$\sigma_0\sigma_1\sigma_1\sigma_1\sigma_0\sigma_1\sigma_0\sigma_0\sigma_0\sigma_1(-6) = -6$
$7/11 = 0.\overline{1010001011}$	$\sigma_1\sigma_1\sigma_0\sigma_1\sigma_0\sigma_0\sigma_0\sigma_1\sigma_0\sigma_1(-7) = -7$
$8/11 = 0.\overline{1011101000}$	$\sigma_0\sigma_0\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1\sigma_1\sigma_0\sigma_1(-8) = -8$
$9/11 = 0.\overline{1101000101}$	$\sigma_1\sigma_0\sigma_1\sigma_0\sigma_0\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1(-9) = -9$
$10/11 = 0.\overline{1110100010}$	$\sigma_0\sigma_1\sigma_0\sigma_0\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1\sigma_1(-10) = -10$

The same information presented diagrammatically.



The action is therefore classified by the string 1011101000.

Example 2.7.4. *The case* $p = 15$.

Fraction	*Binary*
1/15	$0.\overline{0001}$
2/15	$0.\overline{0010}$
3/15	$0.\overline{0011}$
4/15	$0.\overline{0100}$
5/15	$0.\overline{01}$
6/15	$0.\overline{0110}$
7/15	$0.\overline{0111}$
8/15	$0.\overline{1000}$
9/15	$0.\overline{1001}$
10/15	$0.\overline{10}$
11/15	$0.\overline{1011}$
12/15	$0.\overline{1100}$
13/15	$0.\overline{1101}$
14/15	$0.\overline{1110}$

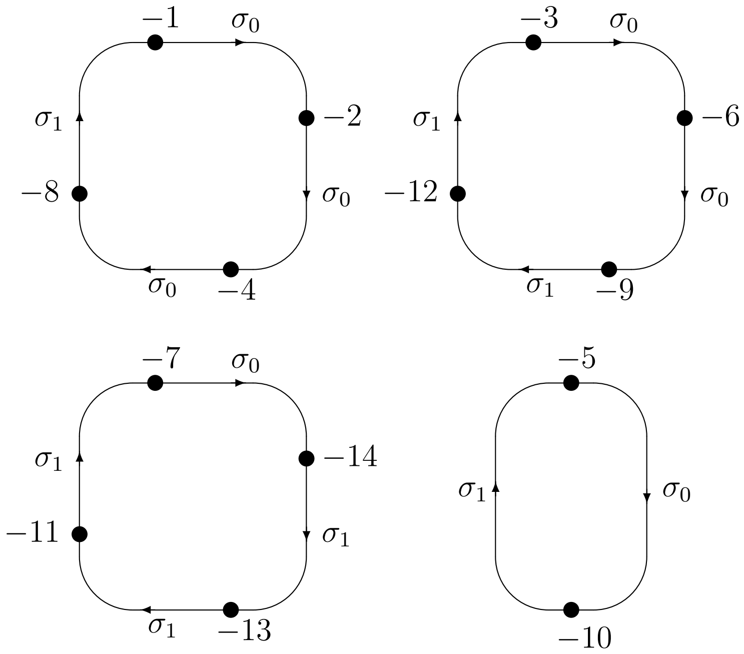
We see that there are four equivalence classes of fractions $\{\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}\}$, $\{\frac{3}{15}, \frac{6}{15}, \frac{9}{15}, \frac{12}{15}\}$,

$\{\frac{7}{15}, \frac{14}{15}, \frac{13}{15}, \frac{11}{15}\}$, *and* $\{\frac{5}{15}, \frac{10}{15}\}$.

The table of fixed points is given by the following.

$1/15 = 0.\overline{0001}$	$\sigma_1\sigma_0\sigma_0\sigma_0(-1) = -1$
$2/15 = 0.\overline{0010}$	$\sigma_0\sigma_1\sigma_0\sigma_0(-2) = -2$
$3/15 = 0.\overline{0011}$	$\sigma_1\sigma_1\sigma_0\sigma_0(-3) = -3$
$4/15 = 0.\overline{0100}$	$\sigma_0\sigma_0\sigma_1\sigma_0(-4) = -4$
$5/15 = 0.\overline{01}$	$\sigma_1\sigma_0(-5) = -5$
$6/15 = 0.\overline{0110}$	$\sigma_0\sigma_1\sigma_1\sigma_0(-6) = -6$
$7/15 = 0.\overline{0111}$	$\sigma_1\sigma_1\sigma_1\sigma_0(-7) = -7$
$8/15 = 0.\overline{1000}$	$\sigma_0\sigma_0\sigma_0\sigma_1(-8) = -8$
$9/15 = 0.\overline{1001}$	$\sigma_1\sigma_0\sigma_0\sigma_1(-9) = -9$
$10/15 = 0.\overline{10}$	$\sigma_0\sigma_1(-10) = -10$
$11/15 = 0.\overline{1011}$	$\sigma_1\sigma_1\sigma_0\sigma_1(-11) = -11$
$12/15 = 0.\overline{1100}$	$\sigma_0\sigma_0\sigma_1\sigma_1(-12) = -12$
$13/15 = 0.\overline{1101}$	$\sigma_1\sigma_0\sigma_1\sigma_1(-13) = -13$
$14/15 = 0.\overline{1110}$	$\sigma_0\sigma_1\sigma_1\sigma_1(-14) = -14$

This information can be presented diagrammatically as follows.



The action is therefore characterized by the strings 1000*,* 1100*,* 1110 *and* 10*.*

The above examples suggest that all the information we need to classify the action of P_2 on X_p is contained in the binary representations of the fractions

$$\frac{1}{p}, \ldots, \frac{p-1}{p}.$$

We now prove this.

Theorem 2.7.5. *The primitive strings that characterize the action of P_2 on X_p are precisely the reverses of the non-conjugate cycle patterns that occur in the binary representations of the fractions $\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, \frac{p-1}{p}$. In particular, the number of cycles of the action is equal to the number of non-conjugate cycle patterns.*

Proof. Let $0 < x < p$ for some odd number $p \geq 3$. Let $\frac{x}{p} = 0 \cdot \overline{\delta_1 \delta_2 \ldots \delta_n}$, where $\delta_i \in \{0, 1\}$. Recall that $\sigma_0(z) = 2z$ and $\sigma_1(z) = 2z + p$. We will show that $\sigma_{\delta_n} \ldots \sigma_{\delta_1}(-x) = -x$. It is immediate that

$$-x = -2^n x + p \sum_{i \in I} 2^{n-i}$$

where $I = \{i\colon 1 \leq i \leq n \text{ and } \delta_i = 1\}$. We show that $\sigma_{\delta_n} \dots \sigma_{\delta_1}(-x)$ is equal to the righthand side of the above equality. We prove the result by induction by n. Suppose that $n = 1$. Then if $\delta_1 = 0$ we have that $\sigma_{\delta_1}(-x) = -2x$ and if $\delta_1 = 1$ we have that $\sigma_{\delta_1}(-x) = -2x + p$. We see that the formula when $n = 1$ gives the same answer. Assume that the result holds for all strings of length n; we prove the result for strings of length $n + 1$. We therefore have to calculate $\sigma_{\delta_{n+1}\delta_n \dots \delta_1}(-x)$. By the induction hypothesis we have that

$$\sigma_{\delta_n} \dots \sigma_{\delta_1}(-x) = -2^n x + p \sum_{i \in I} 2^{n-i}$$

where $I = \{i\colon 1 \leq i \leq n \text{ and } \delta_i = 1\}$. There are now two cases to consider. If $\delta_{n+1} = 0$ then

$$\sigma_{\delta_{n+1}\delta_n \dots \delta_1}(-x) = -2^{n+1} x + p \sum_{i \in I} 2^{n+1-i},$$

whereas if $\delta_{n+1} = 1$ then

$$\sigma_{\delta_{n+1}\delta_n \dots \delta_1}(-x) = -2^{n+1} x + p \sum_{i \in I} 2^{n+1-i} + p.$$

Put $J = \{i\colon 1 \leq i \leq n + 1 \text{ and } \delta_i = 1\}$. Clearly $J = I \cup \{\delta_{n+1}\}$. Then it is easy to check that we have

$$\sigma_{\delta_{n+1}\delta_n \dots \delta_1}(-x) = -2^{n+1} x + p \sum_{i \in J} 2^{n+1-i},$$

as required.

For these actions we can calculate the number of orbits and what the stabilisers are from the binary representations of the corresponding fractions. Let p be a positive odd number and let P_2 act on X_p. If there are n non-conjugate cyclic patterns in the binary representations of the fractions $\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \dots, \frac{p-1}{p}$ then the action of P_2 on X_p will have n orbits. Let $\frac{x}{p} = 0.\overline{\delta_1 \dots \delta_n}$. Then $\sigma_{\delta_n \dots \delta_1}(-x) = -x$. Let y be the element of P_2 that represented by $\sigma_{\delta_n \dots \delta_1}$. As y is a string in only the positive generators of P_2 then for all $r, s \in \mathbb{N}$ the element $y^r y^{-s}$ also fixes $-x$. That is the action of P_2 on X_p is in part equivalent to the action of P_2 on the cosets of P_2^y.

$\square$

Two special cases of the above theorem are worth singling out. Our first is really a binary version of a well-known result about the representation of fractions as decimals. See [21] for background information.

Corollary 2.7.6. *The action of P_2 on X_p is transitive if and only if p is a prime and* 2 *is a primitive root modulo p.*

Proof. The action of P_2 on X_p is transitive if and only if all the cycle patterns in the binary representations of the fractions $\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, \frac{p-1}{p}$ are conjugate to each other. What follows is well-known [21] but we give the proof for the sake of completeness.

Suppose first that p is a prime and that 2 is a primitive root modulo p. Then $2, 2^2, \ldots, 2^{p-1}$ are all distinct modulo p. It follows that the fractions $\frac{1}{p}, \ldots, \frac{p-1}{p}$ have cycle patterns that are conjugate to that of $\frac{1}{p}$.

Now assume that the fractions $\frac{1}{p}, \ldots, \frac{p-1}{p}$ have cycle patterns that are conjugate to that of $\frac{1}{p}$. Suppose that p is not a prime. Then $p = mn$ where $m, n \neq 1$. Then $\frac{m}{p} = \frac{1}{n}$ where $n < p$. However the cycle pattern for $\frac{1}{n}$ will have length at most $n-1$ and so strictly less than $p-1$. This contradicts our assumption. It follows that p is a prime number. By assumption $\frac{1}{p} = 0 \cdot \overline{x}$ where x is a binary string of length $p-1$ and all the other binary fractions can be obtained by taking the fractional parts of $\frac{2^s}{p}$ where $s = 0, 1, \ldots, p-2$. It follows that 2 has order $p-1$ modulo p and so, since p is a prime, it follows that 2 is a primitive root modulo p. □

Example 2.7.7. *The first prime value of p is 3. Modulo 3 we have that $2^1 = 2$ and $2^2 = 1$. Thus* 2 *is a primitive root modulo 3 and therefore X_3 has one orbit.*

The next prime value of p is 5. Modulo 5 we have that $2^1 = 2, 2^2 = 4, 2^3 = 3, 2^4 = 1$. Thus 2 *is a primitive root modulo 5 and therefore X_5 has one orbit.*

Finally, the next prime value of p is 7. Modulo 7 *we have that $2^1 = 2, 2^2 = 4, 2^3 = 1$. Thus* 2 *is* not *a primitive root modulo 7 and so X_7 is not transitive.*

Our second case was described first as Proposition 8.4 of [5] but we give a different proof.

Corollary 2.7.8. *If $p = 2^q - 1$ for some integer $q \geq 2$, that is p is a so-called* Mersenne number, *then the action of P_2 on X_p has every possible cycle whose atomic weight is greater than unity and divides q. The number of cycles with atomic weight k is equal to the number of binary Lyndon words of length k.*

Proof. By Theorem 2.7.5 we have that the number of cycles equals the number of non-conjugate cyclic patterns that appear in the binary representations of $\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, \frac{p-1}{p}$. Therefore we need only show that every binary word of length $k \neq 1$ such that k divides q appears in the binary representations.

Let x be a string in the alphabet $\{0, 1\}$ of length exactly q that is neither 0^q nor 1^q. There are $2^q - 2$ such strings. Then the binary number $0 \cdot \overline{x}$ represents a decimal number r such that

$$\frac{1}{2^q - 1} \leq r \leq \frac{2^q - 2}{2^q - 1}.$$

On the other hand if x and y are two distinct such strings then the binary numbers $0 \cdot \overline{x}$ and $0 \cdot \overline{y}$ are also distinct. If $x \neq 0^q, 1^q$ is a string over the alphabet $0+1$ of length q then it can be written as $x = y^l$ for some natural number l where y is primitive of length k, $k \neq 1$, and $k \mid q$. All possible such strings y arise. We conclude that the binary Lyndon words classify the conjugacy classes of such words from the theory of primitive strings. □

The number of pairwise non-conjugate primitive strings of length n over a two-letter alphabet is given by

$$l_n(2) = \frac{1}{n} \sum_{d \mid n} \mu(d) 2^{\frac{n}{d}},$$

where μ denotes the Möbius function. Below is a table of the first twelve of these numbers (see [4]).

n	1	2	3	4	5	6	7	8	9	10	11	12
$l_n(2)$	2	1	2	3	6	9	18	30	56	99	186	335

Example 2.7.9. *Consider the action of P_2 on X_{15}. The number 15 is a Mersenne number since $15 = 2^4 - 1$. The possible atomic weights that can occur are $2, 4$. From the above table, we see that there is 1 Lyndon word of length 2, and 3 Lyndon words of length 4. It follows that the action has 4 orbits or cycles with total atomic weight equal to $2 + 3 \times 4 = 14$. This, of course, agrees with our Example 2.7.4.*

Example 2.7.10. *Now consider the action of P_2 on X_{4095}. The number 4095 is a Mersenne number since $4095 = 2^{12} - 1$. The possible atomic weights that can occur are $2, 3, 4, 6, 12$. From our table above, we see that there is 1 Lyndon word of length 2, 2 Lyndon words of length 3, 3 Lyndon words of length 4, 9 Lyndon words of length 6, and 335 Lyndon words of length 12. It follows that the action has 350 orbits or cycles with total atomic weight equal to $1 \times 2 + 2 \times 3 + 3 \times 4 + 9 \times 6 + 335 \times 12 = 4094$. This agrees with the calculations carried out in Section 8.1 of [5].*

Chapter 3

Graph inverse semigroups

Graph inverse semigroups generalise the polycyclic inverse monoids and play an important role in the theory of C^*-algebras, outlined in [46]. The main aim of this chapter is to generalise most of the results of the previous chapter. The graph inverse monoid [36], a local submonoid of the graph inverse semigroup with extra conditions, is also studied. It can be considered as a stepping stone between the polycyclic monoids and the graph inverse semigroups.

The connection between wide inverse subsemigroups and right congruences on the underlying structure (this time on the free category of the graph) is formalised and expanded upon for both the graph inverse semigroup and the graph inverse monoid. The graph inverse semigroups are only congruence free when the graph is well behaved in some way. We discuss how the graph structure relates to the congruence structure when it is non-trivial.

The gauge inverse semigroup (on $\mathcal{G}$) $Q(\mathcal{G})$, the generalisation of G_n, is important again but in a different context. The strong representations of the graph inverse semigroups coincide with E-algebraic branching systems as defined in [16] (they use E to denote the underlying graph). These are a generalisation of branching function systems. We do not take this connection as far as we did with the polycyclic monoids.

Many of the results in this chapter, sections 3.5 and 3.6 in particular, will appear in a joint paper with Lawson. This paper will outline how the graph inverse semigroups are a special case of inverse semigroups formed from categories. The origins of this construction can be found in [38].

3.1 Basic properties

For each directed graph $\mathcal{G}$ there exists a free category $\mathcal{G}^*$. To generalise the polycyclic monoids we replace free monoids with free categories. This means replacing pairs of words with pairs of paths that have a common domain. If we look at the free monoid as a free category of an n-rose then we see that all words have the same domain. This is why we can add the common domain condition and still be generalising the polycyclic monoids. We need to generalise the idea of positive and negative generators. To do this we use the idea of an opposite graph $\mathcal{G}^{-1}$. But this doesn't solve the whole problem. Let x be an edge in $\mathcal{G}_1$, the edge set of $\mathcal{G}$, and x^{-1} the corresponding edge in $\mathcal{G}_1^{-1}$. We can extend the definition of $\mathbf{d}, \mathbf{r}$ on $\mathcal{G}_1$ to the opposite graph by saying $\mathbf{d}(x^{-1}) = \mathbf{r}(x)$ and $\mathbf{r}(x^{-1}) = \mathbf{d}(x)$ for all $x^{-1} \in \mathcal{G}_1^{-1}$ with corresponding edge $x \in \mathcal{G}_1$. In our generalised polycyclic monoid we need $x^{-1}x$ to cancel out in some way. We can't have one global identity as we want to consider paths. If $x^{-1}x = 1$ then for any edge y with $\mathbf{r}(y) \neq \mathbf{d}(x)$ we would have

$$y = 1 \cdot y = x^{-1}x \cdot y$$

and xy doesn't make sense when talking about paths.

The solution is to use the idea of an empty path on a vertex. For each vertex v in $\mathcal{G}_0$, the vertex set of $\mathcal{G}$, we have an empty path 1_v from v to v that behaves like an identity in a category. The idea being that 1_v is a right identity to paths from v, a left identity to paths into v. We further extend the definition of $\mathbf{d}, \mathbf{r}$ by letting $\mathbf{d}(1_v) = v = \mathbf{r}(1_v)$ for all $v \in \mathcal{G}_0$. We identify both the set of vertices and the set of identities with $\mathcal{G}_0$, this will not be as confusing as it first appears.

The *graph inverse semigroup* $P(\mathcal{G})$ is the semigroup with zero defined by the following presentation

$$\begin{aligned}P(\mathcal{G}) = \{\mathcal{G}_1 \cup \mathcal{G}_1^{-1} \cup \mathcal{G}_0 \quad : \quad & 1_{\mathbf{r}(x)}x = x = x1_{\mathbf{d}(x)} \ \forall x \in \mathcal{G}_1 \cup \mathcal{G}_1^{-1} \cup \mathcal{G}_0, \\ & xy = 0 \text{ if } \mathbf{d}(x) \neq \mathbf{r}(y) \ \forall x, y \in \mathcal{G}_1 \cup \mathcal{G}_1^{-1} \cup \mathcal{G}_0, \\ & x^{-1}x = 1_{\mathbf{d}(x)} \text{ and } x^{-1}y = 0 \text{ if } x \neq y \ \forall x, y \in \mathcal{G}_1\}.\end{aligned}$$

We shall now briefly discuss this presentation, This presentation is due to Paterson [46] (he does not require $x^{-1}x = 1_{\mathbf{d}(x)}$), where he proves that this is in fact an inverse semigroup. The first relation ensures that identities behave in the way we would expect. We shall now work through the three cases. If $x \in \mathcal{G}_1$ then everything is straight-forward. If $x \in \mathcal{G}_0$ then $x = 1_v$ for some vertex v and $\mathbf{d}(x) = v = \mathbf{r}(x)$. Then the relation says that $1_v 1_v = 1_v$. If $x \in \mathcal{G}_1^{-1}$ then $x = y^{-1}$ for some $y \in \mathcal{G}_1$ and $\mathbf{d}(x) = \mathbf{r}(y)$, $\mathbf{r}(x) = \mathbf{d}(y)$. Thus the relation says that the identities combine with edges in the opposite graph in the appropriate way.

The second relation restricts our attention to paths and not just arbitrary words of edges. There are nine cases to consider here. If $x, y \in \mathcal{G}_1$ then the relation says that combinations of edges that don't form paths are zero. Similarly, if $x, y \in \mathcal{G}_1^{-1}$ then the relation says that combinations of opposite edges that don't form paths in the opposite graph are zero. If x or y are identities that the relation is now straight-forward. If $x \in \mathcal{G}_1$, $y \in \mathcal{G}_1^{-1}$ then the relation says that we will restrict our attention to pairs of paths with common domain. If $y \in \mathcal{G}_1$, $x \in \mathcal{G}_1^{-1}$ then the relation actually becomes redundant in light of the final relation.

The third relation instructs us how to deal with 'cross' terms. For any edge $x \in \mathcal{G}_1$ if we multiply on the left by the corresponding edge from opposite graph x^{-1} the product is the appropriate identity. If we multiply x on the left by any other edge in the opposite graph then the product is 0. We can now picture elements geometrically as pairs of paths; one in $\mathcal{G}$ and one in $\mathcal{G}^{-1}$.

Lemma 3.1.1. *The non-zero elements of $P(\mathcal{G})$ are of the form uv^{-1} where u, v are* *paths (possibly identities) in $\mathcal{G}$ with common domain.*

Proof. We shall now sketch a proof. Let $a = a_1 \dots a_n$ be a non-zero word in $\mathcal{G}_1 \cup \mathcal{G}_1^{-1}$ (we need not consider words containing letters from $\mathcal{G}_0$ as these are identities that we would simply cancel out). As $a \neq 0$ we have $\mathbf{d}(a_i) = \mathbf{r}(a_{i+1})$ for $1 \leq i < n$. Assume $a_1 = x^{-1}$. Then a_2 is either x or y^{-1} for some y such that $\mathbf{d}(y) = \mathbf{r}(x)$. If $a_2 = x$ then we can cancel and look at $a' = a_3 \dots a_n$. If $a_2 \neq x$ then by repeated use of the previous argument we see a is just a path in $\mathcal{G}^{-1}$. By setting $u = 1_{\mathbf{r}(a)}, v = a^{-1}$ we are done. A dual argument says that if $a_n = x$ then a is a path in $\mathcal{G}$. This time we set $u = a, v = 1_{\mathbf{d}(a)}$. This shows that all paths in $\mathcal{G}_1$ and in $\mathcal{G}_1^{-1}$ are elements in $P(\mathcal{G})$. Let u, v be two paths in $\mathcal{G}_1$ such that $\mathbf{d}(u) = \mathbf{d}(v)$. Then $u = u1_{\mathbf{d}(u)}$ and $v^{-1} = 1_{\mathbf{d}(v)}v^{-1}$ are elements of $P(\mathcal{G})$ and so is their product

$$u1_{\mathbf{d}(u)} \cdot 1_{\mathbf{d}(v)}v^{-1} = uv^{-1}.$$

$\square$

With elements in this form multiplication works in the following way.

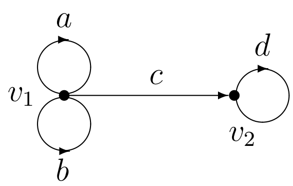
$$xy^{-1} \cdot uv^{-1} = \begin{cases} xzv^{-1} & \text{if } u = yz \text{ for some path } z \\ x\,(vz)^{-1} & \text{if } y = uz \text{ for some path } z \\ 0 & \text{otherwise.} \end{cases}$$

It is now clear that $P(\mathcal{G})$ is an inverse semigroup where $(xy^{-1})^{-1} = yx^{-1}$ and the idempotents are all the elements of the form xx^{-1}.

Example 3.1.2. *Let $\mathcal{G}$ be the graph with two vertices v_1, v_2 and four edges $a, b : v_1 \rightarrow v_1,\ c : v_1 \rightarrow v_2,\ d : v_2 \rightarrow v_2$ (see figure below). If we divide the paths in $\mathcal{G}$ into three types then things become simpler. Paths from v_1 to v_1 look like x, where $x \in \{a, b\}^*$ and $\epsilon = 1_{v_1}$. Paths from v_1 to v_2 look like $d^n cx$ where $n \in \mathbb{N}$, $d^0 = 1_{v_2}$ and x as above. Paths from v_2 to v_2 look like d^n where again $n \in \mathbb{N}$, $d^0 = 1_{v_2}$. Therefore*

$$P(\mathcal{G}) = \{d^{n_1}d^{-n_2} | n_1, n_2 \in \mathbb{N}\} \cup P_2 \cup \{d^n cx(d^m cy)^{-1} | n, m \in \mathbb{N}, x, y \in \{a, b\}^*\}$$

where P_2 is the polycyclic monoid on two generators.



Let xy^{-1} and uv^{-1} be non-zero elements of $P(\mathcal{G})$. Then

$$xy^{-1} \leq uv^{-1} \Leftrightarrow \exists p \in \mathcal{G}^* \text{ such that } x = up \text{ and } y = vp.$$

If $xy^{-1} \leq uv^{-1}$ or $uv^{-1} \leq xy^{-1}$ then we say xy^{-1} and uv^{-1} are *comparable*. An important property of the idempotents in $P(\mathcal{G})$ is the following: if $xx^{-1}yy^{-1} \neq 0$ then xx^{-1} and yy^{-1} are comparable.

Lemma 3.1.3. *In a graph inverse semigroup $P(\mathcal{G})$ Green's relations are defined in the following way:*

1. $xy^{-1} \,\mathcal{L}\, uv^{-1}$ *iff* $y = v$,
2. $xy^{-1} \,\mathcal{R}\, uv^{-1}$ *iff* $x = u$,
3. $xy^{-1} \,\mathcal{H}\, uv^{-1}$ *iff* $xy^{-1} = uv^{-1}$,
4. $xy^{-1} \,\mathcal{D}\, uv^{-1}$ *iff* $\mathbf{d}(y) = \mathbf{d}(u)$,
5. $xy^{-1} \,\mathcal{J}\, uv^{-1}$ *iff* $\mathbf{d}(y)$ *and* $\mathbf{d}(u)$ *are strongly connected (there exist paths from* $\mathbf{d}(y)$ *to* $\mathbf{d}(u)$ *and from* $\mathbf{d}(u)$ *to* $\mathbf{d}(y)$*).*

Proof. (1) Let $xy^{-1} \,\mathcal{L}\, uv^{-1}$. Then there exists some $rs^{-1} \in P(\mathcal{G})$ such that $rs^{-1}xy^{-1} = uv^{-1}$. It is clear that y must be a prefix of v. By symmetry we have $y = v$.

Let $xy^{-1}, uy^{-1} \in P(\mathcal{G})$. Then $\mathbf{d}(x) = \mathbf{d}(y) = \mathbf{d}(u)$ and so $xu^{-1}, ux^{-1} \in P(\mathcal{G})$. Giving $xu^{-1}uy^{-1} = xy^{-1}$, $ux^{-1}uy^{-1} = xy^{-1}$ and so $xy^{-1} \,\mathcal{L}\, uy^{-1}$.

A dual argument proves (2). Then (1) and (2) combine to prove (3).

(4) First we assume $xy^{-1}\,\mathcal{D}\,uv^{-1}$. Let $ab^{-1} \in P(\mathcal{G})$ such that $xy^{-1}\,\mathcal{L}\,ab^{-1}\,\mathcal{R}\,uv^{-1}$. As $xy^{-1}\,\mathcal{L}\,ab^{-1}$ and $ab^{-1}\,\mathcal{R}\,uv^{-1}$ we know $y = b$ and $u = a$. Thus $\mathbf{d}(y) = \mathbf{d}(u)$ as $ab^{-1} \in P(\mathcal{G})$.

Let $\mathbf{d}(y) = \mathbf{d}(u)$ for $xy^{-1}, uv^{-1} \in P(\mathcal{G})$. Then $uy^{-1} \in P(\mathcal{G})$ and $xy^{-1}\,\mathcal{L}\,uy^{-1}\,\mathcal{R}\,uv^{-1}$. Thus $xy^{-1}\,\mathcal{D}\,uv^{-1}$.

(5) Let $xy^{-1}\,\mathcal{J}\,uv^{-1}$, remembering that $\mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{d}(u) = \mathbf{d}(v)$. Then there exists $gh^{-1}, mn^{-1} \in P(\mathcal{G})$ such that

$$xy^{-1} = gh^{-1}uv^{-1}mn^{-1}.$$

As h and u are prefix comparable either $u = hz$ or $h = uz$. If $u = hz$, then

$$xy^{-1} = gh^{-1}uv^{-1}mn^{-1} = gh^{-1}(hz)v^{-1}mn^{-1} = (gz)v^{-1}mn^{-1}$$

and $x = gz$ or $y = n$. Either way $\mathbf{d}(x) = \mathbf{d}(u)$ and we are done. If $h = uz$, then

$$xy^{-1} = gh^{-1}uv^{-1}mn^{-1} = g(uz)^{-1}uv^{-1}mn^{-1} = g(vz)^{-1}mn^{-1}$$

and $x = g$ or $n = y$. If $x = g$ then z is a path from $\mathbf{d}(x)$ to $\mathbf{d}(u)$. If $n = y$ then $m = (vz)t$ and zt is a path from $\mathbf{d}(x)$ to $\mathbf{d}(u)$. A dual argument shows that $\mathbf{d}(y)$ and $\mathbf{d}(u)$ are strongly connected.

Now we prove the converse. Let $xy^{-1}, uv^{-1} \in P(\mathcal{G})$ such that $\mathbf{d}(y)$ and $\mathbf{d}(u)$ are strongly connected. Let r be a path from $\mathbf{d}(y)$ to $\mathbf{d}(u)$ and s be a path from $\mathbf{d}(u)$ to $\mathbf{d}(y)$. Then $x(ur)^{-1}, vry^{-1}, u(xs)^{-1}, ysv^{-1}$ are elements of $P(\mathcal{G})$. Thus

$$xy^{-1} = (x(ur)^{-1})(uv^{-1})(vry^{-1}) \text{ and } uv^{-1} = (u(xs)^{-1})(xy^{-1})(ysv^{-1}).$$

Therefore $xy^{-1}\,\mathcal{J}\,uv^{-1}$.

□

Lemma 3.1.4. *For every vertex $t \in \mathcal{G}$ the local submonoid at $1_t1_t^{-1}$ of $P(\mathcal{G})$ is defined in the following way*

$$1_t1_t^{-1}P(\mathcal{G})1_t1_t^{-1} = P_t(\mathcal{G}) = \{xy^{-1} \in P(\mathcal{G}) : \mathbf{r}(x) = t = \mathbf{r}(y)\} \cup \{0\}.$$

We call this the graph inverse monoid *at the vertex t.*

Proof. Let $P_t(\mathcal{G})$ denote the set $\{xy^{-1} \in P(\mathcal{G}) : \mathbf{r}(x) = t = \mathbf{r}(y)\} \cup \{0\}$. First we show $1_t1_t^{-1}P(\mathcal{G})1_t1_t^{-1} \subseteq P_t(\mathcal{G})$. Let $xy^{-1} \in 1_t1_t^{-1}P(\mathcal{G})1_t1_t^{-1}$. Then there exists some $uv^{-1} \in P(\mathcal{G})$ such that $xy^{-1} = 1_t1_t^{-1}(uv^{-1})1_t1_t^{-1}$. If $1_t1_t^{-1}uv^{-1}1_t1_t^{-1} = 0$ then $xy^{-1} \in P_t(\mathcal{G})$. If $1_t1_t^{-1}uv^{-1}1_t1_t^{-1} \neq 0$ then 1_t and u are prefix comparable, as are 1_t and v. Therefore $\mathbf{r}(u) = t = \mathbf{r}(v)$ and $1_t1_t^{-1}uv^{-1}1_t1_t^{-1} = uv^{-1}$. Thus $xy^{-1} = uv^{-1}$, so $\mathbf{r}(x) = t = \mathbf{r}(y)$ and $xy^{-1} \in P_t(\mathcal{G})$.

Now we show $P_t(\mathcal{G}) \subseteq 1_t1_t^{-1}P(\mathcal{G})1_t1_t^{-1}$. Let $xy^{-1} \in P_t(\mathcal{G})$. If $xy^{-1} = 0$ then $xy^{-1} = 1_t1_t^{-1}(0)1_t1_t^{-1}$ and $xy^{-1} \in 1_t1_t^{-1}P(\mathcal{G})1_t1_t^{-1}$. If $xy^{-1} \neq 0$ then $\mathbf{r}(x) = t = \mathbf{r}(y)$. Therefore $xy^{-1} = 1_t1_t^{-1}(xy^{-1})1_t1_t^{-1}$ and $xy^{-1} \in 1_t1_t^{-1}P(\mathcal{G})1_t1_t^{-1}$. Thus $P_t(\mathcal{G}) \subseteq 1_t1_t^{-1}P(\mathcal{G})1_t1_t^{-1}$.

□

This monoid is of particular interest if t is a root. A distinguished vertex t of a directed graph is a *root* if for every vertex v there exist a path from v to t. We say such a graph is *rooted*.

Proposition 3.1.5. *The graph inverse semigroup is an enlargement of the local submonoid at xx^{-1} if and only if $\mathbf{d}(x)$ is a root of $\mathcal{G}$.*

Proof. Let xx^{-1} be an idempotent such that $P(\mathcal{G})xx^{-1}P(\mathcal{G}) = P(\mathcal{G})$. Then $uv^{-1} \in P(\mathcal{G})xx^{-1}P(\mathcal{G})$ for all $uv^{-1} \in P(\mathcal{G})$. By the previous lemma there exist a path from $\mathbf{d}(u)$ to $\mathbf{d}(x)$. As uv^{-1} was arbitrary there exist a path from every vertex to $\mathbf{d}(x)$, in other words $\mathbf{d}(x)$ is a root.

Now let xx^{-1} be an idempotent such that $\mathbf{d}(x)$ is a root. Then for each $v \in \mathcal{G}_0$ there exist a path t from v to $\mathbf{d}(x)$. Let $rs^{-1} \in P(\mathcal{G})$ and let t be a path from $\mathbf{d}(r)$ to $\mathbf{d}(x)$. Then $r(xt)^{-1}, (xt)s^{-1} \in P(\mathcal{G})$ and

$$rs^{-1} = r(xt)^{-1}xx^{-1}(xt)s^{-1} \in P(\mathcal{G})xx^{-1}P(\mathcal{G}).$$

□

In [36] Lawson proved that $P_t(\mathcal{G})$ is congruence free if and only if $\mathcal{G}$ is strongly connected and the in-degree of each vertex is greater than or equal to two. We shall now prove that this result holds for the graph inverse semigroup too.

Theorem 3.1.6. *The graph inverse semigroup $P(\mathcal{G})$ is congruence free if and only if $\mathcal{G}$ is strongly connected and the in-degree of each vertex is greater than or equal to two.*

Proof. We will use Lawson's proof as a guide. On page 29 of [37] it is shown that an inverse semigroup is congruence free if it is fundamental, 0-simple and and its idempotents are 0-disjunctive (this is also shown in [47]). The graph inverse semigroups are always combinatorial and so fundamental. We require conditions for when they are 0-simple and when $E(P(\mathcal{G}))$ is 0-disjunctive. First we note that $P(\mathcal{G}) \setminus \{0\}$ and $\{0\}$ are the only $\mathcal{J}$-classes if and only if $\mathcal{G}$ is strongly connected by Lemma 3.1.3. Thus $\mathcal{G}$ being strongly connected is necessary and sufficient condition for $P(\mathcal{G})$ to be 0-simple.

Now we show that the in-degree of each vertex of $\mathcal{G}$ is greater then or equal to two if and only if $E(P(\mathcal{G}))$ is 0-disjunctive. Firstly we note that if $\mathcal{G}$ is strongly connected then the in-degree of each vertex is non-zero. Suppose that $E(P(\mathcal{G}))$ is 0-disjunctive. For each $v \in \mathcal{G}_0$ we know there exist an edge a to v. Let x be a path with $\mathbf{d}(x) = v$. Then $xx^{-1}, xa(xa)^{-1} \in E(P(\mathcal{G}))$ and $xx^{-1} \geq xa(xa)^{-1}$. By our assumption there exists a yy^{-1} such that $yy^{-1} \leq xx^{-1}$ and $yy^{-1}xa(xa)^{-1} = 0$. It follows that $y = xz$ and a is not a suffix of z. That is there are at least two edges entering v

Now suppose that the in-degree of each vertex is greater than or equal to two. Let $yy^{-1} < xx^{-1}$ for $yy^{-1}, xx^{-1} \in E(P(\mathcal{G}))$. Then $y = xp$ for some path $p = p_1 \ldots p_{|p|}$ with $p_i \in \mathcal{G}_1$ and $\mathbf{r}(p) = \mathbf{d}(x)$. By our assumption there exist an edge $a \neq p_1$ such that $\mathbf{r}(a) = \mathbf{d}(x)$. Thus $xa(xa)^{-1} \in E(P(\mathcal{G}))$, $xa(xa)^{-1} < xx^{-1}$ and $xa(xa)^{-1}yy^{-1} = 0$. $\square$

Lemma 3.1.7.

1. *$P(\mathcal{G})$ has the Dedekind height property,*

2. $P(\mathcal{G})$ is unambiguous.

Proof. (1) Let $xx^{-1} \in P(\mathcal{G})$. Then $yy^{-1} \geq xx^{-1}$ implies that y is a prefix of x. As the paths which determine idempotents are always finite there can only be finitely many idempotents above any given idempotent. More precisely $|(xx^{-1})^{\uparrow} \cap E(P(\mathcal{G}))| = |(xx^{-1})^{\uparrow}| = |x| + 1 \leq \infty$ for all idempotents $xx^{-1} \in P(\mathcal{G})$

(2) Let $xx^{-1}, yy^{-1} \in E(P(\mathcal{G})) \setminus \{0\}$ such that $xx^{-1}yy^{-1} \neq 0$. Then x and y are prefix comparable. If $x = yr$ then $xx^{-1} \leq yy^{-1}$. If $y = xr$ then $yy^{-1} \leq xx^{-1}$.

□

A *Perrot semigroup* is an inverse semigroup that is unambiguous and has the Dedekind height property.

Corollary 3.1.8. *The graph inverse semigroups are combinatorial Perrot semigroups.*

3.2 Congruences on $P(\mathcal{G})$

In this section we assume that $\mathcal{G}$ is not strongly connected so we can discuss the ideal structure of $P(\mathcal{G})$ when it isn't 0-simple. A subset of vertices $V \subseteq \mathcal{G}_0$ is called a *hereditary* set of vertices if $v \in V$ when there exist $x \in \mathcal{G}^*$ with $\mathbf{r}(x) = v$ and $\mathbf{d}(x) \in V$. This is the terminology that Paterson [46] uses. We need a dual condition. A subset of vertices $V \subseteq \mathcal{G}_0$ is called a *co-hereditary* set of vertices if $v \in V$ when there exist $x \in \mathcal{G}^*$ with $\mathbf{d}(x) = v$ and $\mathbf{r}(x) \in V$. That is the set of vertices forms a hereditary set in the opposite graph.

Lemma 3.2.1. *Let $\mathcal{G}$ be a graph. For any subset of vertices $U \subseteq \mathcal{G}_0$ there exist a smallest co-hereditary set V containing U.*

Proof. Let $U \subseteq \mathcal{G}_0$. We will show that

$$V = \{v \in G_0 : \text{there exist a path from } v \text{ to some } u \in U\}$$

is the smallest co-hereditary set containing U. Firstly we need to show that V is a co-hereditary set. Let $e \in V$ and let there be a path x to e from some other vertex f. As $e \in V$ there exist a path y from e to some $u \in U$. Because $\mathbf{d}(y) = e = \mathbf{r}(x)$ we have that yx is a valid path with $\mathbf{d}(yx) = \mathbf{d}(x) = f$ and $\mathbf{r}(yx) = \mathbf{r}(y) = u$. Therefore we have found a path from f to $u \in U$ and as such $f \in V$.

Let W be a co-hereditary set containing U. We will now show $V \subseteq W$. Let $v \in V$. Then there exists a path from v to some $u \in U$. As W is a co-hereditary set containing u and there exists a path from v to u we have that v is in W. $\square$

The maximal idempotents of $P(\mathcal{G})$ are of the form $1_v 1_v^{-1}$ where v is a vertex. We shall see that these are the elements that define the ideals.

Lemma 3.2.2. *$1_v 1_v^{-1} \leq_J 1_u 1_u^{-1}$ if and only if there exists a path from v to u.*

Proof. Let $1_v 1_v^{-1} \leq_J 1_u 1_u^{-1}$ in $P(\mathcal{G})$. Then there exists $gh^{-1}, pq^{-1} \in P(\mathcal{G})$ such that $1_v 1_v^{-1} = gh^{-1} 1_u 1_u^{-1} pq^{-1}$. So $g = q = 1_v$ and $h = p$ with $\mathbf{r}(p) = u$ and $\mathbf{d}(p) = v$. Thus there exist a path from v to u. We now show the converse. Let p be a path from v to u. Then $1_v p^{-1}, p 1_v^{-1} \in P(\mathcal{G})$ and $1_v 1_v^{-1} = 1_v p^{-1} 1_u 1_u^{-1} p 1_v^{-1}$ $\square$

The strongly connected components of a directed graph $\mathcal{G}$ are its maximal strongly connected subgraphs. Maximal in the sense that these strongly connected subgraphs are not proper subgraphs of any other strongly connected subgraph of $\mathcal{G}$. If $1_v 1_v^{-1} \leq_J 1_u 1_u^{-1}$ and $1_u 1_u^{-1} \leq_J 1_v 1_v^{-1}$ then $1_v 1_v^{-1} \mathcal{J} 1_u 1_u^{-1}$, so u and v are in the same strongly connected component.

Let $\mathcal{G}$ be a graph with strongly connected components $\{\mathcal{G}^i : i \in I\}$. We denote the underlying tree of the strongly connected components by $\mathcal{G}^\dagger$. Define $\mathcal{G}^i \leq_0 \mathcal{G}^j$ if and only if there exists $e \in \mathcal{G}^i_0$ and $f \in \mathcal{G}^j_0$ and a path from e to f.

Lemma 3.2.3. *With the above definitions $\leq_0$ is a partial order on $\mathcal{G}^\dagger$.*

Proof. We need only check antisymmetry. Suppose that $\mathcal{G}^i \leq_0 \mathcal{G}^j$ and $\mathcal{G}^j \leq_0 \mathcal{G}^i$. Then there are arrows in each direction linking vertices in each strongly connected component. This implies that every vertex of $\mathcal{G}^i$ is strongly connected to every vertex in $\mathcal{G}^j$. As the subgraphs $\mathcal{G}^i$ are maximal in terms of strong connectivity it follows that $\mathcal{G}^i = \mathcal{G}^j$, as required. □

With this definition of a partial order we have that the co-hereditary sets of $\mathcal{G}_0$ are in bijective correspondence with the order ideals of $(\mathcal{G}_0^\dagger, \leq_0)$.

Lemma 3.2.4. *Let I be an ideal of $P(\mathcal{G})$. All $xy^{-1} \in P(\mathcal{G})$ with $\mathbf{d}(x) = v$ are in I if $1_v 1_v^{-1} \in I$.*

Proof. Let $1_v 1_v^{-1} \in I$. If $\mathbf{d}(x) = v$ then $x1_v^{-1}, 1_v y^{-1} \in P(\mathcal{G})$. As I is an ideal we have

$$xy^{-1} = x1_v^{-1} 1_v 1_v^{-1} 1_v y^{-1} \in I.$$

□

Proposition 3.2.5. *The poset $P(\mathcal{G})/\mathcal{J}$ of principal ideals is order-isomorphic to the poset $(\mathcal{G}_0^\dagger, \leq_0)$.*

Proof. By lemma 3.2.4 we have that all principal ideals of $P(\mathcal{G})$ are generated by a maximal idempotent. Associate the strongly connected component of the graph containing v, denoted by $\mathcal{G}_v$, with $P(\mathcal{G})^1 1_v 1_v^{-1} P(\mathcal{G})^1$. Observe that, $P(\mathcal{G})^1 1_u 1_u^{-1} P(\mathcal{G})^1 = P(\mathcal{G})^1 1_v 1_v^{-1} P(\mathcal{G})^1$ if and only if u and v are strongly connected. Thus $\mathcal{G}_v = \mathcal{G}_u$. It follows that we have a well-defined function from $P(\mathcal{G})/\mathcal{J}$ to $\mathcal{G}_0^\dagger$. It is evident that this function is injective, since $\mathcal{G}_v = \mathcal{G}_u$ if and only if u and v are strongly connected, and immediate that it is surjective. It remains to show that we have defined an order-isomorphism. Suppose that $P(\mathcal{G})^1 1_u 1_u^{-1} P(\mathcal{G})^1 \subseteq P(\mathcal{G})^1 1_v 1_v^{-1} P(\mathcal{G})^1$. Then $1_u 1_u^{-1} \leq_J 1_v 1_v^{-1}$, and there is an path from u to v and so $\mathcal{G}_u \leq \mathcal{G}_v$. Conversely, if $\mathcal{G}_u \leq \mathcal{G}_v$ then there is an path from a vertex in $\mathcal{G}_u$ to a vertex in $\mathcal{G}_v$. But from the definition of strongly connected component this gives rise to a path from u to v and so we have that $1_u 1_u^{-1} \leq_J 1_v 1_v^{-1}$ and so $P(\mathcal{G})^1 1_u 1_u^{-1} P(\mathcal{G})^1 \subseteq P(\mathcal{G})^1 1_v 1_v^{-1} P(\mathcal{G})^1$. □

Corollary 3.2.6. *The ideals of $P(\mathcal{G})$ have the form*

$$I = \{xy^{-1} \in P(\mathcal{G}) : \mathbf{d}(x) \in V\},$$

where V is a co-hereditary set.

We have shown that co-hereditary sets of vertices determine the ideals of $P(\mathcal{G})$, and the $\mathcal{J}$-classes are determined by the strongly connected components.

Lemma 3.2.7. *Let $E(P(\mathcal{G}))$ be 0-disjunctive and let ρ be a non-trivial congruence on $P(\mathcal{G})$. Then there exist an $ab^{-1} \in P(\mathcal{G})$ such that $ab^{-1} \,\rho\, 0$.*

Proof. Let $xy^{-1} \,\rho\, uv^{-1}$ for $xy^{-1} \neq uv^{-1}$. Then either $x \neq u$, or $y \neq v$, or both. Without loss of generality we assume $x \neq u$. As $xx^{-1} \,\rho\, uu^{-1}$ we have $xx^{-1} \,\rho\, xx^{-1}uu^{-1} \,\rho\, uu^{-1}$. If $xx^{-1}uu^{-1} = 0$ we are done. If not then $u = xz$ or $x = uz$ for some path z. Assume $u = xz$. As $P(\mathcal{G})$ 0-disjunctive each vertex of $\mathcal{G}$ can not have in-degree equal to one, this is a corollary of Theorem 3.1.6. The vertex $\mathbf{d}(x)$ has in-degree of at least one as $\mathbf{r}(z) = \mathbf{d}(x)$. Thus we have an edge t to $\mathbf{d}(x)$ such that z and t are not prefix comparible. Then $xx^{-1}uu^{-1} = uu^{-1}$, $xx^{-1}(xt)(xt)^{-1} = (xt)(xt)^{-1}$, and $uu^{-1}(xt)(xt)^{-1} = 0$. That is

$$(xt)(xt)^{-1} = xx^{-1}(xt)(xt)^{-1} \,\rho\, uu^{-1}(xt)(xt)^{-1} = 0.$$

A similar argument holds if $x = uz$ or if we had $y \neq v$. $\square$

It is important to note here that we have not shown that either of the initial congruent elements are congruent to zero. This would be a much more powerful result from which we could deduce that the only congruences are Rees congruences. What we have shown is that no congruences on $P(\mathcal{G})$ are 0-restricted. We shall now see under which conditions non-Rees congruences exist on $P(\mathcal{G})$.

Let R be a relation on a semigroup S. Denote the smallest congruence on S containing R by $R^\sharp$. In [18] Howie shows $(a, b) \in R^\sharp$ if and only if either $a = b$ or for some $n \in \mathbb{N}$

there exist a sequence

$$a = t_1 \to t_2 \to \ldots \to t_n = b$$

where $t_i = xpy$, $t_{i+1} = xqy$ for $(p, q) \in R \cup R^{-1}$, $x, y \in S$. We call each pair (t_i, t_{i+1}) an *elementary R-transition.*

We say $u \in \mathcal{G}_0$ is a *bridging vertex* if the in-degree of u is greater than or equal to two and exactly one in-edge is from the strongly connected component containing u. These special vertices turn out to be very important to the congruence structure of $P(\mathcal{G})$.

Proposition 3.2.8. *If a graph $\mathcal{G}$ has bridging vertices then there exists non-Rees congruences on $P(\mathcal{G})$.*

Proof. Let u be a bridging vertex and let z be the only in-edge to u from the strongly connected component that contains u. Define a relation $R = \{(zz^{-1}, 1_u 1_u^{-1})\}$ and let $R^\sharp$ denote the smallest congruence containing R. We will show that if a sequence of elementary R-transitions has $t_i = zz^{-1}$ then $t_j \neq 0$ for all $j \geq i$ (we actually show that t_j is either zz^{-1} or $1_u 1_u^{-1}$ for all $j \geq i$). It follows that $(zz^{-1}, 0) \notin R^\sharp$ and as $(zz^{-1}, 1_u 1_u^{-1}) \in R^\sharp$ it is not a Rees congruence.

Let $t_1 \to t_2 \to \ldots \to t_n$ be a sequence of elementary R-transitions and let $t_i = zz^{-1}$. Then there exist $rs^{-1}, xy^{-1} \in P(\mathcal{G})$, $(pq^{-1}, mn^{-1}) \in R$ such that

$$zz^{-1} = t_i = rs^{-1}(pq^{-1})xy^{-1} \text{ and } t_{i+1} = rs^{-1}(mn^{-1})xy^{-1}.$$

Either $pq^{-1} = zz^{-1}, mn^{-1} = 1_u 1_u^{-1}$ or $pq^{-1} = 1_u 1_u^{-1}, mn^{-1} = zz^{-1}$.

Assume the former. Then there are a few options for rs^{-1}, xy^{-1}. As $rs^{-1}(pq^{-1})xy^{-1} = rs^{-1}(zz^{-1})xy^{-1} \neq 0$ we require s and z to be prefix comparable, also x and z. If $s = 1_u$ then $r = 1_u$ as $rs^{-1}(zz^{-1})xy^{-1} = zz^{-1}$, similarly if $x = 1_u$ then $y = 1_u$. In this case

$$t_{i+1} = rs^{-1}(mn^{-1})xy^{-1} = rs^{-1}(1_u 1_u^{-1})xy^{-1} = 1_u 1_u^{-1}(1_u 1_u^{-1})1_u 1_u^{-1} = 1_u 1_u^{-1}.$$

If $s \neq 1_u$ then as s, z are prefix comparable $s = zw$, where w may be an identity. Then

$$zz^{-1} = t_i = rs^{-1}(zz^{-1})xy^{-1} = r(zw)^{-1}(zz^{-1})xy^{-1} = r(zw)^{-1}xy^{-1}.$$

Thus

$$\begin{aligned} t_{i+1} &= rs^{-1}(mn^{-1})xy^{-1} = rs^{-1}(1_u 1_u^{-1})xy^{-1} \\ &= r(zw)^{-1}(1_u 1_u^{-1})xy^{-1} = r(zw)^{-1}xy^{-1} = zz^{-1}. \end{aligned}$$

Now assume $pq^{-1} = 1_u 1_u^{-1}, mn^{-1} = zz^{-1}$ As

$$t_i = zz^{-1} = rs^{-1}(pq^{-1})xy^{-1} = rs^{-1}(1_u 1_u^{-1})xy^{-1},$$

we have that $\mathbf{r}(s), \mathbf{r}(x) = u$. Thus $zz^{-1} = rs^{-1}xy^{-1}$. If $rs^{-1} = 1_u 1_u^{-1}$ then $xy^{-1} = zz^{-1}$ and

$$t_{i+1} = rs^{-1}(mn^{-1})xy^{-1} = rs^{-1}(zz^{-1})xy^{-1} = 1_u 1_u^{-1}(zz^{-1})zz^{-1} = zz^{-1}.$$

Similarly if $xy^{-1} = 1_u 1_u^{-1}$ then $rs^{-1} = zz^{-1}$ and

$$t_{i+1} = rs^{-1}(mn^{-1})xy^{-1} = rs^{-1}(zz^{-1})xy^{-1} = zz^{-1}(zz^{1})1_u 1_u^{-1} = zz^{-1}.$$

If $rs^{-1} \neq 1_u 1_u^{-1} \neq xy^{-1}$ then s and x are prefix comparable with $\mathbf{r}(s) = \mathbf{r}(x) = u$. As $rs^{-1}xy^{-1} = zz^{-1}$ we have $y = z$ if s is a prefix of x, similarly $r = z$ if x is a prefix of s. Without loss of generality we assume $s = xw$ and so $r = z$. Then $rs^{-1} = z(xw)^{-1}$ and $\mathbf{d}(s) = \mathbf{d}(z)$. As u is a bridging vertex and s starts in the same strongly connected component of u we have that z is a prefix of s. Therefore $s = xw = zgw$, where gw is a cycle at $\mathbf{d}(z)$. Thus

$$zz^{-1} = rs^{-1}xy^{-1} = z(zgw)^{-1}(zg)y^{-1}$$

and

$$\begin{aligned} t_{i+1} &= rs^{-1}(mn^{-1})xy^{-1} = rs^{-1}(zz^{-1})xy^{-1} \\ &= z(zgw)^{-1}(zz^{-1})(zg)y^{-1} = z(zgw)^{-1}(zg)y^{-1} = zz^{-1}. \end{aligned}$$

We have shown that if $t_i = zz^{-1}$ then t_{i+1} is either zz^{-1} or $1_u1_u^{-1}$. We will now see that if $t_i = 1_u1_u^{-1}$ then t_{i+1} is either zz^{-1} or $1_u1_u^{-1}$. Let $t_i = 1_u1_u^{-1}$. Then there exist $rs^{-1}, xy^{-1} \in P(\mathcal{G})$, $(pq^{-1}, mn^{-1}) \in R$ such that

$$1_u1_u^{-1} = t_i = rs^{-1}(pq^{-1})xy^{-1} \text{ and } t_{i+1} = rs^{-1}(mn^{-1})xy^{-1}.$$

Either $pq^{-1} = zz^{-1}, mn^{-1} = 1_u1_u^{-1}$ or $pq^{-1} = 1_u1_u^{-1}, mn^{-1} = zz^{-1}$. Assume the former. As $1_u1_u^{-1} = rs^{-1}(zz^{-1})xy^{-1}$ we have $r = 1_u = y$ and $s = zw_1$, $x = zw_2$ where w_1, w_2 are paths from u to $\mathbf{d}(z)$. Then

$$1_u1_u^{-1} = rs^{-1}(zz^{-1})xy^{-1} = 1_u(zw_1)^{-1}(zz^{-1})(zw_2)1_u^{-1} = 1_u(zw_1)^{-1}(zw_2)1_u^{-1},$$

thus $w_1 = w_2$. Therefore

$$\begin{aligned} t_{i+1} &= rs^{-1}(mn^{-1})xy^{-1} = rs^{-1}(1_u1_u^{-1})xy^{-1} \\ &= 1_u(zw_1)^{-1}(1_u1_u^{-1})(zw_1)1_u^{-1} = 1_u(zw_1)^{-1}(zw_1)1_u^{-1} = 1_u1_u^{-1}. \end{aligned}$$

Now assume $pq^{-1} = 1_u1_u^{-1}, mn^{-1} = zz^{-1}$. As

$$t_i = 1_u1_u^{-1} = rs^{-1}(pq^{-1})xy^{-1} = rs^{-1}(1_u1_u^{-1})xy^{-1},$$

we have that $\mathbf{r}(s), \mathbf{r}(x) = u$. Thus $1_u1_u^{-1} = rs^{-1}xy^{-1}$. This all gives that $r = 1_u = y$, and that $s = x$ is a cycle on u. If $s = x = 1_u$ then

$$t_{i+1} = rs^{-1}(mn^{-1})xy^{-1} = rs^{-1}(zz^{-1})xy^{-1} = 1_u1_u^{-1}(zz^{-1})1_u1_u^{-1} = zz^{-1}$$

Now let $s = x \neq 1_u$. As u is a bridging vertex and $s = x$ is a path in the same strongly connected component as u we have that z is a prefix and $s = x = zw$, where w is a path from u to $\mathbf{d}(z)$. Therefore

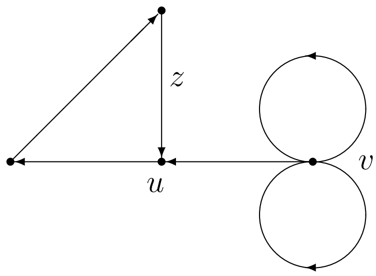
$$\begin{aligned} t_{i+1} &= rs^{-1}(mn^{-1})xy^{-1} = rs^{-1}(zz^{-1})xy^{-1} \\ &= 1_u(zw)^{-1}(zz^{-1})(zw)1_u^{-1} = 1_u(zw)^{-1}(zw)1_u^{-1} = 1_u1_u^{-1}. \end{aligned}$$

We can now use induction on the length of sequences of effective R-transitions that start with $t_1 = zz^{-1}$ to show that t_n is either zz^{-1} or $1_u1_u^{-1}$ for all $n \in \mathbb{N}$. If $n = 2$ and

$t_1 = zz^{-1}$ then $t_n = t_2$ is either zz^{-1} or $1_u 1_u^{-1}$. Now assume all sequences of effective R-transitions of length n with $t_1 = zz^{-1}$ have t_n equal to either zz^{-1} or $1_u 1_u^{-1}$. Let $zz^{-1} = t_1 \to t_2 \to \ldots \to t_{n+1}$ be a sequence of effective R-transitions. Then t_n equals either zz^{-1} or $1_u 1_u^{-1}$, therefore t_{n+1} is either zz^{-1} or $1_u 1_u^{-1}$. That is for all $n \in \mathbb{N}$ there exists no sequence $zz^{-1} = t_1 \to t_2 \to \ldots \to t_n = 0$. Therefore $(zz^{-1}, 0) \notin R^\sharp$ and it is not a Rees congruence. □

We will now see an example of a graph with a bridging vertex

Example 3.2.9. *In the graph below we have labeled two vertices, u, v and an edge z. Note that the set of just v is a co-hereditary set. As such $I_v = \{xy^{-1} \in P(\mathcal{G}) : \mathbf{d}(x) = v\}$ is an ideal and there is an associated Rees congruence ρ_v. We will construct the congruence ρ that is the smallest congruence containing $(zz^{-1}, 1_u 1_u^{-1})$.*



Formally

$$\rho = \rho_v \cup \{(xz(yz)^{-1}, xy^{-1}), (xy^{-1}, xz(yz)^{-1}) : x, y \text{ with } \mathbf{d}(x) = \mathbf{d}(y) = u\}.$$

We shall now show ρ is a congruence explicitly. Reflexivity and symmetry are straightforward. Let $rs^{-1} \, \rho \, pq^{-1}$ and $pq^{-1} \, \rho \, gh^{-1}$. Firstly, if $rs^{-1} \, \rho_v \, pq^{-1}$ and $pq^{-1} \, \rho_v \, gh^{-1}$ then $rs^{-1} \, \rho_v \, gh^{-1}$ and $rs^{-1} \, \rho \, gh^{-1}$.

Secondly, if $rs^{-1} \not\rho_v \, pq^{-1}$ then pq^{-1} is either of the form $xz(yz)^{-1}$ or xy^{-1} where $\mathbf{d}(x) = u$. If $pq^{-1} = xy^{-1}$ then $rs^{-1} = xz(yz)^{-1}$, and gh^{-1} equals xy^{-1} or $xz(yz)^{-1}$. For both possibilities $rs^{-1} \, \rho \, gh^{-1}$. The same argument holds if $pq^{-1} = xz(yz)^{-1}$. Thus ρ is transitive.

Now let $rs^{-1} \, \rho \, pq^{-1}$ and $gh^{-1} \in P(\mathcal{G})$. We will show ρ is a right congruence and a dual argument will show the left congruence property also holds. As ρ is symmetric there are three options;

1. $rs^{-1} = pq^{-1}$,
2. $rs^{-1}, pq^{-1} \in I_v$,
3. $rs^{-1} = xy^{-1}$ *and* $pq^{-1} = xz(yz)^{-1}$.

With the first two options $rs^{-1} \, \rho_v \, pq^{-1}$. Therefore $rs^{-1}gh^{-1} \, \rho_v \, pq^{-1}gh^{-1}$ as ρ_v is a right congruence, and so $rs^{-1}gh^{-1} \, \rho \, pq^{-1}gh^{-1}$.

The third option takes more work. Let $rs^{-1} = xy^{-1}$ and $pq^{-1} = xz(yz)^{-1}$. First note that if $xy^{-1}gh^{-1} = 0$ then $xz(yz)^{-1}gh^{-1} = 0$, which is fine. Also, if $xz(yz)^{-1}gh^{-1} = 0$ and $xy^{-1}gh^{-1} \neq 0$ then $\mathbf{d}(g) = v$ and $xy^{-1}gh^{-1} = gh^{-1}$. Giving

$$xz(yz)^{-1}gh^{-1} = 0 \, \rho \, gh^{-1} = xy^{-1}gh^{-1}$$

However as $\mathbf{d}(g) = v$ we already had $0 \, \rho_v \, gh^{-1}$, thus

$$rs^{-1}gh^{-1} \, \rho \, pq^{-1}gh^{-1}.$$

Now let $xy^{-1}gh^{-1} \neq 0 \neq xz(yz)^{-1}gh^{-1}$. So yz and g are prefix comparable. If yz is a prefix of g or they are equal then $g = yzt$, where t may be an identity, and

$$xy^{-1}gh^{-1} = xy^{-1}(yzt)h^{-1} = xzth^{-1} = xz((yz)^{-1}(yz))th^{-1} = xz(yz)^{-1}gh^{-1}.$$

So $rs^{-1}gh^{-1} \, \rho \, pq^{-1}gh^{-1}$ by reflexivity. If g is a proper prefix of yz then $yz = gt$ where t is not an identity. Thus $y = gt'$, where t' maybe the identity on u, and

$$xy^{-1}gh^{-1} = x(ht')^{-1} \ , \ xz(yz)^{-1}gh^{-1} = xz(ht'z)^{-1}.$$

But then $xy^{-1}gh^{-1} \, \rho \, xz(yz)^{-1}gh^{-1}$ as ht is a path with domain u.

We have shown ρ is a congruence. Now we have $xy^{-1} \, \rho \, (xz)(yz)^{-1} \not\rho\, 0$ and $xy^{-1} \neq (xz)(yz)^{-1}$. Therefore ρ is not a Rees congruence.

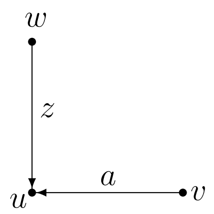
The important factors in this argument are that there exists a co-hereditary set containing just v and u is a bridging vertex. If we added another edge w from $\mathbf{d}(z)$ to u then

$$zz^{-1} = zw^{-1}(1_u 1_u^{-1})wz^{-1} \, \rho \, zw^{-1}(zz^{-1})wz^{-1} = 0$$

and ρ becomes a Rees congruence.

A vertex is *degenerate* if it is not strongly connected to any other vertex and if there are no loops on it. This is equivalent to saying that the strongly connected component containing u is just a vertex with no edges. The next example shows that degenerate vertices can give raise to non-Rees congruences but not that they always will.

Example 3.2.10. *The graph in this example is a simplification of the graph in the earlier example.*



Let $\mathcal{G}$ be the graph above. Then

$$P(\mathcal{G}) = \{1_w 1_w^{-1}, 1_w z^{-1}, z1_w^{-1}, zz^{-1}, 1_v 1_v^{-1}, 1_v a^{-1}, a1_v^{-1}, aa^{-1}, 1_u 1_u^{-1}, 0\}$$

as a set and has the following Cayley table as an inverse semigroup:

	$1_w1_w^{-1}$	1_wz^{-1}	$z1_w^{-1}$	zz^{-1}	$1_v1_v^{-1}$	1_va^{-1}	$a1_v^{-1}$	aa^{-1}	$1_u1_u^{-1}$	0
$1_w1_w^{-1}$	$1_w1_w^{-1}$	1_wz^{-1}	0	0	0	0	0	0	0	0
1_wz^{-1}	0	0	$1_w1_w^{-1}$	1_wz^{-1}	0	0	0	0	0	0
$z1_w^{-1}$	$z1_w^{-1}$	zz^{-1}	0	0	0	0	0	0	0	0
zz^{-1}	0	0	$z1_w^{-1}$	zz^{-1}	0	0	0	0	0	0
$1_v1_v^{-1}$	0	0	0	0	$1_v1_v^{-1}$	1_va^{-1}	0	0	0	0
1_va^{-1}	0	0	0	0	0	0	$1_v1_v^{-1}$	1_va^{-1}	0	0
$a1_v^{-1}$	0	0	0	0	$a1_v^{-1}$	aa^{-1}	0	0	0	0
aa^{-1}	0	0	0	0	0	0	$a1_v^{-1}$	aa^{-1}	0	0
$1_u1_u^{-1}$	0		$z1_w^{-1}$	zz^{-1}	0	0	$a1_v^{-1}$	aa^{-1}	$1_u1_u^{-1}$	0
0	0	0	0	0	0	0	0	0	0	0

Let ρ the smallest congruence containing $(zz^{-1}, 1_u1_u^{-1})$. The congruence classes of ρ are:

$$\rho = \{1_v1_v^{-1}, 1_va^{-1}, a1_v^{-1}, aa^{-1}, 0\}, \{1_w1_w^{-1}\}, \{1_wz^{-1}\}, \{z1_w^{-1}\}, \{zz^{-1}, 1_u1_u^{-1}\}$$

In the previous example we had two strongly connected components, here we have three. However if the strongly connected component containing u was anything more that just one vertex our construction would give just another Rees congruence. Lets add a loop p at u. Then

$$pp^{-1} = pp^{-1}1_u1_u^{-1}\,\rho\, pp^{-1}zz^{-1} = 0.$$

Similarly if there had been another edge from w to u.

We have seen that bridging vertices lead to non-Rees congruences. Now we will see where the existence of non-Rees congruences leads.

Lemma 3.2.11. *Let ρ be a non-Rees congruence on $P(\mathcal{G})$. Then there exist an edge z with $\mathbf{r}(z) = u$ such that $zz^{-1}\,\rho\,1_u1_u^{-1}\,\not\rho\,0$.*

Proof. Let ρ be a non-Rees congruence on $P(\mathcal{G})$. If all pairs of non-equal congruent elements are also congruent to zero then ρ would be a Rees congruence. That is there exists $xy^{-1}, rs^{-1} \in P(\mathcal{G})$ such that $xy^{-1} \neq rs^{-1}$, and $xy^{-1} \,\rho\, rs^{-1} \not\rho\, 0$. Thus $xx^{-1} \,\rho\, rr^{-1}$ and $yy^{-1} \,\rho\, ss^{-1}$. As $xy^{-1} \neq rs^{-1}$ and $P(\mathcal{G})$ is combinatorial either $xx^{-1} \neq rr^{-1}$ or $yy^{-1} \neq ss^{-1}$. Without loss of generality assume $xx^{-1} \neq rr^{-1}$. Then $xx^{-1}rr^{-1} \,\rho\, rr^{-1}rr^{-1} = rr^{-1}$. Note that if $xx^{-1}rr^{-1} \neq 0$ we would have a contradiction as then $0 \,\rho\, rr^{-1}$ and

$$0 = 0\; rs^{-1} \,\rho\, rr^{-1}rs^{-1} = rs^{-1}.$$

As $xx^{-1}rr^{-1} \neq 0$ and $P(\mathcal{G})$ is unambiguous we have that x and r are prefix comparable. Without loss of generality assume $x = rt$ (where t can not be an identity as we assumed $x \neq r$). Let $\bar{x}$ be the prefix of x of length $(|x| - 1)$, that is $x = \bar{x}z$ where z is an edge. As t is not an identity and z is only an edge $t = t'z$ (where t' may be an identity) and $x = \bar{x}z = rt'z$. Thus $\bar{x} = rt'$. Let $u = \mathbf{r}(z) = \mathbf{d}(\bar{x})$, then $1_u\bar{x}^{-1}, \bar{x}1_u^{-1} \in P(\mathcal{G})$ and

$$\begin{array}{rcl}
xx^{-1} & \rho & rr^{-1} \\[1.5em]
1_u\bar{x}^{-1}(xx^{-1})\bar{x}1_u^{-1} & \rho & 1_u\bar{x}^{-1}(rr^{-1})\bar{x}1_u^{-1} \\[1.5em]
1_u(\bar{x}^{-1}x)(x^{-1}\bar{x})1_u^{-1} & \rho & 1_u(\bar{x}^{-1}r)(r^{-1}\bar{x})1_u^{-1} \\[1.5em]
1_u zz^{-1}1_u^{-1} & \rho & 1_u t'^{-1}t'1_u^{-1} \\[1.5em]
zz^{-1} & \rho & 1_u1_u^{-1}.
\end{array}$$

□

The next result is a little wordy but we shall clear things up shortly.

Lemma 3.2.12. *Let ρ be a non-Rees congruence and let z be an edge such that $zz^{-1} \,\rho\, 1_u1_u^{-1} \not\rho\, 0$ where $\mathbf{r}(z) = u$. We denote the other edges into u by $a_1, \ldots, a_n$ and the smallest co-hereditary set containing all $\mathbf{d}(a_i)$ by V. Then $\mathbf{d}(z) \notin V$.*

Proof. Firstly we note that for all a_i we have $a_ia_i^{-1} \leq 1_u1_u^{-1}$ and $a_ia_i^{-1}zz^{-1} = 0$, that is

$$a_i a_i^{-1} = a_i a_i^{-1} 1_u 1_u^{-1} \, \rho \, a_i a_i^{-1} z z^{-1} = 0.$$

Such a set V exists by Lemma 3.2.1. We assume $\mathbf{d}(z) \in V$ for a contradiction. Then there exist a path t from $\mathbf{d}(z) = w$ to some $\mathbf{d}(a_i) = v$. Thus $z(a_i t)^{-1}, (a_i t)z^{-1} \in P(\mathcal{G})$ and

$$\begin{aligned}
a_i a_i \quad & \rho \quad 0 \\
z(a_i t)^{-1}(a_i a_i^{-1})(a_i t)z^{-1} \quad & \rho \quad z(a_i t)^{-1} 0 (a_i t) z^{-1} \\
z t^{-1}(a_i^{-1} a_i)(a_i^{-1} a_i) t z^{-1} \quad & \rho \quad 0 \\
z t^{-1} 1_v 1_v t z^{-1} \quad & \rho \quad 0 \\
z t^{-1} t z^{-1} \quad & \rho \quad 0 \\
z 1_w z^{-1} \quad & \rho \quad 0 \\
z z^{-1} \quad & \rho \quad 0.
\end{aligned}$$

A contradiction, therefore $\mathbf{d}(z) \notin V$. □

Proposition 3.2.13. *Let $\mathcal{G}$ be a graph without degenerate vertices. There exists a bridging vertex u with in-edge z from the same strongly connected component if and only if there exist a non-Rees congruence ρ with $zz^{-1} \, \rho \, 1_u 1_u^{-1} \, \not\rho \, 0$.*

Proof. By proposition 3.2.8 we know the existence of bridging vertices implies there are non-Rees congruences. In the proof we take a bridging vertex u with z the in-edge from the strongly connected component containing u and construct a non-Rees congruence ρ with $zz^{-1} \, \rho \, 1_u 1_u^{-1} \, \not\rho \, 0$. Therefore the forward implication is a direct corollary of that result.

Let ρ be a non-Rees congruence with $zz^{-1} \, \rho \, 1_u 1_u^{-1} \, \not\rho \, 0$. By lemma 3.2.12 we know that $\mathbf{d}(z) \notin V$ where $a_1, \ldots, a_n$ are all the other edges into u and V is the smallest co-hereditary set containing all $\mathbf{d}(a_i)$. Then $\mathbf{d}(z)$, and thus u, are in different strongly connected components to all the $\mathbf{d}(a_j)$. We now have two options. Either u and $\mathbf{d}(z)$ are in the same strongly connected component and therefore u is a bridging vertex. Or u is in a different strongly connected component to $\mathbf{d}(z)$, thus the strongly connected component containing u is just a vertex and therefore u is a degenerate vertex. The second option is a contradiction, therefore u must be a bridging vertex $\square$

Let S be a semigroup, $J(a) = S^1 a S^1$ the principal ideal generated by $a \in S$ and $\mathcal{J}_a$ the $\mathcal{J}$-class containing a. The *principal factor* generated by a is the Rees quotient

$$PF_a = \frac{J(a)}{(J(a) \setminus \mathcal{J}_a)}.$$

We will now construct the principal factors of $P(\mathcal{G})$. The principal ideals of $P(\mathcal{G})$ are defined by the co-hereditary sets and the $\mathcal{J}$-classes are defined by the strongly connected components. Given $xy^{-1} \in P(\mathcal{G})$ the principal factor of xy^{-1} has the following form:

$$PF_{xy^{-1}} = \{rs^{-1} \in P(\mathcal{G}) : \mathbf{d}(r) \text{ and } \mathbf{d}(x) \text{ are strongly connected}\} \cup \{0\},$$

that is as sets $PF_{xy^{-1}} = \mathcal{J}_{xy^{-1}}$. As $PF_{xy^{-1}} \cong PF_{rs^{-1}}$ if $\mathbf{d}(x)$ and $\mathbf{d}(r)$ are in the same strongly connected component $\mathcal{G}^i$ we shall abuse the notation and talk about $PF_{\mathcal{G}^i}$ the principal factor generated by $\mathcal{G}^i$. Let xy^{-1} be in a strongly connected component $\mathcal{G}^i$ such that there are no edges out of $\mathcal{G}^i$, that is $\mathcal{G}^i$ is at the 'bottom'. Then $P(\mathcal{G}^i) \cong PF_{\mathcal{G}^i}$. This only holds if the strongly connected component is at the 'bottom'.

Lemma 3.2.14. *Let $\mathcal{G}^i$ be a strongly connected component at the bottom of a graph which is not just an degenerate vertex. If the graph inverse semigroup of $\mathcal{G}^i$ is congruence free then $\mathcal{G}^i$ contains no bridging vertices.*

Proof. Let $P(\mathcal{G}^i)$ be congruence free. Then by Theorem 3.1.6 the in-degree of each vertex is greater than or equal to two. Therefore all vertices of $\mathcal{G}^i$ have two in-edges from $\mathcal{G}^i$ and thus can not be bridging vertices. $\square$

The other principal factors may not be isomorphic to (well-behaved) graph inverse semigroups however we can get around this.

Theorem 3.2.15. *Let $\mathcal{G}$ be a graph such that the in-degree of each vertex is not equal to one and $\mathcal{G}$ has no degenerate vertices. The semilattices of idempotents of the principal factors of $P(\mathcal{G})$ are all 0-disjunctive if and only the only congruences on $P(\mathcal{G})$ are Rees congruences.*

Proof. Let $\mathcal{G}$ be a graph without degenerate vertices and let the semilattices of idempotents of the principal factors of $P(\mathcal{G})$ all be 0-disjunctive. We will show that there can be no bridging vertices, by assumption there are no degenerate vertices and therefore the only congruences are Rees congruences.

Let $\mathcal{G}^i$ be a strongly connected component of $\mathcal{G}$ and let $\mathcal{G}^i_0$ denote the set of vertices in $\mathcal{G}^i$. As it is not just a degenerate vertex we know there is a vertex $v \in \mathcal{G}^i_0$ with at least one in-edge whose domain is also in $\mathcal{G}^i_0$. Let a be an edge into v with $\mathbf{d}(a) \in \mathcal{G}^i_0$, and let x be a path in $\mathcal{G}^*$ with $\mathbf{d}(x) = v$. Then $xx^{-1}, xa(xa)^{-1} \in E(PF_{\mathcal{G}^i})$ and $xx^{-1} \geq xa(xa)^{-1}$. As $E(PF_{\mathcal{G}^i})$ is 0-disjunctive there exists a $yy^{-1} \in E(PF_{\mathcal{G}^i})$ such that $yy^{-1} \leq xx^{-1}$ and $yy^{-1}xa(xa)^{-1} = 0$. That is x is a prefix of y but xa is not, so $y = xb_1 \dots b_n$ where the b_i are edges and $b_1 \neq a$. As $yy^{-1} \in E(PF_{\mathcal{G}^i})$ we have $\mathbf{d}(y) \in \mathcal{G}^i_0$. Thus all $\mathbf{d}(b_i) \in \mathcal{G}^i_0$. Therefore there exist a second edge, b_1, into v from $\mathcal{G}^i$ and it is not a bridging vertex.

Now we show the converse. Let $\mathcal{G}$ be a graph without degenerate vertices where the in-degree of each vertex is not equal to one and let all congruences on $P(\mathcal{G})$ be Rees congruences. As there are only Rees congruences there are no bridging vertices. We will now show that if there are no bridging vertices or degenerate vertices and the in-degree of each vertex is not equal to one then the semilattices of idempotents of the principal factors of $P(\mathcal{G})$ are all 0-disjunctive.

Let $\mathcal{G}^i$ be a strongly connected component. As $\mathcal{G}^i$ is not an degenerate vertex there exists $xx^{-1}, yy^{-1} \in E(PF_{\mathcal{G}^i})$ with $xx^{-1} \geq yy^{-1}$. That is $\mathbf{d}(x), \mathbf{d}(y) \in \mathcal{G}^i_0$ and x is a prefix of y so we can write $y = xa_1 \dots a_n$, where the a_i are edges. As $\mathbf{d}(y), \mathbf{d}(x)$ are in the same strongly connected component the edge a_1 and its domain must also be in $\mathcal{G}^i$. By our assumption about the in-degrees of the vertices there exist at least one more edge into $\mathbf{d}(x)$. If none of these in-edges originate in $\mathcal{G}^i_0$ then $\mathbf{d}(x)$ would be a bridging vertex, a contradiction. Thus there are at least two edges from $\mathcal{G}^i_0$ to $\mathbf{d}(x)$. Let $b \neq a_1$ be an edge from some vertex of $\mathcal{G}^i_0$ to $\mathbf{d}(x)$. Then $bx(bx)^{-1} \leq xx^{-1}$ and $yy^{-1}xb(xb)^{-1} = 0$. Thus $E(PF_{\mathcal{G}^i})$ is 0-disjunctive.

As $\mathcal{G}^i$ was an arbitrary strongly connected component we have shown the result holds for all semilattices of idempotents of the principal factors $\square$

3.3 Wide inverse subsemigroups

In this section we look to generalise the connection between relations on the free monoid and submonoids of the polycyclic monoid in Section 2.2 to a connection between relations on the free category and subsemigroups of the graph inverse semigroup. From here we will try to specialise to the graph inverse monoid, however this is not as straight forward as one might imagine. For all vertices t in a graph $\mathcal{G}$ we define $\mathcal{G}_t$ to be the subgraph of $\mathcal{G}^*$, the free category on $\mathcal{G}$, consisting of all arrows that terminate at t except the identity arrow. The free category of $\mathcal{G}_t$ is denoted by $\mathcal{G}^*_t$ and its arrow set is the union of the edge set of $\mathcal{G}_t$ and the required identities.

We define a *right congruence* (respectively left) ρ on a category C to be a equivalence relation of arrows such that for all $(x, y) \in \rho$:

- $\mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{r}(x) = \mathbf{r}(y)$,
- $xz \,\rho\, yz$ (resp. $zx \,\rho\, zy$) for all arrows z such that $\mathbf{r}(z) = \mathbf{d}(x) = \mathbf{d}(y)$ (resp. $\mathbf{d}(z) = \mathbf{r}(x) = \mathbf{r}(y)$).

We say a subsemigroup S of $P(\mathcal{G})$ has the *range property* if $xy^{-1} \in S$ implies $\mathbf{r}(x) =$

$\mathbf{r}(y)$.

Theorem 3.3.1. *There is a bijection between right congruences on $\mathcal{G}^*$ and the wide inverse subsemigroups of $P(\mathcal{G})$ with the range property.*

Proof. Let ρ be a right congruence on $\mathcal{G}^*$. Define

$$P_\rho = \{xy^{-1} \in P(\mathcal{G}) : x \,\rho\, y\} \cup \{0\}.$$

We call P_ρ the *subset corresponding* to ρ. We prove that P_ρ is a wide inverse subsemigroup of $P(\mathcal{G})$ with the range property. Let $xy^{-1} \in P_\rho$. Then $x \,\rho\, y$ and $\mathbf{r}(x) = \mathbf{r}(y)$ as ρ is a right congruence. Therefore P_ρ has the range property. Since ρ is reflexive for every path x we have $x \,\rho\, x$. Thus P_ρ contains all the idempotents of $P(\mathcal{G})$. If $xy^{-1} \in P_\rho$ then $x \,\rho\, y$ and so $y \,\rho\, x$, since ρ is symmetric, and thus $yx^{-1} \in P_\rho$. Therefore P_ρ is closed under inverses. To show that P_ρ is closed under products requires a little more work. Let $xy^{-1}, wz^{-1} \in P_\rho$. Suppose $xy^{-1}wz^{-1} \neq 0$. Then either $w = yp$ or $y = wp$. Let $w = yp$. Then $xy^{-1}wz^{-1} = (xp)z^{-1}$. Now $x \,\rho\, y$ and $yp = w \,\rho\, z$. Since ρ is a right congruence $xp \,\rho\, yp$. By transitivity $xp \,\rho\, z$ and $(xp)z^{-1} \in P_\rho$, as required. A similar argument shows that if $y = wp$ then $x(zp)^{-1} \in P_\rho$. We have shown that P_ρ is a wide inverse subsemigroup of $P(\mathcal{G})$ with the range property.

We now prove the converse. Let S be a wide inverse subsemigroup of $P(\mathcal{G})$ with the range property. Define a relation ρ on $\mathcal{G}^*$ by

$$x \,\rho\, y \Leftrightarrow xy^{-1} \in S.$$

We call ρ the *relation corresponding* to S. We will show that ρ is a right congruence on $\mathcal{G}^*$. Firstly we note that if $x \,\rho\, y$ then $\mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{r}(x) = \mathbf{r}(y)$ as $xy^{-1} \in S \subseteq P(\mathcal{G})$ and as S has the range property. Let x be an arrow in $\mathcal{G}^*$. Then as S is a wide inverse subsemigroup $xx^{-1} \in S$. It follows that ρ is reflexive. Let $xy^{-1} \in S$. Then yx^{-1} is also in S as it is closed under inverses. Thus ρ in symmetric. Let $x \,\rho\, y$ and $y \,\rho\, z$. Then $xy^{-1}, yz^{-1} \in S$. Since S is closed under products $xz^{-1} \in S$. Thus ρ is transitive. Let $x \,\rho\, y$, $p \in \mathcal{G}^*$ with $\mathbf{r}(p) = \mathbf{d}(x) = \mathbf{d}(y)$. Then $xy^{-1} \in S$ and $xp, yp \in \mathcal{G}^*$. Therefore

$(xp)(xp)^{-1} \in S$ as S is wide and $(xp)(xp)^{-1}xy^{-1} = (xp)(yp)^{-1} \in S$. Thus $xp\,\rho\,yp$ and ρ is a right congruence.

It is now clear that we can construct a bijection between the two collections. $\square$

We also want to consider the wide inverse subsemigroups without the range property. A *weak* right congruence on a free category has all the properties of a right congruences except we do not require the congruent elements to have a common range.

Corollary 3.3.2. *There is a bijection between weak right congruences on $\mathcal{G}^*$ and the wide inverse subsemigroups of $P(\mathcal{G})$.*

Specialising the previous result to the graph inverse monoid simplifies some aspects while creating new difficulties. As $\mathbf{r}(x) = \mathbf{r}(y) = t$ for all $xy^{-1} \in P_t(\mathcal{G})$ we can look at all wide inverse submonoids. The problem arises in the category not the monoid. Let S be a wide inverse submonoid of $P_t(\mathcal{G})$ and let ρ be the relation corresponding to S. Let x be an arrow in $\mathcal{G}^*$ such that $\mathbf{r}(x) \neq t$. Then $xx^{-1} \notin P_t(\mathcal{G})$, so $(x,x) \notin \rho$ and ρ is not reflexive. However all is not lost. While a wide inverse semigroup may not correspond to a right congruence it does correspond to a relation with right congruence like properties. We can then expand this relation to a family of right congruences in a way that preserves the wide inverse semigroup.

Lemma 3.3.3. *Let ρ be the relation corresponding to S a wide inverse submonoid of $P_t(\mathcal{G})$ and let ι denote the identity congruence. Then $\rho \cup \iota = \sigma$ is the smallest right congruence containing ρ*

Proof. Let S be a wide inverse submonoid of $P_t(\mathcal{G})$. Define a relation ρ on $\mathcal{G}^*$ by

$$x\,\rho\,y \Leftrightarrow xy^{-1} \in S.$$

We will show that $\sigma = \rho \cup \iota$ is a right congruence. By the previous theorem we know that ρ is symmetric and transitive, as is ι. It is straight-forward to show σ also possesses these properties. As ι is reflexive σ is an equivalence. Let $x\,\sigma\,y$. There are two cases to consider: either $x = y$, or $x \neq y$. If $x = y$ then $xp = yp$ for all appropriate p. Thus $xp \, \sigma \, yp$. If $x \neq y$ then $xy^{-1} \in S$. In this case $(xp)(yp)^{-1} \in S$ for all appropriate p. Thus $xp \, \sigma \, yp$ and σ is a right congruence.

Now we will show σ is minimal. Let κ be a right congruence containing ρ such that $\kappa \subseteq \sigma$. Assume $(x, y) \in \sigma \setminus \kappa$. As κ is reflexive $(z, z) \in \kappa$ for all arrows z. Thus $x \neq y$. However if $(x, y) \in \sigma$ and $x \neq y$ then $x \, \rho \, y$ and $x \, \kappa \, y$ as ρ is contained in κ. Therefore no (x, y) exists and $\kappa = \sigma$, thus σ is minimal. $\square$

In the special case of the polycyclic monoid the correspondence is one-to-one. Under which conditions on the graph does this hold? The arrows that do not terminate at t are 'ignored' by $P_t(\mathcal{G})$. In the polycyclic case all edges terminate at t and therefore all arrows in the free category terminate at t too. However for any digraph with two or more vertices the free category will have arrows that do not terminate at t (e.g. the identity for any vertex that isn't t). One approach to this problem is to alter the category in some property preserving way. At the beginning of the section we defined $\mathcal{G}_t$. As $\mathcal{G}_t$ is a graph we can form $P_t(\mathcal{G}_t)$.

Lemma 3.3.4. *Let $\mathcal{G}$ be a graph with root t. Then $P_t(\mathcal{G}_t) = P_t(\mathcal{G})$.*

Proof. Let $xy^{-1} \in P_t(\mathcal{G}_t)$. Then x, y are edges in $\mathcal{G}_t$ such that $\mathbf{r}(x) = \mathbf{r}(y) = t$ and $\mathbf{d}(x) = \mathbf{d}(y)$. Thus x, y are arrows in $\mathcal{G}^*$. The conditions on the domains and ranges of x, y requires that $xy^{-1} \in P_t(\mathcal{G})$. Therefore $P_t(\mathcal{G}_t) \subseteq P_t(\mathcal{G})$.

Let $xy^{-1} \in P_t(\mathcal{G})$. Then x, y are arrows in $\mathcal{G}^*$ such that $\mathbf{r}(x) = t = \mathbf{r}(y)$ and $\mathbf{d}(x) = \mathbf{d}(y)$. Thus $x, y \in \mathcal{G}_t^*$ and $xy^{-1} \in P_t(\mathcal{G}_t)$ because of the conditions on the ranges and domains. $\square$

From here we can strengthen our result.

Lemma 3.3.5. *There is a bijection between right congruences on $\mathcal{G}_t^*$ and the wide inverse submonoids of $P_t(\mathcal{G})$.*

Proof. Using the same argument as in Theorem 3.3.1 we have that every right congruence defines a wide inverse submonoid of $P_t(\mathcal{G})$. The condition that $\mathbf{r}(x) = \mathbf{r}(y)$ for all xy^{-1} in the submonoid is automatic as $xy^{-1} \in P_t(\mathcal{G})$. Now we work the other way. Let S be a wide inverse submonoid of $P_t(\mathcal{G})$. Define a relation ρ on $\mathcal{G}_t^*$ by

$$x \,\rho\, y \Leftrightarrow xy^{-1} \in S.$$

By Lemma 3.3.3 we know that $\sigma = \rho \cup \iota$ is the smallest right congruence corresponding to S when ι is the identity congruence. We will show that σ is the only right congruence corresponding to S. Let κ be another right congruence corresponding to S. Then $\sigma \subseteq \kappa$. Let $(x, y) \in \kappa \setminus \sigma$. As κ and σ corresponds to S we know that $xy^{-1} \notin S$. Thus x, y are arrows in $\mathcal{G}_t^*$ who have common domain and common range which is not equal to t. By the construction of $\mathcal{G}_t^*$ we have $x = y = \mathrm{id}_e$ for some object $e \neq t$ in $\mathcal{G}_t^*$. By reflexivity $(x, y) \in \sigma$, and as such no (x, y) can exist. Thus $\kappa = \sigma$ and the result is proved. $\square$

At this point we remember that in a category C the set of all arrows from e to f is called a hom-set and is denoted by fCe.

Theorem 3.3.6. *Let $\mathcal{G}$ be a graph with a distinguished vertex t. There is a bijection between right congruences on $\mathcal{G}^*$ and the wide inverse submonoids of $P_t(\mathcal{G})$ if and only if $|f\mathcal{G}^*e| \leq 1$ for all pairs of vertices $f \neq t$.*

Proof. For the the purposes of this proof we do not consider $S \times S$ to be a right congruence. This is so we may talk about a largest *meaningful* right congruence that contains the relation associated to a wide inverse submonoid. We begin by defining the largest right congruence κ which contains the relation associated to a wide inverse submonoid S. We then show that κ is equal to the smallest right congruence containing the relation associated to S if and only if the condition on the homsets holds.

Let S be a wide inverse submonoid of $P_t(\mathcal{G})$. Let κ be the relation defined by

$$(x, y) \in \kappa \Leftrightarrow \begin{cases} xy^{-1} \in S, \text{ or} \\ \mathbf{d}(x) = \mathbf{d}(y), \mathbf{r}(x) = \mathbf{r}(y) \neq t. \end{cases}$$

First we note that $x \,\kappa\, y$ implies $\mathbf{r}(x) = \mathbf{r}(y), \mathbf{d}(x) = \mathbf{d}(y)$. Let $x \in \mathcal{G}^*$. If $\mathbf{r}(x) \neq t$ then $x \,\kappa\, x$ by definition. If $\mathbf{r}(x) = t$ then $xx^{-1} \in S$ as it is wide so $x \,\kappa\, x$. Let $x \,\kappa\, y$. If $\mathbf{r}(x) = \mathbf{r}(y) \neq t$ then $y \,\kappa\, x$ by definition. If $\mathbf{r}(x) = \mathbf{r}(y) = t$ then $xy^{-1} \in S$ and so $yx^{-1} \in S$, thus $y \,\kappa\, x$. Let $x \,\kappa\, y$ and $y \,\kappa\, z$. If $\mathbf{r}(x) = \mathbf{r}(y) \neq t$ then $\mathbf{r}(y) = \mathbf{r}(z) \neq t$ and $x \,\kappa\, z$ by definition. If $\mathbf{r}(x) = \mathbf{r}(y) = t$ then $\mathbf{r}(y) = \mathbf{r}(z) = t$ and $xy^{-1}, yz^{-1} \in S$. Thus $xz^{-1} = xy^{-1}yz^{-1} \in S$ and so $x \,\kappa\, z$. Therefore κ is an equivalence.

Now let $x \,\kappa\, y$ and $p \in \mathcal{G}^*$ such that $\mathbf{r}(p) = \mathbf{d}(x) = \mathbf{d}(y)$. If $\mathbf{r}(x) = \mathbf{r}(y) \neq t$ then $\mathbf{r}(xp) = \mathbf{r}(yp) \neq t$, $\mathbf{d}(xp) = \mathbf{d}(yp)$ and $xp \,\kappa\, yp$ by definition. If $\mathbf{r}(x) = \mathbf{r}(y) = t$ then $xy^{-1} \in S$. As S is wide $(xp)(xp)^{-1} \in S$. Thus $(xp)(yp)^{-1} = (xp)(xp)^{-1}xy^{-1} \in S$ and $xp \,\kappa\, yp$. Therefore κ is a right congruence which contains the relation associated to S.

We now show κ is the largest right congruence containing the relation associated to S. Let ν be a right congruence containing the relation associated to S. Let $x \,\nu\, y$. If $\mathbf{r}(x) = \mathbf{r}(y) = t$ then $xy^{-1} \in S$ and $x \,\kappa\, y$. If $\mathbf{r}(x) = \mathbf{r}(y) \neq t$ then $x \,\kappa\, y$ by definition. Thus $\nu \subseteq \kappa$. We have defined the smallest, σ, and the largest, κ, right congruences associated to S. If $\sigma = \kappa$ then all right congruences associated to S will also be equal to both by a sandwich argument.

We will now show that $\kappa = \sigma$ iff $|f\mathcal{G}^*e| \leq 1$ for all pairs of vertices such that $f \neq t$. Let $\kappa = \sigma$. Let $x, y \in \mathcal{G}^*$ with $\mathbf{r}(x) = \mathbf{r}(y) \neq t$ and $\mathbf{d}(x) = \mathbf{d}(y)$. Then $x \,\kappa\, y$ by definition. Thus $x \,\sigma\, y$ and $x = y$. That is that there is at most one arrow from $\mathbf{d}(x)$ to $\mathbf{r}(x) \neq t$. This holds for all ordered pairs of vertices such that the second vertex is not t as x, y were arbitrary.

Let $|f\mathcal{G}^*e| \leq 1$ for all pairs of vertices such that $f \neq t$. For $x \,\kappa\, y$ with $\mathbf{r}(x) = \mathbf{r}(y) = t$ the two right congruences already agree and $\sigma \subseteq \kappa$. Let $x \,\kappa\, y$ for $\mathbf{r}(x) = \mathbf{r}(y) \neq t$.

Then $x = y$ and $x \, \sigma \, y$. Thus $\kappa \subseteq \sigma$ and the two are equal. $\square$

3.4 The gauge inverse subsemigroup

We shall now focus our attention on a certain wide inverse subsemigroup of the graph inverse semigroup. Define

$$Q(\mathcal{G}) = \{xy^{-1} \in P(\mathcal{G}) : |x| = |y|\} \cup \{0\}.$$

The set of pairs $x, y \in \mathcal{G}^*$ such that $|x| = |y|$ and $\mathbf{d}(x) = \mathbf{d}(y)$ is a weak right congruence on $\mathcal{G}^*$ and so $Q(\mathcal{G})$ is a wide inverse subsemigroup of $P(\mathcal{G})$. This subsemigroup is discussed in detail in the polycyclic case in [20] and in section 2.2. We shall now show directly that $Q(\mathcal{G})$ is an inverse subsemigroup.

Lemma 3.4.1. *The subset $Q(\mathcal{G})$ is an inverse subsemigroup of $P(\mathcal{G})$.*

Proof. Let $xy^{-1} \in Q(\mathcal{G})$. Then $|x| = |y|$ and $\mathbf{d}(x) = \mathbf{d}(y)$. Thus $yx^{-1} \in Q(\mathcal{G})$.

Let $xy^{-1}, uv^{-1} \in Q(\mathcal{G})$. If $xy^{-1}uv^{-1}$ is non-zero then y and u are prefix comparable. If $y = uz$ then $xy^{-1}uv^{-1} = x(vz)^{-1}$ and

$$|x| = |y| = |u| + |z| = |v| + |z|.$$

Similarly if $u = yz$. $\square$

We denote the *weight* of $xy^{-1} \in Q(\mathcal{G})$ by $\mu(xy^{-1}) = |x|$. Just as in the special case of the gauge inverse monoid μ is a pre-homomorphism.

In the paper by Ramos et al. [53] it is stated that $\mathcal{F}_A$ is a simple C^*-algebra if the adjacency matrix of the graph is aperiodic. The $\mathcal{F}_A$ algebra is the C^*-algebra analogue of $Q(\mathcal{G})$, *i.e. $\mathcal{F}_A$ is the linear span of all monomials of the form $s_x s_y^*$ with $|x| = |y|$.* We now look to find an analogue of this result for the gauge inverse submonoid. A matrix A is *aperiodic* if there exist some $m \in \mathbb{N}$ such that $(A^m)_{ij} \neq 0$ for all i, j. In the case that an adjacency matrix is aperiodic we have that $(A^n)_{ij} > 0$ for all $n \geq m$.

Lemma 3.4.2. *In the gauge inverse subsemigroup* $xy^{-1}\,\mathcal{J}\,uv^{-1}$ *if and only if* $\mathbf{d}(x) = \mathbf{d}(u)$ *and* $\mu(xy^{-1}) = \mu(uv^{-1})$.

Proof. Let $xy^{-1}\,\mathcal{J}\,uv^{-1}$ in $Q(\mathcal{G})$. Then there exist $pq^{-1}, rs^{-1} \in Q(\mathcal{G})$ such that

$$pq^{-1}xy^{-1}rs^{-1} = uv^{-1}.$$

Note that $\mu(uv^{-1}) \geq \mu(pq^{-1}), \mu(xy^{-1}), \mu(rs^{-1})$. A dual argument shows $\mu(xy^{-1}) \geq \mu(uv^{-1})$, thus $\mu(xy^{-1}) = \mu(uv^{-1})$. It follows that $\mu(pq^{-1}), \mu(rs^{-1}) \leq \mu(xy^{-1})$ so q is a prefix of x and r is a prefix of y. Thus $x = qq'$ and $y = rr'$ and

$$uv^{-1} = pq^{-1}xy^{-1}rs^{-1} = pq'(sr')^{-1}.$$

That is that $\mathbf{d}(u) = \mathbf{d}(pq') = \mathbf{d}(x)$.

Now let $xy^{-1}, uv^{-1} \in Q(\mathcal{G})$ such that $\mathbf{d}(x) = \mathbf{d}(u)$ and $\mu(xy^{-1}) = \mu(uv^{-1})$. Then $ux^{-1}, yv^{-1} \in Q(\mathcal{G})$ as are their inverses. Thus

$$uv^{-1} = ux^{-1}xy^{-1}yv^{-1} \text{ and } xy^{-1} = xu^{-1}uv^{-1}vy^{-1}.$$

Therefore $xy^{-1}\,\mathcal{J}\,uv^{-1}$.

□

Lemma 3.4.3. *In the gauge inverse subsemigroup* $\mathcal{D} = \mathcal{J}$.

Proof. As $\mathcal{D} \subseteq \mathcal{J}$ always holds we are only required to show $\mathcal{J} \subseteq \mathcal{D}$. With the above result about the $\mathcal{J}$ relation we need to prove that $xy^{-1}\,\mathcal{D}\,uv^{-1}$ if $\mu(xy^{-1}) = \mu(uv^{-1})$ and $\mathbf{d}(x) = \mathbf{d}(u)$.

Let $xy^{-1}, uv^{-1} \in Q(\mathcal{G})$ with $\mu(xy^{-1}) = \mu(uv^{-1})$ and $\mathbf{d}(x) = \mathbf{d}(u)$. Then $xv^{-1} \in Q(\mathcal{G})$ and

$$xy^{-1}\,\mathcal{R}\,xv^{-1}\,\mathcal{L}\,uv^{-1}.$$

Thus $xy^{-1}\,\mathcal{D}\,uv^{-1}$. □

Lemma 3.4.4. *In the gauge inverse subsemigroup $xy^{-1} \leq_J rs^{-1}$ if and only if there exists $z \in \mathcal{G}^*$ with $|z| = \mu(xy^{-1}) - \mu(rs^{-1})$ and $\mathbf{d}(z) = \mathbf{d}(x)$, $\mathbf{r}(z) = \mathbf{d}(r)$.*

Proof. We will start by showing there exists of an appropriate z if $xy^{-1} \leq_J rs^{-1}$. Let $xy^{-1} \leq_J rs^{-1}$ in $Q(\mathcal{G})$. Then $Q(\mathcal{G})^1 xy^{-1} Q(\mathcal{G})^1 \subseteq Q(\mathcal{G})^1 rs^{-1} Q(\mathcal{G})^1$. Thus $xy^{-1} \in Q(\mathcal{G})^1 rs^{-1} Q(\mathcal{G})^1$ and there exists $pq^{-1}, mn^{-1} \in Q(\mathcal{G})^1$ such that

$$xy^{-1} = pq^{-1}(rs^{-1})mn^{-1}.$$

Firstly we note that $\mu(rs^{-1}) \leq \mu(xy^{-1})$. We shall now find an appropriate path z. There are two options. If $\mu(pq^{-1}), \mu(mn^{-1}) \leq \mu(rs^{-1})$ then $r = qr', s = ms'$ and

$$\begin{aligned} xy^{-1} &= pq^{-1}(rs^{-1})mn^{-1} = pq^{-1}((qr')(ms')^{-1})mn^{-1} \\ &= (pr')(ns')^{-1}. \end{aligned}$$

Thus $\mathbf{d}(x) = \mathbf{d}(pr') = \mathbf{d}(r)$ and $\mu(xy^{-1}) = \mu((pr')(ns')^{-1}) = \mu(rs^{-1})$. Therefore $z = 1_{\mathbf{d}(x)}$ is a path with $|z| = \mu(xy^{-1}) - \mu(rs^{-1})$ and $\mathbf{d}(x) = \mathbf{d}(z)$, $\mathbf{r}(z) = \mathbf{d}(r)$.

If both $\mu(pq^{-1})$ and $\mu(mn^{-1})$ are not less than or equal to $\mu(rs^{-1})$ then one or both of them is greater than. Let $\mu(pq^{-1}) \geq \mu(mn^{-1})$ and $\mu(pq^{-1}) > \mu(rs^{-1})$. Then $q = rz$ and $sz = mq'$. Therefore

$$\begin{aligned} xy^{-1} &= pq^{-1}(rs^{-1})mn^{-1} = p(rz)^{-1}(rs^{-1})mn^{-1} \\ &= p(sz)^{-1}mn^{-1} = p(mq')^{-1}mn^{-1} = p(nq')^{-1}. \end{aligned}$$

So $\mu(xy^{-1}) = \mu(p(nq')^{-1}) = \mu(pq^{-1})$ and $\mathbf{d}(p) = \mathbf{d}(x)$. Thus

$$|z| = \mu(pq^{-1}) - \mu(rs^{-1}) = \mu(xy^{-1}) - \mu(rs^{-1})$$

and $\mathbf{d}(x) = \mathbf{d}(p) = \mathbf{d}(z)$, $\mathbf{r}(z) = \mathbf{d}(r)$. There is a symmetric argument when $\mu(mn^{-1}) \geq \mu(pq^{-1})$ and $\mu(mn^{-1}) > \mu(rs^{-1})$.

We now proof the converse. Let $xy^{-1}, rs^{-1} \in Q(\mathcal{G})$ such that there exists $z \in \mathcal{G}^*$ with $|z| = \mu(xy^{-1}) - \mu(rs^{-1})$ and $\mathbf{d}(z) = \mathbf{d}(x)$, $\mathbf{r}(z) = \mathbf{d}(r)$. Then rz, sz are paths in $\mathcal{G}^*$ of length $\mu(xy^{-1})$ and both have domain equal to the domain of x. Thus $x(rz)^{-1}, (sz)y^{-1} \in Q(\mathcal{G})^1$ and

$$xy^{-1} = x(rz)^{-1}(rs^{-1})(sz)y^{-1} \in Q(\mathcal{G})^1 rs^{-1} Q(\mathcal{G})^1.$$

Therefore $Q(\mathcal{G})^1 xy^{-1} Q(\mathcal{G})^1 \subseteq Q(\mathcal{G})^1 rs^{-1} Q(\mathcal{G})^1$ and $xy^{-1} \leq_J rs^{-1}$. □

A corollary of the above results is that $Q(\mathcal{G})$ is never 0-simple. We shall now approach the problem from the other direction. What properties does $Q(\mathcal{G})$ have if the adjacency matrix of $\mathcal{G}$ is aperiodic? Firstly we shall classify the principal ideals of $Q(\mathcal{G})$.

Lemma 3.4.5. *The principal ideals of $Q(\mathcal{G})$ take the form*

$$I_{v,n} = \{xy^{-1} \in Q(\mathcal{G}) : \exists z \in \mathcal{G}^* \text{ from } \mathbf{d}(x) \text{ to } v, |z| = \mu(xy^{-1}) - n\} \cup \{0\}$$

where $v \in \mathcal{G}_0$ and $n \in \mathbb{N}$.

Proof. Let $rs^{-1} \in Q(\mathcal{G})$ with $\mathbf{d}(r) = v$ and $\mu(rs^{-1}) = n$. We shall prove that $Q(\mathcal{G}) rs^{-1} Q(\mathcal{G}) = I_{v,n}$.

Let $xy^{-1} \in Q(\mathcal{G}) rs^{-1} Q(\mathcal{G})$. Then there exists $kj^{-1}, pq^{-1} \in Q(\mathcal{G})$ such that

$$xy^{-1} = kj^{-1} rs^{-1} pq^{-1}.$$

If $\mu(kj^{-1}), \mu(pq^{-1}) \leq \mu(rs^{-1})$ then $\mathbf{d}(x) = \mathbf{d}(r)$ and $\mu(xy^{-1}) = \mu(rs^{-1})$. Thus $xy^{-1} \in I_{v,n}$ by taking z equal to the identity on v. Now assume that $\mu(kj^{-1}) \geq \mu(pq^{-1})$ and $\mu(kj^{-1}) > \mu(rs^{-1})$. Then $\mathbf{d}(x) = \mathbf{d}(k)$ and $j = rz$. That is there exist a $z \in \mathcal{G}^*$ with $\mathbf{d}(x) = \mathbf{d}(z), v = \mathbf{r}(z), |z| = \mu(xy^{-1}) - n$. Thus $xy^{-1} \in I_{v,n}$. If $\mu(pq^{-1}) \geq \mu(kj^{-1})$ and $\mu(pq^{-1}) > \mu(rs^{-1})$ then $xy^{-1} \in I_{v,n}$ by a similar argument. Thus $Q(\mathcal{G}) rs^{-1} Q(\mathcal{G}) \subseteq I_{v,n}$.

Now let $xy^{-1} \in I_{v,n}$. Then $\exists z \in \mathcal{G}^*$ with $\mathbf{d}(x) = \mathbf{d}(z), v = \mathbf{r}(z), |z| = \mu(xy^{-1}) - n$. Therefore rz, sz are paths in $\mathcal{G}^*$ both with domain equal to $\mathbf{d}(x)$ and

$$|sz| = |rz| = |r| + |z| = n + |z| = \mu(xy^{-1}).$$

Thus $x(rz)^{-1}, (sz)y^{-1} \in Q(\mathcal{G})$ and

$$xy^{-1} = x(rz)^{-1}rs^{-1}(sz)y^{-1} \in Q(\mathcal{G})rs^{-1}Q(\mathcal{G}).$$

Therefore $I_{v,n} \subseteq Q(\mathcal{G})rs^{-1}Q(\mathcal{G})$. $\square$

Although this definition isn't very pretty we can still prove some very clean results when the adjacency matrix is aperiodic. In a semigroup S an subset I is *co-finite* if $S \setminus I$ is finite.

Theorem 3.4.6. *Let $\mathcal{G}$ be a strongly connected graph. The adjacency matrix of a graph $\mathcal{G}$ is aperiodic if and only if all non-zero ideals of $Q(\mathcal{G})$ are co-finite.*

Proof. Let the adjacency matrix A be aperiodic. Then there exist an m such that $A^m_{ij} > 0$ for all i, j. Therefore A has no zero rows and no zero columns (so $\mathcal{G}$ has no sinks or sources). It follows that $A^n_{ij} > 0$ for all i, j and $n \geq m$. Thus for $v_1, v_2 \in \mathcal{G}_0$, $n \geq m$ there exists $x, y \in \mathcal{G}^*$ such that $\mathbf{d}(x) = v_1 = \mathbf{r}(y)$, $\mathbf{r}(x) = v_2 = \mathbf{d}(y)$ and $|x| = n = |y|$.

Let I be a non-zero ideal of $Q(\mathcal{G})$ and let xy^{-1} be any element of I. As A is aperiodic for each $v \in \mathcal{G}_0$, $n \geq m$ there exists a path $z_{v,n}$ from v to $\mathbf{d}(x) = w$ with $|z| = n$. Let $rs^{-1} \in Q(\mathcal{G})$ with $\mu(rs^{-1}) \geq (\mu(xy^{-1}) + m)$ and $u = \mathbf{d}(r)$. If $l = (\mu(rs^{-1}) - \mu(xy^{-1}))$ then there exist paths $z_{u,l}, z_{u,l}$ as $l \geq m$, therefore $xz_{u,l}, yz_{u,l}$ are also valid paths. Thus $r(xz_{u,l})^{-1}, (yz_{u,l})s^{-1} \in Q(\mathcal{G})$ as $\mathbf{d}(s) = \mathbf{d}(r) = u = \mathbf{d}(xz_{u,l}) = \mathbf{d}(yz_{u,l})$ and $|xz_{u,l}| = |yz_{u,l}| = \mu(rs^{-1})$. Then

$$\begin{aligned} r(xz_{u,l})^{-1}(xy^{-1})(yz_{u,l})s^{-1} &= rz_{u,l}^{-1}(x^{-1}xy^{-1}y)z_{u,l}s^{-1} = rz_{u,l}^{-1}(1_w 1_w)z_{u,l}s^{-1} \\ &= r(z_{u,l}^{-1}z_{u,l})s^{-1} = r1_u s^{-1} = rs^{-1} \end{aligned}$$

and $rs^{-1} \in I$. Therefore all elements with weight greater than of equal to $(|x| + m)$ are in I, that is I is co-finite.

Let all the ideals of $Q(\mathcal{G})$ be co-finite. Therefore all the principal ideals are co-finite, in particular the principal ideals generated by maximal idempotents are. We denote the principal ideal associated with $1_v 1_v^{-1}$ by $I_{v,0}$. As $I_{v,0}$ is co-finite there exists an element of maximal weight in $Q(\mathcal{G})\backslash I$. That is there exists a *basement* of $I_{v,0}$, $m_v \in \mathbb{N}$, that is the smallest natural number such that if $\mu(xy^{-1}) \geq m_v$ then $xy^{-1} \in I$. By lemma 3.4.5 we know that if $xy^{-1} \in I_{v,0}$ then there exists a path z from $\mathbf{d}(x)$ to v with $|z| = \mu(xy^{-1})$. As $\mathcal{G}$ is strongly connected for each vertex u and $l \in \mathbb{N}$ there exist a path p from u of length l. That is for all $l \in \mathbb{N}$ and $u \in \mathcal{G}_0$ then there exists an element $pp^{-1} \in Q(\mathcal{G})$ with $\mathbf{d}(p) = u$ and $\mu(pp^{-1}) = l$. Let u be any vertex and $n \geq m_v$. Then there exist $pp^{-1} \in Q(\mathcal{G})$ with $\mathbf{d}(p) = u$ and $\mu(pp^{-1}) = n$ and as $I_{v,0}$ is co-finite with basement m_v we have $pp^{-1} \in I_{v,0}$. Thus there exist a path z from $u = \mathbf{d}(p)$ to v with $|z| = \mu(pp^{-1}) = n$. As u was arbitrary we have that for any vertex there exist a path to v for each length greater than m_v.

This holds for all $v \in \mathcal{G}_0$ and we define $m = \max_{v \in \mathcal{G}_0}\{m_v\}$. Then for each pair of vertices v_1, v_2 there exists a paths of length m from v_1 to v_2 and from v_2 to v_1 Therefore the adjacency matrix of $\mathcal{G}$ is aperiodic with period m. □

In semigroup with zero S an ideal $I \neq S$ is *essential* is it has non-zero intersection with every other non-zero ideal of S.

Corollary 3.4.7. *Let $\mathcal{G}$ be a strongly connected graph. All the ideals of $Q(\mathcal{G})$ are essential if they are all co-finite.*

Proof. Let I, J be two co-finite ideals of $Q(\mathcal{G})$. We will show that they have non-zero (infinite in fact) intersection. As I, J are arbitrary we have that all pairwise intersections of co-finite ideals are non-empty. Therefore all ideals are essential if they are all co-finite.

As $\mathcal{G}$ is strongly connected $Q(\mathcal{G})$ is infinite. Thus are I, J are infinite as they are co-finite. Let $Q(\mathcal{G}) \setminus I = I^c$. Then

$$J = J \cap Q(\mathcal{G}) = J \cap (I \cup I^c) = (J \cap I) \cup (J \cap I^c).$$

As I^c is finite so is $(J \cap I^c)$. Then $(J \cap I)$ must be infinite because J is. That is I and J have infinite, and thus non-zero, intersection if they are both co-finite. $\square$

3.5 An alternative construction of $P(\mathcal{G})$

The next two sections are joint work with Dr M. V. Lawson. These results are the back bone of a paper that is in preparation and still to be submitted to a journal. Although the author provided assistance this work is mainly that of Lawson. Many results are generalisations of result from earlier in this chapter.

Graph inverse semigroups are constructed as a special case of a general procedure for constructing inverse semigroups from left cancellative categories [25, 26, 28, 29] which has its origins in the work of Leech [38]. The left cancellative categories to which this procedure can be applied are required to satisfy the additional condition that any pair of arrows with a common range that can be completed to a commutative square have a pullback. There is no standard term for such categories so in this paper we shall call them *Leech categories*. The main goal of this section is to prove Theorem 3.5.24.

With each Leech category C, we may associate an inverse semigroup $\mathbf{S}(C)$ as follows; all proofs may be found in [28]. Put

$$U = \{(a, b) \in C \times C \colon \mathbf{d}(a) = \mathbf{d}(b)\}.$$

Define a relation $\sim$ on U as follows

$$(a, b) \sim (a', b') \Leftrightarrow (a, b) = (a', b')u$$

for some isomorphism $u \in C$. This is an equivalence relation on U and we denote the equivalence class containing (a, b) by $[a, b]$. The product $[a, b][c, d]$ is defined as follows: if there are no elements x and y such that $bx = cy$ then the product is defined to be zero; if such elements exist choose such a pair that is a pullback. The product is then defined to be $[ax, dy]$. Define $\mathbf{S}(C)$ to be the set of equivalence classes together with an additional element that plays the role of zero. Then the following can be deduced from [28].

Theorem 3.5.1. *Let C be a Leech category. Then $\mathbf{S}(C)$ is an inverse semigroup with zero.*

The inverse semigroup $\mathbf{S}(C)$ has the following important features: $[a,b]^{-1} = [b,a]$; the non-zero idempotents are the elements of the form $[a,a]$; the natural partial order is given by $[a,b] \leq [c,d]$ if and only if $(a,b) = (c,d)p$ for some arrow p.

Lemma 3.5.2. *Let C be a Leech category. Then the semilattice of idempotents of the inverse semigroup $\mathbf{S}(C)$ is order-isomorphic to the set of principal right ideals of C together with the emptyset under subset inclusion.*

Proof. The non-zero idempotents of $\mathbf{S}(C)$ are the elements of the form $[a,a]$. We have that $[a,a] \leq [b,b]$ if and only if $a = bp$ for some $p \in C$. Define a map from idempotents of $\mathbf{S}(C)$ to principal right ideals of C by $[a,a] \mapsto aC$ and maps the zero to the emptyset. This is well-defined because if $[a,a] = [a',a']$ then $a = a'u$ for some isomorphism u and we have that $aC = a'uC = a'C$. Next observe that $aC = bC$ if and only if $a = bu$ for some isomorphism u using the fact that C is left cancellative. Also $[a,a] \leq [b,b]$ if and only if $a = bp$ if and only if $aC \subseteq bC$. $\square$

Lemma 3.5.3. *Let C be a Leech category. Then in the inverse semigroup $\mathbf{S}(C)$, we have the following:*

1. *$[a,b] \, \mathcal{L} \, [c,d]$ if and only if $b = du$ for some isomorphism $u \in C$.*

2. *$[a,b] \, \mathcal{R} \, [c,d]$ if and only if $a = cu$ for some isomorphism $u \in C$.*

3. *$[a,b] \, \mathcal{D} \, [c,d]$ if and only if $\mathbf{d}(b)$ and $\mathbf{d}(d)$ are isomorphic.*

4. *$[a,b] \leq_J [c,d]$ if and only if there is an arrow in the category from $\mathbf{d}(b)$ to $\mathbf{d}(d)$.*

5. *$[a,b] \, \mathcal{J} \, [c,d]$ if and only if the identities $\mathbf{d}(b)$ and $\mathbf{d}(d)$ are strongly connected.*

Proof. The proofs of (1) and (2) are straightforward.

(3) Suppose that $\mathbf{d}(b)$ and $\mathbf{d}(d)$ are isomorphic where $u\colon \mathbf{d}(d) \to \mathbf{d}(b)$. Then we have $[a,b]\,\mathcal{R}\,[a,d]\,\mathcal{L}\,[c,d]$ and so $[a,b]\,\mathcal{D}\,[c,d]$. Conversely, suppose that $[a,b]\,\mathcal{D}\,[c,d]$. Then for some $[x,y]$ we have that $a = xu$ and $d = yv$ for isomorphisms u and v. Then $v^{-1}u$ is an isomorphism from $\mathbf{d}(b)$ to $\mathbf{d}(d)$.

(4) From [27], this is equivalent to $[b,b]\,\mathcal{D}\,[x,x] \leq [d,d]$ for some $x \in C$. Thus there is an isomorphism u from $\mathbf{d}(x)$ to $\mathbf{d}(b)$ and $x = dp$ for some $p \in C$. Thus pu^{-1} is a path from $\mathbf{d}(b)$ to $\mathbf{d}(d)$. Conversely, let p be a path from $\mathbf{d}(b)$ to $\mathbf{d}(d)$. Put $x = dp$, a well-defined element of C. Then $[x,x] \leq [d,d]$. But $\mathbf{d}(x) = \mathbf{d}(b)$. Thus $[b,b]\,\mathcal{D}\,[x,x]$.

The proof of (5) follows immediately from the proof of (4). $\square$

Define $[a,a]^{\circ} = [\mathbf{r}(a), \mathbf{r}(a)]$ and observe that $[a,a] \leq [a,a]^{\circ}$.

Lemma 3.5.4. *Let C be a Leech category. Then the inverse semigroup $\mathbf{S}(C)$ has maximal idempotents and each non-zero $\mathcal{D}$-class contains a maximal idempotent. Thus these semigroups are Leech semigroups.*

Proof. Let e be an identity of the category C. Then $[e,e]$ is an idempotent. Let $[e,e] \leq [x,x]$. Then $e = xp$ for some arrow p in C. Let $f = \mathbf{d}(x)$. Then $x = xpx$ and $p = pxp$. By left cancellation, $f = px$ and $e = xp$. Thus both x and p are isomorphisms and so $[e,e] = [x,x]$. Suppose that $[a,a] \leq [e,e], [f,f]$ where e and f are both identities in C. Then it follows immediately that $e = f = \mathbf{r}(a)$. We have shown that $\mathbf{S}(C)$ is an inverse semigroup with maximal idempotents. Finally, we show that each $\mathcal{D}$-class contains a maximal idempotent. Let $[a,a]$ be a non-zero idempotent. Observe that $[a,a]\mathcal{D}[\mathbf{d}(a), \mathbf{d}(a)]$. $\square$

The ideal structure of Leech semigroups can be described in terms of certain subsets of the set of maximal idempotents. The following result generalizes what may be found in [51].

Proposition 3.5.5. *Let S be a Leech semigroup. There is an order-isomorphism between the poset of ideals of S and the poset of co-hereditary subsets of the set of maximal idempotents.*

Proof. Let I be an ideal of S. Define $\mathsf{M}(I)$ to be the set of maximal idempotents in I. Let H be an co-hereditary subset of the set of maximal idempotents. Define $\mathsf{I}(H) = SHS$. We prove that these two maps set up an order-isomorphism between the poset of ideals and the poset of co-hereditary subsets under subset-inclusion.

We prove first that $\mathsf{M}(I)$ is a co-hereditary subset. Let $e \in \mathsf{M}(I)$ and let $f \leq_J e$ where f is maximal. Then since I is an ideal we have that $f \in I$. It is immediate from the definition that $\mathsf{I}(H) = SHS$ is an ideal. It is clear that both maps are order-preserving.

We now check what happens when we iterate these maps. We calculate first $\mathsf{I}(\mathsf{M}(I))$ where I is an ideal. Since $\mathsf{M}(I) \subseteq I$ we have that $\mathsf{I}(\mathsf{M}(I)) = S\mathsf{M}(I)S \subseteq SIS \subseteq I$. On the other hand, let $a \in I$. By assumption $a \,\mathcal{D}\, e$ where e is a maximal idempotent. Clearly $e \in I$ and so $e \in \mathsf{M}(I)$. Hence $SeS \subseteq \mathsf{I}(\mathsf{M}(I))$ but $a \in SeS$. It follows that $I \subseteq \mathsf{I}(\mathsf{M}(I))$. Thus $I = \mathsf{I}(\mathsf{M}(I))$.

Finally, we calculate $\mathsf{M}(\mathsf{I}(H))$ where H is a co-hereditary set. Let $e \in H$. Then $e \in SeS \subseteq \mathsf{I}(H)$ and so $e \in \mathsf{M}(\mathsf{I}(H))$. It follows that $H \subseteq \mathsf{M}(\mathsf{I}(H))$. Conversely, let $e \in \mathsf{M}(\mathsf{I}(H))$. Then $e \in \mathsf{I}(H)$. Thus $e \in SHS$. This means that $e \leq_J f$ where $f \in H$. But by assumption, H is an co-hereditary set and so $e \in H$. It follows that $\mathsf{M}(\mathsf{I}(H)) \subseteq H$. Thus $H = \mathsf{M}(\mathsf{I}(H))$. ☐

Let C be a category with strongly connected components $\{C^i : i \in I\}$. Define $C^i \leq C^j$ if and only if there exists $e \in C^i_0$ and $f \in C^j_0$ and an arrow $e \xrightarrow{x} f$.

Lemma 3.5.6. *With the above definitions $\leq$ is a partial order.*

Proof. We need only check antisymmetry. Suppose that $C^i \leq C^j$ and $C^j \leq C^i$. Then there are arrows in each direction linking identities in each strongly connected component. But this implies that every identity of C^i is strongly connected to an identity in C^j. But the sets C^i are supposed to be maximal sets with respect to the property that any two identities in them are strongly connected. It follows that $C^i = C^j$, as required. $\square$

Proposition 3.5.7. *The poset* $\mathbf{S}(C)/\mathcal{J}$ *of principal ideals is order-isomorphic to the poset of strongly connected components of the Leech category* C.

Proof. Put $S = \mathbf{S}(C)$. In an inverse semigroup, each principal ideal is generated by an idempotent, and in a Leech semigroup each principal ideal is generated by a maximal idempotent. Associate with $S[e,e]S$, where $[e,e]$ is a maximal idempotent, the strongly connected component of the category containing e, denoted by C_e. Observe that by Lemma 3.5.3(5), $S[e,e]S = S[f,f]S$ if and only if e and f are strongly connected. Thus $C_e = C_f$. It follows that we have a well-defined function from $S/\mathcal{J}$ to the set of strongly connected components of C. It is evident that this function is injective, since $C_e = C_f$ if and only if e and f are strongly connected, and immediate that it is surjective. It remains to show that we have defined an order-isomorphism. Suppose that $S[e,e]S \subseteq S[f,f]S$. Then by Lemma 3.5.3(4), there is an arrow from e to f and so $C_e \leq C_f$. Conversely, if $C_e \leq C_f$ then there is an arrow from an identity in C_e to an identity in C_f. But from the definition of strongly connected component this gives rise to an arrow from e to f and so by Lemma 3.5.3(4) we have that $e \leq_J f$ and so $S[e,e]S \leq S[f,f]S$. $\square$

Lemma 3.5.8. *Let* C' *be a strongly connected component of the Leech category* C. *Let* D *be the set of all arrows of* C *whose domains lie in* C'. *Then* D *is a Leech category.*

Proof. Let $a, b \in D$ be such that there are elements $x, y \in D$ such that $ax = by$. We prove that a and b have a pullback in D. Let u, v be a pullback which we know exists in C. We prove that in fact $u, v \in D$. By the definition of the pullback there is an arrow $p \in C$ such that $x = up$ and $y = vp$. Observe that $\mathbf{d}(u)$ can be reached from an identity in C' by p and connects to an identity in C' via u. Thus $\mathbf{d}(u) \in C'$ and so $u, v \in D$, as required. $\square$

We say that the category D above is *associated with* the strongly connected component C'. Given such a D, we can form the inverse semigroup $\mathbf{S}(D)$. By Lemma 3.5.3(5), this is nothing other than a principal factor of the inverse semigroup $\mathbf{S}(C)$ and every principal factor is isomorphic to an inverse semigroup constructed in this way.

The proof of the following is immediate.

Lemma 3.5.9. *Let C be a Leech category. Then $\mathbf{S}(C)$ is 0-simple if and only if C is strongly connected.*

We now turn to structural properties of the inverse semigroups $\mathbf{S}(C)$. A category C is said to be *right rigid* if $aC \cap bC \neq \emptyset$ implies that $aC \subseteq bC$ or $bC \subseteq aC$; this terminology is derived from Cohn [8].

Lemma 3.5.10. *Let C be a Leech category. Then in the inverse semigroup $S = \mathbf{S}(C)$, we have the following:*

1. *The semigroup $\mathbf{S}(C)$ is E^*-unitary if and only if the Leech category C is right cancellative.*
2. *The semigroup $\mathbf{S}(C)$ is combinatorial if and only if the invertible elements in each local monoid of C are identities.*
3. *Each $\mathcal{D}$-class of $\mathbf{S}(C)$ contains a unique maximal idempotent if and only if the only invertible elements are in the local monoids of C.*
4. *The groupoid of invertible elements in C is trivial if and only if $\mathbf{S}(C)$ is combinatorial and each $\mathcal{D}$-contains exactly one maximal idempotent.*
5. *The semigroup $\mathbf{S}(C)$ is unambiguous if and only if the category C is right rigid.*

6. *The inverse semigroup* $\mathbf{S}(C)$ *is completely semisimple if and only if for all identities* e *and* f *whenever* eCf *contains an isomorphism then every element of* eCf *is an isomorphism.*

7. *The inverse semigroup* $\mathbf{S}(C)$ *is* 0*-bisimple if and only if* C *is equivalent to a monoid.*

Proof. (1) Suppose that C is right cancellative. Let $[a,a] \leq [x,y]$. Then $a = xp$ and $a = yp$ for some arrow p. But then $xp = yp$. By right cancellation we have that $x = y$ and so $[x,y]$ is an idempotent as required. To prove the converse, suppose that S is E^*-unitary. Let $xp = yp$ in the category C. Put $a = xp = yp$. Then $[a,a] \leq [x,y]$. But $[a,a]$ is a non-zero idempotent. It follows by assumption that $[x,y]$ is an idempotent and so $x = y$, as required.

(2) Suppose that the only invertible elements in the local submonoids are identities. Let $[a,b]\,\mathcal{H}\,[c,c]$. Then there are isomorphisms u and v such that $au = c = bv$. It follows that u and v are isomorphisms that begin and end at the same identities. By assumption $u^{-1}v$ is an invertible element in a local monoid and so must be an identity. It follows that $u = v$. Thus $au = bu$ and u is an isomorphism and so $a = b$. It follows that each subgroup of S is trivial and so S is combinatorial. To prove the converse, suppose that S is combinatorial. Let u be an isomorphism from e to itself. Observe that $[e,u]\,\mathcal{H}\,[e,e]$. Thus since S is combinatorial, we have that $[e,u] = [e,e]$. Thus there is an isomorphism v such that $e = ev$ and $u = ev$. It follows $u = v$ and $v = e$. Thus u is an identity.

(3) Suppose first that the only isomorphisms in C are in the local submonoids. Let $[e,e]\,\mathcal{D}\,[f,f]$ where e and f are identities in C. By Lemma 3.5.3(3), e and f are isomorphic and so by our assumption are equal. Conversely, suppose that each $\mathcal{D}$-class contains a unique maximal idempotent. Let e and f be isomorphic identities. Then $[e,e]\mathcal{D}[f,f]$ by Lemma 3.5.3(3). By assumption $[e,e] = [f,f]$. Thus there is an isomorphism u such that $f = eu$. Thus $f = u$ and so $e = f$, as required.

(4) This is immediate by (2) and (3) above.

(5) This is immediate by Lemma 3.5.2.

(6) Suppose that each hom-set either doesn't contain any isomorphisms or every element is an isomorphism. Let $[a,a]\,\mathcal{D}\,[b,b]$ and $[a,a] \leq [b,b]$. Then there is an isomorphism u from $\mathbf{d}(a)$ to $\mathbf{d}(d)$ and $a = bp$ for some arrow p. Thus p is an arrow from $\mathbf{d}(a)$ to $\mathbf{d}(d)$. By assumption, p must be an isomorphism and so $[a,a] = [b,b]$, as required. We now prove the converse. Suppose that S is completely semisimple. Let a and u be arrows from f to e where u is an isomorphism. Then $[a,a]\,\mathcal{D}\,[e,e]$ and $[a,a] \leq [e,e]$. Thus $[a,a] = [e,e]$ and so there is an isomorphism v such that $a = ev$. It follows that $a = v$ is an isomorphism, as required.

(7) The inverse semigroup with zero S is 0-bisimple if and only if there is an isomorphism between any two identities of C, by Lemma 3.5.3(3), if and only if C is equivalent to a monoid. $\square$

When a Leech category has a trivial groupoid of invertible elements the equivalence class $[a,b]$ is just the singleton set $\{(a,b)\}$. It is convenient in this case to denote $[a,b]$ by ab^{-1}, which is to be understood to be just a notation.

The following result was inspired by [42].

Proposition 3.5.11. *Let C be a Leech category. Then the wide inverse subsemigroups of $\mathbf{S}(C)$ are in bijective correspondence with the weak right congruences on C.*

Proof. Let T be a wide inverse subsemigroup of $\mathbf{S}(C)$. Define

$$\rho_T = \{(a,b) \in C \times C \colon [a,b] \in T\}.$$

Observe that if $a \rho_T b$ then $\mathbf{d}(a) = \mathbf{d}(b)$ since $[a,b] \in \mathbf{S}(C)$. This relation is reflexive because it contains all idempotents, it is symmetric because it is closed under inverses, and it is transitive because it is closed under products. Suppose that $x \rho_T y$ and $\exists xz, yz$. By definition $[x,y] \in T$. Observe that $[xz, yz] \leq [x,y]$. But wide inverse subsemigroups are also order ideals. It follows that $[xz, yz] \in T$ and so $xz\rho_T yz$. Thus ρ_T is a weak right congruence.

Let ρ be a weak right congruence. Define

$$S_\rho = \{[a, b] \colon (a, b) \in \rho\} \cup \{0\}.$$

We prove that S_ρ is a wide inverse subsemigroup of $\mathbf{S}(C)$. It is straightforward to check that S_ρ contains all idempotents, is closed under inverses, and is an order ideal. By transitivity, it is closed under restricted products. Thus S_ρ is a wide inverse subsemigroup of $\mathbf{S}(C)$.

It is now routine to check that the maps $T \mapsto \rho_T$ and $\rho \mapsto S_\rho$ are mutually inverse and so set up a bijection between the two classes of structures. □

We now look at what happens to $\mathbf{S}(C)$ if C is a free category of a graph. Let G be a directed graph and G^* the free category it generates. A *left Rees category* is a left cancellative, right rigid category in which each principal right ideal is properly contained in only finitely many distinct principal right ideals. The proof of the following is straightforward.

Lemma 3.5.12. *Free categories are left Rees categories with trivial groupoids of invertible elements.*

Given a directed graph G, we define P_G to be the inverse semigroup $\mathbf{S}(G^*)$. The free category has no non-trivial invertible elements and so each equivalence class is denoted by xy^{-1}. Thus the non-zero elements of P_G are of the form uv^{-1} where u, v are paths in $\mathcal{G}$ with common domain.

It is clear that P_G are $P(\mathcal{G})$ are the same inverse semigroup. We remember that a poset is said to be pseudofinite if whenever $e > f$ there exists $g \in \hat{e}$ such that $e > g \geq f$, and for which the sets $\hat{e}$ are always finite.

Lemma 3.5.13.

1. *The inverse semigroup $P(\mathcal{G})$ has no 0-minimal idempotents if and only if the in-degree of each vertex is at least one.*

2. *The inverse semigroup $P(\mathcal{G})$ has a 0-disjunctive semilattice of idempotents if and only if the in-degree of each vertex is either 0 or at least 2.*

3. *The semilattice of idempotents of $P(\mathcal{G})$ is pseudofinite if and only if the in-degree of each vertex is finite.*

Proof. (1) Let e be a vertex with in-degree at least 1, and let b be an edge with range e. Let x be a path with source e, where we include the possibility that x is the empty path at e. Then $xb(xb)^{-1} \leq xx^{-1}$. It follows that if the in-degree of each vertex is at least 1 then there can be no 0-minimal idempotents. Now let e be a vertex with in-degree 0. Then $1_e 1_e^{-1}$ is a 0-minimal idempotent.

(2) Suppose that E is 0-disjunctive. Let v be any vertex. Let x be any path that starts at v including the empty path 1_v. Suppose that the in-degree of v is not zero. Then there is at least one edge w into v. It follows that $xw(xw)^{-1} \leq xx^{-1}$. By assumption, there exists $zz^{-1} \leq xx^{-1}$ such that zz^{-1} and $xw(xw)^{-1}$ are orthogonal. Now $z = xp$ for some non-empty path p. It follows that w is not a prefix of p and so there is at least one other edge coming into the vertex v.

Suppose now that the in-degree of each vertex is either zero or at least two. Let $yy^{-1} < xx^{-1}$ where $y = xp$ where the target of x is the vertex v. Since p is a non-empty path that starts at v it follows that there is at least one other edge w with target v that differs from the first edge of p. Thus $xw(xw)^{-1} \leq xx^{-1}$ and $xw(xw)^{-1}$ and yy^{-1} are orthogonal.

(3) Straightforward.

□

The following is now immediate from Lemmas 3.5.2, 3.5.9, 3.5.10 and 3.5.13(2).

Proposition 3.5.14.

1. *A graph inverse semigroup is a combinatorial Perrot semigroup with maximal idempotents such that each $\mathcal{D}$-class contains a unique maximal idempotent.*

2. *A graph inverse semigroup is completely semisimple if and only if the graph contains no non-trivial loops.*

An important class of rooted graphs are the Bratteli diagrams as defined on page 20 of [51]. Such a diagram gives rise to a rooted graph and so to an associated inverse *monoid*. These inverse monoids arise naturally in the construction of AF-algebras; see Proposition 2.12 of [51].

It is also worth noting, although we do not pursue this here, that the multiplier algebra of a C^*-algebra is reminiscent of the *translational hull* of a semigroup; see page 18 of [51] and [47].

3.5.1 Perrot semigroups

We shall begin by obtaining an abstract characterization of free categories in Theorem 3.5.20. First we recall some results that were proved in a much more general frame in [34].

Lemma 3.5.15. *Let C be a left cancellative category.*

1. *If $e = xy$ is an identity then x is invertible with inverse y.*

2. *We have that $aC = bC$ iff $a = bg$ where g is an invertible element.*

3. *$aC = eC$ for some identity e iff a is invertible.*

Proof. (1) We have that $\mathbf{r}(x) = e$ and $\mathbf{d}(y) = e$. Now

$$xyx = ex = x \text{ and } yxy = ye = y.$$

Thus by left cancellation, x is invertible with inverse y.

(2) Suppose that $aC = bC$. Then $a = bx$ and $b = ay$. Thus $a = ayx$ and so by left cancellation $\mathbf{d}(a) = yx$. Thus by (1) above, x is invertible. Conversely, suppose that $a = bg$ where g is invertible with inverse g^{-1}. Then $\mathbf{d}(b) = gg^{-1}$ and $\mathbf{d}(a) = g^{-1}g$. But $ag^{-1} = bgg^{-1} = b$, and so $aC = bC$.

(3) Suppose that $aC = eC$. Then by (2), we have that $a = eg$ for some invertible element g. Thus $a = g$ is invertible. Conversely, if a is invertible then $aC = aa^{-1}aC \subseteq aa^{-1}C \subseteq aC$. Thus $aC = aa^{-1}C$, as required. $\square$

Lemma 3.5.16. *Let S be a left cancellative category. Then the maximal principal right ideals are those generated by identities.*

Proof. Observe that for any element a we have that $aC \subseteq \mathbf{r}(a)C$. It follows that if aC is maximal then $aC = \mathbf{r}(a)C$. By Lemma 3.5.15(3), this implies that a is invertible. Conversely, let e be an identity. Suppose that $eC \subseteq aC$. Then $e = ab$ for some b and so $\mathbf{r}(a) = e$. Thus $eC \subseteq aC \subseteq eC$. Hence $eC = aC$, and so eC is maximal. $\square$

The proof of the following is immediate by the above result.

Lemma 3.5.17. *Let S be a left cancellative right rigid category. Then two maximal principal right ideals either have an empty intersection or are equal.*

An element $a \in C$ is said to be *indecomposable* iff $a = bc$ implies that either a or b is invertible. A principal right ideal aC is said to be *submaximal* if $aC \neq \mathbf{r}(a)C$ and there are no proper principal right ideals between aC and $\mathbf{r}(a)S$.

Lemma 3.5.18. *Let C be a left cancellative category. The non-invertible element a is indecomposable iff aC is submaximal.*

Proof. Suppose that a is indecomposable, and that $aC \subseteq bC$. Then $a = bc$. By assumption either b or c is invertible. If c is invertible then $aC = bcC = bC$ by

Lemma 3.5.15. If b is invertible then bC is a maximal principal right ideal by Lemmas 3.5.15 and 3.5.16. Thus aC is submaximal.

Conversely, suppose that aC is submaximal. Let $a = bc$. Then $aC = bcC \subseteq bC$. By assumption either $aC = bC$ or b is invertible. If the latter we are done; suppose the former. Then $a = bg$ where g is invertible by Lemma 3.5.15. By left cancellation $c = g$ and so c is invertible. It follows that a is indecomposable. □

Lemma 3.5.19. *Let C be a left cancellative category. The set of invertible elements is trivial iff for all identities e we have that $e = xy$ implies that either x or y is an identity.*

Proof. Suppose that the set of invertible elements is trivial. Let e be an identity such that $e = xy$. Then by Lemma 3.5.15(1), x and y are both invertible. By assumption, they must be identities.

Conversely, suppose that for all identities e we have that $e = xy$ implies that either x or y is an identity. If a is invertible then it has an inverse a^{-1}. Thus $e = a^{-1}a$ is an identity. By assumption, either a^{-1} or a is an identity. But the set of invertible elements is a groupoid and so a^{-1} is an identity iff a is an identity. It follows that a is an identity. Thus the set of invertible elements is trivial. □

Theorem 3.5.20. *A category is free if and only if it is a left Rees category having a trivial groupoid of invertible elements.*

Proof. By Lemma 3.5.12, free categories are left Rees categories with trivial groupoids of invertible elements.

Let C be a left Rees category having a trivial groupoid of invertible elements. We prove that it is isomorphic to a free category generated by a directed graph. Let X be a transversal of generators of the submaximal principal right ideals. We may regard X as a directed graph: the set of vertices is C_0 and if $a \in X$ then $\mathbf{r}(a) \xleftarrow{a} \mathbf{d}(a)$. We shall prove that the free category X^* generated by X is isomorphic to C. Let

$a \in C$. If aC is submaximal then a is indecomposable and since we are assuming that the invertible elements are trivial it follows that $a \in X$. Suppose that aC is not submaximal. Then $aC \subseteq a_1 C$ where $a_1 \in X$. Thus $a = a_1 b_1$. We now repeat this argument with b_1. We have that $b_1 C \subseteq a_2 C$ where $a_2 \in X$. Thus $b_1 = a_2 b_2$. Observe that $aC \subseteq a_1 C \subseteq a_1 a_2 C$. Continuing in this way and using the fact that each principal right ideal is contained in only a finite set of principal right ideals we have shown that $a = a_1 \ldots a_n$ where $a_i \in X$. It remains to show that each element of C can be written uniquely as an element of X^*. Suppose that

$$a = a_1 \ldots a_m = b_1 \ldots b_n$$

where $a_i, b_j \in X$. Then $a_1 C \cap b_1 C \neq \emptyset$. But both principal right ideals are submaximal and so $a_1 C = b_1 C$. Hence $a_1 = b_1$. By left cancellation we get that

$$a_2 \ldots a_m = b_2 \ldots b_n.$$

If $m = n$ then $a_i = b_i$ for all i and we are done. If $m \neq n$ then we deduce that a product of indecomposables is equal to an identity. Suppose that $e = c_1 \ldots c_r$ where e is an identity and the c_i are indecomposables. Then $c_1 \ldots c_r C$ is a maximal principal right ideal. But $c_1 \ldots c_r C \subseteq c_1 C$. Thus $c_1 C$ is maximal and so c_1 is invertible which is a contradiction. $\square$

We now return to the main goal of this section that of characterizing graph inverse semigroups. Let S be an inverse semigroup. Put

$$\mathbf{C}(S) = \{(e, s) \in E(S^*) \times S^* \colon \mathbf{r}(s) \leq e\}$$

and define $\mathbf{d}(e, s) = (\mathbf{d}(s), \mathbf{d}(s))$ and $\mathbf{r}(e, s) = (e, e)$. Define a partial product $(e, s)(f, t) = (e, st)$ iff $\mathbf{d}(e, s) = \mathbf{r}(f, t)$. Then $\mathbf{C}(S)$ is a Leech category called the *Leech category associated with S* [28].

Lemma 3.5.21. *An element $(e, s) \in \mathbf{C}(S)$ is an isomorphism if and only if $e = ss^{-1}$.*

Proof. Suppose that (e, s) is an isomorphism. Then there is an element $(f, t) \in \mathbf{C}(S)$ such that $(e, s)(f, t) = (e, e)$ and $(f, t)(e, s) = (s^{-1}s, s^{-1}s)$. Thus $st = e$ and $ts = s^{-1}s$.

But then $sts = s$ and $tst = t$. It follows that $t = s^{-1}$. Thus $ss^{-1} = e$. Conversely, suppose that $e = ss^{-1}$. Then (e, s) is invertible with inverse $(s^{-1}s, s^{-1})$. $\square$

The proof of the following is immediate by the lemma above.

Lemma 3.5.22. *Let S be a combinatorial inverse semigroup. Then the invertible elements of $\mathbf{C}(S)$ are those elements (e, s) where $e = ss^{-1}$ and $s^{-1}s \neq e$.*

Lemma 3.5.23. *If S is a Perrot semigroup then $\mathbf{C}(S)$ is a left Rees category. If, in addition, S is combinatorial then $\mathbf{C}(S)$ has trivial subgroups.*

Proof. Suppose that $(e, s)\mathbf{C}(S) \cap (e, t)\mathbf{C}(S) \neq \emptyset$. Then $(e, s)(i, a) = (e, t)(j, b)$ for some $(i, a), (j, b) \in \mathbf{C}(S)$. Thus $sa = tb$. Observe that $ss^{-1} \cdot sa = sa$. It follows that $ss^{-1}tt^{-1} \neq 0$. But S is unambiguous and so either $ss^{-1} \leq tt^{-1}$ or $tt^{-1} \leq ss^{-1}$. Without loss of generality we assume that $ss^{-1} \leq tt^{-1}$. Thus $ss^{-1} = tt^{-1}ss^{-1}$ and so $s = t(t^{-1}s)$. Observe that $\mathbf{r}(t^{-1}s) \leq \mathbf{d}(t)$. Thus $(\mathbf{d}(t), t^{-1}s) \in \mathbf{C}(S)$. But $(e, s) = (e, t)(\mathbf{d}(t), t^{-1}s)$ and so $(e, s)\mathbf{C}(S) \subseteq (e, t)\mathbf{C}(S)$, as required.

Suppose now that $(e, s)\mathbf{C}(S) \subseteq (e, t)\mathbf{C}(S)$. Then $(e, s) = (e, t)(\mathbf{d}(t), a)$ for some $(\mathbf{d}(t), a) \in \mathbf{C}(S)$. It follows that $s = ta$ and so $\mathbf{r}(s) \leq \mathbf{r}(t)$. Suppose now that $\mathbf{r}(s) = \mathbf{r}(t)$. Then $t = s(s^{-1}t)$. Observe that $\mathbf{r}(s^{-1}t) \leq \mathbf{d}(s)$. Thus $(s^{-1}s, s^{-1}t) \in \mathbf{C}(S)$ and $(e, t) = (e, s)(s^{-1}s, s^{-1}t)$. Thus $(e, s)\mathbf{C}(S) = (e, t)\mathbf{C}(S)$. The result now follows.

When S is combinatorial the claim follows from the above lemma. $\square$

Our characterization theorem can now be stated.

Theorem 3.5.24. *Let S be a combinatorial Perrot semigroup with maximal idempotents such that each $\mathcal{D}$-class contains a unique maximal idempotent. Then there is a free category C such that S is isomorphic to the inverse semigroup $\mathbf{S}(C)$.*

Proof. Let S be an inverse semigroup satisfying the conditions of the theorem. Let $s \in S$ be a non-zero element. By assumption $s\mathcal{D}e$ for a unique maximal idempotent

e. Thus there is an element a such that $s\mathcal{R}a\mathcal{L}e$. Put $b = a^{-1}s$. Thus $s = ab$. Observe that $a^{-1}a = e = bb^{-1}$ and that $\mathbf{r}(s) = \mathbf{r}(a)$ and $\mathbf{d}(s) = \mathbf{d}(b)$. Suppose that $s = a'b'$ where $\mathbf{d}(a') = e = \mathbf{r}(b')$. Then because S is combinatorial we have that $a = a'$ and $b = b'$. We shall say that each element of S can be uniquely factored through the maximal idempotents.

We have seen that the category $\mathbf{C}(S)$ is a left Rees category with trivial subgroups. However, there may be isomorphisms between distinct identities. For this reason, we shall define a full subcategory, denoted by $\mathbf{C}^r(S)$, whose elements are those pairs $(e, s) \in \mathbf{C}(S)$ such that $\mathbf{d}(s)$ and e are maximal idempotents. In other words, we take the full subcategory of $\mathbf{C}(S)$ determined by those identities (e, e) where e is a *maximal idempotent* of S. It follows that $\mathbf{C}^r(S)$ is a left Rees category with only trivial isomorphisms. Thus by Theorem 3.5.20, this category is free.

Put $S' = \mathbf{SC}^r(S)$. We shall prove that S and S' are isomorphic. A typical element of S' is an ordered pair $((e, s), (f, t))$ such that $s^{-1}s = t^{-1}t$ and where e, f, $s^{-1}s$ and $t^{-1}t$ are all maximal identities. We shall map this element to $st^{-1} \in S$. On the other hand the non-zero element $s \in S$ which has the factorization through e of $s = ab$ will be mapped to the element

$$((\mathbf{r}(a)^\circ, a), (\mathbf{d}(b)^\circ, b^{-1})).$$

We denote this map by θ. The zero elements in both cases are paired off. We have therefore shown that there is a bijection between S and S'. It remains to show that this is a homomorphism and we shall have proved the theorem.

Let $s = ab$ be the factorization through e and let $t = cd$ be the factorization through f. The semigroup S is unambiguous and so there are three cases to consider: (1) $\mathbf{d}(b)\mathbf{r}(c) = 0$, (2) $\mathbf{d}(b) < \mathbf{r}(c)$ and (3) $\mathbf{r}(c) < \mathbf{d}(b)$. In case (1), $st = 0$. In case (2), $st = a(bcd)$ is a factorization through e. In case (3), $st = (abc)d$ is a factorization through f.

Now $s \mapsto ((\mathbf{r}(a)^\circ, a), (\mathbf{d}(b)^\circ, b^{-1})) = \theta(s)$ and $t \mapsto ((\mathbf{r}(c)^\circ, c), (\mathbf{d}(d)^\circ, d^{-1})) = \theta(t)$. We now calculate $\theta(s)\theta(t)$ in each of the three cases. In case (1), the product is zero. In case (2), we have that $\mathbf{d}(b)^\circ = \mathbf{r}(c)^\circ$. Observe that

$$(\mathbf{d}(b)^\circ, b^{-1})(\mathbf{r}(b)^\circ, \mathbf{r}(b)^\circ) = (\mathbf{r}(a)^\circ, c)(\mathbf{r}(c)^\circ, c^{-1}b^{-1}).$$

Their product is therefore

$$((\mathbf{r}(a)^\circ, a), (\mathbf{d}(d)^\circ, d^{-1}c^{-1}b^{-1})).$$

Case (3) is similar to case (2). In all three cases, we have that $\theta(st) = \theta(s)\theta(t)$. $\square$

3.6 Completion of the graph inverse semigroups: the Cuntz-Krieger semigroups

Let G be a directed graph satisfying the condition that the in-degree of each vertex is at least 2 and finite. We define the *Cuntz-Krieger inverse semigroup* CK_G in the following way:

1. It is complete.
2. It contains a copy of $P(\mathcal{G})$ and every element of CK_G is the join of a finite subset of $P(\mathcal{G})$.
3. $e = \bigvee_{f' \in \hat{e}} f'$ for each maximal idempotent e of $P(\mathcal{G})$.
4. It is the freest inverse semigroup satisfying the above conditions.

We shall prove that this inverse semigroup exists and show how to construct it. In addition, we shall explain how it is related to the representation theory of the graph inverse semigroup $P(\mathcal{G})$, and explain its relation to the Cuntz-Krieger C^*-algebra via the associated topological groupoid.

In the case where G has one vertex and n loops, the graph inverse semigroup is just the polycyclic monoid P_n and we denote its completion by C_n and call it the

Cuntz semigroup of degree n. This semigroup was constructed in [32]. Our goal is to generalize the techniques described there to the more general case.

3.6.1 The Lenz arrow relation

The following definitions assume that the inverse semigroup is an inverse $\wedge$-semigroup. Since the inverse semigroups to which these definitions will be used are E^*-unitary this will not be a problem by Lemma 1.4.4. The key concept we shall need in this is the *Lenz arrow relation* introduced in [39]. Let $a, b \in S$. We define $a \rightarrow b$ iff for each non-zero element $x \leq a$, we have that $x \wedge b \neq 0$. Observe that $a \leq b \Rightarrow a \rightarrow b$. We write $a \leftrightarrow b$ iff $a \rightarrow b$ and $b \rightarrow a$. More generally, if $a, a_1, \ldots, a_m \in S$ then we define $a \rightarrow (a_1, \ldots, a_m)$ iff for each non-zero element $x \leq a$ we have that $x \wedge a_i \neq 0$ for some i. Finally, we write

$$(a_1, \ldots, a_m) \rightarrow (b_1, \ldots, b_n)$$

iff $a_i \rightarrow (b_1, \ldots, b_n)$ for $1 \leq i \leq m$, and we write

$$(a_1, \ldots, a_m) \leftrightarrow (b_1, \ldots, b_n)$$

iff both $(a_1, \ldots, a_m) \rightarrow (b_1, \ldots, b_n)$ and $(b_1, \ldots, b_n) \rightarrow (a_1, \ldots, a_m)$. A subset $Z \subseteq A$ is said to be a *cover* of A if for each $a \in A$ there exists $z \in Z$ such that $a \wedge z \neq 0$. A special case of this definition is the following. A finite subset $A \subseteq a^{\downarrow}$ is said to be a *cover* of a if $a \rightarrow A$. A homomorphism $\theta \colon S \rightarrow T$ is said to be a *cover-to-join* map if for each element $s \in S$ and each finite cover A of s we have that $\vee\theta(A)$ exists and $\theta(s) = \vee\theta(A)$.

A much more detailed discussion of cover-to-join maps and how they originated from the work of Exel [12] and Lenz [39] can be found in [35] which can be viewed as a substantial generalization of this section.

An inverse $\wedge$-semigroup S is said to be *separative* if and only if the Lenz arrow relation is equality.

Lemma 3.6.1. *Let S be an unambiguous E^*-unitary inverse semigroup. Then S is separative if and only if the semilattice of idempotents $E(S)$ is 0-disjunctive.*

Proof. Suppose first that S is separative. We prove that $E(S)$ is 0-disjunctive. Let $0 \neq e < f$. Then $e \to f$. By assumption, we cannot have that $f \to e$. Thus for some $e' \leq f$ we must have that $e' \wedge e = 0$. It follows that $E(S)$ is 0-disjunctive.

We shall now prove the converse. We shall prove that if $s \not\leq t$ where s and t are non-zero then there exists $0 \neq s' \leq s$ such that $s' \wedge t = 0$. Before we do this, we show that this property implies that S is separative. Suppose that $s \leftrightarrow t$ and that $s \neq t$. Then we cannot have both $s \leq t$ and $t \leq s$. Suppose that $s \not\leq t$. Then we can find $0 \neq s' \leq s$ such that $s' \wedge t = 0$ which contradicts our assumption.

We now prove the claim. We shall use Lemma 1.4.5 that tells us that the inverse semigroup itself is an unambiguous poset. Suppose that $s \wedge t = 0$. But then $0 \neq s \leq s$ and $s \wedge t = 0$. We may therefore assume that $s \wedge t \neq 0$. But then $s \leq t$ or $t < s$. The former cannot occur by assumption and so $t < s$. It follows that $\mathbf{d}(t) < \mathbf{d}(s)$. The semilattice of idempotents is 0-disjunctive and so there exists an idempotent $e < \mathbf{d}(s)$ such that $\mathbf{d}(t)e = 0$. Put $s' = se$. Then $0 \neq s' \leq s$. We have to calculate $s' \wedge t$. Suppose that $a \leq s', t$. Then $\mathbf{d}(a) \leq \mathbf{d}(s')\mathbf{d}(t) = e\mathbf{d}(t) = 0$, as required. $\square$

3.6.2 Orthogonal completions

We begin by recalling some results from [31]. Observe that all orthogonal sets will be assumed finite. The following is Lemma 2.1 of [31].

Lemma 3.6.2. *Let A and B be orthogonal subsets containing zero of an inverse semigroup with zero.*

(i) *AB is a orthogonal subset containing zero.*

(ii) *$AA^{-1} = \{aa^{-1} \colon a \in A\}$ and $A^{-1}A = \{a^{-1}a \colon a \in A\}$.*

(iii) $A = AA^{-1}A$ *and* $A^{-1} = A^{-1}AA^{-1}$.

Let $D(S)$ denote the set of finite orthogonal subsets of the inverse semigroup S that contain zero. The following is Lemma 2.2 and Lemma 2.3 of [31].

Lemma 3.6.3. *With the above definition, $D(S)$ is an inverse semigroup with zero under multiplication of subsets. In addition, the following hold:*

1. *If $A, B \in D(S)$ then $A \leq B$ iff for each $a \in A$ there exists $b \in B$ such that $a \leq b$.*
2. *If $A, B \in D(S)$ then A and B are orthogonal iff $A \cup B$ is an orthogonal subset of S.*
3. *If $A, B \in D(S)$ and A and B are orthogonal then $A \vee B = A \cup B$.*
4. *Multiplication distributes over finite orthogonal joins in $D(S)$.*

Define the function $\iota\colon S \to D(S)$ by $s \mapsto \{0, s\}$. This is an injective homomorphism. The following is Theorem 2.5 of [31] and describes the universal property enjoyed by this map.

Theorem 3.6.4. *Let S be an inverse semigroup with zero. Then $D(S)$ is orthogonally complete. Let $\theta\colon S \to T$ be a homomorphism to an orthogonally complete inverse semigroup T. Then there is a unique orthogonal join preserving homomorphism $\phi\colon D(S) \to T$ such that $\phi\iota = \theta$.*

Finally, the following is Lemma 3.4 of [31].

Lemma 3.6.5. *Let S be an orthogonally complete inverse semigroup. Let ρ be a 0-restricted congruence on S such that if $\rho(a) = \rho(a')$ and $\rho(b) = \rho(b')$ and a and b are orthogonal, and a' and b' are orthogonal then $(a \vee b)\rho(a' \vee b')$. Then S/ρ is also an orthogonally complete inverse semigroup and the natural homomorphism from S to S/ρ preserves finite orthogonal joins.*

3.6.3 Definition of an equivalence relation

In this section, we shall show how the definition of the congruence given in [32] can be phrased in terms of the Lenz arrow relation. This will show that the construction described in the next section really is a generalization of the one to be found in [32].

The idempotents of $D(P_n)$ correspond to prefix codes in the free monoid on n letters by Corollary 3.4 of [31]. By Lemma 4.1 of [31], the maximal prefix codes correspond to the essential idempotents of $D(P_n)$. However, it is immediate from this that these correspond to those sets of orthogonal idempotents of P_n that cover, in the sense this term was defined above, the identity of P_n.

Let G be a directed graph. We may construct the inverse semigroup $P(\mathcal{G})$ and therefore the inverse semigroup $D = D(P(\mathcal{G}))$. The elements of D will be written A^0 where A is a finite set of non-zero orthogonal elements of $P(\mathcal{G})$. For $A^0, B^0 \in D$ define

$$A^0 \preceq B^0$$

if and only if $A^0 \leq B^0$ and $B \to A$.

Lemma 3.6.6. *In a graph inverse semigroup, we have the following.*

1. *Let $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$ be orthogonal sets. Let*

$$(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)$$

and let $a_{i_1}, \ldots, a_{i_q}$ denote all the elements that lie beneath b_i. Then $\{e_1 = \mathbf{d}(a_{i_1}), \ldots, e_q = \mathbf{d}(a_{i_q})\}$ covers $\mathbf{d}(b_i)$.

2. *Let $a_1, \ldots, a_m$ be a set of orthogonal elements below a. If $\{\mathbf{d}(a_1), \ldots, \mathbf{d}(a_m)\}$ covers $\mathbf{d}(a)$ then $\{a_1, \ldots, a_m\}$ covers a.*

Proof. (1) Let $0 \neq e \leq \mathbf{d}(b_i)$. Then $b_i e \leq b_i$. It follows that there exists k such that $0 \neq b_i e \wedge a_k$. By our assumption that the b_j are orthogonal, we must have that $a_k \leq b_i$. Thus $a_k = a_{i_k}$, say. By Lemma 1.4.1, we have that $e \wedge e_{i_k} \neq 0$, as required.

(2) Let $0 \neq b \leq a$. Then $0 \neq \mathbf{d}(b) \leq \mathbf{d}(a)$. By assumption, there exists i such that $\mathbf{d}(a_i) \wedge \mathbf{d}(b) \neq 0$. But $a_i, b \leq a$ implies that a_i and b are compatible. Thus by Lemma 1.4.1, we have that $\mathbf{d}(a_i \wedge b) = \mathbf{d}(a_i) \wedge \mathbf{d}(b) \neq 0$. Thus $a_i \wedge b \neq 0$, as required. $\square$

Lemma 3.6.7. *Let $\{e_1, \ldots, e_m\}$ cover the idempotent e. Suppose that $e = a^{-1}a$ and $f = aa^{-1}$. Then $\{ae_1a^{-1}, \ldots, ae_ma^{-1}\}$ covers f.*

Proof. Let $0 \neq p \leq f$. Then $a^{-1}pa \leq e$ and it is easy to check that $a^{-1}pa \neq 0$. By assumption, there exists an i such that $a^{-1}pa \wedge e_i \neq 0$. But $a^{-1}pa \wedge e_i \leq a^{-1}a$ and so $a(a^{-1}pa \wedge e_i)a^{-1} \neq 0$. But $a(a^{-1}pa \wedge e_i)a^{-1} = p \wedge ae_ia^{-1} \neq 0$, as required, by Lemma 1.4.1. $\square$

We now use the above two lemmas to study the relation $\preceq$ on the semigroup $D(P_n)$. Let $A = \{x_iy_i : 1 \leq i \leq p\}$ and $B = \{u_jv_j : 1 \leq j \leq q\}$ be two elements of $D(P_n)$. We may partition the elements of A according to which elements of B they lie beneath. Thus if we choose $b = u_jv_j^{-1}$ we may consider all the $x_iy_i^{-1}$ that lie below it. Denote these elements by $x_1y_1^{-1}, \ldots, x_ly_l^{-1}$ for some l. Then by Lemma 3.6.6(1), the set of idempotents $\mathbf{d}(a_1), \ldots, \mathbf{d}(a_l)$ covers $\mathbf{d}(b)$. That is, $\{y_1y_1^{-1}, \ldots, y_ly_l^{-1}\}$ covers $v_jv_j^{-1}$. But every non-zero element in a polycyclic monoid is $\mathcal{D}$-related to the identity. It follows that the set of idempotents $\{y_1y_1^{-1}, \ldots, y_ly_l^{-1}\}$ may be obtained from a set of idempotents associated with a maximal prefix code by conjugation. Thus there is a maximal prefix code $\{z_1, \ldots, z_l\}$ where for each k we have that $y_k = v_jz_k$. It follows that $x_ky_k^{-1} = u_jz_kz_k^{-1}v_j^{-1}$. We have proved that if $A \preceq B$ in the sense of this work then $A \preceq B$ in the sense of the definition given in Section 3 of [32]. On the other hand, the converse is true by Lemma 3.6.6(2).

If $A^0, B^0 \neq 0$ define $A^0 \equiv B^0$ if and only if there exists $C^0 \neq 0$ such that $C^0 \preceq A^0$ and $C^0 \preceq B^0$. In addition, define $\{0\} \equiv \{0\}$.

Lemma 3.6.8. *In a graph inverse semigroup, we have the following. We have that $A^0 \equiv B^0$ if and only if $A^0 \leftrightarrow B^0$.*

Proof. Observe that if $(a_1, \ldots, a_m) \leftrightarrow (b_1, \ldots, b_n)$. then

$$(a_1, \ldots, a_m) \rightarrow (a_i \wedge b_j \colon 1 \leq i \leq m, 1 \leq j \leq n)$$

and

$$(b_1, \ldots, b_n) \rightarrow (a_i \wedge b_j \colon 1 \leq i \leq m, 1 \leq j \leq n).$$

Thus if $(a_1, \ldots, a_m) \leftrightarrow (b_1, \ldots, b_n)$ then there is $(c_1, \ldots, c_p)$ such that $(c_1, \ldots, c_p) \preceq (a_1, \ldots, a_m)$ and $(c_1, \ldots, c_p) \preceq (b_1, \ldots, b_n)$.

On the other hand, if $(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)$ then in fact $(a_1, \ldots, a_m) \leftrightarrow (b_1, \ldots, b_n)$. □

It follows that the complicated equivalence relation defined in Section 3 of [32] is nothing other than the relation $\leftrightarrow$. It is therefore this relation we shall use in our main construction in the next section, confident that we are generalizing [32] exactly.

3.6.4 The construction

In this section, we shall construct the Cuntz-Krieger semigroup CK_G.

Lemma 3.6.9. *In a graph inverse semigroup $P(\mathcal{G})$ we have the following. The relation $\equiv$ is a 0-restricted, idempotent pure congruence on $D(P(\mathcal{G}))$. Furthermore, if $A^0 \equiv B^0$ and $C^0 \equiv D^0$ and A^0 and C^0 are orthogonal and B^0 and D^0 are orthogonal then $A^0 \vee C^0 \equiv B^0 \vee D^0$.*

Proof. Observe first that if $a \rightarrow 0$ then $a = 0$. It follows that the relation $\equiv$ will be 0-restricted. From [39, 35] or by direct calculation, the relation $\rightarrow$ is reflexive and transitive. It follows readily from this that $\equiv$ is an equivalence relation. From [39, 35] or by direct calculation, the relation $\rightarrow$ is right and left compatible with the multiplication. It follows that $\equiv$ is a 0-restricted congruence. This congruence is idempotent pure. To see why, let $A^0 \equiv B^0$ where A^0 is a non-zero idempotent. Let $xy^{-1} \in B$. Then there exists $uu^{-1} \in A$ such that $xy^{-1} \wedge uu^{-1} \neq 0$. But this implies that xy^{-1} lies above a non-zero idempotent and $P(\mathcal{G})$ is E^*-unitary. It follows that xy^{-1} is an idempotent. Since xy^{-1} was arbitrary, B^0 is an idempotent as claimed. If A^0 and B^0 are orthogonal then $A^0 \vee B^0 = A^0 \cup B^0$. It readly follows that the last stated property holds. $\square$

Define CK_G to be $D(P(\mathcal{G}))/\equiv$ and define $\delta \colon P(\mathcal{G}) \to CK_G$ by $\delta(s) = [\{0, s\}]$, where $[x]$ denotes the $\equiv$-class containing x.

Proposition 3.6.10. *For any directed graph G, there is an orthogonally complete inverse semigroup CK_G together with a homomorphism $\delta \colon P(\mathcal{G}) \to CK_G$ such that every element of CK_G is a finite join of a finite orthogonal subset of the image of δ. For each maximal idempotent e in $P(\mathcal{G})$, we have that*

$$\delta(e) = \bigvee_{f \in \hat{e}} \delta(f).$$

If G has the additional property that the in-degree of each vertex is either 0 or at least 2, then the homomorphism δ is injective.

Proof. By Lemma 3.6.4, the semigroup CK_G is orthogonally complete. The homomorphism δ is injective if and only if the Lenz arrow relation is equality. Since $P(\mathcal{G})$ is unambiguous and E^*-unitary, it follows by Lemma 3.6.1 that the semilattice of idempotents of $P(\mathcal{G})$ must be 0-disjunctive. By Lemma 3.5.13(2) this means that the in-degree of each vertex of G is either 0 or at least 2. $\square$

Our goal now is to obtain a more concrete description of the inverse semigroup CK_G as well as a description of its semilattice of idempotents. We shall do this by first defining an action of $P(\mathcal{G})$ on the set G^ω of right-infinite paths in the graph G and thereby define a homomorphism $\theta \colon P(\mathcal{G}) \to I(G^\omega)$. Let $xy^{-1} \in P(\mathcal{G})$ and $w \in G^\omega$. We define $xy^{-1} \cdot w$ if and only if we may factorize $w = yw'$ where $w' \in G^\omega$; in which case, $xy^{-1} \cdot w = xw'$. This is well-defined since $\mathbf{d}(x) = \mathbf{d}(y)$. It is easy to check that the two axioms (A1) and (A2) for an action hold. We call this action the *natural action of the graph inverse semigroup on the space of infinite paths.*

Lemma 3.6.11. *Let $\theta\colon P(\mathcal{G}) \to I(G^\omega)$ be the homomorphism associated with the above natural action.*

1. *The action leads to a 0-restricted homomorphism θ if and only if there is no vertex of in-degree 0.*

2. *If the in-degree of each vertex is at least 2 then the homomorphism θ is injective.*

Proof. (1) Suppose that $\theta(xx^{-1}) = 0$ for some x. This means that there are no right-infinite strings with prefix x. This implies that there is some vertex of the graph which has in-degree zero. On the other hand, if each vertex of the graph has in-degree at least one then the action is 0-restricted: given any finite path x we may extend it to an infinite path $w = xw'$. Then $xx^{-1} \cdot w$ is defined.

(2) Suppose that $\theta(xy^{-1}) = \theta(uv^{-1})$. Then $yG^\omega = vG^\omega$ and so $y = v$ by Lemma 1.6.2. Similarly $xG^\omega = uG^\omega$ and so $x = u$ again by Lemma 1.6.2. □

From now on, we shall assume that the in-degree of each vertex of the graph is finite and at least 2. The representation $\theta\colon P(\mathcal{G}) \to I(G^\omega)$ is injective and so $P(\mathcal{G})$ is isomorphic to its image P'. Define O_G to be the inverse subsemigroup of $I(G^\omega)$ consisting of all non-empty finite unions of pairwise orthogonal elements of P'.

Let $X = \{x_1y_1^{-1}, \ldots, x_my_m^{-1}\}$ be an orthogonal sets in $P(\mathcal{G})$. Define a function $f_X \in I(G^\omega)$ as follows:

$$f_X\colon \bigcup_{i=1}^{m} y_iG^\omega \to \bigcup_{i=1}^{m} x_iG^\omega$$

is given by $f_X(w) = x_iw'$ if $w = y_iw'$

Lemma 3.6.12. *Let $X = \{x_1y_1^{-1}, \ldots, x_my_m^{-1}\}$ and $Y = \{u_1v_1^{-1}, \ldots, u_nv_n^{-1}\}$ be two orthogonal sets in $P(\mathcal{G})$. Then $f_X = f_Y$ if and only if $X \leftrightarrow Y$.*

Proof. By definition

$$f_X\colon \bigcup_{i=1}^{m} y_iG^\omega \to \bigcup_{i=1}^{m} x_iG^\omega$$

and

$$f_Y\colon \bigcup_{j=1}^{n} v_j G^\omega \to \bigcup_{j=1}^{n} u_j G^\omega$$

We suppose first that $f_X = f_Y$. Thus

$$\{y_1, \ldots, y_m\}G^\omega = \{v_1, \ldots, v_n\}G^\omega \text{ and } \{x_1, \ldots, x_m\}G^\omega = \{u_1, \ldots, u_n\}G^\omega.$$

Let $0 \neq wz^{-1} \leq x_i y_i^{-1}$. Then for some finite string p we have that $w = x_i p$ and $z = y_i p$. By definition, f_X restricts to define a map from $x_i p G^\omega$ to $y_i p G^\omega$ such that for any infinite string ω for which the product is defined we have that $f_X(y_i p\omega) = x_i p\omega$. By assumption, $f_Y(y_i p\omega) = x_i p\omega$. It follows that there are two possibilities. Either zG^ω has a non-empty intersection with $v_j G^\omega$ with exactly one of the j, in which case $zG^\omega \subseteq v_j G^\omega$ or it intersects a number of them in which case $v_j G^\omega \subseteq zG^\omega$ for a number of the j.

Suppose the first possibility occurs. Then $z = v_j q$ for some finite path q. The map from zG^ω to wG^ω must be a restriction of the map from $v_j G^\omega$ to $u_j G^\omega$. It follows that $wG^\omega = u_j q G^\omega$ and so by the above lemma we have that $w = u_j q$. It follows that $wz^{-1} \leq u_j v_j^{-1}$.

We now suppose that the second possibility occurs. Then for at least one j we have that $v_j = zq$ for some finite path q. In this case, we have that $u_j G^\omega = wqG^\omega$ and so by the above lemma we have that $u_j = wq$. It follows that $u_j v_j^{-1} \leq wz^{-1}$.

We have therefore shown that $X \to Y$. The result follows by symmetry.

We now prove the converse. Suppose that $X \leftrightarrow Y$. We prove that $f_X = f_Y$. Let w be an infinite string in $\mathrm{dom}(f_X)$. Then we may write it as $w = y_i \bar{w}$ for some infinite string $\bar{w}$. By definition $f_X(y_i \bar{w}) = x_i \bar{w}$. Choose a finite prefix p of $\bar{w}$ such that the element $x_i p(y_i p)^{-1}$ cannot be greater than or equal to any element in Y. It follows that there is a j such that $x_i p(y_i p)^{-1} \leq u_j v_j^{-1}$ using the fact that $X \to Y$. Thus $x_i p = u_j q$ and $y_i p = v_j q$ for some finite path q. Put $\bar{w} = pw'$. Then $w = y_i p w'$. Thus

$$f_Y(w) = f_Y(y_i p w') = f_Y(v_j q w') = u_j q w' = x_i p w' = x_i \bar{w} = f_X(w).$$

The result now follows by symmetry. $\square$

It follows by the above lemma that the function $F\colon CK_G \to O_G$ given by $F([A^0]) = f_A$ is well-defined and a bijection.

Theorem 3.6.13. *Let G be a directed graph in which the in-degree of each vertex is at least 2 and is finite. Then the inverse semigroup CK_G is isomorphic to the inverse semigroup O_G defined as an inverse semigroup of partial bijections of the set G^ω. In particular, the semilattice of idempotents of CK_G is a (non-unital, in general) boolean algebra. It follows that CK_G is a complete inverse semigroup.*

Proof. We have defined a bijection from $F\colon CK_G \to O_G$. It remains to show that this is a homomorphism. From [27], we can simplify this proof by splitting it up into three simple cases.

Case 1: If $\mathbf{d}(X) = \mathbf{r}(Y)$ then $F(XY) = F(X)F(Y)$

Let $X = \{x_1y_1^{-1}, \dots, x_ny_n^{-1}\}, Y = \{u_1v_1^{-1}, \dots, u_mv_m^{-1}\}$ be orthogonal sets with $\mathbf{d}(X) = \mathbf{r}(Y)$. Then for all $x_iy_i^{-1} \in X$ there exists $u_jv_j^{-1} \in Y$ such that $y_iy_i^{-1} = u_ju_j^{-1}$. Assume $rs^{-1} \in Y, rs^{-1} \neq u_jv_j^{-1}$ and $y_iy_i^{-1} = rr^{-1}$. As Y is orthogonal we have $u_ju_j^{-1}rr^{-1} = 0$, giving the following contradiction:

$$y_iy_i^{-1} = rr^{-1} \neq u_ju_j^{-1} = y_iy_i^{-1}.$$

By symmetry we have $m = n$.

Let $rs^{-1} \in XY$ be non-zero. Then there exists $x_iy_i^{-1} \in X, u_jv_j^{-1} \in Y$ such that $x_iy_i^{-1}u_jv_j^{-1} = rs^{-1}$. Assume $y_iy_i^{-1} \neq u_ju_j^{-1}$. Then as $\mathbf{d}(X) = \mathbf{r}(Y)$ there is a $u_kv_k^{-1} \in Y$, with $y_iy_i^{-1} = u_ku_k^{-1}$. As Y is orthogonal $u_ju_j^{-1}u_ku_k^{-1} = 0$ if $k \neq j$. Assume $k \neq j$ then by substitution $u_ju_j^{-1}y_iy_i^{-1} = 0$ and thus $x_iy_i^{-1}u_jv_j^{-1} = 0$, a contradiction. Therefore $k = j$ and $y_iy_i^{-1} = u_ju_j^{-1}$. Thus $rs^{-1} = x_iy_i^{-1}u_jv_j^{-1} = x_iv_j^{-1}$ and

$$XY = \{x_1v_1^{-1}, \dots, x_nv_n^{-1}\}.$$

Now we show $f_{XY} = f_X f_Y$. Let $f_{XY}(w) = w'$. Then there exists $x_i v_i^{-1} \in XY$ such that $w = v_i \bar{w}$ and $w' = x_i \bar{w}$. Thus $y_i \bar{w} = u_i \bar{w}$ exists and so

$$\begin{aligned} f_X f_Y(w) &= f_X f_Y(v_i \bar{w}) = f_X(u_i v_i^{-1} \cdot v_i \bar{w}) = f_X(u_i \bar{w}) \\ &= f_X(y_i \bar{w}) = x_i y_i^{-1} \cdot y_i \bar{w} = x_i \bar{w} = w'. \end{aligned}$$

Similarly if $f_X f_Y(w) = w'$ then $w = v_j \bar{w}$, $w' = x_i \hat{w}$ and $u_j \bar{w} = y_i \hat{w}$ for some $x_i y_i^{-1} \in X, u_j v_j^{-1} \in Y$. By Lemma 1.6.1 we have y_i and u_j are prefix comparable. As $\mathbf{d}(X) = \mathbf{r}(Y)$ and X, Y are orthogonal $y_i = u_j$. Thus $x_i v_j^{-1} \in XY$, $\bar{w} = \hat{w}$ and

$$f_{XY}(w) = x_i v_j^{-1} \cdot w = x_i v_j^{-1} \cdot v_j \bar{w} = x_i \bar{w} = w'.$$

Therefore $f_{XY} = f_X f_Y$ and $F(XY) = F(X)F(Y)$ if $\mathbf{d}(X) = \mathbf{r}(Y)$.

Case 2: If $X \leq Y$ then $F(X) \leq F(Y)$.

Let $X \leq Y$. Then for all $xy^{-1} \in X$ there exist $uv^{-1} \in Y$ such that $xy^{-1} \leq uv^{-1}$. Thus $x = up$ and $y = vp$ for some path p. Let

$$T = \{yy^{-1} : xy^{-1} \in X\},$$

so T is idempotent. We will show $f_X = f_Y f_T$. Let $f_X(w) = w'$. Then $w = y\bar{w}$, $w' = x\bar{w}$ for some $xy^{-1} \in X$. Thus $yy^{-1} \in T$ and there exist $uv^{-1} \in Y$, $p \in \mathcal{G}^*$ such that $x = up, y = vp$. Therefore $w = vp\bar{w}$, $w' = up\bar{w}$ and

$$\begin{aligned} f_Y f_T(w) &= f_Y f_T(y\bar{w}) = f_Y(yy^{-1} \cdot y\bar{w}) = f_Y(y\bar{w}) \\ &= f_Y(w) = f_Y(vp\bar{w}) = uv^{-1} \cdot vp\bar{w} = up\bar{w} = w'. \end{aligned}$$

Now let $f_Y f_T(w) = w'$. Then there exist $yy^{-1} \in T$, $uv^{-1} \in Y$ with $w = y\bar{w} = v\hat{w}$ and $w' = u\hat{w}$. As $yy^{-1} \in T$ there exist $xy^{-1} \in X$ and so there exist $rs^{-1} \in Y$ such that $xy^{-1} \leq rs^{-1}$. Then $uv^{-1} = rs^{-1}$ as Y orthogonal and $x = up, y = vp$. Giving $v\hat{w} = y\bar{w} = vp\bar{w}$ and cancelling the v we have $\hat{w} = p\bar{w}$. Thus $w' = up\bar{w} = x\bar{w}$ and

$$f_X(w) = xy^{-1} \cdot w = xy^{-1} \cdot y\bar{w} = x\bar{w} = up\bar{w} = w'.$$

Therefore $f_X = f_Y f_T$, so $f_X \leq f_Y$ and $F(X) \leq F(Y)$ if $X \leq Y$.

Case 3: If X, Y are idempotent then $F(X \wedge Y) = F(X)F(Y)$.

Let X, Y be idempotents. Then $X = \{x_1x_1^{-1}, \ldots, x_nx_n^{-1}\}$, $Y = \{y_1y_1^{-1}, \ldots, y_my_m^{-1}\}$. We have $x_ix_i^{-1} \wedge y_jy_j^{-1} \neq 0$ if and only if $x_ix_i^{-1} \leq y_jy_j^{-1}$ or $y_jy_j^{-1} \leq x_ix_i^{-1}$ if and only if x_i and y_i are prefix comparable. Thus $x_ix_i^{-1} \wedge y_jy_j^{-1} = x_ix_i^{-1}$ or $x_ix_i^{-1} \wedge y_jy_j^{-1} = y_jy_j^{-1}$, and $X \wedge Y$ is idempotent.

We will now show $f_{X\wedge Y} = f_X f_Y$. Let $f_{X\wedge Y}(w) = w$. Then there exists $x_ix_i^{-1} \in X, y_jy_j^{-1} \in Y$ such that $w = (x_ix_i^{-1} \wedge y_jy_j^{-1}) \cdot w$. As meet is commutative we can assume without loss of generality that $x_ix_i^{-1} \wedge y_jy_j^{-1} = x_ix_i^{-1}$. Then $x_i = y_jp$ and $w = x_ix_i^{-1} \cdot w$ so $w = x_i\bar{w} = y_jp\bar{w}$. Therefore

$$\begin{aligned} f_X f_Y(w) &= f_X f_Y(y_j\bar{w}) = f_X(y_jy_j^{-1} \cdot y_jp\bar{w}) = f_X(y_jp\bar{w}) \\ &= f_X(w) = f_X(x_i\bar{w}) = x_ix_i^{-1} \cdot x_i\bar{w} = x_i\bar{w} = w. \end{aligned}$$

Remembering that X, Y are idempotents we now let $f_X f_Y(w) = w$. Then there exist $x_ix_i^{-1} \in X, y_jy_j^{-1} \in Y$ with $w = x_i\bar{w} = y_j\hat{w}$. Thus x_i and y_j are prefix comparable by Lemma 1.6.1. Therefore $x_ix_i^{-1} \wedge y_jy_j^{-1} \neq 0$ and equals $x_ix_i^{-1}$ or $y_jy_j^{-1}$ By the commutativity of meet we can assume without loss of generality that $x_ix_i^{-1} \wedge y_jy_j^{-1} = x_ix_i^{-1}$. Thus

$$f_{X\wedge Y}(w) = (x_ix_i^{-1} \wedge y_jy_j^{-1}) \cdot w = x_ix_i^{-1} \cdot w = x_ix_i^{-1} \cdot x_i\bar{w} = x_i\bar{w} = w.$$

Therefore $f_{X\wedge Y} = f_X f_Y$ and $F(X \wedge Y) = F(X)F(Y)$ if X, Y are idempotents.

The semilattice of idempotents of O_G is in bijective correspondence with the subsets of G^ω of the form XG^ω where X is a finite set of finite paths in G. However, these are precisely the compact-open subsets of the topological space G^ω which has a basis of compact-open subsets and is hausdorff. It follows that the semilattice of idempotents of O_G and so of CK_G is a boolean algebra. Thus by Lemma 1.4.3, CK_G is complete. $\square$

3.6.5 Universal characterization

It remains to show that this inverse semigroup has the right universal property. If xx^{-1} is a non-zero idempotent in $P(\mathcal{G})$ then we define its *weight* to be the number $|x|$.

Lemma 3.6.14. *Let $P(\mathcal{G})$ be a graph inverse semigroup in which the in-degree of each vertex is finite. Let $F = \{e_1, \ldots, e_m\}$ be an orthogonal cover of the maximal idempotent e. Suppose that $e_1 = xx^{-1}$ is an idempotent in F of maximum weight at least 1. Put $x = \bar{x}a_1$ where a_1 is an edge with target f. Let $a_1, \ldots, a_n$ be all the edges with range f.*

1. *Then $f_j = \bar{x}a_j a_j^{-1}\bar{x}^{-1} \in F$ for $1 \leq j \leq n$.*
2. *Put $F' = F \setminus \{f_1, \ldots, f_n\} \cup \{\bar{x}\bar{x}^{-1}\}$. Then F' is a cover of e and $|F'| < |F|$.*
3. *$F = F' \setminus \{\bar{x}\bar{x}^{-1}\} \cup \bar{x}\hat{f}\bar{x}^{-1}$.*

Proof. (1) The string $\bar{x}a_j$ has range the vertex corresponding to e. Thus $f_j = \bar{x}a_j a_j^{-1}\bar{x}^{-1} \leq e$. By assumption $f_j \wedge e_i \neq 0$ for some i. Let $e_i = yy^{-1}$. Then y and $\bar{x}a_j$ are prefix-comparable. By assumption, e_1 has maximum weight amongst all the idempotents in F and so $\bar{x}a_j = yz$ for some path z. If z were not empty, y would be a prefix of $\bar{x}$ and so we would have that $e_1 < e_i$ which is a contradiction. It follows that z is empty and so $f_j = e_i$.

(2) Let $0 \neq f \leq e$ and suppose that $0 = f \wedge e_i$ for all $i > 1$. We must have that $0 \neq f \wedge e_1$. We shall show that $0 \neq \bar{x}\bar{x}^{-1} \wedge f$. Let $f = yy^{-1}$. Then x and y are prefix-comparable. If $|y| < |x|$ then y is a prefix of $\bar{x}$ and $\bar{x}$ and y are prefix-comparable. If $|y| \geq |x|$ then $\bar{x}$ is a prefix of y and again $\bar{x}$ and y are prefix-comparable.

(3) This is immediate. $\square$

Lemma 3.6.15. *Let $\theta\colon P(\mathcal{G}) \to T$ be a homomorphism to a complete inverse semigroup where for each idempotent e and cover F of e we have that $\theta(e) = \bigvee_{f \in F} \theta(f)$. Then θ is a cover-to-join map.*

Proof. Let $\{a_1, \ldots, a_m\}$ be a cover of a. Then $\{\mathbf{d}(a_1), \ldots, \mathbf{d}(a_m)\}$ is a cover of $\mathbf{d}(a)$. By assumption $\theta(\mathbf{d}(a)) = \bigvee_{i=1}^{m} \theta(\mathbf{d}(a_i))$. Now multiplying on the left by $\theta(a)$ and using distributivity we get that $\theta(a) = \bigvee_{i=1}^{m} \theta(a_i)$. $\square$

Lemma 3.6.16. *Let $\theta\colon P(\mathcal{G}) \to T$ be a homomorphism to a complete inverse semigroup where $\theta(e) = \bigvee_{f \in \hat{e}} \theta(f)$ for each maximal idempotent e in $P(\mathcal{G})$. Then θ is a cover-to-join map.*

Proof. Suppose first that we can prove the following claim: for every maximal idempotent e and every cover F of e we have that $\theta(e) = \bigvee_{f \in F} \theta(f)$. Then we can prove that θ is a cover-to-join map. Let $\{e_1, \ldots, e_m\}$ be the cover of the idempotent f. In a graph inverse semigroup, there is a maximal idempotent e such that $f \,\mathcal{D}\, e$. Thus we may find an element a such that $a^{-1}a = f$ and $aa^{-1} = e$. By an argument similar to Lemma 3.6.7, we have that $\{ae_1a^{-1}, \ldots, ae_ma^{-1}\}$ is a cover of e. By assumption,

$$\theta(e) = \bigvee_{i=1}^{m} \theta(ae_ia^{-1}).$$

Multiplying on the left by $\theta(a^{-1})$ and on the right by $\theta(a)$ and using distributivity we get that

$$\theta(f) = \bigvee_{i=1}^{m} \theta(e_i).$$

To prove the lemma, it therefore remains to prove the claim. This can now be achieved using induction and Lemma 3.6.14 and, since the maximal idempotents are pairwise orthogonal, we can fix attention on all covers of a fixed maximal idempotent e. Suppose that our claim holds for all orthogonal covers of e with at most p elements. Let F be an orthogonal cover of e with $p + 1$ elements. By Lemma 3.6.14, we may write

$$F = F' \setminus \{\bar{x}\bar{x}^{-1}\} \cup \bar{x}\hat{f}\bar{x}^{-1}$$

where F' is cover of e and $|F'| < |F|$. By our induction hypothesis, we may write

$$\theta(e) = \bigvee_{f' \in F'} \theta(f').$$

By assumption, we may write

$$\theta(f) = \bigvee_{g \in \hat{f}} \theta(g).$$

Thus using distributivity, we have that

$$\theta(\bar{x}\bar{x}^{-1}) = \bigvee_{g \in \hat{f}} \theta(\bar{x}g\bar{x}^{-1}).$$

It follows that

$$\theta(e) = \bigvee_{f \in F} \theta(f),$$

as required. $\square$

The following theorem can be deduced from the general theory described in [35], but we give a direct proof.

Theorem 3.6.17. *Let $\theta \colon P(\mathcal{G}) \to T$ be a homomorphism to a complete inverse semigroup where $\theta(e) = \bigvee_{f \in \hat{e}} \theta(f)$ for each maximal idempotent e in $P(\mathcal{G})$ Then there is a unique join-preserving homomorphism $\bar{\theta} \colon CK_G \to T$ such that $\bar{\theta}\delta = \theta$.*

Proof. By Lemma 3.6.15, the map θ is a cover-to-join map. The theorem will be proved if we can show that for any two orthogonal sets X and Y in $P(\mathcal{G})$ we have that $X \leftrightarrow Y$ implies that $\bigvee_{x \in X} \theta(x) = \bigvee_{y \in Y} \theta(y)$. By Lemma 3.6.8, it is enough to show this in the special case where $X \preceq Y$ and we may further assume that $Y = \{y\}$. But then the result is immediate by the definition of a cover-to-join map. $\square$

The above theorem justifies our claims made about the semigroup CK_G at the beginning of this section.

3.6.6 The topological connection

This topic is taken up in more depth in [35]. Here we shall just sketch out the key result. This subsection is only included for completeness and is not original work of the author. We shall use the following notation. Let $w = uw'$ where w, w' are infinite strings and u is a finite string. Define $u^{-1}w = w'$. Given a directed graph G a groupoid $\mathcal{G}$ is defined as follows. Its elements consist of triples $(w, k, w') \in G^{\omega} \times \mathbb{Z} \times G^{\omega}$ where there are finite strings u and v such that $u^{-1}w = v^{-1}w'$ and $|v| - |u| = k$. The groupoid product is given by $(w, k, w')(w', l, w'') = (w, k + l, w'')$ and $(w, k, w')^{-1} = (w', -k, w)$. A basis for a topology is given as follows. For each pair $x, y \in G^*$ define $Z(x, y)$ to consist of all groupoid elements (w, k, w') where $x^{-1}w = y^{-1}w'$ and $k = |y| = |x|$. Observe that under our assumptions on G, the sets $Z(x, y)$ are always non-empty. It can be shown that this is a basis, and that with respect to the topology that results the groupoid $\mathcal{G}$ is an étale, hausdorff topological groupoid in which the sets $Z(x, y)$ are compact-open bisections. The space of identities of this groupoid is homeomorphic to the usual topology defined on G^{ω} [24]. With our usual assumptions on the directed graph G this makes $\mathcal{G}$ what we have called a *boolean groupoid* in [35]. The compact-open bisections of the groupoid $\mathcal{G}$ form an inverse semigroup called the *ample semigroup* of $\mathcal{G}$. We shall prove that this semigroup is the Cuntz-Krieger semigroup CK_G. The following two lemmas are the key to our main theorem.

Lemma 3.6.18.

$$Z(x, y)Z(u, v) = \begin{cases} Z(x, vz) & \textit{if } y = uz \\ Z(yz, v) & \textit{if } u = yz \\ \emptyset & \textit{else} \end{cases}$$

Proof. It is easy to check that the product is empty if y and u are not prefix-comparable. We shall therfore assume, without loss of generality, that $y = uz$ for some finite path z. It is straightforward to check that $Z(x, y)Z(u, v) \subseteq Z(x, vz)$. We prove the reverse inclusion. Let $(w_1, m, w_2) \in Z(x, vz)$ where $m = |vz| - |x|$. Let

$w_1 = x\bar{w}$ and $w_2 = vz\bar{w}$. Then a routine calculation shows that (w_1, m, w_2) is equal to the product

$$(x\bar{w}, |y| - |x|, y\bar{w})(uz\bar{w}, |vz| - |uz|, vz\bar{w})$$

where the first element is from $Z(x, y)$ and the second is from $Z(u, v)$. □

The following is Lemma 2.5 of [24].

Lemma 3.6.19.

$$Z(x,y) \cap Z(u,v) = \begin{cases} Z(x,y) & \textit{if } xy^{-1} \leq uv^{-1} \\ Z(u,v) & \textit{if } uv^{-1} \leq xy^{-1} \\ \emptyset & \textit{else} \end{cases}$$

Denote by $\mathsf{B}(\mathcal{G})$ the inverse semigroup of compact-open bisections of the topological groupoid $\mathcal{G}$.

Theorem 3.6.20. *The Cuntz-Krieger semigroup CK_G is the ample semigroup of the topological groupoid $\mathcal{G}$ constructed from the directed graph G.*

Proof. Define $\theta \colon P(\mathcal{G}) \to \mathsf{B}(\mathcal{G})$ by $\theta(xy^{-1}) = Z(x, y)$ and map the zero to the emptyset. Then by Lemma 3.6.18, this map is a homomorphism. We claim that it is injective. Suppose that $Z(x, y) = Z(u, v)$ Since these sets are non-empty, we have that xy^{-1} and uv^{-1} are comparable since by Lemma 1.4.5 the poset $P(\mathcal{G})$ is unambiguous. It is now immediate by Lemma 3.6.19 that $xy^{-1} = uv^{-1}$. Let e be a vertex of G and let $a_1, \ldots, a_m$ be the edges of G with source e. Then

$$Z(1_e, 1_e) = \bigcup_{i=1}^{m} Z(a_i, a_i).$$

The conditions of Theorem 3.6.17 hold and so θ may be extended to a homomorphism $\bar{\theta} \colon CK_G \to \mathsf{B}(\mathcal{G})$. Lemma 3.6.19 implies that each element of $\mathsf{B}(\mathcal{G})$ is a finite disjoint union of basis elements and so θ is surjective.

It remains to show that $\bar{\theta}$ is injective. We shall prove that if

$$\bigcup_{i=1}^{m} Z(x_i, y_i) = \bigcup_{j=1}^{n} Z(u_j, v_j)$$

then

$$(x_1y_1^{-1}, \ldots, x_my_m^{-1}) \leftrightarrow (u_1v_1^{-1}, \ldots, u_nv_n^{-1}).$$

By symmetry it is enough to prove that if

$$Z(x, y) \subseteq \bigcup_{j=1}^{n} Z(u_j, v_j)$$

then

$$xy^{-1} \to (u_1v_1^{-1}, \ldots, u_nv_n^{-1}).$$

Let $wz^{-1} \leq xy^{-1}$. Then $w = xp$ and $z = yp$ for some finite path p. Let w' be any infinite path so that xpw' and ypw' are defined. Then

$$(xpw', |y| - |x|, ypw') \in Z(x, y)$$

and so belongs to $Z(u_j, v_j)$ for some j. It is now easy to show that wz^{-1} and $u_jv_j^{-1}$ are comparable where we make essential use of the central number in the triple. $\square$

3.7 Representations of the graph inverse semigroup

In the section we aim to classify the representations of the graph inverse semigroups. By Theorem 1.5.2 we do this by looking at the proper closed inverse subsemigroups up to conjugacy, with the maximal subsemigroups being of particular interest. We use the work of Lawson [30] and Section 2.4 on the representation of the polycyclic monoids as a guide. Again we notice that zero lies at the bottom of the order and thus only subsemigroups in which all the elements are compatible are non-trivial. In the graph inverse monoid case many things work exactly the same as in the graph inverse semigroup case or with some small alteration. We shall discuss the semigroup case first before highlighting the differences in the monoid case.

Let H be a proper closed inverse subsemigroup of $P(\mathcal{G})$. As all the elements of H have to be compatible then they take the form xpx^{-1} and $xp^{-1}x^{-1}$. As $\mathbf{d}(p) = \mathbf{d}(xp) = \mathbf{d}(x) = \mathbf{r}(p)$ we have that p is a cyclic (possibly an identity). If p is an identity then $xpx^{-1} = xx^{-1}$ is idempotent. Let us consider one such element $xpx^{-1} \in H$. Then $xp^nx^{-1} = (xpx^{-1})^n \in H$ for all $n \in \mathbb{Z}$. This gives us our first meaningful note. A proper closed inverse subsemigroup H is finite if and only if H consists entirely of idempotents.

Let us consider such a finite closed subsemigroup H. For all $xx^{-1} \in H$ we have $\bar{x}\bar{x}^{-1} \in H$, where $\bar{x}$ is a prefix of x, as H is closed. Assume $xx^{-1} \in H$ is minimal in terms of the natural order. Then $yy^{-1} \in H$ only if yy^{-1} and xx^{-1} are compatible. Therefore y is a prefix of x by the minimality of xx^{-1}. We have proved the following:

Lemma 3.7.1. *Let x be a path in $\mathcal{G}$. Then*

$$(xx^{-1})^{\uparrow} = \{yy^{-1} \in P(\mathcal{G}) : y \textit{ a prefix of } x\}$$

is a proper closed inverse subsemigroup of $P(\mathcal{G})$.

We will call this the finite case and each subsemigroup is defined by its unique smallest element in terms of the natural order.

Let us consider an infinite proper closed inverse subsemigroup H. If H contains elements of the form xp^nx^{-1} then it also contains $xp^n(xp^m)^{-1}$, for all $n, m \in \mathbb{N}$. By closure H therefore contains elements of the form $xp^n\bar{p}(xp^m\bar{p})^{-1}$ where $\bar{p}$ is a prefix of p. Finally H will also contain $\bar{x}\bar{x}^{-1}$ as $xx^{-1} \in H$.

Lemma 3.7.2. *Let p be a cycle on some vertex e and x any path from e. Then*

$$P(\mathcal{G})^{x,p} = \{xp^r\bar{p}(xp^s\bar{p})^{-1} : r, s \in \mathbb{N}, \bar{p} \textit{ is a prefix of } p\} \cup (xx^{-1})^{\uparrow}$$

is a proper closed inverse subsemigroup of $P(\mathcal{G})$.

We call this the cyclic case. These subsemigroups are uniquely defined by two strings x and p such that x and p share no common suffix.

From this subsemigroup we can get a new proper closed inverse subsemigroup for free. The idempotents of $P(\mathcal{G})^{x,p}$ are of the form $\bar{x}\bar{x}^{-1}$ or $xp^n\bar{p}(xp^n\bar{p})^{-1}$. Any element above $xp^n\bar{p}(xp^n\bar{p})^{-1}$ is of the same form or $\bar{x}\bar{x}^{-1}$. Therefore the idempotents of $P(\mathcal{G})^{x,p}$ form a proper closed inverse subsemigroup which we denote by $(xp^\omega(xp^\omega)^{-1})^\uparrow$ and call it the periodic case. These subsemigroups are also uniquely define by a path x and cycle p such that they share no common prefix. We use the ultimately periodic right infinite paths to uniquely define an infinite subsemigroup of idempotents of $P(\mathcal{G})$. We can also use aperiodic paths in the same way. For all $w \in \mathcal{G}^\omega$ the set

$$(ww^{-1})^\uparrow = \{xx^{-1} \in P(\mathcal{G}) : x \text{ is a prefix of } w\}$$

is a proper closed inverse subsemigroup. When w is aperiodic then we call $(ww^{-1})^\uparrow$ the aperiodic case.

These infinite subsemigroups rely on the graph containing cycles. Let $p = p_1 \dots p_n$ be a cycle, we say v is a *vertex of* p if for some $1 \leq i \leq n$ we have $v = \mathbf{d}(p_i)$. We say two cycles p, q are *strongly connected* if there is a path from a vertex of p to a vertex of q and visa versa. Only when $\mathcal{G}$ has at least two strongly connected cycles can we find aperiodic paths in $\mathcal{G}$.

We shall now look to classify the conjugacy of these proper closed inverse subsemigroups. From the work of Lawson on polycyclic monoids we can deduce that finite idempotent type can only be conjugate to finite idempotent type, infinite idempotent type to infinite idempotent type and thus cyclic type to cyclic type.

Lemma 3.7.3. *Two finite proper closed inverse subsemigroups are conjugate if and only if their defining elements have a common domain. In other words, $(xx^{-1})^\uparrow$ and $(yy^{-1})^\uparrow$ are conjugate if and only if $xy^{-1} \in P(\mathcal{G})$.*

Proof. Let $(xx^{-1})^\uparrow$ and $(yy^{-1})^\uparrow$ be finite closed inverse subsemigroups of $P(\mathcal{G})$ and $xy^{-1} \in P(\mathcal{G})$. Then $yx^{-1}(xx^{-1})^\uparrow xy^{-1} = \{yy^{-1}\}$ and $xy^{-1}(yy^{-1})^\uparrow yx^{-1} = \{xx^{-1}\}$. Thus $(xx^{-1})^\uparrow$ and $(yy^{-1})^\uparrow$ are conjugate.

Let $(xx^{-1})^{\uparrow}$ and $(yy^{-1})^{\uparrow}$ be conjugate. Then there exist an $rs^{-1} \in P_t(\mathcal{G})$ such that $rs^{-1}(yy^{-1})^{\uparrow}sr^{-1} \subseteq (xx^{-1})^{\uparrow}$ and $sr^{-1}(xx^{-1})^{\uparrow}rs^{-1} \subseteq (yy^{-1})^{\uparrow}$. We note that s is a prefix of y and r is a prefix of x. If $x = r$ and $y = s$ then we are done. Assume r and s are proper prefixes of x and y respectively, that is $x = r\hat{x}$ and $y = s\hat{y}$. Then

$$rs^{-1}yy^{-1}sr^{-1} = r\hat{y}(r\hat{y})^{-1} \in (xx^{-1})^{\uparrow}.$$

Thus

$$\begin{aligned} r\hat{y}(r\hat{y})^{-1} &\geq xx^{-1} \\ sr^{-1}r\hat{y}(r\hat{y})^{-1}rs^{-1} &\geq sr^{-1}xx^{-1}rs^{-1} \\ s\hat{y}\hat{y}^{-1}s^{-1} &\geq s\hat{x}\hat{x}^{-1}s^{-1} \\ yy^{-1} &\geq s\hat{x}(s\hat{x})^{-1}. \end{aligned}$$

As $s\hat{x}(s\hat{x})^{-1} \in sr^{-1}(xx^{-1})^{\uparrow}rs^{-1} \subseteq (yy^{-1})^{-1}$ and yy^{-1} is minimal in $(yy^{-1})^{\uparrow}$ we have $s\hat{x} = y$. Thus x and y have a common domain and therefore $xy^{-1} \in P(\mathcal{G})$. $\square$

Lemma 3.7.4. *Two infinite idempotent type proper inverse subsemigroups are conjugate iff their defining strings are ultimately the same. In otherwords, $(xx^{-1})^{\uparrow}$ and $(yy^{-1})^{\uparrow}$ (for $x, y \in \mathcal{G}^{\omega}$) are conjugate if and only if there exists some paths u, v and $z \in \mathcal{G}^{\omega}$ such that $x = uz$ and $y = vz$.*

Proof. Let $(xx^{-1})^{\uparrow}$ and $(yy^{-1})^{\uparrow}$ (for $x, y \in \mathcal{G}^{\omega}$) be conjugate. Then there exists $uv^{-1} \in P(\mathcal{G})$ such that

$$uv^{-1}(xx^{-1})^{\uparrow}vu^{-1} \subseteq (yy^{-1})^{\uparrow} \text{ and } vu^{-1}(yy^{-1})^{\uparrow}uv^{-1} \subseteq (xx^{-1})^{\uparrow}.$$

Firstly we note $x = vz_1$ and $y = uz_2$ for $z_1, z_2 \in \mathcal{G}^{\omega}$. As the conjugated sets are subsets of the original subsemigroups $z_1 = z_2$. Thus $x = vz$ and $y = uz$ for some $z \in \mathcal{G}^{\omega}$.

Now assume $x = vz$ and $y = uz$ for some paths u, v and $z \in \mathcal{G}^\omega$. Let $\bar{x}, \bar{z}$ denote prefixes of x and z respectively. Then

$$uv^{-1}\bar{x}\bar{x}^{-1}vu^{-1} = \begin{cases} uu^{-1} & \text{if } \bar{x} \text{ is a prefix of } v \\ u\bar{z}(u\bar{z})^{-1} & \text{otherwise.} \end{cases}$$

Both uu^{-1} and $u\bar{z}(u\bar{z})^{-1}$, for all $\bar{z}$, are in $(yy^{-1})^\uparrow$. Thus $uv^{-1}(xx^{-1})^\uparrow vu^{-1} \subseteq (yy^{-1})^\uparrow$ and a dual argument shows $vu^{-1}(yy^{-1})^\uparrow uv^{-1} \subseteq (xx^{-1})^\uparrow$. □

An obvious corollary of the above result is that periodic subsemigroups are only conjugate to periodic subsemigroups and aperiodic to aperiodic.

Finally we see under which conditions two cyclic subsemigroups are conjugate.

Lemma 3.7.5. *Two cyclic proper closed inverse subsemigroups $P(\mathcal{G})^{x,p}, P(\mathcal{G})^{y,q}$ are conjugate iff p and q are conjugate strings.*

Proof. The details of the proof are the same as in polycyclic case. See [30] for details. □

A cyclic subsemigroup $P(\mathcal{G})^{x,p}$ is purely cyclic if $x = \epsilon$.

Just as for the polycyclic monoids we say that a transitive action of a graph inverse semigroup is *primitive* if the stabilizer of any point is a maximal proper closed inverse monoid. We can now classify the conjugacy classes of proper closed inverse subsemigroups that correspond to primitive actions.

Theorem 3.7.6. *Every proper closed inverse subsemigroup of the graph inverse semigroup that corresponds to a primitive action is conjugate to a closed inverse subsemigroup of one of the following types:*

1. Finite type *where the defining string originates at a source.*

2. Aperiodic type.

3. Pure cyclic type *where p is a primitive string.*

Proof. We already know which closed inverse subsemigroups are conjugate. We now look at the cases when they are maximal.

In the polycyclic case there are no maximal finite subsemigroups. Let x be a path such that $\mathbf{d}(x)$ doesn't have in-arrows. Then x is not the prefix of any other path. Thus $(xx^{-1})^{\uparrow}$ is maximal. It is straight forward to see that this is a necessary and sufficient condition for $(xx^{-1})^{\uparrow}$ to be maximal.

The periodic subsemigroup defined by x, p is simply the idempotents of the cyclic subsemigroup defined by x, p and therefore can not be maximal. This is not the case for aperiodic subsemigroups nor can one aperiodic subsemigroup be contained in another unless they are equal.

To see that cyclic subsemigroups are maximal we add any element not of the form $xp^r\bar{p}(xp^s\bar{p})^{-1}$ or $\bar{x}\bar{x}^{-1}$. This new element will not be comparable to all other elements of $P_t(\mathcal{G})^{x,p}$ and thus the new set can not be proper. Finally if $p = q^r$, $r > 1$ then $P(\mathcal{G})^{x,p} \subseteq P(\mathcal{G})^{x,q}$. Therefore we require p to be primitive for $P(\mathcal{G})^{x,p}$ to be maximal. $\square$

3.8 Representations of the graph inverse monoid

We now quickly discuss the representations of the graph inverse monoid, highlighting the differences to the semigroup case. All results in this section are effectively simple corollaries of the results in Section 3.7. It is straight-forward to see that if S is proper in $P(\mathcal{G})$ then it must be proper in $P_t(\mathcal{G})$. The same is true of upwardly closed and closure under inverses.

Lemma 3.8.1. *Let x be a path in $\mathcal{G}$ terminating at t. Then*

$$(xx^{-1})^{\uparrow} = \{yy^{-1} \in P_t(\mathcal{G}) : y \text{ a prefix of } x\}$$

is a proper closed inverse submonoid of $P_t(\mathcal{G})$.

Proof. Let x is a path in $\mathcal{G}$ terminating at t. As x terminates at t so do all the prefixes of x. Thus $(xx^{-1})^{\uparrow} \subseteq P_t(\mathcal{G})$. It is clear by comparison to the finite case of the graph inverse semigroup that $(xx^{-1})^{\uparrow}$ is also proper, closed, inverse and that it contains the identity of $P_t(\mathcal{G})$. □

Lemma 3.8.2. *Let $w \in \mathcal{G}_t^{\omega}$. Then*

$$(ww^{-1})^{\uparrow} = \{xx^{-1} \in P_t(\mathcal{G}) : x \text{ a prefix of } w\}$$

is a proper closed inverse submonoid of $P_t(\mathcal{G})$.

Proof. If w terminates at t the all its prefixes also terminate at t and so $(ww^{-1})^{\uparrow} \subseteq P_t(\mathcal{G})$. As 1_t is a prefix of w it is now clear that $(ww^{-1})^{\uparrow}$ is a proper closed inverse submonoid of $P_t(\mathcal{G})$. □

We now look at the non-idempotent pure case.

Lemma 3.8.3. *Let p be a cycle on some vertex e and x any path from e to t. Then*

$$P_t(\mathcal{G})^{x,p} = \{xp^r\bar{p}(xp^s\bar{p})^{-1} : r, s \in \mathbb{N}, \bar{p} \text{ is a prefix of } p\} \cup (xx^{-1})^{\uparrow}$$

is a proper closed inverse submonoid of $P_t(\mathcal{G})$.

Proof. As x terminates at t so do all prefixes of x and all paths of the form $xp^r\bar{p}$. Thus $P_t(\mathcal{G})^{x,p} \subseteq P_t(\mathcal{G})$ for all appropriate p and x and thus is a proper closed inverse monoid. □

The conjugacy of these submonoids is exactly the same as in the graph inverse semigroup.

The classification the proper closed inverse submonoid that correspond to actions is almost identical.

Theorem 3.8.4. *Every maximal proper closed inverse submonoid of the graph inverse semigroup which corresponds to a primitive action is conjugate to a closed inverse monoid of one of the following types:*

1. Finite type *where the defining string originates at a source.*

2. Aperiodic type.

3. Cyclic type *where p is a primitive string.*

Proof. Unlike the graph inverse semigroup case cyclic submonoids are not necessarily conjugate to a purely cyclic submonoid. If the defining path p does not pass over t then x can not be empty. With the exception of this fact the proof is the same as the proof in the graph inverse semigroup case give in Theorem 3.7.6. $\square$

3.9 Strong representations

We now describe the actions associated with each type of conjugacy class of proper closed inverse submonoids of both $P(\mathcal{G})$ and $P_t(\mathcal{G})$. Define an action of $P(\mathcal{G})$ on the free category $\mathcal{G}^*$ as follows:

$$xy^{-1} \cdot u = \begin{cases} xp & \text{if } u = yp \text{ for some path } p \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We call this the *natural action* of $P(\mathcal{G})$ on $\mathcal{G}^*$. We define the natural action of $P_t(\mathcal{G})$ on $\mathcal{G}_t^*$ in the obvious way. Both actions are transitive and it is obvious that the stabiliser of the path u is $(uu^{-1})^{\uparrow}$.

Proposition 3.9.1. *The natural action of $P(\mathcal{G})$ ($P_t(\mathcal{G})$) on $\mathcal{G}^*$ ($\mathcal{G}_t^*$) corresponds to the finite proper closed inverse subsemigroups (submonoids).*

Proof. It is clear that the stabiliser of any point $x \in \mathcal{G}^*(\mathcal{G}_t^*)$ is the finite proper closed inverse subsemigroup (submonoid) $(xx^{-1})^{\uparrow}$. $\square$

Define an action of $P(\mathcal{G})$ on $\mathcal{G}^\omega$ as follows:

$$xy^{-1} \cdot u = \begin{cases} xp & \text{if } u = yp \text{ for some infinite path } p \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We call this the natural action of $P(\mathcal{G})$ on $\mathcal{G}^\omega$. Earlier we defined $\mathcal{G}^\omega$ as the set of right infinite paths from $\mathcal{G}$, $\mathcal{G}_t^w$ is the subset of paths that terminate at t. Again we define the natural action of $P_t(\mathcal{G})$ on $\mathcal{G}_t^w$ in the obvious way.

Neither action is transitive so we look to the orbits.

Proposition 3.9.2. *The natural action of $P(\mathcal{G})$ ($P_t(\mathcal{G})$) on $\mathcal{G}^\omega$ ($\mathcal{G}_t^\omega$) corresponds to the disjoint union of all the aperiodic proper closed subsemigroups (submonoids) and all the cyclic proper closed subsemigroups (submonoids) with each one appearing exactly once.*

Proof. Again it is clear that the stabiliser of any point $w \in \mathcal{G}^\omega(\mathcal{G}_t^\omega)$ is the infinite proper closed inverse subsemigroup (submonoid) $(ww^{-1})^\uparrow$. These subsemigroups (submonoids) are in bijective correspondence with the elements of $\mathcal{G}^\omega(\mathcal{G}_t^\omega)$, as such each such subsemigroup (submonoid) appears exactly once. □

A representation of P_n on X is said to be strong if

$$X = \bigcup_{i=1}^{n} a_i \cdot X.$$

This case was introduced in the book [27]. It was independently formulated by Kawamura [22, 23] in terms of *branching function systems*, who then classified what we would call the transitive strong representations. In [30], the second author reproved Kawamura's results using the theory of transitive inverse semigroup actions. In [20] (and chapter two) the authors showed that the monograph [5] could be understood purely in terms of strong representations of polycyclic monoids. In this section we generalise this idea to the graph inverse semigroups.

For each vertex v of $\mathcal{G}$ let X_v be a non-empty set. We put X equal to the disjoint union of the X_v. A *strong action* of $P(\mathcal{G})$ translates into the following condition:

$$X_v = \bigcup_{\mathbf{r}(a)=v} a \cdot X_{\mathbf{d}(a)}, \forall v \in \mathcal{G}_0.$$

In this case there is only one vertex so the relation becomes

$$X = \bigcup_{a \in \mathcal{G}_1} a \cdot X.$$

However we can derive this simpler relation for any graph.

Lemma 3.9.3. *If $P(\mathcal{G})$ acts strongly on X then*

$$X = \bigcup_{a \in \mathcal{G}_1} a \cdot X.$$

Proof. Define $Y_v = 1_v \cdot X_v = \{x \in X_v : 1_v \cdot x \text{ is defined}\}$. By showing $X_v \subseteq Y_v$ we will have that the two sets are equal. Let $x_0 \in X_v$. Then $x_0 = a \cdot x_1$ for some $a \in \mathcal{G}$ such that $\mathbf{r}(a) = v$ and $x_1 \in X_{\mathbf{d}(a)}$. As representations are homomorphisms we have

$$1_v \cdot x_0 = 1_v \cdot (a \cdot x_1) = (1_v a) \cdot x_1 = a \cdot x_1 = x_0.$$

Thus $1_v \cdot x_0$ is defined and $X_v \subseteq Y_v$, giving $X_v = Y_v$. With this characterisation of the X_v we have

$$\bigcup_{\mathbf{r}(a)=v} a \cdot X = \bigcup_{\mathbf{r}(a)=v} a \cdot 1_{\mathbf{d}(a)} \cdot X = \bigcup_{\mathbf{r}(a)=v} a \cdot X_{\mathbf{d}(a)} = X_v.$$

Also $X = \bigcup_{v \in \mathcal{G}_0} X_v$ as our actions are effective and

$$\bigcup_{v \in \mathcal{G}_0} \bigcup_{\mathbf{r}(a)=v} a = \bigcup_{a \in \mathcal{G}_1} a.$$

Therefore

$$X = \bigcup_{v \in \mathcal{G}_0} X_v = \bigcup_{v \in \mathcal{G}_0} \bigcup_{\mathbf{r}(a)=v} a \cdot X = \bigcup_{a \in \mathcal{G}_1} a \cdot X.$$

$\square$

The natural action of $P(\mathcal{G})$ on $\mathcal{G}^*$ is not strong for the same reason that natural action of P_n on A_n^* is not strong. The identities in $\mathcal{G}^*$ can not be expressed as and edge acting on a path. Now let us look at the natural action on $\mathcal{G}^\omega$.

Proposition 3.9.4. *The natural action of $P(\mathcal{G})$ on $\mathcal{G}^\omega$ is strong.*

Proof. For each $w \in \mathcal{G}^\omega$ we can write $w = w_1 w_2 \dots$ for $w_i \in \mathcal{G}_1$. We can partition $\mathcal{G}^\omega$ into the following sets:

$$\mathcal{G}_v^\omega = \{w \in \mathcal{G}^\omega : \mathbf{r}(w_1) = v\}.$$

From here we can let a act. If $\mathbf{d}(a) = v$ then $a \cdot \mathcal{G}_v^\omega = \{w \in \mathcal{G}^\omega : w_1 = a\}$. If $\mathbf{d}(a) \neq v$ then $a \cdot \mathcal{G}_v^\omega$ is undefined. The set $a \cdot \mathcal{G}_v^\omega$ is a subset of $\mathcal{G}_{\mathbf{r}(a)}^\omega$. We can obtain $\mathcal{G}_{\mathbf{r}(a)}^\omega$ by taking the appropriate union. That is

$$\mathcal{G}_{\mathbf{r}(a)}^\omega = \bigcup_{\mathbf{r}(b)=\mathbf{r}(a)} b \cdot \mathcal{G}_{\mathbf{d}(b)}^\omega.$$

$\square$

The notion of a branching function system can be generalized to graph inverse semigroups. It coincides with what are called *E-algebraic branching systems* in [16], where E is a graph. In our terms, this takes the following form. For each vertex v of G let X_v be a non-empty set. We put X equal to the disjoint union of the X_v. For each edge $v \xleftarrow{e} u$ there is an injective function $X_u \xrightarrow{\theta(e)} X_v$. This strongly suggests developing the theory of strong representations of graph inverse semigroups along the lines of [20] and [5]. McClure [41] and Bartholdi *et al* [3] discuss digraph iterated function systems. These systems have strong connections with fractal tilings. A *digraph iterated function system* $(\mathcal{G}, F_v, f_e)$ with *structural graph* $\mathcal{G}$ is a collection of sets $\{F_v\}_{v \in \mathcal{G}_0}$ together with a collection of injective maps $\{f_e : F_{\mathbf{d}(e)} \to F_{\mathbf{r}(e)}\}_{e \in \mathcal{G}}$, such that for every $v \in \mathcal{G}_0$

$$F_v = \bigcup_{\mathbf{r}(e)=v} f_e(F_{\mathbf{d}(e)}).$$

It is clear that these systems are simply E-algebraic branching systems under another name.

By Lemma 3.6.16 and Theorem 3.6.17 and the fact that the symmetric inverse monoids are complete, every strong representation $\theta\colon P(\mathcal{G}) \to I(X)$ gives rise to a join-preserving homomorphism $\bar{\theta}\colon CK_G \to I(X)$. The relationship between CK_G and $P(\mathcal{G})$ is therefore analogous to the relationship between the group-ring of a group and the group itself. The partial join operation on complete inverse semigroups gives them a ring-like character.

The semigroup CK_G may also explain why the role of inverse semigroups in the theory of graph C^*-algebras is not as apparent as maybe it should be. Although Paterson makes an explicit connection in [45, 46], they are not mentioned in [51] or [55]. We believe that one explanation is that the *Cuntz-Krieger relations* $e = \sum_{f' \in \hat{e}} f'$ cannot be expressed in pure inverse semigroup theory and these would seem to be the essence of Cuntz-Krieger algebras. However, we have shown that by working with a suitable completion of the graph inverse semigroup, these relations can be naturally expressed.

There are therefore three algebraic structures that arise in studying algebras, in the most general sense, arising from directed graphs:

Cuntz-Krieger semigroups, Leavitt path algebras, graph C^*-algebras.

From the theory developed in subsection 3.6.6, we know that the Cuntz-Krieger semigroups can be used to construct, and may be constructed from, the topological groupoids usually associated with the graph C^*-algebras. As a result, we believe that they are basic structures to study.

3.10 Unambiguous semigroups

The notion of unambiguity was first formalized within semigroup theory in [6] and briefly considered in the context of inverse semigroup theory in [26]. However, it was the paper [33] by Lawson that highlighted just how important unambiguous inverse semigroups were. This paper shows that the self-similar groups of [43] are in bijective correspondence with what we would now call 0-bisimple Perrot monoids. This latter terminology was introduced to record the fact that this result is implicit in Perrot's thesis [48] even though this predates the formal introduction of self-similar groups. In its turn, Perrot's work can be seen as a wide-ranging generalization of Rees's [52] pioneering paper, which led to the terminology for the categories we have used in this work. Further evidence for the importance of the class of unambiguous semigroups comes from the work of [19] who studies topological groupoids associated with certain kinds of ultra-metric spaces. The poset of open balls in such a space is unambiguous.

If the 0-bisimple Perrot monoids are constructed from self-similar groups, the question arises of what we can say about more general kinds of Perrot semigroup. The starting point is Theorem 3.5.24 where we described graph inverse semigroups as those Perrot semigroups that are combinatorial and strictly Leech. By Lemmas 3.5.2, 3.5.4, 3.5.10, we have that C is a left Rees category if and only if $\mathbf{S}(C)$ is a Perrot semigroup that is also Leech. The structure of left Rees categories is described in [34]: every left Rees category is isomorphic to a Zappa-Szép product of a free category by a groupoid.

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